UNIVERSITÀ DI PISA

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

CORSO DI LAUREA MAGISTRALE IN MATEMATICA

TESI DI LAUREA MAGISTRALE

Bounded cohomology and the simplicial volume of the product of two surfaces

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Anno Accademico 2010/2011

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Introduction

The simplicial volume is a topological invariant of closed connected oriented manifolds that was first defined by Gromov in the seminal article "Volume and Bounded Cohomology" [Gro82]. This invariant is defined in the context of singular homology: the l^1-norm on singular chains naturally induces a seminorm on homology and the simplicial volume is the value taken by this seminorm on the fundamental class of the manifold.

An interesting feature of the simplicial volume is that it is related to metric properties of the manifold even if its definition is purely topological. Gromov and Thurston proved (respectively in [Gro82] and in [Thu79]) what is known as Gromov's proportionality principle: for every manifold that admits a Riemannian structure, the simplicial volume is proportional to the Riemannian volume and the proportionality constant depends only on the metric covering of the manifold.

The purpose of the thesis is to compute the value of the simplicial volume of the product of two surfaces following Bucher-Karlsson [Buc08B]. Using Gromov's proportionality principle, we will reduce to compute the proportionality constant for the Riemannian manifolds covered by the product of two hyperbolic planes. It is worth remarking that this is the unique nonvanishing proportionality constant that is known, apart from the case of hyperbolic manifolds.

A fundamental step in the computation of this proportionality coefficient, and, in general, in the study of the simplicial volume is a duality theorem (due to Gromov) that translates the problems related to the simplicial volume in a cohomological language. The bounded cohomology of a topological space is the homology of the complex of the bounded singular cochains, i.e. the singular cochains with finite l^∞ norm. It can be proved (Theorem 3.2.2) that the duality at the cochain level between the l^1 norm and the l^∞ norm descends to a duality between the induced seminorms in homology. This implies that the computation of the simplicial volume (i.e. the seminorm of the fundamental class) is equivalent to the computation of the seminorm of the fundamental coclass.

The cohomological translation is useful not only because of the richer structure on the cohomological ring, but also because of the relationships between the bounded cohomology of topological spaces and of groups. The the-

ory of bounded cohomology of discrete groups, studied by Ivanov in [Iva87] is, indeed, a valid tool: many results from classical homological algebra generalize to this context, and this fact allows to choose simpler resolutions whose homology is isometrically isomorphic to the group cohomology making the simplicial volume explicitly computable. This sort of argument will be used in order to compute the proportionality coefficient for manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$.

Recently Burger and Monod (see [Mon01]) generalized the bounded group cohomology to the context of Lie groups (or, more generally, of locally compact topological groups). In this case it is important to consider also the topology on the group and hence it is natural to study the continuous bounded cohomology (i.e. the homology of the complex of the continuous bounded G-invariant cochains). The continuous bounded cohomology has strict, yet not fully understood, relationships with a third cohomological theory (namely the continuous cohomology first studied by Mostow in [Mos61]). All this theories (and the subtle relationships among them) have important geometric applications, particularly in the study of the simplicial volume.

For example, when we restrict to the class of locally simmetric spaces (e.g. manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$), the bounded group cohomology is a key tool for an easy proof of the proportionality principle (this proof is due to Bucher-Karlsson, [Buc08A]). A symmetric space (e.g. $\mathbb{H}^2 \times \mathbb{H}^2$) is the quotient of its isometry group (that we denote by G) with respect to a compact subgroup. Moreover the choice of a metric covering of a manifold M induces an inclusion of the fundamental group of M in G and hence induces, in cohomology, a map res : $H^*(G, \mathbb{R}) \to H^*(\pi_1(M), \mathbb{R})$. Using arguments of group cohomology, it can be proved (Theorem 3.3.10) that, whenever we consider the continuous cohomology of G, the map res is an isometric inclusion. Then, in order to prove the proportionality principle it is sufficient to study the preimage of the fundamental coclass.

An useful tool for this purpose is van Est's Theorem (Theorem 1.7.5) on continuous cohomology. This fundamental theorem implies that the continuous cohomology of a Lie group G can be computed from the complex of the G-invariant differential forms on the homogeneous space G/K where K is a maximal compact subgroup. In particular, starting from this description, it can be easily proved that res maps the class of the volume form of $\mathbb{H}^2 \times \mathbb{H}^2$, belonging to $H_c^*(G, \mathbb{R})$, to the class $\mathrm{vol}(M) \cdot [M]^{\mathbb{R}}$. The proportionality principle follows from this fact, moreover the proof explains what should be computed in order to get the proportionality constant.

In this thesis we will introduce all the ingredients necessary to the study of simplicial volume of locally symmetric spaces following the approach of Bucher-Karlsson with a particular interest towards the computation of the proportionality constant for manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$.

In the first part of the thesis we collect the algebraic prerequisites concerning continuous (bounded) group cohomology. In the first chapter we describe the theory of continuous cohomology for locally compact topological groups that was first introduced by Mostow and Hochschild in 1960 (see [Mos61] and [HoMo62]). We follow the exposition of Borel and Wallach that can be found in [BoWa00]. After the combinatorial definition we develop the useful homological approach that allows us to find many different (and useful) complexes whose homology is the continuous cohomology of a given Lie group. In particular we discuss the resolutions provided by the continuous functions, the locally integrable functions and the functions from the homogeneous space associated to G. All these resolutions will be useful in the study of the simplicial volume in the second part of the thesis. At the end of Chapter 1 we prove Van Est's Theorem providing also an explicit description (at the cochain level) of the map that induces the isomorphism in cohomology.

In the second chapter we focus on the theory of continuous bounded cohomology as described in Monod's monography [Mon01]. We find analogies and differences with respect to the theory of continuous cohomology described in the first chapter and we follow, at least in the first sections, the structure of Chapter 1. In the second part of the chapter we depart from that treatment: no analogue of Van Est's Theorem is known to hold in the context of continuous bounded cohomology. Instead we describe the theory of amenability for topological groups that is deeply related to continuous bounded cohomology. A topological group is amenable if there exists a G-invariant projection $m : L^{\infty}(G; \mathbb{R}) \to \mathbb{R}$. This theory is a very useful tool since, on one hand it provides many unexpected vanishing results (for example the continuous bounded cohomology of any abelian group is null), on the other hand it will allow us to choose, when computing the continuous bounded cohomology, smaller complexes: taking advantage of the theory of amenability we show, for example, that the continuous cohomology of $\mathrm{PSL}_2(\mathbb{R})$ can be computed from the complex of the invariant functions from S^1 (regarded as the quotient of $\mathrm{PSL}_2(\mathbb{R})$ with respect to an amenable subgroup). In the last section (Section 2.7) we define the bounded cohomology for topological spaces and discuss a deep theorem of Gromov that relates the bounded cohomology of a topological space and that of its fundamental group.

The second part of the thesis is devoted to the study of simplicial volume. In the third chapter we focus on general properties of the simplicial volume: we begin by giving the definiton of this topological invariant and studying its first properties (e.g. the behaviour under finite coverings and some vanishing results); then we translate the problem in a cohomological setting proving the duality principle. As a first application we give extimates on the simplicial volume of the product of manifolds. Starting from Section 3.3 we focus on locally symmetric spaces: after briefly recalling their defini-

tion and their properties, we prove the proportionality principle for locally symmetric spaces following the approach of Bucher-Karlsson ([Buc08A]). As an example we consider the hyperbolic case in which the cohomological proof of the proportionality principle can be used in order to easily compute the simplicial volume. We compute the simplicial volume also a homological setting and compare the two approaches.

In the last chapter of the thesis we compute the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. Let us denote by $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ the connected component of the identity of the isometries of $\mathbb{H}^2 \times \mathbb{H}^2$. As a consequence of the discussion of the third chapter, it is sufficient to compute the seminorm of the image, under van Est's isomorphism, of the volume form in $H_c^4(G; \mathbb{R})$. In the first sections of the chapter (Sections 4.1 to 4.3) we take advantage of the homological algebra developed in the first chapters in order to find a small complex in which the seminorm is actually computable combinatorically. The suitable complex is the one given by the bounded measurable alternating functions from $S^1 \times S^1$, that we regard as the product of the boundaries of the two hyperbolic factors. This complex is useful for two reason: the first is that G acts transitively on the triples of points, the other is that an even permutation of the vertices of a 4-simplex can be realized by an odd isometry; both this properties will be crucial in the proof of the fact that our chosen representative of the class is, indeed, of minimal norm. However $S^1 \times S^1$ is the quotient of G with respect to a minimal parabolic subgroup that is amenable but not compact. This implies that our chosen complex is suitable for computing the continuous bounded cohomology of G but not the continuous cohomology. In order to avoid this difficulty we will have to show (in the whole Section 4.5) that the comparison map, i.e. the map induced by the inclusion of the bounded cochains in the classical cochains, induces an isomorphism $c : H_{cb}^4(G; \mathbb{R}) \to H_c^4(G; \mathbb{R})$.

Chapter 1

Continuous cohomology of topological groups

Let G be a group and R be a ring. The cohomology of G with ring of coefficients R can be defined as the cohomology of the complex formed by the functions which are invariant with respect to a natural action that will be properly introduced in the next section:

$$C^n(G, R)^G = \{\phi : G^{n+1} \to R \,|\, \phi \text{ is } G\text{-invariant}\}.$$

This definition has been first given by Eilenberg and Mac Lane in in 1943 and is now classical (see, for example, the first Chapter of [Gui80]).

When G is a topological group, this definition can be slightly modified (taking in account only continuous cochains) to obtain the *continuous cohomology* of G. The aim of this chapter is to define the continuous cohomology: it will be a fundamental tool in the study of the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$.

A central theorem in the theory of continuous cohomology is van Est's Theorem (Theorem 1.7.5) that describes the continuous cohomology of a Lie group G as a quotient of the G-invariant differential forms on a adequate symmetric space. This resul will be crucial in our geometric applications and makes sense only when the coefficients' module is a vector space.

In order to prove van Est's Theorem (and many other theorems in this context), we will need subtle results on continuous cohomology that we will deduce from some hard algebraic machinery. Since the coefficients we will be interested in later are finite dimensional vector spaces, and homological algebra works better in the context of topological vector spaces, we will restrict to this class of coefficients (even if the definition of continuous cohomology makes sense for a broader class of coefficients). Anyway we will not restrict to the case of finite dimensional vector spaces: infinite dimensional vector spaces naturally arise in the proofs even when one starts with finite dimensional vector spaces. On the contrary the category of Frechet separa-

ble vector spaces is closed under the needed constructions, and this is the reason why we work in this category.

1.1 Combinatorial definition

Let G be a Hausdorff, locally compact topological group that admits an exhaustion of compacts. Since in our applications G will be indeed a Lie group, more precisely the Lie group of the isometries of a symmetric space, this assumption is useful but not restrictive. A *topological G-module* is a Frechet separable vector space V endowed with a representation $\pi : G \to \mathrm{Aut}(V)$ that is strongly continuous: this means that the induced map $G \times V \to V$ has to be jointly continuous. When this does not cause confusion, we will omit explicit reference to the representation and we will simply write $g \cdot_V v$, $g \cdot v$ or also gv instead of $\pi(g)(v)$. We will often call the representation a G-action.

Given a G-module V we will denote by V^G the elements of V that are invariant with respect to the G action:

$$V^G = \{ v \in V \mid \quad g \cdot v = v, \quad \forall g \in G \}.$$

Let us fix a topological G-module V (that will usually be $\mathbb{R}$ with the trivial G-action). The module of the *continuous n-cochains* from G to V is

$$C_c^n(G, V) = \{ \phi : G^{n+1} \to V \mid \phi \text{ is continuous} \}. \tag{1.1}$$

We can consider, on $C_c^n(G, V)$, the G-action given by:

$$(g \cdot \phi)(g_0, \ldots, g_n) = g\phi(g^{-1}g_0, \ldots, g^{-1}g_n).$$

A coboundary operator $\delta : C_c^n(G, V) \to C_c^{n+1}(G, V)$ is defined by the formula:

$$\delta(\phi)(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \phi(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).$$

Since the projection

$$\pi_i : G^{n+2} \to G^{n+1}$$
$$(g_0, \ldots, g_{n+1}) \mapsto (g_0, \ldots, \hat{g}_i, \ldots, g_{n+1})$$

is continuous, δ is well defined (i.e. it maps continuous cochains to continuous cochains), moreover it is obvious that δ commutes with the G-action. This implies that we may define the subcomplex of $C_c^*(G, V)$ given by:

$$0 \longrightarrow C_c^0(G, V)^G \overset{\delta}{\longrightarrow} C_c^1(G, V)^G \overset{\delta}{\longrightarrow} C_c^2(G, V)^G \longrightarrow \cdots.$$

Definition 1.1.1. The *continuous cohomology* of the topological group G with coefficients in the topological G-module V is the homology of the complex of continuous cochains:

$$H_c^*(G, V) = H_*(C_c^*(G, V)^G).$$

It is worth remarking that the classical cohomology of abstract groups can be comprehended in this theory: if G is an abstract group, we can endow it with the discrete topology and consider its continuous cohomology. Since G is discrete, the continuity condition in (1.1) is empty, and hence

$$H_c^*(G, V) = H^*(G, V)$$

where $H^*(G, V)$ is the classical cohomology of G mentioned in the introduction of this chapter.

1.2 Some homological algebra

In this Section we will develop concepts coming from homological algebra that are useful when dealing with the continuous cohomology of locally compact groups. The notion of injectivity is classical in group cohomology and has been adapted to continuous cohomology by Mostow [Mos61] and Hochschild [HoMo62] around 1960. We will follow the approach of Borel and Wallach described in Chapter IX of [BoWa00].

We work in the category of topological G-modules whose definition was given in Section 1.1:

Definition 1.2.1. A *topological G-module* is a Frechet vector space V endowed with a strongly continuous representation $\pi : G \to GL(V)$, where $GL(V)$ denotes the continuous linear automorphisms of V.

Lemma 1.2.2. *The representation is strongly continuous if and only if, for every v in V, the map $G \to V$ defined by $g \mapsto gv$ is continuous (if this second condition holds, we will call the representation separately continuous).*

Proof. Obviously, if the representation is strongly continuous, it is also separately continuous: the map $g \mapsto gv$ is the composition of the continuous inclusion $G \to G \times V$ given by $g \mapsto (g, v)$ and of the map $G \times V \to V$ induced by the representation (that is continuous by the assumption that the representation is strongly continuous).

Let us prove the converse implication. We first show that, provided the representation is separately continuous, the image, under π, of any compact subset K of G is equicontinuous. Indeed, since π is separately continuous, the map $\pi : G \to GL(V)$ is continuous when we endow $GL(V)$ with the topology of the pointwise convergence. This implies that, for every compact subset

K of G, its image is compact in that topology. Banach-Steinhaus Theorem implies that, under this assumption, $\pi(K)$ is equicontinuous (since, for every $v \in V$, the set $\pi(K)v$ is bounded).

Let us now prove that the representation is strongly continuous. Both G (being Hausdorff and locally compact) and V (being Frechet) are metrizable. Let us choose $g_n \to g$ in G and $v_n \to v$ in V, it is enough to show that $g_n v_n$ converges to gv. By triangular inequality we get $d(g_n v_n, gv) \leq d(g_n v_n, g_n v) + d(g_n v, gv)$. Since G is locally compact, we can assume that every g_n belongs to a fixed compact neighborhood of e and hence $\pi(g_n)$ are equicontinuous. This implies that the first term is small if v_n is sufficiently close to v. The continuity of $g \mapsto gv$ ensures that the second term is small when g_n is close to g. $\qquad\square$

The natural arrows in the category of topological G-modules are morphisms that preserve the representations: a *G-morphism* between two G-modules $\phi : V \to W$ is a continuous linear map that commutes with the G action. This means that, for every $v \in V$ and $g \in G$, we have $\phi(g \cdot_V v) = g \cdot_W \phi(v)$.

A G-morphism between two topological G-modules $\phi : V \to W$ is said to be *continuously strongly injective* (or simply strongly injective) if it has a continuous left inverse, i.e. if there exists a continuous linear map $\sigma : W \to V$ such that $\sigma \cdot \phi = \mathrm{id}_V$. Note that σ ought not to be a G-map.

Obviously, if ϕ is continuously strongly injective, ϕ is also injective. Moreover, if ϕ is continuously strongly injective, $(\phi\sigma)^2 = \phi\sigma$ (since $\sigma\phi = \mathrm{id}$) so $\phi\sigma$ is a continuous projector onto a closed subspace W'. This implies that W splits as a direct sum $W = W' \oplus W''$, where $W'' = \ker(\phi\sigma)$ and both W', W'' are closed vector subspaces of W. Indeed $W' = \mathrm{im}(\phi) \cong V$ (because ϕ is injective). Since ϕ is a topological G-morphism, W' is G-invariant, but W'' ought not to be a G-module and so σ, that coresponds to the projection on the first factor of the direct sum, is not necessarily a G-morphism.

The definition of continuously strongly injective morphism was necessary to introduce the notion of relative injectivity for a topological G-module. This notion will allow us to adapt tools from classical homological algebra to the context of continuous cohomology.

Definition 1.2.3. A topological G-module Z is *continuously relatively injective* if, for every continuously strongly injective G-morphism $\phi : V \to W$, for every G-morphism $\alpha : V \to Z$, there exists a G-morphism $\beta : W \to Z$ such that $\beta\phi = \alpha$.

$$
\begin{array}{ccc}
V & \overset{\sigma}{\underset{\phi}{\rightleftarrows}} & W \\
\big\downarrow{\alpha} & \overset{\beta}{\diagdown} & \\
Z & &
\end{array}
$$

We will sometimes call the map β an extension of α: we have already remarked that, since ϕ is continuous strongly injective, it corresponds to an inclusion of V as a G-subspace of W, the morphism β extends the morphism α already defined on this subspace. The following lemma is useful for proving the relative injectivity of many modules:

Lemma 1.2.4. *Let W be a continuously relatively injective G-module. Assume that $\alpha : V \to W$ and $\beta : W \to V$ are G-morphisms such that $\beta \circ \alpha = id_V$. Then V is relatively injective.*

Proof.

$$
\begin{array}{c}
A \underset{\phi}{\overset{\sigma}{\rightleftarrows}} B \\
\gamma \downarrow \quad \diagdown \\
V \quad \diagdown \delta \\
\alpha \downarrow \uparrow \beta \\
W
\end{array}
$$
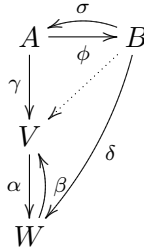
Let A and B be two G-modules, let $\phi : A \to B$ be a strongly injective G-morphism, and let $\gamma : A \to V$ be any G-morphism. Since $\alpha\gamma : A \to W$ is a G-morphism, as a consequence of the relatively injectivity of W, there exists a G-morphism $\delta : B \to W$ such that $\delta\phi = \alpha\gamma$. The composition $\beta\delta : B \to V$ is a G-morphism such that $(\beta\delta)\phi = \beta(\alpha\gamma) = \gamma$.

$\qquad\square$

If V is a topological G-module, a *G-resolution* of V is an exact sequence (F^i, d) of topological G-modules and G-morphisms.

A resolution is *strong* if there exist continuous maps $k^i : F^i \to F^{i-1}$ such that $k^{i+1}d^i + d^{i-1}k^i = \mathrm{id}_{F^i}$. Note that, as in the definition of strongly injective G-morphism, we do not require the contracting homotopy k^* to be made of G-morphisms.

$$0 \longrightarrow V \underset{}{\overset{k^0}{\rightleftarrows}} F^0 \underset{d^0}{\overset{k^1}{\rightleftarrows}} F^1 \underset{d^1}{\overset{k^2}{\rightleftarrows}} F^2 \underset{d^2}{\overset{k^3}{\rightleftarrows}} F^3 \underset{d^3}{\longrightarrow} \cdots .$$

Remark 1.2.5. As in the definition of strongly injective morphism, a contracting homotopy induces a splitting of the topological vector spaces F^i into a direct sum of closed subspaces (of whom only one must be G-invariant).

To be more precise, but omitting apices for the sake of brevity, let us consider the continuous automorphism kd of F^i, it is a projector on a closed subspace V^i:

$$
\begin{aligned}
(kd)^2 \ &= k(dk)d = \\
&= k(\mathrm{id})d - kkdd = \\
&= kd
\end{aligned}
$$

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and hence induces a splitting of F^i as the direct sum $V^i \oplus Q^i$ where we denoted by Q^i the kernel of kd. The subspace Q^i is indeed the kernel of the boundary operator d: obviously $\ker d \subseteq Q^i$, moreover $dQ^i = kddQ^i + dkdQ^i = 0$. Since d is a G-morphism Q^i is G-invariant.

We have just proved that, if (F^*, d) is a strong resolution, each vector space F^i splits as $Q^i \oplus V^i$ where $Q^i = \ker d$ is a G-invariant closed subspace. Moreover $d : V^i \to Q^{i+1}$ is a topological isomorphism (it is bijective because $V^i \cap \ker d = \emptyset$, and $Q^{i+1} = \mathrm{im} d$ since the resolution is exact; a continuous inverse of d is given by the restriction $k| : Q^{i+1} \to V^i$).

The next theorem is a continuous version of of the fundamental theorem of homological algebra:

Theorem 1.2.6 (Uniqueness of the resolution). *Let (F_1^*, d_1), (F_2^*, d_2) be two strong G-resolutions of the topological G-module V. Assume also that F_1^i and F_2^i are relatively injective for all $i \in \mathbb{N}$. Then, for every i, there exists a G-morphism $h_i : F_1^i \to F_2^i$ such that the diagram below commutes. Moreover the h_i's are unique up to continuous G-chain homotopy.*

$$
\begin{array}{ccccccccc}
V & \longrightarrow & F_1^0 & \xrightarrow{\ d\ } & F_1^1 & \xrightarrow{\ d\ } & F_1^2 & \xrightarrow{\ d\ } & \cdots \\
\Big\| {\scriptstyle id} & & \Big\downarrow {\scriptstyle h_0} & & \Big\downarrow {\scriptstyle h_1} & & \Big\downarrow {\scriptstyle h_2} & & \\
V & \longrightarrow & F_2^0 & \xrightarrow{\ d\ } & F_2^1 & \xrightarrow{\ d\ } & F_2^2 & \xrightarrow{\ d\ } & \cdots
\end{array}
$$

Proof. Consider the short strongly exact sequence of topological G-modules:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q_1^i & \underset{j_1}{\overset{dk}{\rightleftarrows}} & F_1^i & \xrightarrow{\ d\ } & Q_1^{i+1} & \longrightarrow & 0 \\
& & \Big\downarrow {\scriptstyle g_i} & & \Big\downarrow {\scriptstyle h_i} & & \Big\downarrow {\scriptstyle g_{i+1}} & & \\
0 & \longrightarrow & Q_2^i & \underset{j_2}{\overset{dk}{\rightleftarrows}} & F_2^i & \xrightarrow{\ d\ } & Q_2^{i+1} & \longrightarrow & 0
\end{array}
$$

where, accordingly with the notations of the previous remark, $Q_k^i = \ker d_k^i \subset F_k^i$. The inclusions j_k are G-morphisms since Q_k^i is a G-submodule. Moreover $j_1 : Q_1^i \to F_1^i$ is strongly injective (an inverse is provided by $d^{i-1} \circ k^i$). Let us prove, by induction on i, that there exist G-morphisms $h_i : F_1^i \to F_2^i$ unique up to G-chain homotopy and these maps induce G-morphisms $g_{i+1} : Q_1^{i+1} \to Q_2^{i+1}$.

To begin the induction, note that $Q_k^0 = V$ so that $g_0 = \mathrm{id}$ is already defined. Suppose then that g_i has been defined. Since F_2^i is relatively injective by assumption and $j_2 \circ g_i$ is a G-morphism, there exists a map h_i making the diagram commutative (we have already pointed out that j_1 is strongly injective). Moreover $Q_k^{i+1} \cong F_k^i / Q_k^i$ as a topological G-module (Q_k^i is G-invariant and closed). Since $h^i(Q_1^i) \subseteq Q_2^i$, the map h^i induces a continuous G-morphism $g^{i+1} : Q_1^{i+1} \to Q_2^{i+1}$.

The proof of the uniqueness up to chain homotopy is similar: suppose that h_*, h'_* are chain G-morphisms that extend the identity in degree -1, we have to prove that there exists a chain homotopy $T : F_1^i \to F_2^{i-1}$ such that $h - h' = Td + dT$. Again (by induction on i) suppose that such a homotopy has been defined for every $n \le i + 1$.

$$\xymatrix{ \longrightarrow F_1^{i-1} \ar[r]^{d} & F_1^i \ar[r]^{d} & F_1^{i+1} \ar[r]^{d} & F_1^{i+2} \\ \longrightarrow F_2^{i-1} \ar[r]^{d} & F_2^i \ar[r]^{d} & F_2^{i+1} }$$

Consider the morphism $h - h' - dT : F_1^{i+1} \to F_2^{i+1}$, it is enough to show that there exists a G-morphism $T : F_1^{i+2} \to F_2^{i+1}$ such that $h - h' - dT = Td$. By inductive hypothesis $h - h' - dT|_{Q^{i+1}} = 0$: we have already pointed out that $Q_1^{i+1} = \mathrm{im}(d)$ and

$$\begin{aligned} hd - h'd - dTd &= dh - dh' - dTd = \\ &= d(h - h' - Td) = \\ &= ddT = 0. \end{aligned}$$

As in the first part of the proof, $h - h' - dT$ induces a G-morphism $a : Q_1^{i+2} \cong F_1^{i+1}/Q_1^{i+1} \to F_2^{i+1}$. Since F_2^{i+1} is relatively injective and j is strong, there exists the desired map T:

$$\xymatrix{ Q_1^{i+2} \ar[r]^{j} \ar[d]_{a} & F_1^{i+2} \ar@{.>}[dl]^{T} \\ F_2^{i+1} & }.$$

$\square$

Corollary 1.2.7. *Fix (F_1^*, d_1), (F_2^*, d_2) two strong relatively injective resolutions of the topological G-module V, and consider $((F_1^*)^G, d_1)$, $((F_2^*)^G, d_2)$ the subcomplexes of the G-invariants. Then the homology groups of the two complexes are isomorphic:*

$$H_*((F_1^*)^G, d_1) \cong H_*((F_2^*)^G, d_2)$$

Proof. Theorem 1.2.6 ensures the existence of G-morphisms $h_i : F_1^i \to F_2^i$ and, arguing by symmetry, $h'_i : F_2^i \to F_1^i$. The compositions $h_i \circ h'_i$ and $h'_i \circ h_i$ are chain G-morphisms that extend the identity of V. Since also the identity $\mathrm{id} : F_j^i \to F_j^i$ is a chain G-morphism that extends the identity on V, the second statement of Theorem 1.2.6 ensures that $h_i \circ h'_i$ and $h'_i \circ h_i$ are G-chain homotopic to the identity.

Since all the maps just constructed are G-morphisms, they restrict to the subcomplexes of the G-invariants inducing, in homology, the required isomorphisms.

$\square$

Theorem 1.2.6 will allow us to define the continuous cohomology of a topological group starting from different resolutions. Infact we will prove that the complex $(C_c^*(G, V), d)$ is strong and relatively injective. Once this result is estabilished, we will conclude that the continuous cohomology of a topological group can be computed from an arbitrary strong, relatively injective resolution of the G-module of the coefficients.

1.3 The standard resolution

We will consider the topological vector space of the continuous functions from G to V

$$C_c(G, V) = \{\phi : G \to V \,|\, \phi \text{ is continuous}\}$$

endowed with the compact-open topology. It can be proved that, provided that V is a Frechet separable space, also $C_c(G, V)$ is. It is a standard fact that, since G is locally compact and σ-compact and V is metrizable, the compact-open topology coincides with the topology of the uniform convergence on compact subsets.

We can endow this space with two different (but both useful) G-actions:

- the regular left representation

$$(l_g \cdot \phi)(g_0) = g \cdot_V \phi(g^{-1} g_0)$$

- the regular right representation:

$$(r_g \cdot \phi)(g_0) = \phi(g_0 g).$$

Lemma 1.3.1. *The regular left representation is continuous.*

Proof. As a consequence of Lemma 1.2.2, it is sufficient to show that the map $g \mapsto l_g \phi$ is continuous for every ϕ in $C_c(G, V)$.

The sets of the form

$$\mathcal{U}(K, B(x, \epsilon)) = \{f \in C_c(G, V) \,|\, d(f(k), x) \le \epsilon,\ k \in K\},$$

where K is a compact subset of G and x is a point in V, are a prebasis of the compact open topology of $C_c(G, V)$. Since a basis of the compact-open topology can be obtained considering the finite intersections of elements of the form $\mathcal{U}(K, B(x, \epsilon))$, it is sufficient to show that, if ϕ belongs to $\mathcal{U}(K, B(x, \epsilon))$, and $\bar{g}$ is sufficiently small, then $\bar{g}\phi$ belongs to $\mathcal{U}(K, B(x, \epsilon))$.

$$\begin{aligned} d((\bar{g}\phi)(k), x) &= d(\bar{g}\phi(\bar{g}^{-1}k), x) \\ &\leq d(\bar{g}\phi(\bar{g}^{-1}k), \bar{g}x) + d(\bar{g}x, x) \\ &\leq cd(\phi(\bar{g}^{-1}k), x) + d(\bar{g}x, x). \end{aligned}$$

In the last inequality we used a refinement of Banach-Steinhaus Theorem on the representation π of G on V: we can choose a sufficiently small compact neighborhood W of e in G such that, not only the image $\pi(W)$ is equicontinuous, but also the constant of equicontinuity (on the compact set $Wk - x$) can be choosen to be smaller than any constant $c > 1$. The thesis follows from the continuity of ϕ (that allows to control the first summand) and that of the representation of G on V. $\qquad\square$

It is even easier to verify that the regular right representation is continuous (since the action of G on V is not involved) and hence both actions make $C_c(G, V)$ a topological G-module.

Indeed the two representations are equivalent: there exists an operator intertwining the two representations, i.e. a topological isomorphism T making the diagram

$$\begin{array}{ccc} C_c(G,V) & \xrightarrow{\ T\ } & C_c(G,V) \\ \Big\downarrow{l_g} & & \Big\downarrow{r_g} \\ C_c(G,V) & \xrightarrow{\ T\ } & C_c(G,V) \end{array}$$

commutative for every g in G.

Let us consider the map $T : C_c(G, V) \to C_c(G, V)$ defined by $(T\phi)(x) = x\phi(x^{-1})$. It is an involution since $TT\phi(x) = x(T\phi(x^{-1})) = xx^{-1}\phi(x)$. Moreover it intertwins the two representations:

$$\begin{aligned} l_g T f(x) &= g((Tf)(g^{-1}x) \\ &= gg^{-1}xf(x^{-1}g)) \\ &= x(r_g f(x^{-1})) \\ &= T(r_g f)(x). \end{aligned}$$

Proposition 1.3.2. *$C_c(G, V)$ endowed with the regular right representation is relatively injective.*

Proof.

Let b denote a generic element in B, define β by putting $\beta b(g) = \alpha \sigma g b(e)$.

- The map βb is continuous (and hence β is well defined): since the action $G \times B \to B$ is continuous, the map $G \to B$ sending g to gb is also continuous. This implies that $\beta b : G \to V$ is continuous because it is a composition of continuous maps.

- The G-invariance is obvious: $\beta(hb)(g) = \alpha \sigma ghb(e) = \beta b(gh) = r_h \beta b(g)$.

- The map β is continuous: let us consider an open neighborhood $\mathcal{U}(K, U)$ of $\beta(\bar{b})$ in the compact open topology. We want to find an open neighborhood W of $\bar{b}$ such that $\beta(W) \subseteq \mathcal{U}$. The existence of such a neighborhood is guaranteed from the continuity of the map $K \times B \to V$, $(g, b) \mapsto \beta(b)(g)$ and the compactness of K: since $\beta(\bar{b})$ belongs to $\mathcal{U}(K, U)$, the image of $K \times \{\bar{b}\}$ is contained in the open set U and, since K is compact, we can find a product neighborhood of $K \times \{\bar{b}\}$ with the same property.

- Also the fact that $\beta \phi = \alpha$ follows from an easy computation:

$$
\begin{aligned}
\alpha \sigma g \phi(a)(e) \ &= \alpha \sigma \phi g(a)(e) \\
&= \alpha ga(e) \\
&= g\alpha a(e) \\
&= \alpha a(e).
\end{aligned}
$$

$\square$

Remark 1.3.3. Since T is an involution that conjugates the regular left representation with the regular right representation, also $C_c(G, V)$ endowed with the left G-module structure is relatively injective (Lemma 1.2.4).

We will now turn back to the modules $C_c^n(G, V) = C_c(G^{n+1}, V)$ on which we will always consider the (diagonal) left representation already defined in Section 1.1:
$$(g\phi)(g_0, \ldots, g_n) = g\phi(g^{-1}g_0, \ldots, g^{-1}g_n).$$

Proposition 1.3.4. *There exists an isomorphism of G-modules $C_c^n(G, V) = C_c(G^{n+1}, V) \cong C_c(G, C_c^{n-1}(G, V))$, where $C_c(G, C_c^{n-1}(G, V))$ is endowed with the regular left representation.*

Proof. Let us consider the map

$$
\begin{aligned}
\Psi : C_c^n(G, V) &\to C_c(G, C_c^{n-1}(G, V)) \\
\Psi(\phi)(g_0)(g_1, \ldots, g_n) &= \phi(g_0, g_1, \ldots, g_n)
\end{aligned}
$$

It is easy to show that it is well defined and bijective: since G is locally compact, a map $\phi : G^{n+1} \to V$ is jointly continuous if and only if, for every g in G, the map $\Psi(\phi)(g)$ is continuous and $\Psi(\phi) : G \to C_c^{n-1}(G, V)$ is a continuous map (the proof is analogue to that of Lemma 1.2.2).

$\bullet\,\Psi$ is also bicontinuous: we will show that the compact open topologies coincide under the identification Ψ. A prebasis for the compact-open topology on $C_c(G^n; V)$ is given by the sets

$$\mathcal{U}(K_1 \times \ldots \times K_{n+1}, U) = \{f \in C_c(G^n; V) \,|\, f(K_1 \times \ldots \times K_{n+1}) \subseteq U\}$$

where K_i is a compact in G and U belongs to a basis of the open sets of V. Clearly Ψ restricts to a bijection of the sets

$$\mathcal{U}(K_1 \times \ldots \times K_{n+1}, U) \mapsto \mathcal{U}(K_1, \mathcal{U}(K_2 \times \ldots \times K_{n+1}, U))$$

and hence Ψ is bicontinuous.

$\bullet\,\Psi$ is G-invariant: on one hand

$$\begin{aligned}\Psi(g\phi)(g_0)(g_1, \ldots, g_n) &= (g\phi)(g_0, \ldots, g_n) \\ &= g\phi(g^{-1}g_0, \ldots, g^{-1}g_n),\end{aligned}$$

on the other hand:

$$\begin{aligned}(g\Psi(\phi))(g_0)(g_1, \ldots, g_n) &= (g\Psi(\phi)(g^{-1}g_0))(g_1, \ldots, g_n) \\ &= g(\Psi\phi(g^{-1}g_0)(g^{-1}g_1, \ldots, g^{-1}g_n)) \\ &= g\phi(g^{-1}g_0, \ldots, g^{-1}g_n).\end{aligned}$$

This concludes the proof: the map Ψ is a bijective G-morphism that provides the required isomorphism. $\qquad\square$

Lemma 1.3.5. *The resolution:*

$$0 \longrightarrow V \longrightarrow C_c^0(G, V) \underset{k^1}{\overset{d}{\rightleftarrows}} C_c^1(G, V) \underset{k^2}{\overset{d}{\rightleftarrows}} C_c^2(G, V) \longrightarrow \cdots$$

is strong.

Proof. We will construct the homotopy using the map dual to the continuous inclusion

$$\begin{aligned}i_e : G^n &\to G^{n+1} \\ (g_1, \ldots, g_n) &\to (e, g_1, \ldots, g_n)\end{aligned}$$

This means that we are considering the homotopy

$$k^n : C_c^{n+1}(G, V) \to C_c^n(G, V)$$
$$k^n(\phi)(v_1, \ldots, v_n) = \phi(e, v_1, \ldots, v_n)$$

- It is well defined ($k^n\phi$ is continuous being the composition of continuous maps);

- k^* is continuous with respect to the compact-open topologies (the proof is analogue to that of the continuity of Ψ in Proposition 1.3.4);

- it is a contracting homotopy (an easy computation shows that $kd + dk =$ id).

$\square$

Proposition 1.3.4 combined with Proposition 1.3.2 imply that $C_c^*(G, V)$ is relatively injective and thus, by Lemma 1.3.5, $(C_c^*(G, V), d)$ provides a strong relatively injective resolution of V. A consequence of this fact and of the foundamental Theorem of homological algebra (Theorem 1.2.6) is the following theorem:

Theorem 1.3.6. *Let (F^*, d) be any strong relatively injective resolution of the topological G-module V. Then*

$$H_*((F^*)^G, d) \cong H_c^*(G, V).$$

1.4 Resolution via locally p-integrable functions

As a first application of the algebraic machinery we have developed, we exhibit a different strong relatively injective resolution of a topological G-module V that will be useful in Section 3.4. As a consequence of Theorem 1.3.6, the homology of the subcomplex of G-invariants is canonically isomorphic to $H_c^*(G, V)$.

As usual we will only sketch the proofs omitting many details. Complete proofs can be found in P. Blanc original paper [Bla79] in wich locally p-integrable class functions were first introduced to study continuous cohomology.

In order to construct maps and homotopies, it will be necessary to compute integrals. Since G is, by assumption, Hausdorff and locally compact, we will consider the left Haar measure μ on G, i.e. the unique (up to scaling) left invariant measure on G that is finite on compact sets. We address to [Gui80, Chapter D.2] for definition and properties of integrals of functions with values in a Frechet vector space.

The modules that we will study are:

$$L_{loc}^p(G^{n+1}, V) = \{f : G^{n+1} \to V \,|\, \forall K \subseteq G^{n+1}, \int_K \|f\|_i^p d\mu^{\otimes n+1} \leq \infty\}$$

where $\{\|\cdot\|_i\}$ are the seminorms that define the Frechet space. It can be proved that, if V is a Frechet separable vector space, and G is σ-compact, then $L_{loc}^p(G^n, V)$ is a Frechet separable vector space with the seminorms given by the integration on a family of compact subsets.

The space $C_c(G^n; V)$ is continuously included in $L_{loc}^p(G^n, V)$: the topology of $C_c(G^n; V)$ is finer than the one of $L_{loc}^p(G^n, V)$. Moreover $C_c(G^n; V)$ is dense in $L_{loc}^p(G^n, V)$ as a consequence of standard approximation theorems. The density of $C_c(G^n; V)$ allows to define a G-morphism $d : L_{loc}^p(G^n, V) \to$

$L_{loc}^p(G^{n+1}, V)$ by continuously extending the coboundary operator defined on continuous cochains.

The same arguments imply that the regular left representation of G on $L_{loc}^p(G^{n+1}, V)$ is well defined and continuous: it is the continuous extension of the continuous representation of G on the dense subset $C_c^n(G, V)$.

An application of Fubini-Tonelli's Theorem is the following useful lemma:

Lemma 1.4.1.

$$L_{loc}^p(G^{n+1}, V) \cong L_{loc}^p(G, L_{loc}^p(G^n, V)).$$

Proof. See [Gui80, D.2.2 vii]. □

We will now prove that $(L_{loc}^p(G^{n+1}, V), d)$ is a strong relatively injective resolution of V:

Proposition 1.4.2. *The resolution* $(L_{loc}^p(G^{n+1}, V), d)$ *is strong.*

Proof. We define the contracting homotopy averaging on a compact neighborhood of the identity the cone defined for continuous cochains. Fix a continuous function $\chi : G \to \mathbb{R}$ with compact support K and mean equal to one. For every $\phi \in L_{loc}^p(G^n, V)$, we denote by $\widetilde{\phi}$ the element of $L_{loc}^p(G, L_{loc}^p(G^{n-1}, V))$ corresponding to ϕ under the isomorphism of Lemma 1.4.1 and we set

$$k^n : L_{loc}^p(G^n, V) \to L_{loc}^p(G^{n-1}, V)$$
$$k^n(\phi) = \int_K \chi(g)\widetilde{\phi}(g)d\mu.$$

This provides the contracting homotopy we were looking for. □

It remains to prove that the modules $L_{loc}^p(G^n, V)$ are relatively injective.

Proposition 1.4.3. *The topological G-module* $L_{loc}^p(G^{n+1}, V)$ *is relatively injective.*

Proof. As a consequence of Lemma 1.4.1, it is enough to show the thesis for the module $L_{loc}^p(G, V)$. Since we already know (see Proposition 1.3.2) that $C_c(G, L_{loc}^p(G, V))$ is relatively injective, by Lemma 1.2.4, it is sufficient to exhibit G-morphisms $\alpha : L_{loc}^p(G, V) \to C_c(G, L_{loc}^p(G, V))$, $\beta : C_c(G, L_{loc}^p(G, V)) \to L_{loc}^p(G, V)$ such that $\beta\alpha = \mathrm{id}$.

We set α equal to the constant inclusion, namely $\alpha\phi(g) = \phi$. An easy computation shows that it is a G-morphism (its continuity is obvious). To define the map β let us consider the diagonal inclusion $\Delta : G \to G \times G$. Δ induces a G-morphism $\Delta^* : L_{loc}^p(G \times G, V) \to L_{loc}^p(G, V)$ that induces the required G-morphism β by composition:

$$\beta : C_c(G, L_{loc}^p(G, V)) \hookrightarrow L_{loc}^p(G, L_{loc}^p(G, V)) \cong L_{loc}^p(G \times G, V) \to L_{loc}^p(G, V)$$

The composition $\beta\alpha$ is the identy by definition. □

We have thus proved the main theorem of this section:

Theorem 1.4.4. *Let V be a topological G-module, the homology of the complex*

$$0 \longrightarrow L_{loc}^p(G,V)^G \longrightarrow L_{loc}^p(G^2,V)^G \longrightarrow L_{loc}^p(G^3,V)^G \longrightarrow \cdots.$$

is naturally isomorphic to $H_c^*(G,V)$.

Explicit formulas for the isomorphims can be given: the inclusion of $C_c^n(G,V)$ in $L_{loc}^p(G^{n+1},V)$ induces the isomorphism in cohomology. The homotopical inverse to this map (at the cochain level) has been explicitly defined by P. Blanc in [Bla79] by regularizing L^p functions.

1.5 Other useful complexes

Let K be a compact subgroup of the locally compact group G. We consider the space G/K of left cosets $\{gK \,|\, g \in G\}$ of K in G i.e the quotient of G with respect to the right multiplication of K. We will denote by $\pi : G \to G/K$ the projection. In this section we will study the topological G-module $C_c(G/K;V)$ endowed with the regular left representation.

Proposition 1.5.1. *The topological G-module $C_c(G/K;V)$ is relatively injective*

Proof. As a consequence of Lemma 1.2.4 and Proposition 1.3.2, we need only to construct a left G-inverse α to π^*

$$C_c(G/K;V) \xrightarrow{\pi^*} C_c(G;V) \xrightarrow{\alpha} C_c(G/K;V).$$

To construct the map α it is sufficient to observe that, since the projection π is surjective and G/K is endowed with the quotient topology, the map π^* identifies $C_c(G/K;V)$ with the submodule of $C_c(G;V)$ invariant with respect to the action of K on G via the right multiplication (i.e. the restriction to K of the regular right representation).

Let dx denote the left invariant Haar measure on K, we consider the map $\alpha : C_c(G;V) \to C_c(G;V)$ defined by

$$\alpha\phi(g) = \frac{1}{\mu(K)} \int_K r_x\phi(g)dx = \frac{1}{\mu(K)} \int_K \phi(gx)dx.$$

The map α is well defined (i.e. $\alpha\phi$ is continuous) since we are averaging continuous functions on a compact set. It is G-invariant (with respect to the left action of G) since

$$
\begin{aligned}
l_y\alpha\phi(g) \quad &= y\alpha\phi(y^{-1}g)\\
&= \tfrac{1}{\mu(K)} \int_K yr_x\phi(y^{-1}g)dx\\
&= \tfrac{1}{\mu(K)} \int_K r_xy\phi(y^{-1}g)dx\\
&\alpha(l_y\phi)(g).
\end{aligned}
$$

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Since the Haar measure on K is left invariant, the image of α is contained in the submodule of the right K-invariant functions and hence α induces a map $\widetilde{\alpha} : C_c(G; V) \to C_c(G/K; V)$. It is moreover obvious from the definition of $\widetilde{\alpha}$ that $\widetilde{\alpha}$ is a left inverse to π^*: if ϕ lies in the image of π^* (and hence is K-invariant), we get $\alpha\phi = \phi$. $\qquad\square$

Theorem 1.5.2. *The resolution*

$$0 \longrightarrow V \longrightarrow C_c(G/K; V) \longrightarrow C_c((G/K)^2; V) \longrightarrow \cdots$$

is a strong relatively injective resolution of the topological G-module V.

Proof. The proof of Proposition 1.3.4 applies verbatim (we only used the fact that G was locally compact and that all the spaces were endowed with the compact open topology) in this context to show that $C_c((G/K)^{n+1}, V) \cong C_c(G/K, C_c((G/K)^n, V))$ where both vector spaces are endowed with the regular left representation. As a consequence of Proposition 1.5.1 each module is continuously relatively injective.

It thus remains only to prove that the resolution is strong. The cone operator described in the proof of Lemma 1.3.5 provides the required contracting homotopy. $\qquad\square$

Corollary 1.5.3. *Let G be a Lie group, K a maximal compact subgroup, $M = G/K$ the associated homogeneous space. The continuous cohomology of G can be computed from the complex $(C_c(M^k, V)^G, d)$.*

We will use this result in Theorem 3.4.1 where we compute the continuous cohomology of $\mathrm{Isom}^+(\mathbb{H}^n)$ from the complex $(C_c((\mathbb{H}^n)^k, \mathbb{R})^G, d)$.

1.6 Closed subgroups

In the previous section we have considered a smaller complex than $C_c^*(G; V)^G$ (namely $C_c((G/K)^*; V)^G$) whose homology is isometric to the continuous cohomology of G. It is sometimes useful to consider also bigger complexes with the same homology, expecially when we need to compare the continuous cohomology of different groups.

For this purpose let Γ be a closed subgroup of a topological group G. The aim of this section is to prove the following theorem:

Theorem 1.6.1. *The vector space $C_c(G; V)$ is relatively injective as a topological Γ-module and hence the continuous cohomology of*

$$0 \longrightarrow C_c^0(G; V)^\Gamma \longrightarrow C_c^1(G; V)^\Gamma \longrightarrow C_c^2(G; V)^\Gamma \longrightarrow \cdots$$

is isomorphic to $H_c^*(\Gamma; V)$.

The proof or the relative injectivity of $C_c(G; V)$ as a Γ-module is based on the existence of a generalized Bruhat function (compare [Mon01, page 42])

Definition 1.6.2. Let Γ be a closed subgroup of the locally compact group G. A *generalized Bruhat function* for the action (via left multiplication) of Γ on G is a continuous function $h : G \to \mathbb{R}^+$ such that

- for every x in G, $\int_\Gamma h(\gamma^{-1}x)d\gamma = 1$;

- for every compact subset K of G, $\mathrm{supp}(h) \cap \overline{\Gamma K}$ is compact.

Generalized Bruhat functions exist for every closed subgroup of G provided G is locally compact. This is a non-elementary consequence of the fact that the homogeneous space of right cosets $\Gamma \backslash G$ is paracompact (cfr. [Mon01, Lemma 4.5.4]). We begin proving the existence of such a function in the two easy cases that we will need.

Lemma 1.6.3. *Let $\Gamma < G$ be a subgroup of finite index $n = [G : \Gamma]$, then there exists a Bruhat function for Γ in G.*

Proof. Since Γ is closed and has finite index, the group G is disconnected and is the union of n disjoint copies Γ_i of Γ: the right lateral classes of Γ in G. It is sufficient to consider the sum h of functions h_i compactly supported in Γ_i with mean 1. The first property of a Bruhat function follows from the fact that, for any point x in Γ_i, we have $\Gamma x = \Gamma_i$; the second frome the fact that, since $\mathrm{supp}(h)$ is compact, also $\mathrm{supp}(h) \cap \overline{\Gamma K}$ is compact. $\square$

Lemma 1.6.4. *Let Γ be a discrete subgroup of a locally compact group G. Then there exists a Bruhat function for the action of Γ on G.*

Proof. Since Γ is a discrete subgroup of a locally compact group, the projection $\pi : G \to \Gamma \backslash G$ is a covering map. Let us choose a locally finite trivializing covering $\{U_i\}$ of $\Gamma \backslash G$ (with the additional property that $\bar{U}_i$ is compact) and, for every i, a preimage $\widetilde{U}_i$ of U_i (so the restriction of π is a homeomorphism between $\widetilde{U}_i$ and U_i).

Let us moreover fix on the quotient space $\Gamma \backslash G$ a continuous partition of unity ϕ_i (i.e. continuous functions $\phi_i : \Gamma \backslash G \to \mathbb{R}^+$ such that $\sum \phi_i = 1$), adapted to the covering $\{U_i\}$, i.e. with the property that $\mathrm{supp}(\phi_i) \subseteqq U_i$.

For every i let us consider the function $\widetilde{\phi}_i : \widetilde{U}_i \to \mathbb{R}$ satisfying $\widetilde{\phi}_i = \phi\pi$. The functions $\widetilde{\phi}_i$ extend to continuous functions $\widetilde{\phi}_i : G \to \mathbb{R}$. The map $h = \sum_i \widetilde{\phi}_i$ is the generalized Bruhat function we were looking for: it is continuous since it is a locally finite sum of continuous functions, and it is easy to check that, since $\{\phi_i\}$ is a partition of unity, the sum h satisfies the properties of a Bruhat function. $\square$

Proposition 1.6.5. *Let Γ be a closed subgroup of the locally compact group G, let V be a topological G-module (and hence also a topological Γ-module with the restricted action), then $C_c(G, V)$ is relatively injective as a topological Γ-module.*

Proof.

$$
\begin{array}{ccc}
A & \underset{\phi}{\overset{\sigma}{\rightleftarrows}} & B \\
{\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\beta} & \\
C_c(G, V) & &
\end{array}
$$

Let us consider the fuction β defined by

$$b \mapsto \int_\Gamma h(\gamma^{-1}g)\alpha(\gamma\sigma(\gamma^{-1}b))(g)d\gamma.$$

The function $\beta(b)$ is continuous since we have proved in Proposition 1.3.2 that the function $(\gamma, g) \mapsto \alpha(\gamma\sigma(\gamma^{-1}b))(g)$ is continuous and the second property of a Bruhat function implies that, for every g in G, the function $\gamma \mapsto h(\gamma^{-1}g)\alpha(\gamma\sigma(\gamma^{-1}b))(g)$ is different from 0 only on a compact subset of Γ. With similar arguments it can be proved that the linear map β is continuous (see [Mon01, Lemmma 4.5.5]).

Moreover β is G-invariant:

$$
\begin{aligned}
\beta(g_0 b) & = \int_\Gamma h(\gamma^{-1}g)\alpha(\gamma\sigma(\gamma^{-1}g_0 b))(g)d\gamma = \\
& = \int_\Gamma h(\gamma^{-1}g_0 g_0^{-1}g)\alpha(g_0 g_0^{-1}\gamma\sigma(\gamma^{-1}g_0 b))(g)d\gamma = \\
& = \int_\Gamma h(\gamma^{-1}g_0^{-1}g)g_0\alpha(\gamma\sigma(\gamma^{-1}b))(g_0^{-1}g)d\gamma = \\
& = g_0\beta(b).
\end{aligned}
$$

And β satisfies $\beta \circ \phi = \alpha$:

$$
\begin{aligned}
\beta(\phi(a)) & = \int_\Gamma h(\gamma^{-1}g)\alpha(\gamma\sigma(\phi\gamma^{-1}a))(g)d\gamma = \\
& = \int_\Gamma h(\gamma^{-1}g)\alpha(a)(g)d\gamma = \alpha(a).
\end{aligned}
$$

$\square$

Theorem 1.6.6. *Let Γ be a closed subgroup of the topological group G, the continuous cohomology of*

$$0 \longrightarrow C_c^0(G; V)^\Gamma \longrightarrow C_c^1(G; V)^\Gamma \longrightarrow C_c^2(G; V)^\Gamma$$

is isomorphic to $H_c^*(\Gamma; V)$.

Proof. The isomorphism $C_c^n(G, V) \cong C_c(G, C_c^{n-1}(G, V))$ is also an isomorphism of Γ modules (since the representation of Γ is the restriction of that of G) and hence Proposition 1.6.5 implies that the resolution is relatively injective. Since the fact that a resolution is strong doesn't depend on the representation, Lemma 1.3.5 implies that we are dealing with a strong resolution. The thesis of the theorem is hence a consequence of the fundamental Theorem of homological algebra. $\square$

We will use this result in Theorem 3.3.10 (resp. in Proposition 4.1.1) to show that, if Γ is a discrete subgroup of G (resp. a finite index subgroup), there is an injection

$$H_c^*(G;V) \to H_c^*(\Gamma;V).$$

1.7 Van Est's Theorem

We can now turn to the main theorem of this chapter: van Est's Theorem that relates continuous cohomology (with real coefficients) of a Lie group to the differential forms on the homogeneous space given by the quotient of G with respect to any maximal compact subgroup K of G. Indeed this theorem is much more general and an analogue statement is valid for a broad class of coefficient (namely the integrable differential G-modules) but for the purposes of this thesis we will need only this elementary statement and the loss of generality prevents us from a rather long functional analitic detour.

Let H be any compact subgroup of the Lie group G. The space of left cosets $G/H = \{gH \,|\, g \in G\}$ can be endowed with a unique smooth structure such that the projection $\pi : G \to G/H$ is C^∞ and there exist local smooth sections of G/H in G [War83, Theorem 3.58]. If we consider this smooth structure on G/H, the group G acts transitively on G/H via diffeomorphisms by left translations. We are interested in the complex formed by the modules

$$\Omega^i(G/H;\mathbb{R}) = \{\mathbb{R}\text{-valued differential } i\text{-forms on } G/H\}.$$

with the external differential. As usual we topologize this space with the topology of the uniform convergence on compact sets.

Just as in the previous section, we are going to show that the differential forms provide a relatively injective strong resolution of $\mathbb{R}$ (with the trivial G-action) and hence the homology of the complex is $H_c(G;\mathbb{R})$. We will prove the result in several steps: it is rather easy to show that $\Omega^i(G;\mathbb{R})$ is relatively injective, however, in general, the complex of the differential forms is not even exact and hence, in particular, is not a strong resolution. We will restrict to K-invariant differential forms to get a strong resolution, here K denotes a maximal compact subgroup.

The action of G on $M = G/H$ via diffeomorphisms induces a representation of G on the differentiable functions on M: if we fix f in $C^\infty(M;\mathbb{R})$ and g in G, we set

$$(g \cdot f)(x) = f(g^{-1}x).$$

It is naturally defined also an action of G on vector fields: let us fix $X \in \mathcal{T}(M)$, if we denote by l_g the diffeomorphism of M induced by the left multiplication by g, we have

$$(g \cdot X)_m = d(l_g)_{g^{-1}m} X_{g^{-1}m}.$$

A vector field on M is G-invariant if $g \circ X = X$. Finally we consider the action on $\Omega^i(M, \mathbb{R})$ defined by

$$(g \circ \omega)_m = ((l_{g^{-1}})^* \omega)_m = \omega_{g^{-1}m} \circ (\Lambda^i d(l_{g^{-1}})_m).$$

All these actions are compatible: $(g \cdot X)(f) = g \circ (X(g^{-1} \circ f))$, and, in the same way,

$$\langle g \circ \omega, X \rangle = g \langle \omega, g^{-1} \circ X \rangle.$$

Let us now consider the case $H = \{e\}$ and hence work with the differential forms on the group G itself. We denote by $\mathfrak{g}$ the Lie algebra of the left invariant vector fields on G. Since $\mathfrak{g}$ is a basis (over $C^\infty(G; \mathbb{R})$) of the vector fields in G, we have $\Omega^i(G, \mathbb{R}) \cong \operatorname{Hom}(\Lambda^i \mathfrak{g}, \mathbb{R}) \otimes C^\infty(G; \mathbb{R})$, moreover, the action on $\operatorname{Hom}(\Lambda^i \mathfrak{g}, \mathbb{R}) \otimes C^\infty(G; \mathbb{R})$ is trivial on the first factor and the regular left representation on the second.

The first step of the proof of van Est's Theorem is to show that $C^\infty(G; \mathbb{R})$ is relatively injective. This result will allow us to deduce that the modules $\Omega^i(G; \mathbb{R})$ are, in turn, relatively injective.

Proposition 1.7.1. *The G-module $C^\infty(G, \mathbb{R})$ is relatively injective.*

Proof. The proof is very similar to the proof of Proposition 1.4.3. The only difference is due to the fact that we do not have an immersion $C_c(G, \mathbb{R}) \to C^\infty(G, \mathbb{R})$ and thus we need to regularize continuous cochains. Fix a function $\chi \in C^\infty(G, \mathbb{R}^+)$ with compact support, and with mass one (that is $\int_G \chi(g^{-1}) d\mu = 1$, where μ is the left Haar measure on G). Let us consider the right convolution with χ:

$$\alpha : C_c(G, C^\infty(G, \mathbb{R})) \to C^\infty(G \times G, \mathbb{R})$$
$$\alpha\phi(g_0, g_1) = \int_G \chi(g^{-1}g_0)\phi(g)(g_1)d\mu(g).$$

Since the convolution of the smooth map χ with the continuous map $\phi(\cdot)(g_1)$ is smooth, $\alpha\phi$ is smooth in the first variable. One can easily see that the map α is well defined ($\alpha\phi$ is smooth in both its variables). The continuity of α descends from the definition of the topologies via some computations. We will now prove that the left invariance of the Haar measure μ implies the G-invariance of α:

$$
\begin{aligned}
\alpha(l_x\phi)(g_0, g_1) &= \int_G \chi(g^{-1}g_0)((l_x\phi)(g))(g_1)d\mu(g) \\
&= \int_G \chi(g^{-1}g_0)(\phi(x^{-1}g))(x^{-1}g_1)d\mu(g) \\
&= \int_G \chi(g^{-1}xx^{-1}g_0)\phi(x^{-1}g)(x^{-1}g_1)d\mu(g) \\
&= \int_G \chi(h^{-1}x^{-1}g_0)\phi(h)(x^{-1}g_1)d\mu(h) \\
&= (\alpha\phi)(x^{-1}g_0)(x^{-1}g_1) \\
&= l_x\alpha\phi(g_0).
\end{aligned}
$$

Since we have chosen χ such that $\int_G \chi(g^{-1}) d\mu = 1$, if $\phi(g) = \phi_0$ is an element of $C_c(G; C^\infty(G, \mathbb{R}))$ that doesn't depend on g, then $\alpha(\phi)(g_0, g_1) = \phi_0(g_1)$. We thus have that the composition:

$$C^\infty(G, V) \hookrightarrow C_c(G, C^\infty(G, V)) \to C^\infty(G \times G, V) \to C^\infty(G, V)$$

is the identity, where the first G-morphism is the inclusion as constant functions, the second is the just defined map α and the last G-morphism is dual to the diagonal map. All the involved maps are G-morphisms, and the composition is the identity of $C^\infty(G, V)$. We already know that $C_c(G, C^\infty(G, V))$ is continuously relatively injective, so we get the thesis from Lemma 1.2.4. $\quad\square$

Proposition 1.7.2. *The G-module $\Omega^i(G; \mathbb{R})$ is continuously relatively injective.*

Proof. We have already pointed out that

$$\Omega^i(G; \mathbb{R}) \cong C^\infty(G; \mathbb{R}) \otimes \mathrm{Hom}(\Lambda^i \mathfrak{g}, \mathbb{R})$$

and that G acts only on the first factor. This implies that $\Omega^i(G; \mathbb{R})$ is a finite direct sum of relatively injective G-modules (it is a consequence of Proposition 1.7.1) and so is itself relatively injective. $\quad\square$

Proposition 1.7.3. *Let H be any compact subgroup of G, then $\Omega^p(G/H; \mathbb{R})$ is relatively injective.*

Proof. Consider the projection $\pi : G \to G/H$. The pullback via π gives an inclusion

$$\Omega^p(G/H; \mathbb{R}) \hookrightarrow \Omega^p(G; \mathbb{R})$$

whose image are precisely the differential forms that are invariant with respect to the action of H via right multiplication, i.e. the differential forms ω such that $r_h^* \omega = \omega$ for every $h \in H$. Since H is compact, $\mu(H) < \infty$ hence we can define the map

$$\alpha : \Omega(G; \mathbb{R}) \to \Omega(G; \mathbb{R})$$
$$\alpha\omega = \frac{1}{\mu(H)} \int_H r_g^* \omega \, d\mu(g)$$

By definition, α is a projection on $\pi^* \Omega^p(G/H; \mathbb{R})$. Moreover it is easy to show that α is continuous and that it is G-invariant (since the actions of G on itself via left and right multiplication commute). This is enough to ensure that $\Omega^p(G/H; \mathbb{R})$ is relatively injective, since we already know that $\Omega^p(G; \mathbb{R})$ is (see Proposition 1.7.2). $\quad\square$

The first part of the proof of van Est's Theorem doesn't require that the compact subgroup K is maximal. This is however necessary to guarantee that the resolution of the trivial G-module $\mathbb{R}$ provided by the differential forms is strong:

Proposition 1.7.4. *Let K be a maximal compact subgroup of G. Then the resolution*

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(G/K; \mathbb{R}) \longrightarrow \Omega^1(G/K; \mathbb{R}) \longrightarrow \Omega^2(G/K; \mathbb{R}) \longrightarrow$$

is strong.

Proof. This is a consequence of Poincaré's Lemma: the quotient of a Lie group G by a maximal compact subgroup is diffeomorphic to $\mathbb{R}^n$ (see [Hel62, IV, Theorem 2.2(iii)]). The usual proof of Poincaré's Lemma (that can be found, for example, in [War83, 4.18]) provides a continuous contracting homotopy that ensures that the resolution is strong. □

Combining Proposition 1.7.3 and Proposition 1.7.4, we have shown van Est's Theorem:

Theorem 1.7.5 (van Est)**.** *Let G be a Lie group, $K < G$ a maximal compact subgroup, M the associated homogeneous space. Then*

$$H_c^*(G; \mathbb{R}) \cong H^*(\Omega^*(M; \mathbb{R})^G, d).$$

1.8   Explicit isomorphism

In Section 3.4 we will need explicit formulas for van Est's isomorphism. These formulas were first given by Dupont in [Dup76, Section 5]. We will need the fact that the quotient of a Lie group with respect to a maximal compact subgroup has non-positive sectional curvature at every point and hence is a uniquely geodesic space (and geodesics depend continuously on their endpoints).

In view of Theorem 1.2.6 a G-chain map $I^* : \Omega^*(G/K; \mathbb{R}) \to C_c^*(G; \mathbb{R})$ that extends the identity of $\mathbb{R}$ is unique up to G-chain homotopy; thus it is sufficient to exhibit a particular chain map.

Let us fix a point $x \in M = G/K$. We will need, for every i and for every $(i+1)$-uple $(g_0, \ldots, g_i)$, a simplex spanning $(g_0 x, \ldots, g_i x)$ such that its choice is invariant with respect to the action of G on G^{i+1} via left multiplication and on G/K via isometries. Since G/K is uniquely geodesic and geodesics depend continuously on their endpoints, we can define, by induction on i, a geodesic simplex $\triangle(g_0, \ldots, g_i)$, coning on the first vertex. This means that we are considering the function from the linear parametrization of the standard i-simplex with values in X given by the formulas:

$$\triangle(g_0, \ldots, g_i)(t(1, 0, \ldots, 0) + (1 - t)s) = [g_0 x, \triangle(g_1, \ldots, g_i)(s)](t)$$

where we denote by $[a,b] : [0,1] \to X$ the unique geodesic with $[a,b](0) = a$ and $[a,b](1) = b$. Since X is uniquely geodesic, and G acts on X by isometries, for every g in G we get

$$g \cdot \triangle(g_0, \dots, g_i) = \triangle(gg_0, \dots, gg_i).$$

We can now exhibit explicit formulas for van Est's isomorphism:

Proposition 1.8.1. *The chain G-morphism:*

$$I^n : \Omega^n(G/K; \mathbb{R}) \to C_c^n(G; \mathbb{R})$$
$$I^n(\omega)(g_0, \dots, g_n) = \int_{\triangle(g_0, \dots, g_n)} \omega$$

induces, in homology, van Est's isomorphism.

Proof. Since in X geodesics depend continuously on their endpoints (with respect to the $\mathcal{C}^1$ topology), the map is well defined and continuous. Moreover, it is G-invariant:

$$
\begin{aligned}
l_g(I^n(\omega))(g_0, \dots, g_n) \quad &= I^n(\omega)(g^{-1}g_0, \dots, g^{-1}g_n) \\
&= \int_{\triangle(g^{-1}g_0, \dots, g^{-1}g_n)} \omega \\
&= \int_{g^{-1}\triangle(g_0, \dots, g_n)} \omega \\
&= \int_{\triangle(g_0, \dots, g_n)} l_{g^{-1}}^* \omega \\
&= I^n l_{g^{-1}}^* \omega.
\end{aligned}
$$

The fact that I^* is a chain map descends from Stokes theorem and the choice of the geodesic simplex spanning $(g_0, \dots, g_{n+1})$:

$$
\begin{aligned}
(dI^n(\omega)(g_0, \dots, g_{n+1})) \quad &= \sum_{i=0}^n (-1)^i \int_{\triangle(g_0, \dots, \hat{g}_i, \dots, g_n)} \omega \\
&= \int_{\partial\triangle(g_0, \dots, g_{n+1})} \omega \\
&= \int_{\triangle(g_0, \dots, g_{n+1})} d\omega \\
&= I^{n+1} d\omega.
\end{aligned}
$$

$\square$

Chapter 2

Continuous bounded cohomology

The purpose of this chapter is to develop another cohomological theory for locally compact groups that will have deep geometric applications in the next chapters. More precisely, we will study the cohomology of the complex

$$C_{cb}^n(G; V) = \{f : G^{n+1} \to V \mid f \text{ is continuous and bounded}\}$$

where V is a normed vector space.

The functorial approach to continuous bounded cohomology is less transparent then the one we dealt with in the unbounded setting: it is natural to put on $C_{cb}^n(G; V)$ the topology of the uniform convergence (induced by the operatorial norm), unfortunately the regular actions of G on $C_{cb}^n(G; V)$ are not continuous with respect to this topology, while, in Chapter 1, we used widely the fact that the regular actions of G on $C_c^n(G; V)$ are continuous with respect to the compact-open topology (see Lemma 1.3.1). However it is possible, with some efforts, to adapt the algebraic tools we have developed in the first chapter to the context of continuous bounded cohomology.

In the first part of the chapter we will describe the necessary ingredients that make the algebraic machinery work in this context. Then we will closely follow the exposition of Chapter 1 and discuss the analogies and the differences of the two theories. We will refer to the monograph [Mon01] for complete proofs. In particular we will study the resolutions provided by continuous bounded functions and measurable bounded functions (these are the analogues in the bounded context of continuous and locally integrable functions that we have studied in the first chapter).

A big difference with respect to continuous cohomology is that no analogue of Van Est's Theorem is known to hold in the bounded setting and hence continuous bounded cohomology is in general much more difficult to compute. One of the main advantages of this theory is that the continuous bounded cohomology of a group can be endowed with a seminorm that has

a deep geometric meaning, as we will show in the next chapters.

Another key feature of the theory of continuous bounded cohomology is that there is a large class of groups (namely the amenable groups that we will define in Section 2.6) that, roughly speaking, can be ignored in the study of continuous bounded cohomology. This means that, if $H < G$ is amenable, the continuous bounded cohomology of G can be computed from the complex $C_{cb}((G/H)^n; V)^G$, in particular the continuous bounded cohomology of amenable groups vanishes. This feature will play a crucial role in Chapter 4. Section 2.6 will be devoted to the study of amenable groups and of quotients by amenable groups.

The theory of continuous bounded cohomology of locally compact groups has been developed by Monod in [Mon01] and much of the material of this chapter can be found in that monograph. However the theory of bounded cohomology of discrete groups (that can be considered as a subcase of locally compact groups) is older: it had already been deeply studied by Ivanov in the late eighties (see [Iva87]).

The theory of bounded cohomology of discrete groups has interesting geometric motivations first discovered by Gromov in the seminal article "Volume and bounded cohomology" [Gro82]. It is possible to define also the bounded cohomology of a topological space X and if the discrete group G is the fundamental group of X, a deep result of Gromov states that the bounded cohomology of G is isometrically isomorphic to that of X. In turn the bounded cohomology of a topological space carries informations about interesting geometric invariants such as the simplicial volume.

In the last section of this chapter we will properly define the bounded cohomology of a topological space and discuss the relations with the bounded cohomology of its fundamental group. In the following chapters we will study the simplicial volume and we will see many applications of the theory of continuous (bounded) cohomology in that setting.

2.1 Normed chain complexes

Let G be a locally compact group. A *Banach G-module* is a Banach vector space V endowed with a representation π of G in the group of the isometric linear automorphisms of V. A Banach G-module is *continuous* if the representation is jointly continuous. The analogue of Lemma 1.2.2 holds in this context: it is sufficient to verify that, for every v in V, the map $g \mapsto gv$ is continuous.

Lemma 2.1.1. *Let E be a Banach space on which G acts by isometries, the following statements are equivalent*

   1. the action is continuous

   2. the map $g \mapsto gv$ is continuous for every $v \in E$.

Proof. Clearly the first assertion implies the second and hence we have only to show that the function

$$G \times E \to E$$
$$(g, v) \mapsto gv$$

is continuous. For this purpose let $g_n \to g$ and $v_n \to v$ be convergent sequences in G and E respectively. Then

$$\|g_n v_n - gv\| \leq \|g_n v_n - g_n v\| + \|g_n v - gv\| \leq \|v_n - v\| + \|(g_n - g)v\|.$$

$\square$

As usual we will denote by V^G the submodule of V fixed by G.

$$V^G = \{v \in V \mid gv = v, \forall g \in G\}.$$

Since a continuous Banach G-module is in particular a topological G-module, we can consider the complex of the continuous n-cochains from G to V (cfr Section 1.1)

$$C_c^n(G, V) = \{\phi : G^{n+1} \to V \mid \phi \text{ is continuous}\}.$$

The norm on V induces a norm on $C_c^n(G, V)$:

$$\|\phi\|_\infty = \sup_{(g_0, \dots, g_n) \in G^{n+1}} \|\phi(g_0, \dots, g_n)\|_V \in [0, \infty].$$

The continuous bounded cochains are the elements of the subspace of $C_c^n(G, V)$

$$C_{cb}^n(G; V) = \{\phi \in C_c^n(G, V) \mid \|\phi\|_\infty < +\infty\}.$$

The $\|\cdot\|_\infty$ norm makes $C_{cb}^n(G, V)$ a Banach space.

Since G acts on V via isometries, the regular left (resp. right) representations of G on $C_c^n(G; V)$ restrict to well defined isometric actions on $C_{cb}^n(G; V)$.

Remark 2.1.2. Unfortunately this action is, in general, not continuous: indeed let us fix ϕ in $C_{cb}^n(G; V)$, the map $g \mapsto g\phi$ is continuous with respect to the l^∞ topology if the map ϕ is uniformly continuous with respect to diagonal left translations:

$$\begin{aligned}
\|g\phi - \phi\|_\infty &= \sup_{x \in G^{n+1}} \|g\phi(g^{-1}x) - \phi(x)\|_V \\
&\leq \sup_{x \in G^{n+1}} \|g\phi(g^{-1}x) - g\phi(x)\|_V + \|g\phi(x) - \phi(x)\|_V
\end{aligned}$$

where the first term is small, provided g is close to e if ϕ is uniformly continuous, the second is small since the module V is continuous and ϕ is bounded.

In particular, when $n = 0$, if the map $g \mapsto g\phi$ is continuous then ϕ is uniformly continuous:

$$\begin{aligned}
\sup_{x \in G} \|\phi(g^{-1}x) - \phi(x)\|_V &= \sup_{x \in G} \|g\phi(g^{-1}x) - g\phi(x)\|_V \\
&\leq \sup_{x \in G} \|g\phi(g^{-1}x) - \phi(x)\|_V + \|\phi(x) - g\phi(x)\|_V \\
&\leq \|g\phi - \phi\|_\infty + \|\phi(x) - g\phi(x)\|_V,
\end{aligned}$$

the first term tends to zero (when g is small) since the representation is continuous, the second since the module V is continuous and ϕ is bounded (and hence $\|\phi(x)\|_V \leq \|\phi\|_\infty$.

This problem will make the functorial approach described in the next section far more complicated than the one described in Chapter 1.

The coboundary operator $\delta : C_c^n(G, V) \to C_c^{n+1}(G, V)$ defined in Section 1.1 restricts to a bounded operator $\delta : C_{cb}^n(G, V) \to C_{cb}^{n+1}(G, V)$ that is G-invariant. This implies that it restricts to the subcomplex of the G-invariants $C_{cb}^n(G, V)^G$.

Definition 2.1.3. The *continuous bounded cohomology* of G with coefficients in V is the cohomology of this complex:

$$H_{cb}^*(G, V) = H_*(C_{cb}^*(G, V)^G).$$

The continuous G-invariant bounded cochains form a normed chain complex.

Definition 2.1.4. A *normed chain complex* is a complex $(C^*, \| \cdot \|_{C^*}, d^*)$ where $(C^*, \| \cdot \|_{C^*})$ is a Banach G-module, and $d^i : C^i \to C^{i+1}$ is a bounded linear operator.

If $(C^*, \| \cdot \|_{C^*}, d^*)$ is a normed chain complex, its homology is naturally endowed with a seminorm: if $a \in H_i(C^*)$ is a class, we can define its norm as

$$\|a\| = \inf_{[\alpha]=a} \|\alpha\|_{C^i}.$$

It is easy to see that the seminorm is subadditive (i.e. $\|a + b\| \leq \|a\| + \|b\|$) and homogeneous (i.e. $\|\lambda a\| = \lambda\|a\|$) while, in general, it is not a norm: even if the spaces C^i are Banach spaces and the maps d^i are bounded, there is no reason for the image dC^i to be closed in C^{i+1}.

If we consider the normed complex $(C_{cb}^i(G, V)^G, \| \cdot \|_\infty, \delta)$, the seminorm induced in cohomology is called the *canonical seminorm* on $H_{cb}^*(G, V)$. We will see in Section 3.2 that this seminorm has interesting geometric meanings.

In the next section we will adapt the relative homological algebra to the context of continuous bounded cohomology in order to find other complexes whose homology is isomorphic to $H_{cb}^*(G; V)$. Since the seminorm has a deep importance in geometric applications we will always need to show that the induced isomorphisms are isometric.

The inclusion $i : C_{cb}^*(G; V) \to C_c^*(G; V)$ induces a map in cohomology

$$c : H_{cb}^*(G; V) \to H_c^*(G; V)$$

that is known as the *comparison* map. This map is, in general, not injective nor surjective, however we will show in Section 4.5 that, if $G = \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$, the comparison map

$$c : H^4_{cb}(G;V) \to H^4_c(G;V)$$

is an isomorphism (this result is due to Bucher-Karlsson [Buc08B]).

2.2 Some homological algebra

We have pointed out in Remark 2.1.2 that the Banach G-modules $C^*_{cb}(G;V)$ are, in general, not continuous. However the algebraic machinery works only in the category of continuous G-modules: we will need to prove that $C_{cb}(G;V)$ is relatively injective, but if the Banach G-modules on the first row of the diagram below are not continuous, there is no hope that $\beta(b)$ is continuous, when b belongs to B.

$$
\begin{array}{ccc}
A & \underset{\phi}{\overset{\sigma}{\rightleftarrows}} & B \\
\downarrow{\scriptstyle\alpha} & \swarrow{\scriptstyle\beta} & \\
C_{cb}(G,V) & &
\end{array}
$$

However it is worth remarking that the action of G on $C_{cb}(G;V)^G$ is continuous (being trivial). We can hence consider, instead of $C_{cb}(G;V)$, its maximal continuous submodule:

Definition 2.2.1. Let V be a Banach G-module, the *maximal continuous submodule* of V is

$$\mathcal{C}V = \{v \in V \,|\, G \to V, \, g \mapsto gv \text{ is continuous}\}.$$

As a consequence of Lemma 2.1.1, the maximal continuous submodule of a Banach G-module is a continuous Banach G-module.

Since we are working in the category of Banach G-modules, all the maps will be linear and bounded, unless otherwise stated. Moreover, if V and W are Banach G-modules, we will call a linear bounded morphism

$$\phi : V \to W$$

a G-morphism if it commutes with the representations, i.e. if $\phi(gv) = g\phi(v)$, for every $v \in V$. A useful lemma is the following

Lemma 2.2.2. *Let A be a continuous Banach G-module, let B be a Banach G-module, and $\phi : A \to B$ be a G-morphism. Then $\mathrm{im}(\phi) \subseteq \mathcal{C}B$.*

Proof. Let $b = \phi(a)$, the map $g \mapsto gb$ is continuous since $gb = g\phi(a) = \phi(ga)$ and hence it is a composition of continuous functions. $\square$

Since we will need to control the seminorms, the appropriate notion of strongly injective morphism in this context is slightly different from the one we gave in Chapter 1 (moreover we call it *admissible injective* following the notation of [Mon01] and thus reserving the adjective *strong* for the continuous submodules).

Definition 2.2.3. A G-morphism between two Banach G-modules $\phi : V \to W$ is *admissible injective* if it has a linear left inverse $\sigma : W \to V$ with $\|\sigma\| \leq 1$.

$$V \underset{\phi}{\overset{\sigma}{\rightleftarrows}} W \ .$$

As in Chapter 1 we do not require σ to be a G-morphism. This definition is more restrictive than the one we gave in Chapter 1. An admissible injective G-morphism is a continuously strongly injective G-morphism, but the converse is not true: the requirement that the image of V is complemented in W is necessary but not anymore sufficient to guarantee that the inclusion $V \hookrightarrow W$ is admissible injective.

Definition 2.2.4. A Banach G-module Z is *relatively injective* if, for every pair of continuous Banach G-modules V and W, for every admissible injective G-morphism $\phi : V \to W$, for every G-morphism $\alpha : V \to Z$, there exists a G-morphism $\beta : W \to Z$ with $\|\beta\| \leq \|\alpha\|$, making the diagram commutative.

$$
\begin{array}{c}
V \underset{\phi}{\overset{\sigma}{\rightleftarrows}} W \\
\alpha \downarrow \quad \diagdown \beta \\
Z
\end{array}
$$

This definition shades light on the reason why we have required that the norm of the inverse of a strongly injective G-morphism is smaller than one: without that assumption, we wouldn't have been able to control the norm of β.

Lemma 2.2.5. *If Z is relatively injective, also $\mathcal{C}Z$ is.*

Proof. Lemma 2.1.1 implies that, since V and W are continuous Banach G-modules, and α and β are G-morphisms, the image of α and β is indeed contained in $\mathcal{C}Z$. $\qquad\square$

The analogue of Lemma 1.2.4 holds with these definitions:

Lemma 2.2.6. *Let W be a relatively injective Banach G-module. Assume that α, β are G-morphisms such that $\beta \circ \alpha = id_V$, $\|\beta\| \leq 1$. Then V is relatively injective.*

$$A \underset{\phi}{\overset{\sigma}{\rightleftarrows}} B$$
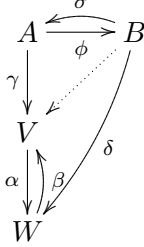
Let us now fix a continuous Banach G-module V. A Banach resolution (F^i, d^i) of the Banach G-module V is a resolution (i.e. an exact complex null in negative dimension different from -1 and with $F^{-1} = V$) made by Banach G-modules and G-morphisms. As a consequence of Lemma 2.2.2, the differentials restrict to the maximal continuous subresolution $(\mathcal{C}F^i, d^i)$.

The Banach resolution (F^i, d^i) is *strong* if its maximal continuous subresolution $(\mathcal{C}F^i, d^i)$ admits a contracting homotopy, i.e. a collection of linear maps $k^i : F^i \to F^{i-1}$ such that $\|k^i\| \leq 1$ and $k^{i+1}d^i + d^{i-1}k^i = \mathrm{id}_{F^i}$.

$$0 \longrightarrow V \underset{d^0}{\overset{k^0}{\rightleftarrows}} F^0 \underset{d^1}{\overset{k^1}{\rightleftarrows}} F^1 \underset{d^2}{\overset{k^2}{\rightleftarrows}} F^2 \overset{k^3}{\rightleftarrows} F^3 \rightleftarrows \cdots$$

This definition has two important differences from the definition of strong resolution of a continuous G-module: the first is that we require the contracting homotopy to have norm bounded by one, the second is that whether a resolution is strong depends on the representation: the continuous submodule of a Banach G-module depends on the representation. Indeed, in general, a contracting homotopy of the Banach resolution (F^*, d) doesn't induce a contracting homotopy of the continuous subresolution (since k is not G-invariant).

We are now ready to restate the fundamental theorem of homological algebra in the context of continuous bounded cohomology. We will omit its proof that is analogue to that of Theorem 1.2.6, even though extra care should be taken in passing from the Banach resolutions to the continuous subresolutions. It can be found in [Mon01, Theorem 7.2.1(ii)].

Theorem 2.2.7. *Let (F_1^*, d_1) and (F_2^*, d_2) be two strong Banach G-resolutions of the Banach G-module V. If F_1^i and F_2^i are relatively injective Banach G-modules for every i, there exist G-morphisms $h_i : \mathcal{C}F_1^i \to \mathcal{C}F_2^i$ making the diagram commutative. The h_i are unique up to G-chain homotopy.*

A consequence of Theorem 2.2.7 is that the homology of the invariants of two strong relatively injective Banach resolutions of a Banach G-module are isomorphic:

Corollary 2.2.8. *Let (F_1^*, d_1), (F_2^*, d_2) two strong relatively injective Banach resolutions of the Banach G-module V. The homology group of the subcomplexes $((F_1^*)^G, d_1)$, $((F_2^*)^G, d_2)$ are isomorphic:*

$$H_*((F_1^*)^G, d_1) \cong H_*((F_2^*)^G, d_2)$$

Proof. It is sufficient to remark that the G-invariants $(F_i^*)^G$ provide a G-submodule of the continuous subresolutions and then apply the same arguments of Corollary 1.2.7 $\qquad\square$

The homological methods just introduced are powerful tools to show the isomorphism of the homologies but cannot ensure that this isomorphism is isometric. The following example shows that in general we cannot assume that it is:

Example 2.2.9. Let us consider any strong, relatively injective Banach resolution (F^*, d) of a Banach G-module V. We denote by $\|\cdot\|_k$ the norm on F^k. Let us now consider the resolution $(\widetilde{F}^*, d)$ where $\widetilde{F}^k$ is the vector space F^k endowed with the norm $k\|\cdot\|_k$. Since we have rescaled the norm with increasing factors also the resolution $\widetilde{F}^*$ is strong (the maximal continuous submodules coincide and the contracting homotopy has norm bounded by 1). The identity is an isomorphism between the two resolutions but its operatorial norm in degree k is k and hence the isomorphism it induces in homology isn't isometric.

Since we will be interested in the seminorms, we will need extra efforts to show that, in each relevant case, the isomorphisms we will be considering are isometric.

2.3 The standard resolution

One big difference with respect to the unbounded case is that, in the context of continuous bounded cohomology, the restriction of the map

$$\Psi : C_c^n(G, V) \to C_c(G, C_c^{n-1}(G, V))$$
$$\Psi(\phi)(g_0)(g_1, \dots, g_n) = \phi(g_0, g_1, \dots, g_n)$$

to bounded cochain is not well defined since the map $\Psi(\phi)$ isn't, in general, continuous with respect to the norm of uniform convergence: the continuity of $\Psi(\phi)$ corresponds to a uniform continuity of ϕ with respect to translations in the first coordinate:

$$\|\Psi(\phi)(g) - \Psi(\phi)(e)\|_\infty = \sup_{(g_1,\dots,g_n) \in G^{n+1}} \|\phi(g, g_1, \dots, g_n) - \phi(e, g_1, \dots, g_n)\|_V.$$

In this section we will show that the modules $\bar{C}_{cb}^n(G;V)$ defined inductively by

$$\bar{C}_{cb}^{n+1}(G,V) = C_{cb}(G,\bar{C}_{cb}^n(G,V))$$

provide a strong relatively injective resolution of the Banach G-module V: in the unbounded case we used this result to get that the same result is true for $C_c^n(G,V)$ (since $\bar{C}_c^n(G;V) \cong C_c^n(G;V)$).

In the bounded context the fact that $C_{cb}^n(G,V)$ is relatively injective is not anymore a straightforward consequence of the relative injectivity of $C_{cb}(G,V)$ but is still true provided G is locally compact. We will show how it is possible to get this result in Section 2.5 using a generalized Bruhat function.

Let us consider the Banach space $C_{cb}(G;V)$ with respect to the sup norm as defined in Section 2.1. Both the regular left representation and the regular right representation are isometric and hence make $C_{cb}(G;V)$ a Banach G-module.

The operator T we have defined in Section 1.3 requiring that

$$(T\phi)(x) = x\phi(x^{-1})$$

is isometric (since G acts on V via isometries) and intertwins the two representations. It is hence sufficient to show that the regular right representation is relatively injective.

Proposition 2.3.1. $C_{cb}(G,V)$ *endowed with the regular right representation is relatively injective.*

Proof.

Let us fix an admissible injective morphism of continuous Banach G-modules $\phi : A \to B$. We have already pointed out that an admissible injective morphism of Banach G-modules is in particular a strongly injective morphism of topological G-modules and hence we can consider the extension $\delta : B \to C_c(G;V)$ of $i \circ \gamma$ constructed in the proof of Proposition 1.3.2.

We recall that $\delta b(g) = \gamma(\sigma(gb))(e)$ for every $b \in B$, g in G. We have already proved that δ is continuous and G-invariant, it only remains to prove that its image lies in $C_{cb}(G;V)$:

$$\|\delta b\| = \sup_g \|\gamma(\sigma(gb))\|_V = \|\gamma\|\|\sigma\|\|b\| \leq \|\gamma\|\|b\|.$$

This concludes the proof since $\|\delta\| \leq \|\gamma\|$ and hence δ satisfies the required properties. $\qquad\square$

Let us now consider the coboundary operator inductively defined by

$$\bar{d} : \bar{C}^n_{cb}(G, V) \to \bar{C}^{n+1}_{cb}(G, V)$$
$$(\bar{d}\phi)(g) = \phi - \bar{d}(\phi(g)).$$

In the unbounded setting it holds $\bar{d}\Psi = \Psi d$ with d the natural coboundary operator. It is easy to verify that $\bar{d}$ is a G-morphism. We will denote by $\bar{H}^*_{cb}(G, V)$ the homology of the complex

$$0 \longrightarrow \bar{C}^0_{cb}(G, V) \longrightarrow \bar{C}^1_{cb}(G, V) \longrightarrow \bar{C}^2_{cb}(G, V) \longrightarrow$$

In Section 2.5 we will get the isometric isomorphism $H^*_{cb}(G; V) \cong \bar{H}^*_{cb}(G; V)$.

Theorem 2.3.2. *Let (F^*, d) be any strong relatively injective Banach resolution of the Banach G-module V. Then*

$$H_*((F^*)^G, d) \cong \bar{H}^*_{cb}(G, V).$$

Moreover the isomorphism doesn't increase the norm.

Proof. The Banach resolution $(\bar{C}^n_{cb}(G; V), \bar{d}^n)$ is strong because the operators $k_n : \bar{C}^n_{cb}(G, V) \to \bar{C}^{n-1}_{cb}(G, V)$, $\phi \mapsto \phi(e)$ have norm bounded by one and satisfy $k\bar{d} + \bar{d}k = \text{id}$. Moreover the image of the restriction of k to the continuous submodule $\mathcal{C}\bar{C}^n_{cb}(G, V)$ is contained in $\mathcal{C}\bar{C}^{n-1}_{cb}(G, V)$ (this is analogue to Lemma 2.1.2). Hence the first part of the theorem descends from Theorem 2.2.7.

In order to prove the second statement it is sufficient to exhibit a chain G-morphism $\alpha : F^* \to \bar{C}^n_{cb}(G, V)$ that extends the identity on V and has norm, in each dimension, bounded by one.

$$
\begin{array}{ccccccccc}
V & \longrightarrow & F^0 & \underset{h}{\overset{d_0}{\rightleftarrows}} & F^1 & \underset{h}{\overset{d_1}{\rightleftarrows}} & F^2 & \overset{d_2}{\longrightarrow} & \cdots \\
\Big\| {\scriptstyle \text{id}} & & \Big\downarrow {\scriptstyle \alpha_0} & & \Big\downarrow {\scriptstyle \alpha_1} & & \Big\downarrow {\scriptstyle \alpha_2} & & \\
V & \longrightarrow & \bar{C}^0_{cb}(G; V) & \overset{\bar{d}_0}{\longrightarrow} & \bar{C}^1_{cb}(G; V) & \overset{\bar{d}_1}{\longrightarrow} & \bar{C}^2_{cb}(G; V) & \overset{\bar{d}_2}{\longrightarrow} & \cdots
\end{array}
$$

Let us define inductively the map α_i by the formula

$$\alpha_i(f)(g) = \alpha_{i-1}gh(g^{-1}f)$$

Since G acts on F^* via isometries and $\|h\| \leq 1$, it is obvious that the norm of α_i is not greater than the norm of α_{i-1} and hence is not greater than one (since $\alpha_{-1} = \text{id}$). Since we have already justified the continuity

and the G invariance of α (for example in Proposition 2.3.1), it only remains to prove that α_* commute with the differentials:

$$
\begin{aligned}
\bar{d}_i(\alpha_i(f))(g) \quad &= \alpha_i(f) + \bar{d}_{i-1}(\alpha_i(f)(g)) = \\
&= \alpha_i(f) + \bar{d}_{i-1}(\alpha_{i-1}gh(g^{-1}f)) = \\
&= \alpha_i(f) + \alpha_i g d_{i-1}h(g^{-1}f) = \\
&= \alpha_i(f) - \alpha_i(f) + \alpha_i g h d(g^{-1}f) = \\
&= \alpha_i g h g^{-1} df = \\
&= \alpha_{i+1} df(g)
\end{aligned}
$$

where we used the definition, the inductive hypothesis, the fact that both α and d are G-morphisms and the fact that $hd - dh = \mathrm{id}$.

$\square$

Remark 2.3.3. Theorem 2.3.2 explains the terminology we have used in Section 2.1: the seminorm on $H_{cb}^*(G, V)$ defined in Section 2.1 (that coincides with the seminorm just defined on $\bar{H}_{cb}^*(G, V)$, as we will prove in the next section) is canonical as much as it can be described as the infimum of all the seminorms induced by any Banach strong relatively injective resolution of V. The requirement that $\|\sigma\| \leq 1$ for any contracting homotopy is important at this point: if we drop that requirement, we would only get, with the notations of Proposition 2.3.1 that $\|\delta\| \leq \|\gamma\| \|\sigma\|$. This wouldn't guarantee that the canonical seminorm is indeed the infimum.

2.4 Resolution via measurable bounded functions

The purpose of this section is the study of an analogue, in the bounded context, of the locally integrable functions we have studied in Section 1.4. If we restrict to the coefficient module $\mathbb{R}$, every bounded measurable function is locally integrable with respect to the Haar measure (that is finite on compact subset): this because we are restricting to locally compact groups. This implies that, for trivial coefficients, the measurable functions are the bounded version of locally integrable functions. In the more general context of Banach G-module coefficients, we need to find a suitable version of measurability (it will be weak*-measurability), that has good properties. In order to properly defining the concept of weak*-measurability we need to restrict ourselves to a (broad) class of coefficient, the *coefficient G-modules*, that are, roughly speaking, G-modules for wich a preferred predual that has good properties is fixed:

Definition 2.4.1. A *coefficient G-module* is a Banach G-module (π, E) contragradient to some separable continuous Banach G-module $(\pi^\flat, E^\flat)$.

The contragradient of a Banach G-module V is its topological dual endowed with the G-action defined by $g \cdot \phi = \phi \circ g^{-1}$. This corresponds to the G-module $\mathrm{Hom}(V, \mathbb{R})$ if $\mathbb{R}$ is endowed with the trivial G-structure.

A first, elementary example of coefficient G-modules are the finite dimensional G-modules (that are, indeed, the only coefficients in which we will be interested for geometric applications). Another example are separable Hilbert spaces with any continuous G-action.

Let now (S, μ) be a measured space (for example the group G itself with the Haar measure) endowed with a measure preserving G-action. Let, moreover, V be a coefficient G-module. We will denote by $L^\infty(S, V)$ the G-module

$$L^\infty(S; V) = \{f : S \to V \mid f \text{ is bounded, weak*-measurable}\}.$$

Here, by the weak*-measurability condition, we mean that, for every $v \in V^\flat$, the function $s \to f(s)(v)$ is a measurable function from S to $\mathbb{R}$. As usual we endow $L^\infty(S; V)$ with the regular left representation: $(gf)(s) = g(f(g^{-1}s))$. An important feature of the theory of bounded weak*-measurable function is that, if V is a coefficient G-module, also $L^\infty(S; V)$ is:

Proposition 2.4.2. *The module $L^\infty(S; V)$ is a coefficient G-module in a canonical way.*

Proof. It can be proved that it is contragradient to $L^1(S, V)$. See [Mon01, Corollary 2.3.2]. $\square$

One of the main advantages of weak*-measurability is that it can be proved that, for weak*-measurabile functions, holds the exponential law:

Proposition 2.4.3. *Let (S_1, μ_1) and (S_2, μ_2) be two measured G-spaces, and let V be a coefficient G-module, then*

$$L^\infty(S_1 \times S_2; V) \cong L^\infty(S_1, L^\infty(S_2; V)).$$

Proof. See [Mon01, Corollary 2.3.3] $\square$

The thesis of Proposition 2.4.3 is the analogue of a statement that we have widely used in Chapter 1 (and doesn't hold for continuous bounded functions).

In the rest of the section we will study the Banach modules $L^\infty(G^n; V)$, with the sup norm, and endowed with the regular left representation. As already happened for the continuous bounded functions, the action on the bounded measurable functions is clearly isometric, but it isn't continuous. As one can easily guess, we want to show that the complex $(L^\infty(G^*, V), d)$ with the usual coboundary operator provides a strong relatively injective resolution of the Banach G-module V.

Proposition 2.4.4. *The module $L^\infty(G^{n+1}; V)$ is a relatively injective Banach G-module.*

Proof.

$$\begin{array}{ccc} A & \underset{\phi}{\overset{\sigma}{\rightleftarrows}} & B \\ \gamma \downarrow & \swarrow{\scriptstyle\delta} & \\ L^\infty(G^n, V) & & \end{array}$$

We have already pointed out in Proposition 2.4.3 that, for bounded weak*-measurable functions, we can rely on the isomorphism

$$L^\infty(G^{n+1}; V) \cong L^\infty(G, L^\infty(G^n; V)).$$

and hence, we can reduce to prove that $L^\infty(G; V)$ is relatively injective.

Moreover we rely on the fact that

$$\mathcal{C}L^\infty(G; V) = \mathcal{C}C_{cb}(G; V)$$

(also this result comes from hard functional analysis and we refer to [Mon01, Lemma 4.4.3] for a proof). Since we have already pointed out that a module is relatively injective if and only if its maximal continuous submodule is, we get the thesis of the proposition from Proposition 2.3.1.

$\square$

Proposition 2.4.5. *The Banach resolution of the continuous Banach G-module V*

$$0 \longrightarrow V \longrightarrow L^\infty(G; V) \longrightarrow L^\infty(G^2; V) \longrightarrow \cdots$$

is strong.

Proof. The result is proved in [Mon01, Lemma 7.5.5]. The contracting homotopy

$$k_i : \mathcal{C}L^\infty(G; L^\infty(G^n; V)) \rightarrow \mathcal{C}L^\infty(G^n; V)$$

can be constructed averaging with respect to any probability measure on G. Explicit formulas for k_i and the proof that $\mathrm{im}(k_i)$ lies in $\mathcal{C}L^\infty(G^n; V)$ are given in [Mon01, Proposition 5.5.1]. $\square$

Theorem 2.4.6. *The cohomology of the complex*

$$0 \longrightarrow L^\infty(G; V)^G \longrightarrow L^\infty(G^2; V)^G \longrightarrow L^\infty(G^3; V)^G \longrightarrow \cdots$$
(2.1)

is isometrically isomorphic to $H^*_{cb}(G; V)$.

Proof. We have already proved that $(L^\infty(G^i; V), d)$ is a strong relatively injective Banach resolution of V and hence the homology of the complex (2.1) is isomorphic to $\bar{H}^*_{cb}(G; V)$.

To prove that the isomorphism is isometric it is sufficient to exhibit a chain G-morphism $\mathcal{C}\bar{C}^i_{cb}(G, V) \rightarrow \mathcal{C}L^\infty(G^{i+1}; V)$ with norm bounded by one

in each dimension: this would imply that the norm induced on $\bar{H}_{cb}(G;V)$ by the resolution $(\mathcal{C}L^\infty(G^i;V),d)$ is not greater than the canonical seminorm and hence is equal to it (as a consequence of Remark 2.3.3).

To obtain such a chain G-morphism, we consider the composition

$$\bar{C}_{cb}^i(G;V) \hookrightarrow C_{cb}^i(G;V) \hookrightarrow L^\infty(G^{i+1};V)$$

that is obviously G-invariant and isometric. $\qquad\square$

2.5 Other useful resolutions

One of the main problems in generalizing the results we have obtained in Section 1.5 to the bounded context is that we cannot anymore rely on a statement of the form

$$C_{cb}(S^2, V) \cong C_{cb}(S, C_{cb}(S, V)),$$

and hence the proof of the relative injectivity of the various modules is far more complicated. One of the easiest way to overcome this difficulty is to use a generalized Bruhat function. We have already defined (in Chapter 1) a Bruhat function for the action of a closed subgroup on a locally compact topological group via left multiplication (see Definition 1.6.2). We now need a slightly more general definition that comprehends the one we gave in Section 1.5:

Definition 2.5.1. Let G be a group acting on a topological space X. A *generalized Bruhat function* for the action of G on X is a continuous function $h : X \to \mathbb{R}^+$ such that

- for every x in X, then $\int_G h(g^{-1}x)dg = 1$;

- for every compact subset K of X, the set $\mathrm{supp}(h) \cap \overline{GK}$ is compact.

We recall that a Hausdorff topological space is said *paracompact* if any open cover admits a locally finite open refinement; it is a mild assumption since, for example, every metrizable space is paracompact (see [Mon01, Remark 4.5.3], and any locally compact, σ-finite topological space is paracompact. The following proposition assures that generalized Bruhat functions exist if the action of G is proper.

Proposition 2.5.2. *Let X be a locally compact topological space such that G acts continuously and properly on X and such that the quotient is paracompact. Then there exists a generalized Bruhat function for the G-action on X.*

Proof. See [Mon01, Lemma 4.5.4] $\qquad\square$

Proposition 2.5.2 is useful to show that many modules are relatively injective:

Theorem 2.5.3. *Let G be a locally compact group that acts continuously and properly on a locally compact space such that the quotient $G\backslash X$ is paracompact. Then $C_{cb}(X,V)$ is a relatively injective Banach G-module.*

Proof. The proof is analogue to the proof of Proposition 1.6.5: the generalized Bruhat function allows to average on the G-orbits the value of $\alpha\sigma$ and hence to define the extension β by the formula

$$\beta(b)(x) = \int_G h(g^{-1}x)\alpha(g\sigma(g^{-1}b))(x).$$

The properties of the Bruhat function imply all the required properties (i.e. that β is well defined, continuous, invariant, extends α and has norm smaller then $\|\alpha\|$). See [Mon01, Theorem 4.5.2] for more details. $\square$

Theorem 2.5.4. *Let G be a locally compact group, let X be a locally compact topological space with a continuous proper G-action such that the quotient $G\backslash X^n$ is paracompact for every n. Then the cohomology of the complex*

$$0 \longrightarrow V \longrightarrow C_{cb}(X;V)^G \longrightarrow C_{cb}(X^2;V)^G \longrightarrow \cdots$$

is isometrically isomorphic to $\bar{H}_{cb}(G,V)$.

Proof. We have already proved that, under these hypoteses, the modules $C_{cb}(X^n;V)$ are relatively injective (using a Bruhat function for the action of G on X^n).

It is not obvious that the complex is strong: the cone operator doesn't induce a map between the maximal continuous submodules since the cone of a function that is uniformly continuous with respect to the diagonal left multiplication needs not to be uniformly continuous with respect to the left multiplication. However the existence of a Bruhat function for the action of G on the first factor X allows to construct a contracting homotopy that restricts to the maximal continuous submodules. Let us define the map σ:

$$\sigma : C_{cb}(X^{n+1};V) \to C_{cb}(X^n;V)$$
$$\sigma\phi(x_1,\ldots,x_n) = \int_G h(g^{-1}x)\phi(gx,x_1,\ldots,x_n)dx.$$

It is easy to verify (see [Mon01, Theorem 7.4.5]) that σ is continuous (since h is continuous) and that σ restricts to the maximal continuous submodules. Moreover the first property of a Bruhat function implies that σ has norm at most one and hence the complex is indeed strong.

It only remains to prove that the isomorphism is isometric. As a consequence of the minimality property of the canonical seminorm, it is sufficient to exhibit a norm decreasing chain map

$$V \longrightarrow \mathcal{C}\bar{C}_{cb}^0(G;V) \longrightarrow \mathcal{C}\bar{C}_{cb}^1(G;V) \longrightarrow \mathcal{C}\bar{C}_{cb}^2(G;V) \longrightarrow \cdots$$

$$\Big\| \mathrm{id} \qquad \Big\downarrow \alpha \qquad \qquad \Big\downarrow \alpha \qquad \qquad \Big\downarrow \alpha$$

$$V \longrightarrow \mathcal{C}C_{cb}(X;V) \longrightarrow \mathcal{C}C_{cb}(X^2;V) \longrightarrow \mathcal{C}C_{cb}(X^3;V) \longrightarrow \cdots$$

Again this can be defined using a Bruhat function for the action of G on X: if we identify the functions in $\bar{C}_{cb}^n(G,\mathbb{R})$ with bounded functions $\phi : G^{n+1} \to \mathbb{R}$, we can define

$$\alpha\phi(x_0,\ldots,x_n) = \int_{G^n} h(g_0^{-1}x_0)\ldots h(g_n^{-1}x_n)\phi(g_0,\ldots,g_n)dg_0\ldots dg_n.$$

Since, under the identification of $\bar{C}_{cb}^n(G,\mathbb{R})$ with a subspace of $C_{cb}^n(G,\mathbb{R})$, the inhomogeneous differential $\bar{d}$ is precisely the restriction of the homogeneous differential d, the maps α^* commute with the differential. It is also easy to verify that they are continuous and G-invariant and hence provide the required chain G-morphism. $\qquad\square$

Remark 2.5.5. Let us denote by $C_{cb}(X^n,V)_{alt}$ the subcomplex of the alternating functions. The alternating operator

$$\mathrm{Alt} : C_{cb}(X^n,V) \to C_{cb}(X^n,V)_{alt}$$
$$\mathrm{Alt}(\phi)(x_0,\ldots,x_n) = \tfrac{1}{(n+1)!}\sum_{\sigma\in\mathcal{S}_{n+1}}\phi(x_{\sigma(0)},\ldots,x_{\sigma(n)})$$

is a chain G-morphism that extends the identity and has norm, in every dimension, not greater than one. Since the canonical seminorm is the smallest norm on $\bar{H}_{cb}^*(G,V)$ (cfr. Remark 2.3.3), and since we have proved in Theorem 2.5.4 that the canonical seminorm is induced by the sup norm on $C_{cb}(X^n,V)^G$, the homology of $C_{cb}(X^n,V)_{alt}^G$ is isometrically isomorphic to $\bar{H}_{cb}^*(G,V)$.

Theorem 2.5.4 has many applications and will allow us to find analogues, in the bounded context, of the results of Section 1.3, Section 1.5 and Section 1.6.

A first important application is to show that the continuous bounded cohomology of a locally compact topological group can be computed from the homogeneous resolution:

Theorem 2.5.6. *If G is a locally compact group, there exist an isometric isomorphism*

$$H_{cb}^n(G,V) \cong \bar{H}_{cb}^n(G,V).$$

Proof. The topological space G^n satisfies the hypotesis of Theorem 2.5.4 since the action of G on G^{n+1} via diagonal left multiplication is obviously continuous and proper (since the diagonal is closed). Moreover the quotient is paracompact by Proposition 13 in [Bou52, Chapter 4, Paragraph 6]. $\qquad\square$

Another important application of Theorem 2.5.4 is the bounded version of Theorem 1.5.2 i.e. that we can factor a compact subgroup without changing the continuous bounded cohomology. As we have already set in Section 1.5, G will denote a locally compact group, K is a compact subgroup, G/K the topological space of left cosets as a quotient of G with respect to the right K action. We consider on G/K (and also on $(G/K)^i$) the action of G via left multiplication.

Theorem 2.5.7. *For every Banach G-module V, the cohomology of*

$$0 \longrightarrow C_{cb}(G/K; V)^G \longrightarrow C_{cb}((G/K)^2; V)^G \longrightarrow \cdots$$

is isometrically isomorphic to $H_{cb}^*(G; V)$.

Proof. We have only to prove that $(G/K)^n$ satisfies the hypotesis of Theorem 2.5.4. Clearly G/K is localy compact and the action of G on it is continuous and proper. It remains to prove that $G\backslash(G/K)^n$ is paracompact. Since left and right actions commute, we can first factor the diagonal action $\Delta(G)\backslash G^n$ and then consider the quotient with respect to the action of the compact group K^n: if X is paracompact and $X \to X/R$ is proper, also X/R is paracompact (see [Mon01, Lemma 4.5.9]). $\square$

As a consequence of Theorem 2.5.4, we also get that the seminorm induced in cohomology from the resolution $C_{cb}((G/K)^n; V)$ is the canonical seminorm. Since the morphisms

$$\pi^* : C_{cb}((G/K)^n; V) \to C_{cb}(G^n; V)$$

form a chain G-morphism that extend the identity and have norm smaller than one, we get that they induce isometric isomorphisms in cohomology.

Corollary 2.5.8. *Let $G = Isom^+(\mathbb{H}^n)$, let K be a maximal compact subgroup (corresponding to the stabilizer of a point $x \in \mathbb{H}^n$). The cohomology $H_{cb}^*(G; \mathbb{R})$ can be computed from the complex $C_{cb}^*((\mathbb{H}^n)^k, \mathbb{R})_{alt}^G$. Moreover the sup norm on this complex induces the canonical seminorm.*

Proof. It is joint corollary of Theorem 2.5.7 and Remark 2.5.5. $\square$

A last important application of Theorem 2.5.4 is the generalization of Theorem 1.6.1 to the bounded context. Namely if H is a closed subgroup of G we can compute the continuous bounded cohomolgy of H from the subcomplex of the H invariants of the resolution $(C_{cb}^n(G, V), d)$.

Theorem 2.5.9. *Let H be a closed subgroup of the locally compact group G. The homology of*

$$0 \longrightarrow C_{cb}(G; V)^H \longrightarrow C_{cb}(G^2; V)^H \longrightarrow C_{cb}(G^3; V)^H$$

is isometrically isomorphic to $H_{cb}^*(H, V)$.

Proof. We have to prove that, for every n, the quotient $\Delta(H)\backslash G^n$ is para-compact and this fact follows from the fact that, since H is closed in H, also $\Delta(H)$ is closed in G^n. $\qquad\square$

The thesis of Theorem 2.5.9 is that the homology of $C_{cb}(G^*;V)^H$ is isometric to $H^*_{cb}(H;V)$ we will need an explicit description of a map at the level of cochains that induces the isometry in cohomology. Let us consider the restriction map

$$C^n_{cb}(G,V) \to C^n_{cb}(H,V)$$

induced by the inclusion of H in G. It is obviously norm non-increasing and we have constructed, in the proof of Theorem 2.5.4, a homotopical inverse that is norm not increasing; this implies that the restriction map induces an isometry between the homologies of the subcomplexes of the H-invariants of the two complexes. We will use this result in Theorem 3.3.10 and in Proposition 4.1.1.

2.6 Amenable groups and amenable actions

A corollary of Theorem 1.5.2 is that the continuous cohomology of a compact group is trivial. An interesting feature of continuous bounded cohomology is that this fact can be generalized to a larger class, the *amenable groups*, that will be the subject of this section. The definition and the first properties of amenable topological groups can be found in [Gre69].

Definition 2.6.1. A topological group G is *amenable* if there exist a G-invariant *mean* on $L^\infty(G,\mathbb{R})$ i.e. a linear, norm one function $m : L^\infty(G,\mathbb{R}) \to \mathbb{R}$ such that

- $m(1) = 1$, $m(f) \geq 0$ if $f \geq 0$

- $(l_g)_* m = m$.

An useful tool in the study of amenable groups is the following theorem:

Theorem 2.6.2. *The following statements are equivalent:*

1. *there exists a left invariant mean on $L^\infty(G;\mathbb{R})$;*

2. *there exists a left invariant mean on $C_{cb}(G;\mathbb{R})$.*

Proof. Since $C_{cb}(G;\mathbb{R})$ is a closed subspace of $L^\infty(G;\mathbb{R})$, the invariant mean on $L^\infty(G;\mathbb{R})$ restricts to an invariant mean on $C_{cb}(G;\mathbb{R})$, hence the proof of the implication $1 \Rightarrow 2$ is trivial.

In order to prove the converse implication, let us recall that the (right) convolution of the functions $\phi \in L^1(G;\mathbb{R})$ and $f \in L^\infty(G;\mathbb{R})$ is defined by the formula

$$f * \phi(s) = \int f(t)\phi(t^{-1}s)dt$$

where dt denotes the left invariant Haar measure on G. The convolution has the property that, if ϕ has compact support, then, for every $f \in L^\infty(G; \mathbb{R})$, the function $f * \phi$ belongs to $C_{cb}(G; \mathbb{R})$; this follows from the continuity of the action (via left multiplication) of G on $L^1(G)$:

$$
\begin{aligned}
f * \phi(s) - f * \phi(sx) &= \int f(t)\phi(t^{-1}s) - f(t)\phi(t^{-1}sx)dt \\
&\leq \|f\|_\infty \int \phi(t^{-1}s) - \phi(t^{-1}sx)dt.
\end{aligned}
$$

Moreover, another useful property of the (right) convolution is that $(l_g f) * \phi = l_g(f * \phi)$:

$$
\begin{aligned}
(l_g f) * \phi(s) &= \int f(gt)\phi(t^{-1}s)dt \\
&= \int f(gt)\phi(t^{-1}g^{-1}gs)dt \\
&= \int f(t)\phi(t^{-1}gs)dt \\
&= l_g(f * \phi).
\end{aligned}
$$

Let us now fix a left invariant mean m on $C_c(G; \mathbb{R})$, we want to construct a mean on $L^\infty(G; \mathbb{R})$. Let us choose a compact neighborhood K of e and consider ϕ the characteristic function of K normalized so that $\int \phi = 1$. The first property of convolution assures that, for every $f \in L^\infty(G; \mathbb{R})$, the function $f * \phi$ belongs to $C_c(G; \mathbb{R})$. We can thus define a mean $\widetilde{m}$ on $L^\infty(G; \mathbb{R})$ by requiring that $\widetilde{m}(f) = m(f * \phi)$. The left invariance of m implies that also $\widetilde{m}$ is invariant: $\widetilde{m}(l_g f) = m((l_g f) * \phi) = m(l_g(f * \phi)) = m(f^* \phi) = \widetilde{m}(f)$. $\square$

A first application of Theorem 2.6.2 is that it enables to deduce result on the amenablity of topological groups from the amenability of discrete groups. Here by a discrete amenable group G we mean a group that admits a G-invariant mean on the bounded functions from G to $\mathbb{R}$; the theory of amenability for discrete groups is somehow easier than that for topological groups. When we deal with topological group G, we can consider G as a discrete group (forgetting its topology). Theorem 2.6.2 implies that if G is amenable as a discrete group, it is also amenable as a topological group (since the continuous bounded functions are a subspace of the bounded functions). The converse implication is false, we will now prove that compact groups are amenable as topological groups, but some ortogonal groups are not amenable as discrete groups (see [Gre69, p.26]).

Proposition 2.6.3. *Any compact group K is amenable.*

Proof. The map

$$
\begin{aligned}
L^\infty(K, V) &\to V \\
f &\mapsto \tfrac{1}{\mu(K)} \int_K f d\mu
\end{aligned}
$$

is a desired mean (since the Haar measure is left invariant). $\square$

Another class of groups that are amenable are the abelian groups:

Theorem 2.6.4. *Let A be abelian, then A is amenable.*

Proof. Any discrete group that is abelian is amenable (see [Gre69, Theorem 1.2.1]). In particular this implies that a topological abelian group is amenable since the continuous bounded functions are a subspace of the bounded ones (see [Gre69, Theorem 2.2.1]). $\qquad\square$

The class of amenable groups is closed with respect to quotients, closed subgroups and extensions:

Theorem 2.6.5. *Homomorphic images of amenable groups are amenable, closed subgroups of amenable groups are amenable, extensions of amenable groups via amenable groups are amenable.*

Proof. The assertions are [Gre69, Theorem 2.3.1], [Gre69, Theorem 2.3.2] and [Gre69, Theorem 2.3.3] respectively. $\qquad\square$

The importance of amenable groups in the theory of continuous bounded cohomology is due to the fact that their cohomology is zero.

Proposition 2.6.6. *If G is amenable, then $H_{cb}^*(G;\mathbb{R}) = 0$.*

Proof. It is sufficient to show that $\mathbb{R}$ is relatively injective as G-module. However, since there exists a G-invariant mean

$$m : L^\infty(G,\mathbb{R}) \to \mathbb{R}$$

that is left inverse to the inclusion of $\mathbb{R}$ as coefficients, we get that $\mathbb{R}$ is relatively injective (since $L^\infty(G,\mathbb{R})$ is). $\qquad\square$

Proposition 2.6.6 is far more general: it can be proved that, for every coefficient G-module V, the mean $m : L^\infty(G,\mathbb{R}) \to \mathbb{R}$ allows to construct a mean on $L^\infty(G,V)$ and thus that V is relatively injective; this implies that $H_{cb}^*(G,V) = 0$. Since we will need that means on $L^\infty(G,V)$ exist we state this result for further reference.

Proposition 2.6.7. *Let G be an amenable group an V a separable reflexive Banach G-module. Then there exists a G-invariant mean $L^\infty(G,V) \to V$.*

Proof. The result follows from [Mon01, Example 5.7.3] and [Mon01, Theorem 5.6.1]. $\qquad\square$

One of the most interesting features of amenable groups is that we can use the invariant mean as an analogue of Haar measure in oreder to average measurable cochains:

Proposition 2.6.8. *Let P be an amenable group, then there exists a G-invariant linear map*

$$L^\infty(G,V) \overset{\alpha}{\longrightarrow} L^\infty(G/P,V)$$

such that $\|\alpha\| \leq 1$ and such that α is left inverse to the inclusion $i : L^\infty(G/P,V) \to L^\infty(G,V)$.

Proof. We identify $L^\infty(G/P, V)$ with the submodule of $L^\infty(G, V)$ invariant with respect to right translations of elements in P. The invariant mean on $L^\infty(P, L^\infty(G, V))$ allows to generalize the proof of Proposition 1.5.1 in this context interchanging the role of the integral with that of the mean.

Let us fix $f \in L^\infty(G, V)$, we define the function

$$\Phi_f : \quad \begin{aligned} P &\to L^\infty(G, V) \\ p &\mapsto r_p f \end{aligned}$$

and we put $\alpha(f) = m(\Phi_f)$. The map α is obviously linear (since m is) and has norm one. Moreover it is a G-morphism since left and right actions commute. It only remains to prove that its image is contained in the space of (right) P-invariant functions.

$$\begin{aligned} r_p \alpha(f)(g) = \; & \alpha(f)(gp) \\ & = m(\Phi_f)(gp) \\ & = m(l_{p^{-1}} \Phi_f)(g) \\ & = \alpha(f)(g). \end{aligned}$$

Here above we used the fact that $\Phi_f(x)(gp) = r_x f(gp) = f(gpx) = r_{px} f(g) = (l_{p^{-1}} \Phi_f(x))(g)$. $\qquad\qquad\square$

The theory of amenable groups has been widely developed by Zimmer (see [Zim84]), who constructed the theory of amenable actions (of groups on topological spaces). This theory has deep implications on continuous bounded cohomology but relies on non-elementary results of ergodic theory and goes behind the purposes of this thesis. We will only give a definition of an amenable action (different from Zimmer's original but equivalent to that) and draw an elementary consequence.

Definition 2.6.9. Let (S, μ) be a G-space with a quasiinvariant measure. The action of G on (S, μ) is *amenable* if there exists a projection $m : L^\infty(G \times S; \mathbb{R}) \to L^\infty(S; \mathbb{R})$ that is

- G-equivariant;

- $L^\infty(S; \mathbb{R})$ linear,

- $m(1_{G \times S}) = 1_S$, and $m(f) \geq 0$ for every $f \geq 0$.

Proposition 2.6.10. *If the action of G on S is amenable, then $L^\infty(S; \mathbb{R})$ is relatively injective*

Proof. We have seen in Proposition 2.4.3 that $L^\infty(G \times S; \mathbb{R}) \cong L^\infty(G; L^\infty(S; \mathbb{R}))$. Under this identification, the mean is the norm one G-invariant left inverse to the coefficient inclusion

$$L^\infty(S; \mathbb{R}) \hookrightarrow L^\infty(G; L^\infty(S; \mathbb{R})).$$

By Lemma 1.2.4 this implies that $L^\infty(S; \mathbb{R})$ is relatively injective (since $L^\infty(G; L^\infty(S; \mathbb{R}))$ is. $\qquad\qquad\square$

The following fundamental Theorem relates the amenability of a closed subgroup P of G and that of the action of G on the coset space:

Theorem 2.6.11. *Let G be a locally compact group and H a closed subgroup. The action of G on G/H is amenable if and only if H is an amenable group.*

Proof. We refer to [Zim84, Proposition 4.3.2]. $\qquad\qquad\square$

We can now come back to the continuous bounded cohomology and use the properties of amenable subgroups to find another useful resolution (that will be fundamental in Chapter 4).

Proposition 2.6.12. *The module $L^\infty(G/P, V)$ is relatively injective as Banach G module if P is amenable.*

Proof. It is a consequence of Proposition 2.6.8. $\qquad\qquad\square$

Theorem 2.6.13. *Let G be a locally compact second countable group. Let P be an amenable subgroup, V a continuous Banach G-module. The cohomology of the complex*

$$0 \longrightarrow L^\infty(G/P; V)^G \longrightarrow L^\infty((G/P)^2; V)^G \longrightarrow L^\infty((G/P)^3; V)^G$$

is isometrically isomorphic to $H_{cb}(G, V)$.

Proof. Since $L^\infty((G/P)^{n+1}, V) \cong L^\infty(G/P, L^\infty((G/P)^n, V))$, every module in the associated resolution is relatively injective. We need to prove that the resolution is strong, but, by the same arguments of Proposition 2.4.5, a contracting homotopy corresponds to an inverse of the coefficient inclusion $L^\infty((G/P)^n, V) \hookrightarrow L^\infty(G/p, L^\infty((G/P)^n, V))$ and can be constructed by averaging with respect to a probability measure on G/P (we do not require any type of invariance on the contracting homotopy).

With the aid of the usual homological algebra this implies that the cohomology of the complex is isomorphic to the continuous bounded cohomology of G.

It remains to show that this complex induces the canonical seminorm. We have already pointed out that there exists a norm one G-morphism $L^\infty(G, V) \to L^\infty(G/P, V)$ (in Proposition 2.6.8). This allows to construct inductively a G-morphism $\alpha : L^\infty(G^n, V) \to L^\infty((G/P)^n, V)$ using twice the isomorphism $L^\infty(X, V) \cong L^\infty(X, L^\infty(X^{n-1}, V))$, where X is any topological space. The proof that the constructed morphism has the required properties is rather complicated and we refer to [Mon01, Lemma 7.5.6] for the details. $\qquad\square$

2.7 Bounded cohomology of topological spaces

The aim of this section is to define another cohomological theory that is closer to the geometric applications and that will allow us, in the next chapters, to interpret geometric problems using the group cohomology theories we have just defined. Let us consider a topological space X. We will denote by

$$S_i(X) = \{\sigma : \triangle^i \to X \,|\, \sigma \text{ is continuous}\}$$

the set of singular simplices of X. Moreover we will denote by

$$C^i(X, \mathbb{R}) = \{f : S_i(X) \to \mathbb{R}\}$$

the singular cochains with real coefficent. We consider on these vector spaces the l^∞ norm with respect to the basis provided by the simplices and we are interested in the subcomplex of the bounded functions. Obviously the coboundary operator restricts to the submodules of the bounded cochains:

$$C_b^i(X, \mathbb{R}) = \{f \in C^i(X, \mathbb{R}) \,|\, \sup_{\sigma \in S_i} |f(\sigma)| = \|f\|_\infty < \infty\}.$$

Definition 2.7.1. The *bounded cohomology* of X is the cohomology of the complex of the bounded cochains.

We will denote by $H_b^i(X, \mathbb{R})$ the ith group of bounded cohomology of the topological space X.

As one can easily guess, since we are talking about a normed chain complex, the bounded cohomology is endowed with a seminorm (induced by the l^∞ norm on cochains) and we will be particularly interested in its properties.

It is rather easy to prove the fact that the bounded cohomology of a topological space is a homotopy invariant, however it doesn't satisfy any analogue of Mayer-Vietoris sequence and hence it is very difficult to compute and has some strange behaviours: for example, even when we restrict to a manifold M, the bounded cohomology of M can be an infinite dimensional vector space even in dimension higher than the dimension of the manifold.

The inclusion of the bounded cochains in the singular cochains induces, in cohomology, the *comparison* map that is, in general, not injective nor surjective but, as we will see, has very interesting geometric applications. We will always denote the comparison map by

$$c : H_b^*(X; \mathbb{R}) \to H^*(X; \mathbb{R}).$$

A deep result by Gromov (see [Gro82, Section 3.3]), proved in detail by Ivanov in [Iva87, Theorem 4.3] relates the bounded cohomology of a space with that of its fundamental group:

Theorem 2.7.2 (Gromov's mapping Theorem). *Let X be a connected countable CW-complex. There exists an isomorphism*

$$H_b^i(X; \mathbb{R}) \cong H_b^i(\pi_1(X, x); \mathbb{R})$$

that is isometric.

Let us consider $\widetilde{X}$ the universal covering of the topological space X. The action of $\pi_1(X)$ on $\widetilde{X}$ via deck transformation induces an action of $\pi_1(X)$ on $C_b^i(\widetilde{X}, \mathbb{R})$. A crucial step in the proof of Gromov's mapping Theorem is the observation that the projection $p : \widetilde{X} \to X$ induces an isomorphism

$$p^* : C_b^i(X, \mathbb{R}) \to C_b^i(\widetilde{X}, \mathbb{R})^{\pi_1(X)}.$$

This implies that the bounded cohomology of X with real coefficients is the homology of the complex

$$0 \longrightarrow C_b^0(\widetilde{X}, \mathbb{R})^{\pi_1(X)} \longrightarrow C_b^1(\widetilde{X}, \mathbb{R})^{\pi_1(X)} \longrightarrow C_b^2(\widetilde{X}, \mathbb{R})^{\pi_1(X)} \longrightarrow \cdots$$

The latter is the complex of the $\pi_1(X)$ invariants of the resolution of $\mathbb{R}$ provided by the modules $C_b^i(\widetilde{X}, \mathbb{R})$. So, in order to prove Gromov's mapping Theorem, it is sufficient to show that that resolution is strong and relatively injective. It is not difficult to prove directly that the modules are relatively injective:

Proposition 2.7.3. *The modules $C_b^i(X, \mathbb{R})$ are relatively injective as $\pi_1(X)$-modules.*

Proof. The proof of the relative injectivity of $C_b^i(\widetilde{X}, \mathbb{R})$ is very easy. This is because $\pi_1(X)$ is a discrete group and hence we do not have to deal with any continuity problem. Let us consider a fundamental region F for the action of $\pi_1(X)$ on $\widetilde{X}$ and consider the subset $\bar{S}_i(\widetilde{X})$ of $S_i(\widetilde{X})$ made of the simplexes with first vertex belonging to F. The set $\bar{S}_i(\widetilde{X})$ meets any $\pi_1(X)$-orbit of $S_i(\widetilde{X})$ in exactly one point. Moreover the action of $\pi_1(X)$ on $S_i(\widetilde{X})$ is free. This means that, for every $\sigma \in S_i(\widetilde{X})$, there exist a unique $\bar{\sigma} \in \bar{S}_i(\widetilde{X})$, and a unique $g \in \pi_1(X)$ such that $\sigma = g\bar{\sigma}$.

Let us consider, as usual, an extension problem

$$\begin{array}{c} A \xrightarrow{\gamma \atop \alpha} B \\ {\scriptstyle\beta}\downarrow \quad \swarrow {\scriptstyle\delta} \\ C_b^i(\widetilde{X}, \mathbb{R}) \end{array}$$

Let us define the function $\delta(\sigma)$ by requiring

$$\delta(b)(\sigma) = \beta\gamma g^{-1}(b)(\bar{\sigma}).$$

It is easy to verify that the map δ is well defined and G-invariant and that makes the diagram commutative. $\square$

Unfortunately the proof that the resolution is strong is, in general, very difficult and based on results on the amenability of abelian groups (an analogue statement is false in the unbounded setting whenever the universal covering of X is not contractible). Since we will not need this statement in the general case, we omit the proof of the theorem in full generality.

Anyway, if the universal covering of X is contractible, the proof of the fact that the resolution is strong is simple: let us consider the standard proof of the fact that the singular cohomology of a contractible space is trivial (see e.g. [Hat02, page 201]). The cone operator defined in that proof restricts to the bounded cochains providing a contracting homotopy. This result is sufficient for our applications since we will deal only with spaces whose universal covering is diffeomorphic to $\mathbb{R}^n$ and hence, in particular, contractible.

We will need (in the proof of Theorem 3.3.11) that the homology of the complex $(C_b^i(\widetilde{X}; \mathbb{R})^{\pi_1(X)}, d)$ is isometrically isomorphic to the bounded cohomology of $\pi_1(X)$.

Proposition 2.7.4. *Let X be a topological space whose universal covering is contractible, then $H_b^*(\pi_1(X), \mathbb{R})$ and $H_b^*(X, \mathbb{R})$ are isometrically isomorphic.*

Proof. We have already justified that the two groups are isomorphic. Let us prove that the isomorphism is isometric. For an argument that we have widely used in the chapter, it is sufficient to exhibit a chain $\pi_1(X)$-morphism $\alpha : C_b^i(\pi_1(X); \mathbb{R}) \to C_b^i(\widetilde{X}; \mathbb{R})$ that extends the identity on $\mathbb{R}$ and has norm not greater than one.

Let us fix again a fundamental region F for the action of $\pi_1(X)$ on the universal covering $\widetilde{X}$. For every vertex x_j of a simplex σ, there exists a unique $g_j \in \pi_1(X)$ such that $x_j \in g_j F$. We define the morphism α by the formula:
$$\alpha(\phi)(\sigma) = \phi(g_0, \ldots, g_i).$$

Obviously α is a chain morphism that extends the identity and has norm not greater than one. Moreover it is easy to verify that α is $\pi_1(X)$-invariant. $\square$

More details of the proofs of Proposition 2.7.3 and Proposition 2.7.4, toghether with the complete proof of Gromov's mapping Theorem can be found in Ivanov's paper [Iva87].

Chapter 3

Simplicial volume

In this chapter we will introduce the central object in the thesis: the simplicial volume. This invariant of closed connected oriented manifolds first appeared in Gromov's seminal article "Volume and Bouded Cohomology" [Gro82] and in Thurston's lecture notes [Thu79].

The simplicial volume, even if being a homotopical invariant, is deeply related to metric properties. An example of these relations is given by Gromov's Proportionality Principle: let us fix on M a Riemannian structure (and hence a metric universal covering $\widetilde{M}$), the rate $\|M\|/\mathrm{vol}(M)$ depends only on the isometry type of the Riemannian covering $\widetilde{M}$. This fundamental Theorem has many applications in different areas: for example, in case of hyperbolic manifolds it is a fundamental tool in Gromov's simple proof of Mostow rigidity Theorem [BePe92, Section C.5] and it is, in general, useful in proving degree theorems [LoSa09].

In the next chapters we will study properties of the simplicial volume with particular care towards the proportionality principle and, more precisely, to the explicit computation of the proportionality constant involved in that theorem. In this chapter, in addition to the definition and the first properties of simplicial volume, we will recall Bucher-Karlsson's proof of proportionality principle for locally symmetric spaces that relies on the theory of continuous and continuous and bounded cohomolgy introduced in the first two chapters. In the last section we will study in detail the case of hyperbolic manifolds comparing the cohomological approach of Bucher-Karlsson to Gromov's original approach.

Chapter 4 will be devoted to the case of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ where the introduced cohomological machinery can be used to give the exact computation of the proportionality constant (note, however, that in this case no homological version of the proof is known).

3.1   Simplicial volume

In order to define simplicial volume we will put a seminorm on singular homology. Let M be a m-dimensional manifold. The singular homology of M with real coefficients is the homology of the complex

$$\mathbb{R} \xleftarrow{\ \partial_0\ } C_0(M;\mathbb{R}) \xleftarrow{\ \partial_1\ } C_1(M;\mathbb{R}) \xleftarrow{\ \partial_2\ } C_2(M;\mathbb{R}) \xleftarrow{\quad} \cdots$$

where the i-singular chains are the free $\mathbb{R}$-module over i-singular simplices: if we denote by $\triangle^i$ the standard i-simplex, we call

$$S_i(M) = \{\sigma_i : \triangle^i \to M \,\|\, \sigma \text{ is continuous}\}.$$

With this convention, $C_i(M;\mathbb{R})$ is the free $\mathbb{R}$-module on $S_i(M)$:

$$C_i(M;\mathbb{R}) = \left\{ \sum_{j=1}^n a_j\sigma_j \,\Big|\, a_j \in \mathbb{R}, \ \sigma_j \in S_i(M) \right\}.$$

If we denote by $f_i^k : \triangle^i \to \triangle^{i+1}$ the linear parametrization of the k-th face of $\triangle^{i+1}$, the boundary operator is defined by linearly extending the operator

$$\delta_{j+1}(\sigma) = \sum_{i=0}^{j+1} (-1)^i \sigma \circ f_j^i.$$

We can endow every vector space $C_i(M;\mathbb{R})$ with the l_1-norm with respect to the basis formed by the simplices:

$$\Big\| \sum_{j=1}^n a_j\sigma_j \Big\|_1 = \sum_{j=1}^n |a_j|.$$

It is easy to prove that this is actually a norm and that the boundary operator is continuous with respect to this norm. This means that the l_1-norm makes $(C_*(M;\mathbb{R}), \delta)$ a normed chain complex.

   As usual when dealing with a normed chain complex (see Section 1.1), the norm we have just described at the cochain level induces a seminorm in homology taking the infimum over the norms of the representatives of a homology class:
$$\|\beta\| = \inf_{\substack{a \in C_i(M;\mathbb{R}) \\ [a] = \beta}} \|a\|_1.$$

Note that, since the image of δ oughts not to be closed, there is no reason, in general, for $\|\cdot\|$ to be a norm, as we will see in the next example:

Example 3.1.1. Let us consider $M = S^1 = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ and let $[\beta] = [S^1]$ the class in $H_1(S^1; \mathbb{R})$ of the simplex parametrized by $t \to e^{2\pi i t}$ (i.e. the fundamental class of S^1). If we denote by β_n the simplex

$$\begin{aligned} \beta_n : \quad I &\to S^1 \\ t &\to e^{2\pi i n t}, \end{aligned}$$

an easy application of Hurewicz Theorem shows that $[\beta_n] = n[\beta]$. Since, obviously, $\|[\beta_n]\| \leq 1$, we get that $\|[S^1]\| \leq \inf \frac{1}{n} = 0$.

The simplicial volume of a manifold M is the l_1-seminorm of its fundamental class with real coefficients. To make this statement more precise we need to recall some other definitions from algebraic topology. The following fact is well known [Hat02, Theorem 3.26]:

Theorem 3.1.2. *Let M be a closed connected orientable manifold, with $m = \dim(\mathrm{M})$. Let $\mathcal{T}$ be an arbitrary triangulation of M. If we consider the singular homology with integral coefficients, the following facts hold:*

- *$H_m(M; \mathbb{Z}) \cong \mathbb{Z}$;*

- *for an appropriate choice of signs k_i, the class of $\sum_{\sigma_i \in \mathcal{T}} k_i \sigma_i$ is a generator of $H_m(M; \mathbb{Z})$ that we will call $[M]$;*

- *for every $x \in M$, we have $H_m(M, M \backslash \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ and the projection*

$$H_m(M; \mathbb{Z}) \to H_m(M, M \backslash \{x\}; \mathbb{Z})$$

  maps $[M]$ to a generator of $H_m(M, M \backslash \{x\}; \mathbb{Z})$.

Note that the choice of an orientation for M corresponds to the choice of one of the two possible fundamental classes $[M]$ of the manifold.

An easy formulation of the Theorem of universal coefficients for homology implies that $H_*(M; \mathbb{R}) \cong H_*(M; \mathbb{Z}) \otimes \mathbb{R}$ and, moreover, that the isomorphism is induced by the inclusion

$$i : C_*(M, \mathbb{Z}) \to C_*(M; \mathbb{R}).$$

The real fundamental class of M (that we will denote by $[M]_{\mathbb{R}}$ or also, when this will not cause confusion, by $[M]$) is the generator of $H_*(M; \mathbb{R}) \cong \mathbb{R}$ defined by:

$$[M]_{\mathbb{R}} = i_*[M].$$

Definition 3.1.3. Let M be a closed, connected, orientable manifold of dimension m, the *simplicial volume* of M is

$$\|M\| = \|[M]_{\mathbb{R}}\| = \inf_{\substack{a \in C_m(M; \mathbb{R}) \\ [a] = [M]_{\mathbb{R}}}} \|a\|_1.$$

We have defined the simplicial volume only for closed connected orientable manifolds, and, moreover, it is easier to deal with a fundamental class when a manifold is triangulable. Since we are interested in the study of this invariant for symmetric spaces that are smooth manifolds (hence, in particular, triangulable), from now on all manifolds will be understood to be smooth, closed, connected and orientable, unless otherwise stated. Moreover we will often denote the dimension of a manifold with an apex: with M^m we mean that $\dim(M) = m$.

Since in the definition of the simplicial volume only the singular homology of a manifold is involved, the object we have just defined is a homotopy invariant of closed manifolds:

Proposition 3.1.4. *Let M and N be two manifolds of dimension n, let moreover $f : M \to N$ be a homotopy equivalence, then $\|M\| = \|N\|$.*

Proof. The homotopy invariance of singular homology [Hat02, Corollary 2.11], implies that $f_n : H_n(M; \mathbb{Z}) \to H_n(N; \mathbb{Z})$ is an isomorphism, in particular $f_n([M]) = \pm[N]$. Now the definition of simplicial volume as infimum of the l_1-norm of the representatives of the real fundamental class implies that $\|N\| \leq \|M\|$: if the chain $\alpha = \sum a_i \sigma_i$ represents $[M]$, its image via f_*, the chain $\sum a_i f \circ \sigma_i$, represents $\pm[N]$ and has norm equal to $\|\alpha\|_1$. Applying the same argument to the homotopical inverse of f we get the desired equality. $\square$

Recall that the *degree* of a map between two manifolds of the same dimension n can be defined as the integer d such that $f_n([M]) = d[N]$. The ideas at the base of the proof of Proposition 3.1.4 lead to vanishing results for the simplicial volume:

Proposition 3.1.5. *If a manifold M^m admits a self map of degree greater than 1, then its simplicial volume vanishes:*

$$\|M\| = 0.$$

Proof. We have already pointed out that, for every selfmap $f : M \to M$ and for every class a in $H_*(M; \mathbb{R})$, we have $\|f_*(a)\| \leq \|a\|$. Moreover the seminorm induced in homology obviously satisfies $\|d\alpha\| = d\|\alpha\|$ for d in $\mathbb{Z}$, and α in $H_i(M; \mathbb{R})$. This implies that, if the map $f : M \to M$ has degree d,

$$d\|M\| = \|d[M]\| = \|f_n([M])\| \leq \|M\|,$$

that forces $\|M\|$ to be zero. $\square$

In the next proposition we will examine another consequence of the definition of simplicial volume, namely its behaviour under finite coverings:

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Proposition 3.1.6. *Let $\pi : \widetilde{M} \to M$ be a d-sheeted covering of a manifold M. Then $\|\widetilde{M}\| = d\|M\|$.*

Proof. Since π has degree d, we have already shown that $\|\widetilde{M}\| \geq d\|M\|$. To prove the equality, we construct a representative for the fundamental class of $\widetilde{M}$ starting from one of M.

Since π is a d-sheeted covering, every simplex σ in $S_m(M)$ has d different lifts $\{\sigma^i\}_{i=1}^d$ in $S_m(\widetilde{M})$. Moreover, given a representative $\alpha = \sum a_j \sigma_j$ of the fundamental class $[M]$, the chain $\widetilde{\alpha} = \sum a_j (\sum_{i=1}^d \sigma_j^i)$ is closed and represents the fundamental class $[\widetilde{M}]$. This implies our conclusion since

$$\|\widetilde{\alpha}\|_1 = d\|\alpha\|_1.$$

$\square$

Note that also the volume has the same behaviour under finite coverings: if $\widetilde{M}$ is a d-sheeted metric covering of the Riemannian manifold M, it is easy to verify that $\mathrm{vol}\, \widetilde{M} = d\, \mathrm{vol}\, M$. This implies that

$$\frac{\mathrm{vol}\, \widetilde{M}}{\|\widetilde{M}\|} = \frac{\mathrm{vol}\, M}{\|M\|}.$$

This fact is far more general: Gromov's Proportionality Principle (Theorem 3.3.14) states that the hypotesis that the covering is finite sheeted is unnecessary.

A joint Corollary of Proposition 3.1.5 and Proposition 3.1.6 is that the simplicial volume of flat and elliptic manifolds vanishes:

Proposition 3.1.7. *Let M^m be a closed, connected oriented manifold that admits a metric with constant sectional curvature equal to 0 or 1. Then*

$$\|M\| = 0.$$

Proof. The simplicial volume of the sphere S^m vanishes since this manifold admits a selfmap of degree 2. To give an example of such a map we can view the m-sphere as a subset of the product $\mathbb{C} \times \mathbb{R}^{m-1}$

$$S^m = \{(x, y) \in \mathbb{C} \times \mathbb{R}^{m-1} \,|\, |x|^2 + \|y\|^2 = 1\}$$

and consider the map
$$\phi : S^m \to S^m$$
$$(\rho e^{it}, y) \mapsto (\rho e^{2it}, y)$$

that has degree 2. This implies that $\|S^m\| = 0$ for every m.

Let us now consider M^m a Riemannian closed manifold with sectional curvature constant and equal to 1. Its universal metric covering is isomorphic

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to the sphere S^m (since the manifold M is complete). Thus M is the quotient of S^m with respect to a group of isomorphisms of S^m whose action is free and properly discontinuous. Since S^m is compact, this group must be finite and hence $\pi : S^m \to M$ is a finite covering. We have shown that $\|S^m\| = 0$, this implies that also $\|M\| = 0$.

A similar argument applies for flat manifolds: a corollary of Bieberbach Theorem [Rat06, Theorem 8.2.5] is that every compact n-dimensional Euclidean manifold (whose universal metric covering is, hence, $\mathbb{R}^n$) is finitely covered by an n-torus. Since, again, every torus admits self-maps of degree strictly greater than 1, we get as a result that the simplicial volume of flat manifolds is null.

$\square$

The behaviour of hyperbolic manifolds is far different: we will show in Theorem 3.4.1 that if the manifold M^m admits a Riemannian structure with constant sectional curvature equal to -1, its simplicial volume satisfies

$$\|M\| = \frac{1}{v_m}\mathrm{vol}(M),$$

where v_m denotes the volume of the ideal regular simplex in $\mathbb{H}^m$. In particular the simplicial volume of M is different from zero.

3.2   Relations with bounded cohomology

We have defined in Section 2.7 the bounded cohomology of a topological space as the cohomology of the normed chain complex $(C_b^*(M;\mathbb{R}), \delta)$ of bounded cochains from $S_*(X)$ to $\mathbb{R}$. Indeed the bounded cochains $(C_b^i(M;\mathbb{R}), \|\cdot\|_\infty)$ form the topological dual of the normed vector space $(C_i(M;\mathbb{R}), \|\cdot\|_1)$: from the definition of l^∞-norm on bounded cochains it follows that, for every cochain $\beta \in C_b^i(M;\mathbb{R})$,

$$\|\beta\|_\infty = \sup\{\beta(\alpha)|\, \alpha \in C_i(M;\mathbb{R}), \|\alpha\|_1 = 1\}.$$

A standard consequence of Hahn-Banach Theorem is that also the converse is true:

Proposition 3.2.1. *For every chain $\alpha \in C_i(M;\mathbb{R})$,*

$$\|\alpha\|_1 = \sup\{\beta(\alpha)|\, \beta \in C_b^i(M;\mathbb{R}), \|\beta\|_\infty = 1\}.$$

We now aim to a result first pointed out by Gromov in [Gro82]: an analogue of Proposition 3.2.1 holds also in cohomology. Since the coboundary operator on bounded cochains is the dual to the boundary operator on singular homology, it is easy to verify that the Kronecker product:

$$H_*(M;\mathbb{R}) \otimes H_b^*(M;\mathbb{R}) \to \mathbb{R}$$
$$\langle [\alpha], [\beta] \rangle = \beta(\alpha).$$

is well defined.

The Kronecker product allows to generalize the duality between the l^1-norm on chains and the l^∞-norm on cochains highlighted in Proposition 3.2.1 to a duality between the l^1-seminorm induced in homology and the l^∞-seminorm on bounded cohomology we studied in Section 2.7. The following theorem makes this statement more precise and will be crucial since it relates the simplicial volume and the bounded cohomolgy:

Theorem 3.2.2. *Let us fix $a \in H_i(M; \mathbb{R})$. If $\|a\|_1 \neq 0$,*

$$\|a\|_1^{-1} = \inf\{\|b\|_\infty \mid b \in H_b^i(M; \mathbb{R}), \langle a, b \rangle = 1\},$$

moreover $\|a\|_1 = 0$ if and only if the set on the right-hand side is empty.

Proof. The inequality $\leq$ descends from the definition of Kronecker product: let us fix an element b in $H_b^i(M; \mathbb{R})$ such that $\langle a, b \rangle = 1$. If $\alpha \in C_i(M; \mathbb{R})$ represents a and $\beta \in C_b^i(M; \mathbb{R})$ represents b, then

$$1 = \langle a, b \rangle = \beta(\alpha) \leq \|\beta\|_\infty \|\alpha\|_1.$$

If we take the infimum over the representatives at the right-hand side, we get that

$$1 \leq \|b\|_\infty \|a\|_1.$$

This means that, if $\|a\|_1 = 0$, then the set on the right-hand side in the statement is empty, and, otherwise, that

$$\|a\|_1^{-1} \leq \inf\{\|b\|_\infty \mid b \in H_b^i(M; \mathbb{R}), \langle a, b \rangle = 1\}.$$

The opposite inequality is more subtle and descends from Hahn-Banach Theorem: let, again, $\alpha \in C_i(M; \mathbb{R})$ be a representative for a and let V be the subspace of $C_i(M; \mathbb{R})$ generated by α and the i-boundaries, i.e.

$$V = \langle \alpha \rangle \oplus \partial C_{i+1}(M; \mathbb{R}).$$

Since we can assume $\|\alpha\|_1 > 0$, the cycle α doesn't belong to $\partial C_{i+1}(M; \mathbb{R})$. Let us consider the element $\beta \in V^*$ defined by

$$\beta(\partial C_{i+1}(M; \mathbb{R})) = 0,$$
$$\beta(\alpha) = 1. \tag{3.1}$$

Hahn-Banach Theorem states that there exists an element $\widetilde{\beta} \in C_i(M; \mathbb{R})^* = C_b^i(M; \mathbb{R})$ that extends β and such that $\|\widetilde{\beta}\|_\infty = \|\beta\|_\infty$. Clearly $\widetilde{\beta}$ is a coboundary: $d\widetilde{\beta} = 0$ since $\beta(\partial C_{i+1}(M; \mathbb{R})) = 0$. Moreover the class $[\widetilde{\beta}]$ satisfies $\langle a, [\widetilde{\beta}] \rangle = 1$. Let us compute its norm:

$$\|\widetilde{\beta}\|_\infty = \|\beta\|_\infty = \sup_{c \in C_{i+1}} \frac{\beta(a + \partial c)}{\|a + \partial c\|_1} = \frac{1}{\inf \|a + \partial c\|_1} = \frac{1}{\|a\|_1}.$$

$\square$

If a is the fundamental class of a manifold M, Theorem 3.2.2 can be restated as follows:

Corollary 3.2.3. *Let M^m be a closed connected oriented manifold. Then*

$$\|M\|^{-1} = \inf\{\|b\|_\infty \mid b \in H_b^m(M; \mathbb{R}), \langle [M], b \rangle = 1\},$$

where we consider $\|M\| = 0$ if and only if the infimum is taken over the empty set.

We can restate Corollary 3.2.3 in term of a *fundamental coclass* of M, i.e. the generator $[M]^{\mathbb{R}}$ of $H^n(M; \mathbb{R}) \cong \mathbb{R}$ such that $\langle [M]_{\mathbb{R}}, [M]^{\mathbb{R}} \rangle = 1$, where we denoted by $\langle \ , \ \rangle$ the Kronecker product on singular cohomology that induces (as a consequence of Universal Coefficient Theorem for cohomolgy) an isomorphism between the dual of $H_n(M; \mathbb{R})$ and $H^n(M; \mathbb{R})$.

The sup norm on singular cochains induces a seminorm also on singular cohomology (if we let $\|\cdot\|_\infty$ assume also the value ∞). Namely, if β belongs to $C^i(M; \mathbb{R})$, we put

$$\|\beta\|_\infty = \sup\{|\beta(\sigma)| \,\big|\, \sigma \in S_i(M)\} \in [0, \infty].$$

The seminorm induced in cohomology is, as usual, the infimum of the norms of the representatives, that is, if b belongs to $H^i(M; \mathbb{R})$,

$$\|b\|_\infty = \inf\{\|\beta\|_\infty \,\big|\, [\beta] = b\}.$$

It is worth remarking that the just defined seminorm on singular cohomology can also be expressed by

$$\|b\|_\infty = \inf\{\|\bar{b}\|_\infty \mid \bar{b} \in H_b^i(M; \mathbb{R}), \, c(\bar{b}) = b\}$$

where c denotes the comparison map defined in Section 2.7, and, as usual, we consider ∞ the value of the infimum over an empty set. Corollary 3.2.3 states that $\|M\| = \|[M]^{\mathbb{R}}\|_\infty^{-1}$ where we put $\|M\| = 0$ if $\|[M]^{\mathbb{R}}\|_\infty = \infty$.

A first application of the duality between simplicial volume and bounded cohomology proved in Theorem 3.2.2 is the estimate of the simplicial volume of a product of two manifolds.

Proposition 3.2.4. *Let M^m and N^n be two manifolds. Then*

$$\|M\|\|N\| \leq \|M \times N\| \leq \binom{n+m}{n}\|M\|\|N\|.$$

Proof. We will prove the two inequalities separately.
•The fact that $\|M \times N\| \leq \binom{n+m}{n}\|M\|\|N\|$ is an application of the homological cross product: recall that the product of two standard simplices $\triangle^n \times \triangle^m$ can be triangulated by $\binom{n+m}{n}$ simplices [EiSt52, page 68], we call

$\sigma_i : \triangle^{m+n} \to \triangle^m \times \triangle^n$ the simplices in this triangulation. This allows to define a cross product on chains extending by linearity the map:

$$\times : S_k(M) \otimes S_j(N) \to C_{k+j}(M \times N)$$
$$(f,g) \to \sum f \times g|_{\sigma_i}.$$

It is well known that this product induces a well defined cross product in homology, i.e. a bilinear map

$$H_k(M; \mathbb{R}) \otimes H_j(N; \mathbb{R}) \to H_{k+j}(M \times N; \mathbb{R}).$$

Moreover, the product of a triangulation of M and a triangulation of N, if considered as a sum of simplices (after subdividing in the way we have just shown) is a triangulation of the product manifold. This implies, because of the characterization of fundamental classes in terms of triangulations, that

$$[M] \times [N] = [M \times N].$$

It follows from the definition of cross product that, if $a \in C_m(M; \mathbb{R})$ is a cycle representing the class $[M] \in H_m(M; \mathbb{R})$ of norm $\|a\|_1$ and $b \in C_n(M; \mathbb{R})$ is a representative of the class $[N] \in H_n(N; \mathbb{R})$ of norm $\|b\|_1$, $a \times b \in C_{m+n}(M \times N; \mathbb{R})$ is a representative of the fundamental class of $M \times N$ of norm $\binom{n+m}{n}\|a\|_1\|b\|_1$. This implies the inequality we were looking for.

•The first inequality, on the contrary, descends from the cohomological cup product: it is a consequence of the fact that the fundamental coclass $[M \times N]^{\mathbb{R}}$ is the cup product of the fundamental classes of the two factors, more precisely
$$[M \times N]^{\mathbb{R}} = p_1^*[M]^{\mathbb{R}} \cup p_2^*[N]^{\mathbb{R}}$$
where $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are the projections.

To see that this is true, let us call a simplex belongin to $S_i(X)$ *degenerate* if it can be obtained as the composition

$$\triangle^i \xrightarrow{\rho} \triangle^{i-1} \xrightarrow{\tau} X$$

where the first map $\rho_k : \triangle^i \to \triangle^{i-1}$ corresponds to the map linearly collapsing $\triangle^i$ onto one of its faces and τ is a simplex in $S_{i-1}(X)$. We can reduce without loss of generality to the representatives α of $[M]^{\mathbb{R}}$ and β of $[N]^{\mathbb{R}}$ vanishing on degenerate simplices (for example by restricting to the subcomplex of alternating cochains), in this case, if we consider the triangulation of $\triangle^m \times \triangle^n$ given in [EiSt52, page 68], only in the first simplex the projection on M of the first m-face is a nondegenerate m-simplex and

the projection on N of the last n-face is a nondegenerate n simplex. This implies that
$$(p_1^*\alpha \cup p_2^*\beta)(f \times g) =$$
$$= \sum_i p_1^*\alpha \cup p_2^*\beta(f \times g)|_{\sigma_i} =$$
$$= p_1^*\alpha \cup p_2^*\beta(f \times g)|_{\sigma_0} = \alpha(f)\beta(g).$$

This means, applying this formula to a representative of the fundamental class constructed as the cross product of fundamental classes, that

$$p_1^*\alpha \cup p_2^*\beta([M \times N]) = \alpha([M])\beta([N]) = 1,$$

i.e. that $p_1^*\alpha \cup p_2^*\beta$ is a fundamental coclass. Moreover $\|p_1^*\alpha \cup p_1^*\beta\|_\infty \leq \|p_1^*\alpha\|_\infty \|p_2^*\beta\|_\infty = \|\alpha\|_\infty \|\beta\|_\infty$ and then

$$\|[M \times N]^{\mathbb{R}}\|_\infty \leq \|[M]^{\mathbb{R}}\|_\infty \|[N]^{\mathbb{R}}\|_\infty.$$

Since we have proved that $\|M\| = \|[M]^{\mathbb{R}}\|_\infty^{-1}$ (Corollary 3.2.3), we can conclude that $\|M\|\|N\| \leq \|M \times N\|$. $\qquad\square$

We have just seen one of the advantages of the use of bounded cohomology in the treatment of simplicial volume: the presence of cup product allowed us to deduce a lower estimate for the simplicial volume of a product of two manifolds. One another powerful cohomological tool is Gromov's mapping Theorem (Theorem 2.7.2) that allows to reformulate the problem of the computation of simplicial volume in terms of group cohomology and then to take advantage of the machinery we have developed in the first two chapters of the thesis.

Let us fix a manifold M and let $\pi = \pi_1(M)$ its fundamental group. If we consider $K = K(\pi, 1)$ a CW-complex with $\pi_1(K) = \pi$, $\pi_i(K) = 0$ for every i different from 1 (such a space exists and is unique up to homotopy equivalence), a *classifying map* for the manifold M is a continuous map $f : M \to K$ with the property that $\pi_1(f)$ is an isomorphism. The classifying map exists and is well defined up to homotopy as a consequence of cellular approximation Theorems and of the fact that, since M is a manifold, it is also a cellular space. Since the universal covering of K is contractible, the group (bounded) cohomology of π is isomorphic to the singular (bounded) cohomology of K, moreover the isomorphism is isometric. If we consider the commutative diagram

$$
\begin{array}{ccc}
H^*(\pi_1(M); \mathbb{R}) & \xrightarrow{f^*} & H^*(M; \mathbb{R}) \\
{\scriptstyle c}\big\uparrow & & {\scriptstyle c}\big\uparrow \\
H_b^*(\pi_1(M); \mathbb{R}) & \xrightarrow[\cong]{f_b^*} & H_b^*(M; \mathbb{R})
\end{array}
$$

Gromov's mapping Theorem (Theorem 2.7.2) states that f_b^* is an isometric isomorphism. This implies that the simplicial volume of a manifold M depends only on the classifying map of the manifold. This approach provides many vanishing results, for example when the cohomology of the fundamental group of M is null in dimension m:

Proposition 3.2.5. *Let M^m be a manifold. Suppose that $H^m(\pi_1(M); \mathbb{R}) = 0$, then $\|M\| = 0$.*

This applies, for example, when the manifold is simply connected:

Corollary 3.2.6. *Let M be a manifold, if $\pi_1(M) = 0$ or, more generally, if $\pi_1(M)$, then $\|M\| = 0$.*

We have proved the vanishing of simplicial volume for simply connected manifolds translating the problem of its study to a group cohomology problem via the classifying map. A class of manifolds for which this approach is particularly effective is the class of locally symmetric spaces of noncompact type: on one way, since the universal covering of these manifolds is diffeomorphic to $\mathbb{R}^n$ and hence contractible, the classifyng map is an isomorphism, on the other, for these spaces, we can rely on Van Est's Theorem (and similar statements for $\pi_1(M) < \mathrm{Isom}(\widetilde{M})$). This theorems are very useful in the study of the (continuous) cohomology of the relevant groups. In the next section we will see how this idea can be used to give a simple proof of Gromov's proportionality principle for locally symmetric spaces. In Chapter 4 we will use it to give the explicit computation of the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$.

3.3 Proportionality principle for locally symmetric spaces

The aim of this section is to state and prove Gromov's Proportionality Principle for locally symmetric spaces following the proof of Bucher-Karlsson appeared in [Buc08C]. We begin recalling definitions, caracterizations and properties of locally symmetric spaces. We refer to [Hel62] for proofs and details.

To define symmetric spaces we will need the concept of a *geodesic symmetry* with respect to a fixed point $p \in M$. Let us choose a normal neighborhood of p, i.e. a an open neighborhood of p in M diffeomorphic, via the exponential map, to a star-shaped neighborhood U of 0 in T_pM. We moreover require that U is symmetric with respect to the origin. The geodesic symmetry $s_p : V \to V$ is the diffeomorphism of V conjugated, by the exponential map, to the involutive diffeomorphism of $\mathbb{R}^n$ given by $x \mapsto -x$. This

means that, in normal cohordinates $(x_1, \ldots, x_n)$, the diffeomorphism s_p has the expression

$$(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n).$$

Note that, as a consequence of the definition of the exponential map, the diffeomorphism we have just defined is a geodesic symmetry in the sense that, if γ is a geodesic with the property that $\gamma(0) = p$ and $\gamma(1) = q$, we have $s_p(q) = \gamma(-1)$.

Definition 3.3.1. A (Riemannian) *locally symmetric space* is a Riemannian manifold such that every point p has a normal neighborhood V such that the geodesic symmetry with respect to p restricts to an isometry of V.

Indeed any involutive isometry ϕ that has an isolated fixed point p is a geodesic symmetry. Infact its differential in p equals to $d\phi_p = -\mathrm{Id}$: the minimal polynomial of $d\phi_p$ divides $x^2 - 1$ and hence $d\phi_p$ is diagonalizable, moreover a geodesic tangent to an eigenvector of $+1$ eigenvalue would be left fixed by ϕ but p is an isolated fixed point of ϕ. This implies that ϕ maps the geodesic with tangent vector v in p isometrically in the geodesic with tangent vector in p equal to $-v$ and hence it is a geodesic symmetry. From now on we will talk about involutive isometries with an isolated fixed point instead of geodesic symmetries.

Definition 3.3.2. A *globally symmetric space* (that we will also call just *symmetric space*) is a Riemannian manifold M such that, for every point $p \in M$, there exists an involutive isometry s_p of M that has p as an isolated fixed point.

We have already pointed out out that a globally symmetric space is a locally symmetric space. Moreover, if a locally symmetric space is complete and simply connected, the local involutive isometry given by the geodesic symmetry extends to an involutive isometry of the whole manifold that has p as an isolated fixed point:

Theorem 3.3.3. *Let M be a complete, simply connected, locally symmetric space. Then M is a globally symmetric space.*

Proof. See [Hel62, Theorem 5.6, page 187]. □

From this result it immediately follows that the universal covering of a complete locally symmetric space is a symmetric space.

Some examples of symmetric spaces are the spaces of constant curvature $\mathbb{H}^n$, $\mathbb{R}^n$ and S^n: since the isometries of these spaces act transitively on them it suffices to exhibit an involutive isometry that has a specific isolated fixed point: the inversion $(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n)$ provides such a map for the point 0 in the Euclidean space $\mathbb{R}^n$; if we consider the hyperboloid model of $\mathbb{H}^n$, i.e.

$$\mathbb{H}^n = \{(x_0, \ldots, x_n) \in R^{(n,1)} | \langle x, x \rangle_{(n,1)} = -1\},$$

the isometry of $\mathbb{R}^{(n,1)}$ given by $(x_0,\ldots,x_{n-1},x_n) \mapsto (-x_1,\ldots,-x_{n-1},x_n)$ restricts to an isometry of $\mathbb{H}^n$ having $(0,\ldots,0,1)$ as unique fixed point. The same example can be given for the sphere S^n: we consider its model given by

$$S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$$

and restrict to S^n the ambient isometry given by

$$(x_0,\ldots,x_{n-1},x_n) \mapsto (-x_1,\ldots,-x_{n-1},x_n)$$

to get an isometry of S^n that has the north pole $(0,\ldots,0,1)$ as isolated fixed point.

It is also clear that the product of two symmetric spaces is a symmetric space: if $\gamma_1 : M \to M$ and $\gamma_2 : N \to N$ are two involutive isometries having x_1 and x_2 respectively as isolated fixed points, the product $(\gamma_1, \gamma_2) : M \times N \to M \times N$ is an involutive isometry of $M \times N$ that has (x_1, x_2) as an isolated fixed point. We will consider in the next chapter the symmetric space $\mathbb{H}^2 \times \mathbb{H}^2$.

Symmetric spaces are, in particular, homogeneous spaces: if we denote by $G = \mathrm{Isom}_0(M)$ the connected component of the identity in the group of the isometries of M, we get

Theorem 3.3.4. *If M is a symmetric space, then G is a locally compact Lie group that acts transitively on M, and, for every x in M, the subgroup $K = \mathrm{stab(x)}$ composed by the isometries that fix x is compact. Moreover M is diffeomorphic to the quotient G/K.*

Proof. It is proved in [Hel62]: the first assertion is Lemma 3.2 on page 170; the second is Theorem 3.3 on page 173. $\qquad\square$

Let us fix a point x in M, where M is a globally symmetric space, and set $K = \mathrm{stab}(x) \subseteq G = \mathrm{Isom}_0(M)$. Since M is a symmetric space, there exists an involutive isometry s_x with x as an isolated fixed point. This property of the space reflects into an analogous property of the group G: the group of isometries of M admits an involutive automorphism σ that leaves K fixed, defined by the formula

$$\sigma : G \to G$$
$$\gamma \mapsto s_x \gamma s_x.$$

It is not difficult to show that, if we denote by K_σ the subgroup of G of the elements fixed by σ and by $(K_\sigma)_0$ the component of the identity in this group, $(K_\sigma)_0 < K < K_\sigma$ (obviously K is contained in K_σ and the two groups have the same lie algebra: the eigenspace of the eigenvector $+1$ of $d\sigma_0$). The pairs with this properties are called symmetric pairs:

Definition 3.3.5. Let G be a connected Lie group and let K be a compact subgroup of G. If there exists an involutive isomorphism of G such that $(K_\sigma)_0 < K < K_\sigma$, the pair (G, K) is said a *(Riemannian) symmetric pair*.

It is equivalent to study symmetric spaces or Riemannian symmetric pairs: Theorem 3.3.4 assigns to every symmetric space a Riemannian symmetric pair; viceversa, given a Riemannian symmetric pair (G, K), any G-invariant Riemannian metric on the quotient G/K (such a metric exists) makes G/K a symmetric space. Moreover, the Riemannian connection on G/K is independent from the choice of the metric [Hel62, Proposition 3.4, Chapter IV; Corollary 4.3, Chapter IV].

Another step is necessary towards the classification of symmetric spaces: we need to study the Lie algebra of the group G of the symmetric pair associated to a symmetric space M. We will denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathfrak{k}$ the Lie algebra of K. The Lie algebra $\mathfrak{g}$ is endowed with an involutive automorphism $(s = d\sigma)$ with good properties that we will enumerate in the next definition:

Definition 3.3.6. A pair $(\mathfrak{g}, s)$ is an *effective orthogonal symmetric Lie algebra* if the following properties hold:

1. $\mathfrak{g}$ is a real Lie algebra

2. s is an involutive automorphism of $\mathfrak{g}$

3. the set of fixed point of s, $\mathfrak{k}$, is a subalgebra of $\mathfrak{g}$ such that $\mathrm{Ad}_{\mathfrak{g}}(\mathfrak{k})$ is a compact subgroup of $\mathrm{Int}(\mathfrak{g})$

4. $\mathfrak{k}$ doesn't intersect the center of $\mathfrak{g}$.

Theorem 3.3.7. *There is a corrispoondence between simply connected symmetric spaces and effective orthogonal symmetric Lie algebras.*

Proof. We refer to [Hel62, Proposition 3.6, Chapter IV], see also [Loo69, Theorem 4.12, page 116] $\qquad\square$

A fundamental result in the theory of symmetric spaces is the classification of effective orthogonal symmetric Lie algebra [Hel62, Theorem 1.1, Chapter V] that leads to the classification of symmetric spaces:

Theorem 3.3.8 (Classification of symmetric spaces)**.** *Let M be a simply connected symmetric space, then M splits as a product $K \times E \times N$ where*

- *K has sectional curvature everywhere greater than 0 (and is compact)*

- *E has sectional curvature everywhere equal to 0 (and hence is isometric to $\mathbb{R}^k$)*

- *N has sectional curvature everywhere ≤ 0. Moreover the Lie algebra of the isometry group of N is semisimple of noncompact type.*

Proof. This follows from Theorem 3.1 of Chapter V and Theorem 1.1 of Chapter V in [Hel62]. See also [Loo69, Corollary 1, page 147] □

We can now turn back to the study of the simplicial volume for locally symmetric spaces. We begin with another vanishing result:

Proposition 3.3.9. *Let M^m be a locally symmetric space whose universal covering has a nontrivial compact factor, then $\|M\| = 0$*

Proof. The result will follow from Proposition 3.2.5 once we have shown that $H^m(\pi_1(M); \mathbb{R}) = 0$. Indeed, we will prove that, if $\widetilde{M} \cong K \times H$ where K is the compact factor in the decomposition of Theorem 3.3.8, it is easy to show that $(\Omega^i(H; \mathbb{R}), d)$ is a strong relatively injective resolution of the trivial $\pi_1(M)$-module $\mathbb{R}$.

The fact that the resolution is strong is a consequence of Poincarè Lemma: since H is a locally simmetric space with no non-trivial compact factor, it is diffeomorphic to $\mathbb{R}^n$.

To prove that $\Omega^i(H, \mathbb{R})$ is relatively injective as a $\pi_1(M)$-module, let us recall that, in the proof of van Est's Theorem (Theorem 1.7.5), we have seen that $\Omega^i(H; \mathbb{R})$ is relatively injective as G-module where G denotes the identity component of the isometry group of $\widetilde{M}$: this is a consequence of the fact that H is the quotient of G with respect to a compact subgroup (the preimage of the compact factor K). Since $\pi_1(M)$ is a closed subgroup of G, the differential forms on H are relatively injective as $\pi_1(M)$-module as a consequence of Theorem 1.6.6. This implies that the cohomology of $\pi_1(M)$ can be computed from the complex $(\Omega^i(H; \mathbb{R})^{\pi_1(M)}, d)$. Since $\widetilde{M}$ has a nontrivial compact factor, the dimension of H is strictly lower than m and hence $H^m(\pi_1(M); \mathbb{R}) = 0$.

□

A consequence of Proposition 3.3.9 is that, when we study the simplicial volume of a locally symmetric space, we can assume that $\widetilde{M}$ hasn't any nontrivial compact factor, so it is diffeomorphic to $\mathbb{R}^n$. Moreover Theorem 3.3.4 ensures that the universal covering of M is isometric to the quotient $\widetilde{M} = G/K$ where G denotes the connected component of the identity in $\mathrm{Isom}(\widetilde{M})$, and hence it is a connected Lie group; furthermore $K = \mathrm{stab}(x)$ is a maximal compact subgroup (the maximality follows from the fact that $\widetilde{M}$ has no nontrivial compact factor).

Moreover, since M is compact, its fundamental group, $\Gamma = \pi_1(M)$, sits in G as a cocompact latex. A key step in the proof of Gromov's Proportionality Principle (and hence in the study of the simplicial volume of locally symmetric spaces) is the following:

Theorem 3.3.10. *It is naturally defined an embedding*

$$H_c^*(G; \mathbb{R}) \to H^*(\Gamma; \mathbb{R})$$

isometric with respect to the seminorms induced via the comparison map from the canonical seminorms on $H_{cb}^*(G; \mathbb{R})$ and $H_b^*(\Gamma; \mathbb{R})$ respectively.

Proof. We consider $H_c^*(G; \mathbb{R})$ as the cohomology of the complex $(C_c^*(G; \mathbb{R})^G, d)$. By definition of canonical seminorm, we are interested precisely in the seminorm induced in cohomology by the l^∞-norm on this complex.

We have seen in Theorem 1.6.6 that the restriction isomorphism

$$\rho : C_c^*(G; \mathbb{R})^\Gamma \to C^*(\Gamma; \mathbb{R})^\Gamma$$

induces an isomorphism in cohomology since Γ is closed in G. Moreover the restriction of ρ to the bounded cochains is isometric (Theorem 2.5.9). This implies that the canonical seminorm on $H^*(\Gamma; \mathbb{R})$ is induced by the sup norm on the complex $(C_c^*(G)^\Gamma, d)$.

We now want to define a left inverse, trans, to the map res

$$C_c^*(G; \mathbb{R})^G \xrightarrow{\text{res}} C_c^*(G; \mathbb{R})^\Gamma \xrightarrow{\text{trans}} C_c^*(G; \mathbb{R})^G$$

where the map res denotes the inclusion of the G-invariant cochains in the Γ invariant (the action of the subgroup $\Gamma < G$ on $C_c^*(G; \mathbb{R})$ is the restriction of the diagonal left action of G). The idea is to average on a *fundamental domain F* for the action of Γ on G.

We choose a domain $F \subset G$ with the properties that

- F is an open connected set, $F \cap \gamma_i F = \emptyset$ for all $\gamma_i \in \Gamma$;

- $\overline{F}$ is compact, and $\mu(\bar{F} \backslash F) = 0$;

- $\sqcup \gamma_i \overline{F} = G$.

Let us denote by μ the biinvariant Haar measure on the group G (since G is the group of the isometries of a Riemannian manifold, G is unimodular). We define the map trans by requiring that

$$\text{trans } c(g_0, \ldots, g_i) = \frac{1}{\mu(F)} \int_{\bar{F}} c(fg_0, \ldots, fg_n) d\mu(f)$$

We will now prove that the chain trans(c) is G-invariant whenever we start with a Γ-invariant chain c.

Let us fix g in G and let us consider the family $\{F_i = \gamma_i F g^{-1} \cap F\}$. Since F is a fundamental domain for F, we have $F_i \cap F_j = \emptyset$:

$$F_i \cap F_j \subseteq \gamma_i F g^{-1} \cap \gamma_j F g^{-1} = (\gamma_i F \cap \gamma_j F) g^{-1} = \emptyset.$$

The family is also finite: let us consider the projection $\pi : G \to \widetilde{M}$, since the action of $\Gamma = \pi_1(M)$ on $\widetilde{M}$ is proper, also the action of Γ on G is proper and hence the set $\{\gamma_i \,|\, \gamma_i F \cap \overline{Fg}\}$ is finite since $\overline{Fg}$ is compact. To prove that the family $\{F_i\}$ gives a finite partition of the fundamental domain F (negletting a null-measure subset) it remains to prove that $\sqcup \overline{F_i} = \overline{F}$ and this follows from the fact that the left-hand member is obviously contained in $\overline{F}$ and

$$\bigsqcup_{\gamma_i \in \Gamma} \gamma_i \overline{F} g^{-1} = (\bigsqcup_{\gamma_i \in \Gamma} \gamma_i \overline{F}) g^{-1} = G g^{-1} = G.$$

Note that, for the same reason, also the set $\{\gamma_i^{-1} F_i g = F \cap \gamma_i F g\}$ gives a partition of F. We than have

$$
\begin{aligned}
l_{g^{-1}} \text{trans} \, c(g_0, \dots, g_i) \;\; &= \tfrac{1}{\mu(F)} \int_{\overline{F}} c(f g g_0, \dots, f g g_n) d\mu(f) = \\
&= \tfrac{1}{\mu(F)} \sum_i \int_{\overline{F}_i} c(f g g_0, \dots, f g g_n) d\mu(f) = \\
&= \tfrac{1}{\mu(F)} \sum_i \int_{\gamma_i^{-1} \overline{F}_i g} c(\gamma_i f g_0, \dots, \gamma_i f g_n) d\mu(f) = \\
&= \tfrac{1}{\mu(F)} \sum_i \int_{\gamma_i^{-1} \overline{F}_i g} c(f g_0, \dots, f g_n) d\mu(f) = \\
&= \tfrac{1}{\mu(F)} \int_{\overline{F}} c(f g_0, \dots, f g_n) d\mu(f).
\end{aligned}
$$

where we have used the fact that μ is biinvariant, the Γ-invariance of c and the observation that both $\{F_i\}$ and $\{\gamma_i^{-1} F_i g\}$ are partitions of F.

Moreover trans c is continuous since we are averaging on a compact set. From the very definition of the map trans follows that it is a chain map (i.e. commutes with the differential) and hence induces a map in cohomology. Clearly the composition trans $\circ$ res is the identity and both trans and res are norm non-increasing at the cochain level. This implies that res induces the required isometric embedding.

<div align="right">□</div>

In the proof of this theorem we have never used the fact that M is a locally symmetric space. However, in general, the continuous cohomology of G can be null in top dimension and hence, in the general case, a result analoguous to this theorem is not useful towards the study of simplicial volume.

We have already pointed out that, if M is a locally symmetric space whose universal covering has no noncompact factor, the cohomology of M and of $\pi_1(M)$ are isometrically isomorphic (this is because the universal covering $\widetilde{M}$ of M is contractible) and hence the top dimensional cohomology group of $\Gamma = \pi_1(M)$ is one dimensional and generated by the preimage (via the classifying map) of the fundamental coclass. We are now going to exhibit an element in $H_c^*(G; \mathbb{R})$ whose restriction is the fundamental coclass.

Theorem 3.3.11. *Let M^m be a locally simmetric space, let G be the connected component of the identity in the group of the isometries of its universal*

covering $\widetilde{M}$, *and let* $\omega \in H_c^m(G; \mathbb{R})$ *be the image under van Est's isomorphism of the volume form* $\omega_{\widetilde{M}}$ *on* $\widetilde{M}$. *Then*

$$\mathrm{res}(\omega) = \mathrm{vol(M)} \cdot [\mathrm{M}]^{\mathbb{R}}.$$

Proof. We have seen in Van Est Theorem (Theorem 1.7.5) that $H_c^m(G; \mathbb{R})$ is isomorphic to $\Omega^m(\widetilde{M}; \mathbb{R})^G$ (since $\widetilde{M} = G/K$ where G is a connected Lie group and K is the maximal compact subgroup of G corresponding to the stabilizer of a once and for all fixed point p). Moreover $\Omega^m(\widetilde{M}; \mathbb{R})^G$ is one dimensional: since G acts transitively on $\widetilde{M}$, the vector space $\Omega^m(\widetilde{M}; \mathbb{R})^G$ can be viewed as a subspace of $\wedge^m T_p\widetilde{M} \cong \mathbb{R}$, furthermore $\Omega^m(\widetilde{M}; \mathbb{R})^G$ is not empty because the volume form $\omega_{\widetilde{M}}$ is obviously G-invariant (because the group G acts on $\widetilde{M}$ via isometries). This shows that $H_c^m(G; \mathbb{R})$ is one dimenisonal and generated by ω.

We will now track down the isomorphisms $\Omega^m(\widetilde{M}; \mathbb{R})^G \cong H^m(M; \mathbb{R})$ in order to understand in which multiple of the fundamental coclass ω is mapped.

In Proposition 1.8.1 we have seen that, once a a point $p \in \widetilde{M}$ is fixed, a representative for ω in $C_c^m(G; \mathbb{R})$ is given by the cocycle $I^m(\omega_{\widetilde{M}}) \in C_c^m(G; \mathbb{R})^G$ defined by

$$I^m(\omega_{\widetilde{M}})(g_0, \dots, g_m) = \int_{\triangle(g_0, \dots, g_n)} \omega_{\widetilde{M}}$$

where $\triangle(g_0, \dots, g_n)$ is the geodesic simplex with vertices $(g_0 p, \dots, g_n p)$ (we defined the meaning of *geodesic simplex* in Section 1.8). Moreover the image of $I^m \omega_{\widetilde{M}}$ in the complex $C^*(\Gamma; \mathbb{R})^\Gamma$ has the same expression (where we assume that the elements g_i belong to Γ instead of G): we have seen in the proof of Theorem 3.3.10 that the embedding $H_c^*(G; \mathbb{R}) \to H^*(\Gamma; \mathbb{R})$ is induced by the restriction $C_c^*(G; \mathbb{R})^G \to C^*(\Gamma; \mathbb{R})^\Gamma$.

It remains only to make explicit (at the cochain level) the isomorphism between the singular cohomology $H^m(M; \mathbb{R})$ and the group cohomology $H^m(\Gamma; \mathbb{R})$. The covering map $p : \widetilde{M} \to M$ induces an isomorphism of cochains

$$p^* : C^*(M; \mathbb{R}) \to C^*(\widetilde{M}; \mathbb{R})^\Gamma.$$

Since $\widetilde{M}$ is contractible, the latter is the subcomplex of the Γ-invariants of the strong relatively injective Γ-resolution of $\mathbb{R}$ given by $(C^*(\widetilde{M}; \mathbb{R}), d)$. We need to give an explicit description of the chain morphism $C^i(\widetilde{M}; \mathbb{R}) \to C^i(\Gamma; \mathbb{R})$ that induces the isometry in cohomology. We have proved in proposition 2.7.4 that the map

$$\beta : C^i(\widetilde{M}; \mathbb{R}) \to C^i(\Gamma; \mathbb{R})$$
$$\beta(c)(g_0, \dots, g_i) = c(\triangle_{(g_0 p, \dots, g_i p)}).$$

is a norm decreasing chain map that extends the identity in dimension 0, and hence induces an isomorphism in cohomology that is moreover isometric.

Clearly res $I^m(\omega_{\widetilde{M}})$ is the image, under β, of the cocycle $\widetilde{\gamma}$ that assigns to a given simplex $\sigma \in S_m(\widetilde{M})$ the integral of $\omega_{\widetilde{M}}$ over the simplex σ (that we can assume, up to homotopy, smooth or even straight). Moreover, if we call ω_M the volume form on M, $\omega_{\widetilde{M}} = p^*\omega_M$ and hence, in the identification $p^* : C^*(M;\mathbb{R}) \to C^*(\widetilde{M};\mathbb{R})^\Gamma$, $\widetilde{\gamma}$ corresponds to the cochain $\gamma \in C^*(M;\mathbb{R})$ that assumes on a simplex $\sigma \in S_m(M)$ the value of the integral of ω_M over the simplex. We can now compute what multiple of the fundamental coclass γ is. If we choose a representative $\alpha = \sum_i \sigma_i$ for the fundamental class $[M]$ that comes from a smooth triangulation, we get:

$$\langle [M], \gamma \rangle = \gamma(\alpha) = \sum_i \int_{\sigma_i} \omega_M = \int_M \omega_M = \mathrm{vol}(M).$$

This implies our thesis: γ, the image of ω under res, is $\mathrm{vol}(M) \cdot [M]^{\mathbb{R}}$. $\qquad \square$

An immediate consequence of Theorem 3.3.11 is the following result:

Corollary 3.3.12 (Proportionality constant for locally symmetric spaces). *Let ω be the class in $H_c^m(Isom_0(M);\mathbb{R})$ of the volume form on $\widetilde{M}$, then*

$$\|M\| = \frac{vol(M)}{\|\omega\|_\infty}.$$

where we understand $\|M\| = 0$ if and only if $\|\omega\|_\infty = \infty$ as an element of $H_c^*(G;\mathbb{R})$.

Proof. We have proved (see Corollary 3.2.3) that the simplicial volume of the manifold M is the inverse of the seminorm of the fundamental coclass. Since the restriction map $\mathrm{res} : H_c^m(G,\mathbb{R}) \to H^m(M,\mathbb{R})$ is isometric (as proved in Theorem 3.3.10) and $\mathrm{res}(\omega) = \mathrm{vol}(M) \cdot [M]^{\mathbb{R}}$, we get that

$$\|M\| = \frac{1}{\|[M^{\mathbb{R}}]\|_\infty} = \frac{vol(M)}{\|\omega_{\widetilde{M}}\|_\infty}.$$

$\qquad \square$

Since the group G depends only on the universal covering of the manifold M, another important consequence of Theorem 3.3.11 is the Proportionality Principle for locally symmetric spaces:

Corollary 3.3.13 (Gromov Proportionality Principle for Locally Symmetric Spaces). *Let M be a closed connected orientable Riemannian manifold that is a locally symmetric space, the rate*

$$\frac{\|M\|}{vol(M)}$$

depends only on the universal cover of M.

In the next section we will show that the proportionality constant for hyperbolic manifolds equals v_n, the volume of the ideal regular simplex in $\mathbb{H}^n$. Purpose of the next chapter is to study the case of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$, we will see that in that case the constant is $2/3\pi^2$. In general these are the only cases in which the proportionality constant is known explicitly. It is worth remarking that Lafont and Schmidt proved in [LaSch06] that the norm of the volume form is finite (and hence the proportionality constant is nonzero) for every locally symmetric space whose covering is a symmetric space of noncompact type. Instead it is easy to prove that the simplicial volume of a symmetric space whose universal covering has a nontrivial Euclidean factor vanishes: Eberlein proved that, in this case, M admits a finite covering $\overline{M}$ that is a product of a torus with some closed manifold [Ebe83]. Since we have proved that the simplicial volume of a torus vanishes (Proposition 3.1.7) and that the simplicial volume of a product vanishes if the simplicial volume of one of its factors is null (Proposition 3.2.4), the simplicial volume of M vanishes as a consequence Proposition 3.1.6.

Corollary 3.3.13 is far more general: it is valid for every closed compact oriented manifold (and also for finite volume manifold provided an appropriate definition of simplicial volume is given in that case):

Theorem 3.3.14 (Gromov Proportionality Principle). *Let M be a closed connected oriented manifold. Then the rate $c(\widetilde{M}) = \frac{\|M\|}{vol(M)}$ depends only on the universal covering of M.*

In this case $c(\widetilde{M})$ can be computed as the norm of the volume form seen in a suitable cohomology group. Despite this result is far more general and hence very interesting, it is less suited for explicit computations since, in that case, one cannot rely on the fully developed tool of group (continuous, bounded) cohomology. For this reason we will not prove it in full generality and refer to [Gro82, Section 0.4], that first gave a sketch of the proof of this theorem, [Löh06, Theorem 6.3] for a full proof relying on the theory of measure homology introduced by Thurston, [Buc08A, Section 6] for a detailed yet partially uncorret version of Gromov's arguments and [Fri11, Section 9] where the remaining gap is filled in.

3.4 Hyperbolic manifolds

As a first application of the techniques introduced in the previous section, we compute the simplicial volume of hyperbolic manifolds via a cohomological argument: we will compute the norm of the class in $H_c^n(G;\mathbb{R})$ that is the image, under Van Est's isomorphism, of the hyperbolic volume form. Let us fix once and for all the dimension n of $\mathbb{H}^n$. Moreover G will always denote the connected component of the identity of the group of the isometries of $\mathbb{H}^n$, that is the subgroup of the isometries of $\mathbb{H}^n$ that preserve orientation.

Theorem 3.4.1. *Let v_n be the volume of an ideal regular simplex in $\mathbb{H}^n$, if ω is the class of the volume form of $\mathbb{H}^n$, then*

$$\|\omega\|_\infty = v_n.$$

Proof. We have proved in Corollary 1.5.3 and in Theorem 1.5.2 that the continuous cohomology of G can be computed as the cohomology of the complex of

$$C_c^k(\mathbb{H}^n)_{\text{alt}}^G = \{f : (\mathbb{H}^n)^{k+1} \to \mathbb{R} \mid f \text{ is continuous, alternating, } G\text{-invariant}\}$$

and that the sup norm on this complex induces the canonical seminorm on $H_c^*(G; \mathbb{R})$. We recall that, if we fix a point x in $\mathbb{H}^n$ and hence a projection $\pi : G \to G/K = \mathbb{H}^n$, the isometric isomorphism between the cohomology of the complexes $(C_c^k(\mathbb{H}^n)_{\text{alt}}; d)$ and $(C_c^k(G; \mathbb{R}); d)$ is induced by the chain map

$$\pi^* : C_c^k(\mathbb{H}^n)_{\text{alt}} \to C_c^k(G)$$
$$\pi^* \phi(g_0, \ldots, g_k) = \phi(g_0 x, \ldots, g_k x).$$

Let us consider the cocycle $\gamma \in C_c^n(\mathbb{H}^n)_{\text{alt}}^G$ that assignes to an $(n+1)$-uple $(x_0, \ldots, x_n)$ in $\mathbb{H}^n$ the signed volume of the straight simplex with vertices $(x_0, \ldots, x_n)$.

$$\pi^* \gamma(g_0, \ldots, g_n) = \gamma(g_0 x, \ldots, g_n x) = \int_{\triangle(g_0 x, \ldots, g_n x)} \omega = I^n \omega,$$

where $I^n \omega$ is the representative of the image under Van Est isomorphism of the volume form we constructed in Section 1.8. It follows from the definition of γ that its norm, as an element of the normed vector space $C_c^k(\mathbb{H}^n)_{\text{alt}}^G$, is v_n, the sup of the volumes of the geodesic simplices in $\mathbb{H}^n$.

It remains only to show that γ has minimal norm among the representatives of $[\gamma]$, in other words we shall compute the norm of $\gamma + \delta\beta$ with $\beta \in C_c^{n-1}(\mathbb{H}^n)_{\text{alt}}^G$ and show that it is at least v_n. Let us consider the vertices $(x_0^R, \ldots, x_n^R)$ of a regular simplex σ^R in $\mathbb{H}^n$ of radius R. For every face σ_j^R of σ^R, any permutation of its vertices can be realized by an orientation preserving isometry of $\mathbb{H}^n$: infact, since σ is regular, there exists an isometry of $\mathbb{H}^n$ fixing the vertex opposite to σ_j^R and realizing the prescribed permutation of the remaining vertices. Whenever the permutation is odd, this isometry is orientation reversing and hence doesn't belong to G, but in this case it is enough to compose this isometry with the reflexion with respect to the hyperplane through the face: the obtained element of G restricts to an isomety of σ_j^R that realizes the fixed permutation.

Since the cochain β is alternating and G-invariant, $\beta(\sigma_j^R) = 0$ for every face σ_j^R of σ^R: let g be the orientation-preserving isometry that realizes the permutation of the first two vertices of the j-th face σ_j^R and fixing all the other vertices,

$$\begin{aligned}
\beta(x_0, x_1, \ldots, \hat{x}_j, \ldots, x_n) &= \beta(gx_0, gx_1, \ldots, \hat{x}_j, \ldots, gx_n)) \\
&= \beta(x_1, x_0, \ldots, \hat{x}_j, \ldots, x_n) \\
&= -\beta(x_0, x_1, \ldots, \hat{x}_j, \ldots, x_n)
\end{aligned}$$

where in the first equality we have used the G-invariance of β, in the last one the fact that β is alternating. This allows us to conclude the proof: the volume of σ^R tends to v_n and $\delta\beta(\sigma^R) = 0$, hence

$$\begin{aligned}
\|\gamma + \delta\beta\|_\infty &\geq \sup_R |(\gamma + \delta\beta)(\sigma^R)| \\
&= \sup_R |\gamma(\sigma^R)| = v_n.
\end{aligned}$$

$\square$

To conclude the Chapter we will see how the proportionality principle for hyperbolic spaces can be proved in a purely homological way (i.e. without recourse to bounded cohomology). Gromov, in [Gro82], attributed the idea of this proof to N. H. Kupier, we will follow the approach of [BePe92, Chapter C].

Proposition 3.4.2. *Let M^n be a hyperbolic closed connected oriented manyfold. If v_n is the volume of the regular ideal simplices in $\mathbb{H}^n$,*

$$\|M\| \geq \frac{\mathrm{vol(M)}}{v_n}.$$

Proof. We will consider the straightening operator $\mathrm{str} : C^*(M, \mathbb{R}) \to C^*(M, \mathbb{R})$ i.e. the operator that assigns to a simplex σ the projection of the straight simplex of $\mathbb{H}^n$ with the same vertices of an arbitrary lifting $\widetilde{\sigma}$ of σ. It is well known that the straightening is well defined and chain homotopic to the identity.

Let us choose a representative $\alpha = \sum_i a_i \sigma_i$ of the fundamental class $[M]$; since the straightening operator has norm one and is chain homotopic to the identity, we can assume (without changing the class of α nor its norm) that every simplex σ_i in the sum is the projection of a straight simplex $\widetilde{\sigma}_i$ of $\mathbb{H}^n$.

Let us now consider the cocycle $\omega \in C^n(M; \mathbb{R})$ that assigns to every simplex σ the integral over σ of the volume form ω_M. We have already pointed out that $\omega([M]) = \mathrm{vol}(M)$ (this fact can be easily verified choosing a representative of the fundamental class coming from a triangulation). So we get (recalling that $\mathbb{H}^n$ is the metric covering of M) that

$$\mathrm{vol}(M) = \omega([M]) = \left| \sum_i a_i \int_{\sigma_i} \omega_M \right| \leq \sum_i |a_i| \left| \int_{\widetilde{\sigma}_i} \omega_{\mathbb{H}^n} \right| \leq v_n \sum_i |a_i|.$$

This formula gives the desired inequality. $\square$

The other inequality is more difficult, but it is very interesting since it involves the explicit exhibition of a sequence of minimizing cycles, i.e. cycles representing the fundamental class $[M]$ whose l_1-norm is arbitrarily colose to $\|M\|$. To gain this result we fix ϵ and look for a representative $\sum_i a_i \sigma_i$ of $[M]$ such that $|\text{vol}(\sigma_i)| > v_n - \epsilon$ and that $\text{sgn}(a_i)$ is positive if σ_i is orientation preserving, negative otherwise: this will be necessary since $\int_{\sigma_i} \omega_{\mathbb{H}^n} = -\text{vol}(\sigma_i)$ if and only if σ_i is orientation reversing.

In the proof of this theorem also orientation reversing isometries will be important and will also be crucial the distinction between orientation preserving and orientation reversing isometries. For this reason we slightly modify our notation and we denote by G^+ (resp. G^-) the group of orientation preserving (resp. reversing) isometries of $\mathbb{H}^n$. As usual Γ will be the fundamental group of M that we regard as a subgroup of G^+, and μ the biinvariant Haar measure on $G = \text{Isom}(\mathbb{H}^n) = G^+ \cup G^-$.

Theorem 3.4.3. *Under the same hypothesis of Proposition 3.4.2 it holds*

$$\|M\| \leq \frac{vol(M)}{v_n}.$$

Proof. If we could consider an uniformly distributed chain with support on the projections of all the regular simplices of radius R (with a sign reflecting the orientation of the simplex), we would have a chain made of simplices of almost maximal volume (when R goes to infinity). Such a chain would be closed since, once a face f_i of a simplex σ is fixed, if γ is the reflection with respect to the hyperplane through f_i, $\gamma\sigma$ is a regular simplex that has f_i as ith face and opposite orientation (and hence would be counted with opposite sign).

However this chain would not be a finite combination of simplices and hence wouldn't be a singular chain (a precise definition of a chain that is uniformly distributed can be given within the theory of measure homology, see [Thu79, Chapter 6] and [Löh04, Chapter 5] for more details). We will give a discrete version of this chain splitting the set of straight simplices in some suitable classes and taking a representative for each (with the right weight).

Let us fix a convex fundamental domain F for the action of Γ on $\mathbb{H}^n$ and a point x in F, and let d be the diameter of F. Let us moreover fix a regular simplex σ^R of radius R in $\mathbb{H}^n$. The vertices of σ will be the points $(x_0^R, \ldots, x_n^R)$. Since G acts transitively and faithfully on the set S^R of regular simplices of radius R, the choice of σ^R corresponds to an identification of G with S^R, where we associate to the element g in G the simplex $g\sigma^R$.

The function $a_R = a_R^+ - a_R^-$ we will now define measures the amount of regular simplices whose vertices $(gx_0^R, \ldots, gx_n^R)$ lie in the region $F \times \gamma_1 F \times$

$\ldots \times \gamma_n F \subseteq (\mathbb{H}^n)^{n+1}$:

$$a_R^{\pm} : \Gamma^n \to \mathbb{R}$$
$$a_R^+(\gamma_1, \ldots, \gamma_n) = \mu\{g \in G^+ | g(x_0^R) \in F,\ g(x_i^R) \in \gamma_i F\},$$
$$a_R^-(\gamma_1, \ldots, \gamma_n) = \mu\{g \in G^- | g(x_0^R) \in F,\ g(x_i^R) \in \gamma_i F\}.$$

Note that we are considering only the simplices whose first vertex belongs to F, this is because we are implicitly identifying the straight simplices of M with their unique lift with first vertex in the region F.

The function a_R is almost everywhere zero: since σ_R is a regular simplex of radius R, we have $d(x_0, x_i) = R$. This means that $\alpha_R(\gamma_1, \ldots, \gamma_n) \neq 0$ only if $\gamma_i F \cap \overline{B(x, R+d)} \neq \emptyset$ for every i between 1 and n (both gx and x_0 belong to the region F whose diameter is d and hence $d(g(x_i), x) \leq d(g(x_i), g(x_0)) + d(g(x_0), x) \leq R + d$). Since the ball $\overline{B(x, R+d)} \subset \mathbb{H}^n$ is compact and the action of Γ on $\mathbb{H}^n$ is proper, the set $\{j | \gamma_j F \cap \overline{B(x, R+d)} \neq \emptyset\}$ is finite.

Let us now consider, for every n-uple $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, the straight simplex $\widetilde{\sigma}_{(\gamma_1, \ldots, \gamma_n)} \in S_n(\mathbb{H}^n)$ that has vertices in the points $(x, \gamma_1 x, \ldots, \gamma_n x)$. Let us moreover denote by $\sigma_{(\gamma_1, \ldots, \gamma_n)}$ the projection $\pi_* \widetilde{\sigma}_{(\gamma_1, \ldots, \gamma_n)} \in S_n(M)$. The simplex $\sigma_{(\gamma_1, \ldots, \gamma_n)}$ is our representative for the straight simplices that admit a lift with vertices in the region $F \times \gamma_1 F \times \ldots \times \gamma_n F$. Despite σ is not regular, we will see that, provided the class it represents is not empty, it is close to a regular simplex and then its volume is close to v_n whenever we make this construction with a sufficiently big R. We choose, as a coefficient for $\sigma_{(\gamma_1, \ldots, \gamma_n)}$, the mass of the simplices it represents in order to obtain the class

$$\alpha^R = \sum a_R(\gamma_1, \ldots, \gamma_n) \sigma_{(\gamma_1, \ldots, \gamma_n)} \in C_n(M; \mathbb{R}).$$

We have to show that it is a cycle and that its class is a nonzero multiple of the fundamental class $[M]$.

We have already pointed out that $a_R(\gamma_1, \ldots, \gamma_n)$ is almost always null and hence α^R is well defined (it is a finite combination of singular simplices of M).

$\bullet \alpha^R$ *is closed*

It is easy to see that, denoting by $\sigma_{(\gamma_1, \ldots, \gamma_{n-1})}$ the projection of the straight simplex with vertices in $(x, \gamma_1 x, \ldots, \gamma_{n-1} x)$, we have

$$\partial \alpha^R \ = \sum a_R(\gamma_1, \ldots, \gamma_n)(\sum_{i=1}^n (-1)^i \sigma_{(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_n)} + \pi_* \gamma_1 \widetilde{\sigma}_{(\gamma_1^{-1}\gamma_2, \ldots, \gamma_1^{-1}\gamma_{n-1})})$$
$$= \sum a_R(\gamma_1, \ldots, \gamma_n)(\sum_{i=1}^n (-1)^i \sigma_{(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_n)} + \sigma_{(\gamma_1^{-1}\gamma_2, \ldots, \gamma_1^{-1}\gamma_{n-1})}).$$

Let us choose a simplex $\sigma_{(\gamma_1, \ldots, \gamma_{n-1})}$ and show that its coefficient c in this sum is zero. We have

$$c = \sum_{\gamma \in \Gamma} \sum_{i=1}^n (-1)^i a_R(\gamma_1, \ldots, \gamma, \gamma_i, \ldots, \gamma_{n-1}) + \sum_{\gamma \in \Gamma} a_R(\gamma, \gamma\gamma_1, \ldots, \gamma\gamma_{n-1}).$$

Let us show that for every i, the sum $\sum_\gamma a_R(\gamma_1, \ldots, \gamma, \gamma_i, \ldots, \gamma_{n-1}) = 0$.

$$\sum_{\gamma \in \Gamma} a_R(\gamma_1, \ldots, \gamma_i, \gamma, \ldots, \gamma_{n-1}) =$$
$$= \sum_{\gamma \in \Gamma} \mu\{g \in G^+ | gx_0 \in F,\ gx_j \in \gamma_j F,\ gx_{i+1} \in \gamma F\}$$
$$- \mu\{g \in G^- | gx_0 \in F,\ gx_j \in \gamma_j F,\ gx_{i+1} \in \gamma F\} =$$
$$= \sum_{\gamma \in \Gamma} \mu\{g \in G^+ | gx_0 \in F,\ gx_j \in \gamma_j F\} - \mu\{g \in G^- | gx_0 \in F,\ gx_j \in \gamma_j F\} =$$
$$= 0$$

where in the first equality we used the fact that the measure is σ-additive, and $\bigcup_{\gamma \in \Gamma} \gamma \overline{F} = \mathbb{H}^n$, in the second the fact that the two sets have equal measure since they correspond via the right multiplication of the reflection with respect to the iperplane through $(x_0, \ldots, \hat{x}_i, \ldots, x_n)$ and the measure μ is right invariant.

In a similar way we have $\sum_{\gamma \in \Gamma} a_R(\gamma, \gamma\gamma_1, \ldots, \gamma\gamma_{n-1}) = 0$:

$$\sum_{\gamma \in \Gamma} a_R(\gamma, \gamma\gamma_1, \ldots, \gamma\gamma_{n-1}) =$$
$$= \sum_{\gamma \in \Gamma} \mu\{g \in G^+ | gx_0 \in F,\ gx_1 \in \gamma F,\ gx_i \in \gamma\gamma_i F\}$$
$$- \mu\{g \in G^- | gx_0 \in F,\ gx_1 \in \gamma F,\ gx_i \in \gamma\gamma_i F\} =$$
$$= \sum_{\gamma \in \Gamma} \mu\{g \in G^+ | \gamma^{-1}gx_0 \in \gamma^{-1}F,\ \gamma^{-1}gx_1 \in F,\ \gamma^{-1}gx_i \in \gamma_i F\}$$
$$- \mu\{g \in G^- | \gamma^{-1}gx_0 \in \gamma^{-1}F,\ \gamma^{-1}gx_1 \in F,\ \gamma^{-1}gx_i \in \gamma_i F\} =$$
$$= \sum_{\gamma \in \Gamma} \mu(\gamma\{\bar{g} \in G^+ | \bar{g}x_0 \in \gamma^{-1}F,\ \bar{g}x_1 \in F,\ \bar{g}x_i \in \gamma_i F\})$$
$$- \mu(\bar{\gamma}\{g \in G^- | \bar{g}x_0 \in \gamma^{-1}F,\ \bar{g}x_1 \in F,\ \bar{g}x_i \in \gamma_i F\}) = 0.$$

This means that $c = 0$ and hence α^R is closed.

• $[\alpha^R]$ *is different from 0 in* $H^n(M; \mathbb{R})$
If R is big enough (namely $R > 2d$), then $a_R(\gamma_1, \ldots, \gamma_n)$ is greater than 0 (resp. ≤ 0) only if the simplex with vertices $(x, \gamma_1 x, \ldots, \gamma_n x)$ is positively oriented (resp. negatively oriented). Indeed, if $a_R(\gamma_1, \ldots, \gamma_n) \geq 0$, there exists a positively oriented regular simplex $g\sigma^R$ with the property that $d(gx_i, \gamma_i x) < d$ for every i; this means that the $(n+1)$-uple $(x, \gamma_1 x, \ldots, \gamma_n x)$ is positively oriented (since $d(x_i, x_j) = R$). This implies that $\widetilde{\sigma}_{(\gamma_1, \ldots, \gamma_n)}$ is positively oriented and hence also $\sigma_{(\gamma_1, \ldots, \gamma_n)}$ is.

This means that

$$\mathrm{vol}(M)[M]^{\mathbb{R}}[\alpha^R] = \sum a_R(\gamma_1, \ldots, \gamma_n) \int_{\sigma_{(\gamma_1, \ldots, \gamma_n)}} \omega =$$
$$= \sum |a_R(\gamma_1, \ldots, \gamma_n)| \mathrm{vol}(\sigma_{(\gamma_1, \ldots, \gamma_n)}) = a \neq 0.$$

This allows us to conclude that $\mathrm{vol}(M)[\alpha^R] = a[M]$ i.e. that $\mathrm{vol}(M)[\alpha_R/a] = [M]$, moreover

$$\frac{1}{\|\alpha^R/a\|_1} \geq \min_{a_R(\gamma_1, \ldots, \gamma_n) \neq 0} \mathrm{vol}(\sigma_{(\gamma_1, \ldots, \gamma_n)}) = c_R.$$

Since, if $a_R(\gamma_1, \ldots, g_n) \neq 0$, $\widetilde{\sigma}_{(\gamma_1,\ldots,\gamma_n)}$ is a straight simplex whose vertices have distance at most d from a regular simplex of radius R and $\mathrm{vol}(\sigma^R)$ tends to v_n when $R \to \infty$, also c_R tends to v_n: this implies that we can choose R such that $c_R \geq v_n - \epsilon$. This means that

$$\|M\| \leq \mathrm{vol}(M)\|\alpha_R/a\|_1 \leq \frac{\mathrm{vol}(M)}{v_n - \epsilon}.$$

That leads to the desired inequality letting ϵ go to zero.

$\square$

Chapter 4

Manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$

The aim of this chapter is to describe the method introduced by Bucher-Karlsson in [Buc08B] in order to compute the proportionality constant between the Riemannian volume and the simplicial volume for manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. As we have already pointed out this is the only case, apart from hyperbolic manifolds, in which a nonzero proportionality constant is known.

The proof will follow the line of the proof of Theorem 3.4.1. In particular, in Section 4.2 we give an explicit description of an expecially simple representative of the volume form (this is someway a more sophisticated analogue of Dupont's description of the cocycle $I_x\omega$ we studied in Section 1.8).

In Section 4.3 we will reduce to a suitable complex in which the computation can be made with the aid of some combinatorics. Also in the proof of Theorem 3.4.1 we have made something similar when we took advantage of the homological algebra developed in the first two chapters in order to reduce to the complex $(C_c^k(\mathbb{H}^n)_{\mathrm{alt}}; d)$: it is not enough to find a good representative for the volume form whose norm can be combinatorially computed (as a cocycle), it is also necesary to show that the chosen representative is indeed minimal and for this purpose is useful to find a small and tractable complex to which our cocycle (or, more precisely, a slight modification of our cocycle) belongs.

There are two basic differences in the present approach with respect to what we have already done: the first (and less important) is that we will need the full group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ instead of its identity component. In order to show that our representative of the fundamental coclass has minimal norm, we will have to prove that the coboundaries vanish on objects that are the equivalent (in this new context) of the ideal regular simplices. In the proof of this fact we will need extra symmetries that are isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ but do not belong to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, the component of the identity of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$. We will trace back to the full group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ in Section 4.1. In particular we will show in Proposition 4.1.1 that there exists

an isometric inclusion $H_c^4(H; \widetilde{\mathbb{R}}) \hookrightarrow H_c^4(G; \mathbb{R})$ where $H = \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$, and G is its identity component. Proposition 4.1.1 is analogue to Theorem 3.3.10 and in its proof we will use similar techniques.

The main difference respect to the hyperbolic case is, however, the fact that in $\mathbb{H}^2 \times \mathbb{H}^2$ we cannot rely on the theory of regular simplices. The clever idea of Bucher-Karlsson in order to avoid this difficulty is to work with the topological space $S^1 \times S^1$ that is the quotient of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$ with respect to an (amenable) minimal parabolic subgroup and that should be regarded, geometrically, as the product of the boundaries $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. However, in order to reduce to such an easier context (in which combinatorics is much more simple and, as we will see, we can rely on useful extra isometries), it is necessary to use bounded cohomology. We will see how this can be done in Section 4.3.

In particular we will be able only to compute the norm of the volume form as an element of $H_{cb}^4(H; \widetilde{\mathbb{R}})$ and we will actually do the computations in Section 4.4. To complete the proof we will then need to show that the comparison map $c : H_{cb}^4(H; \widetilde{\mathbb{R}}) \to H_c^4(H; \widetilde{\mathbb{R}})$ is indeed an isomorphism: this will be the topic of Section 4.5.

4.1   The group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$

We want to study the group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ that we will denote, in the whole chapter, with the letter H.

Clearly $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is a subgroup of H: since the metric on $\mathbb{H}^2 \times \mathbb{H}^2$ is the product metric, the product of two isometries acting separately on the two factors is an isometry of the product. Moreover the identity belongs to G and, since $PSL_2(\mathbb{R})$ is connected, also G is.

We claim that G has index 8 in H and is the component of the identity of H. Indeed we already know that $PSL_2(\mathbb{R})$ is a closed subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ of index 2. From this follows that $G < \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ is a closed subgroup of index 4. It only remains to show that $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ is a closed subgroup of H of index 2.

Indeed let us consider the isometry (of order 2) switching the two factors:

$$\sigma : \mathbb{H}^2 \times \mathbb{H}^2 \quad \to \mathbb{H}^2 \times \mathbb{H}^2$$
$$(x, y) \quad \to (y, x).$$

Let us denote by $\widetilde{H}$ the group generated by $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ and σ. It is easy to verify that $\widetilde{H} = H$. Indeed the group $\widetilde{H}$ acts transitively on $\mathbb{H}^2 \times \mathbb{H}^2$ (since it contains G that acts transitively itself), moreover it contains the stabilizer of a generic point x: the tangent $T_x(\mathbb{H}^2 \times \mathbb{H}^2)$ splits as a direct sum (where each factor corresponds to the image of the tangent of one factor $\mathbb{H}^2$) and any isometry ϕ of $\mathbb{H}^2 \times \mathbb{H}^2$ fixing x must preserve this

splitting. Up to precomposing with σ we can assume that $d\phi_x$ maps each factor in itself and hence ϕ has the form (ϕ_1, ϕ_2) with $\phi_i \in \text{Isom}(\mathbb{H}^2)$.

Since G is closed and has finite index in H, it is also open. Hence the fact that G is the connected component of the identity follows from the fact that G itself is connected (since $PSL_2(\mathbb{R})$ is).

In what follows, it will be useful to find explicit representatives for the lateral classes of G in H. In particular, if we chose a reflection τ with respect to any geodesic of $\mathbb{H}^2$ (i.e. an orientation reversing isometry of $\mathbb{H}^2$ that has order 2), we can consider the isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ defined by

$$\begin{array}{llll} \tau_1 : & \mathbb{H}^2 \times \mathbb{H}^2 & \to & \mathbb{H}^2 \times \mathbb{H}^2 \\ & (x,y) & \to & (\tau(x), y) \end{array} \qquad \begin{array}{lll} \tau_2 : & \mathbb{H}^2 \times \mathbb{H}^2 & \to \mathbb{H}^2 \times \mathbb{H}^2 \\ & (x,y) & \to (x, \tau(y)). \end{array}$$

clearly $\sigma\tau_1 = \tau_2\sigma$. And, moreover, $H = \langle G, \sigma, \tau_1 \rangle$. Since $F = \langle \sigma, \tau_1 \rangle$ is a discrete subgroup of order 8, H is the semidirect product $H = G \rtimes F$.

We are now ready to relate the cohomology of G with that of H. As we have already done in the proof of Theorem 3.4.1 the idea is to construct an inverse to the restriction $C_c^*(H; \mathbb{R})^H \to C_c^*(G; \mathbb{R})^G$ by averaging on the lateral classes (indeed in this case the space of cosets is finite and hence we will substitute the integral with a finite mean). This will allow us to deduce that $H_c^*(H; \mathbb{R})$ injects in $H_c^*(G; \mathbb{R})$ and that the injection is isometric.

As we have done for the isometry group of a symmetric space, we will then need to show that the cocycle we are interested in lies in the image of the map res. Unfortunately, even though the representative of the volume form (as a function in $C_c^4(G; \mathbb{R})^G$) has a natural extension in $C_c^4(H; \mathbb{R})$, such an extension is not H-invariant: the function

$$\begin{array}{ll} \phi : H^5 & \to \mathbb{R} \\ (g_0, \ldots, g_4) & \mapsto \int_{\triangle(g_0 x, \ldots, g_4 x)} \omega_{\mathbb{H}^2 \times \mathbb{H}^2} \end{array}$$

satisfies $g \cdot \phi = -\phi$ if g is any orientation reversing isometry of $\mathbb{H}^2 \times \mathbb{H}^2$. For this reason we need to work with the cohomology of H with twisted coefficients.

Let us denote by $\widetilde{\mathbb{R}}$ the H-module $(\mathbb{R}, \pi)$ where

$$\begin{array}{l} \pi : H \to \text{Aut}(\mathbb{R}) \\ g \mapsto \text{sgn}(g), \end{array}$$

and $\text{sgn}(g)$ denotes the multiplication by $+1$ if g is orientation preserving, by -1 otherwise. Since G consists only of orientation preserving isometries of $\mathbb{H}^2 \times \mathbb{H}^2$, the restriction of π to G is trivial.

Proposition 4.1.1. *There exists an isometric inclusion*

$$H_c^*(H; \widetilde{\mathbb{R}}) \to H_c^*(G; \mathbb{R}).$$

Proof. We have shown in Theorem 1.6.6 that the continuous cohomology of G can be computed from the complex $(C^*(H;\mathbb{R})^G, d)$. Moreover we have proved in Theorem 2.5.9 that the sup norm on $C^*(H;\mathbb{R})^G$ induces the canonical seminorm. We first describe some maps at the cochain level that induce the isometric isomorphism in cohomology.

The group H can be written as the union of right lateral classes of G: we have already chosen preferred representatives

$$F = \{\text{id}, \tau_1, \tau_2, \tau_1\tau_2, \sigma, \tau_1\sigma, \tau_2\sigma, \tau_1\tau_2\sigma\},$$

and $H = \bigcup_{f \in F} Gf$. This allows us to define a chain map $a : C^*(G;\mathbb{R}) \to C^*(H;\mathbb{R})$ obtained gluing copies of a cochain ϕ on the various lateral classes. The map a preserves the submodules of continuous functions since G is closed in H.

It is moreover obvious that, if we denote by b the map induced by the inclusion $G \hookrightarrow H$, the composition

$$C_c^*(G;\mathbb{R}) \xrightarrow{\ a\ } C_c^*(H;\mathbb{R}) \xrightarrow{\ b\ } C_c^*(G;\mathbb{R})$$

is the identity (and both maps are norm decreasing). Moreover, by construction, the maps a and b are chain G-morphisms (if we consider on H the diagonal left action of G) that extend the identity on $\mathbb{R}$ and hence induce an isometric isomorphism in cohomology.

We have already observed that the restriction to G of the twisted representation of H on $\widetilde{\mathbb{R}}$ is the trivial one, this means that $C_c^*(H;\widetilde{\mathbb{R}})^H$ is naturally a subcomplex of $C_c^*(H;\mathbb{R})^G$.

We will now define the left inverse to the map (that we will denote by res) corresponding to the inclusion

$$\text{res} : C_c^*(H;\widetilde{\mathbb{R}})^H \to C_c^*(H;\widetilde{\mathbb{R}})^G.$$

The required inverse is provided by the map

$$
\begin{aligned}
\text{trans}(\phi)(g_0,\ldots,g_n) &= \frac{1}{8}\sum_{f \in F} f \cdot \phi(g_0,\ldots,g_n) \\
&= \frac{1}{8}\sum_{f \in F} \text{sgn}(f) \cdot \phi(f^{-1}g_0,\ldots,f^{-1}g_n).
\end{aligned}
$$

The proof that the map trans is well defined is anologous to that of Theorem 3.4.1 (even if in this case one must take extra care since the coefficients are twisted). We will do it again for completeness. Since in this particular case we have an explicit description of H as an extension of G, we can show the invariance explicitly. Indeed in order to prove that $\text{trans}(\phi)$ is H-invariant it is sufficient to show that it is G-invariant, σ-invariant and τ_1-invariant.

The G-invariance of $\mathrm{trans}(\phi)$ follows, almost immediately, from the G-invariance of ϕ: since G is normal in H, for every $g \in G$, $f \in F$ there exists an element $g_f \in G$ such that $fg = g_f f$, hence

$$
\begin{aligned}
g^{-1} \cdot \mathrm{trans}(\phi)(g_0, \dots, g_n) \ &= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}\,(f)\, \phi(fgg_0, \dots, fgg_n) = \\
&= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f)\, \phi(g_f fg_0, \dots, g_f fg_n) = \\
&= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f)\, (g_f^{-1} \cdot \phi)(fg_0, \dots, fg_n) = \\
&= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f)\, \phi(fg_0, \dots, fg_n) = \\
&= \mathrm{trans}(\phi)(g_0, \dots, g_n).
\end{aligned}
$$

We will prove the σ and τ_1 invariance together. Let ρ be any element of the set $\{\sigma, \tau_1\}$. The right multiplication by ρ induces a permutation of the set F. Furthermore $\mathrm{sgn}(f\rho) = \mathrm{sgn}(f)\mathrm{sgn}(\rho)$. This implies that:

$$
\begin{aligned}
(\rho^{-1} \cdot \mathrm{trans}(\phi))(g_0, \dots, g_n) \ &= \mathrm{sgn}(\rho^{-1})\, \mathrm{trans}(\phi)(\rho g_0, \dots, \rho g_n) = \\
&= \mathrm{sgn}(\rho) \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f)\, \phi(f\rho g_0, \dots, f\rho g_n) = \\
&= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f)\, \mathrm{sgn}(\rho)\, \phi(f\rho g_0, \dots, f\rho g_n) = \\
&= \tfrac{1}{8} \sum_{f \in F} \mathrm{sgn}(f\rho)\, \phi(f\rho g_0, \dots, f\rho g_n) = \\
&= \mathrm{trans}(\phi)(g_0, \dots, g_n)
\end{aligned}
$$

(the sign of a permutation is the same of its inverse).

Furthermore it is obvious that the obtained function is continuous (being a finite sum of continuous functions), that trans is a chain map and that it provides a left inverse to res. Summarizing we have the commutative diagram

$$
C_c^*(H;\widetilde{\mathbb{R}})^H \xrightarrow{\ \mathrm{res}\ } C_c^*(H;\widetilde{\mathbb{R}})^G \xrightarrow{\ \mathrm{trans}\ } C_c^*(H;\widetilde{\mathbb{R}})^H
$$
$$
\underbrace{\phantom{C_c^*(H;\widetilde{\mathbb{R}})^H \xrightarrow{\ \mathrm{res}\ } C_c^*(H;\widetilde{\mathbb{R}})^G}}_{\mathrm{id}}
$$

that, since both res and trans are norm decreasing and their composition is the identity at chain level, gives the isometric injection we were looking for. $\square$

4.2 A representative for the volume form

The purpose of this section is to find a suitable representative for the volume form whose norm can be computed with the aid of some combinatorics. At the beginning of the section we will consider a representative in the complex $C_c^*(G;\mathbb{R})$. Then we will extend our cocycle in $C_c^*(H;\widetilde{\mathbb{R}})$.

Our first goal is to express the volume form as the cup product of 2-cocycles (that are evidently easier to deal with combinatorically). In order to do this, we first note that an analogue of the cup product can be defined also in the context of continuous (bounded) cohomology of groups. Namely whenever we fix a group G, the continuous bounded cohomology $H_{cb}^*(G;\mathbb{R})$ is the cohomology of the complex $C_{cb}^*(G;\mathbb{R})^G$.

The bilinear pairing (at the cochain level) given by

$$C_c^i(G;\mathbb{R}) \otimes C_c^j(G;\mathbb{R}) \to C_c^{i+j}(G;\mathbb{R})$$
$$\phi \cup \psi(g_0,\ldots,g_{i+j}) = \phi(g_0,\ldots,g_i)\psi(g_i,\ldots,g_{i+j})$$

is well defined since the projections $G^{i+j} \to G^i$ are continuous, this implies that the cup $\phi \cup \psi$ is a product of continuous functions, in particular is continuous. Moreover, since $\|\phi \cup \psi\|_\infty \leq \|\phi\|_\infty \|\psi\|_\infty$, the cup product of bounded cochains is a bounded cochain and hence the cup product restricts to a product of the bounded subcomplex. The verifications that $d(\phi \cup \psi) = d\phi \cup \psi + (-1)^i\phi \cup d\psi$ are formally the same of the corresponding in singular cohomology and lead to the conclusion that the just defined pairing induces a well defined map

$$\cup : H^i(G;\mathbb{R}) \otimes H^j(G;\mathbb{R}) \to H^{i+j}(G;\mathbb{R}).$$

Let us now came back to the isometries of $\mathbb{H}^2 \times \mathbb{H}^2$. We fix notations for the projections, in particular we will denote by $\pi_i : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2$ the projection on the i-th factor, and similarly by $p_i : PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ the i-th projection. Let now M be a symmetric space (that can be either $\mathbb{H}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$) and $I_0(M)$ the connected component of the identity in the isometry group of M (respectively $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$). If $\alpha \in \Omega^i(M;\mathbb{R})^{I_0(M)}$ is an invariant form, we denote by $[\alpha]$ the image of α in $H_c^i(I_0(M);\mathbb{R})$ under van Est isomorphism. The following lemma follows from the functoriality of van Est isomorphism:

Lemma 4.2.1. *For every $\alpha \in \Omega^i(\mathbb{H}^2;\mathbb{R})$,*

$$[\pi_j^*\alpha] = p_j^*[\alpha]$$

as classes in $H_c^i(G;\mathbb{R})$.

Proof. The statement is true even at cocycle level, provided we choose suitable representatives: let us fix a point (x,y) in $\mathbb{H}^2 \times \mathbb{H}^2$ and consider the Dupont representative of α described in Section 1.8. According to the notation of Section 1.8, we denote by $I_{(x,y)}\beta$ the representative of β with respect to the point (x,y). We are going to show that $I_{(x,y)}\pi_1^*\alpha = p_1^*I_x\alpha$ (the proof for π_2^* and p_2^* being similar). Note that an element g of G is a product of two isometries acting separately on the two factors, $g = (g^1, g^2)$.

$$
\begin{aligned}
I_{(x,y)}\pi_1^*\alpha((g_0^1,g_0^2),\ldots,(g_i^1,g_i^2)) &= \int_{\triangle((g_0^1x,g_0^2y),\ldots,(g_i^1x,g_i^2y))} \pi_1^*\alpha = \\
&= \int_{\triangle(g_0^1x,\ldots,g_i^1x)} \alpha = \\
&= I_x\alpha(g_0^1,\ldots,g_i^1) = \\
&= p_1^*I_x\alpha((g_0^1,g_0^2),\ldots,(g_i^1,g_i^2))
\end{aligned}
$$

$\square$

Another important property of van Est isomorphism is that it is multiplicative with respect to wedge product:

Proposition 4.2.2.

$$[\omega_{\mathbb{H}^2 \times \mathbb{H}^2}] = p_1^*[\omega_{\mathbb{H}^2}] \cup p_2^*[\omega_{\mathbb{H}^2}].$$

Proof. Obviously $[\omega_{\mathbb{H}^2 \times \mathbb{H}^2}] = [\pi_1^*\omega_{\mathbb{H}^2} \wedge \pi_2^*\omega_{\mathbb{H}^2}]$ hence it remains to prove that van Est isomorphism is multiplicative. We will get this result as a consequence of the multiplicativity of de Rham isomorphism between de Rham cohomology and singular cohomology. Indeed let us consider the product $\Sigma_2 \times \Sigma_2$ of two surfaces of genus 2. We have seen in Theorem 3.3.10 that there exists an isometric inclusion:

$$H_c^*(G; \mathbb{R}) \hookrightarrow H^*(\pi_1(\Sigma_2 \times \Sigma_2); \mathbb{R}) \cong H^*(\Sigma_2 \times \Sigma_2; \mathbb{R}).$$

Moreover, as a consequence of the explicit description at the cochain level of Van Est isomorphism and de Rham isomorphism, the following diagram is commutative

$$
\begin{array}{ccc}
\Omega^4(\mathbb{H}^2 \times \mathbb{H}^2)^G & \overset{\rho}{\hookrightarrow} & \Omega^4(\Sigma_2 \times \Sigma_2) \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \lambda} \\
H_c^4(G; \mathbb{R}) & \overset{\mathrm{res}}{\hookrightarrow} & H^4(\Sigma_2 \times \Sigma_2).
\end{array}
$$

The arrow ρ corresponds to the pushforward of a $\pi_1(\Sigma_2 \times \Sigma_2)$-invariant form and hence is injective and preserves the wedge product. We have proved in Proposition 4.1.1 that the arrow res is injective, moreover it preserves the cup product (since the definition at the cochain level of the cup product we have given in group cohomology is formally the same to that on singular cohomology). The second vertical arrow λ corresponds to de Rham isomorphism that is multiplicative [War83, Theorem 5.45], this implies that also van Est isomorphism I is multiplicative. $\qquad\square$

Our next step is to find a more suitable representative for the class $[\omega_{\mathbb{H}^2}] \in H_c^2(PSL_2(\mathbb{R}))$. The idea is to look for a cocycle that takes into account only the action of $PSL_2(\mathbb{R})$ on the boundary of $\mathbb{H}^2$.

Let us consider the cocycle $or : (S^1)^3 \to \{-\pi, 0, \pi\}$ defined by

$$or(x_0, x_1, x_2) = \begin{cases} +1 & \text{if } (x_0, x_1, x_2) \text{ are distinct and positively oriented} \\ -1 & \text{if } (x_0, x_1, x_2) \text{ are distinct and negatively oriented} \\ 0 & \text{if } \#\{x_0, x_1, x_2\} \le 2 \end{cases}$$

Since or takes values in a discrete set, it is not a continuous function from $(S^1)^3$ in $\mathbb{R}$ and hence it is not useful to define a continuous cocycle from $PSL_2(\mathbb{R})^3$ in $\mathbb{R}$. Anyway we have proved in Section 1.4 that the continuous

cohomology of G (for every Lie group G) can be computed as the cohomology of the complex

$$L^1_{loc}(G^i; \mathbb{R}) = \{f : G^i \to \mathbb{R} \mid f \text{ is locally integrable}\}.$$

And, since or is a measurable bounded function on $(S^1)^3$, it induces a measurable bounded (and hence locally integrable) function on $PSL_2(\mathbb{R})^3$. Indeed let us fix any point $\xi \in S^1 = \partial \mathbb{H}^2$, since $PSL_2(\mathbb{R}) = \text{Isom}^+ \mathbb{H}^2$ acts on $S^1 = \partial H^2$, we can assign to every triple (g_0, g_1, g_2) in $PSL_2(\mathbb{R})$ a triple $(g_0\xi, g_1\xi, g_2\xi) \in (S^1)^3$. Precomposing with this action we can define a cocycle $or_\xi \in L^1_{loc}(G^3; \mathbb{R})$ defined by

$$or_\xi(g_0, g_1, g_2) = or(g_0\xi, g_1\xi, g_2\xi).$$

The volume of any ideal triangle in the hyperbolic plane is equal to π, moreover the ideal simplex with vertices $(g_0\xi, g_1\xi, g_2\xi)$ is degenerate (and hence has volume 0) if there exists $i \neq j$ such that $g_i\xi = g_j\xi$, otherwise is positively (resp. negatively) oriented if and only if the triple of its endpoints is. This means that, once we have multiplied it by π, the cocycle or_ξ is somehow the analogous of the Dupont representative of the class of the volume form, provided we allow the point x be in the boundary $\partial \mathbb{H}^2$ instead of in $\mathbb{H}^2$.

The fact that $dor_\xi = 0$ descends from the definition (and the characterization in term of volume of convex hulls of ideal points). We will show that $[\pi or_\xi] = [\omega_{\mathbb{H}^2}]$.

Proposition 4.2.3. *We can choose a point ξ in $S^1 = \partial \mathbb{H}^2$ for which the cocycle $\pi or_\xi \in L^1_{loc}(PSL_2(\mathbb{R})^3; \mathbb{R})$ belongs to the class of the volume form in $H^2_c(PSL_2(\mathbb{R}); \mathbb{R})$:*

$$[\pi or_\xi] = [\omega_{\mathbb{H}^2}].$$

Proof. Let us fix a hyperbolic surface Σ_2 and a metric covering $\pi : \mathbb{H}^2 \to \Sigma_2$ (and hence an embedding of $\Gamma = \pi_1(\Sigma_2)$ as a discrete subgroup of $PSL_2(\mathbb{R})$). We have proved in Theorem 3.3.10 that the group $H^*_c(PSL_2(\mathbb{R}); \mathbb{R})$ injects isometrically in $H^*(\Gamma; \mathbb{R})$ and that the injection can be realized, at the cochain level, b the restriction

$$\text{res} : C^*_c(PSL_2(\mathbb{R}); \mathbb{R}) \to C^*(\Gamma; \mathbb{R}).$$

Indeed the map res extends to a well defined map from $L^1_{loc}(PSL_2(\mathbb{R})^3; \mathbb{R})$: the fundamental group Γ is discrete, and hence any continuity condition is empty. Since the map induced in cohomology by res is injective, in order to verify that $[\pi or_\xi] = [\omega_{\mathbb{H}^2}]$, it is sufficient to show that the two classes have the same image in the cohomology of Γ. Moreover $H^*(\Gamma; \mathbb{R})$ is isomorphic to $H^*(\Sigma_2, \mathbb{R})$ because $\mathbb{H}^2$ is contractible, and, in dimension two, these groups are one dimensional. This implies that a cohomology class is identified by its value on the fundamental class of Σ_2.

We can choose a representative of the fundamental class $[\Sigma_2]$ made of six straight simplices that have in Σ_2 only one vertex (the cycle corresponds to an appropriate triangulation of a straight convex fundamental domain of Σ_2). If we choose a lifting x of the common vertex, every simplex σ_i of this triangulation can be identified with a triple $(1, g_1^i, g_2^i)$ in $PSL_2(\mathbb{R})^3$ such that $(x, g_1^i x, g_2^i x)$ are the vertices of the lifting of σ_i.
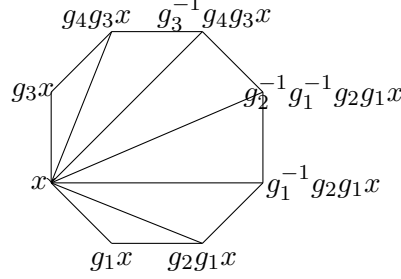
As a consequence of the explicit description (at the cochain level) of the isomorphisms between $H_c^2(PSL_2(\mathbb{R}), \mathbb{R})$ and $H^2(\Sigma_2, \mathbb{R})$, we can represent the fundamental class $[\Sigma_2]$ with the sum $\gamma = \sum_{i=1}^6 (1, g_1^i, g_2^i)$ and compute the values of the cocycles on this sum.

We have only to show that, once any representative α of the volume form $[\omega_{\mathbb{H}^2}]$ is fixed,
$$\pi or_\xi(\gamma) = \alpha(\gamma).$$

We will get this conclusion via a limiting process. Let us consider a pont $\xi \in \partial \mathbb{H}^2$ that doesn't belong to the set of the fixed points of $\{g_j^i, g_1^i(g_2^i)^{-1}\}$:

$$\xi \notin \bigcup_i (\mathrm{fix} g_1^i \cup \mathrm{fix} g_2^i \cup \mathrm{fix} g_1^i (g_2^i)^{-1}).$$

Since the set on the right hand side is finite this assumption is harmless but will be useful in the limiting process (we are asking that, for every i, the ideal simplex $(\xi, g_1^i \xi, g_2^i \xi)$ is nondegenerate).

Let us now consider a sequence of points $y_j \in \mathbb{H}^2$ that tends to ξ, and fix the following Dupont representatives α_{y_j} of the volume form:

$$\alpha_{y_j}(g_0, g_1, g_2) = \int_{\triangle(g_0 y_j, g_1 y_j, g_2 y_j)} \omega_{\mathbb{H}^2}.$$

For every i the value $\pi or_\xi(1, g_1^i, g_2^i)$ is the limit of $\alpha_{y_j}(1, g_1^i, g_2^i)$: the sequence $\{g_k^i y_j\}$ tends to the point $g_k^i \xi$ in the boundary of $\mathbb{H}^2$. Hence they definitely belong to sufficiently small neighborhoods of the (distinct) points $g_k^i \xi$. This means that the angles of the triangles with vertices $(y_j, g_1^i y_j, g_2^i y_j)$ have limit 0 when j goes to infinity.

It is now sufficient to remark that we already know that, for every j, the cocycle α_{y_j} represents the volume form and hence $\alpha_{y_j}(\gamma) = \mathrm{vol}(\Sigma_2) = 4\pi$ for every j: summarizing we have

$$\pi\mathrm{or}_\xi(\gamma) = \sum_i \pi\mathrm{or}(\xi, g_1^i\xi, g_2^i\xi) = \lim_{j\to\infty} \sum \alpha_{y_j}(1, g_1^i, g_2^i) = \mathrm{vol}(\Sigma_2).$$

This means that $[\pi\mathrm{or}_\xi] = [\omega_{\mathbb{H}^2}]$ and hence concludes the proof. $\qquad\square$

We need one more step in order to get the desired cocycle. As we have already pointed out we want to extend the cocycle $\pi^2 p_1^*\mathrm{or}_\xi \cup p_2^*\mathrm{or}_\xi$ to an element that defines a class in $H_c^4(H; \widetilde{\mathbb{R}})$ and hence we want to see $\pi^2 p_1^*\mathrm{or}_\xi \cup p_2^*\mathrm{or}_\xi$ as the restriction of an element of $L_{loc}^1(H^5; \widetilde{\mathbb{R}})$.

Indeed the original Dupont representative had a natural extension to $C_c^4(H; \widetilde{\mathbb{R}})$: the volume form is invariant with respect to the action of the whole isometry group of $\mathbb{H}^2 \times \mathbb{H}^2$ hence, if $\widetilde{\mathbb{R}}$ is considered as a twisted coefficient module, the same formula of $I_x\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ would provide an H-invariant cocycle in $C_c^4(H; \widetilde{\mathbb{R}})$:

$$I_x\omega_{\mathbb{H}^2\times\mathbb{H}^2} : \quad\begin{array}{ll} H^5 & \to \mathbb{R} \\ (g_0,\dots,g_4) & \mapsto \int_{\triangle(g_0 x,\dots,g_4 x)} \omega_{\mathbb{H}^2\times\mathbb{H}^2}. \end{array}$$

However, when we reduced to the cup product of more elementary cocycles, we lost the H-invariance: for example even if we consider a Dupont representative $\alpha = I_x\omega_{\mathbb{H}^2}$ of $[\omega_{\mathbb{H}^2}]$, the cocycle $p_1^*\alpha \cup p_2^*\alpha$ is not σ-invariant (where σ is the isometry of $\mathbb{H}^2 \times \mathbb{H}^2$ switching the factors). This is a consequence of the fact that the formula we gave for the cup product is highly asymmetric.

For this reason we will now alternate the cocycle. We already used in Remark 2.5.5 that the alternating operator:

$$\mathrm{Alt} : C^*(G; \mathbb{R}) \to C^*(G; \mathbb{R})$$
$$\mathrm{Alt}\phi(g_0,\dots,g_n) = \tfrac{1}{n+1!} \sum_{\rho\in\mathcal{S}_{n+1}} \mathrm{sgn}(\rho)\, \phi(g_{\rho(0)},\dots,g_{\rho(n)})$$

is chain homotopic to the identity. We can define with the same formula a G-morphism of the complexes of locally integrable functions $(L_{loc}^1(H^5; \widetilde{\mathbb{R}}))$, that induces the identity in cohomology also in this context.

Since Alt is chain homotopic to the identity, if we define the cocycle Θ_ξ by requiring

$$\Theta_\xi = \mathrm{Alt}(p_1^*\mathrm{or}_\xi \cup p_2^*\mathrm{or}_\xi),$$

$\pi^2\Theta_\xi$ represents the class of the volume form in $H_c^4(G; \mathbb{R})$.

The last step of the section is to show how Θ_ξ can be naturally extended to a class in $H_c^4(H; \widetilde{\mathbb{R}})$. To prove this result we first notice that there is a well defined action of H on $S^1 \times S^1 = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. We have already seen that $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ is a subgroup of H of index 2 and obviously acts on $S^1 \times S^1 = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ separately on the two factors. Moreover we can define

the action of σ on $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ analogously to its action on $\mathbb{H}^2 \times \mathbb{H}^2$, namely with the homeomorphism of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ that exchanges the two factors

$$\sigma: \quad S^1 \times S^1 \to S^1 \times S^1$$
$$(x, y) \to (y, x).$$

It descends from the structure of H as an extension of G that the action is well defined.

Let us now extend the cocycle Θ_ξ: if we call $\pi_i : S^1 \times S^1 \to S^1$ the ith projection, we define the function

$$\pi_1^* \text{or} \cup \pi_2^* \text{or} : (S^1 \times S^1)^5 \to \mathbb{R}$$
$$(\pi_1^* \text{or} \cup \pi_2^* \text{or})((x_0, y_0), \dots (x_4, y_4)) = \text{or}(x_0, x_1, x_2)\text{or}(y_2, y_3, y_4).$$

Clearly Θ_ξ satisfies

$$\Theta_\xi(g_0, \dots, g_4) = \text{Alt}(\pi_1^* \text{or} \cup \pi_2^* \text{or})(g_0(\xi, \xi), \dots, g_4(\xi, \xi)).$$

This formula extends to a cocycle Θ_ξ in $L^1_{loc}(H^5; \mathbb{R})$ (since we have defined the action of H on $S^1 \times S^1$) and the following proposition holds:

Proposition 4.2.4. *The extension Θ_ξ in $L^1_{loc}(H^5; \mathbb{R})$ is H-invariant and hence lies in the image of the map res defined in Proposition 4.1.1.*

Proof. We first remark that, provided $\mathbb{R}$ is endowed with the twisted H-module structure, the cocycle or_ξ is $\text{Isom}(\mathbb{H}^2)$-invariant. From this fact it follows that the function Θ_ξ is $(\text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{H}^2))$-invariant.

It only remains to prove that the cocycle Θ_ξ is σ-invariant. This descends from the fact that we alternated the cocycle and the observation that $\sigma \cdot \pi_1^* \text{or} = \pi_2^* \text{or}$. If we denote by $\bar{\xi}$ the point (ξ, ξ) in $S^1 \times S^1$, we get

$$\sigma \cdot \Theta_\xi(h_0, \dots, h_4) = \Theta_\xi(\sigma h_0, \dots, \sigma h_4) =$$
$$= \text{Alt}(\pi_1^* \text{or} \cup \pi_2^* \text{or})(\sigma h_0(\bar{\xi}), \dots, \sigma h_4(\bar{\xi})) =$$
$$= \sum_\rho \text{sgn}(\rho)\, \pi_1^* \text{or}(\sigma h_{\rho(0)}(\bar{\xi}), \sigma h_{\rho(1)}(\bar{\xi}), \sigma h_{\rho(2)}(\bar{\xi}))\pi_2^* \text{or}(\sigma h_{\rho(2)}(\bar{\xi}), \sigma h_{\rho(3)}(\bar{\xi}), \sigma h_{\rho(4)}(\bar{\xi})) =$$
$$= \sum_\rho \text{sgn}(\rho)\, \pi_2^* \text{or}(h_{\rho(0)}(\bar{\xi}), h_{\rho(1)}(\bar{\xi}), h_{\rho(2)}(\bar{\xi}))\pi_1^* \text{or}(h_{\rho(2)}(\bar{\xi}), h_{\rho(3)}(\bar{\xi}), h_{\rho(4)}(\bar{\xi})) =$$
$$= \sum_\rho \text{sgn}(\rho)\, \pi_2^* \text{or}(h_{\rho(2)}(\bar{\xi}), h_{\rho(0)}(\bar{\xi}), h_{\rho(1)}(\bar{\xi}))\pi_1^* \text{or}(h_{\rho(3)}(\bar{\xi}), h_{\rho(4)}(\bar{\xi}), h_{\rho(2)}(\bar{\xi})) =$$
$$= \sum_\rho \text{sgn}(\rho)\, \pi_2^* \text{or}(h_{\rho(2)}(\bar{\xi}), h_{\rho(3)}(\bar{\xi}), h_{\rho(4)}(\bar{\xi}))\pi_1^* \text{or}(h_{\rho(0)}(\bar{\xi}), h_{\rho(1)}(\bar{\xi}), h_{\rho(2)}(\bar{\xi})) =$$
$$= \Theta_\xi(h_0, \dots, h_4).$$

where in the fourth equality we used that or is cyclic, in the fifth that the precomposition with the (even) permutation $(03)(14)$ is a bijection of the set $\mathcal{S}_5$. $\square$

4.3 A suitable complex

In the previous section we found a suitable representative for the volume form belonging to the complex $L^1_{loc}(H^5; \widetilde{\mathbb{R}})$. The norm of the cocycle Θ_ξ is indeed computable (and we will compute it in Section 4.4). However the complex $L^1_{loc}(H^5; \widetilde{\mathbb{R}})$ is far too big and we wouldn't be able to prove the minimality of the norm of Θ_ξ in its class. The aim of this section is to show that the cocycle Θ_ξ belongs to a smaller complex that has the same (bounded) cohomology of $L^1_{loc}(H^5; \widetilde{\mathbb{R}})$.

Indeed, since the cocycle Θ_ξ takes into account only the action of H on the product $S^1 \times S^1$ of the boundaries of the two factors of $\mathbb{H}^2 \times \mathbb{H}^2$, we would like to reduce ourselves to a complex of the form $C^*(S^1 \times S^1; \widetilde{\mathbb{R}})$ (we have done something similar considering the complex $C^*_c(\mathbb{H}^n; \mathbb{R})$ to compute the norm of the volume form on hyperbolic spaces). However this is not possible in the context of continuous cohomology since $S^1 \times S^1$ is not the quotient of H with respect to a compact subgroup.

However, the group H acts transitively on $S^1 \times S^1$ as a group of homeomorphisms, this implies that $S^1 \times S^1$ is the quotient of H with respect to the stabilizer P of a point (for example the point (ξ, ξ)). We will now describe this stabilizer.

The stabilizer in $PSL_2(\mathbb{R})$ of a point $\xi \in \partial \mathbb{H}^2$ is the group $\mathbb{R} \rtimes \mathbb{R}^+ = \mathrm{Aff}^+(\mathbb{R})$: we identify the orientation preserving isometries of the upper half space model of $\mathbb{H}^2$ that fix the point ∞ with their trace on the boundary. The normal subgroup of the parabolic isometries correspond to translations of the real line, whereas the hyperbolic isometries fixing ∞ and 0 correspond to dilations of $\mathbb{R}$. Together they generate the stabilizer of ∞ that hence corresponds to the group of homeomorphisms of $\mathbb{R}$ generated by translation and dilations, i.e. the positively-oriented affinities of the real line. This implies that $\mathrm{Aff}^+(\mathbb{R}) \times \mathrm{Aff}^+(\mathbb{R}) \subseteq PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is contained in the stabilizer of (ξ, ξ) and is the intersection $\mathrm{stab}(\xi, \xi) \cap G$. Moreover $\mathrm{Aff}^+(\mathbb{R}) \times \mathrm{Aff}^+(\mathbb{R})$ is a normal subgroup of index 8 of $\mathrm{stab}(\xi, \xi)$ (if we choose τ the reflection with respect to a geodesic having ξ as endpoint, both (τ, id) and σ belongs to $\mathrm{stab}(\xi, \xi)$ and they generate a group of order 8).

The groups $\mathbb{R}$ and $\mathbb{R}^+$ are abelian and hence amenable. This implies that also $\mathrm{Aff}^+(\mathbb{R})$ is amenable since it is an extension of an amenable group by another amenable group (see Theorem 2.6.5). The same argument implies that also $\mathrm{Aff}^+(\mathbb{R}) \times \mathrm{Aff}^+(\mathbb{R})$ is amenable. The latter group is normal and has finite index in $\mathrm{stab}(\xi, \xi)$. We get that $\mathrm{stab}(\xi, \xi)$ is an amenable subgroup of H. Hence $S^1 \times S^1$ can be seen as the quotient of H with respect to an amenable subgroup.

A consequence of Theorem 2.6.13 is that the continuous bounded cohomolgy of H can be computed from the complex of bounded measurable functions on $S^1 \times S^1$ with values in $\widetilde{\mathbb{R}}$. In the whole Chapter we will change the notation of Chapter 2 and denote by $M^n_b(S^1 \times S^1; \widetilde{\mathbb{R}})$ the modules of

that complex. This is useful in order to shorten the notation and make the dimension more visible.

$$M^n(S^1 \times S^1; \widetilde{\mathbb{R}}) = \{f : (S^1 \times S^1)^{n+1} \to \widetilde{\mathbb{R}} | \, f \text{ is bounded, measurable}\}.$$

We have already pointed out in Remark 2.5.5 that the continuous bounded cohomology of H can be computed also from the complex

$$M^n_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}}) = \{f : (S^1 \times S^1)^{n+1} \to \widetilde{\mathbb{R}} | \, f \text{ is bounded, measurable, alternating}\}$$

and the l^∞-norm on this complex induces the canonical seminorm in cohomology; this is a consequence of the fact that the alternating operator is chain homotopic to the identity, it is norm decreasing and has image in the alternating functions.

We will consider the cocycle belonging to $M^4_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}})$ given by the formula

$$\Theta_b = \text{Alt}(\pi_1^* \text{or} \cup \pi_2^* \text{or}).$$

The function Θ_b indeed belongs to $M^4_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}})$ since it is always smaller than 1 and hence bounded.

If we denote by c the comparison map $c : H_{cb}(H; \widetilde{\mathbb{R}}) \to H_c(H; \widetilde{\mathbb{R}})$ we get

Theorem 4.3.1.

$$c[\Theta_b] = [\Theta_\xi]$$

Proof. We recall that a map

$$M^n_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}}) \xrightarrow{\ \alpha\ } L^\infty(H^{n+1}; \widetilde{\mathbb{R}})$$

that induces an isomorphism in cohomology can be described, at the cochain level, by choosing a point $(\xi, \xi) \in S^1 \times S^1$ and requiring that

$$\alpha(\phi)(h_0, \ldots, h_n) = \phi(h_0(\xi, \xi), \ldots, h_4(\xi, \xi)) :$$

Since the cocycle Θ_ξ is clearly bounded ($\|\Theta_\xi\|_\infty \leq 1$), we can consider it as an element of $L^\infty(H^5; \widetilde{\mathbb{R}})$. It is now an obvious consequence of the explicit description of the map α that $\alpha(\Theta_b) = \Theta_\xi$ and this concludes the proof. $\square$

In Section 4.4 we will compute the seminorm of $[\Theta_b]$ in $H^4_{cb}(H; \widetilde{\mathbb{R}})$. Since the seminorm of a class $[\phi]$ in continuous cohomology is the infimum of the seminorms of the classes $[\phi]$ in continuous bounded cohomology that satisfy $c([\phi]) = [\psi]$, a consequence of Theorem 4.3.1 is that $\|\Theta_\xi\|_\infty \leq \|\Theta_b\|_\infty$. To get the opposite inequality we will show, in Section 4.4, that the comparison map is indeed an isomorphism and hence $\|\Theta_\xi\|_\infty = \|\Theta_b\|_\infty$.

4.4 Computation of the norm

The aim of this Section is the computation of the norm of Θ_b as a cocycle in $M^4_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ (Proposition 4.4.1) and as a cohomology class in $H^4_{cb}(H; \widetilde{\mathbb{R}})$ (Theorem 4.4.2).

Proposition 4.4.1. *The norm of the cocycle $\Theta_b \in M^4_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ is*

$$\|\Theta_b\|_\infty = \frac{2}{3}.$$

Proof. The proof that $\|\Theta_b\|_\infty \leq \frac{2}{3}$ is a combinatorial computation: we will explicitly write down the formulas for Θ_b (indeed, studying the permutation group and grouping some different summands, we will write only six terms instead of 120), and then we will use the cocycle relation (4.2) to reduce the number of the summands and get the desired inequality.

Let us fix a 5-uple $a = ((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))$ belonging to $(S^1 \times S^1)^5$, it follows from the definitions that

$$\Theta_b(a) = \frac{1}{120} \sum_\sigma \text{sgn}(\sigma)\text{or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})\text{or}(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}).$$

We want to reduce the number of the factors in this sum. First of all note that every permutation can be written uniquely as a product $\tau^k \alpha$ where $\tau = (01234)$ is the 5-cycle, $k \in \{0, 1, 2, 3, 4\}$, an α is a permutation such that $\alpha(2) = 0$. Moreover, since or is alternating, given two permutations such that $\rho(1) = \sigma(2)$, $\rho(2) = \sigma(1)$ and $\rho(i) = \sigma(i)$ for every other index (i.e. $\rho = \sigma(12)$), we get that

$$\text{sgn}(\sigma)\text{or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) = \text{sgn}(\rho)\text{or}(x_{\rho(0)}, x_{\rho(1)}, x_{\rho(2)}).$$

This is a consequence of the fact that the sign of a permutation is multiplicative (and hence $\text{sgn}(\sigma) = -\text{sgn}(\rho)$) and that the cocycle or is alternating (this fact implies that $\text{or}(x_a, x_b, x_c) = -\text{or}(x_a, x_c, x_b)$). These considerations allow us to group the possible permutations in six classes of four elements that give the same value and get

$$
\begin{aligned}
\Theta_b(a) = \frac{1}{120} \sum_{k=0}^{4} \quad & 4\text{or}(x_{\tau^k(1)}, x_{\tau^k(2)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(3)}, y_{\tau^k(4)}) + \\
+ \quad & 4\text{or}(x_{\tau^k(3)}, x_{\tau^k(4)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(2)}) + \\
- \quad & 4\text{or}(x_{\tau^k(1)}, x_{\tau^k(3)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(4)}) + \quad (4.1) \\
+ \quad & 4\text{or}(x_{\tau^k(1)}, x_{\tau^k(4)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(3)}) + \\
+ \quad & 4\text{or}(x_{\tau^k(2)}, x_{\tau^k(3)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(4)}) + \\
- \quad & 4\text{or}(x_{\tau^k(2)}, x_{\tau^k(4)}, x_{\tau^k(0)})\text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(3)}).
\end{aligned}
$$

In order to show that
$$\Theta_b(a) \le \frac{2}{3}$$
we have to consider two different cases depending on whether the x_i are all distinct or $\#\{x_0, \ldots, x_4\} \le 4$.

•x_i *distinct*

Let us assume that the x_i are all distinct, we can suppose that they are cyclically ordered according to their numbering: the cocycle Θ_b is alternating and we can change the indices precomposing with the proper permutation. This can only change the sign of Θ_b, but we are only interested in the absolute value.

This means that $\operatorname{or}(x_i, x_j, x_k) = 1$ for all the summands in the expression 4.1: we have chosen representatives such that, before applying τ, the values were positive and, since τ is a 5-cycle, it doesn't change the orientation.

Thus we have to compute the value of

$$\frac{1}{30} \sum_{k=0}^{4} \quad \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(3)}, y_{\tau^k(4)}) + \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(2)}) +$$
$$- \quad \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(4)}) + \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(3)}) +$$
$$+ \quad \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(4)}) - \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(3)}).$$

We can now apply the cocycle relation

$$0 = \delta\operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(3)}, y_{\tau^k(4)}) = \qquad (4.2)$$
$$= \operatorname{or}(y_{\tau^k(2)}, y_{\tau^k(3)}, y_{\tau^k(4)}) - \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(3)}, y_{\tau^k(4)})$$
$$+ \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(4)}) - \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(2)}, y_{\tau^k(3)}).$$

This implies that we can rewrite the expression (4.2) as

$$\frac{1}{30} \sum_{k=0}^{4} \operatorname{or}(y_{\tau^k(2)}, y_{\tau^k(3)}, y_{\tau^k(4)}) + \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(2)}) \qquad (4.3)$$
$$+ \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(4)}) - \operatorname{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(3)}).$$

Since 20 summands appear in this sum and each of them is bounded by one, we get
$$|\Theta_b(a)| \le \frac{20}{30} = \frac{2}{3}.$$

•$\#\{x_0, \ldots, x_4\} \le 4$

If the x_i are not all distinct we can assume that neither the y_i are: otherwise we can exchange the role of x_i and y_i taking advantage of the invariance of Θ_b with respect to the isometry σ.

We assume, up to precomposing with a permutation (Θ_b is alternating), that $x_0 = x_1$. If $y_0 = y_1$, then $\Theta_b(a) = 0$:to prove this fact we recall that Θ_b is alternating, hence, if we call ρ the permutation (01), we get

$$
\begin{aligned}
\Theta_b(a) \ &= \Theta_b((x_1, y_1), (x_0, y_0), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = \\
&= \Theta_b((x_{\rho(0)}, y_{\rho(0)}), (x_{\rho(1)}, y_{\rho(1)}), (x_{\rho(2)}, y_{\rho(2)}), (x_{\rho(3)}, y_{\rho(3)}), (x_{\rho(4)}, y_{\rho(4)})) = \\
&= \operatorname{sgn}(\rho)\, \Theta_b((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = \\
&= -\Theta_b((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = -\Theta_b(a).
\end{aligned}
$$

In the whole paragraph, as we have already done previously, we denoted by a the generic 5-uple $((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))$. In order to conclude the proof of the fact that

$$
\|\Theta_b\|_\infty \leq \frac{2}{3},
$$

we need only to deal with the case $x_0 = x_1$, and $y_2 = y_k$ for some $k \in \{0, 3, 4\}$. Let us consider the expression (4.3). We will find 10 different summands that vanish in that expression. Among the 30 summands there are exactly 9 that have the factor $or(x_0, x_1, x_j)$ and hence vanish: if $k = 0$ there are three of them, if $k = 1$, then $\tau(0) = 1$ and $\tau(4) = 0$ hence $or(x_0, x_1, x_j)$ appears in three summands. In the other cases $k = 2, 3, 4$ there is only one summand with the factor $or(x_0, x_1, x_j)$. Moreover the summand $or(x_0, x_3, x_4) or(y_1, y_2, y_k)$ that occurs (up to a permutation of the entries) in the sum (4.3) vanishes (since $y_2 = y_k$) and we haven't counted it among the 9 vanishing summands (since it doesn't have a factor $or(x_0, x_1, x_j)$).

It remains only to show that the value $2/3$ is achieved. We choose a 5-uple

$$
((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))
$$

where the x_i and the y_i are all distinct and ordered as in the drawing, i.e. in a way such that the x_i are cyclically ordered according to their numbering and the y_i are ordered so that $(y_0, y_3, y_1, y_4, y_2)$ are cyclically ordered. The 5-uples of this form plays (here and in the proof of the next theorem) the role of a simplex of maximal volume.
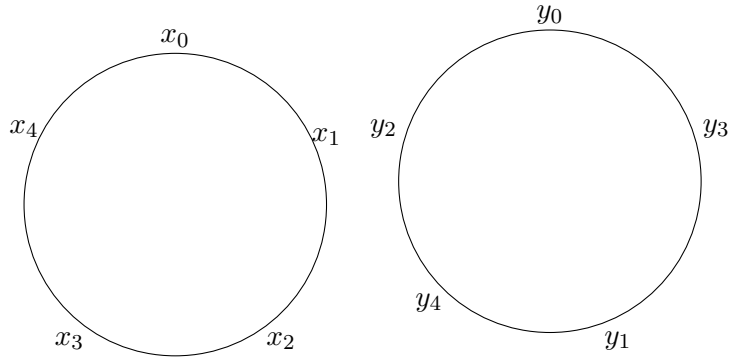
If we compute Θ_b on this 5-uple using the formula (4.3), we get

$$\Theta_b((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) \tag{4.4}$$
$$= \tfrac{1}{30} \sum_{k=0}^{4} \text{or}(y_{\tau^k(2)}, y_{\tau^k(3)}, y_{\tau^k(4)}) + \text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(2)})$$
$$+ \text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(4)}) - \text{or}(y_{\tau^k(0)}, y_{\tau^k(1)}, y_{\tau^k(3)}) = \tfrac{2}{3}.$$

$\square$

The last step of the section is to show that $2/3$ is actually the norm of $[\Theta_b]$ in $H^4_{cb}(H; \widetilde{\mathbb{R}})$, more precisely we will show that, for every β in $M^3_b(S^1 \times S^1; \widetilde{\mathbb{R}})$,

$$\delta\beta((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = 0$$

whenever the points respect the condition given at the end of the proof of Proposition 4.4.1, that is the x_i are cyclically ordered and the y_i are ordered so that $(y_0, y_3, y_1, y_4, y_2)$ are cyclically ordered. Since in the proof of Proposition 4.4.1 we have shown that, on this specific 5-uple a, Θ_b has the value $2/3$, we will get that

$$\|\Theta_b + \delta\beta\|_\infty \geq |(\Theta_b + \delta\beta)(a)| = \frac{2}{3}.$$

Theorem 4.4.2. *The seminorm of the class* $[\Theta_b]$ *in* $H^4_{cb}(H; \widetilde{\mathbb{R}})$ *satisfies*

$$\|[\Theta_b]\| = 2/3.$$

Proof. As we have already pointed out, we want to show that $\delta\beta(a) = 0$. We first show that

$$\beta((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = 0.$$

Let us consider the geodesic α_1 in $\mathbb{H}^2$ having as endpoints the points x_1 and x_2 belonging to $\partial\mathbb{H}^2$ and the geodesic α_2 having as endpoints x_3 and x_4. Let α_3 be the common orthogonal geodesic. Let us consider the orientation reversing isometry γ_1 of $\mathbb{H}^2$ corresponding to the reflection with respect to α_3. Clearly this isometry leaves the points $\{x_i\}_{i=1}^4$ invariant and acts on this set as the even permutation $(12)(34) = \sigma$.

Let us now construct an orientation preserving isometry of $\mathbb{H}^2$ that realizes the same (even) permutation of the points $\{y_i\}_{i=1}^4$. We consider the geodesic α_4 in $\mathbb{H}^2$ having as endpoints the points y_1 and y_2 and the geodesic α_5 having as endpoints y_3 and y_4. The points y_i are oriented in such a way that the two geodesics meet in a unique point that we call $\bar{y}$. Let us consider the orientation preserving isometry γ_2 of $\mathbb{H}^2$ corresponding to the geodesic symmetry in $\bar{y}$. Also γ_2 leaves the points y_i invariant and acts as the permutation $(12)(34)$.

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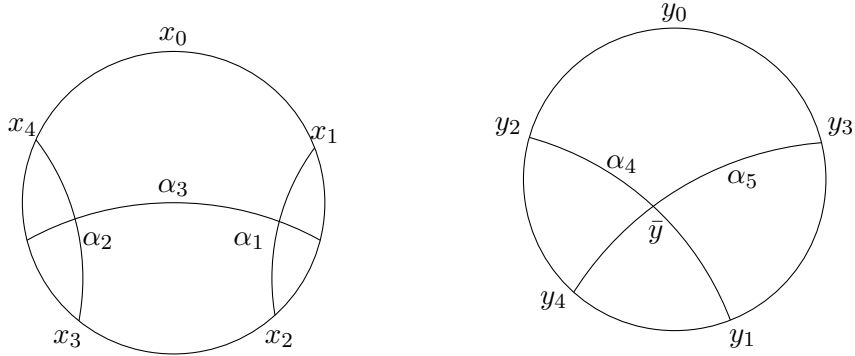
Let us consider the isometry $\gamma = (\gamma_1, \gamma_2)$ of $\mathbb{H}^2 \times \mathbb{H}^2$. It realizes, on the 4-uple $((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))$, the permutation σ and hence

$$
\begin{aligned}
\beta((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) &= \\
= \ \ \mathrm{sgn}(\sigma)\beta(((x_{\sigma(1)}, y_{\sigma(1)}), (x_{\sigma(2)}, y_{\sigma(2)}), (x_{\sigma(3)}, y_{\sigma(3)}), (x_{\sigma(4)}, y_{\sigma(4)}))) &= \\
= \ \ +\beta((x_2, y_2), (x_1, y_1), (x_4, y_4), (x_3, y_3)) &= \\
= \ \ -(\gamma \cdot \beta)((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) &= \\
= \ \ -\beta((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))
\end{aligned}
$$

where the first equality is a consequence of the fact that β is alternating (and σ is even), the third descends from the definition of the action of H on $M_b^4(S^1 \times S^1; \widetilde{\mathbb{R}})$ and the fact that γ is orientation reversing, the last one follows from the H-invariance of β. This implies that β vanishes on the first "face" of the 5-uple α.

With similar arguments, we can prove that β vanishes on every "face", namely, as we have done in this case, we can construct an orientation reversing isometry that realizes the permutation $(04)(23)$ for the second "face", $(01)(34)$ for the third, $(04)(12)$ for the fourth and $(01)(23)$ for the fifth. This concludes the proof of Theorem 4.4.2 $\qquad\qquad\square$

4.5   The comparison map is an isomorphism

We are now ready to finish the proof of the central theorem of the chapter:

Theorem 4.5.1. *The proportionality constant for the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ is $\frac{2}{3}\pi^2$.*

We have already pointed out in Theorem 3.3.11 that the proportionality constant for the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ is one over the norm $\|\omega_{\mathbb{H}^2 \times \mathbb{H}^2}\|_\infty$ as a class in $H_c^4(\mathrm{Isom}_0(\mathbb{H}^2 \times \mathbb{H}^2), \mathbb{R})$ and we have shown in Proposition 4.1.1 that we could compute its norm as an element of $H_c^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}})$. Moreover we have described, in Theorem 4.3.1, a class $[\pi^2 \Theta_b]$ in $H_{cb}^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}})$ whose image, via the comparison map, is $[\omega_{\mathbb{H}^2 \times \mathbb{H}^2}]$ and we have proved, in Theorem 4.4.2, that the norm of

$[\pi^2 \Theta_b]$ is $\frac{2}{3}\pi^2$. In order to finish the proof, it is now sufficient to show that the comparison map

$$c : H_{cb}^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}}) \to H_c^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}})$$

is an isomorphism. We split the proof of the theorem in two parts. We will start by showing an analogous result in the case of $\mathbb{H}^2$ instead of $\mathbb{H}^2 \times \mathbb{H}^2$ and then deduce the thesis from this fact.

Proposition 4.5.2. *The comparison map*

$$c : H_{cb}^2(Isom(\mathbb{H}^2), \widetilde{\mathbb{R}}) \to H_c^2(Isom(\mathbb{H}^2), \widetilde{\mathbb{R}})$$

is an isomorphism.

Proof. We already know that the group $H_c^2(\mathrm{Isom}(\mathbb{H}^2), \widetilde{\mathbb{R}})$ is one dimensional and generated by the class of the volume form (this is a consequence of van Est's Theorem 1.7.5). Moreover, we have already pointed out that Dupont's representative of the volume form is bounded. This implies that the comparison map is surjective. In order to prove the proposition, we need only to show that c is injective.

Theorem 2.6.13 ensures that the continuous bounded cohomology of $L = \mathrm{Isom}(\mathbb{H}^2)$ can be computed from the complex of measurable, bounded, alternating, L-invariant functions on $S^1 = \partial \mathbb{H}^2$. Let us choose a cocycle f belonging to $M_{b,\mathrm{alt}}^2(S^1; \widetilde{\mathbb{R}})^L$ and let us assume that $c([f]) = 0$. We want to show that $[f] = 0$. Since f is, by assumption, alternating, we get that $f(x, x, y) = 0$ for every x, y in $\partial \mathbb{H}^2$. Moreover L acts transitively on the positively oriented triples of elements of $\partial \mathbb{H}^2$ and hence, since f is L invariant, the value of f on a triple (x, y, z) depends only on the orientation $or(x, y, z)$ and hence $f = \lambda \, or$. Since the image of or via the comparison map is a nonnull multiple of the class of the volume form, and since we assumed that $c([f]) = 0$, we get that $\lambda = 0$ and hence $f = 0$. $\qquad \square$

Let us now consider the commutative diagram

$$
\begin{array}{ccc}
H_c^2(\mathrm{Isom}(\mathbb{H}^2); \widetilde{\mathbb{R}}) \otimes H_c^2(\mathrm{Isom}(\mathbb{H}^2); \widetilde{\mathbb{R}}) & \xrightarrow{\cup} & H_c^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}}) \\
\Big\uparrow {\scriptstyle c \otimes c} & & \Big\uparrow {\scriptstyle c} \\
H_{cb}^2(\mathrm{Isom}(\mathbb{H}^2); \widetilde{\mathbb{R}}) \otimes H_{cb}^2(\mathrm{Isom}(\mathbb{H}^2); \widetilde{\mathbb{R}}) & \xrightarrow{\cup} & H_{cb}^4(\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2), \widetilde{\mathbb{R}}).
\end{array}
$$

Proposition 4.5.2 ensures that the first vertical arrow is an isomorphism (because it is an isomorphism on each factor of the tensor product). Moreover the description of the continuous cohomology of $\mathrm{Isom}(\mathbb{H}^2)$ via $\mathrm{Isom}(\mathbb{H}^2)$-invariant differential forms on $\mathbb{H}^2$ together with the multiplicativity of van Est's Theorem (see Proposition 4.2.2) implies that the above arrow is an

isomorphism: the group $H_c^4(H, \widetilde{\mathbb{R}}) \cong H^4(\Omega^i(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R}))^G$ is one dimensional and generated by $\omega_{\mathbb{H}^2 \times \mathbb{H}^2} = p_1^* \omega_{\mathbb{H}^2} \wedge p_2^* \omega_{\mathbb{H}^2}$.

In order to finish the proof of Theorem 4.5.1 by showing that the comparison map (in dimension four) is an isomorphism, it is sufficient to show that the cup product on continuous bounded cohomology is surjective:

Proposition 4.5.3. *Every class ψ in $H_{cb}^4(Isom(\mathbb{H}^2 \times \mathbb{H}^2); \widetilde{\mathbb{R}})$ can be written as a product $p_1^* \alpha \cup p_2^* \beta$ where α and β belong to $H_{cb}^2(Isom(\mathbb{H}^2); \widetilde{\mathbb{R}})$.*

Proof. Keeping the notation from the preceding section we will denote by H the group $\text{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$. We compute the continuous bounded cohomology of H (resp. of $\text{Isom}(\mathbb{H}^2)$) from the complex of the H-invariant (resp. $\text{Isom}(\mathbb{H}^2)$-invariant), alternating, bounded, measurable functions defined on $S^1 \times S^1$ (resp. S^1) with values in $\widetilde{\mathbb{R}}$. Let us fix $f \in M_{b,\text{alt}}^4(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ representing the class ψ.

Let us fix three points x, y, z in S^1 that are cyclically ordered and consider the value $f((x,x),(x,y),(x,z),(y,x),(z,y)) = b$. It is worth remarking that, since $\text{Isom}(\mathbb{H}^2)$ acts transitively on the 3-uples of S^1 and since f is, by assumption, H-invariant, b doesn't depend on the choice of the three points, provided they are distinct and cyclically ordered.

Let us consider a 5-uple $a = ((x_0, y_0), \ldots, (x_4, y_4))$ in $(S^1 \times S^1)^5$ and define $n_1(a) = \#\{x_0, \ldots, x_4\}$ (resp. $n_2(a) = \#\{y_0, \ldots, y_4\}$). We want to show that the cocycle $f_0 = f - 6b\text{Alt}(p_1^*[\text{or}] \cup p_2^*[\text{or}])$ is cohomologous to 0 as an element of $M_{b,\text{alt}}^4(S^1 \times S^1; \widetilde{\mathbb{R}})^H$. In order to get this result, we will use an induction on the couple (n_1, n_2) (with respect to an adequate ordering that we will introduce in the proof) as follows: once we have constructed a cochain f_{i-1} cohomologous to f_0 and vanishing on the uples smaller than (n_1, n_2), we will add a coboundary h_i that is, too, null on the uples smaller than (n_1, n_2) but has also the property that $f_{i-1} - h_i = f_i$ is null also on the uples that have (n_1, n_2) distinct elements.

■ $n_i \leq 2$

In this case f_0 vanishes because it is H-invariant: let us assume, without loss of generality, that $n_1 = 2$. If we consider the geodesic with endpoints x_i and x_j, the reflection with respect to that geodesic is an orientation reversing isometry that leaves the 5-uple a fixed.

■ $(n_1, n_2) = (3, 3)$.

From now on we will denote by c a 3-uple of the form $((x_0, y_0), \ldots, (x_3, y_3))$
Let us consider the element h_1 in $M_{b,\text{alt}}^3(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ that has the value

$$h_1(c) = \begin{cases} f_0((x_i, y_k), c) & \text{if } x_i = x_j, \ y_k = y_l \ (\text{with } i \neq j, k \neq l) \\ 0 & \text{otherwise.} \end{cases}$$

The cochain h_1 is well defined since, if there exist $x_i = x_j \neq x_k = x_l$, we get $n_1((x_i, y_s), c) \leq 2$ and hence $h_1(c) = 0$ for both choices (similarly for y_m).

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Moreover, since f_0 is H-invariant, alternating, bounded, measurable, h_1 has the same properties. Since we have proved that f_0 vanishes on every 5-uple with $n_i(a) \leq 2$ and h_1 is defined from f_0, the same result holds for h_1.

Let us now consider the representative $f_1 = f_0 - dh_1$ (that is obviously cohomologous to f_0). We want to prove that f_1 vanishes on every 5-uple a with $n_i(a) = 3$. Since f_1 is H-invariant and alternating, we need only to show that $f_1(a) = 0$ when a ranges among representatives of the 5-uples in $S^1 \times S^1$ with $n_i(a) \leq 3$ up to permutations and isometries. We have that, if two elements of the 5-uple a coincide, then $f_1(a) = 0$ since f_1 is alternating. Therefore it is easy to verify that only three possible choice for the 5-uple a have to be considered (once three cyclically ordered points x, y, z in S^1 are fixed):

$$a_1 = ((x,x)(x,y)(x,z)(y,x)(z,x))$$
$$a_2 = ((x,x)(x,y)(x,z)(y,x)(z,y))$$
$$a_3 = ((x,x)(x,y)(y,x)(y,z)(z,y))$$

using the expression (4.1) for $g = \mathrm{Alt}(p_1^*[\mathrm{or}] \cup p_2^*[\mathrm{or}])$ it is easy to verify that $g(a_1) = \frac{1}{6}$, $g(a_2) = \frac{1}{6}$, $g(a_3) = -\frac{1}{6}$. Let us denote by $\lambda_i = f_0(a_i)$, since $f_0 = f - 6bg$ we get $\lambda_1 = 0$ as a consequence of the choice of b. Moreover, applying the cocycle relation we get:

$$0 = df_0(a_1, (y,y)) = 2\lambda_1 + \lambda_2 - \lambda_0 = \lambda_2 - \lambda_0.$$

Moreover it is easy to verify that

$$
\begin{aligned}
dh_1(a_0) &= f_0((x,x)(x,y)(x,z)(y,x)(z,x)) = \lambda_0 \\
dh_1(a_2) &= f_0((y,y)(x,y)(y,x)(y,z)(z,y)) = \\
&= f_0((x,x)(y,x)(x,y)(x,z)(z,x)) = \lambda_0.
\end{aligned}
$$

where the first equality follows from the definition of λ_0, the second from the fact that f_0 is invariant with respect to the orientation preserving isometry that realizes the permutation (xy) on each factor, i.e. the isometry of $\mathbb{H}^2 \times \mathbb{H}^2$ that, on each factor, is the reflection with respect to the geodesic γ with endpoint z orthogonal to the geodesic with endpoints x, y.
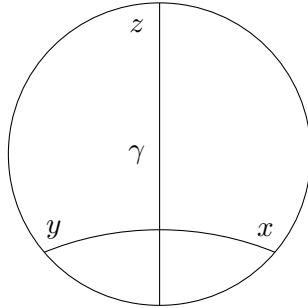
This proves that $f_1 = f_0 - dh_1$ vanishes on every 5-uple with $n_i(a) \leq 3$.

■ $n_1 + n_2 \leq 7$

Let us consider the element h_2 in $M^3_{b,\text{alt}}(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ that has the value

$$
h_2(c) = \begin{cases}
\frac{1}{2}(f_1((x_i, y_k), c) + f_1((x_i, y_l), c)) & \text{if } x_i = x_j, \ \#\{i,j,k,l\} = 4 \\
\frac{1}{2}(f_1((x_k, y_i), c) + f_1((x_l, y_i), c)) & \text{if } y_i = y_j, \ \#\{i,j,k,l\} = 4 \\
0 & \text{otherwise.}
\end{cases}
$$

The function h_2 is well defined (since, if c satisfies both conditions, then $n_1(c) + n_2(c) \leq 6$ and hence $f_1((x_m, y_n), c) = 0$ for every m, n, and if there are more than two entries that are equal on one factor, then $n_i(c) \leq 2$ and hence $f_1((x_m, y_n), c) = 0$). Moreover h_2 is bounded, measurable, alternating and H-invariant since f_1 is.

We will show that $f_2 = f_1 - h_2(a) = 0$ provided $(n_1(a), n_2(a)) = (3, 4)$; the case $(n_1, n_2) = (4, 3)$ follows by symmetry. Let us call $\{x_0, x_1, x_2\}$ the three distinct points on the first factor, $\{y_0, y_1, y_2.y_3\}$ the four distinct points in the second factor. There are, up to the action of H and up to permutations, four possibilities for the 5-uple a and we will treat them separately:

♦ $a_1 = ((x_0, y_0)(x_0, y_1)(x_0, y_2)(x_1, y_3)(x_2, y_0))$

An easy computation implies that (all the other summands vanish)

$$
dh_2(a_1) = \quad \frac{1}{2}(f_1((x_0, y_0)(x_0, y_1)(x_0, y_2)(x_1, y_3)(x_2, y_0)) +
$$
$$
f_1((x_0, y_3)(x_0, y_1)(x_0, y_2)(x_1, y_3)(x_2, y_0))).
$$

There are two different possibilities depending on the configuration of the points y_i. If the geodesics γ_1 with endpoints (y_0, y_3) and γ_2 with endpoints (y_1, y_2) don't intersect, then the permutation $(y_0 y_3)(y_1 y_2)$ can be realized by an orientation reversing isometry: the reflection with respect to the geodesic γ orthogonal to γ_1 and γ_2. Also the permutation $(x_1 x_2)$ can be realized by an orientation reversing isometry, the reflection with respect to the geodesic from x_0 orthogonal to the geodesic from x_1 and x_2.
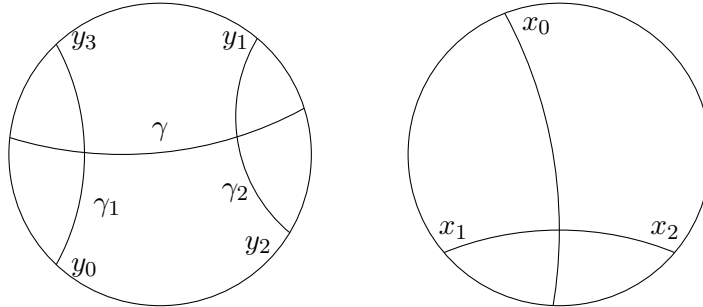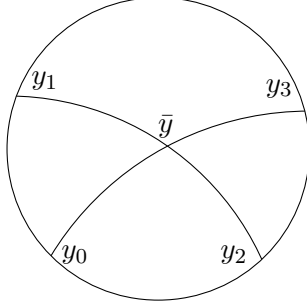

In this case $dh_2(a_1) = f_1(a_1)$ and hence $f_2(a_1) = 0$.

Otherwise $dh_2(a_1) = 0$ and we can obtain the desired result appliing the cocycle relation

$$
0 = df_1((x_0, x_3), a_1) = 2f_1(a_1)
$$

where the last equality is obtained using the invariance with respect to the orientation preserving isometry corresponding to the geodesic symmetry in $\bar{y}$.

In both cases we get $f_2(a_1) = f_1(a_1) - dh_2(a_1) = 0$.

♦ $a_2 = ((x_0, y_0)(x_0, y_1)(x_0, y_2)(x_1, y_3)(x_2, y_3))$

In this case it is sufficient to observe that $df_2((x_0, x_3), a_2) = f_2(a_2)$ since all the other terms involved in the expression of df_2 vanish because they correspond to f_2 computed on a 5-uple with $n_1 + n_2 \leq 6$. Since f_2 is a cocycle, follows that $f_2(a_2) = 0$.

♦ $a_3 = ((x_0, y_0)(x_0, y_1)(x_1, y_2)(x_1, y_3)(x_2, y_0))$

Let us compute the cocycle relation $df_2 = 0$ on the 6-uple $((x_1, y_0), a_3)$. We get

$$f_2(a_3) = f_2((x_1, y_0)(x_0, y_1)(x_1, y_2)(x_1, y_3)(x_2, y_0))$$

and the second 5-uple has the form a_1 and hence, as desired, $f_2(a_3) = 0$.

♦ $a_4 = ((x_0, y_0)(x_0, y_1)(x_1, y_0)(x_1, y_2)(x_2, y_3))$

In this case we compute the cocycle relation $0 = df_2$ on the 6-uple $((x_1, y_3), a_4)$. It implies

$$f_2(a_4) = f_2((x_1, y_3)(x_0, y_1)(x_1, y_0)(x_1, y_2)(x_2, y_3))$$

and the second 5-uple has the form a_1 and hence, as desired, $f_2(a_4) = 0$.

■ $(n_1, n_2) = (3, 5)$

In this case we do not even need to sum another coboundary. If the 5-uple has three equal entries in the first factor (say x) it is sufficient to compute the cocycle relation on the 6-uple $(a, (x, y_0))$ obtaining $df_2(a, (x, y_0)) = f_2(a)$, otherwise

$$a = ((x_0, y_0)(x_0, y_1)(x_1, y_2)(x_1, y_3)(x_2, y_4)),$$

and, in this case, $df_2(a, (x_0, y_4)) = f_2(a)$ and hence $f_2(a)$ vanishes.

■ $(n_1, n_2) = (4, 4)$

Let us define the function

$$h_3((x_0, y_0)(x_1, y_1)(x_2, y_2)(x_3, y_3)) = f_2((x_i, x_j)(x_0, y_0)(x_1, y_1)(x_2, y_2)(x_3, y_3))$$

for any $i \neq j$. By computing the value of $0 = df_2$ on the 6-uple

$$((x_{i'}, y_{j'})(x_i, x_j)(x_0, y_0)(x_1, y_1)(x_2, y_2)(x_3, y_3)),$$

it is indeed easy to verify that h_3 doesn't depend on the choice of i, j As usual h_3 is well defined, bounded, measurable, alternating and H-invariant. In order to show that $f_3 = f_2 - dh_3$ vanishes on the 5-uples with $n_i(a) \leq 4$, we need to consider two subcases:

♦ $a_1 = ((x_0, y_0)(x_0, y_1)(x_1, y_1)(x_2, y_2)(x_3, y_3))$

In this case we have $dh_3(a_1) = f_2(a_1)$. This implies that $f_3(a_1) = f_2(a_1) - h_3(a_1) = 0$.

♦ $a_2 = ((x_0, y_0)(x_0, y_1)(x_1, y_2)(x_2, y_3)(x_3, y_3))$

In this case it is sufficient to compute the value of df_3 on the uple $(a_2, (x_0, y_3))$ and show that every term, except from $f_3(a_2)$, vanishes.

■ $(n_1, n_2) = (4, 5)$, $a = ((x_0, y_0)(x_0, y_1)(x_1, y_2)(x_2, y_3)(x_3, y_4))$

It is sufficient to compute the value of df_3 on the 6-uple $(a, (x_0, y_2))$.

■ $(n_1, n_2) = (5, 5)$

If one computes the value of df_3 on the 6-uple $(a, (x_0, y_1))$, the only a priori non vanishing term is $f_3(a)$ and hence also $f_3(a)$ vanishes since f_3 is a coboundary.

We have thus proved that the cocycle f_3 belonging to $M_{b,\mathrm{alt}}^4(S^1 \times S^1; \widetilde{\mathbb{R}})^H$ is null and cobordant to $f - 6b\mathrm{Alt}(p_1^*[\mathrm{or}] \cup p_2^*[\mathrm{or}])$. In particular we get that $[f] = 6b\, p_1^*[\mathrm{or}] \cup p_2^*[\mathrm{or}]$ and hence the cup product is surjective on the 4th continuous bounded cohomology group of H.

□

In order to conclude the chapter we give the computation of the simplicial volume of the product of two surfaces.

Corollary 4.5.4. *Let $\Sigma_g \times \Sigma_h$ be the product of two surfaces of genus $g \geq 1$ and $h \geq 1$ respectively, then*

$$\|\Sigma_g \times \Sigma_h\| = 24(g-1)(h-1)$$

Proof. If g or h is equal to 1 the simplicial volume of Σ_g (resp. Σ_h) is equal to zero and hence the thesis follows from Proposition 3.2.4. Otherwise let us fix a hyperbolic structure on each factor. The metric universal covering of $\Sigma_g \times \Sigma_h$ is $\mathbb{H}^2 \times \mathbb{H}^2$. Moreover, Gauss Bonnet formula implies that $\mathrm{vol}(\Sigma_g) = 2\pi\chi(\Sigma_g) = 4\pi(g-1)$. Hence we get, as a consequence of the multiplicativity

of Riemannian volume, that:

$$
\begin{aligned}
\|\Sigma_g \times \Sigma_h\| &= \frac{\mathrm{vol}(\Sigma_g \times \Sigma_h)}{\|\omega_{\mathbb{H}^2 \times \mathbb{H}^2}\|_\infty} \\
&= \frac{3}{2\pi^2}\mathrm{vol}(\Sigma_g)\mathrm{vol}(\Sigma_h) \\
&= \frac{3}{2\pi^2}4\pi(g-1)4\pi(h-1) \\
&= 24(g-1)(h-1).
\end{aligned}
$$

$\square$

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