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Stochastic differential equations with rough coefficients

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Introduction

This thesis deals with the study of the stochastic continuity equation (SCE) on $\mathbb{R}^d$

$$d\mu_t + \left[-\frac{1}{2}\sum_{i,j=1}^{d}(a_{ij}D^2\mu_t)\mathrm{div}(b\mu_t)\right]dt + \sum_{k=1}^{\infty}\mathrm{div}(\sigma_k\mu_t)dW_t^k = 0, \qquad (1)$$

under low regularity assumptions on the coefficients. Here b and σ_k, $k \in \mathbb{N}^+$, are deterministic vector fields, $a = \sum_k \sigma_k\sigma_k^*$, W is a cylindrical Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$. The solution $(\mu_t)_t$ is a family of random measures on $\mathbb{R}^d$; the equation must be understood in the distributional sense. We will assume $a \equiv I_d$ (the d-dimensional identity matrix).

We use also the SCE for observables or generalized flows

$$S_t\varphi = \varphi + \int_0^t S_r[b\cdot\nabla\varphi + \frac{1}{2}\mathrm{tr}(aD^2\varphi)]dr + \sum_{k=1}^{\infty}\int_0^t S_r[\sigma_k\cdot\nabla\varphi]dW_r^k, \qquad (2)$$

for every φ in $C_c^\infty(\mathbb{R}^d)$; the solution $(S_t)_t$ is a family of random operators on $L^2(\mathbb{R}^d)$.

The importance of these equations is in their link with the stochastic differential equation (SDE) on $\mathbb{R}^d$

$$dX_t = b(X_t)dt + \sum_{k=1}^{\infty}\sigma_k(X_t)dW_t^k. \qquad (3)$$

To fix the ideas, consider first the deterministic case, i.e. when $\sigma \equiv 0$. Suppose that all the objects are regular (for instance, when b is regular bounded), let X be a flow which solves the ODE, that is

$$\frac{dX_t(x)}{dt} = b(X_t(x)), \qquad (4)$$
$$X_0(x) = x,$$

for every t in $[0,T]$, x in $\mathbb{R}^d$. Let φ be a regular observable, which represents a certain feature of the flow, and let μ_0 be a positive finite measure on $\mathbb{R}^d$, which represents the distribution of the mass at time 0. We are interested in the evolution of these feature and mass, i.e. in $S_t\varphi(x) := \varphi(X_t(x))$ and $\mu_t := (X_t)_\#\mu_0$. By the chain rule, it holds

$$d(\varphi(X_t)) = \nabla\varphi(X_t) \cdot b(X_t)dt \tag{5}$$

and so

$$dS_t\varphi = S_t[b \cdot \nabla\varphi]dt, d(\langle \mu_t, \varphi\rangle) = \langle \mu_t, b \cdot \nabla\varphi\rangle dt, \tag{6}$$

which are the continuity equations (CEs) for observables and measures.

The same reasoning is valid in the stochastic context. Precisely, in the regular case, one takes the stochastic flow X solution of the SDE, which is a family X^ω of flows parametrized by ω, such that

$$\begin{aligned} d(X_t(x)) = b(X_t(x))dt + \sum_k \sigma_k(X_t(x))dW_t^k, \\ X_0(x) = x. \end{aligned} \tag{7}$$

Then, given φ regular observable and μ_0 initial measure, one defines, ω by ω, $S_t\varphi(\omega) = \varphi(X_t(\omega))$ and $\mu_t^\omega = (X_t^\omega)_\#\mu_0$ and finally checks the SCEs, using Itô's formula, which is the chain rule for stochastic differentials.

The SCE for observables and for measures are morally the same equation: if S is a flow solution and S^* is its dual on the space of measure, then $S^*\mu_0$ solved the SCE.

In the first chapter Wiener pathwise uniqueness (i.e. uniqueness for solutions adapted to Brownian filtration) is proved for the SCE, among L^2 solutions; we suppose only some integrability conditions on b and low regularity assumptions on σ. The proof is in two steps. In the first step, we reduce Wiener uniqueness for the SCE to uniqueness for the Fokker-Planck equation (FPE)

$$d\nu_t + \mathrm{div}(b\nu_t)dt = \frac{1}{2}\Delta\nu_t. \tag{8}$$

The method is presented in a more general framework, for stochastic linear equations on Hilbert spaces, and is based on Wiener chaos decomposition. This states that, given the Brownian filtration $(\mathcal{F}_t = \sigma(W_s|s \le t))_t$ on Ω, $L^2(\Omega, \mathcal{F}_t, P)$ is decomposed in the orthogonal sum of the Wiener chaos spaces, where the n-th Wiener chaos is the space of stochastic n-time iterated integrals of deterministic functions. The method consists of projecting the equation on the Wiener chaos spaces and using the shift effect of the projectors in order to discard the Itô integral. The second step is a self-contained proof of uniqueness for the FPE.

The second chapter deals with the SCE for flows, following Le Jan-Raimond's approach. Here we need that the FPE admits a particular semigroup as solution, which is guaranteed by the theory of Dirichlet forms under mild assumptions on the coefficients. Wiener chaos gives Wiener uniqueness (as before) and also existence, defining inductively the projections of the solution S on Wiener chaos spaces. Another method of existence is based on filtering a weak solution X of the associated SDE with respect to a certain cylindrical Brownian motion W:

$$S_t\varphi(x,\omega) = \int_{\Gamma_T} \varphi(\pi_t(\gamma))K^{x,\omega}(d\gamma), \tag{9}$$

where $\Gamma_T := C([0,T])^d$, π_t is the evaluation at t and $K^{x,\omega}$ is the conditional law of X with respect to W and X_0.

In the third chapter, we consider the case b rough and $\sigma \equiv I_d$. Here a phenomenon of regularization by noise can be observed: the results in the first chapter give immediately Wiener uniqueness for the SCE, while uniqueness does not hold in the deterministic case (that is with $\sigma \equiv 0$) without additional hypotheses on b. We cite an example of this phenomenon.

We prove also that, in many cases, strong uniqueness (i.e. uniqueness with respect to every filtration, not only Brownian filtration) holds for the SCE. This is not surprising since a strong uniqueness result (due to Krylov-Röckner) holds for the SDE. First, extending Ambrosio's approach, we associate to every measure-valued solution of the SCE a superposition solution N, i.e.

$$\mu_t^\omega = \int_{\mathbb{R}^d} (\pi_t)_\# N^{x,\omega}\mu_0(dx), \tag{10}$$

where $N^{x,\omega}$ is roughly an "adapted" kernel from $\mathbb{R}^d \times \Omega$ to Γ_T, concentrated on solutions of the SDE with ω blocked and initial datum x. Then, starting from N, we build a weak solution of the SDE. This correspondence and Krylov-Röckner's result imply strong uniqueness for the SCE.

The last chapter is about a particular class of generalized flows, the isotropic Brownian flows (IBFs). An IBF is a family of Brownian motions, indexed by their starting points in $\mathbb{R}^d$, which are invariant in law for translation and rotation; it can be found as (possibly generalized) solution S of an SCE with $b \equiv 0$ and isotropic infinitesimal covariance function $K(x,y) := \sum_k \sigma_k(x)\sigma_k(y)^*$. Here we consider K's driven by two parameters α and η, related respectively to the correlation of the two-point motion and to the compressibility of the flow. Studying the distance between the motions of two points (which is a 1-dimensional diffusion), we find that coalescence and/or splitting occur, depending of the values of α, η and d. By coalescence we mean roughly that the mass initially on more points can coalesce in one point (it cannot be $S_t\varphi = \varphi(X_t)$ with X injective flow of maps); by splitting we mean that the mass initially concentrated on one point can split (it cannot be $S_t\varphi = \varphi(X_t)$ with X flow of maps).

This analysis makes rigorous the results for a simple model of turbulence and shows that this situation cannot be described classically, thus motivating the theory of generalized flows.

Finally two appendices recall preliminaries and technical results.

In the end, we make some remarks on notation. When not specified, $\langle\cdot,\cdot\rangle$ usually denotes the duality "scalar product" in L^2 (for instance between a measure and a continuous function). The quadratic variation for martingales is denoted by $[\cdot]_t$. $\ell^2(H)$ is the space of sequences (a_n) in a Hilbert space H with $\sum_n \|a_n\|_H^2 < +\infty$. For the Sobolev spaces we use the notation $W^{k,p}$ (k is the number of weak derivatives). Γ_T is $C([0,T];\mathbb{R}^d)$. For a matrix a, a^* denotes its adjoint matrix.

Chapter 1

Uniqueness by Wiener chaos method

1.1 Introduction

In this chapter we consider the linear stochastic partial differential equation (SPDE) on an Hilbert space H

$$du_t = Bu_t + \sum_{k=1}^{\infty} Cu_t dW_t^k. \tag{1.1}$$

We want to study the problem of uniqueness for this SPDE. We will concentrate on Wiener pathwise uniqueness, that is uniqueness among solutions adapted to Brownian filtration. Using Wiener chaos we will be able to reduce Wiener uniqueness to the corresponding Kolmogorov equation

$$dv_t = Bv_t, \tag{1.2}$$

which is obtained by (1.1) by taking the expectation.

As we will see, this is a generalization of the SCE on $\mathbb{R}^d$ for measures (and also for flows in some sense). Nevertheless, this setting allows to cover other cases on $\mathbb{R}^d$, such as stochastic transport equation; we hope it could be applied succefully to SCEs associated to infinite-dimensional SDEs and to other linear equations.

The technique developed in this chapter are inspired by [16] and used in [17].

Precisely, we are given the following spaces: $V \subseteq H \simeq H^* \subseteq V^*$, where H is a separable Hilbert space, V is a Fréchet space and the injection $V \subseteq H$ is dense. We assume B, C_k, $k \in \mathbb{N}^+$, are linear bounded operators from H to V^*, such that $\sum_{k=1}^{\infty} \|C_k u\|_H^2 \leq c\|u\|_H$ for every u in H.

In many applications, H is an L^2 space, V is a subspace of functions with two or more (weak) derivatives (e.g. $W^{2,2}$, $W^{2,\infty}$ or C_c^∞).

We fix a probability space $(\Omega, \mathcal{A}, P)$ and a cylindrical Brownian motion W on it. We call $(\mathcal{F}_t)_t$ the natural (completed) Brownian filtration. We say that a filtration $(\mathcal{G}_t)_t$ is a W-filtration if W remains a Brownian motion with respect to $\mathcal{G}$.

Definition 1. *An H-valued (weak) solution of (1.1) is a process u in $L^2([0,T]\times\Omega; H)$, progressively measurable with respect to a W-filtration $(\mathcal{G}_t)_t$, such that, for every φ in V, it holds*

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle u_r, B^*\varphi\rangle dr + \sum_{k=1}^\infty \int_0^t \langle u_r, C_k^*\varphi\rangle dW_r^k. \tag{1.3}$$

This solution is said to be Wiener (or strong) if it is adapted to the Brownian filtration $\mathcal{F}$.

By progressive measurability we mean here that $\langle u_t, \varphi\rangle$ must be progressively measurable for every φ in V.

Definition 2. *The SPDE (1.1) has Wiener pathwise (resp. strong) uniqueness if uniqueness holds in the class of Wiener (resp. weak) solutions.*

Definition 3. *An H-valued solution of (1.2) is a function v in $L^2([0,T]; H)$, such that, for every $\varphi \in V$, it holds*

$$\langle v_t, \varphi\rangle = \langle v_0, \varphi\rangle + \int_0^t \langle u_r, B^*\varphi\rangle dr. \tag{1.4}$$

1.2 Wiener chaos

The next result is classical, see e.g. [5]. In the following, we consider as Ω the classical Wiener space $\Omega = C([0,+\infty[, \mathbb{R})^{\mathbb{N}^+}$, with its Wiener measure P (i.e. such that the identity W on Ω is a cylindrical Brownian motion); $(\mathcal{F}_t)_t$ will denote the Brownian filtration (generated by the projections up to time t). H will be a separable Hilbert space.

Definition 4. *For n in $\mathbb{N}^+$, call $\Delta_n(T) = \{(t_1, \dots t_n) | 0 \le t_1 \le \dots t_n \le T\}$. For f in $L^2(\Delta_n(T); \ell^2(H)^{\otimes n})$ define*

$$\int_{\Delta_n(T)} f_r d^n W_r = \sum_{k_1,\dots k_n} \int_0^t \int_0^{r_n} \cdots \int_0^{r_2} f^{k_1,\dots k_n}(r_1, \dots r_n) dW_{r_1}^{k_1} \dots dW_{r_n}^{k_n}. \tag{1.5}$$

The process $(\Omega, (\mathcal{F}_t)_t, (\int_{\Delta(t)} f_r d^n W_r)_t, P)$ is a continuous square integrable martingale with mean value 0. Besides, the above integral is an isometry between the Hilbert spaces $L^2(\Delta_n(T); \ell^2(H)^{\otimes n})$ e $L^2(\Omega, \mathcal{F}_T, P; H)$.

Theorem 5. *Call* $\Pi_0 = H$, $\Pi_n = \left\{ \int_{\Delta_n(T)} f_r d^n W_r | f \in L^2(\Delta_n(T); \ell^2(H)^{\otimes n}) \right\}$ *for* $n \in \mathbb{N}^+$. *Then* $L^2(\Omega, \mathcal{F}_T, P; H)$ *admits the following orthogonal decomposition:*

$$L^2(\Omega, \mathcal{F}_T, P; H) = \oplus_{n=0}^{\infty} \Pi_n. \tag{1.6}$$

Definition 6. *The decomposition above is called Wiener chaos decomposition. The subspace* Π_n *is called n-th Wiener chaos.*

Proof. Orthogonality and closure of the Wiener chaos spaces and completeness of the sum follow easily from the definition.

It remains to prove that the sum in 1.6 is dense in $L^2(\mathcal{F}_T)$. It is enough to prove that, if $X \in (\oplus_{n=0}^{\infty} \Pi_n)^{\perp}$, then $X = 0$, i.e., for every basis $(e_j)_j$ of H, for every j and $0 \le t_1 \le \ldots \le t_n$, the image measure of $(X \cdot e_j) dP$ under the map $(W_{t_1}^{k_1}, \ldots W_{t_m}^{k_m})$ is null (here we use the adaptedness of X to $\mathcal{F}_T$). We will show that its Fourier transform is null, i.e. for every ξ in $\mathbb{R}^d$,

$$P[\exp(i\xi \cdot (W_{k_1}(t_1), \ldots W_{k_m}(t_m))) X \cdot e_j] = 0.$$

Lemma 7. *Take* f *in* $L^\infty([0,T]; \ell^2(H))$, h *in* H, λ *in* $\mathbb{C}$. *Then it holds (with convergence in* $L^2(\Omega)$*)*

$$\exp\left(\lambda \int_0^T \langle f_r, h \rangle dW_r - \frac{\lambda^2}{2} \int_0^T \|\langle f_r, h\rangle\|_{\ell^2}^2 dr\right) = 1 + \lambda \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \langle f, h\rangle_s^{\otimes n} d^n W_s,$$

where $\langle f(r), h\rangle$ *denotes the element in* $\ell^2(\mathbb{R})$ *with k-th coordinate* $\langle f^k(r), h\rangle$.

Proof. The convergence is ensured by the following estimates (proved by induction)

$$E\left[\left(\int_{\Delta_n(T)} \langle f, h\rangle_s^{\otimes n} d^n W_s\right)^2\right] \le \frac{T^n \|\langle f, h\rangle\|_{L^\infty}^{2n}}{n!}.$$

Let M_t, N_t resp. the LHS and the RHS of the above equality with T replaced by t. Then by Ito formula, M solves the SDE

$$M_t = 1 + \int_0^t \lambda M_r \langle f_r, h\rangle dW_r.$$

Since N satisfies the same equation, we have $M = N$ by strong uniqueness. □

Because of the orthogonality condition of X, we have $E[X] = 0$ and, for every n and f in $L^2([0,t];\ell^2(H))$,

$$P\left[\sum_q \int_{\Delta_n(T)} \langle f, e_q\rangle^{\otimes n}(r) d^n W(r) \langle X, e_q\rangle\right] = 0.$$

Now fix ξ in $\mathbb{R}^m$ and choose f with $f_r^k = \sum_{p=1}^m \delta_{k,k_p}\xi_p 1_{r\le t_p} e_j$. Summing over n the above equality and using the previous lemma, we get

$$\begin{aligned} &P[\exp(i\xi\cdot(W_{t_1}^{k_1},\dots W_{t_m}^{k_m}))\langle X, e_j\rangle] = \\ &= \; e^{-\frac{1}{2}mT|\xi|^2}\left(P[\langle X, e_j\rangle] + i\sum_{n=1}^{\infty} E[\int_{\Delta_n(t)} \langle f, e_j\rangle_s^{\otimes n} d^n W_s \langle X, e_k\rangle]\right) = 0. \end{aligned}$$

Thus the Fourier transform of $(W_{t_1}^{k_1},\dots W_{t_m}^{k_m})_\#(\langle X, e_j\rangle)dP$ is null. The proof is complete. □

1.3 Wiener pathwise uniqueness

The main idea is the following. The stochastic (standard) integral acts like a shift for the Wiener chaos, i.e. formula (1.7). Then, if u is a solution of (1.1), $Q_n u$ solves an equation which is (1.2) but for the stochastic part, which is driven by $Q_{n-1}u$ and thus can be regarded as a random external force, fixed a priori by inductive hypothesis. So the equation for $Q_n u$ is morally the Kolmogorov equation for (1.1).

Lemma 8. *Let X be a stochastic process, progressively measurable with respect to $(\mathcal{F}_t)_t$, in $L^2([0,T]\times\Omega;\ell^2(H))$; for $n\in\mathbb{N}$, let Q_n be the projector on the n-th Wiener chaos. Then it holds the following shift property:*

$$Q_{n+1}\sum_k \int_0^t X_k(r) dW_r^k = \sum_k \int_0^t Q_n X_k(r) dW_r^k. \tag{1.7}$$

Proof. Since $\sum_k \int_0^t X_k(r)dW_r^k = \sum_n \sum_k \int_0^t Q_n X_k(r) dW_r^k$ (in $L^2(\Omega;H)$), it is enough to prove that the LHS of (1.7) belongs to the $(n+1)$-th Wiener chaos.

But, for every k, $Q_n X_k(r)$ is an n-times iterated stochastic Wiener integral, so that $\sum_k \int_0^t Q_n X_k(r) dW^k(r)$ is an $(n+1)$-times iterated stochastic Wiener integral (with the isometry property, it is easy to check the measurability and the square-integrability of the integrated function). We are done. □

Now we state the main result.

Theorem 9. *Suppose uniqueness, in the class of H-valued solutions, for Kolmogorov equation (1.2). Then there is Wiener pathwise uniqueness, in the class of H-valued solutions, for the SPDE (1.1).*

Proof. Let u be a solution of (1.1) with $u_0 = 0$. By Wiener chaos decomposition, it is enough to show $Q_n u \equiv 0$ for every $n \in \mathbb{N}$. We will prove it inductively.

Projecting equation (1.3) on the n-th Wiener chaos, by the previous lemma, we obtain, for every φ in V,

$$\langle Q_n u_t, \varphi\rangle = \int_0^t \langle Q_n u_r, B^*\varphi\rangle dr + \sum_k \int_0^t \langle Q_{n-1} u_r, C_k^*\varphi\rangle dW_r^k,$$

where we have posed $Q_{-1} \equiv 0$. By inductive hypothesis $Q_{n-1}u \equiv 0$, this equation becomes equation (1.2), which has uniqueness property among H-valued solutions (by hypothesis). The proof is complete. □

1.4 An application to stochastic linear hyperbolic equations on $\mathbb{R}^d$

In this section we apply the previous result to stochastic linear hyperbolic equations (SLHEs) on $\mathbb{R}^d$

$$du_t + [-\frac{1}{2}\sum_{i,j=1}^d (a_{ij}D^2 u_t) + \mathrm{div}(bu_t) - cu_t]dt + \sum_{k=1}^\infty \mathrm{div}(\sigma_k u_t)dW_t^k = 0. \quad (1.8)$$

Here b, σ_k, $k \in \mathbb{N}^+$ are deterministic fields of vectors, $a = \sum_k \sigma_k\sigma_k^*$, c is a deterministic function, Ω and W are as in the previous section. The solution is a random scalar function. Notice that $\sum_k |\sigma_k|^2 = \mathrm{tr}(a)$.

In the case $c = 0$, we find the stochastic continuity equation (SCE). In the case $c = \mathrm{div} b$, we find the stochastic transport equation (STE), which expresses the evolution of a certain feature of the flow with solves the SDE (3). Indeed, in the regular case, if X is such a flow, for every function u_0, $u_0(X_t(\omega)^{-1})$ solves the STE.

The name "hyperbolic" comes from this two cases: with the representation formulae $u_t = u_0(X_t^{-1})|\det(DX_t)|$ for SCE and $u_t = u_0(X_t^{-1})$ for STE, we see that no regularization effect is expected if u_0 is not regular, even where the coefficients are smooth. Thus it is not clear that the precence of the Laplacian term gives some benefits for uniqueness: in principle we could have ill-posed problems, as in the deterministic case (see Chapter 3). It will give, however, for Wiener chaos method.

For an SLHE, the Kolmogorov equation takes the form of

$$dv_t + [\mathrm{div}(bu_t) + cu_t]dt = \frac{1}{2}\sum_{i,j=1}^{d} D^2(a_{ij}u_t)dt. \tag{1.9}$$

In the case $c = 0$, this equation is called Fokker-Planck equation (FPE).

Definition 10. *Suppose u_0 is in $L^p(\mathbb{R}^d)$, $2 \le p \le +\infty$. Suppose also a, b, c are in $L^{p'}_{loc}(\mathbb{R}^d)$, where $\frac{1}{p}+\frac{1}{p'} = 1$, $a = \sum_k \sigma_k\sigma_k^*$ with covergence in $L^2_{loc}(\mathbb{R}^d)$. A distributional L^p solution is a function u in $L^p([0,T]\times\mathbb{R}^d\times\Omega)$, progressively measurable with respect to a W-filtration $(\mathcal{G}_t)_t$, such that it holds, for every φ in $C^\infty_c(\mathbb{R}^d)$,*

$$\langle u_t,\varphi\rangle = \langle u_0,\varphi\rangle + \int_0^t \langle u_r, [\frac{1}{2}\mathrm{tr}(aD^2)+b\cdot\nabla+c]\varphi\rangle dr + \sum_{k=1}^{\infty}\int_0^t \langle u_r, \sigma_k\cdot\nabla\varphi\rangle dW^k_r. \tag{1.10}$$

A similar definition can be given for Kolmogorov equation (1.9) (without the requirement of progressive measurability). Wiener and strong uniqueness are defined obviously.

It is easy to see that, in the case $p = 2$, the above definition coincides with the general Definition 1, with $H = L^2(\mathbb{R}^d)$, $V = C^\infty_c(\mathbb{R}^d)$, $B\varphi = \frac{1}{2}\sum_{i,j=1}^d(a_{ij}D^2\varphi) - \mathrm{div}(b\varphi) + c\varphi$, $C_k\varphi = -\mathrm{div}(\sigma_k\varphi)$.

Theorem 11. *Suppose a, b, c, σ_k, $k \in \mathbb{N}^+$, are in $L^2_{loc}(\mathbb{R}^d)$ (resp. in $L^1_{loc}(\mathbb{R}^d)$), $a = \sum_k \sigma_k\sigma_k^*$ with covergence in $L^2_{loc}(\mathbb{R}^d)$. Suppose uniqueness for Kolmogorov equation (1.9) in the class of distributional L^2 (resp. L^∞) solutions. Then Wiener pathwise uniqueness holds for the SLHE (1.8) in the class of distributional L^2 (resp. L^∞) solutions.*

Proof. In the L^2 case, the result is a direct application of Theorem 9.

In the L^∞ case, we use the projections on the Wiener chaos spaces of $L^2(\Omega,\mathcal{F}_t,P;\mathbb{R})$ of the solution valued in x. Again we obtain inductively that Q_nu satisfies Kolmogorov equation (1.9), but is not a priori in L^∞. However, we know that $Q_nu_t(x) = \int_{\Delta_n(t)} f(r,t,x)d^nW_r$, for a certain deterministic f; for the L^∞ bound on u, $\int_{\Delta_n(t)} \|f(r,t,x)\|^2_{\ell^2}d^nr \le C$ for a.e. (t,x). Now take g in $L^\infty(\Delta_n(T);\ell^2(\mathbb{R})^n)$ and define $h(t,x) = \int_{\Delta_n(t)}\sum_k f_k(r,t,x)\cdot g_k(r)d^nr$; obviously h is in L^∞. Starting from the equation satisfies by Q_nu, with some exchanges of integrals, we get that h is a distributional solution of (1.9), so, by uniqueness hypothesis, $h \equiv 0$. Since this happens for all g, it must be $f \equiv 0$, i.e. $Q_nu \equiv 0$. □

To state a uniqueness result, we must have uniqueness for the Kolmorogov equation (1.9).

Condition 12. *b is in $L^p(\mathbb{R}^d)$ and c is in $L^q(\mathbb{R}^d)$ for some $p > d$, $q > d/2$ ($p = +\infty$, $q = +\infty$ are allowed). $a \equiv id$.*

Lemma 13. *Suppose Condition 12; suppose also, in the L^2 case, $p, q \geq 2$. Then equation (1.9) has uniqueness property in the class of distributional L^2 solutions and in the class of distributional L^∞ solutions.*

We first need the following technical lemma, which extends the distributional formulation to time-dependent test functions; its proof is posponed in Appendix B.

Lemma 14. *Let G be a domain of $\mathbb{R}^d$ and let A^* be a linear operator defined on a dense domain of $L^2(G)$, suppose that it is bounded as an operator $C_c^\infty(G) \to L^2(G)$. Let u be a function in $L^2([0,T] \times G)$ such that, for every ψ in C_c^∞, for a.e. t it holds*

$$\langle u_t, \psi \rangle = \langle u_0, \psi \rangle + \int_0^t \langle u_r, A^* \psi \rangle dr. \tag{1.11}$$

Then, for every ϕ in $C_c^\infty(G)$, it holds

$$\langle u_t, \phi_t \rangle = \langle u_0, \phi_0 \rangle + \int_0^t \langle u_r, A^* \phi_r \rangle dr + \int_0^t \langle u_r, \frac{\partial \phi_r}{\partial t} \rangle dr. \tag{1.12}$$

Proof. By the previous lemma with $A^* = B^*$, if v is a solution of (1.9), then, for every φ in $C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned} \langle v_t, \varphi_t \rangle = \langle v_0, \varphi_0 \rangle &+ \int_0^t \langle v_r, c\varphi_r \rangle dr + \int_0^t \langle v_r, b \cdot \nabla \varphi_r \rangle dr + \\ &+ \int_0^t \langle v_r, \frac{1}{2} \Delta \varphi_r \rangle dr + \int_0^t \langle v_r, \frac{\partial \varphi_r}{\partial t} \rangle dr. \end{aligned} \tag{1.13}$$

Now consider functions

$$\varphi^{s,x}(r,y) = \psi(s-r, x-y) = (2\pi(s-r))^{-d/2} \exp\left(-\frac{|x-y|^2}{2(s-r)}\right),$$

with $x \in \mathbb{R}^d$, $0 \leq r < s$. It is not difficult to see that, for every s, x, $\varphi^{s,x}$ can be approximated with functions φ_n in $C_c^\infty([0,t] \times \mathbb{R}^d)$, if $t < s$, in such a way that $B^* \phi_n$, $\frac{\partial \varphi_n(r)}{\partial r}$ converge in $L^\infty([0,t]; L^\gamma(\mathbb{R}^d))$ respectively to $B^* \varphi$,

$\frac{\partial \varphi}{\partial t}$, for every γ in $[1, +\infty]$; thus (1.13) holds also for $\varphi^{s,x}$ (here we use, in the L^2 case, $p, q \geq 2$).

With an easy computation, we have

$$\begin{aligned} \nabla\psi(\epsilon, z) &= -\frac{z}{\epsilon}\psi(\epsilon, z), \\ \Delta\psi(\epsilon, z) = 2\frac{\partial\psi}{\partial\epsilon}(\epsilon, z) &= \left(\frac{|z|^2}{\epsilon^2} - \frac{d}{\epsilon}\right)\psi(\epsilon, z), \\ \|\psi(\epsilon, \cdot)\|_{L^{p'}} &= C_{d,p}\epsilon^{-d/(2p)}, \\ \|\partial_i\psi(\epsilon, \cdot)\|_{L^{p'}} &= C_{d,p}\epsilon^{-d/(2p)-1/2}. \end{aligned}$$

So, if $t < s$, (1.13) becomes

$$\langle u_t, \varphi_t \rangle = \langle u_0, \varphi_0 \rangle + \int_0^t \langle u_r, c\varphi_r \rangle dr + \int_0^t \langle u_r, b \cdot \nabla\varphi_r \rangle dr. \tag{1.14}$$

We start with an L^∞ estimate. We analyze the two integrals in (1.14) separately. For the first integral, we have

$$\begin{aligned} &\left| \int_0^t \langle u_r, c\varphi_r \rangle dr \right| \\ &\leq \int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)} \|c\|_{L^q(\mathbb{R}^d)} \|\varphi(r, \cdot)\|_{L^{q'}(\mathbb{R}^d)} dr \\ &\leq C\|c\|_{L^q(\mathbb{R}^d)} \int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)} (s-r)^{-d/(2q)} dr. \end{aligned} \tag{1.15}$$

For the second integral, we have

$$\begin{aligned} &\left| \int_0^t \langle u_r, b \cdot \nabla\varphi_r \rangle dr \right| \\ &\leq \int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)} \|b\|_{L^p(\mathbb{R}^d)} \|\nabla\varphi(r, \cdot)\|_{L^{p'}(\mathbb{R}^d)} dr \\ &\leq C\|b\|_{L^p(\mathbb{R}^d)} \int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)} (s-r)^{-d/(2p)-1/2} dt. \end{aligned} \tag{1.16}$$

For the L^2 estimates, we take $\tilde{q}$ such that $\frac{1}{\tilde{q}} + \frac{1}{q'} = 1 + \frac{1}{2}$ and we use Young inequality for convolutions; the same with p in place of q. Then for the first integral we have

$$\left\| \int_0^t \langle u_r, c\varphi_r^{s,\cdot} \rangle dr \right\|_{L^2(\mathbb{R}^d)}$$

$$\begin{aligned}
&\leq \ t^{1/2}\int_0^t \|(u_r c) * \psi(s-r,\cdot)\|_{L^2(\mathbb{R}^d)}dr \\
&\leq \ t^{1/2}\int_0^t \|(u_r c)\|_{L^{\tilde{q}}(\mathbb{R}^d)}\|\psi(s-r,\cdot)\|_{L^{q'}(\mathbb{R}^d)}dr \\
&\leq \ C\|c\|_{L^q(\mathbb{R}^d)}t^{1/2}\int_0^t \|u_r\|_{L^2(\mathbb{R}^d)}(s-r)^{-d/(2q)}dr. \qquad (1.17)
\end{aligned}$$

For the second integral we have

$$\begin{aligned}
&\left\|\int_0^t \langle u_r, b\cdot\nabla\varphi_r^{s,\cdot}\rangle dr\right\|_{L^2(\mathbb{R}^d)} \\
&\leq \ t^{1/2}\int_0^t \|(u_r b) * \nabla\psi(s-r,\cdot)\|_{L^2(\mathbb{R}^d)}dr \\
&\leq \ t^{1/2}\int_0^t \|u_r b\|_{L^{\tilde{p}}(\mathbb{R}^d)}\|\nabla\psi(s-r,\cdot)\|_{L^{p'}(\mathbb{R}^d)}dr \\
&\leq \ C\|b\|_{L^p(\mathbb{R}^d)}t^{1/2}\int_0^t \|u_r\|_{L^2(\mathbb{R}^d)}(s-r)^{-d/(2p)-1/2}dr. \qquad (1.18)
\end{aligned}$$

Now we take $u_0 \equiv 0$, $s = t+h$. Then for (1.15) and (1.16), since $p > d$ and $q > d/2$, it holds for some $\alpha \in]0,1[$

$$\|u_t * \psi_h\|_{L^\infty(\mathbb{R}^d)} \leq C\int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)}(t+h-r)^{-\alpha}dr.$$

Since $\|u_t\|_{L^\infty(\mathbb{R}^d)} \leq \overline{\lim}_{h\to 0}\|u_t * \psi_h\|$, for $h \to 0$ the previous bound becomes

$$\|u_t\|_{L^\infty(\mathbb{R}^d)} \leq C\int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)}(t-r)^{-\alpha}dr. \qquad (1.19)$$

The conclusion follows from a Gronwall-type argument. Precisely, we iterate formula (1.19):

$$\begin{aligned}
\|u_t\|_{L^\infty(\mathbb{R}^d)} &\leq C\int_0^t\int_0^s \|u_r\|_{L^\infty(\mathbb{R}^d)}(s-r)^{-\alpha}(t-s)^{-\alpha}drds \\
&\leq \ C\int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)}(t-r)^{-2\alpha+1}dr;
\end{aligned}$$

interating again, we obtain

$$\|u_t\|_{L^\infty(\mathbb{R}^d)} \leq C\int_0^t \|u_r\|_{L^\infty(\mathbb{R}^d)}(t-r)^{-2^k\alpha+2^k-1}dr.$$

Since $\alpha < 1$, we can take k such that $-2^k\alpha + 2^k - 1 > 0$. Now the thesis in the L^∞ case follows from Gronwall lemma. The same reasoning works in the L^2 case. □

Now we can conclude:

Corollary 15. *Suppose Condition 12; suppose also, in the L^2 case, $p, q \geq 2$. Then the SHLE (1.8) has Wiener pathwise uniqueness in the class of distributional L^2 solutions and in the class of distributional L^∞ solutions.*

Chapter 2

Generalized stochastic flows

2.1 Introduction

This chapter is devoted to the study of the SCE for observables

$$S_t\varphi = \varphi + \int_0^t S_r[b \cdot \nabla\varphi + \frac{1}{2}\mathrm{tr}(aD^2\varphi)]dr + \sum_{k=1}^{\infty} \int_0^t S_r[\sigma_k \cdot \nabla\varphi]dW_r^k, \quad (2.1)$$

mainly as a tool for the corresponding possibly ill-posed SDE

$$dX_t = b(X_t)dt + \sum_{k=1}^{\infty} \sigma_k(X_t)dW_t^k. \quad (2.2)$$

Here we follow and readapt the famous results by Le Jan-Raimond [16], while some of the ideas of this introduction are inspired by [2].

The idea is that, in some rough situations, at fixed ω, the mass initially concentrated on a point can split under the action of the SDE. This corresponds to the absence of strong solutions and the presence of more than one weak solution of the SDE. To understand this fact, we refer to the following intuition about the concepts of strong and weak solution of the SDE.

Suppose $\Omega = \Gamma_T := C([0,T];\mathbb{R}^d)$ is the canonical Wiener space. Roughly speaking, with an analytical point of view, an SDE can be seen as a collection of ODEs parametrized by ω in Ω, plus a mass assigned to every ODE (the probability measure P) and a constraint on the information brought by the solutions of such ODEs (the filtration $(\mathcal{G}_t)_t$).

A strictly weak solution (i.e. a solution which is not strong) represents in some sense a non-uniqueness situation. Indeed, suppose that the SDE with ω blocked has more than one solution, for ω in a non-P-null set of Ω. For every ω, we can take a measure N^ω on Γ_t concentrated on the solutions of the SDE with ω blocked (this is what is called a superposition solution); by teh non-uniqueness hypothesis, this measure N^ω will be non trivial (i.e. not concentrated on a Dirac delta) for ω in a non-P-null set of Ω. The advantage of the weak formulation is that now we can enlarge the probability space to create a solution. Precisely, we observe that, under N^ω, the canonical process on Γ_T solves the SDE with ω blocked. Thus, assuming suitable adaptedness conditions and taking $\tilde{\Omega} = \Omega \times \Gamma_T$ and $\tilde{P} = P \otimes N^\omega$, it is clear that the canonical process $X(\omega, \gamma) = \gamma$ on Γ_T is a weak solution of the SDE.

Now we consider a situation of possible non-uniqueness and see how the probability P and the filtration $\mathcal{F}_t$ play a fundamental role. The precence of a probability measure allows us to use the law of the solution. This must satisfy the Fokker-Planck equation (FPE), i.e. its marginals ν_t's verify

$$d\nu_t + \operatorname{div}(b\nu_t)dt = \frac{1}{2}\Delta\nu_t. \tag{2.3}$$

This equation has often uniqueness property, for the precence of the Laplacian term. Thus uniqueness in law holds for the SDE.

Having a filtration $(\mathcal{G}_t)_t$, we can "filter" a weak solution with respect to $\mathcal{G}$, thus obtaining flow of stochastic kernels; in some sense, these kernels are measures concentrated to solutions of the SDE at fixed ω and initial datum x. The new fact that happens in the stochastic case is that all weak solutions reduce to the same (stochastic) kernel, if we filter with respect to the Brownian filtration. This Wiener uniqueness result is achieved using Wiener chaos decomposition, which reduces the SCE to the FPE. This way has also a physical meaning, since, among all the possible kernels, it choose the only one which is a function of the random perturbation up to that moment.

Thus, in the stochastic case, even if we do not have strong uniqueness, there is a way to select a kernel uniquely. As we will see in Chapter 4, this is particularly useful, since it captures a possible split of the mass (a situation which corresponds to flows of kernels).

2.2 Existence and uniqueness

We work under the following hypotheses.

Condition 16. *b is in $L^\infty(\mathbb{R}^d)$; $[\operatorname{div} b]^-$ is in $L^\infty(\mathbb{R}^d)$; $a \equiv I_d$; σ_k's are measurable functions with $\sum_k \sigma_k\sigma_k^* = a$ in $L^2(\mathbb{R}^d)$.*

We will use the following notation. K is the measurable function on $\mathbb{R}^{2d}$ with values in the $d \times d$ matrices, defined by

$$K = \sigma_k \otimes \sigma_k^*; \tag{2.4}$$

it is called infinitesimal covariance and it is bounded (since a is bounded): $v \cdot K(x,y)w \leq [\mathrm{tr}(a(x))\mathrm{tr}(a(y))]^{1/2}|v||w|$ for every x, y in $\mathbb{R}^d$ and v, w vectors in $\mathbb{R}^d$. In particular, the σ_k's are bounded.

The operators A, D_k, $k \in \mathbb{N}^+$, are the closures in $L^2(\mathbb{R}^d)$ of the following operators, defined for φ in $C_c^\infty(\mathbb{R}^d)$:

$$A\varphi = b \cdot \nabla\varphi + \frac{1}{2}\mathrm{tr}(aD^2\varphi) = b \cdot \nabla\varphi + \frac{1}{2}\Delta\varphi,$$
$$D_k\varphi = \sigma_k \cdot \nabla\varphi.$$

We also define, for φ, ψ in $C_c^\infty(\mathbb{R}^d)$,

$$\Gamma(\varphi,\psi) = A(\varphi\psi) - \varphi A\psi - \psi A\varphi = \sum_k D_k\varphi \cdot D_k\psi = \nabla\varphi \cdot a\nabla\psi$$

and extend it to φ, ψ in the Sobolev space $W^{1,2}(\mathbb{R}^d)$.

Of course, Hypotheses (16) are not the most general ones. One could take a general bounded uniformly elliptic covariance function K (i.e. $K(x,x) = a(x) \geq cI_d$ for some $c > 0$ independent of x), such that a is continuous. We suppose $a \equiv I_d$ because we only deal with examples in this case. Notice that both b and σ can be very rough.

We fix the probability space $(\Omega, \mathcal{A}, P)$ and a cylindrical Brownian motion W on it, with its natural filtration $(\mathcal{F}_t)_t$. We recall that a filtration $(\mathcal{G}_t)_t$ is a W-filtration if W remains a Brownian motion with respect to $\mathcal{G}$.

Definition 17. *A solution of the SCE for observables is a family $(S_t)_{t\in[0,T]}$ of linear bounded operators from $L^2(\mathbb{R}^d, \mathcal{L}^d)$ with values in $L^2(\mathbb{R}^d \times \Omega, \mathcal{L}^d \otimes P)$, adapted to a certain W-filtration $(\mathcal{G}_t)_t$, with uniformly (in time) bounded operator norms, such that, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, it holds in $L^2(\mathbb{R}^d)$*

$$S_t\varphi = \varphi + \int_0^t S_r A\varphi dr + \sum_{k=1}^\infty \int_0^t S_r D_k\varphi dW_r^k. \tag{2.5}$$

This solution is called Markovian if, for every t and every positive f in $L^2(\mathbb{R}^d)$, $S_t f$ is positive and if $S_t 1 = 1$.

The concepts of Wiener solution and Wiener and strong uniqueness are defined naturally as in Chapter 1.

A Markovian solution expresses the evolution, in the values of the observable, of a "family of solutions $(X(x))_x$" (x being the initial point) to the given SDE (2.2). Indeed we recall that, in the regular case, if X is such a family and φ is a regular function, then by Ito formula $S_t\varphi = \varphi(X_t)$ satisfies (2.5) and is Markovian.

However, if we want a proper substitute of a flow, we must require the cocycle law. In the regular case, this states that a stochastic flow X satisfies, outside a P-null set (independent of space and time), $X_{u,t}(X_{s,u}(x)) = X_{s,t}(x)$. We cannot hope in such a strong property: the random operators S_t are defined up to a P-null set, which depends on the argument ϕ and on time t. So we will obtain an analogue of cocycle property but, in some sense, at fixed space and time.

To do this, we use the Wiener space $\Omega = \Gamma_T = C([0,T];\mathbb{R}^d)$ and we notice that, in the homogeneous case, $X_{t,t+s}(x,\omega) = X_{0,s}(x,\theta_t\omega)$, where θ_t is the shift operator $\theta_t\omega = \omega(t+\cdot) - \omega(t)$ (i.e. $W_s(\theta_t) = W_{t+s} - W_t$): indeed we have the following simple rule (which can be proved first for elementary processes and then in the general case by approximation):

$$\left(\int_{\alpha+t}^{\beta+t} Z_{r-t}(\theta_t)dW_r\right)(\omega) = \left(\int_{\alpha}^{\beta} Z_r dW_r\right)(\theta_t\omega). \tag{2.6}$$

Then the following definition appears natural.

Definition 18. *A statistical solution (or generalized stochastic flow solution) of the SDE is a Markovian Wiener solution of the SCE for observables which verifies the cocycle law: for every $s,t \geq 0$,*

$$S_{t+s} = S_t(S_s \circ \theta_t). \tag{2.7}$$

This property seems to be a refinement of the concept Markov process: we find the Markov property by taking the expectiation. The last formula must be read as: for every $s,t \geq 0$, for every f in $L^2(\mathbb{R}^d)$, for $\mathcal{L}^d$-a.e. x, for P-a.e. ω,

$$S_{s+t}f(x,\omega) = S_t[S_s f(\cdot,\theta_t\omega)](x,\omega), \tag{2.8}$$

Notice that the RHS makes sense, since $S_s f \circ \theta_t$ and $S_t f$ are independent; more precisely, they can be defined resp. on $C([s,t])$ and on $C([0,t])$.

The SCE for observable is morally a usual SCE (for measures), because, for every positive finite measure μ_0 on $\mathbb{R}^d$, "$\mu_t^\omega = S_t(\omega)^*\mu_0$" (in the sense of $\langle \mu_t, \varphi\rangle = \langle \mu_0, \S_t\varphi\rangle$) solves the SCE. Thus, in order to find existence and (Wiener) uniqueness, we want to expoit again Wiener chaos method; so we need some well-posedness properties for the associated Fokker-Planck equation (FPE)

$$dv_t = Av_t dt. \tag{2.9}$$

In particular, we hope to have existence by building the projections on each Wiener chaos. But, to ensure the convergence of the sum of this projections, we need some control, in L^2, L^∞ and L^1 of the solution v of the FPE: as we will see, we would like to have $v_t = P_t^* v_o$, where $(P_t)_t$ is a suitable semigroup. All the technical assumptions holds under our hypotheses, as the following proposition says.

Proposition 19. *Suppose Condition 16. Then A is the infinitesimal generator of a semigroup $(P_t)_t$ on $L^2(\mathbb{R}^d)$. This semigroup is Markovian, i.e., for every t, $P_t f \geq 0$ for every f positive in $L^2(\mathbb{R}^d)$ and $P_t 1 = 1$. The domain $\mathcal{D}(A)$ is in $W^{1,2}(\mathbb{R}^d)$. For every $t > 0$, P_t maps $L^2(\mathbb{R}^d)$ into $\mathcal{D}(A)$. Finally $(P_t)_t$ extends to a family of L^1 operators with $\|P_t\|_{L^1} \leq C$ for every t.*

Now we come to the desired existence and uniqueness result. It is valid whenever the thesis of the previous proposition holds; this happens when A is associated to a certain Dirichlet form (see Appendix B).

Theorem 20. *There exists a unique Wiener solution S of the SCE for observable and it is a generalized flow. It satisfies, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$P[(S_t\varphi)^2] \leq P_t(\varphi^2), \tag{2.10}$$

$$S_t\varphi = P_t\varphi + \sum_{k=1}^{\infty} \int_0^t S_r D_k P_{t-r}\varphi dW_r^k. \tag{2.11}$$

Proof. The proof of Markovian property will be given in the next section. Make the projection of equation (2.5) on the n-Wiener chaos: calling Q_n the projector and $J_t^n = Q_n S_t$, then the shift effect gives, for every n positive integer and every regular φ,

$$J_t^0\varphi = \varphi + \int_0^t J_r^0 A\varphi dr, \tag{2.12}$$

$$J_t^n\varphi = \int_0^t J_r^n A\varphi dr + \sum_k \int_0^t J_r^{n-1} A\varphi dW_r^k. \tag{2.13}$$

Uniqueness: Uniqueness for equation (2.5) can be reduced to uniqueness for SCE using the adjoint S_t^* of the random operator S_t. To avoid the theory of random operators, we can proceed directly. Let S, T two generalized flows solutions of the SCE (2.5), let $K_t^n = Q_n(S_t - T_t)$. we will prove inductively that $K^n \equiv 0$ for all n. For the above formulae, K^0 satisfies

$$K_t^0\varphi = \int_0^t K_r^0 A\varphi dr. \tag{2.14}$$

Fix $t > 0$ and, for every regular φ, consider the function $s \mapsto K_s Pt - s\varphi$, $s \in [0, t]$. Using the uniform (in s) L^2 boundedness of K_s^0, we apply the chain rule to find $\frac{d}{dt}(K_s P_{t-s}\varphi) = 0$, so that $K_t\varphi = K_O P_t \varphi \equiv 0$. Now suppose inductively that $K^{n-1} \equiv 0$, then K^n satisfies the same equation (2.14), so is 0. Uniqueness is proved.

Existence: For every regular φ, define

$$J_t^0 \varphi = P_t \varphi,$$

$$J_t^n \varphi = \int_0^t J_r^{n-1} D_k P_{t-r} \varphi dW_r^k, \; n \in \mathbb{N}^+.$$

Call $S_t^n = \sum_{j=0}^n J_t^j$; it satisfies

$$S_t^n \varphi = P_t \varphi + \int_0^t S_r^{n-1} D_k P_{t-r} \varphi dW_r^k. \tag{2.15}$$

Claim 21. *For every n and t, J_t^n and S_t^n can be extended to bounded adapted operators $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \Omega)$; it holds a.e., for every f in $L^2(\mathbb{R}^d)$,*

$$P[(S_t^n f)^2] \leq P_t(f^2). \tag{2.16}$$

Then the family S of random operators defined by $S_t f = L^2 - \lim_n S^n f = \sum_{j=0}^\infty J_t^j f$ is well defined and uniformly bounded, because P_t is uniformly bounded in L^1. It is a solution of the SCE for observables and (2.10), (2.11) hold passing to the limit in (2.16), (2.15).

claim. It is enough to prove inductively the claim for $f = \varphi$ in $C_c^\infty(\mathbb{R}^d)$. In the case $n = 0$ we must prove $(P_t f)^2 \leq P_t(f^2)$ a.e.. Note that $\frac{\partial}{\partial v} P_u (P_v f)^2 = 2P_u((P_v f)(AP_v f))$, so that

$$\frac{d}{dr}(P_r(P_{t-r}f)^2) = P_r \Gamma(P_{t-r}f, P_{t-r}f).$$

Then we have

$$P_t(f^2) = (P_t f)^2 + \int_0^t P_r \Gamma(P_{t-r}f, P_{t-r}f) dr. \tag{2.17}$$

Since Γ is positive semi-definite and P_t is monotone, the integrand in (2.17) is non-negative. The proof for $n = 0$ is complete.

In the case $n+1$, it is clear that $S^{n+1}\varphi$ is $(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t)$-measurable. To verify the well posedness and (2.16), note that

$$P[(S^{n+1}\varphi)^2] = (P_t f)^2 + \sum_k \int_0^t P[(S_r^n D_k P_{t-r} \varphi)^2] dr$$

$$\begin{aligned}
&\le (P_t\varphi)^2 + \sum_k \int_0^t P_r((D_k P_{t-r}\varphi)^2)dr \\
&\le (P_t\varphi)^2 + \int_0^t P_r \Gamma(P_{t-r}\varphi, P_{t-r}\varphi)dr \\
&\le P_t(\varphi^2)
\end{aligned}$$

where I have used the inductive hypothesis and (2.17). □

It remains to prove the cocycle law. For this, we use the Wiener chaos decomposition of S_t, which can be obtained by induction, using the fact that $P_t(L^2)$ is contained in $\mathcal{D}(A)$ and so in $W^{1,2}$: we can extend (2.11), so that, for every φ in $C_C^\infty(\mathbb{R}^d)$,

$$J_t^n\varphi = \sum_{k_1,\dots k_n} \int_0^t \int_0^{r_n} \dots \int_0^{r_2} P_{r_1} D_{k_1} P_{r_2-r_1} \dots D_{k_n} P_{t-r_n}\varphi dW_{r_1}^{k_1} dW_{r_2}^{k_2} \dots dW_{r_n}^{k_n}.$$

Using formula (2.6), one sees that $\sum_{j=0}^n J_t^j(J_s^{n-j}\varphi(\theta_t\omega))(\omega) = J_{t+s}^n\varphi(\omega)$ for a.e. ω. □

2.3 A representation formula

In this section we give a representation formula for the Wiener generalized flow S: S is the conditional law of a weak solution X of the SDE (2.2), with respect to the initial datum X_0 and the Brownian filtration. This will make rigorous the intuition in the introduction and prove the Markovian property of S.

To do this, we need first the existence of a weak solution.

Theorem 22. *Under Condition 16, for a.e. x in $\mathbb{R}^d$, there exists a probability P^x on $(\mathbb{R}^d)^{[0,T]}$ such that the canonical process on $(\mathbb{R}^d)^{[0,T]}$ is a Markov process associated to the semigroup $(P_t)_t$.*

This theorem is a consequence of the fact that A generates a regular local conservative Dirichlet form (see Appendix B). We will suppose also that the process is a.e. continuous, so that we work with $\Gamma_T = C([0,T];\mathbb{R}^d)$ in place of $(\mathbb{R}^d)^{[0,T]}$. Uned our hypotheses (Condition 16), this is always true a posteriori (for other results), and in many (if not all) cases it is a consequence of the theory of Dirichlet form.

The canonical process above is a natural candidate for a weak solution, since every solution of the SDE has marginal laws which verifies the FPE with generator A (this is implication 1 → 2 in the theorem below). In the regular case, when W is replaced by a d-dimensional Brownian motion B, it is well known that the converse holds, i.e. every Markov process associated to $(P_t)_t$ is a weak solution of the SDE. The following result extends this equivalence to our case, thus proving the existence of a weak solution.

Theorem 23. *Let $(\Omega, \mathcal{A}, (\mathcal{G}_t)_t, X, P)$ be a continuous Markov process. Then the following facts are equivalent:*

1. *on an enlarged probability space, there exists a cylindrical Brownian motion W such that (X, W) is a weak solution of the SDE;*
2. *X is a Markov process associated to the semigroup $(P_t)_t$.*
3. *the law $(X)_\# P$ is a martingale solution of the SDE, i.e., for every φ in $C_c^\infty(\mathbb{R}^d)$, $M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t A\varphi(X_r)dr$ is a $\mathcal{G}$-martingale.*

Proof. The implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are standard. $1 \Rightarrow 2$ follows from Ito formula: indeed, if X is a weak solution, then, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\varphi(X_t) = \varphi(X_0) + \int_0^t A\varphi(X_r)dr + \sum_k \int_0^t D_k\varphi(X_r)dW_r^k,$$

and the stochastic integral is a martingale.

As for $2 \Rightarrow 3$, notice that, for $s < t$,

$$P[M_t^\varphi|\mathcal{G}_s] = P_{t-s}\varphi(X_s) - \varphi(X_0) - \int_s^t P_{r-s}A\varphi(X_s)dr - \int_0^s A\varphi(X_r)dr = M_s^\varphi.$$

The implication $3 \Rightarrow 1$ is more complex and requires the construnction of a cylindrical Brownian motion such that the SDE is satisfied. Theorem 26 will complete the proof. □

Lemma 24. *For every φ, ψ in C_c^∞, it holds*

$$[M^\varphi, M^\psi]_t = \int_0^t \Gamma(\varphi, \psi)(X_r)dr. \tag{2.18}$$

In particular, $Y = M^{id}$ is a d-dimensional Brownian motion.

Proof. It is enough to prove that Y is a Brownian motion, i.e. (by Lévy theorem) a centered local martingale with quadratic variation $[Y^i, Y^j]_t = t\delta_{ij}$. We will use the notation $Z \sim Z'$ to say that $Z - Z'$ is a local martingale. We suppose for the sake of simplicity that $X_0 = 0$. Using first $\varphi(x) = x$ and then $\varphi(x) = x^i x^j$, we find that $Y_t = M_t^{id} = X_t - \int_0^t b(X_r)dr$ and

$$M^{x^i x^j} = X_t^i X_t^j - \int_0^t X_r^i b^j(X_r)dr - \int_0^t X_r^j b^i(X_r)dr + t\delta_{ij}$$

are local martingales. So, calling $A_t = \int_0^t b(X_r)dr$,

$$\begin{aligned} Y_t^i Y_t^j - t\delta_{ij} &= X_t^i X_t^j - t\delta_{ij} - X_t^i A_t^j - X_t^j A_t^j + A_t^i A_t^j \\ &\sim \int_0^t X_r^i dA_r^j + \int_0^t X_r^j dA_r^i - X_t^i A_t^j - X_t^j A_t^i + A_t^i A_t^j \\ &\sim -\int_0^t A_r^j dX_r^i - \int_0^t A_r^i dX_r^j + A_t^i A_t^j \\ &\sim -\int_0^t A_r^j b dA_r^i - \int_0^t A_r^i b dA_r^j + A_t^i A_t^j = 0. \end{aligned}$$

where we have use Ito formula applied to the product $X_t^i \int_0^t b^j(X_r)dr$. The proof is complete. □

We now define our desired Brownian motion. Consider the space $\tilde{\Omega} = \Omega \times \Gamma_T^{\mathbb{N}^+}$. Call $\tilde{W}$ the canonical process on $\Gamma_T^{\mathbb{N}^+}$, with its natural completed filtration $\tilde{\mathcal{F}}$ and the Wiener measure Q which makes $\tilde{W}$ a cylindrical Brownian motion. On $\tilde{\Omega}$ put the filtration $\tilde{\mathcal{G}}$ defined by $\tilde{\mathcal{G}}_t = \mathcal{G} \otimes \tilde{\mathcal{F}}_t$ and the probability measure $\tilde{P} = P \otimes Q$. Define, for $k \in \mathbb{N}^+$,

$$dW_t^k = d\tilde{W}_t^k + \sigma_k(X_t) \cdot dY_t - \sum_{l=1}^{\infty} \sigma_k(X_t) \cdot \sigma_l(X_t) d\tilde{W}_t^l. \tag{2.19}$$

Lemma 25. *W is a cylindrical Brownian motion with respect to $\tilde{\mathcal{G}}$ under the probability $\tilde{P}$.*

Proof. For every k, W^k is a local martingale with respect to $\tilde{\mathcal{G}}$; by Levy theorem, it is enough to verity $\langle W^h, W^k \rangle_t = t\delta_{hk}$. This is true because, by the previous lemma,

$$\begin{aligned} &\frac{d}{dt}[W^h, W^k]_t \\ &= \delta_{hk} - 2\sigma_h \cdot \sigma_k(X_t) + \sum_l \sigma_h^* \sigma_l \sigma_l^* \sigma_k(X_t) + \sigma_h^* \sigma_k(X_t) = \delta_{hk} \end{aligned}$$

□

Theorem 26. *For every φ in $C_c^\infty(\mathbb{R}^d)$, it holds under the probability $\tilde{P}$*

$$\varphi(X_t) = \varphi(X_0) + \int_0^t A\varphi(X_r)dr + \sum_{k=1}^{\infty} \int_0^t D_k\varphi(X_r)dW_r^k \tag{2.20}$$

In particular $(\tilde{\Omega}, \tilde{\mathcal{G}}, W, X, \tilde{P})$ is a weak canonical solution of the SDE starting from $(X_0)_\# P$.

Proof. It is enough to prove that $M^\varphi_t = \sum_{k=1}^\infty \int_0^t D_k\varphi(X_r)dW^k_r$. The two sides of this equality are continuous square-integrable martingales with respect to $\tilde{\mathcal{G}}$, so it is enough

$$[M^\varphi - \sum_{k=1}^\infty \int_0^\cdot D_k\varphi(X_r)dW^k_r] \equiv 0.$$

This holds since, for (2.18),

$$\begin{aligned}
&[M^\varphi - \sum_{k=1}^\infty \int_0^\cdot D_k\varphi(X_r)dW^k_r]_t \\
&= \int_0^t [\Gamma(\varphi,\varphi) + \Gamma(\varphi,\varphi) - 2\sum_k D_k\varphi \cdot \sum_{j=1}^d [\sigma_k]_j \Gamma(\varphi, x_j)](X_r)dr = 0.
\end{aligned}$$

To prove that X is a weak solution of the SDE, it is enough to apply the result to $\varphi = id$: precisely, we approximate id with φ_n regular with compact support and we pass to the limit in (2.20). □

A similar reasoning holds also for more general (regular) diffusion matrices a. Indeed, Lemma 24 holds with a similar proof and again a cylindrical Brownian motion W can be defined:

$$dW^k_t = d\tilde{W}^k_t + \sigma_k(X_t)\cdot a^{-1}(X_t)dY_t - \sum_{l=1}^\infty \sigma_k(X_t)\cdot a^{-1}(X_t)\sigma_l(X_t)d\tilde{W}^l_t,$$

where $a^{-1}(x)$ is the pseudo-inverse matrix of $a(x)$.

Now we have $(\tilde{\Omega}, \tilde{\mathcal{A}}, (\tilde{\mathcal{G}}_t)_t, W, X, \tilde{P}^\lambda)$ a weak solution of the SDE with initial measure λ; we proceed to filter. We say that a (right-continuous completed) filtration $(\mathcal{H}_t)_t$ on $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P}^\lambda)$ is a $(W, \tilde{\mathcal{G}})$-filtration if W remains a Brownian motion with respect to $\mathcal{H}$ and, for every t, for every Y $\tilde{\mathcal{G}}_t$-measurable, $P[Y|X_0, \mathcal{H}_T] = P[Y|X_0, \mathcal{H}_t]$. Of course, $\tilde{\mathcal{G}}$ and the Brownian filtration $\mathcal{F}$ are $(W, \tilde{\mathcal{G}})$-filtrations.

Consider $(\mathcal{H}_t)_t$ a $(W, \tilde{\mathcal{G}})$-filtration. Consider a version N of the conditional law of X with respect to X_0 and $\mathcal{H}_T$ (where X takes values in Γ_T). For t in $[0,T]$, define, for every f measurable bounded function on $\mathbb{R}^d$,

$$T_t f(x,\omega) = \int_0^t f(\gamma_t)N^{x,\omega}(d\gamma). \tag{2.21}$$

It is clear that T_t is the conditional law of X_t with respect to X_0 and $\mathcal{H}_T$ (in the sense that is the operator associated to such conditional law). Now take $\lambda(dx) = h(x)dx$, with h strictly positive function (so that λ and $\mathcal{L}^d$ are equivalent).

Theorem 27. *$(T_t)_t$ can be extended to a Markovian solution of the SCE for observables.*

We call T a solution associated to $\mathcal{H}$.

Proof. The Markovian property is clear. Adaptedness follows from the condition of $(W, \tilde{\mathcal{G}})$-filtration, since $P[f(X_t)|X_0, \mathcal{H}_t] = P[f(X_t)|X_0, \mathcal{H}_T]$. The uniform L^2 bound comes from Jensen inequality:

$$P[(T_t f)^2] \le P[f(X_t)^2] = P_t(f^2).$$

Finally the SCE for ooblservales is obtained filtering the SDE with respect to $\mathcal{H}_t$ and using the condition of $(W, \tilde{\mathcal{G}})$-filtration. □

Finally we find the desired result.

Corollary 28. *Let S be the solution associated to the Brownian filtration $\mathcal{F}$. Then S is the generalized flow which solves the SDE.*

Note that this representation gives also a Wiener existence result for the SCE for measures. We call $\mathcal{M}_+$ the set of positive finite measure on $\mathbb{R}^d$.

Definition 29. *Given μ_0 in $\mathcal{M}_+$, a distributional $\mathcal{M}_+$ solution of the SCE is a family $\mu = (\mu_t^\omega)_{t,\omega}$ of measures in $\mathcal{M}_+$, weakly progressively measurable, with total mass $\mu(\mathbb{R}^d)$ in $L^\infty([0,T] \times \Omega)$, such that, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, it holds*

$$\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_r, A\varphi \rangle dr + \sum_{k=1}^d \langle \mu_r, D_k \varphi \rangle dW_r^k. \tag{2.22}$$

Corollary 30. *Let μ_0 be in $\mathcal{M}_+$. Then the family of random measures $(\mu_t)_t$ defined by*

$$\langle \mu_t^\omega, f \rangle = \langle \mu_0, T_t f(\omega) \rangle = \int_0^t f(\gamma_t) N^{x,\omega}(d\gamma) \mu_0(dx) \tag{2.23}$$

is a Wiener distributional $\mathcal{M}_+$ solution of the SCE.

2.4 Flows of maps and flows of kernels

For n positive integer, define, for $f_1, \dots f_n$ in $L^\infty(\mathbb{R}^d)$,

$$P_t^{(n)}(f_1 \otimes \ldots \otimes f_n) = P[S_t f_1 \otimes \ldots \otimes S_t f_n]. \tag{2.24}$$

Proposition 31. *$(P_t^{(n)})_t$ is a Markovian semigroup associated to the canonical process on $\Gamma_T^n = C([0,T];\mathbb{R}^{nd})$ with probability measure $P[\otimes_{j=1}^n N^{x_j,\cdot}]$.*

Proof. The fact that $P_t^{(n)}$ is Markovian can be checked using the cocycle law and the Markovian property of S. Using representation formula (2.21), we get, for $\mathcal{L}^{nd}\otimes P$-a.e. (x,ω) in $\mathbb{R}^{nd}\times\Omega$,

$$\begin{aligned} S_t^{\otimes n}(\otimes_{j=1}^n f_j)(x,\omega) &:= (\otimes_{j=1}^n S_t f_j)(x,\omega) \\ &= \Pi_{j=1}^n \int_{\Gamma_T} f_j(\gamma_t) N^{x_j,\omega}(d\gamma) = \int_{\Gamma_T^n} (\otimes_{j=1}^n f_j)(\gamma_t)[\otimes_{j=1}^n N^{x_j,\omega}](d\gamma), \end{aligned}$$

Thus the canonical process on $(\Gamma_T^n, P[\otimes_{j=1}^n N^{x_j,\cdot}])$ is associated to $(P_t^{(n)})_t$. □

We can see this process as the projection X^n on Γ_T^n in the space $(\Omega\times\Gamma_T^n, P\otimes\otimes_{j=1}^n N^{x_j,\cdot}$. At fied ω, X^n represents the motion of n particles driven by the SDE blocked at ω. This explains the following definition.

Definition 32. *X^n is called the n-point motion of S.*

Proposition 33. *For every $\varphi_1,\dots\varphi_n$ in $\mathcal{D}(A)\cap L^\infty(\mathbb{R}^d)$, it holds*

$$d[S_t^{\otimes n}(\otimes_{j=1}^n\varphi_j)] = S_t^{\otimes n}A^{(n)}(\otimes_{j=1}^n\varphi_j)dt + \sum_{k=1}^\infty S_t^{\otimes n}D_k^{(n)}(\otimes_{j=1}^n\varphi_j)dW_t^k. \quad (2.25)$$

Proof. It is enough to apply Ito formula to (2.5). □

Definition 34. *$(S_t)_t$ is a flow of maps if it exists a family of measurable maps $(\Phi_t)_t$ from $\mathbb{R}^d\times\Omega$ to $\mathbb{R}^d$ such that $S_t f = f\circ\Phi_t$ for every t and f in $L^2(\mathbb{R}^d)$. It is a flows of non-trivial kernels otherwise.*

In the previous section, we have seen that S is a kernel. Thus it is clear that S is a flow of maps if and only if this kernel is "trivial", i.e. a Dirac delta.

In the following, we denote by $(X_t,Y_t)_t$ the two-point motion starting from (x,y) in $\mathbb{R}^{2d}$.

Definition 35. *$(S_t)_t$ is an injective flow of maps if it is a flow of maps and, for every $x\neq y$ and $t>0$, P-a.s. $\Phi_t(x)\neq\Phi_t(y)$.*
$(S_t)_t$ is a coalescing flow of maps if it is a flow of maps and, for every x,y, P-a.s. there exists $t\geq 0$ such that $\Phi_t(x)=\Phi_t(y)$ for all $u\geq t$.
$S_t)_t$ is a splitting flow without hitting if it is a flow of non-trivial kernels and, for every $x=y$, $X_t\neq Y_t$ for all $t>0$.
$S_t)_t$ is a splitting flow with hitting (or coalescence) if it is a flow of non-trivial kernels and, for every x,y, there exists $t>0$ such that $X_t=Y_t$ with positive probability.

Lemma 36. *S is a flow of maps if and only if, for every f in $L^2(\mathbb{R}^d)$ and every $t \geq 0$, it holds*

$$P[(S_t f)^2] = P_t f^2. \tag{2.26}$$

Proof. Notice that $P_t(f^2)(x) - P[(S_t f(x))^2] = P[S_t(f^2)(x) - (S_t f(x))^2]$. Since S_t is a kernel, $S_t(f^2)(x) - (S_t f(x))^2 \geq 0$ and equality holds for all x if and only if S is concentrated on a Dirac delta. □

Intuitively, a flow is of maps if and only if the two-point motion strating from (x, x) remains on the diagonal. Thus it is natural to use this motion in the classification of flows. In particular, we have two useful results (see Chapter 4).

Proposition 37. *Suppose that, for every $t, r > 0$, for a.e. x,*

$$\lim_{y \to x} P^{(2)}_{(x,y)}\{d(X_t, Y_t) \geq r\} = 0$$

Then $(S_t)_t$ is a flow of maps.

Proposition 38. *Suppose that there exists $r, t, p > 0$ such that, for a.e. (x, y),*

$$P^{(2)}_{(x,y)}\{d(X_t, Y_t) \geq r\} \geq p.$$

Then $(S_t)_t$ is a flow of non-trivial kernels.

Chapter 3

The case of a rough drift

3.1 Introduction

In this chapter we study the SCE with an irregular drift b and with $\sigma \equiv 0$. The interest in this case is due to the so-called phenomenon of regularization by noise.

Precisely, we are interested in the SDE

$$dX_t = b(X_t)dt + dW_t \tag{3.1}$$

(where, for this chapter, W will be a d-dimensional Brownian motion) and in the corresponding SCE

$$d\mu_t + \mathrm{div}(b\mu_t)dt + \sum_{k=1}^{d} \partial_{x_k}\mu_t dW_t^k = \frac{1}{2}\Delta\mu_t. \tag{3.2}$$

The classical theory for the deterministic CE and TE (transport equation), developed by DiPerna-Lions ([10]) and Ambrosio ([1]) and based on renormalized solutions, gives existence and uniqueness in the class of weak L^∞ solutions under hypotheses (a bit simplified for brevity) $b \in L^\infty(\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}^d)$ and $\mathrm{div}b \in L^\infty(\mathbb{R}^d)$; such hypotheses cannot be relaxed too much. As Flandoli et al. have shown, the introduction of noise allows some improvements: in the case of stochastic transport equation, existence and uniqueness hold asking b Hölder continuous, $\mathrm{div}b \in L^q(\mathbb{R}^d)$ for $q > 2$ ([12]) or $b \in L^\infty(\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}^d)$, $\mathrm{div}b \in L^1(\mathbb{R}^d)$ ([3].

Following this line, we will state two uniqueness results for the SCE and recall an example when the corresponding deterministic case has no uniqueness. This is what is called regularization by noise.

3.2 An example of regularization by noise

The following uniqueness result is a particular case of Theorem 9 (see [17]):

Theorem 39. *Suppose b is in $L^p(\mathbb{R}^d)$ for some $p > d$ and, in the L^2 case, $p \geq 2$. Then there is Wiener uniqueness in the class of distributional L^2 solutions and in the class of distributional L^∞ solutions.*

This result is an example of regularization by noise, since the corresponding deterministic result is false. A counterexample is due to Depauw ([9]), who shows non-uniqueness (among L^∞ solutions) for the CE with a drift b in $L^\infty([0,T] \times \mathbb{R}^d)$ with $\mathrm{div} b \equiv 0$. In the stochastic case, Wiener existence holds for the SCE (which is also a transport equation) by approximations method, while uniqueness holds by the above result, extended (easily) to the case of bounded time-dependent drifts.

3.3 Uniqueness by superposition solutions

The previous uniqueness result is limited to Brownian filtrations. However, uniqueness for the corresponding SDE holds among all the filtrations, as proved by Krylov-Röckner ([13]) and Fedrizzi-Flandoli ([11]):

Theorem 40. *Suppose b is in $L^p(\mathbb{R}^d)$. Then strong existence and uniqueness holds for the SDE (3.1). Furthermore there is a stochastic flow of homeomorphisms which solves the SDE.*

Now one imagines a solution of the SCE as induced by a solution of the SDE, that is $\mu_t = (X_t)_\# \mu_0$, and so is brought to believe that strong uniqueness holds also for the SCE. We will prove this (under slightly different hypotheses), making rigorous the intuition above. Our approach is inspired by the deterministic superposition principle ([2]).

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a cylindrical Brownian motion W on it, with its natural filtration $\mathcal{F}$. Call $\Gamma_T = C([0,T];\mathbb{R}^d)$, with the canonical filtration $\mathcal{H}_t = \sigma(\pi_s | s \leq t)$ (π_t being the evaluation map at time t).

Definition 41. *A superposition solution is a measure $\mu_0 \otimes P \otimes N^{x,\omega}$ on $\mathbb{R}^d \times \Omega \times \Gamma_T$, where:*

- *μ_0 is a measure on $\mathbb{R}^d$;*
- *$N^{x,\omega}$ is a kernel from $\mathbb{R}^d \times \Omega$ to Γ_T such that, for every t, $(\pi_{[0,t]})_\# N^{x,\omega}$ is (weakly) measurable with respect to $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t$, where $\mathcal{G}$ is a certain W-filtration;*

- *for $\mu_0 \otimes P$-a.e. (x,ω), $N^{x,\omega}$ is concentrated on solutions of the SDE with ω fixed and initial datum x.*

The second condition is an adaptedness condition. It can be easily restated as follows: for every t, for every B in $\mathcal{H}_t$, $N^{x,\omega}(B)$ is measurable with respect to $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t$.

Superposition solutions are particular solutions of the SCE: given N and μ_0,

$$\int_{\mathbb{R}^d} (\pi_t)_\# N^{x,\omega} \mu_0(dx) \tag{3.3}$$

is a solution of the SCE starting from μ_0.

We now state the uniqueness result.

Theorem 42. *Suppose b in $L^\infty(\mathbb{R}^d)$. Then strong existence and strong uniqueness holds for the SCE among $\mathcal{M}_+$ solutions.*

Proof. Existence follows by taking $\mu_t^\omega = (X_t(\omega))_\# \mu_0$, where X_t is a flow solution of the SDE. We will prove uniqueness in two steps:

1. strong uniqueness for the SDE implies strong uniqueness among superposition solutions;

2. every $\mathcal{M}_+$ solution can be represented as a superposition solution.

□

3.3.1 First step

Take $\mu_0 \otimes P \otimes N$ superposition solution. Fix the intial datum x and define

- $\tilde{\Omega} = \Omega \times \Gamma_T$,
- $\tilde{\mathcal{G}}_t = \mathcal{G}_t \otimes \mathcal{H}_t$ (where $\mathcal{H}_t = \sigma(\pi_s | s \leq t)$),
- $\tilde{P}^x = P \otimes N^{x,\omega}$,
- $W_t(\omega,\gamma) = W_t(\omega)$,
- $X(\omega,\gamma) = \gamma$.

Proposition 43. *The process W remains a Brownian motion under $\tilde{P}^x$ with respect to $(\tilde{\mathcal{G}}_t)_t$ and $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P}^x, \tilde{W}, X)$ is a solution of the SDE with initial datum x.*

Proof. Take $s < t$ and $A \times B$ in $\tilde{\mathcal{G}}_s$. Then

$$\tilde{P}^x[1_{A\times B}(W_t - W_s)] = P[1_A N^{x,\cdot}(B)(W_t - W_s)]$$
$$= P[1_A N^{x,\cdot}(B)]P[W_t - W_s] = \tilde{P}^x(A \times B)\tilde{P}^x[W_t - W_s],$$

where in the third equality we have used the adaptedness condition. So $W_t - W_s$ is independent of $\tilde{\mathcal{G}}_s$ and obviously it has law $\mathcal{N}(0, t-s)$. The first part of the statement is proved.

The second part follows from the fact that X is $(\tilde{\mathcal{G}}_t)_t$-progressively measurable and that for P-a.s. ω, for $N^{x,\omega}$-a.s. γ, $X(\omega,\gamma) = \gamma$ solves the SDE with ω blocked. □

For the following, the next remark will be useful: a continuous process Z is adapted to a certain filtration $\mathcal{G}$ if and only if, for every t, for every B in $\mathcal{H}_t$, the event $\{Z \in B\}$ is in $\mathcal{G}_t$. One verifies easily this fact, using the definition $\mathcal{H}_t = \sigma\{\pi_s | s \le t\}$. We also notice that Γ_T is a complete metric space with the metric of uniform convergence and that its Borel σ-algebra is precisely $\mathcal{H}_T$.

Theorem 44. *Suppose strong existence and strong uniqueness for the SDE with initial datum x, for μ_0-a.e. x. Then there is strong existence and strong uniqueness among superposition solutions starting from μ_0. The superposition solution is given by $\mu_0 \otimes P \otimes \delta_{Y(x,\omega)}$, where Y is the solution of the SDE.*

Proof. Fix x in the set where strong existence and uniqueness holds. With the previous notation, X must be the unique strong solution of the SDE; in particular X must be adapted to $(\mathcal{F}_t)_t$ (with a little abuse of notation, we use again $\mathcal{F}_t$ for $\mathcal{F}_t \otimes \mathcal{T}$, where $\mathcal{T}$ is the trivial σ-algebra generated by $\tilde{P}^x$-null sets).

Lemma 45. *For P-a.s. ω, $N^{x,\omega}$ must be concentrated on a single $\gamma = \gamma^\omega$.*

Proof. It is enough to verify that, for every B in $\mathcal{H}_T$, for P-a.e. ω, $N^{x,\omega}(B)$ is 0 or 1. Indeed, suppose this is the case and consider a partition of Γ_T in countably many sets $(B_j)_j$ with diameter less than ϵ, $\epsilon > 0$ (here we take Γ_T with the metric of uniform convergence). Then, for P-a.e. ω, $N^{x,\omega}$ is concentrated on a single $B_{j(\omega)}$; since this holds for every $\epsilon > 0$, $N^{x,\omega}$ must be a Dirac delta.

Since X is $\mathcal{F}_T$-measurable, the event $\{X \in B\} = \Omega \times B$ must be equivalent to a set $A \times \Gamma_T$, for some A in $\mathcal{F}_T$, modulo $\tilde{P}^x$. This means that $P[1_{A^c} N^x(B)] = P[1_A N^x(B^c)] = 0$, that is $N^x(B)$ is equivalent to 1_A modulo P, in particolar it assumes values 0 or 1 a.s.. □

Thus the superposition solution must be concentrated on a process. Since the SDE has strong uniqueness among processes, the superposition solution is strong and coincides with $\mu_0 \otimes P \otimes \delta_{X(x,\omega)}$. The proof is complete. □

3.3.2 Second step

Theorem 46 (Superposition principle)**.** *Suppose b in L^∞. Let μ be a distributional $\mathcal{M}_+$ solution of SCE. Then μ is representable as a superposition solution (as in (3.3)).*

This theorem generalizes a deterministic result. The idea of the proof is the following: first we mollify μ with μ^ϵ's regular measures which are solutions of regular SCEs, thus superposition solutions; then we pass to the limit with a compactness method, finding the required superposition solution. In the stochastic case we have two difficulties: the regular SCEs have stochastic coefficients (but one can reduce these SCEs to a family of deterministic CEs) and a compactness method made ω by ω does not preserve adaptedness, so we have to use Crauel theory for random measures. We can suppose μ_0 a probability measure, without loss of generality. We will need the supeposition principle for regular deterministic cases:

Lemma 47. *Suppose v in $L^1([0,T];W^{1,\infty}_{loc}(\mathbb{R}^d))$. Then every regular solution of the deterministic CE*

$$d\lambda_t + \operatorname{div}(v_t\lambda_t)dt = 0 \tag{3.4}$$

is a deterministic superposition solution, i.e. $\lambda_t = (\pi_t)_\# K^x \mu_0(dx)$ and, for μ_0-a.e. x, K^x is a measure on Γ_T concentrated on the solutions of the ODE $dy_t = v_t(y_t)dt$.

Proof. (of superposition principle). Step 1: regularization. Take

$$\mu_t^{\omega,\epsilon} = \mu_t^\omega * \rho_\epsilon, \tag{3.5}$$

$$b_t^{\omega,\epsilon} = \frac{(b\mu_t) * \rho_\epsilon}{\mu_t^\epsilon}. \tag{3.6}$$

Lemma 48. *Fix $\epsilon > 0$. Then*

$$\mu_t^{\omega,\epsilon} dx = \int_{\mathbb{R}^d} (\pi_t)_\# \delta_{X^\epsilon(x,\omega)} \mu_0(dx) \tag{3.7}$$

where X^ϵ is a regular flow solution of the SDE

$$dX_t^\epsilon = b_t^\epsilon(X_t^\epsilon)dt + dW_t. \tag{3.8}$$

Here, by "regular flow solution of the SDE", we mean that, for every t, X^ϵ_t is measurable with respect to $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$ and that, for every x, $X^\epsilon(x)$ is a solution of the given SDE.

Proof. Using ρ^ϵ as test function, it is easy to see that μ^ϵ is a regular solution of the SCE with b as stochastic drift, i.e. it safisfies

$$d\mu^\epsilon_t + \operatorname{div}(b^\epsilon \mu^\epsilon_t)dt + \sum_k (\partial_{x_k}\mu^\epsilon_t)dW^k_t = \frac{1}{2}\Delta\mu^\epsilon_t$$

in the strong (spatial) sense. Call $\tilde{\mu}^{\omega,\epsilon}_t = \mu^{\omega,\epsilon}_t(\cdot + W_t(\omega))$ (this corresponds, in terms of measures, to $\tilde{\mu}^{\omega,\epsilon}_t = (\cdot - W_t(\omega))_\# \mu^{\omega,\epsilon}_t$). To obtain the expression of $\tilde{\mu}^{\omega,\epsilon}_t$ as a stochastic differential, we could use a generalized Ito formula, but we proceed following the interpretation in terms of measures. Take φ in $C^\infty_c(\mathbb{R}^d)$, then, by Ito formula and integration by parts,

$$d(\langle \mu^\epsilon_t, \varphi(\cdot - W_t)\rangle) + \langle \operatorname{div}(b^\epsilon_t \mu^\epsilon_t), \varphi(\cdot - W_t)\rangle dt = 0.$$

Changing variable and using the spatial regularity of $\tilde{\mu}^\epsilon_t$, the previous formula reads

$$\tilde{\mu}^\epsilon_t - \mu^\epsilon_0 + \int_0^t \operatorname{div}(\tilde{b}^\epsilon_t \mu^\epsilon_r)dr = 0,$$

where $\tilde{b}^{\omega,\epsilon}_t(x) = b^{\omega,\epsilon}_t(x + W_t(\omega))$.

Fix the parameter ω. Then the above formula is a deterministic CE with regular velocity field $\tilde{b}^{\omega,\epsilon}$ and regular solution $\tilde{\mu}^{\omega,\epsilon}$. Thus, by the deterministic lemma above, $\tilde{\mu}^{\omega,\epsilon}$ is representable as a deterministic superposition solution. Precisely, the ODE

$$d\tilde{X}^{\omega,\epsilon}_t = \tilde{b}^{\omega,\epsilon}_t(\tilde{X}^{\omega,\epsilon}_t)dt$$

admits a unique regular flow $\tilde{X}^\epsilon(\omega)$ as solution and it holds

$$\tilde{\mu}^{\omega,\epsilon}_t dx = \int_{\mathbb{R}^d} (\pi_t)_\# \delta_{\tilde{X}^\epsilon(x,\omega)} \mu_0(dx), \tag{3.9}$$

or equivalently

$$\mu^{\omega,\epsilon}_t dx = \int_{\mathbb{R}^d} (\pi_t)_\# \delta_{X^\epsilon(x,\omega)} \mu_0(dx),$$

where $X^\epsilon(x,\omega) = \tilde{X}^\epsilon(x,\omega) + W_t(\omega)$. Note that X^ϵ "solves the SDE" (we do not have yet adaptedness)

$$dX^\epsilon_t = b^\epsilon_t(X^\epsilon_t)dt + dW_t.$$

In order to conclude, we must show the progressive measurability of X^ϵ or equivalently for $\tilde{X}^\epsilon$. But $\tilde{X}^\epsilon(x,\omega)$ is the pointwise limit of $(\tilde{X}^{\epsilon,n}(x,\omega))_n$, where $\tilde{X}^{\epsilon,0}(x,\omega) \equiv x$ and

$$\tilde{X}_t^{\epsilon,n+1}(x,\omega) = x + \int_0^t \tilde{b}_r^{\omega,\epsilon}(\tilde{X}_r^{\epsilon,n}(x,\omega))dr.$$

Since both $\tilde{X}^{\epsilon,n}$ and $\tilde{b}^\epsilon$ are measurable with respect to $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$, so is $\tilde{X}^\epsilon$. The lemma is proved. □

Step 2: tightness and adaptedness. Being $(\mu_0 * \rho_\epsilon)_\epsilon$ tight, we can choose a function $\psi : \mathbb{R} \to [0,+\infty[$, with $\psi(x) \to +\infty$ as $|x| \to +\infty$, such that $\int_{\mathbb{R}^d} \psi \mu_0 * \rho_\epsilon dx \leq 1$. Applying the characterization of uniform integrability given by Dunford-Pettis to the singleton $\{b\}$, we can find a convex nondecreasing function $\Theta : [0,+\infty[\to [0,+\infty[$, with more than linear growth at infinity, such that

$$\int_0^T P\left[\int_{\mathbb{R}^d} \Theta(|b_t|)d\mu_t\right] dt < +\infty.$$

Consider the functional on $\{(x,\gamma) \in \mathbb{R}^d \times \Gamma_T | \gamma_0 = x\}$ defined by

$$\Psi(x,\gamma) = \psi(x) + \int_0^T \Theta(|\frac{d}{dt}\gamma_t|)dt.$$

It is a coercive functional. Indeed, given $\alpha > 0$, if $\Psi(x,\gamma) \leq \alpha$, then:

- $x = \gamma_0$ belongs to a compact set of $\mathbb{R}^d$;
- the family $\frac{d}{dt}\gamma$ is uniformly integrable on $[0,T]$, so that the functions γ's are equicontinuous;

so, by Arzelà-Ascoli theorem, the set $\{\Psi \leq \alpha\}$ is compact.

Tightness of $(\tilde{\eta}^\epsilon := P[\mu_0 \otimes \tilde{N}^\epsilon])_\epsilon$ follows from the fact that $\int \Psi d\tilde{\eta}^\epsilon$ is uniformly bounded:

$$\begin{aligned}
&\int \Psi d\tilde{\eta}^\epsilon \\
&= \int \psi(x)\mu_0(dx) + \int_0^T \int_\Omega \int_{\mathbb{R}^d} \Theta(|\tilde{b}_t^{\omega,\epsilon}(x)|)d\tilde{\mu}_t^{\omega,\epsilon} P(d\omega)dt \\
&= \int \psi(x)\mu_0(dx) + \int_0^T \int_\Omega \int_{\mathbb{R}^d} \Theta(|b_t^{\omega,\epsilon}(x)|)d\mu_t^{\omega,\epsilon} P(d\omega)dt \\
&\leq \int \psi(x)\mu_0(dx) + (1+\epsilon M)\int_0^T P\left[\int_{\mathbb{R}^d} (\Theta(|b_t(x)|)\mu_t) * \rho_\epsilon dx\right] dt
\end{aligned}$$

$$\leq \int \psi(x)\mu_0(dx) + (1+\epsilon M)\int_0^T P\left[\int_{\mathbb{R}^d} \Theta(|b_t(x)|)\mu_t(dx)\right] dt \quad (3.10)$$

We have used the inequality $\Theta(|b_t^{\omega,\epsilon}(x)|)d\mu_t^{\omega,\epsilon} \leq (\Theta(|b_t(x)|)\mu_t) * \rho_\epsilon$. It follows by Jensen inequality applied to the convex l.s.c. function $(t,z) \mapsto t\Theta(|z|/t)$ and the measure $\rho_\epsilon(x-\cdot)\mathcal{L}^d$.

By Crauel theory, the family of random measures $(\mu_0 \otimes \tilde{N}^{\omega,\epsilon})_\epsilon$ converges narrowly to a random measure, which can be written as $\mu_0 \otimes \tilde{N}^\omega$. Passing to the limit in (3.9), we obtain

$$\tilde{\mu}_t = \int_{\mathbb{R}^d} (\pi_t)_\# \tilde{N}^{x,\omega}\mu_0(dx);$$

calling $N^{x,\omega} = (\cdot + W(\omega))_\# \tilde{N}^{x,\omega}$, the previous equation reads

$$\mu_t = \int_{\mathbb{R}^d} (\pi_t)_\# N^{x,\omega}\mu_0(dx). \quad (3.11)$$

We now verify adaptedness of N (that is of $\tilde{N}$). Note this is equivalent to show that, for every f in $C_b(\mathbb{R}^d)$, G in $L^1(\Omega, \mathcal{G}_T, P)$, φ in $C_b(\Gamma_T)$ measurable with respect to $\mathcal{H}_t$, t in $[0,T]$, it holds

$$\int_{\mathbb{R}^d}\int_\Omega f(x)G(\omega)N^{x,\omega}(\varphi)P(d\omega)\mu_0(dx)$$
$$= \int_{\mathbb{R}^d}\int_\Omega f(x)P[G|\mathcal{G}_t](\omega)N^{x,\omega}(\varphi)P(d\omega)\mu_0(dx).$$

Since this formula is true for N^ϵ, it is enough to pass to the limit.

Step 3: η is concentrated on solutions of the SDE. We will prove that, for every t in $[0,T]$,

$$\Lambda(b,N) := \int_{\mathbb{R}^d}\int_\Omega\int_{\Gamma_T} \frac{\left|\gamma_t - x - W_t(\omega) - \int_0^t b(\gamma_r)dr\right|}{1+\max_{[0,T]}|\gamma|} N^{x,\omega}(d\gamma)P(d\omega)\mu_0(dx) = 0.$$

To fix the ideas, suppose first b countinuous with compact support. Since $\Lambda(b^\epsilon, N^\epsilon) = 0$ by step 1, we have

$$\Lambda(b,N) \leq |\Lambda(b,N) - \Lambda(b,N^\epsilon)| + |\Lambda(b,N^\epsilon) - \Lambda(b^\epsilon,N^\epsilon)|.$$

Now $\Lambda(b,N) - \Lambda(b,N^\epsilon)$ goes to 0, by definition of narrow convergence of measure, since, in this case, the integrand is in $C_b(\mathbb{R}^d \times \Gamma_T; L^1(\Omega))$. Also $\Lambda(b^\epsilon,N^\epsilon) - \Lambda(b,N^\epsilon)$ goes to 0, using the triangle inequality and the uniform convergence of $b - b^\epsilon$ towards 0 uniformly (since b is uniformly continuous).

In the general case, we approximate b with continuous functions with compact support; precisely, if c is such a function on $\mathbb{R}^d$, it holds

$$\begin{aligned}
&\Lambda(b, N) \\
&\quad\leq \; |\Lambda(b, N) - \Lambda(c, N)| + |\Lambda(c, N) - \Lambda(c, N^\epsilon)| + \\
&\qquad +|\Lambda(c, N^\epsilon) - \Lambda(c^\epsilon, N^\epsilon)| + |\Lambda(c^\epsilon, N^\epsilon) - \Lambda(b^\epsilon, N^\epsilon)|.
\end{aligned}$$

As before, $|\Lambda(c, N) - \Lambda(c, N^\epsilon)| + |\Lambda(c, N^\epsilon) - \Lambda(c^\epsilon, N^\epsilon)|$ goes to 0. We notice that

$$|\Lambda(c^\epsilon, N^\epsilon) - \Lambda(b^\epsilon, N^\epsilon)| \leq \int_0^t \int_{\mathbb{R}^d} P\left[|c^\epsilon_r - b^\epsilon_r| d\mu^\epsilon_r\right] dr$$

and that the same holds without the ϵ's. The RHS in the previous inequality converges, as $\epsilon \to 0$, towards

$$\int_0^t \int_{\mathbb{R}^d} |c - b| d\nu_r dr,$$

where $\nu_t = P[\mu_t]$. Now it is enough to approximate b with functions c^n in $L^1(\mathbb{R}^d, \int_0^T \nu_r dr)$. □

Chapter 4

Isotropic Brownian flows

4.1 Definition and properties

In this chapter we consider a special case of generalized flows. This kind of flow is suitable for a rigorous description of certain simple models in fluid dynamics (essentially for turbulence), where splitting and/or coalescence occur. Again, we follow [16] for the generalized case, while we refer to [4] and [15] for the classical regular results.

Definition 49. *A generalized isotropic Brownian flow (IBF) is a generalized flow which solves an SCE with $b \equiv 0$ and K isotropic Brownian covariance, i.e.:*

1. *$K(x, x) = id$ for every x in $\mathbb{R}^d$;*
2. *K omogeneous (invariant under translation): $K(x, y) = K(x - y)$ for every x, y in $\mathbb{R}^d$;*
3. *K invariant under rotation: $G^* K(Gx, Gy)G = K(x, y)$ for every x, y in $\mathbb{R}^d$.*

Proposition 50. *A measurable isotropic covariance K can be written as*

$$K(z) = (K_L(|z|) - K_N(|z|))\frac{zz^*}{|z|^2} + K_N(|z|)I_d, \tag{4.1}$$

where

$$K_L(r) = K_{pp}(re_p),$$
$$K_N(r) = K_{pp}(re_q), \; p \neq q.$$

K_L and K_N do not depend on the choise of the basis $(e_n)_n$ of $\mathbb{R}^d$.

Theorem 51. *There exist isotropic Brownian flows of diffeomorphisms. Every regular IBF Φ is characterized by the following properties (besides regularity):*

1. *$\Phi(x)$ is a Brownian motion starting from x;*
2. *the law of Φ is omogeneous;*
3. *the law of Φ is invariant under rotation.*

Furthermore, for every x, y in $\mathbb{R}^d$, the process $dist(\Phi(x), \Phi(y))$ is a diffusion with drift b and diffusion coefficient σ given by

$$b(r) = (d-1)\frac{1 - K_N(r)}{r}, \tag{4.2}$$

$$\sigma(r)^2 = 2(1 - K_L(r)). \tag{4.3}$$

We will prove a similar result also for generalized IBFs.

Theorem 52. *A generalized IBF enjoys the following properties:*

1. *every one-point motion is a Brownian motion;*
2. *the law of every two-point motion at fixed time is omogeneous;*
3. *the law of every two-point motion at fixed time is invariant under rotation;*
4. *for every two-point motion (X, Y), the process $dist(X, Y)$ is a diffusion with drift and diffusion coefficient given by the above formulas.*

Proof. The first statement follows from the fact that the infinitesimal generator of the one-point motion is $\frac{1}{2}\Delta$, since $K(x, x) = id$.

The second and the third statement can be proven using the Wiener chaos expansion of the generalized flow S: indeed, for every φ, ψ regular functions in $L^2(\mathbb{R}^d)$, it holds

$$P[S_t^{n+1}\varphi \otimes S_t^{n+1}\psi] = P_t\varphi \otimes P_t\psi + \int_0^t P[S_r^n \otimes S_r^n(K(P_{t-r}\varphi, P_{t-r}\psi))]dr,$$

so one proves inductively that the law of $S_t^n \otimes S_t^n$ depends only on K; with similar computations we get that the law of $S^n \otimes S^n$ depends only on K, and the conclusion follows passing to the limit.

As for the fourth statement, we apply (2.25) first to the process $dist(X, Y)^2$ (which is in the domain of the infinitesimal generator of the two-point motion) and then to its square root; in this way we get that $dist(X, Y)$ satisfies an SDE with b and σ as required. □

In Chapter 2 we have seen some criteria, based on the two-point motion, to determine the nature of a generalized flow. Now we can apply these results, because the two-point motion $dist(X, Y)$ is a diffusion. Indeed, we will see in the next section that there are two tools which characterize the behaviour of $dist(X, Y)$ near 0.

4.2 Singular one-dimensional diffusions

Let Z be a diffusion with an interval I as state space, let l, r be the two (possibly non-finite) extrema of I; for y in I, call T_y the time when Z first reach y; for $J = [c, d]$ interval in I, x in J, call $m(x, J) = P^x[\min\{T_c, T_d\}]$. We want to analyze the behaviour of the diffusion near c. The following results are in [6].

Theorem 53. *There exists a continuous, strictly increasing function $s : I \to \mathbb{R}$, unique up to a linear transformation, such that, for every c, d, x in I with $c < x < d$,*

$$P^x\{T_d < T_c\} = \frac{s(x) - s(c)}{s(d) - s(c)}. \tag{4.4}$$

Proof. Uniqueness of s is clear. As for existence, suppose first $I = [l, r]$. In this case the thesis follows from

$$P^x\{T_r < T_l\} = P^x\{T_d < T_c\}P^d\{T_r < T_l\} + P^x\{T_d > T_c\}P^c\{T_r < T_l\},$$

which is a consequence of the strong Markov property. The general case can be obtained by approximation of l, r. □

Definition 54. *The function s is called the scale function of Z.*

Let J be an interval in $\int(I)$; define G_J as

$$G_J(x, y) = \frac{2(s(x \wedge y) - s(c))(s(d) - s(c \vee y))}{s(d) - s(c)}$$

if x, y in J, $G_J = 0$ otherwise.

Theorem 55. *There exists a unique Radon measure on $int(I)$ such that, for every interval $J =]c, d[$ with $\bar{J}$ in I and every x in J, it holds*

$$m(x, J) = \int_I G_J(x, y) m(dy). \tag{4.5}$$

This measure m (or better $m(dx)/dx$) represents in some sense the average speed of the process Z. This explains the folllowing definition.

Definition 56. *The measure m is called the speed measure of Z.*

Definition 57. *The point c is said to be open if c does not belong to I, closed otherwise.*

Definition 58. *If c is open, c is said to be natural if, for every x in int(I) and every $t > 0$,*

$$\lim_{y \to c} P^y[T_x < t] = 0;$$

otherwise, c is said to be entrance.

Definition 59. *If c is closed, c is said to be exit if, for every x in int(I),*

$$m(]c, x[) = +\infty;$$

otherwise, c is said to be regular.

Proposition 60. *Suppose c closed. It is possible to extend m on $[c, d[$ such that (4.5) remains true for $J = [c, g[$ in I.*

Definition 61. *If c is closed regular, c is said to be:*

- *absorbing: if $m\{c\} = +\infty$;*
- *slowly reflecting: if $0 < m\{c\} < +\infty$;*
- *instantaneously reflecting: if $m\{c\} = 0$.*

Proposition 62. *The point c is closed if and only if, for every x in int(I),*

$$\int_c^x |s(y) - s(c)| m(dy) < +\infty.$$

Proposition 63. *If c is open, c is entrance if and only if $s(c)$ is infinite and, for every x in int(I),*

$$\int_c^x |s(y)| m(dy) < +\infty.$$

Proposition 64. *If c is exit closed, then it holds, for every $x > c$ and $t > 0$,*

$$P^c[T_x < t] = 0.$$

Proposition 65. *Let Z be a diffusion on I with infinitesimal generator*

$$L\varphi = b\varphi' + \frac{1}{2}\sigma^2\varphi''.$$

Then the following formulas hold for s and m on I (x_0 is a point in I):

$$s(x) = \int_{x_0}^{x} \exp\left(-\int_{x_0}^{y} \frac{2b(z)}{\sigma(z)^2} dz\right) dy, \tag{4.6}$$

$$m(dx) = \frac{2}{s'(x)\sigma(x)^2} dx. \tag{4.7}$$

4.3 Classification of generalized IBFs

Now we come back to our diffusion $dist(X, Y)$, the two-point motion of a generalized IBF starting from (x, y); here the space state is the interval with extrema 0 and $+\infty$. The previous propositions allow us to study its behaviour by means of b and σ and so by means of K. So we suppose:

$$b(r) = c_b r^{\alpha-1}(1 + o(1)), \tag{4.8}$$

$$\sigma(r)^2 = 2c_\sigma r^\alpha(1 + o(1)). \tag{4.9}$$

for some α, c_b, c_σ positive real numbers. This means that

$$K_L(r) = 1 - c_\sigma r^\alpha(1 + o(1)), \tag{4.10}$$

$$K_N(r) = 1 - \frac{c_b}{d-1} r^\alpha(1 + o(1)). \tag{4.11}$$

Proposition 66. *The following facts hold.*

1. *The point 0 is closed if and only if hitting occurs.*

2. *If 0 is natural open, then the flow is an injective flow of maps.*

3. *If 0 is entrance open, then the flow is diffusive without hitting.*

4. *If 0 is exit closed, then the flow is a coalescent flow of maps.*

5. *If 0 is instantaneously reflecting regular closed, then the flow is diffusive with hitting.*

Proof. The first statement is clear from the definition of closed point.

Suppose 0 is natural open. Then, for every $t > 0$, $r > 0$, we have

$$\lim_{y\to x} P\{dist(X_t, Y_t) \geq r\} = 0,$$

so that the flow is of maps by Proposition 37; it is injective since 0 is open.

Suppose 0 is entrance open. Then there esist $t > 0$, $r > 0$, $p > 0$ such that, for every distinct points x, y in $\mathbb{R}^d$,

$$P\{dist(X_t, Y_t) \geq r\} \geq p,$$

so that the flow is not of maps by Proposition 38; it is without hitting since 0 is open.

Suppose 0 exit closed. Then, if $x = y$, by Proposition 64 it must be $X_t = Y_t$, so that the flow is of maps; it is coalescent since 0 is closed.

The last stetement follow from the definition of instantaneously reflecting regular closed point. □

Proposition 67. *The following facts hold.*

1. *If $\alpha > 2$, then 0 is natural open and so the flow is an injective flow of maps.*
2. *If $\alpha < 2$ and $\frac{c_b}{c_\sigma} > 1$, then 0 is entrance open and so the flow is splitting without hitting.*
3. *If $\alpha < 2$ and $\alpha - 1 < \frac{c_b}{c_\sigma} < 1$, then 0 is regular closed.*
4. *If $\alpha < 2$ and $\frac{c_b}{c_\sigma} < \alpha - 1$, then 0 is exit closed and so the flow is a coalescent flow of maps.*

Proof. With some computations, using (4.7) and (4.7). □

It remains to analyze the case $\alpha < 2$, $\alpha - 1 < \frac{c_b}{c_\sigma} < 1$, where we know that hitting must occur but we still have to check if 0 is absorbing or reflecting. Here b and σ do not give any information on $m\{0\}$.

4.4 Interpretation of the results

In this section we analyze the intuitive meaning of the parameters and the results obtained, making use of a concrete example.

First, we note that the infinitesimal convariance K can be interpreted as the correlation function of a "generating" random field U. Precisely,

$$K(x, y) = P[U(x)U(y)], \tag{4.12}$$

where $U(x) = \sum_k \sigma_k(x)B^k$ and $(B^k)_k$ is a sequence of independent $\mathbb{R}^d$-valued $\mathcal{N}(0, I_d)$ r.v.'s. The field U is generating in the following sense. Take $(U^n)_n$

a sequence of indepenent r.v.'s with the same law as U; for j positive integer, take $X^j(x)$ the flow, with $\frac{1}{j}\mathbb{N}$ as time set, defined by $X^j_0(x,\omega) = x$, $X^j_{(n+1)/j}(x,\omega) = \frac{1}{\sqrt{j}}U^{n+1}(X^j_{n/j}(x,\omega),\omega)$. The intuitive idea is that, as j goes to $+\infty$, X converge in law to the (generalized) flow S; so U is a sort of velocity with a different rescaling of space and time in the approximation.

Thus α is related to the correlation of the random field U: if α is close to 0, then $U(x)$ and $U(y)$ have low correlation even if they are closed each other.

To introduce our examples, we need reproducing kernel Hilbert spaces. Denote with L the linear bounded operator on $L^2(\mathbb{R}^d)^d$ given by

$$LU(x) = \int_{\mathbb{R}^d} K(x-y)U(y)dy. \tag{4.13}$$

Let H_0 be the image of $L^2(\mathbb{R}^d)^d$ under L; let H be the complection of H_0 with respect to the scalar product

$$\langle LU, LV\rangle_H = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x)\cdot K(x-y)U(y)dxdy = \int_{\mathbb{R}^d} V(x)\cdot LU(x)dx. \tag{4.14}$$

Note that $\|LU\|_H \le \|U\|_{L^2}$ (so H is separable) and $\|LU\|_{L^2} \le \|LU\|_H$. Take $(\sigma_k)_{k\in\mathbb{N}^+}$ a basis in H.

Definition 68. *H is called the reproducing kernel Hilbert space associated to K.*

Proposition 69. *It holds $K(x-y) = \sum_k \sigma_k(x)\sigma_k(y)^*$ (in the sense that, for every y the sum converges in H as a function of x).*

Proof. It is enough to observe that, for every U, V in $L^2(\mathbb{R}^d)^d$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x)\cdot\sum_k \sigma_k(x)\sigma_k(y)^*U(y)dxdy\\ =\ &\sum_k(\langle\sigma_k, LU\rangle_H)(\langle\sigma_k, LV\rangle_H) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x)\cdot K(x-y)U(y)dxdy.\end{aligned}$$

□

A similar reasoning can be done for a general $L^2(\mathbb{R}^{2d})$ covariance function. Now consider the Fourier representation of L:

$$\widehat{LU}(\xi) = \widehat{K}(\xi)\widehat{U}(\xi). \tag{4.15}$$

In our case, this will allow to identify the space H. Indeed, take the covariance K with the following Fourier transform:

$$\widehat{K}(\xi) = c(|\xi|^2+1)^{-(d+\alpha)/2}\left(a\frac{\xi\otimes\xi}{|\xi|^2} + \frac{b}{d-1}\left(I_d - \frac{\xi\otimes\xi}{|\xi|^2}\right)\right), \tag{4.16}$$

where $a, b > 0$. It is well known that the Fourier representation of the Laplacian operator is the multiplication by the function $-|\xi|^2$. For the other term, we use the following result.

Proposition 70. *The space $L^2(\mathbb{R}^d)^2$ can be decomposed in an orthogonal sum of the spaces of gradient and divergence-free vector fields. If Π is the projection on the space of gradient vector fields, then the Fourier representation of Π is the operator $\widehat{U} \to \frac{\xi\cdot\widehat{U}(\xi)}{|\xi|^2}\xi$.*

Proof. The orthogonality of the two spaces follows from the formula $\langle V, \nabla\psi\rangle_{L^2} = \langle \mathrm{div}V, \psi\rangle_{L^2}$. As for completeness, let U be a $W^{1,2}(\mathbb{R}^d)$ vector field. Let φ be the unique $W^{2,2}(\mathbb{R}^d)$ solution to the problem $\Delta\varphi = \mathrm{div}U$; then clearly $U - \nabla\varphi$ is divergence-free, so $U = (\nabla\varphi) + (U - \nabla\varphi)$ is the decomposition required.

Since the Fourier representations of ∂_j and of Δ^{-1} are resp. the multiplication by $i\xi_j$ and the multiplication by $-1/|\xi|^2$, then Π admits the representation above. Finally Π is bounded because so is its Fourier representation. □

Then $(-\Delta+1)^{(d+\alpha)/2}LU = a\Pi U + \frac{b}{d-1}(1-\Pi)U$ and so

$$\begin{aligned}
\|LU\|_H^2 &= \langle U, LU\rangle_{L^2} \\
&= \frac{1}{a}\langle a\Pi U, \Pi LU\rangle_{L^2} + \frac{d-1}{b}\langle\frac{b}{d-1}(1-\Pi)U, (1-\Pi)LU\rangle_{L^2} \\
&= \frac{1}{a}\langle(-\Delta+1)^{(d+\alpha)/2}\Pi LU, \Pi LU\rangle_{L^2} + \\
&\quad + \frac{d-1}{b}\langle(-\Delta+1)^{(d+\alpha)/2}(1-\Pi)LU, (1-\Pi)LU\rangle_{L^2}.
\end{aligned} \tag{4.17}$$

Thus the space H is the Sobolev space $W^{s,2}(\mathbb{R}^d)^d$, where $s = (d+\alpha)/2$, equipped with the norm $\|V\|_H^2 = \frac{1}{a}\|\Pi V\|_{W^{s,2}}^2 + \frac{d-1}{b}\|(1-\Pi)V\|_{W^{s,2}}^2$, where $\|V\|_{W^{s,2}} = \langle(-\Delta+1)^{(d+\alpha)/2}V, V\rangle_{L^2}$ is the usual Sobolev norm.

The parameter $\eta = \frac{b}{a+b}$ measures the "grade of compressibility" of the RKHS H. Indeed, given an element V of fixed H norm, the smaller is η, the smaller will be the zero-divergence component $(1-\Pi)V$. Since the σ_k's are in H, η is an indicator of the compressibility of the generating random field U and α an indicator of its regularity.

Expression (4.16) for $\widehat{K}$ allows to compute the behaviour of K_L and K_N near 0. It can be proved the following representation in spherical coordinates:

$$\widehat{K}(\xi) = u \otimes u\lambda(du)(F_L - F_N)(d\rho) + I_d\lambda(du)F_N(d\rho), \tag{4.18}$$

where $(\rho, u) = (|\xi|, \xi/|\xi|)$, λ is the normalized Lebesgue measure on S^{d-1}, F_L and F_N are positive measures on $[0, +\infty[$ with all finite moments. Given such measures, one can produce any isotropic covariance K with known formulas. Finally, with some real analysis, one gets that

$$c_b = C_{\alpha,\eta}(d - 1 + \alpha\eta),$$
$$c_\sigma = C_{\alpha,\eta}(\alpha + 1 - \alpha\eta),$$

so that

$$\frac{c_b}{c_\sigma} = \frac{d - 1 + \alpha\eta}{\alpha + 1 - \alpha\eta}. \tag{4.19}$$

Fixed α, this ratio is an increasing function of η; thus, fixed α, c_b/c_σ can be seen as a parameter for the compressibility.

Now we can obtain a classification result for this example:

Theorem 71. *The following facts hold.*

1. *If $\alpha > 2$, then the flow is an injective flow of maps.*
2. *If $\alpha < 2$ and $\eta > \frac{1}{2} - \frac{d-2}{2\alpha}$, then the flow is splitting without hitting.*
3. *If $\alpha < 2$ and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{d-2}{2\alpha}$, then the flow is splitting with hitting.*
4. *If $\alpha < 2$ and $\eta < 1 - \frac{d}{\alpha^2}$, then the flow is a coalescent flow of maps.*

Note that, for $\alpha < 2$ and $d \geq 4$, the flow is always splitting without hitting.

Proof. For the cases 1,2 and 4, it is enough to apply Propositions 67, using (4.19) for c_b/c_σ. In the case 3, a rahter technical approximation method can be used to show that 0 is instantaneously reflecting for the diffusion $dist(X, Y)$, so that the flow is splitting; it is coalescent since 0 is closed. □

Roughly speaking, this theorem tells that, if $\alpha < 2$,

- the increase of α brings less splitting for $d = 2, 3$ and more coalescence for $d = 3$;
- the increase of η brings more splitting for $d = 2, 3$ and less coalescence for $d = 2, 3$;

- the increase of d brings more splitting and less coalescence.

This facts can be understood as follows.

- If the motions of two points x and y are very uncorrelated (i.e. if α is small), then, in the limit $x = y$, the mass initially at x cannot stay together in one point, causing splitting.
- The increase of compressibility (i.e. the decrease of η) brings the mass to be "compressed" and stay concentrated, causing coalescence and preventing splitting.
- The increase of the dimension d brings more degrees of freedom, so that the mass can easily split and rarely come back together.

We must say this explanation is not complete: for example, one can expect that, in the gradient case $U = \nabla\psi$, there is splitting around the points wherethe minimum of ψ is reached, but this does not happen. Furthermore, we do not know for instance if a mass concentrated in a point splits in countably many parts or if it spreads over a whole region. Thus, generalized IBFs require more investigation.

Appendix A: preliminary results

Here we recall some preliminaries; one can find these classical results on [14], [18].

Definition 72. *A stochastic process (with values in a measurable space $(E, \mathcal{E})$) the object*

$$X = (\Omega, \mathcal{A}, (\mathcal{G}_t)_{t\in[\alpha,\beta]}, (X_t)_{t\in T}, P) \tag{4.20}$$

with $(\Omega, \mathcal{A}, P)$ probability space, $(\mathcal{G}_t)_{t\in[\alpha,\beta]}$ filtration (i.e. family of increasing σ-algebras) in $\mathcal{A}$ and, for every t in $[\alpha, \beta]$, X_t r.v. $(\Omega, \mathcal{G}_t)$-measurable with values in $(E, \mathcal{E})$.

We will always consider completed right-continuous filtrations (i.e. $\mathcal{G}_0$ contains all the P-null sets and, for all t, $\mathcal{G}_t = \cap_{\epsilon>0}\mathcal{G}_{t+\epsilon}$). We will also suppose E metric space and $\mathcal{E} = \mathcal{B}(E)$.

A process X is said to be progressively measurable if, for every u, the map $(t, \omega) \mapsto X_t(\omega)$ is measurable on $([0, u] \times \Omega, \mathcal{B}([0, u]) \otimes \mathcal{F}_u)$. X is said to be continuous if, for every (or a.e.) ω in Ω, the map $t \mapsto X_t(\omega)$ is conitnuous. A right-continuous process is progressively measurable.

Definition 73. *A real Brownian motion (BM) is a stochastic process $B = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0}, P)$ such that:*

1. *$B(0) = 0$;*
2. *for every $0 \leq s \leq t$, the r.v. $B_t - B_s$ is independent of $\mathcal{F}_s$;*
3. *for every $0 \leq s \leq t$, the r.v. $B_t - B_s$ has law $N(0, (t - s))$ (where $N(m, a)$ is the Gaussian law with mean value m e covariance a);*
4. *B has continuous trajectories.*

A cylindrical Brownian motion is a stochastic process $W = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, P)$, with $W_t = (W_t^k)_{k\in\mathbb{N}^+}$ $\mathbb{R}^{\mathbb{N}^+}$-valued, such that the coordinates B_k's are independent and real Brownian motions (with respect to $(\mathcal{F})$).

Theorem 74. *There exists a cylindrical Brownian motion.*

Definition 75. *A martingale is an $\mathbb{R}^d$-valued stochastic process $M = (\Omega, (\mathcal{G}_t)_t, (M_t)_t, P)$ such that, for every $s <$, M_t is integrable and $P[M_t|\mathcal{G}_s] = M_s$.*

A local martingale is an $\mathbb{R}^d$-valued stochastic process $M = (\Omega, (\mathcal{G}_t)_t, (M_t)_t, P)$ such that there exists an increasing sequence of stopping times $(\tau_n)_n$, with $\tau_n \nearrow +\infty$ as $n \to \infty$, such that $M^{\tau_n} := (M_{t\wedge\tau_n})_t$ is a martingale for all n.

A semimartingale is the sum of a local martingale and a process with bounded total variation (i.e. whose trajectories have bounded total variation).

Definition 76. *Let X, Y be $\mathbb{R}^d$-valued processes. For every partition $\Delta = \{0 = t_0 < t_1 < \ldots < t_l = T\}$ define*

$$[X,Y]_t^\Delta := \sum_{j=0}^{l-1} (X(t_{j+1}\wedge t) - X(t_j \wedge t))(Y(t_{j+1}\wedge t) - Y(t_j\wedge t))^*$$

The joint quadratic variation of X and Y, if it exists, is the process $[X,Y]$ such that, for every t, $[X,Y]_t = P\text{-}\lim_{\|\Delta\|\to 0}[X,Y]_t^\Delta$ (limit in probability). The quadratic variation of X is the process $[X] = [X,X]$.

Theorem 77 (Doob-Meyer)**.** *Let M, N be two continuous real (local) martingales with filtration $(\mathcal{G}_t)_t$. Then there exists a unique increasing continuous process A, with $A_0 = 0$, such that $MN - A$ is a (local) martingale. It holds $A = [M,N]$.*

Theorem 78 (Lévy)**.** *The cylindrical Brownian motion W is the only infinite-dimentional process starting from 0 such that its coordinates W^k's are continuous square-integrable martingales with $[W^h, W^k] = \delta_{hk}$ for every h, k.*

In the following, H is a separable Hilbert space with $\cdot$ as scalar product; if $H = \mathbb{R}^d$, $\cdot$ denotes the canonical scalar product. We define $\ell^2(H)^{\otimes n}$ as the set of multi-sequences $(a_k)_{k\in(\mathbb{N}^+)^n}$, indexed by $(\mathbb{N}^+)^n$, with values in H, such that $\sum_k \|a_k\|_H^2 < +\infty$. $\ell^2(H)^{\otimes n}$ is a Hilbert space with the scalar product $\langle a, b\rangle_{\ell^2} = \sum_k (a_k \cdot b_k)$.

We fix a cilindrical Brownain motion W on a probability space $(\Omega, \mathcal{A}, P)$, we call $(\mathcal{F}_t)_t$ the natural completed Brownian filtration. We also fix $(\mathcal{G}_t)_t$ a W-filtration (i.e. $\mathcal{G}_t \supseteq \mathcal{F}_t$).

Definition 79. *Fix $T > 0$. A process in $\Lambda^p(H)$ is a process with values in $\ell^2(H)$, progressively measurable with respect to $(\mathcal{G}_t)_t$, such that, for P-a.e. ω, $\int_0^T \|X_r(\omega)\|_{\ell^2}^p dr < +\infty$. Such a process is in $M^p(H)$ if $P[\int_0^T \|X_r\|_{\ell^2}^p dr] < +\infty$. Sometimes we will use the same notation for a process with value in H (not in $\ell^2(H)$).*

Definition 80. *A process in $\Lambda^2(\mathbb{R}^d)$ is elementary if X_k's are definitively null and $X(t) = \sum_{j=0}^{n-1} Z^j 1_{[t_j,t_{j+1}]}(t)$, with $0 = t_0 < t_1 < \ldots < t_n = T$, where Z^j is a $\mathcal{G}_{t_j}$-measurable r.v. with values in $\ell^2(\mathbb{R}^d)$ for every j (since the X_k's are definitively null in k, so are the Z_k^j's).*
We define stochastic integral of X with respect to W the r.v.

$$\int_0^T X_r dW_r = \sum_{j=0}^{n-1} \langle Z^j, W(t_{j+1}) - W(t_k)\rangle_{\ell^2}. \tag{4.21}$$

Proposition 81. *A process X in $\Lambda^2(\mathbb{R}^d)$ (resp. in $M^2(H)$) can be approximated by processes $(X^n)_n$ in $\Lambda^2(\mathbb{R}^d)$ (resp. in $M^2(H)$), in such a way that there exists the limit in probability (resp. in L^2) of $(\int_0^T X_r^n dW_r)_n$, and does not depend on the choise of the approximated sequence $(X^n)_n$.*

Definition 82. *The above limit is called stochastic integral of X with respect ot W and is denoted by $\int_0^T X_r dW_r$.*

Definition 83. *Given a process X in $M^2(H)$, we define the stochastic integral of X with respect to W by*

$$\int_0^T X_r dW_r = \sum_n \int_0^T X_r \cdot e_n dW_r, \tag{4.22}$$

where $(e_n)_n$ is a basis of H (one can prove that the definition is independent of the choise of the basis).

Theorem 84. *Given X, Y in $\Lambda^2(\mathbb{R})$, the process $(I(X)_t = \int_0^t X_r dW_r)_t$ is (up to modifications) a continuous $\mathcal{G}$-local martingale. The joint quadratic variation $[I(X), I(Y)]$ is given by*

$$[I(X), I(Y)]_t = \int_0^t \langle X_r, Y_r\rangle_{\ell^2} dr. \tag{4.23}$$

In particular, if X is in $M^2(\mathbb{R})$, $I(X)$ is a continuous square-integrable $\mathcal{G}$-martingale which satisfies the Ito isometry

$$P[I(X)_t^2] = \int_0^t P[\|X_r\|_{\ell^2}^2] dr. \tag{4.24}$$

Given X in $M^2(H)$, the H-valued process $(I(X)_t = \int_0^t X_r dW_r)_t$ is a weakly continuous square-integrable $\mathcal{G}$-martingale which satisfies the Ito isometry

$$P[\|I(X)\|_t^2] = \int_0^t P[\|X_r\|_{\ell^2(H)}^2] dr. \tag{4.25}$$

Definition 85. *We say that a process X with values in $\mathbb{R}^d$ admits stochastic differential if $dX_t = F_t dt + G_t dW_t$, with G in $\Lambda^1(\mathbb{R}^d)$ and G in $\Lambda^2(\mathbb{R}^d)$, if it holds*

$$X_t = X_0 + \int_0^t F_r dr + \int_0^t G_r dW_r. \tag{4.26}$$

Theorem 86 (Ito formula)**.** *Let X be a process as above and φ a function in $C^2(\mathbb{R}^d)$. Then the process $\varphi(X)$ admits stochastic differential*

$$\begin{aligned} &d(\varphi(X))_t \\ &= F_t \cdot \nabla\varphi(X_t)dt + (G_k)_t \cdot \nabla\varphi(X_t)dW_t^k + \frac{1}{2}\mathrm{tr}[(G_k)_t(G_k)_t^*(D^2\varphi)(X_t)]dt \end{aligned} \tag{4.27}$$

(the sum over k being omitted).

More in general, one can define an integral with respect to a martingale M. In this case we have

Theorem 87 (Ito formula)**.** *Let X be a $\mathbb{R}^d$ process with $dX_t = F_t dt + dM_t$ and φ a function in $C^2(\mathbb{R}^d)$. Then the process $\varphi(X)$ admits stochastic differential*

$$\begin{aligned} &d(\varphi(X))_t \\ &= F_t \cdot \nabla\varphi(X_t)dt + \nabla\varphi(X_t) \cdot dM_t + \frac{1}{2}\mathrm{tr}[(D^2\varphi)(X_t)d[M,M]_t]. \end{aligned} \tag{4.28}$$

Definition 88. *A weak solution of the stochastic differential equation (SDE)*

$$dX_t = b(X_t,t)dt + \sum_k \sigma_k(X_t,t)dW_t^k \tag{4.29}$$

is a pair of stochastic processes $(\Omega, (\mathcal{G}_t)_t, (W_t)_t (X_t)_t, P)$, where W is a Brownian motion with respect to $\mathcal{G}$, the processes $(b(X_t,t))_t$, $(\sigma(X_t,t))_t$ are in $\Lambda^1([0,T])$, $\Lambda^2([0,T])$ resp. and, for every t, it holds

$$X_t = X_0 + \int_0^t b(X_r,r)dr + \sum_k \int_0^t \sigma_k(X_r,r)dW_r^k.$$

b and σ are called resp. drift and diffusion coefficient.

Definition 89. *A weak solution is a strong solution if X is adapted to the Brownian completed filtration $(\mathcal{F}_t)_t$.*

Definition 90. *The SDE has strong uniqueness if every two weak solutions of the SDE with the same initial datum coincide.*

In the following, $(\mathcal{F}_{s,t})_{0 \leq s \leq t}$ will be a double-indices filtration, that is a family of σ-algebras with $\mathcal{F}_{s,t} \subseteq \mathcal{F}_{s',t'}$ for every $0 \leq s' \leq s \leq t \leq t'$.

Definition 91. *A stochastic flow of C^k-diffeomorphisms is a measurable map $\Phi : \{(s,t) | 0 \leq s \leq t \leq T\} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that, for every s and x in $\mathbb{R}^d$, the process $(\Phi(s,t,x,\cdot))_{t \in [s,T]}$ is adapted to $(\mathcal{F}_{s,t})_{t \in [s,T]}$ and, for P-a.e. ω in Ω the following properties hold:*

1. *for every $0 \leq s \leq t \leq u \leq T$, we have $\Phi(s,u,\cdot,\omega) = \Phi(t,u,\cdot,\omega) \circ \Phi(s,t,\cdot,\omega)$;*
2. *for every s in $[0,T]$, we have $\Phi(s,s,\cdot,\omega) = id_{\mathbb{R}^d}$;*
3. *for every $0 \leq s \leq t \leq T$, $\Phi(s,t,\cdot,\omega)$ is a C^k-diffeomorphism of $\mathbb{R}^d$;*
4. *for every h in $\mathbb{N}^d$ with $\sup_i h_i \leq k$, $D^h \Phi(\cdot,\cdot,\cdot,\omega)$ is continuous.*

Definition 92. *A stochastic flow Φ of diffeomorphisms is a weak (resp. strong) solution of an SDE if, for all s in $[0,T]$ and x in $\mathbb{R}^d$, $(\Phi(s,t,x,\cdot))_{t \in [s,T]}$ is a weak (resp. strong) solution of the SDE with initial datum x and initial time s.*

Theorem 93. *Suppose b and σ bounded and smooth functions (with values resp. in $\mathbb{R}^d$ and $\ell^2(\mathbb{R}^d)$). Then, for every initial datum x in $\mathbb{R}^d$, the SDE has strong existence and strong uniqueness. Furthermore there exists a (unique) stochastic flow of diffeomorphisms which is a strong solution to the SDE.*

Definition 94. *Let $(P_t)_t$ be a semigroup on $L^2(\mathbb{R}^d)$. A process $(\Omega, (\mathcal{G}_t)_t, X, P)$ is said to be Markov associated to $(P_t)_t$ if, for every s,t and every f in $L^2(\mathbb{R}^d)$, $P[f(X_{t+s}) | \mathcal{G}_t] = P_s f(X_t)$.*

Proposition 95. *In the regular case, every stochastic flow solution of an SDE is a family of Markov processes.*

Appendix B: technical results

4.5 Dirichlet forms

The following exposition is mainly adapted from [7] and [19].

Definition 96. *We are given a measure space $(X, \mathcal{A}, m)$, with m σ-finite. A functional space on X is a subspace H of $L^2(X, \mathcal{A}, m)$, which is an Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_H$, such that $f \wedge 1$ is in H for every f in H and the injection of H in L^2 is continuous.*

Definition 97. *A Dirichlet space is a pair $(\mathcal{E}, H)$, where*

- *H is a functional space on $L^2(X, \mathcal{A}, m)$.*
- *$\mathcal{E}$ is a bilinear form on H;*
- *there exists α in $\mathbb{R}$ suhc that $\mathcal{E} + \alpha\langle \cdot, \cdot \rangle$ is coercive and continuous in H;*
- *for every φ in H, $c > 0$, it holds $\mathcal{E}(\varphi \wedge c, [\varphi - c]^+) \geq 0$.*

Definition 98. *Now suppose X a topological locally compact a base numerabile space, with Borel σ-algebra $\mathcal{A} = \mathcal{B}(X)$. A Dirichlet space $(\mathcal{E}, H)$ is called:*

- *regular: if $C_c(X) \cap H$ is dense both in C_c and in H;*
- *local: if, for every φ, ψ in H with disjoint supports, $\mathcal{E}(\varphi, \psi) = 0$.*

We will always deal with regular local Dirichlet forms.

Definition 99. *A semigroup $(P_t)_t$ on $L^2(X)$ is sub-Markovian if, for every f in L^2 with $0 \leq f \leq 1$, it holds $0 \leq P_t f \leq 1$. It is Markovian if it is sub-Markovian and conservative (i.e. $P_t 1 = 1$).*

The conservative condition makes sense also if m is not finite, because P_t can be extended to positive functions by the sub-Markovian condition.

Definition 100. *Let A be an unbounded operator on $L^2(X)$, with domain $\mathcal{D}(A)$. We say that $(\mathcal{E}, H)$ is associated with A if $\mathcal{D}(A)$ is contained in H and, for every φ in $\mathcal{D}(A)$, ψ in H, $\mathcal{E}(\varphi, \psi) = \langle -A\varphi, \psi \rangle$.*

Theorem 101. *Let $(\mathcal{E}, H)$ a regular Dirichlet form. Then there exists a unique sub-Markovian semigroup $(P_t)_t$ with infinitesimal generator A associated to $(\mathcal{E}, H)$. Besides, for every $t > 0$, P_t maps L^2 into $\mathcal{D}(A)$.*

Proposition 102. *Suppose that the dual form $\mathcal{E}$ is a (regular) Dirichlet form. Then $(P_t)_t$ extends to a family of L^1 operators with norms $\|P_t\|_{L^1} \leq 1$.*

Theorem 103. *Let $(\mathcal{E}, H)$ and $(P_t)_t$ as above. Suppose local and conservative conditions. Then there exist a Markov process associated to $(P_t)_t$ and with state space a set G with $\mathcal{L}^d(G^c) = 0$.*

In many cases (if not all), this process is continuous.

Proposition 104. *Let A be the differential operator defined on $\mathcal{D}(A) = W^{2,2}(\mathbb{R}^d)$ by*

$$A\varphi = \frac{1}{2}\mathrm{tr}(aD^2\varphi) + b \cdot \nabla\varphi + c\varphi. \tag{4.30}$$

Suppose: a is a field of symmetric matrices, uniformly continuous, buonded, unformly elliptic ($v \cdot av \geq \nu|v|$ for every v in $\mathbb{R}^d$); b is a bounded vector field; c is a bounded measurable function with $c \leq 0$. For φ in $W^{2,2}(\mathbb{R}^d)$, ψ in $W^{1,2}(\mathbb{R}^d)$, define

$$\mathcal{E}(\varphi, \psi) = \langle -A\varphi, \psi \rangle. \tag{4.31}$$

Then the natural extension of $\mathcal{E}$ to $H = W^{1,2}(\mathbb{R}^d)$ gives a regular local Dirichlet form. Furthermore, if $c = 0$, the associated semigroup is Markovian; if $c - \mathrm{div} b \leq 0$ (in distributional sense), the dual form is a Dirichlet form.

Many results about semigroups can be applied also to generators A such that $A - \lambda I$ is associated to a Dirichlet form for some real λ: indeed, if $(Q_t)_t$ is the semigroup with generator $A - \lambda I$, then $(P_t = e^{\lambda t} Q_t)_t$ is the semigropu with generator A.

4.6 Convergence of (random) measures

The following results are classical or taken from [8].

Theorem 105 (Prokhorov)**.** *Let E be a Polish space. A sequence of measures $(\mu_n)_n$ on E is tight (i.e., for every $\epsilon > 0$, there exists a compact set K_ϵ such that, for every n, $\mu_n(K_\epsilon^c) < \epsilon$) if and only if, possibly passing to a subsequence, μ_n converges weakly to a measure μ on E (i.e., for every f in $C_b(E)$, $\mu_n[f]$ tends to $\mu[f]$).*

Proposition 106. *A sequence* $(\mu_n)_n$ *is tight if and only if there exists a coercive positive functional* Ψ *on* E *which verifies* $\mu_n[\Psi] \leq C$ *for every* n.

Definition 107. *Given a Polish space* E*, a random measure on* E *is a family of probability measures on* E $(\mu^\omega)_\omega$*, parametrized by* ω *in* Ω*, such that, for every* f *in* $C_b(E)$*, the function* $\mu[f]$ *is a r.v. on* $(\Omega\,\mathcal{A}, P)$.

Definition 108. *We say that a sequance of random measures* $(\mu_n)_n$ *converges towards a random measure* μ *if, for every* f *in* $C_b(E)$ *and* Z *in* $L^1(\Omega)$*,* $(P[Z\mu_n[f]])_n$ *tends to* $P[Z\mu[f]]$.

This implies the convergence of $\int_\Omega \mu_n^\omega(F(\cdot,\omega))P(d\omega)$ to $\int_\Omega \mu^\omega(F(\cdot,\omega))P(d\omega)$ for every F in $C_b(E; L^1(\Omega))$.

Theorem 109. *Let* $(X_n)_n$ *be a sequence of r.v.'s, with values in a Polish space* E*, call* ρ_n *the law of* X_n*,* n *positive integer. Suppose* (ρ_n) *is tight. Then, possibly passing to a subsequence, the random measures* $(\delta_{X_n(\omega)})_\omega$ *converges to a random measure* μ*, with* $P[\mu] = \rho$*, where* ρ *is a limit point of* $(\rho_n)_n$.

4.7 Proof of Lemma 14

Proof. For $n \in \mathbb{N}^+$, we take $h = \frac{t}{n}$, $t_i = ih$, $i = 0, ...n$; then we write

$$\langle u_t, \phi_t\rangle - \langle u_0, \phi_0\rangle - \int_0^t \langle u_r, A^*\phi_r\rangle dr - \int_0^t \langle u_r, \frac{\partial \phi_r}{\partial r}\rangle dr =$$
$$= \sum_{i=1}^n \langle u_{t_i}, \phi_{t_i}\rangle - \langle u_{t_{i-1}}, \phi_{t_{i-1}}\rangle - \int_{t_{i-1}}^{t_i} \langle u_r, A^*\phi_r\rangle dr - \int_{t_{i-1}}^{t_i} \langle u_r, \frac{\partial \phi_r}{\partial r}\rangle dr,$$

so that it is enough to show $\langle u_{t_i}, \phi_{t_i}\rangle - \langle u_{t_{i-1}}, \phi_{t_{i-1}}\rangle - \int_{t_{i-1}}^{t_i} \langle u_r, A^*\phi_r\rangle dr - \int_{t_{i-1}}^{t_i} \langle u_r, \frac{\partial \phi_r}{\partial r}\rangle dr = o(h)$ uniformly in i as $h \to 0$.

Using equation (1.11) for u and identity $\phi_{t_i} - \phi_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \frac{\partial \phi_r}{\partial t} dr$, we have

$$\langle u_{t_i}, \phi_{t_i}\rangle - \langle u_{t_{i-1}}, \phi_{t_{i-1}}\rangle - \int_{t_{i-1}}^{t_i} \langle u_r, A^*\phi_r\rangle dr - \int_{t_{i-1}}^{t_i} \langle u_r, \frac{\partial \phi_r}{\partial r}\rangle dr =$$
$$= \int_{t_{i-1}}^{t_i} \langle u_{t_i} - u_r, \frac{\partial \phi_r}{\partial t}\rangle dr + \int_{t_{i-1}}^{t_i} \langle u_r, A^*(\phi_r - \phi_{t_{i-1}})\rangle dr$$

For the first integral, using Hölder inequality and $\int_{t_{i-1}}^s \|\frac{\partial \phi_r}{\partial t}\| dr \leq (s - t_{i-1})\|\frac{\partial \phi}{\partial t}\|$, we have

$$\left|\int_{t_{i-1}}^{t_i} \langle u_{t_i} - u_r, \frac{\partial \phi_r}{\partial t}\rangle dr\right| = \left|\int_{t_{i-1}}^{t_i}\int_r^{t_i} \langle u_s, A^*\frac{\partial \phi_r}{\partial t}\rangle ds dr\right| \leq$$

$$\begin{aligned}
&\leq \int_{t_{i-1}}^{t_i}\int_r^{t_i} \|u_s\|\|A^*\|\|\frac{\partial \phi_r}{\partial t}\|dsdr = \int_{t_{i-1}}^{t_i} \|u_s\|\|A^*\| \left(\int_{t_{i-1}}^{s} \|\frac{\partial \phi_r}{\partial t}\|dr\right) ds \leq \\
&\leq C\int_{t_{i-1}}^{t_i} (s-t_{i-1})\|u_s\|ds \leq C\left(\int_{t_{i-1}}^{t_i} \|u_r\|^2 dr\right)^{1/2} \left(\int_{t_{i-1}}^{t_i} (r-t_{i-1})^2 dr\right)^{1/2} \leq \\
&\leq Ch^{3/2}\left(\int_0^T \|u_r\|^2 dr\right)^{1/2}.
\end{aligned}$$

For the second integral, using Hölder inequality and $\|\phi_r - \phi_{t_{i-1}}\| \leq (r - t_{i-1})\|\frac{\partial \phi}{\partial t}\|$, we have

$$\begin{aligned}
&\left|\int_{t_{i-1}}^{t_i} \langle u_r, A^*(\phi_r - \phi_{t_{i-1}})\rangle dr\right| \leq \int_{t_{i-1}}^{t_i} \|u_r\|\|A^*\|\|\phi_r - \phi_{t_{i-1}}\|dr \leq \\
&\leq C\int_{t_{i-1}}^{t_i} (r-t_{i-1})\|u_r\|dr \leq C\left(\int_{t_{i-1}}^{t_i} \|u_r\|^2 dr\right)^{1/2} \left(\int_{t_{i-1}}^{t_i} (r-t_{i-1})^2 dr\right)^{1/2} \leq \\
&\leq Ch^{3/2}\left(\int_0^T \|u_r\|^2 dr\right)^{1/2}.
\end{aligned}$$

So it holds $\langle u_{t_i}, \phi_{t_i}\rangle - \langle u_{t_{i-1}}, \phi_{t_{i-1}}\rangle - \int_{t_{i-1}}^{t_i}\langle u_r, A^*\phi_r\rangle dr - \int_{t_{i-1}}^{t_i}\langle u_r, \frac{\partial \phi_r}{\partial r}\rangle dr = o(h)$ uniformly in i and we are done. □

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