Università di Pisa

Facoltà di Scienze Matematiche, Fisiche e Naturali

Corso di Laurea Magistrale in Matematica

Tesi di Laurea Magistrale

Lagrangians of Hypergraphs

Candidato
Alessandra Caraceni

Relatore
Prof. Roberto Dvornicich
Università di Pisa

Corelatore
Prof. Imre Leader
University of Cambridge

Controrelatore
Prof. Giovanni Gaiffi
Università di Pisa

Anno Accademico 2010/2011

Contents

About

The central topic of this thesis is the Lagrange function for graphs and hypergraphs, together with its applications to Extremal Graph (and Hypergraph) Theory.

Chapter 1 is a rather generic introduction to the subject of extremal problems for graphs and hypergraphs; it contains all the basic definitions that are relevant to us and anticipates most of the results which will be proved in the chapters that follow. A reader who is already familiar with the setting and main characters of our account may choose to skip such an introduction and refer to the Index of Definitions in case of need; it should be noted, however, that – since this “About” section will only consist of brief descriptions and notes – the fist chapter is where any actual presentation of the content and statement of purpose is to be found.

Chapter 2 introduces Lagrangians, and fully investigates their behavior in the case of graphs. The very first application that we recount was also the first to be found (it dates back to 1963), and consists of a rapid proof of Turán’s Theorem.

Chapter 3 sees some important additions to our set of tools: the interplay between Lagrangians and blowups is described. The theory developed will enable us to easily obtain a Theorem of Erdős and Stone as a Corollary of Turán’s Theorem; it will then be possible to state, discuss and disprove a famous conjecture of Erdős regarding the so-called *jumps* for hypergraphs, with arguments from Frankl and Röld [7]: this is the central result we shall be dealing with in this work.

Chapter 4 discusses a conjecture about Lagrangians which was proposed by Frankl and Füredi; we write a complete account of the proof given by Talbot for some particular cases, and state some more partial results; to this day, the conjecture in its full generality is still open.

Chapter 5 goes back to the matter of jumps. We feel that our account would not have been complete without due attention being given to a very recent proof (by Baber and Talbot [1]) of the fact that non-trivial jumps for hypergraphs do exist. The method used is of a different nature than those employed in the preceding chapters, and will provide the occasion for a (partial) introduction to Razborov’s *flag algebras*.

Notation is usually defined within the text at the times when it is first needed; however, since we are aware that some symbols are not universally adopted in the literature, we refer the reader to the table of Appendix A, which contains a list of the symbols used.

Each chapter (save for the first and fifth) is preceded by the drawing of a graph which depicts all important statements from the chapter in question, pairwise connected by an edge whenever one is explicitly invoked in the proof of another; generally, the result being invoked is drawn in a higher position on the page.

We have also included labels with summarized statements which are placed upon the nodes: we warn the reader that such summaries are not meant to be entirely accurate, and that they very often lack some hypotheses or definitions which are present in the text: they are meant as quick reminders, so that the the graphs may be comfortably adopted as maps to guide the reader through the chapter, hopefully providing a broader perspective on the structure of the proofs whenever the need for it may be felt.

Now that we have introduced all the tools provided for a safe and comfortable navigation though this work, nothing is left but to wish the reader a pleasant journey.

CHAPTER 1

An Introduction to Turán Type Problems

where we learn about graphs and hypergraphs, ask many questions, provide a few answers, and thus make our very first steps into the realm of Extremal Graph and Hypergraph Theory.

> Albert grunted. "Do you know what happens to lads who ask too many questions?"
> Mort thought for a moment.
> "No," he said eventually, "what?"
> There was silence.
> Then Albert straightened up and said, "Damned if I know. Probably they get answers, and serve 'em right."
>
> *Mort*,
> Terry Pratchett

1.1 Extremal Problems for Graphs

A *graph* G is a couple $(V(G), E(G))$, where $V(G)$ is a finite set which we will sometimes refer to as the *ground set* of the graph, and whose elements we call *vertices* or *nodes*; $E(G)$, the set of *edges* of G, is a subset of $V(E)^{(2)}$ (the unordered pairs of vertices). Generally speaking, the ground set of a graph may be any finite set: whenever there exists a bijection between $V(G)$ and another set V', such that the induced bijection between $V(G)^{(2)}$ and $V'^{(2)}$ sends $E(G)$ to some E', the graphs G and $G' = (V', E')$ are *isomorphic*; we will, however, from now on intentionally confuse graphs with their isomorphism classes, thus treating G and G' as effectively *the same graph*. We take advantage of this by usually identifying the ground set of a graph with an initial segment of the positive integers, and systematically writing $[n]$ for the ground set of a graph on n vertices.

There are many kinds of questions we may be tempted to ask while investigating the structural properties of graphs. One first concept we are compelled to introduce, as with most mathematical structures, is an appropriate notion of a substructure: namely, a *subgraph*. A subgraph of a graph G is defined as a graph G' with $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G) \cap V(G')^{(2)}$. We will write $G' \subset G$ meaning G' *is a subgraph of* G.

Alongside that of subgraph, we shall sometimes consider the notion of *induced subgraph*. Given a subset W of the set of vertices of a graph G, *the induced subgraph on* W is the graph $(W, E(G) \cap W^{(2)})$; we will sometimes denote such a graph by $G[W]$. To avoid any future confusion, we draw the reader's attention to the obvious difference between the two objects, and to the fact that – with the given definitions – subgraphs need not be induced.

Armed with only those basic concepts, we can nevertheless entertain ourselves with a number of very natural (and not necessarily easy) problems.

Given a graph G, can we gain any information on its subgraphs by merely counting its vertices and edges? Clearly, fixing the number of vertices and increasing the number of edges forces the appearance of more and more subgraphs, until all of the graphs on $|V(G)|$ vertices or less do appear as subgraphs of G (this happens in the trivial case where G is a *complete graph*, *i.e.* a graph with all possible edges; K_n shall be our symbol for a complete graph on n vertices).

Let us be more specific, in hopes of obtaining some precise results.

We take a small graph, say the triangle – K_3 – and ask ourselves

> ? *At most how many edges can a graph on n vertices have, if K_3 is not present as its subgraph?*

This question is answered by Mantel in [16]: graphs on $[n]$ with no triangles are bipartite (a graph is *bipartite* if the ground set is $A \cup B$, and the set of edges is disjoint from $A^{(2)} \cup B^{(2)}$); thus the maximum possible number of edges is attained by the graph which is complete bipartite under a partition of its ground set into two parts of $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices.

If we write $ex(n, K_3)$ for the maximum number of edges in a graph on n vertices which is "triangle-free", we thus have

Theorem 1.1 (Mantel). $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$

There are many proofs we could exhibit for this result, some of which quite short. We will, however, carry on and dare to ask ourselves a more general question: could we solve the problem in the case of any graph F in place of K_3? How would the answer depend on F? In our notation,

? *Given a graph F, what is $ex(n,F)$?*

This question turns out to be way more difficult than we might have been expecting; in general, we are not able to compute $ex(n,F)$ exactly. One result we can obtain is the classic theorem of Turán:

Theorem 1.2 (Turán)**.** *If the positive integer $t-1$ divides the positive integer n, then*

$$ex(n,K_t) = \left(1 - \frac{1}{t-1}\right)\frac{n^2}{2}$$

and the only extremal example (i.e. *the only K_t-free graph on n vertices with $ex(n,K_t)$ edges) is the complete $(t-1)$-partite graph with each part consisting of $\frac{n}{t-1}$ vertices, sometimes called the* Turán graph $T(n,t-1)$.

Notice how Turán's Theorem naturally extends Mantel's: extremal examples are now $(t-1)$-partite; the general case of n not being divisible by t is a very unremarkable extension of the Theorem as stated above: extremal examples are still complete $(t-1)$-partite, with parts of either $\lfloor \frac{n}{t-1} \rfloor$ or $\lceil \frac{n}{t-1} \rceil$ vertices, hence $ex(n,K_t)$ is easily computed.

In memory of Pál Turán, instances of the general question we stated earlier are often referred to as *Turán type problems*.

Since exactly computing extremal numbers $ex(\cdot,F)$ turns out to be very difficult in general, the next step we take is to investigate their behavior when we indefinitely enlarge the ground set: we consider

$$\pi(n,F) = \frac{ex(n,F)}{\binom{n}{2}},$$

that is the *edge density* of an extremal example for an F-free graph on $[n]$.

The sequence $\pi(n,F)$ is decreasing in n. This can be established as an immediate consequence of an easy averaging argument (found in Katona et altera [12]) which we do write explicitly (in little more generality than needed here), since we shall sometimes exploit more elaborate versions of the same kind of reasoning.

Lemma 1.3. *Let n and m be positive integers such that $n \geq m$. The edge density $d(G)$ of a graph G on $[n]$ is the average edge density of its induced subgraphs $G[W]$ with $|W| = m$. As a consequence, a graph G on $[n]$ with edge density at least d has an induced subgraph on m vertices which has edge density at least d.*

Proof. Let d' be the average edge density of all induced subgraphs $G[W]$ with $|W| = m$. Then we can compute the edge density of G as

$$d(G) = \frac{|E(G)|}{\binom{n}{2}} = \frac{\sum_{e\in E(G)} 1}{\binom{n}{2}} = \frac{\sum_{W\in[n]^{(m)}} \sum_{e\in E(G[W])} 1}{\binom{n}{2}\binom{n-2}{m-2}} = \frac{1}{\binom{n}{m}} \sum_{W\in[n]^{(m)}} \frac{|E(G[W])|}{\binom{m}{2}} = d'$$

since
$$\binom{n}{2}\binom{n-2}{m-2} = \frac{n!(n-2)!}{2(n-2)!(m-2)!(n-m)!} = \binom{n}{m}\binom{m}{2}.$$
□

Thus a graph on $[n+1]$ with edge density $d > \pi(n, F)$ has a subgraph on n vertices which is not F-free: that is, $\pi(n+1, F) \leq \pi(n, F)$. Consequently, there exists the limit
$$\pi(F) = \lim_{n \longrightarrow \infty} \pi(n, F)$$
which we call *Turán density* of F.

In terms of densities, Turán's Theorem tells us that
$$\pi(K_t) = 1 - \frac{1}{t-1}$$
and this result we are able to extend: Erdős and Stone [5] proved that

Theorem 1.4 (Erdős-Stone). *If, for an integer $t > 1$, a graph F has chromatic number t (*i.e. *it is t-partite but not $(t-1)$-partite), then*
$$\pi(F) = \pi(K_t) = 1 - \frac{1}{t-1}.$$

All of the results stated so far will be proved in due time in the course of this thesis: we postpone the proofs since we intend to use these classic theorems to gauge the effectiveness of the more modern tools to be introduced in the next chapers.

What we will do now is conclude this section with a few further observations.

The theorem of Erdős and Stone, one might argue, does not answer our original question as fully as we might have wanted: in the case of F being bipartite, in particular, the only information we have collected about its extremal numbers is
$$ex(n, F) = o(n^2).$$

The computation of $ex(n, F)$ still seems to be out of our possibilities (in fact, though we do have some exact results – see the survey [13] for some – Turán type problems are thought to be very hard and are still for the most part unsolved).

What we do have, though, is a rather complete global picture, and a good amount of information about Turán densities. One thing we are immediately able to notice is that the range of possible Turán densities is a discrete set, namely
$$\left\{1 - \frac{1}{r} \mid r \in \mathbb{Z}^+\right\} \subset [0, 1).$$

1.2 Hypergraphs

This thesis will be concerned with hypergraphs rather than graphs. Since the reader might not be as familiar with the former as with the latter, we devote this section to a brief presentation of hypergraphs as combinatorial objects, in hopes of increasing familiarity with the concept and providing a few tools for visualization, which we will occasionally make use of throughout this work.

Hypergraphs are very general objects:

Definition 1.1. A hypergraph G is a couple $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a subset of $\mathcal{P}(V(G))$ (where by $\mathcal{P}(X)$ we denote the *power set* of X).

We are looking at a hypergraph as a *set system*: a family of subsets of $[n]$.

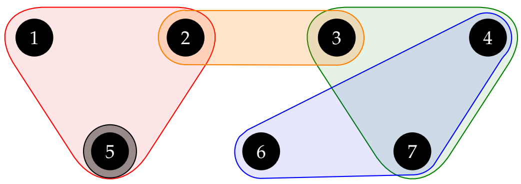


Figure 1.1: A hypergraph on [7].

What we are dealing with is indeed a generalization of a graph, and most terminology can be inherited without much ado: in the example from Figure 1.1, the sets $\{1,2,5\}$, $\{2,3\}$, $\{3,4,7\}$, $\{6,4,7\}$ are *hyperedges*.

The hypergraph $(\{3,4,6,7\}, \{347, 467\})$, for example, is a *subhypergraph* of the original one (furthermore, it is *induced* by its vertex set) – notice we started denoting hyperedges by the juxtaposed symbols for the vertices involved, thus dropping the rather cumbersome "set" notation.

Another way we could look at the same objects is as subsets of the discrete hypercubes: a hypergraph is nothing but a set of points in $\{0,1\}^n$, where n is the cardinality of the ground set: simply identify each hyperedge with the vector having the i-th component equal to either zero (if the i-th vertex is not part of the hyperedge) or 1 (if it is).

We shall rarely adopt this point of view since it does not particularly help immediate visualization: it does, however, bring concepts into a rather more geometrical light, and give rise to quite a few applications of hypergraphs in discrete geometry.

A tool we will be using more often is that of *incidence structures*.

An *incidence structure* is a triple (P, L, I) where $I \subseteq P \times L$; P is interpreted as a set of *points*, L a set of *lines*, and we read "$(p,l) \in I$" as "*point* p *lies on line* l", so that I is the set of *incidences* between points and lines, often called *flags*.

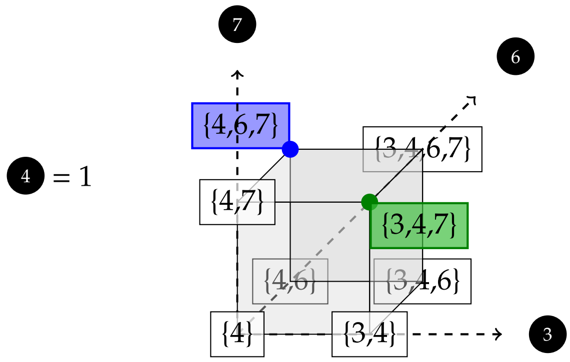


Figure 1.2: The induced hypergraph on $\{3, 4, 6, 7\}$ (from Figure 1.1) seen as projected on the cube obtained by setting the coordinate for vertex 4 equal to 1.

Of course, nothing keeps us from interpreting a hypergraph H as an incidence structure: let the ground set $V(H)$ be the set of points, the edge set $E(H)$ be the set of lines, and let the couple $(v, e) \in V(H) \times E(H)$ belong to the set I of flags if and only if $v \in e$. Drawing the corresponding incidence structure (as a set of points and curves on the plane), though admittedly rather similar to drawing a set system as we did earlier, can be a rather neater way of visually organizing the information that makes up a hypergraph. We might draw the example from Figure 1.1 as:

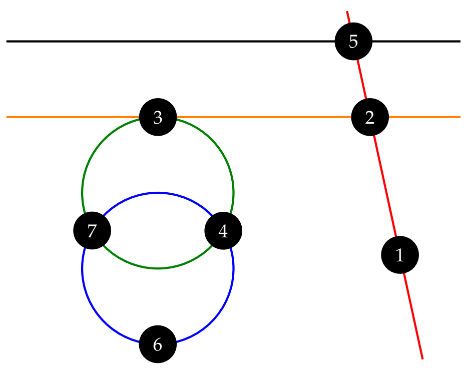


Figure 1.3: The same graph as in Figure 1.1 visualized as an incidence structure.

A graph is but a hypergraph whose (hyper)edges each have exactly 2 elements. In a similar way we may consider k-uniform hypergraphs: hypergraphs whose hyperedges are k-sets (in our previous notation, hypergraphs of the form (V, E) with $E \subseteq V^{(k)}$).

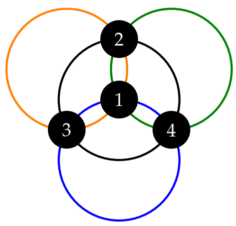


Figure 1.4: $K_4^{(3)}$ (the tetrahedron).

Since those are the objects we shall be dealing with from now on, we lighten our terminology by simply calling them *k-graphs* (thus a graph is a 2-graph) and writing "subgraph" for "subhypergraph", "edge" for "hyperedge", etc.

Again, we generally speak of hypergraphs and k-graphs as *isomorphism classes* rather than specific instances: thus $K_n^{(k)}$ (where $k \leq n$) will be our notation for the *complete k-graph on n vertices*.

Now that we have the necessary language we may finally consider extremal problems for hypergraphs rather than graphs.

1.3 Extremal Problems for Hypergraphs

What we intend to do is try our luck with the same questions we asked about graphs, this time in the setting of general k-graphs. Let us start by adapting the ones we were able to answer more readily, thus asking

? *At most how many edges does a $K_t^{(k)}$-free k-graph on n vertices ($n \geq t \geq k$) have?*

In other words, what is $ex(n, K_t^{(k)})$?

This time we're in for disappointment: no exact values for $ex(n, K_t^{(k)})$, with $n > t > k > 2$, are known.

Turán himself conjectured an extremal example for $K_4^{(3)}$-free 3-graphs on $[n]$, inspired by Turán 2-graphs: split the ground set into 3 parts V_1, V_2, V_3 of either $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$ vertices. Consider the family E_n of all 3-sets e such that $e \cap V_1 = e \cap V_2 = e \cap V_3 = 1$, or $e \cap V_i = 2$ and $e \cap V_{i+1 \bmod 3} = 1$ for some i. Then

Conjecture 1 (Turán). *The 3-graph $T_{4,n}^{(3)} = ([n], E_n)$ has $ex(n, K_4^{(3)})$ edges; thus, if $3\,|\,n$,*

$$ex(n, K_4^{(3)}) = \binom{n/3}{2} n + (n/3)^3 \,.$$

If this conjecture ever turned out to be true, however, it would not only establish similarities between the 2-graph and general k-graph Turán problems. In fact, since Brown [3] and Kostochka [15] subsequently found a number of $K_4^{(3)}$-free 3-graphs on $[n]$ with exactly $|E(T_{4,n}^{(3)})|$ edges, a prominent aspect of the solution in the 2-graph case – namely, the uniqueness of the extremal example – would necessarily fall apart for $k = 3$.

This is no small matter. The existence of a unique extremal example (up to isomorphisms), together with the fact that "near-optimal" solutions tend to be very "similar" to the extremal graph, give the problem a property that we might call *stability*, which has recently been exploited even in the hypergraph case to prove some of the very few exact results we possess at this time. We shall not concern ourselves with such methods here, but the reader may be interested in [9] and [14], where the problem of computing extremal numbers for the *fano planes* is solved for $k = 3, 4$.

In the same way as with 2-graphs, we can define the Turán density of a k-graph F.

The proof that

$$\frac{ex(n+1, F)}{\binom{n+1}{k}} \leq \frac{ex(n, F)}{\binom{n}{k}}$$

is the same as that we discussed in the Section 1.1: substituting k for 2 in the statement and proof of Lemma 1.3 yields corresponding results for k-uniform hypergraphs.

As before, we take

$$\pi(F) = \lim_{n\to\infty} \frac{ex(n, F)}{\binom{n}{k}}$$

but again, very little is known about $\pi(F)$ when $k \geq 3$, with the exception of very few specific graphs. In fact, the problem is considered to be extremely difficult in general.

Going back to Conjecture 1, the value it predicts for $\pi(K_4^{(3)})$ is easily computed by taking a limit, and amounts to 5/9. Even this has not been confirmed: the best available bound at this time is that of [19], obtained by means of a method that Razborov introduced in [18], based on the concept of *flag algebras*, which we will discuss in some detail in Chapter 5.

When Turán densities for k-graphs were first considered, similarities and differences with the $k = 2$ case were explored. Erdős, in investigating the graph case and discovering the discrete structure of the set of graph Turán densities, remarked the following property of 2-graphs (see Section 3.4): for any $\alpha \in [0, 1)$ there is $c > 0$ such that, for any integer $m \geq 2$ and any $\epsilon > 0$, a graph G with density at least $\alpha + \epsilon$ will – provided its ground set is larger than some $n(m, \epsilon)$ – have a subgraph on m vertices with density at least $\alpha + c$.

He then asked if the same property is true for general k-graphs, making the well-known "jumping" conjecture (see [4]).

The positive conjecture has been disproved for $k \geq 3$. We will go through a complete proof of the fact that "hypergraphs do not jump" in Sections 3.4 through 3.6: the proof was originally developed by Frankl and Rödl [7] and heavily relies on the concept of Lagrangian of a hypergraph, to which Chapters 2 to 4 will be mostly devoted.

Hypergraph Lagrangians have been introduced for 2-graphs by Motzkin and Straus, and have subsequently found very interesting and valuable uses in extremal hypergraph theory; they seem to measure how "tightly packed" a subgraph it is possible to find for a given hypergraph (such a concept will be formalized in due time, verified for 2-graphs in Chapter 2 and generally discussed in Chapter 4), and relate to many important properties of hypergraphs. Chapter 3 will explain their close relationship with jumps and with the "jump conjecture" of Erdős.

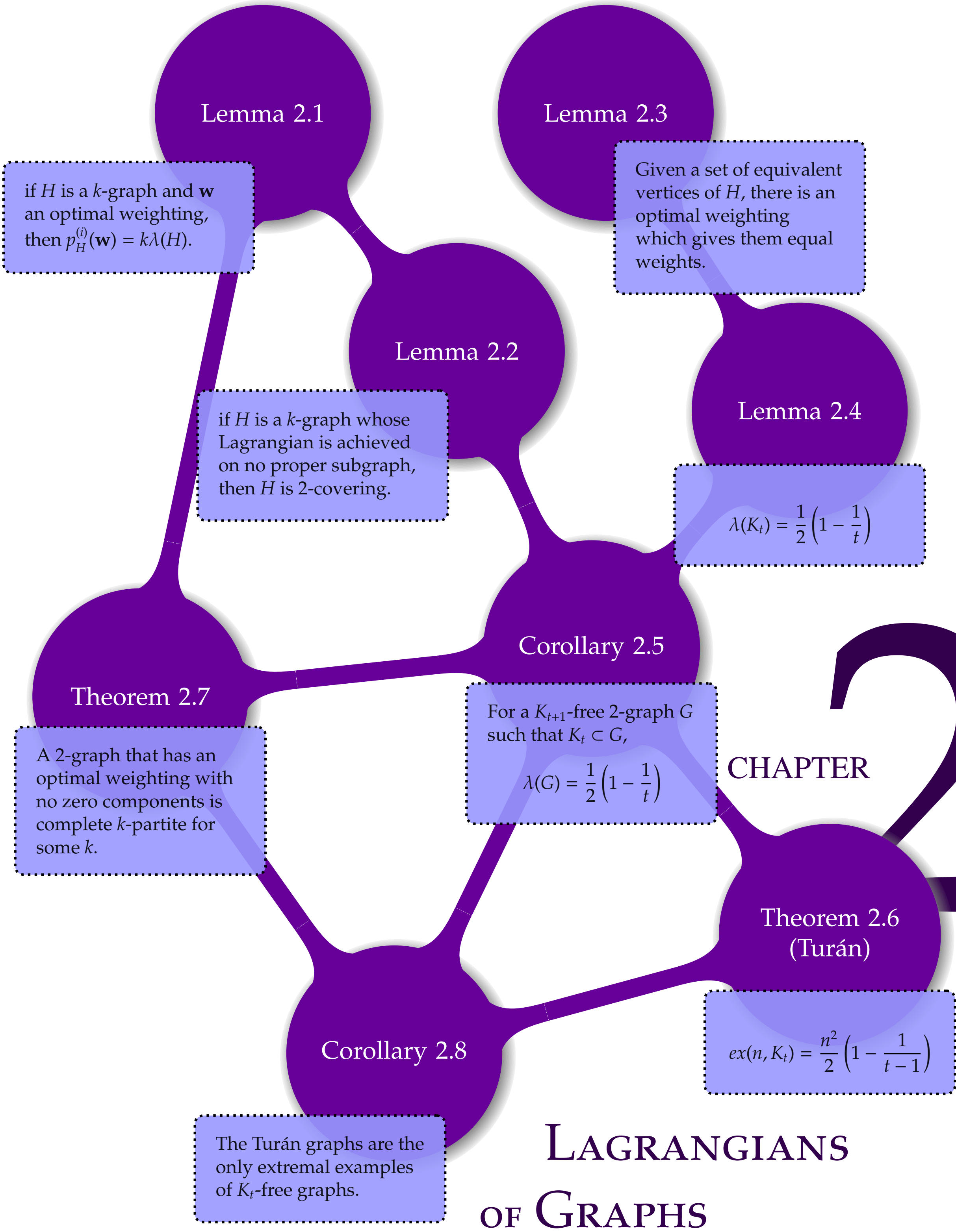
Though quite a few years later than the first examples of "non-jumps" (all of which were exhibited thanks to arguments similar to those of Frankl and Rödl), "jumps" for $k = 3$ were found by means of *flag algebras*: see Chapter 5 for an account of Baber and Talbot's proof [1].

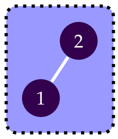
Flag algebras were first introduced in 2007: they provide a way of formalizing rather complex counting arguments, and have recently been employed to transform Turán type questions into instances of problems which are computationally treatable by means of semidefinite programming techniques. They have given rise to some of the latest and most accurate estimates for Turán densities, as well as some exact results [19]; also, they may still prove useful in ways which have not yet been fully investigated (for an account of flag algebras and their application to extremal hypergraph theory, see Sections 5.2 and 5.4).

Still, how jumps and non-jumps are distributed, and even whether the first nonzero Turán density is a jump or not (see Section 3.4) are entirely open questions: our understanding of hypergraph Turán problems is, on the whole, extremely partial.

CHAPTER 2

Lagrangians of Graphs and Hypergraphs



CHAPTER 2 

Lagrangians of Graphs and Hypergraphs

where we first meet hypergraph Lagrangians, and put them to good use, obtaining a new proof of Turán's Theorem by Motzkin and Straus [17].

2.1 The Lagrangian of a Hypergraph

Consider a k-graph H on $[n]$ with edge set E. A *weighting* of H is a map

$$w : [n] \longrightarrow \mathbb{R}_{\geq 0}$$

such that $\sum_{i \in [n]} w(i) = 1$.

The *Lagrange polynomial* of H is $p_H(\mathbf{x}) \in \mathbb{Z}[x_1, ..., x_n]$ defined as

$$p_H(\mathbf{x}) = \sum_{e \in E} \prod_{i \in e} x_i \, .$$

Thus Lagrange polynomials are homogeneous of degree k in the case of k-graphs, hence quadratic forms for of 2-graphs; $p_{K_t^{(k)}}$, for example, is the elementary symmetric homogeneous polynomial of degree k in t variables.

Write $\mathbf{w}$ for the vector $(w(1), \dots, w(n))$ in $\mathbb{R}^n$ and define the Lagrangian of H as

$$\lambda(H) = \sup_w p_H(\mathbf{w})$$

where the sup is taken over all weightings of H.

In fact, it is immediately clear that we can consider the maximum in place of the supremum, since the latter is taken on the standard simplex of $\mathbb{R}^n$ (which is compact) and the expression being evaluated is a polynomial.

We call a weighting w such that $\lambda(H) = p_H(\mathbf{w})$ an *optimal weighting* for H.

We now make a few general remarks about Lagrangians which will be extremely useful for the applications to be discussed in the next sections.

Let $H = ([n], E(H))$ be a k-graph, p_H its Lagrange polynomial, w an optimal weighting for H. Let $W \subseteq [n]$ be the set $\{i \in [n] \mid w(i) \neq 0\}$ and consider $H[W]$; without loss of generality assume $W = [h]$ for some $h \leq n$. We denote by $p^{(i)}$ the derivative of the polynomial p with respect to the variable x_i.

Lemma 2.1. *For each i in W, $p_H^{(i)}(\mathbf{w}) = k\lambda(H)$.*

Proof. Firstly, observe that $p_H^{(i)}(\mathbf{w}) = p_{H[W]}^{(i)}(\mathbf{w})$: each monomial appearing in $p_H(\mathbf{x}) - p_{H[W]}(\mathbf{x})$ has degree 1 in a variable x_j such that $j \notin W$. Differentiating with respect to x_i (since i is in W) yields a polynomial with the same property. Thus, as $w(j) = 0$ for all j not in W, evaluating in $\mathbf{w}$ finally yields 0.

We may then assume, without loss of generality, that $W = [n]$ and $H = H[W]$.

Consider the rational function $\frac{p_H}{s^k}$, where $s(\mathbf{x}) = x_1 + \cdots + x_n$. Since $W = [n]$, such a function attains a maximum at $\mathbf{w}$; thus

$$\left(\frac{p_H}{s^k}\right)^{(i)}(\mathbf{w}) = 0$$

which yields

$$p_H^{(i)}(\mathbf{w}) = p_H(\mathbf{w}) k s^{k-1}(\mathbf{w}) = k\lambda(H)\,.$$

□

Now let w be an optimal weighting such that $|W|$ is minimal. Then we have

Lemma 2.2. *The hypergraph $H[W]$ covers pairs: that is, for all $i, j \in W$ there is an edge e of $H[W]$ such that $\{i, j\} \subseteq e$.*

Proof. Take $i < j$ in W and suppose there are no edges involving both i and j.

Then

$$p_H(\mathbf{x}) = x_i p_H^{(i)}(\mathbf{x}) + x_j p_H^{(j)}(\mathbf{x})\,,$$

and $p_H^{(i)}$ and $p_H^{(j)}$ have degree 0 in both x_i and x_j.

Thus, denoting by $\mathbf{e}_i$ the vector having the i-th component equal to 1 and all others to 0,

$$p_H(\mathbf{w} + w(j)(\mathbf{e}_i - \mathbf{e}_j)) = p_H(\mathbf{w}) + w(j)(p_H^{(i)}(\mathbf{w}) - p_H^{(j)}(\mathbf{w})) = p_H(\mathbf{w}) = \lambda(H)$$

by Lemma 2.1.

Since $\mathbf{w} + w(j)(\mathbf{e}_i - \mathbf{e}_j) = (\ldots, w(i) + w(j), \ldots, 0, \ldots)$ is still a weighting and has only $|W| - 1$ nonzero components in spite of being optimal, by minimality of $|W|$ we get a contradiction. □

Consider two vertices i, j in $[n]$; we call them *equivalent* if, for each $e \in ([n] \setminus \{i, j\})^{(k-1)}$, $i \cup e \in E(H)$ if and only if $j \cup e \in E(H)$.

Thus the two vertices are equivalent if the permutation of $[n]$ that exchanges i and j and fixes all other vertices, together with the induced map on $[n]^{(k)}$, sends the k-graph G to one that is isomorphic to it (or rather to itself, if we are considering isomorphism classes): i and j have the same neighborhoods (if we do not count any edges involving both i and j), thus they play equivalent roles inside the k-graph.

The last general fact that we state is the following, which will prove useful in determining Lagrangians of k-graphs with a high degree of symmetry:

Lemma 2.3. *Let $S = \{i_1, \dots, i_s\} \subseteq [n]$ be a set of s equivalent vertices of H. Then there is a weighting z such that $p_H(\mathbf{z}) = \lambda(H)$ and $z(i_1) = \cdots = z(i_s)$.*

Proof. Let w be an optimal weighting for H and set $\mu = \frac{1}{s}(w(i_1) + \cdots + w(i_s))$.

We make an inductive construction for z: let $z_0 = w$, and suppose we have an optimal weighting z_m ($m < s$) such that

- $z_m(i) = \mu$ for (at least) m elements of S;
- z_m coincides with w on $[n] \setminus S$;
- we still have $\frac{1}{s}(z_m(i_1) + \cdots + z_m(i_s)) = \mu$.

Consider $a, b \in S$ such that $z_m(a) < \mu < z_m(b)$ (if there are no such a, b we set $z_{m+1} = z_m$, and all the above conditions are satisfied for z_{m+1}).

Take $\alpha = \mu - z_m(a)$ and take z_{m+1} so that $\mathbf{z}_{m+1} = \mathbf{z}_m + \alpha(\mathbf{e}_a - \mathbf{e}_b)$. Clearly, z_{m+1} is still a weighting and the cumulative weight of S is unchanged; also, since $z_{m+1}(a) = \mu \neq z_m(a)$, there are at least $m + 1$ elements of S with weight μ.

We show that z_{m+1} is optimal.

By a simple computation

$$p_H(\mathbf{z}_{m+1}) - p_H(\mathbf{z}_m) = \alpha(z_m(b) - z_m(a))p_H^{(a)(b)}(\mathbf{z}_m)$$

which is nonnegative since $\alpha(z_m(b) - z_m(a)) > 0$ and $p_H^{(a)(b)}(\mathbf{z}_m) \geq 0$.

Thus, since $p_H(\mathbf{z}_m) = \lambda(H)$, we get $p_H(\mathbf{z}_{m+1}) \geq \lambda(H)$; by maximality of $\lambda(H)$ equality – *i.e.* optimality for z_{m+1} – follows.

We can now set $z = z_s$, which is optimal and assigns to all vertices in S equal weight μ. □

Before discussing applications of Lagrangians in Graph Theory we give a few examples of how they are actively computed for some very particular k-graphs. The example that first ought to come to mind is the following:

Example 2.1. **Lagrangian of the complete k-graphs.** As we have remarked before, the Lagrange polynomial for the complete k-graph $K_t^{(k)}$ is the elementary symmetric polynomial of degree k in t variables:

$$p_{K_t^{(k)}}(x_1, \ldots, x_t) = \sum_{sym} x_1 \ldots x_k.$$

All vertices of a complete k-graph are equivalent, so by Lemma 2.3 the uniform weighting is optimal. Hence we simply get

$$\lambda(K_t^{(k)}) = p_{K_t^{(k)}}(1/t, \ldots, 1/t) = \binom{t}{k}\frac{1}{t^k}.$$

Example 2.2. **Lagrangian of a path.** Consider another extremely simple case, namely that of paths (2-graphs of the form $P_n = ([n], \{12, 23, \ldots (n-1)n\})$. Their Lagrange polynomials are

$$p_{P_n}(x_1, \ldots, x_n) = x_1x_2 + \cdots + x_{n-1}x_n;$$

This time the Lagrangian may be computed thanks to Lemma 2.2: take an optimal weighting with minimal number of nonzero components; then its support must be 2-covering. This cannot be true if it contains more than 2 vertices (or two non-adjacent ones), hence it must of the form $\{i, i+1\}$ for some i in $[n-1]$. Also, we must have $w(i) = w(i+1) = 1/2$ (by AM-GM), so the Lagrangian of any path is 1/4.

The problem of maximizing the quadratic form p_{P_n} on the standard simplex was, incidentally, the very origin for the use of Lagrangians. Actually, we are now able to compute the Lagrangian of any 2-graph via Lemma 2.2, but a discussion of this fact we postpone to the next section.

Example 2.3. **Lagrangian of K_4^-.** One Lagrangian that will prove useful in the last chapter of this thesis, which we can compute easily enough with the help of Lemma 2.3, is that of the 3-graph on 4 vertices with 3 edges (notice that there is only one such graph up to isomorphisms, which is represented in Figure 2.1).

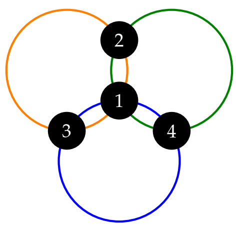


Figure 2.1: K_4^-.

Clearly, we have

$$p_{K_4^-}(\mathbf{x}) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4;$$

notice (either by observing symmetry in the polynomial or in Figure 2.1) that vertices 2, 3, 4 are equivalent. Hence we may look for an optimal weighting such that $w(2) = w(3) = w(4) = \alpha$ and $w(1) = 1 - 3\alpha$.

All we need to do is to maximize

$$p_{K_4^-}(1-\alpha, \alpha, \alpha, \alpha) = 3(1-3\alpha)\alpha^2.$$

This is very easily done, for example by noticing that

$$(1-3\alpha)\frac{3\alpha}{2}\frac{3\alpha}{2} \leq \left(\frac{1}{3}\right)^3$$

by GM-AM, so $\lambda(G) = 4/81$, obtained by setting $\alpha = 2/9$ (so that $1 - 3\alpha = \frac{3\alpha}{2}$).

2.2 Lagrangians and Turán's Theorem

Lagrangians in the case of 2-graphs were first introduced by Motzkin and Straus [17], who were interested in maximizing certain square-free quadratic forms on simplices. They were able to use Lagrangians to easily obtain Turán's Theorem. In fact, Lagrangians have a very simple behavior in the case of graphs, thanks to the fact that "covering pairs" is indeed very restrictive when edges are 2-sets: it is the same as being complete.

Throughout this section and part of the next chapter, we will use Lagrangians of graphs to obtain the classic results from extremal graph theory that we discussed in Section 1.1.

First and foremost restate the result obtained in Example 3.1:

Lemma 2.4.

$$\lambda(K_n) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

Proof. As in Example 3.1, apply Lemma 2.3: $(1/n, \ldots, 1/n)$ is optimal, so

$$\lambda(K_n) = p_{K_n}(1/n, \ldots, 1/n) = \frac{1}{n^2}\binom{n}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

□

We can now compute the Lagrangian for any graph (cfr. Example 2.2). In fact we have

Corollary 2.5. *If G is a K_{t+1}-free graph such that $K_t \subset G$, then*

$$\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

Proof. Let w be an optimal weighting with minimum number of nonzero components and let $W = \{i \in [n] \,|\, w(i) \neq 0\}$. By Lemma 2.2, the graph $G[W]$ is two-covering, hence complete. Since $\lambda(G) = \lambda(G[W])$, the result is proved thanks to Lemma 2.4. □

We can now prove

Theorem 2.6 (Turán). *Suppose n, t are positive integers such that $t-1$ divides n. Then*

$$ex(n, K_t) = \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right)$$

Proof. Suppose the graph G on $[n]$ is K_t-free, hence any complete graph appearing as a subgraph of G has no more than $t-1$ vertices.

By Corollary 2.5, this implies

$$\lambda(G) \leq \frac{1}{2}\left(1 - \frac{1}{t-1}\right).$$

On the other hand, $\lambda(G) \geq p_G(1/n, \ldots, 1/n) = \frac{|E(G)|}{n^2}$. Combining the two gives

$$|E(G)| \leq \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right)$$

hence $ex(n, K_t) \leq \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right)$.

The opposite inequality is obtained by considering the complete $(t-1)$-partite graph on $[n]$ with parts of equal cardinality (the Turán graph $T(n, t-1)$), which is K_t-free and has

$$\binom{t-1}{2}\left(\frac{n}{t-1}\right)^2 = \frac{1}{2}n^2\frac{t}{t-1}$$

edges. □

For the uniqueness part of Turán's theorem we need some information on which graphs can achieve their Lagrangian by giving each of their vertices positive weight. We prove

Theorem 2.7. *Let G be a graph. Suppose w is an optimal weighting for G with no zero components. Then G is a complete k-partite graph for some k.*

Proof. Suppose $\lambda(G) = \frac{1}{2}(1 - \frac{1}{k})$; we will show G is then complete k-partite.

If $k = n = |V(G)|$, then the result is trivial.

We proceed by induction on n, and suppose $k < n$.

Clearly, G cannot be complete, so there are two vertices – without loss of generality assume they are vertices 1 and n – which are not connected by an edge.

Consider $G[[n-1]]$: we show that its Lagrangian (which is the same as $\lambda(G)$ by Corollary 2.5) is also attainable via a weighting with no zero components. By Lemma 2.1, as in the proof of Lemma 2.2,

$$p_G(\mathbf{w} + w(n)(\mathbf{e}_1 - \mathbf{e}_n)) = p_G(\mathbf{w}) + w(n)(p_G^{(1)}(\mathbf{w}) - p_G^{(n)}(\mathbf{w})) = \lambda(G)\,;$$

in other words, the Lagrangian of the subgraph can be obtaining by simply shifting the weight of vertex n onto vertex 1.

By induction hypothesis, $G[[n-1]]$ is then complete k-partite under a partition of the ground set, say $[n-1] = V_1 \cup \cdots \cup V_k$.

We show that there is $i \in [k]$ such that vertex n is not connected to any of the vertices in V_i: suppose this weren't the case; then we would have some $V = \{n, v_1, \ldots, v_k\}$ such that (for $i = 1, \ldots, k$) $v_i \in V_i$ and $v_i n \in E(G)$; clearly, $G[V] = K_{k+1}$, which contradicts our hypothesis about the Lagrangian of G.

Let v be any vertex in V_i. Since by Lemma 2.1

$$p_G^{(n)}(\mathbf{w}) = \sum_{nj \in E(G)} w(j) = p_G^{(v)}(\mathbf{w}) = \sum_{vj \in E(G)} w(j) = \sum_{j \in [n-1] \setminus V_i} w(j)\,,$$

where the values $w(j)$ are nonzero and $\{j \mid nj \in E(G)\} \subseteq [n-1] \setminus V_i$, we are forced to conclude that vertex n is joined to all of the vertices in $[n-1] \setminus V_i$. Hence G is complete k-partite, with vertex n in the same part of the ground set as vertex v. □

Remark 2.1. *Notice that the converse of Theorem 2.7 is also true: since the Lagrangian of a complete k-partite graph on ground set $V_1 \cup \cdots \cup V_k$ is $\frac{1}{2}(1 - \frac{1}{k})$, it can be obtained by means of any weighting w with $\sum_{j \in V_i} w(j) = \frac{1}{k}$.*

We can now prove

Corollary 2.8. *Suppose $(t-1)|n$. The complete $(t-1)$-partite graph on n vertices having parts of equal cardinality is the only K_t-free graph on $[n]$ with $ex(n, K_t)$ edges.*

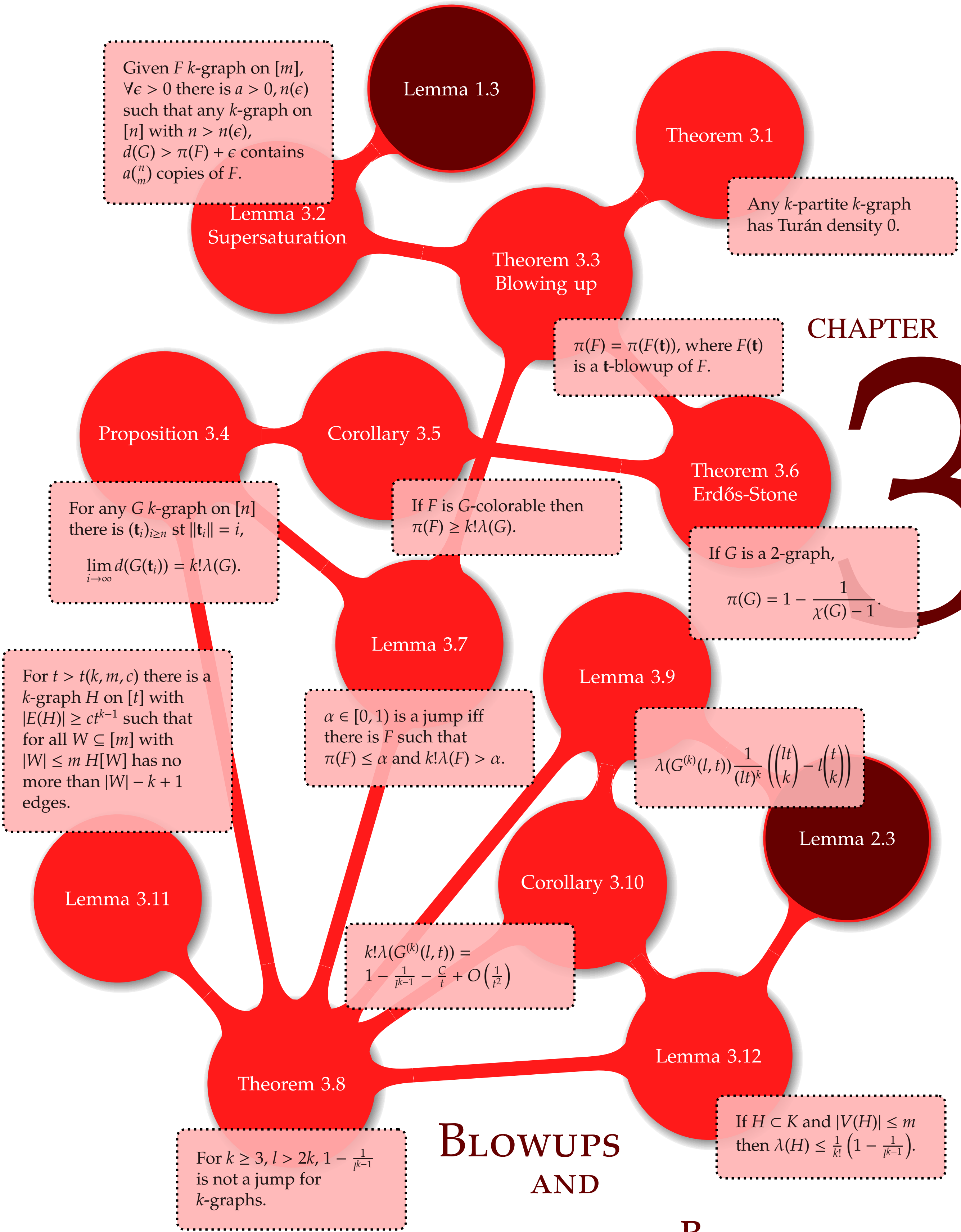
Proof. Let G be a K_t-free graph on $[n]$ with $ex(n, K_t)$ edges; since $ex(n, K_t) > ex(n, K_{t-1})$ we have $K_{t-1} \subset G$, which implies

$$\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{t-1}\right) = \frac{ex(n, K_t)}{n^2} = p_G(1/n, \ldots, 1/n)\,;$$

thus the uniform weighting is optimal, and Theorem 2.7 ensures G is complete and k-partite for some $k < t$; among the graphs on $[n]$ with this property, the one with the most edges is indeed the $(t-1)$-partite graph on equal parts, which is K_t-free. □

CHAPTER 3

Blowups and Jumps



CHAPTER 3

Blowups and Jumps

which is the real core of this work: k-graph homomorphisms and blowups come into play; the Theorem of Erdős and Stone is proved, and we finally get to disprove Erdős's conjecture about jumps, thanks to Frankl and Röld [7].

3.1 Homomorphisms and Blowups

We now go back to the more general setting of k-graphs and introduce some notions that will prove very useful throughout the rest of this work.

Given two k-graphs F, G a *homomorphism* from F to G is a map

$$\varphi : V(F) \longrightarrow V(G)$$

such that, if we still denote by φ the induced map from $V(F)^{(k)}$ to $V(G)^{(\leq k)}$, then we have $\varphi(E(F)) \subseteq E(G)$.

If there exists a homomorphism from F to G, we will say that F *is G-colorable*.

We follow this the definition with a simple but important example.

? *What does it mean for a k-graph F to be $K_t^{(k)}$-colorable?*

Example 3.1. Take a homomorphism $\varphi : F \longrightarrow K_t^{(k)}$, and consider the (disjoint, possibly empty) sets $V_i = \varphi^{-1}(i)$ for $i = 1, \ldots, t$, which make up a t-partition of the ground set of F. Any edge of F cannot involve two vertices from the same V_i, since its image would certainly not be an edge in $K_t^{(k)}$ (it would have cardinality strictly less than k); conversely, any edge involving vertices from k distinct parts is allowed, thanks to $K_t^{(k)}$ being complete.

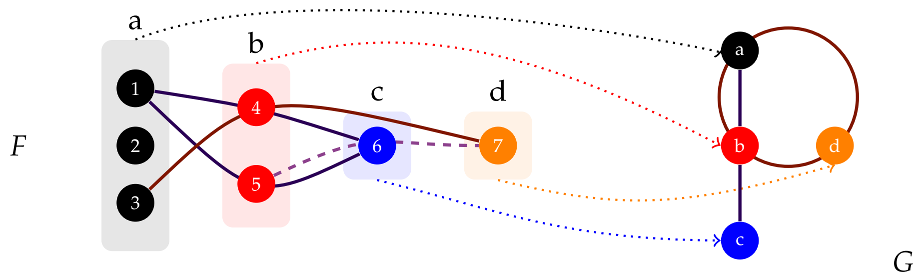


Figure 3.1: The 3-graph $F = ([7], \{146, 156, 347\})$ is G-colorable, where $G = ([4], \{123, 124\})$. The picture shows a G-coloring φ of F. Notice φ is not complete and F is not a blowup of G; also observe φ would not be a G-coloring if 567 (the dashed line) were an edge of F.

We will use the term *t-partite* for any $K_t^{(k)}$-colorable k-graph. Notice the notion of being t-partite for 2-graphs is indeed equivalent to that of being K_t-colorable.

If $\varphi : F \longrightarrow G$ is a homomorphism between k-graphs such that φ is surjective on vertices, and for all $e \in V(F)^{(k)}$

$$e \in E(F) \Leftrightarrow \varphi(e) \in E(G)\,,$$

we call φ a *complete homomorphism*.

Example 3.2. A k-graph F is completely homomorphic to $K_t^{(k)}$ if and only if it is complete t-partite.

If F is completely homomorphic to G, we say F is a *blowup* of G.

In fact, we will now explicitly define an operation called *blowing up*, in a sense the inverse of complete homomorphisms, which is of the utmost importance for the sections to come.

Definition 3.1. Let G be a k-graph on $[n]$, and let $\mathbf{t}$ be a vector in $(\mathbb{Z}^+)^n$. We define the $\mathbf{t}$-blowup of G as the graph $G(\mathbf{t})$ on $t_1 + \cdots + t_n$ vertices such that

$$V(G(\mathbf{t})) = V_1 \cup \cdots \cup V_n$$

with $|V_i| = t_i$, and that the map sending each element of V_i to vertex i in $V(G)$ is a complete homomorphism.

3.2 The Blowing up Theorem

We might expect to be able to relate the Turán density of a k-graph to that of its blowups; thus the first question we ask ourselves is

? *What is $\pi(F(\mathbf{t}))$, where F is a k-graph on $[n]$, $\mathbf{t} \in (\mathbb{Z}^+)^n$, and $F(\mathbf{t})$ is a $\mathbf{t}$-blowup of F?*

In order to be able to answer we first need to introduce two tools. One is a very important – though quite simple – result about the number of copies of a k-graph F which appear in large k-graphs with high enough edge density. The other is a fact proved by Erdős in [5], namely

Theorem 3.1. *Any k-partite k-graph has Turán density 0.*

This we shall not attempt to prove here. Clearly, the graph $K_k^{(k)}$ (the "k-edge") has Turán density 0, since any k-graph with positive edge density does contain a hyperedge. What we are saying is that blowups $K_k^{(k)}(\mathbf{t})$ of $K_k^{(k)}$ (in other words, complete k-partite graphs) also have Turán density 0: whatever $\mathbf{t}$ and $\epsilon > 0$, there is $n(\mathbf{t}, \epsilon)$ such that any k-graph on $n > n(\mathbf{t}, \epsilon)$ vertices with density at least ϵ has $K_k^{(k)}(\mathbf{t})$ as a subgraph.

The other ingredient we need is the following lemma:

Lemma 3.2 (Supersaturation)**.** *Let F be a k-graph on $[m]$, and $\epsilon > 0$ be any positive real number. Then there is $a > 0$ such that for $n > n(\epsilon)$ any k-graph G on $[n]$ with $d(G) > \pi(F) + \epsilon$ contains at least $a\binom{n}{m}$ distinct[1] copies of F.*

Proof. Pick n_0 such that $ex(n_0, F) \leq (\pi(F) + \frac{\epsilon}{2})\binom{n_0}{k}$, and let G be a k-graph on $n > n_0$ vertices with edge density $d(G) > \pi(F) + \epsilon$. Then we shall prove that the fraction of n_0-sets W in $[n]^{(n_0)}$ such that $d(G[W]) > \pi(F) + \frac{\epsilon}{2}$ is at least $\frac{\epsilon}{2}$.

Set
$$I = \left\{ W \in [n]^{(n_0)} \,|\, d(G[W]) > \pi(F) + \frac{\epsilon}{2} \right\} ;$$
then the average density of all induced subgraphs on n_0 vertices is at most
$$\frac{1}{\binom{n}{n_0}} \left[\#I + \left(\binom{n}{n_0} - \#I \right) \left(\pi(F) + \frac{\epsilon}{2} \right) \right]$$
which, if $\#I$ were less than $\frac{\epsilon}{2}\binom{n}{n_0}$, would in turn be (strictly) less than
$$\frac{1}{\binom{n}{n_0}} \left[\frac{\epsilon}{2}\binom{n}{n_0} + \binom{n}{n_0} \left(\pi(F) + \frac{\epsilon}{2} \right) \right] = \pi(F) + \epsilon \, .$$

Since, by Lemma 1.3, the average density of subgraphs induced by n_0-sets is equal to the edge density $d(G)$, which we took to be at least $\pi(F) + \epsilon$, this yields a contradiction.

[1]By *distinct* we mean that they do not share the same ground set.

Now for all $W \in I$, since $|E(G[W])| > ex(n_0, F)$, we must have $F \subset G[W]$. Each copy of F present as a subgraph of G has a vertex set of cardinality m, which appears as a subset of at most $\binom{n-m}{n_0-m}$ n_0-sets in I.

G must then contain at least

$$\frac{\#I}{\binom{n-m}{n_0-m}} \geq \frac{\epsilon\binom{n}{n_0}}{2\binom{n-m}{n_0-m}} = \frac{\epsilon}{2\binom{n_0}{m}}\binom{n}{m}$$

copies of F.

Choosing $a = \frac{\epsilon}{2\binom{n_0}{m}}$ yields the desired result. □

We are now ready to prove the following Theorem (for an example which might improve the readability of the proof, also see Table 3.1 from the next page):

Theorem 3.3 (Blowing up). *For any k-graph F on $[m]$ and any $\mathbf{t} \in (\mathbb{Z}^+)^m$,*

$$\pi(F) = \pi(F(\mathbf{t}))\,.$$

Proof. Let G be a k-graph on $[n]$ such that $d(G) > \pi(F) + \epsilon$. We show that there is $n(\epsilon)$ such that, provided $n > n(\epsilon)$, $F(\mathbf{t}) \subset G$, hence $\pi(F(\mathbf{t})) \leq \pi(F)$ (the opposite inequality is trivial since $F \subset F(\mathbf{t})$).

We know by the Supersaturation Lemma that, for some a not depending on n, as long as $n > n_0(\epsilon)$ there are $a\binom{n}{m}$ distinct copies of F appearing as subgraphs of G.

Define an m-graph H on $[n]$ such that, for each $W \in [n]^{(m)}$, W is in $E(H)$ if and only if $F \subset G[W]$. The preceding remark then ensures $d(H) \geq a$.

By Theorem 3.1, for each $T \in \mathbb{Z}^+$ there is $n_1(a, T) \geq n_0(\epsilon)$ such that $K_m^{(m)}(T, \ldots, T) \subset H$ if $n > n_1(a, T)$; suppose the latter is true, and call K a subgraph of H which is indeed complete m-partite with parts of cardinality T; number the m parts of the ground set of K as $V_1, \ldots, V_m$.

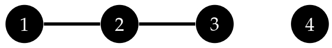
Given an edge W of K, consider the map φ from $[m]$ to W sending i to the vertex of W belonging to V_i and let ψ be the map sending each vertex in W to the corresponding vertex of F. The resulting map $\sigma_W = \psi\varphi : [m] \longrightarrow [m]$ is then a permutation of $[m]$.

Assigning to each edge W its corresponding m-permutation σ_W gives an $(m!)$-coloring of the edges of K.

Let $t = \|\mathbf{t}\|_1$; by a generalized Ramsey's Theorem [10] there is $T(t)$ such that, provided $T > T(t)$, we can find $K_m^{(m)}(\mathbf{t})$ as a monochromatic subgraph of K; let $V_1 \cup \cdots \cup V_m$ be the ground set of such a copy V of $K_m^{(m)}(\mathbf{t})$: then all m-sets whose elements form an m-tuple of $V_1 \times \cdots \times V_m$ are edges of H that are colored the same way, hence they all span copies of F "in the same position"; more precisely, V spans a copy of $F(\mathbf{t})$ as a subgraph of G (see also Table 3.1).

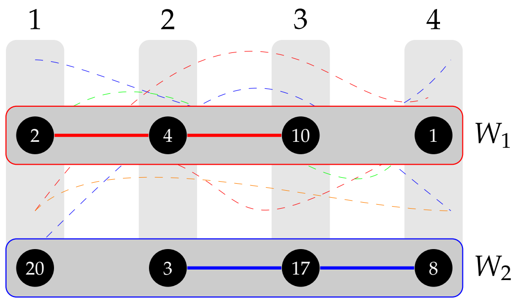
We can then take $n(\epsilon) = n_1(a, T(t))$ (t is constant and a only depends on ϵ), and thus the Theorem is proved. □

Suppose F is the 3-graph $([4], \{123\})$ and we want to find $F(3, 1, 1, 1)$ in some big 3-graph G.



F

We construct the 4-graph H and find a complete 4-partite 4-graph K, with parts of cardinality T for a suitable T, as a subgraph of H. We then color each 4-edge W of K in one of 4 colors (thanks to the high simmery of F we don't need 4!), depending on which 3 vertices span the 3-edge in the copy of F represented by W.



Find a monochromatic subgraph of K isomorphic to $K_4^{(4)}(3, 1, 1, 1)$; its vertex set will then span a copy of $F(3, 1, 1, 1)$.

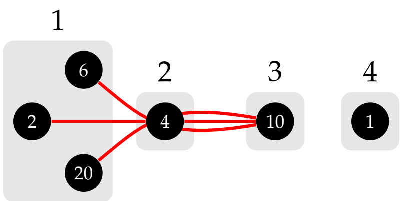


Table 3.1: An example relating to the proof of the Blowing up Theorem.

3.3 Applications of Blowing up

How does blowing up relate to Lagrangians?

Let G be a k-graph; it is quite apparent that there is a very nice expression for the Lagrange polynomial of blowups of G in terms of the polynomial $p_G(\mathbf{x})$. In fact, let $\mathbf{t} = (t_1, \ldots, t_n)$ be a vector in $(\mathbb{Z}^+)^n$ and consider a set of $\|\mathbf{t}\|_1$ variables indexed as $y_{1,1} \ldots y_{1,t_1} \ldots y_{n,1} \ldots y_{n,t_n}$, arranged in a vector $\mathbf{y}$. Then

$$p_{G(\mathbf{t})}(\mathbf{y}) = p_G((y_{1,1} + \cdots + y_{1,t_1}), \ldots, (y_{n,1} + \cdots + y_{n,t_n}))\,.$$

One consequence of this is that the Lagrangian of $G(\mathbf{t})$ is exactly the same as that of G; furthermore, it can be achieved by a weighting w whose support is a copy of G seen as a subgraph of the blowup (induced by a set containing one vertex for each part of the ground set); w is effectively an optimal weighting for G.

Another consequence relating to the edge density of blowups we now state as the following proposition:

Proposition 3.4. *For any k-graph G on $[n]$ there is a sequence of vectors of $(\mathbb{Z}^+)^n$ $(\mathbf{t}_i)_{i\geq n}$ such that $\|t_i\| = i$, and*

$$\lim_{i\to\infty} d(G(\mathbf{t}_i)) = k!\lambda(G)\,.$$

Proof. Consider a weighting w such that $\lambda(G) = p_G(\mathbf{w})$.

Take n sequences of positive integers $(t_i^1)_{i\geq n}, \dots, (t_i^n)_{i\geq n}$ such that for all i $t_i^1 + \cdots + t_i^n = i$, and

$$\lim_{i\to\infty} \frac{t_i^j}{i} = w(j)$$

for $j = 1, \dots, n$. (This can be done since we know that $\sum_{j\in[n]} w(j) = 1$.)

Let then $\mathbf{t}_i$ be the vector $(t_i^1, \dots, t_i^n)$ and consider the k-graph $G(\mathbf{t}_i)$. This, being a blowup of G on i vertices, has edge density

$$\frac{p_{G(\mathbf{t}_i)}(1,\dots,1)}{\binom{i}{k}} = \frac{p_G(\mathbf{t}_i)}{\binom{i}{k}}$$

whilst

$$\lambda(G) = p_G(\mathbf{w}) = \lim_{i\to\infty} p_G\left(\frac{1}{i}\mathbf{t}_i\right) = \lim_{i\to\infty} \frac{p_G(\mathbf{t}_i)}{i^k}\,.$$

This implies

$$\lim_{i\to\infty} \frac{d(G(\mathbf{t}_i))}{\lambda(G)} = \lim_{i\to\infty} \frac{i^k}{\binom{i}{k}} = k!$$

hence

$$\lim_{i\to\infty} d(G(\mathbf{t}_i)) = k!\lambda(G)\,.$$

□

The result we have proved allows to establish one first way to apply Lagrangians to the computation of Turán densities for general k-graphs, by providing the following as an easy consequence:

Corollary 3.5. *Let F be a k-graph which is not G-colorable (for some k-graph G on $[n]$). Then $\pi(F) \geq k!\lambda(G)$.*

Proof. If F is not G-colorable, then for all vectors $\mathbf{t}$ in $(\mathbb{Z}^+)^n$ the k-graph $G(\mathbf{t})$ is F-free; this is quite straightforward: if F were a subgraph of $G(\mathbf{t})$, composition of the inclusion with the complete homomorphism sending $G(\mathbf{t})$ to G would give a G-coloring of F (notice the converse is also true: if there were a homomorphism from F to G then the k-graph $G(|V(F)|, \dots, |V(F)|)$, for example, would have F as a subgraph).

Consider the sequence $\mathbf{t_i}$ for G as defined in Proposition 3.4.

Since $G(\mathbf{t}_i)$ is an F-free k-graph on $[i]$,

$$\pi(i, F) > d(G(\mathbf{t}_i));$$

As a consequence,

$$\pi(F) = \lim_{i\to\infty} \pi(i, F) \geq \lim_{i\to\infty} d(G(\mathbf{t}_i)) = k!\lambda(G).$$

□

As a last application of blowups in this section we give a very rapid way of deducing the Erdős-Stone Theorem from Turán's Theorem. We remind the reader of

Theorem 3.6 (Erdős-Stone)**.** *Let G be a 2-graph; then $\pi(G) = 1 - \frac{1}{\chi(G)-1}$, where $\chi(G)$ is the chromatic number of G.*

Proof. Suppose $\chi(G) = r$; this implies G is r-partite, *i.e.* it is a subgraph of a complete r-partite graph $K_r(\mathbf{t})$ for some $\mathbf{t}$ (r-partite k-graphs are blowups of $K_r^{(k)}$, see Example 3.1).

Thus, by the Blowing up Theorem, $\pi(G) \leq \pi(K_r(\mathbf{t})) = \pi(K_r) = 1 - \frac{1}{r-1}$.

On the other hand, $\chi(G) = r$ implies that G is not $(r-1)$-partite, thus it is not K_{r-1}-colorable.

The result we just proved (Corollary 3.5) then gives

$$\pi(G) \geq 2\lambda(K_{r-1}) = 1 - \frac{1}{r-1}$$

and the theorem is proved. □

3.4 Turán Densities and Jumps

We have already remarked in Section 1.3 (and just proved thanks to Theorem 3.6) that the set $\Gamma^{(2)}$ of possible Turán densities for 2-graphs has the rather unexpected property of being countable and well-ordered, thus made up of isolated points in $[0, 1)$: we know that

$$\Gamma^{(2)} = \left\{1 - \frac{1}{t} \mid t \in \mathbb{Z}^+\right\}.$$

This is a manifestation of a very interesting general phenomenon: choose any real number $\alpha \in [0, 1)$; then there exists $\Delta(\alpha) > 0$ such that any graph with edge density strictly higher than α (by any arbitrarily small amount), provided it has a large enough ground set, will have "large" subgraphs of density at least as high as $\alpha + \Delta(\alpha)$.

The reason for this is quite clear: let t be such that $1 - \frac{1}{t-1} \leq \alpha$, *i.e.* $\pi(K_t) \leq \alpha$; then, thanks to the Blowing up Theorem, given any $\epsilon > 0$ and $m \in \mathbb{Z}^+$ there is n_0 such that any graph on $n > n_0$ vertices with density $\alpha + \epsilon$ contains the blowup $K_t(m, \ldots, m)$, which has mt vertices and edge density

$$\frac{\binom{t}{2}m^2}{\binom{mt}{2}} = \frac{(t-1)m}{mt-1} = \frac{t-1}{t-1/m} > 1 - \frac{1}{t} > \alpha + \left(\frac{1}{t-1} - \frac{1}{t}\right).$$

Inspired by this, we can give the following definition of a *jump*:

Definition 3.2. A real number $\alpha \in [0, 1)$ is a *jump* for k-graphs if there exists $\Delta(\alpha) > 0$ such that for all $\epsilon > 0$, $m \geq k$, any k-graph on $n > n(m, \epsilon)$ vertices with edge density at least $\alpha + \epsilon$ has a subgraph on m vertices which has edge density at least $\alpha + \Delta(\alpha)$.

In other words, α is a jump if, given a sequence $(H_n)_{n>0}$ of k-graphs such that $\lim_{n\to\infty} |V(H_n)| = \infty$ and $\lim_{n\to\infty} d(H_n) > \alpha$, for all $m \geq k$

$$\liminf_{n} \max_{W \subseteq V(G_n)^{(m)}} d(G_n[W]) > \alpha + \Delta(\alpha)\,.$$

Though we have somewhat changed the point of view, jumps are still very closely related to the set of possible Turán densities; in fact, we shall shortly make use of some of the theory developed in the preceding sections to prove a very important Lemma from [7], which gives a characterization of jumps; in order ti state it (and prove it) we need to give a new (very unsurprising) definition, and generalize our results a little.

Definition 3.3. Let $\mathcal{F} = \{F_1, \ldots, F_r\}$ be a finite family of k-graphs. Then we define the Turán density of the family, $\pi(\mathcal{F})$, as the limit

$$\lim_{n\to\infty} \frac{ex(n, \mathcal{F})}{\binom{n}{k}},$$

where $ex(n, \mathcal{F})$ is the maximum number of edges of a k-graph on $[n]$ that is F-free for all F in $\mathcal{F}$.

We can now make the following

Remark 3.1. *The Blowing up Theorem can be restated in the case of a* finite family *of k-graphs; the proof we do not repeat: it does need to be adapted somewhat (we can easily prove a version of the Supersaturation Lemma which applies to finite families* *of k-graphs, ensuring the existence of some positive real $a(\epsilon)$ such that a large enough k-graph G with density $\pi(\mathcal{F}) + \epsilon$ will contain at least $a\binom{|V(G)|}{m}$ distinct copies of one member F of $\mathcal{F}$, where $m = \max_{F\in\mathcal{F}} |V(F)|$). By repeating the arguments of the proof we have discussed, it can be shown that for any $\mathbf{t} \in (\mathbb{Z}^+)^{|V(F_1)|}$*

$$\pi(\{F_1(\mathbf{t}), \ldots, F_r\}) = \pi(\{F_1, \ldots, F_r\})$$

and hence

$$\pi(\{F_1(\mathbf{t}_1), \ldots, F_r(\mathbf{t}_r)\}) = \pi(\{F_1, \ldots, F_r\})$$

for any $\mathbf{t}_1, \ldots, \mathbf{t}_r$ such that $\mathbf{t}_i \in (\mathbb{Z}^+)^{|V(F_i)|}$.

Finally, we are ready for

Lemma 3.7. *Let α be a real number in $[0, 1)$. Then α is a jump for k-graphs if and only if there is a finite family of k-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) \leq \alpha$ and $k!\lambda(F) > \alpha$ for all F in $\mathcal{F}$.*

Proof. Suppose α is a jump, and let $\Delta(\alpha)$ be as in Definition 3.2. Take m to be such that

$$\frac{k!}{m^k}\binom{m}{k}(\alpha + \Delta(\alpha)) > \alpha$$

(which is easily done since $\lim_{m\to\infty} \frac{k!}{m^k}\binom{m}{k} = 1$).

Let $\mathcal{F}$ be the finite family of k-graphs F on $[m]$ with $d(F) \geq \alpha + \Delta(\alpha)$. By definition of $\Delta(\alpha)$ we have $\pi(\mathcal{F}) \leq \alpha$. Also, by simply considering the uniform weighting on $[m]$, we get

$$\lambda(F) \geq \frac{1}{m^k}(\alpha + \Delta(\alpha))\binom{m}{k} > \frac{\alpha}{k!}$$

for all F in $\mathcal{F}$.

For the converse, consider a family $\mathcal{F}$ as in the statement of the Lemma, and suppose

$$\mathcal{F} = \{F_1, \ldots, F_r\};$$

we know that for all $\mathbf{t}^1, \ldots, \mathbf{t}^r$ such that $\mathbf{t}^i \in (\mathbb{Z}^+)^{|V(F_i)|}$ (for $i = 1, \ldots, r$) we have

$$\pi(\{F_1(\mathbf{t}^1), \ldots, F_r(\mathbf{t}^r)\}) = \pi(\mathcal{F}) \leq \alpha$$

thanks to a corollary of the Blowing up Theorem.

We also know by Proposition 3.4 from the preceding section that for each i in $[r]$ there is a sequence $(\mathbf{t}^i_n)_{n\geq|V(F_i)|}$ of vectors in $(\mathbb{Z}^+)^{|V(F_i)|}$ such that $\lim_{n\to\infty} d(F_i(\mathbf{t}^i_n)) = k!\lambda(F_i) > \alpha$, and also $\|\mathbf{t}^i_n\|_1 = n$ (so that $|V(F_i(\mathbf{t}^i_n))| = n$).

Take $0 < \Delta(\alpha) < \min_{i\in[r]}(\alpha - k!\lambda(F_i))$; choose any $\epsilon > 0$, $m \geq k$, and take $j \geq m$ such that $d(F_i(\mathbf{t}^i_j)) \geq \alpha + \Delta(\alpha)$ for $i = 1, \ldots, r$.

Since $\pi(\{F_i(\mathbf{t}^i_j)|i = 1, \ldots, r\}) \leq \alpha$, any k-graph G on at least $n(\epsilon)$ vertices which has edge density $d(G) \geq \alpha + \epsilon$ will have some $F_i(\mathbf{t}^i_j)$ – which is a k-graph on m or more vertices with density at least $\alpha + \Delta(\alpha)$ – as a subgraph.

In other words, α is a jump. □

3.5 Hypergraphs do not Jump

Our discussion of *jumps* was originated by observing that

Remark 3.2. *For 2-graphs, every α in $[0, 1)$ is a jump.*

It seemed reasonable, at the time when such matters first came to light, to conjecture

Conjecture 2 (Erdős)**.** *For any $k \geq 2$, every α in $[0, 1)$ is a jump for k-graphs.*

In fact, it is immediately apparent from Theorem 3.1 that 0 is a jump for k-graphs: any k-graph of positive density on a big enough ground set will have suitable blow-ups of the k-graph $K_k^{(k)}$ (that is, complete k-partite graphs with parts of any cardinality we require) as subgraphs; since $K_k^{(k)}(t, \dots, t)$ has kt vertices and edge density $\frac{t^k}{\binom{kt}{k}} \geq \frac{k!}{k^k}$, 0 is indeed a jump, and furthermore we can take $\Delta(0) = \frac{k!}{k^k}$.

Conjecture 2, however, is untrue.

It was disproved by Frankl and Rödl in 1983, and their approach – strongly based on the Lemma we proved in the preceding section – has been exploited in many subsequent papers exhibiting non-jumps for k-graphs.

We will include a complete proof of the following, stated and proved in [7]:

Theorem 3.8. *For $k \geq 3$, $l > 2k$, $1 - \frac{1}{l^{k-1}}$ is not a jump for k-graphs.*

Sketch of proof. Here we give a very brief sketch of the proof we are about to put into being; all details will be covered throughout this section and the next.

Set $\alpha = 1 - \frac{1}{l^{k-1}}$.

By Lemma 3.7, if α were a jump for k-graphs, there would be a finite family of k-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) \leq \alpha$ and $k!\lambda(F) > \alpha$ for all F in $\mathcal{F}$.

We shall find an appropriate k-graph K such that $k!\lambda(K) > \alpha$. This, for t big enough, will ensure that big blowups $K(\mathbf{t})$ of K have density strictly greater than α by at least some fixed amount (by Proposition 3.4), thus (provided they are big enough) they contain some member F of $\mathcal{F}$ as a subgraph.

F is a small subgraph of the blowup $K(\mathbf{t})$: in fact, it can be recovered as a subgraph of a blowup of its image (via the projection of $K(\mathbf{t})$ on K) H in K, and we have $|V(H)| \leq |V(F)|$. Since the Lagrangian of blowups is the same as that of the original k-graph, we know that $\lambda(F) \leq \lambda(H)$.

We would have contradiction if we were able to prove that $k!\lambda(H) \leq \alpha < k!\lambda(F)$ for all F in $\mathcal{F}$ (by exploiting the fact that $V(H)$ is small, and we are – hopefully – able to choose K to be big).

Hence this is the crucial property that we require of the k-graph K: if we fix $m = \max_{F \in \mathcal{F}} |V(F)|$, all of the subgraphs of K that are induced by vertex sets with cardinality no greater than m must have a "small" Lagrangian.

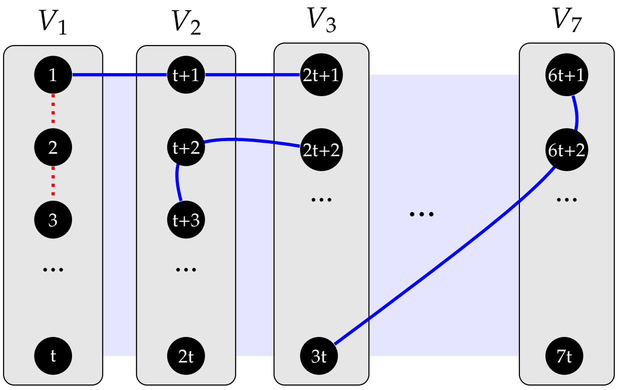


Figure 3.2: The k-graph $G^{(k)}(l,t)$ is the maximal k-graph whose ground set is $[lt]$ with no edges entirely contained in one of the l sets $\{j | \lfloor (j-1)/t \rfloor = i-1\}$ $(i = 1, \ldots, l)$. The picture represents $G^{(3)}(7,t)$: the red dotted edge is prohibited (since it is contained in V_1); the blue edges are present, and so are all those involving vertices from at least two distinct parts.

Our main aim is then finding a suitable K. We shall discover that this can be done by taking, for some big t, the graph $G^{(k)}(l,t)$ from Figure 3.2 (described in the statement of Lemma 3.9) and adding $O(t^{k-1})$ edges in such a way that the Lagrangian becomes higher than the required amount, but subgraphs induced by small vertex sets have very few "extra" edges, hence they are similar to subgraphs of $G^{(k)}(l,t)$; this will allow us to keep control of their Lagrangian and show the added edges do not raise it significantly. □

Lemma 3.9 and Corollary 3.10 discuss the Lagrangians of the k-graphs $G^{(k)}(l,t)$, showing they are big enough for the edges we will be adding to raise them over the value we need, but small enough that taking small subgraphs with few added edges will not do so.

Lemma 3.11 shows we can add a fair number of edges to $G^{(k)}(l,t)$ in such a way that small vertex sets induce subgraphs with few extra edges.

After finishing the proof of Lemma 3.11 we will go through the arguments already sketched, but in greater detail; finally, in Section 3.6, we conclude the proof by showing that small subgraphs of our final k-graph K have indeed small enough Lagrangians, drawing from the results of Corollary 3.10 and Lemma 3.11.

Lemma 3.9. *Fix $t, k, l \geq k$. Let $\bar{E} \subseteq [lt]^{(k)}$ be the family of k-sets*

$$\bar{E} = \{\{i_1, \ldots, i_k\} \mid \lfloor (i_1 - 1)/t \rfloor = \cdots = \lfloor (i_k - 1)/t \rfloor\}$$

and let $G^{(k)}(l,t) = ([lt], [lt]^{(k)} \setminus \bar{E})$ be the maximal k-graph on lt vertices whose edges, for a partition of the ground set into l parts of cardinality t, do not lie entirely inside *any of the parts.*[2] *Then*

$$\lambda(G^{(k)}(l,t)) = \frac{1}{(lt)^k}\left(\binom{lt}{k} - l\binom{t}{k}\right).$$

Proof. We write G for $G^{(k)}(l,t)$ throughout the proof.

The uniform weighting on G immediately gives the inequality

$$\lambda(G) \geq \frac{1}{(lt)^k}\left(\binom{lt}{k} - l\binom{t}{k}\right);$$

in order to prove the opposite inequality, notice that

$$p_G(\mathbf{x}) = p_{K_{lt}^{(k)}}(\mathbf{x}) - \sum_{0\leq i<l} p_{K_t^{(k)}}(x_{it+1},\ldots,x_{(i+1)t}).$$

Let w be an optimal weighting for G. Take any two distinct i and j belonging to the same part of the ground set (*i.e.* such that $\lfloor (i-1)/t \rfloor = \lfloor (j-1)/t \rfloor$); then they are equivalent, thus by Lemma 2.3 we may assume $w(i) = w(j)$.

So we have

$$p_G(\mathbf{w}) = p_{K_{lt}^{(k)}}(\mathbf{w}) - \binom{t}{k}\sum_{1\leq i\leq l}(w(it))^k$$

and by Jensen's inequality together with the fact that $\sum_i w(it) = 1/t$, we get

$$\frac{1}{l}\sum_{1\leq i\leq l}(w(it))^k \geq \left(\frac{1}{l}\sum_{1\leq i\leq l}(w(it))\right)^k = \frac{1}{(lt)^k}. \tag{3.1}$$

Also,

$$p_{K_{lt}^{(k)}}(\mathbf{w}) \leq \lambda(K_{lt}^{(k)}) = \frac{1}{(lt)^k}\binom{lt}{k}. \tag{3.2}$$

Using both (3.1) and (3.2) gives

$$p_G(\mathbf{w}) \leq p_{K_{lt}^{(k)}}(\mathbf{w}) - \binom{t}{k}\frac{l}{(lt)^k} \leq \frac{1}{(lt)^k}\left(\binom{lt}{k} - l\binom{t}{k}\right)$$

which was our aim. □

Since what will be most useful is the asymptotic behavior in t of the Lagrangian $\lambda(G^{(k)}(l,t))$ we also state the following corollary:

Corollary 3.10. *There is a positive constant C depending only on k and l such that for all $t \in \mathbb{Z}^+$*

$$k!\lambda(G^{(k)}(l,t)) = 1 - \frac{1}{l^{k-1}} - \frac{C}{t} + O\left(\frac{1}{t^2}\right).$$

[2] Frankl and Rödl call this an l-partite k-graph; we remark that it is not l-partite according to our definitions, since for $k > 2$, $l > 1$ it has edges involving more than one vertex from a single part of the ground set.

Proof. All that we need to do is to examine the two terms of highest degree in t in the expression given for $\lambda(G^{(k)}(l,t))$ by Lemma 3.9; direct computation gives

$$\frac{1}{(lt)^k}\left(\frac{(lt)^k}{k!}+\frac{(lt)^{k-1}(-1-\cdots-(k-1))}{k!}-l\frac{t^k}{k!}-l\frac{t^{k-1}(-1-\cdots-(k-1))}{k!}\right)=$$

$$=\frac{1}{k!}\left(1-\frac{1}{l^{k-1}}-\frac{1}{t}\binom{k}{2}\left(\frac{1}{l}-\frac{1}{l^{k-1}}\right)\right)$$

so we can take C to be

$$\binom{k}{2}\left(\frac{1}{l}-\frac{1}{l^{k-1}}\right).$$

□

Lemma 3.11. *We fix k, m, c. Then for each $t > t(k,m,c)$ there is a k-graph H on $[t]$ with $|E(H)| \geq ct^{k-1}$ such that all induced subgraphs $H[W]$ with $|W| \leq m$ have no more than $|W| - k + 1$ edges.*

Proof. The proof of this Lemma employs very typical probabilistic arguments to establish the existence of H.

Consider a k-graph H^* on $[t]$ whose edge set is obtained by choosing elements of $[t]^{(k)}$ independently at random with probability p.

The expected number I of edges for H^* is $p\binom{t}{k}$.

We now call a set W in $[t]^{(\leq m)}$ *bad* if the k-graph $H^*[W]$ has at least $|W|-k+2$ edges, and compute the expected number J of bad sets.

We have

$$J=\sum_{1\leq j\leq m}\binom{t}{j}\binom{\binom{j}{k}}{j-k+2}p^{j-k+2}.$$

Removing all edges of $H^*[W]$ for each bad W yields a graph with no bad sets; the expected number of edges after this operation is

$$E \geq I - J\binom{m}{k};$$

if $E \geq ct^{k-1}$ (as long as t is big enough) then we're done, since there will be a k-graph on $[t]$ for which removing edges from all bad subsets of the ground set leaves a k-graph H which satisfies all requirements of the lemma.

Take then $p = \frac{2ck!}{t}$ (which is less than 1 for $t > 2ck!$) so that

$$I=\frac{2ck!}{t}\binom{t}{k}=2ct^{k-1}+O(t^{k-2});$$

then $J = O(t^{k-2})$, since m, k, c are fixed constants not depending on t, and the leading term in t (in the expression for J) has degree $k-2$.

Thus $E = 2ct^{k-1} + O(t^{k-2})$ so

$$E \geq ct^{k-1}$$

for t greater than some $t(k, m, c)$ and we're done. □

We now get to the core of the proof of Theorem 3.8.

Take $k \geq 3$, $l > 2k$.

Suppose $1 - \frac{1}{l^{k-1}}$ is a jump for k-graphs: by Lemma 3.7 we have a finite family of k-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) \leq 1 - \frac{1}{l^{k-1}}$ and $k!\lambda(F) > 1 - \frac{1}{l^{k-1}}$ for all F in $\mathcal{F}$. Set $m = \max_{F\in\mathcal{F}} |V(F)|$.

Consider the k-graph $G = G^{(k)}(l, t)$ as defined in the statement of Lemma 3.9 and set $c = C + 2$ where C is the constant from Corollary 3.10, so that

$$k!\frac{|E(G)|}{(lt)^k} \geq 1 - \frac{1}{l^{k-1}} - \frac{1}{t}(c-2) + O\left(\frac{1}{t^2}\right).$$

Suppose t is such that we can take a k-graph H^* on $[t]$ satisfying the requirements of Lemma 3.11 for constants k, m, $l^k c/k!$. Let K be the k-graph $([lt], E(G) \cup E(H^*))$. Then

$$\lambda(K) \geq \frac{1}{(lt)^k}|E(K)| \geq \frac{1}{k!}\left(1 - \frac{1}{l^{k-1}} + \frac{2}{t}\right) + O\left(\frac{1}{t^2}\right).$$

Thus for t big enough

$$\lambda(K) \geq \frac{1}{k!}\left(1 - \frac{1}{l^{k-1}} + \frac{1}{t}\right). \tag{3.3}$$

By Proposition 3.4 there is a sequence $(\mathbf{t}_n)_{n\geq lt}$ of vectors in $(\mathbb{Z}^+)^{lt}$, such that $||\mathbf{t}_n||_1 = n$ and

$$\lim_{n\to\infty} d(K(\mathbf{t}_n)) = k!\lambda(K).$$

Because of (3.3) we can find n_0 such that, for $n > n_0$,

$$d(K(\mathbf{t}_n)) \geq 1 - \frac{1}{l^{k-1}} + \frac{1}{2t}$$

But then $d(K(\mathbf{t}_n)) \geq \pi(\mathcal{F}) + \frac{1}{2t} > \pi(\mathcal{F})$, so for each $n > n_1(t)$ we have $F \subset K(\mathbf{t}_n)$ for some member F of $\mathcal{F}$. Since $|V(F)| \leq m$, if we denote by φ the complete homomorphism which sends $K(\mathbf{t}_n)$ to K, $\varphi(F)$ (where by F we mean the copy of F present in $K(\mathbf{t}_n)$) is a subgraph H of K on no more than m vertices, such that $F \subset H(\mathbf{t}_n)$.

Then $\lambda(F) \leq \lambda(H(\mathbf{t}_n)) = \lambda(H)$ (the identity between the Lagrangian of a k-graph and that of its blowups was remarked at the beginning of Section 3.3). We get a contradiction by finally showing

Lemma 3.12. *For all subgraphs H of K with $|V(H)| \leq m$ we have*

$$\lambda(H) \leq \frac{1}{k!}\left(1 - \frac{1}{l^{k-1}}\right).$$

3.6 One Last Lemma

Let $V_i = \{j \mid \lfloor (j-1)/t \rfloor = i-1\}$ for $i = 1, \ldots, l$ be the l parts of the ground set of $G^{(k)}(l,t)$ (and therefore K, of which H is a subgraph) as defined in the statement of Lemma 3.9.

Let then W_i be $V_i \cap V(H)$; for our purpose, we can suppose H is induced, thus with edge set

$$E(H) = \left(V(H)^{(k)} \setminus \left(\bigcup_{i \in [l]} W_i^{(k)} \right) \right) \cup (E(H^*) \cap E(H)) .$$

If $E(W_1)$ were empty H would be a subgraph of $G^{(k)}(l,t)$, hence $\lambda(H) \leq \lambda(G)$, and by Corollary 3.10 we'd be finished. Notice we're now using Lemma 3.9 in its full strength (the mere upper bound on the Lagrangian given by the uniform weighting is not sufficient), which we haven't needed before.

Suppose then $E(W_1) \neq \emptyset$, in which case W_1 must have at least k vertices.

Without loss of generality we may assume $W_1 = [s]$ for some $s \geq k$, and $w(1) \geq \cdots \geq w(s)$, where w is an optimal weighting for H.

For ease of notation, given any edge e in $E(H)$, we write $w[e]$ for $\prod_{i \in e} w(i)$.

Let us now number the edges in $E(W_1)$ as $e_1, \ldots, e_m$ (notice that according to Lemma 3.11 $m \leq s - k + 1$) in such a way that, if $1 \leq i \leq j \leq m$, then $w[e_i] \geq w[e_j]$.

For each $p = 1, \ldots, m$ consider the set of vertices $P = e_1 \cup \cdots \cup e_p$; by Lemma 3.11 and by definition of P, we have

$$p \leq |E(H[P])| \leq |P| - k + 1 ,$$

hence $|P| \geq p + k - 1$.

Thus (by the pigeonhole principle) there must be one edge e out of $e_1, \ldots, e_p$ which involves a vertex not in $[p + k - 2]$.

This edge is then such that

$$w[e] = \prod_{i \in e} w(i) \leq w(1) \ldots w(k-1) w(k-1+p) ;$$

Since $w[e] \geq w[e_p]$, we get

$$\sum_{p \in [m]} w[e_p] \leq w(1) \ldots w(k-1) \left(\sum_{p \in [m]} w(k-1+p) \right) =$$

$$= w(1) \ldots w(k-1) \left(\sum_{k \leq p \leq s} w(p) \right)$$

(where we have used the fact that $s \geq m + k - 1$).

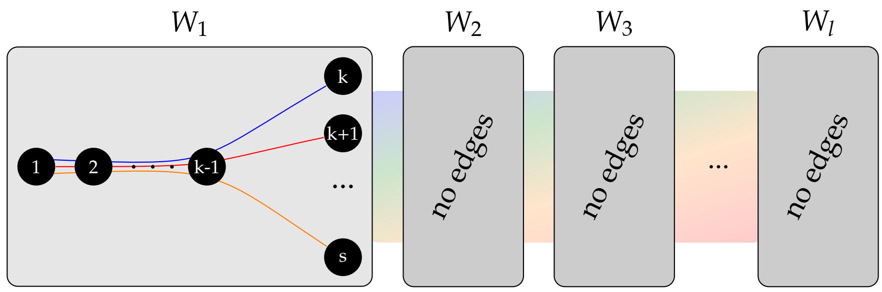


Figure 3.3: The structure of the k-graph $\hat{H}$: the edges inside each of the sets $W_1, \dots, W_l$ are as drawn; all other edges are present.

Thus $\lambda(H) \le \lambda(\hat{H})$, where $\hat{H}$ is the graph with ground set $W_1 \cup \dots \cup W_l$ and edge set

$$E(\hat{H}) = \left(|V(\hat{H})|^{(k)} \setminus \bigcup_{i \in [l]} W_i^{(k)} \right) \cup \{1 \dots (k-1)p \,|\, k \le p \le s\}.$$

We shall from now on suppose H has the form of $\hat{H}$ (see also Figure 3.3), since exchanging $\hat{H}$ for H would not decrease the Lagrangian.

Observe that vertices 1 to $k-1$ are equivalent, as are vertices k to s (again, see Figure 3.3), and also all vertices inside of W_i for $i = 2, \dots, l$. We may than assume that the optimal weighting w for H has $w(1) = \dots = w(k-1) = \rho$, $w(k+1) = \dots = w(s)$ and for $i = 2, \dots, l$

$$w(W_i) = \left\{ \frac{\alpha_i}{|W_i|} \right\};$$

set then $\alpha_1 = 1 - \sum_{2 \le i \le l} \alpha_i = \sum_{i \in [s]} w(i)$ so that $(s-k+1)w(k) = \alpha_1 - (k-1)\rho$.

Consider the Lagrange polynomial of H on the set of variables $\mathbf{x} = \{x_i \,|\, i \in V(H)\}$, and the polynomial $S(\mathbf{x}) = (\sum_{i \in V(H)} x_i)^k$; clearly, $S(\mathbf{x})$ contains:

- $k!$ times each monomial in $p_H(\mathbf{x})$;
- for $i = 2, \dots, l$, all monomials in $\left(\sum_{j \in W_i} x_j\right)^k$, none of which can appear in $p_H(\mathbf{x})$;
- all monomials in $(x_1 + \dots + x_s)^k$, except for those which correspond to edges in $E(H[W_1])$ (they have already been counted);
- for $1 \le a < b < k$, all monomials in $\binom{k}{2} x_a x_b \left(\sum_{i \in W_2 \cup \dots \cup W_l} x_i \right)^{k-2}$.

Evaluating in $\mathbf{w}$, since $S(\mathbf{w}) = 1$, yields the inequality

$$1 \geq k!\lambda(H) + \sum_{i\in[l]} \alpha_i^k - k!\rho^{k-1}(\alpha_1 - (k-1)\rho) + \binom{k}{2}(k-1)\rho^2(1-\alpha_1)^{k-2}\,;$$

since our aim is showing

$$\lambda(H) \leq \frac{1}{k!}\left(1 - \frac{1}{l^{k-1}}\right),$$

it would be enough to prove

$$\sum_{i\in[l]} \alpha_i^k - k!\rho^{k-1}(\alpha_1 - (k-1)\rho) + \binom{k}{2}(k-1)\rho^2(1-\alpha_1)^{k-2} \geq \frac{1}{l^{k-1}}\,. \tag{3.4}$$

The rest of the proof involves no new concepts or important ideas, but we do include it for the sake of completeness: it entirely consists of computations and elementary inequalities. We distinguish two cases ($k = 3$ and $k > 3$) and deal with each separately.

Case I: $k = 3$

We need to show that

$$\sum_{i\in[l]} \alpha_i^3 - 6\rho^2(\alpha_1 - 2\rho) + 6\rho^2(1-\alpha_1) \geq \frac{1}{l^2}$$

that is

$$\sum_{i\in[l]} \alpha_i^3 - 6\rho^2(2\alpha_1 - 2\rho - 1) \geq \frac{1}{l^2}\,.$$

By Jensen's inequality,

$$\frac{1}{l}\sum_{i\in[l]} \alpha_i^3 \geq \frac{1}{l^3}$$

so we are done in the case of $2\alpha_1 - 2\rho - 1 \leq 0$, *i.e.* $\alpha_1 - \rho \leq \frac{1}{2}$.

Suppose $\alpha_1 > \rho + \frac{1}{2}$. We may then apply the AM-GM inequality to the three terms $\rho, \rho, 2\alpha_1 - 2\rho - 1$, which yields

$$\frac{2\alpha_1 - 1}{3} \geq (\rho^2(2\alpha_1 - 2\rho - 1))^{1/3}\,,$$

hence $6\rho^2(2\alpha_1 - 2\rho - 1) \leq \frac{2}{9}(2\alpha_1 - 1)^3$.

We then have

$$\sum_{i\in[l]} \alpha_i^3 - 6\rho^2(2\alpha_1 - 2\rho - 1) \geq \alpha_1^3 - \frac{2}{9}(2\alpha_1 - 1)^3$$

and the RHS is minimized, within the relevant range for α_1, when $\alpha_1 = 1/2$ (since it is increasing in α_1 for $1 \geq \alpha_1 \geq 1/2$).

Thus

$$\alpha_1^3 - \frac{2}{9}(2\alpha_1 - 1)^3 > \frac{1}{8} > \frac{1}{l^2}$$

(we even have $l > 6$). This concludes the case of $k = 3$.

Case II: $k > 3$

Suppose $k > 3$. Take ϵ such that

$$\alpha_1 = \frac{1}{l} + (l-1)\epsilon$$

and thus $1 - \alpha_1 = (l-1)(\frac{1}{l} - \epsilon)$.

Consider the case where $\epsilon \geq \frac{1}{lk}$; the AM-GM inequality on k terms, of which $k-1$ terms equal to ρ and one to $\alpha_1 - (k-1)\rho$, yields

$$k!\rho^{k-1}(\alpha_1 - (k-1)\rho) \leq k!\left(\frac{\alpha_1 - (k-1)\rho + (k-1)\rho}{k}\right)^k = \frac{k!}{k^k}\left(\frac{1}{l} + (l-1)\epsilon\right)^k$$

which, since $\frac{1}{l} \leq k\epsilon \leq \frac{l-1}{2}\epsilon$ (we use $l > 2k$), is bounded from above by

$$\frac{k!}{k^k}\left(\frac{3}{2}(l-1)\epsilon\right)^k . \tag{3.5}$$

Since for $k \geq 4$ we have $k! \leq \left(\frac{2}{3}k\right)^k$, (3.5) is in turn no greater than $(l-1)^k\epsilon^k$.

Now go back to (3.4). We get

$$\text{LHS } \geq \alpha_1^k - (l-1)^k\epsilon^k = \left(\frac{1}{l} + (l-1)\epsilon\right)^k - (l-1)^k\epsilon^k \geq \frac{1}{l^k} + \frac{k(l-1)\epsilon}{l^{k-1}} \geq$$

$$\geq \frac{1}{l^{k-1}}\left(\frac{1}{l} + k(l-1)\epsilon\right) \geq \frac{1}{l^{k-1}}\left(\frac{1}{l} + 1 - \frac{1}{l}\right) = \frac{1}{l^{k-1}} .$$

Only the case where $k > 3$ and $\epsilon < \frac{1}{lk}$ is left.

As before, $\sum_{i\in[l]} \alpha_i^k \geq \frac{1}{l^{k-1}}$ by Jensen's inequality. Thus showing

$$\binom{k}{2}(k-1)\rho^2(1-\alpha_1)^{k-2} \geq k!\rho^{k-1}(\alpha_1 - (k-1)\rho)$$

would yield the desired result. Applying the AM-GM inequality on $k-3$ terms ρ and one term equal to $(k-3)(\frac{\alpha_1}{k-1}-\rho)$ gives

$$\left(\frac{\frac{(k-3)\alpha_1}{k-1}}{k-2}\right)^{k-2} \geq \left(\frac{k-3}{k-1}\right)\rho^{k-3}(\alpha_1-(k-1)\rho)$$

so it would be sufficient to show

$$\frac{k!(k-1)}{k-3}\left(\frac{\frac{(k-3)\alpha_1}{k-1}}{k-2}\right)^{k-2} \leq \binom{k}{2}(k-1)(1-\alpha_1)^{k-2}. \tag{3.6}$$

By rearranging the terms in (2.6) our aim becomes showing that

$$\left(\frac{\alpha_1}{1-\alpha_1}\right)^{k-2} \leq \binom{k}{2}\frac{((k-2)(k-1))^{k-2}}{k!(k-3)^{k-3}}$$

which is quite straightforward.

The LHS is increasing in α_1, which in turn is increasing in ϵ; we can then take $\epsilon = \frac{1}{lk}$, giving

$$\alpha_1 = \frac{1}{l} + \frac{1}{k}\left(\frac{l-1}{l}\right) \leq \frac{2}{k},$$

hence all we need to show is

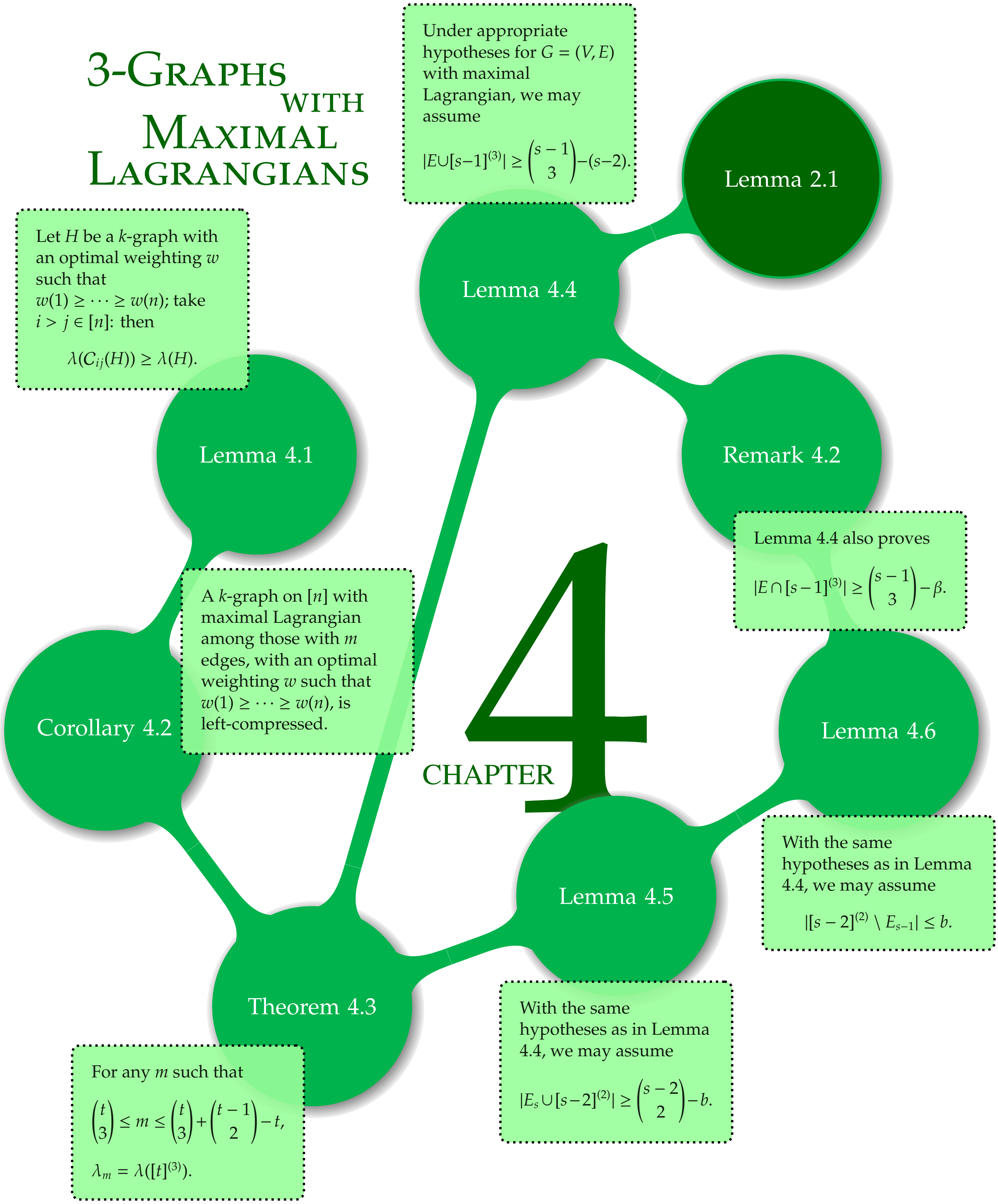
$$1 \leq \frac{(k-2)^{2k-4}(k-1)^{k-1}}{2^{k-1}(k-3)^{k-3}(k-1)!}$$

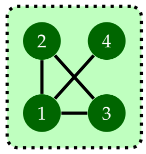
which is trivially true for $k > 3$.

This finally completes the proof of Theorem 3.8. $\square$

Chapter 4

3-Graphs with Maximal Lagrangians



CHAPTER 4 

3-Graphs with Maximal Lagrangians

in which we try to get to know Lagrangians better, and do succeed to some extent.

4.1 Bounds on Lagrangians

Let us now step back from hypergraph Turán type problems and consider Lagrangians of k-graphs in themselves.

As we have seen, computing the Lagrangian of a given k-graph H exactly is in general no easy task: unless we are able to exploit some kind of symmetry and use Lemma 2.3 as in Sections 3.5 and 3.6, and often even then, we have to resort to bounds of some kind.

A trivial lower bound for the Lagrangian in terms of the number of vertices and edges in H – which we have oftentimes used – is given by evaluating the Lagrange polynomial in the uniform weighting: clearly,

$$\lambda(H) \geq \frac{1}{|V(H)|^k} |E(H)| \,.$$

Finding a suitable upper bound is much harder. In fact, what we are asking ourselves is

> ? *Given positive integers n and $m \leq \binom{n}{k}$, which k-graph(s) on $[n]$ with m edges have maximal Lagrangian?*

In the case of 2-graphs, since the Lagrangian of G is the same as that of the largest complete graph appearing as its subgraph, the answer is immediate.

For $\binom{t}{2} \leq m < \binom{t+1}{2}$, we have that

$$\lambda(G) \leq \lambda(K_t) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

Notice that the bound does not depend on the number of vertices: one of the 2-graphs with m edges and largest Lagrangian is the largest complete graph we can put together with the edges available (with possibly a few extra vertices of zero weight to accommodate any exceeding edges).

This is compatible with the idea that a graph with m edges and maximal Lagrangian will have to "pack" its edges as "tightly" as possible.

The Lagrangian of general k-graphs bears enough resemblance to that of 2-graphs that we might expect a similar principle to hold.

Notice, however, that attempts at trivial generalizations of the 2-graph results blatantly fail: for example, for $k > 2$ it is no longer true that the maximal Lagrangian for k-graphs with m edges is that of the largest complete k-graph with no more than m edges.

This can be seen by taking any m such that $\binom{t}{k} + \binom{t-1}{k} < m < \binom{t+1}{k}$; as an example, take $k = 3$ and $m = 8$.

The largest complete 3-graph with no more than 8 edges is $K_4^{(3)}$, which has Lagrangian $\lambda(K_4^{(3)}) = \frac{1}{4^3}4 = \frac{1}{16}$ (all vertices are equivalent).

Consider now the 3-graph H on $[5]$ with edge set

$$E(H) = [4]^{(3)} \cup \{125, 135, 235, 145\},$$

and the weighting w on H such that $w(1) = \cdots = w(4) = 2/9$, $w(5) = 1/9$.

We have $p_H(\mathbf{w}) = 4\frac{8}{9^3} + 4\frac{4}{9^3} = \frac{16}{3^5}$, hence

$$\lambda(H) \geq \frac{16}{243} > \frac{1}{16} = \lambda(K_4^{(3)})$$

(it's also not hard to see that, in fact, $\lambda(H) = \frac{16}{243}$).

It still seems reasonable, however, that the k-graphs with largest Lagrangians should not "spread" their edges over a ground set that is larger than necessary, and that they should tend to involve the same vertices in as many edges as possible. The 3-graph H, for example, seems likely to be the one with highest Lagrangian among those with 8 edges (and it is). In the next section we formalize these thoughts and state an ensuing conjecture.

4.2 Colex and Compressions

The notion of "packing k-sets together" can be formalized by means of the colex order (or reverse lexicographic order) on $\mathbb{N}^{(k)}$.

Definition 4.1. Given $e_1, e_2 \in \mathbb{N}^{(k)}$, we say e_1 *comes before* e_2 *in the colex order*, and write $e_1 < e_2$, if

$$\max\{i \mid i \in e_1 \setminus e_2\} < \max\{i \mid i \in e_2 \setminus e_1\}.$$

Listing elements of $\mathbb{N}^{(k)}$ in the colex order is easily done by following the informal rule: *avoid big numbers*. For example, $\{2,4,6\} < \{1,5,6\}$ in colex, since the second 3-set contains a 5 which the first does not have.

Also notice that, if we write k-sets as vectors in $\{0,1\}^{\mathbb{N}}$ as we did in Section 1.1, and read those vectors as binary numbers, the colex order corresponds to the usual ordering of the naturals: we have just observed

Remark 4.1. *$e_1 < e_2$ in colex if and only if*

$$\sum_{i \in e_1} 2^i < \sum_{i \in e_2} 2^i.$$

Any k-set containing $i+1$ comes after all k-sets of $[i]^{(k)}$ in the colex order: the first m k-sets in colex are "packed" on the smallest possible ground set.

As an example, the first $\binom{5}{3}$ elements of $\mathbb{N}^{(3)}$ in the colex order are those of $[5]^{(3)}$, and can be listed as

$$\overbrace{\underbrace{123}_{[3]^{(3)}} < 124 < 134 < 234}^{[4]^{(3)}} < 125 < 135 < 235 < 145 < 245 < 345.$$

Notice that the edge set of the 3-graph G from the preceding section was made up of 8 3-sets forming an initial segment of the colex order.

For integers n, m, k such that $k \geq 2$, $n > 0$, and $0 \leq m \leq \binom{n}{k}$, define $\lambda^{(k)}_{n,m}$ as

$$\lambda^{(k)}_{n,m} = \max\{\lambda(H) \mid H \text{ is a } k\text{-graph on } [n] \text{ with } m \text{ edges}\}.$$

Arguments from Section 4.1 lead us to conjecture, as Frankl and Füredi did in [6]:

Conjecture 3 (Frankl, Füredi). *Let m be a positive integer. The k-graph*

$$C^{(k)}_m = (e_1 \cup \cdots \cup e_m, \{e_1, \ldots, e_m\}),$$

where $e_1 < \cdots < e_m$ form the m-th initial segment in the colex order of $\mathbb{N}^{(k)}$, is such that for any positive integer n with $\binom{n}{k} \geq m$,

$$\lambda^{(k)}_{n,m} = \lambda(C^{(k)}_m).$$

A very natural step we may take in trying to prove Conjecture 3 is considering *colex compressions*.

Take two positive integers i and j such that $i < j$, and a k-set e; we may *ij-compress e* by taking the k-set

$$C_{ij}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } j \in e, i \notin e; \\ \\ e & \text{otherwise.} \end{cases}$$

Clearly, $C_{ij}(e) \leq e$ in the colex order.

We now define the ij-compression of a family $E \subseteq \mathbb{N}^{(k)}$ as the family

$$C_{ij}(E) = \left\{C_{ij}(e) \,|\, e \in E\right\} \cup \left\{e \,|\, C_{ij}(e) \in E\right\}.$$

Also, we say E is ij-compressed if $C_{ij}(E) = E$.

Notice the ij-copression of E has two fundamental properties:

1. $|C_{ij}(E)| = |E|$: the ij-compression is done by substituting k-sets e (such that $j \in e$, $i \notin e$) with their compression $C_{ij}(e)$, unless it is already present (in which case both the original k-set and its compression are left alone), thus it does not affect cardinality;

2. $\sum_{e \in C_{ij}(E)} c(e) \leq \sum_{e \in E} c(e)$, where $c(e)$ denotes the position of e in the colex order of $\mathbb{N}^{(k)}$.

Finally, the ij-compression (for $i, j \in [n]$, $i < j$) of a k-graph $H = ([n], E(H))$ is the k-graph

$$C_{ij}(H) = ([n], C_{ij}(E(H)))$$

and H is ij-compressed if its edge set $E(H)$ is.

The graph $C_{ij}(H)$ has the same ground set and the same number of edges as H (although some vertices that did belong to edges of H might belong to no edge in $C_{ij}(H)$). Property 2 of ij-compressions tells us that the edge set $C_{ij}(H)$ is in some way "more packed" than that of H, "more similar" to an initial segment of colex.

Thus, if Conjecture 3 is true, we expect (appropriate) colex compressions not to decrease the Lagrangian of a k-graph. In fact, we can state and easily prove the following lemma:

Lemma 4.1. *Let H be a k-graph on $[n]$ such that there is an optimal weighting w for H with $w(1) \geq \cdots \geq w(n)$. Suppose there are positive integers $i, j \in [n]$ with $i < j$ such that H is not ij-compressed. Then*

$$\lambda(C_{ij}(H)) \geq \lambda(H).$$

Proof. We have
$$p_{C_{ij}(H)}(\mathbf{w}) - p_H(\mathbf{w}) = (w(i) - w(j))q(\mathbf{w})$$
where $x_j q(\mathbf{x})$ is the polynomial containing exactly those monomials m of p_H that have degree 1 in x_j and degree 0 in x_i, and such that $\frac{x_i}{x_j} m$ is not a monomial of p_H.

Since $q(\mathbf{w}) \geq 0$ and (by our hypothesis on w) $w(i) \geq w(j)$, we get
$$\lambda(C_{ij}(H)) \geq p_{C_{ij}(H)}(\mathbf{w}) \geq \lambda(H).$$
□

As a consequence we have

Corollary 4.2. *Let $G = ([n], E)$ ($|E| = m$) be a k-graph that has maximal Lagrangian among all k-graphs with m edges, such that there is an optimal weighting w for G with $w(1) \geq \cdots \geq w(n)$.*

Then for all $i, j \in [n]$ such that $i < j$, G is ij-compressed.

We call a family of k-sets *left-compressed* if it is ij-compressed for all positive integers $i < j$, and we say a k-graph is left-compressed if its edge set is.What Corollary 4.2 is stating is that for all m there is a k-graph H such that $\lambda_m^{(k)} = \lambda(H)$, $|E(H)| = m$, and H is left-compressed.

Clearly, an initial segment of colex is left-compressed. If the converse were true then Conjecture 3 would be proved.

Unfortunately, this is not the case at all: for an example of a left-compressed set that is not an initial segment of colex, take
$$\{123, 124, 125, 126\} \subseteq [6]^{(3)}$$
which is left-compressed, but is not $[4]^{(3)}$.

There is still much work to be done before we can establish any actual result about k-graphs with maximal Lagrangians; in fact, Conjecture 3 is still open for $k > 3$, and even the available proof for $k = 3$ does not cover all possible values of m.

In the next section we will go through the proof given by Talbot [20], who settled the conjecture for $k = 3$ and $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - t$. Thanks to Lemma 4.1, we will consistently assume that our candidate extremal examples are left-compressed.

4.3 Talbot's Proof

Suppose we have
$$\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - t \tag{4.1}$$

and let $G = ([s], E)$ be a 3-graph with maximal Lagrangian among those with no more than m edges, such that all of its vertices are given nonzero weight by any optimal weighting for G.

We shall show that G is actually supported on t vertices (*i.e.* $s = t$), which implies $\lambda^{(3)}_{n,m} = \lambda([t])^{(3)}$ for all $n \geq t$. Our result will therefore be

Theorem 4.3 (Talbot)**.** *For any integer m as in (4.1) and any $n \geq t$,*

$$\lambda^{(3)}_{n,m} = \lambda([t])^{(3)}.$$

More precisely, we claim that we may assume

$$|E| \geq \binom{s-1}{3} + \binom{s-2}{2} - (s-2) \tag{4.2}$$

which would yield

$$\binom{s-1}{3} + \binom{s-2}{2} - (s-2) < \binom{t}{3} + \binom{t-1}{2} - (t-1)$$

hence $s - 1 < t$, *i.e.* $s \leq t$, and we get Talbot's Theorem.

For $i = 1, \dots, s$ we define the 2-graphs $G_i = ([s], E_i)$, with

$$E_i = \{e \in [s]^{(2)} | e \cup \{i\} \in E\}$$

(thus $p_{G_i}(\mathbf{x}) = p^{(i)}_G(\mathbf{x})$); also, for $i, j \in [s]^{(2)}$, define

$$E_{i,j} = \{x \,|\, xij \in E\}.$$

Let w be an optimal weighting for G: we can assume that $w(1) \geq \cdots \geq w(s) > 0$ and that G is left-compressed.

Consider $E_{s-1,s}$. Since $w(s-1) \geq w(s) > 0$, by Lemma 2.2 the set $E_{s-1,s}$ is nonempty. Together with fact that G is left-compressed, this implies that $E_{s-1,s} = [b]$ for some $b > 0$.

Notice now that, for all i in $[b]$, since $\{s-1, s\} \in E_i$, we must have $E_i = ([s] \setminus i)^{(2)}$; this implies that all vertices in $[b]$ are equivalent and, by the same argument as in Lemma 2.3, we may assume $w(1) = \cdots = w(b)$.

Also observe that $b \geq s - 3$ would imply that $G[[s] \setminus \{s-2\}]$ is a complete 3-graph, and $[s-2]^{(2)} \subseteq E_{s-1}$, so

$$|E| \geq \binom{s-1}{3} + \binom{s-2}{2}$$

which is stronger than (4.2). We will therefore assume that $b \leq s - 4$.

Our claim of (4.2) shall be proved through the two lemmas that follow (given all assumptions stated so far on G):

Lemma 4.4.
$$|E \cap [s-1]^{(3)}| \geq \binom{s-1}{3} - (s-2)$$

Lemma 4.5.
$$|E_s \cap [s-2]^{(2)}| \geq \binom{s-2}{2} - b$$

Proof of claim (4.2).
$$m \geq |E| = |E \cap [s-1]^{(3)}| + |E_s \cap [s-2]^{(2)}| + |E_{s-1,s}|$$

hence, by Lemma 4.4 and Lemma 4.5,
$$m \geq \binom{s-1}{3} + \binom{s-2}{2} - (s-2) - b + b$$

which is (4.2). □

Proof of Lemma 4.4. Our aim is to show that there are no more than $s-2$ triples in $[s-1]^{(3)}$ which are missing from the edge set E. We proceed by contradiction, showing that, if Lemma 4.4 were to fail, then by shifting the weight of vertex s to vertex $s-1$ (which allows us to destroy all edges in E_s) and adding less than $|E_s|$ 3-sets from $[s-1]^{(3)}$ to the edge set of G, we would obtain a new 3-graph G^* and a weighting z, with $|E(G^*)| \leq |E(G)|$ and $p_{G^*}(\mathbf{z}) > \lambda(G)$.

This would contradict our choice of G.

Consider the weighting z as described, so that $\mathbf{z} = \mathbf{w} + w(s)(\mathbf{e}_{s-1} - \mathbf{e}_s)$.

We compute the difference
$$p_G(\mathbf{z}) - p_G(\mathbf{w}) = w(s)\left(p_G^{(s-1)}(\mathbf{w}) - p_G^{(s)}(\mathbf{w})\right) - w(s)^2 p_G^{(s)(s-1)}(\mathbf{w});$$

we know that $p_G^{(s-1)}(\mathbf{w}) = p_G^{(s)}(\mathbf{w})$ by Lemma 2.1, hence
$$p_G(\mathbf{z}) - p_G(\mathbf{w}) = -w(s)^2 p_G^{(s)(s-1)}(\mathbf{w}).$$

Now, since $p_G^{(s)(s-1)}(\mathbf{w}) = \sum_{i \in E_{s,s-1}} w(i) = w(1) + \cdots + w(b) = bw(1)$, we have
$$p_G(\mathbf{z}) - p_G(\mathbf{w}) = -bw(1)w(s)^2. \tag{4.3}$$

Suppose there are more than $s-2$ triples in $[s-1]^{(3)}$ which are not in E. What we shall do is find $E(F) \subseteq [s-1]^{(3)} \setminus E$ such that the 3-graph $F = ([s], E(F))$ has the two properties

1. $|E(F)| \leq |E_s|$;
2. $p_F(\mathbf{z}) > bw(1)w(s)^2$.

The graph $G^* = ([s], E \cup E(F))$ will then yield the contradiction we seek.

First we need to give further estimates for the expression (4.3).

This we do by using the fact that $p_G^{(s-1)}(\mathbf{w}) = p_G^{(1)}(\mathbf{w})$, which yields (if we bring all terms containing $w(s-1)$ or $w(1)$ to the LHS)

$$(w(1) - w(s-1))\, p^{(1)(s-1)}(\mathbf{w}) = -\sum_{\substack{i,j\neq 1\\ ij\in E_{s-1}\setminus E_1}} w(i)w(j) + \sum_{\substack{i,j\neq s-1\\ ij\in E_1\setminus E_{s-1}}} w(i)w(j).$$

Since E is left-compressed, $ij \in E_{s-1}$ (with $i, j \neq 1$) implies $ij \in E_1$, hence the first sum from the RHS is empty. Split the second sum as

$$w(s) \sum_{\substack{i\neq s-1\\ i\in E_{1,s}\setminus E_{s-1,s}}} w(i) + \sum_{\substack{i,j\neq s-1,s\\ ij\in E_1\setminus E_{s-1}}} w(i)w(j).$$

Set $C = ([s], [s-2]^{(2)} \setminus E_{s-1})$, and we finally have

$$w(1) \leq w(s-1) + \frac{w(s)\sum_{i=b+1}^{s-2} w(i) + p_C(\mathbf{w})}{\sum_{i\neq 1,s-1} w(i)}.$$

We multiply both sides of the identity by $bw(s)^2$ and give an upper bound for the RHS, using the inequality between arithmetic means

$$\frac{1}{s-2-b}\sum_{i=b+1}^{s-2} w(i) \leq \frac{1}{s-3}\sum_{i\neq 1,s-1,s} w(i) \leq \frac{1}{s-3}\sum_{i\neq 1,s-1} w(i)$$

which descends from the fact that the second sum contains all terms in the first sum, plus some larger weights.

We get

$$bw(1)w(s)^2 \leq bw(s)^2 w(s-1)\left(1 + \frac{s-2-b}{s-3}\right) + \frac{bw(s)^2 p_C(\mathbf{w})}{w(s)(s-2)}. \tag{4.4}$$

Now set

$$\alpha = \left\lceil \frac{b|E(C)|}{s-2} \right\rceil$$

and

$$\beta = b\left\lceil 1 + \frac{s-2-b}{s-3} \right\rceil$$

so that (4.4) can read as

$$bw(1)w(s)^2 \leq \beta w(s)^2 w(s-1) + \frac{\alpha w(s) p_C(\mathbf{w})}{|E(C)|}. \tag{4.5}$$

Since, as remarked, we can assume $b < s-3$, we have $\alpha \leq |E(C)|$ (in fact, it is sufficient to notice $b < s-2$, which is trivial by definition of b).

Set $F_1 = ([s], E(F_1))$, with $E(F_1)$ consisting of the α heaviest edges in

$$\{\{s-1\} \cup e \,|\, e \in E(C)\}.$$

By definition of C, edges in $E(F_1)$ do not belong to E (remember $E(C) \cap E_{s-1} = \emptyset$).

We defined z so that $z(s-1) = w(s-1) + w(s)$. Thus we have

$$p_{F_1}(\mathbf{z}) \geq (w(s-1) + w(s)) \frac{\alpha}{|E(C)|} p_C(\mathbf{w}).$$

Therefore

$$p_{F_1}(\mathbf{z}) \geq w(s-1)\alpha w(s)^2 + w(s) \frac{\alpha}{|E(C)|} p_C(\mathbf{w})$$

which, combined with (4.5), gives

$$p_{F_1}(\mathbf{z}) - bw(1)w(s)^2 \geq (\alpha - \beta) w(s-1) w(s)^2. \tag{4.6}$$

We show that, if $\alpha > \beta$, we can take $F = F_1$.

case $\alpha > \beta$

What we need to prove for F_1 are the two properties

1. $p_{F_1}(\mathbf{z}) > bw(1)w(s)^2$;
2. $|E(F_1)| \leq |E_s|$.

(1) is immediate from (4.6).

(2) is also easily proved as follows.

The fact that $\{b, s-1\} \in E_s$ implies that $[b]^{(2)} \cup \{\{1, i\} \,|\, b < i \leq s-1\} \subseteq E_s$ (because E is left-compressed).

Thus (since we have assumed $b \leq s-4$)

$$|E_s| \geq \binom{b}{2} + b(s-1-b) = b\frac{2s-b-3}{2} \geq b\frac{s+1}{2} > \frac{b}{s-2}\binom{s-1}{2}$$

which implies $|E_s| \geq \alpha = |E(F_1)|$, as required.

case $\beta \geq \alpha$

Only the case where $\beta \geq \alpha$ is left.

Let F_1 be as before and take $E(F_2) \subseteq ([s-1]^{(3)} \setminus E) \setminus E(F_1)$ such that $|E(F_2)| \geq \beta - \alpha + 1$.

This can be done thanks to our initial hypothesis that $|[s-1]^{(3)} \setminus E| > s-2$, since $\beta \leq s-2$ (to check this it is sufficient to notice that β is increasing in b, and thus it is no larger than the value obtained for $b = s-4$).

We take $F = ([s], E(F_1) \cup E(F_2))$ and prove properties (1) and (2) for F.
(1) descends from (4.6):

$$p_F(\mathbf{z}) - bw(1)w(s)^2 = p_{F_1}(\mathbf{z}) - bw(1)w(s)^2 + p_{F_2}(\mathbf{z}) \geq$$

$$\geq (\alpha - \beta)w(s-1)w(s)^2 + (\beta + 1 - \alpha)w(s-1)w(s)^2 = w(s-1)w(s)^2 > 0.$$

(2) is also quite straightforward: as before,

$$|E_s| \geq b\frac{2s-b-3}{2} > b\frac{2s-b-5}{s-3} = b\left(1 + \frac{s-b-2}{s-3}\right)$$

hence $|E_s| > \beta$. □

Remark 4.2. *Notice that the exact same proof of Lemma 4.4 also shows that*

$$|E \cap [s-1]^{(3)}| \geq \binom{s-1}{3} - \beta,$$

since $|[s-1]^{(3)} \setminus E| > \beta$ *was all that we actually needed to reach a contradiction.*

Proof of Lemma 4.5. This proof follows closely that of Lemma 4.4, the difference being that this time we assume there are at least b 2-sets in $[s-2]^{(2)}$ that are missing from E_s, and prove that shifting the weight of vertex $s-1$ onto vertex s allows us to exchange edges in E_{s-1} for edges containing s that do not belong to E, thus raising the Lagrangian.

Set $\mathbf{z} = \mathbf{w} + w(s-1)(\mathbf{e}_s - \mathbf{e}_{s-1})$.
This time

$$p_G(\mathbf{z}) - p_G(\mathbf{w}) = -w(s-1)^2 \sum_{i=1}^{b} w(i) = -bw(1)w(s-1)^2$$

in perfect analogy with (4.3).
As before, we need to find $F = ([s], E(F))$ such that $E(F) \cap E = \emptyset$ and

1. $|E(F)| \leq |E_{s-1}|$;

2. $p_F(\mathbf{z}) > bw(1)w(s-1)^2$.

Proceed as we did to obtain (4.4), interchanging the roles of s and $s-1$: $p_G^{(1)}(\mathbf{w}) = p_G^{(s)}(\mathbf{w})$, together with the fact that G is left-compressed, implies that

$$w(1) = w(s) + \frac{w(s-1)\sum_{b+1}^{s-2} w(i) + p_D(\mathbf{w})}{\sum_{i \neq 1,s} w(i)}.$$

where $D = ([s], [s-2]^{(2)} \setminus E_s)$.

Multiply by $bw(s-1)^2$ to get the analogue of (4.3):

$$bw(1)w(s-1)^2 \leq bw(s)w(s-1)^2 + \frac{bw(s-1)^3(s-b-2)}{s-3} + \frac{bw(s-1)p_D(\mathbf{w})}{s-2}. \quad (4.7)$$

Now set

$$F = ([s], \{x \cup s \mid x \in E(D)\});$$

if we suppose that the statement of Lemma 4.5 fails to hold, we have $|E(F)| \geq b+1$.

We prove condition (2) first.

We have $p_F(\mathbf{z}) = (w(s-1)+w(s))p_D(\mathbf{w})$, so that $p_F(\mathbf{z}) - bw(1)w(s-1)^2$, thanks to (4.7), is bounded below by

$$w(s)w(s-1)^2 + |E(D)|w(s-1)^3\left(1 - \frac{b}{s-2}\right) - bw(s-1)^3\frac{s-b-2}{s-3}$$

which is positive since

$$|E(D)|\left(1 - \frac{b}{s-2}\right) > b\frac{s-b-2}{s-3}.$$

As for condition (1), we claim the following:

Claim.

$$|[s-2]^{(2)} \setminus E_{s-1}| \leq b$$

This would imply that $|E_{s-1}| \geq \binom{s-2}{2} - b + |E_{s,s-1}| = \binom{s-2}{2}$. Thus the fact that $E(D) \subseteq [s-2]^{(2)}$ would give

$$|E(F)| = |E(D)| \leq \binom{s-2}{2} \leq |E_{s-1}|,$$

and Lemma 4.5 would be entirely proved.

We now restate Claim 4.3 separately as a Lemma, so that its proof will finally establish Talbot's Theorem. □

Lemma 4.6.

$$|[s-2]^{(2)} \setminus E_{s-1}| \leq b$$

Proof. Again, the structure of the proof is very similar to that of the preceding Lemmas, and it draws from the proof of Lemma 4.4 in various places.

Suppose by contradiction $|[s-2]^{(2)} \setminus E_{s-1}| \geq b+1$.

As before, set $\mathbf{z} = \mathbf{w} + w(s)(\mathbf{e}_{s-1} - \mathbf{e}_s)$ and

$$C = ([s], [s-2]^{(2)} \setminus E_{s-1});$$

take

$$F = ([s], \{x \cup \{s-1\} \mid x \in E(C)\}).$$

We verify the usual conditions (1) and (2) for F.

$$p_F(\mathbf{z}) = (w(s) + w(s-1))p_F^{(s-1)}(\mathbf{w}) \geq 2w(s)p_F^{(s-1)}(\mathbf{w}).$$

Since $p_F^{(s-1)}(\mathbf{w}) = p_C(\mathbf{w})$, thanks to (4.4) we get

$$p_F(\mathbf{z}) - bw(1)w(s)^2 \geq w(s)\left(p_C(\mathbf{w})\left(2 - \frac{b}{s-2}\right) - w(s)w(s-1)\beta\right).$$

We have

- $p_C(\mathbf{w})\left(2 - \frac{b}{s-2}\right) \geq w(s)w(s-1)|E(C)|\left(2 - \frac{b}{s-2}\right)$;
- $|E(C)| \geq b + 1$ (our hypothesis by contradiction);
- $2 - \frac{b}{s-2} \geq 1 + \frac{s-b-2}{s-3}$

and three inequalities combined give

$$p_F(\mathbf{z}) - bw(1)w(s)^2 > 0.$$

Condition (1) is verified since $|F| \leq |[s-1]^{(3)} \setminus E| \leq \beta$ by Remark 4.2, and $|E_s| > \beta$ as established at the very end of the proof of Lemma 4.4. □

This completes the proof of Theorem 4.3.

4.4 Some Further Remarks

We conclude this chapter with a few further remarks about the relation between Theorem 4.3 and Conjecture 4.

One first very trivial thing we may notice is the following:

Remark 4.3. *Talbot's Theorem implies Frankl and Füredi's Conjecture for the values of m it covers.*

The 3-graph whose edge set is the m-th initial segment of colex contains the complete 3-graph $[t]^{(3)}$. Hence its Lagrangian is no less than that of $[t]^{(3)}$, and this implies (thanks to Talbot's Theorem) that such a 3-graph has maximal Lagrangian among those with m edges.

Also, an easy argument allows to obtain Conjecture 4 for a few more values of m.

In fact, suppose

$$\binom{t}{3} - 2 \leq m \leq \binom{t}{3} - 1;$$

in such cases, since $m \leq \binom{t}{3}$, a 3-graph with highest Lagrangian will have no more than t vertices, since the proof of Theorem 4.3 does apply (the upper bound for m from (4.1), which is all we need to control the number of vertices, is valid).

Now for the two specific values of m stated above, being a left-compressed subset of $[t]^{(3)}$ actually implies being an initial segment of colex.

This proves Conjecture 4.

As we have seen, Talbot's Theorem answers Frankl and Füredi's question for the value of m which is perhaps the most natural, namely $m = \binom{t}{3}$ for some t; for the values not covered by the Theorem an approximate result can be stated:

Theorem 4.7. *Let m, t, a be such that $-(t-1) \le a \le t-4$, and*

$$m = \binom{t}{3} + \binom{t-1}{2} + a;$$

Let n be the minimum positive integer such that $\lambda^{(3)}_{n,m} = \lambda^{(3)}_{N,m}$ for all $N > n$, and let G be a 3-graph on n vertices having m edges and satisfying $\lambda(G) = \lambda^{(3)}_{n,m}$. Let $C^{(3)}_m$ be the 3-graph on n vertices whose edge set is the m-th initial segment of colex. Then

$$|E(G) \triangle E(C^{(3)}_m)| \le 2(t-a-1)$$

where $A \triangle B$ denotes the symmetric difference between sets A and B.

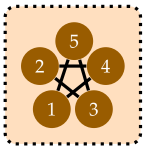
For $k > 3$ the arguments we have used throughout the preceding section do not seem to yield the results we would wish for. We end this chapter with the statement of a last Theorem which Talbot obtains by adapting the proof from Section 4.3, and again we refer to [20] for full details.

We have

Theorem 4.8. *For any $k \ge 4$ there exist γ_k and $t_0(k)$ such that, if m satisfies*

$$\binom{t}{k} \le m \le \binom{t}{k} + \binom{t-1}{k-1} - \gamma_k t^{k-2}$$

for some $t \ge t_0(k)$, then $\lambda^{(k)}_{t+1,m} = \lambda(C^{(k)}_m)$. Thus in this case the k-graph $[t]^{(k)}$ has maximal Lagrangian among those supported on (no more than) $t+1$ vertices, having $\binom{t}{k}$ edges.

CHAPTER 5 

Hypergraphs do Jump

in which flag algebras are first introduced; they will work in combination with Frankl and Rödl's Lemma to show that jumps for 3-graphs do, after all, exist.

5.1 The Idea

Flag algebras, first introduced by Razborov in [18], are the objects of a calculus whose aim is to formalize typical arguments in Extremal Combinatorics, controlling their interactions through a powerful algebraic framework. Flag algebras, when defined with a more abstract, model theoretic approach, are very general objects; they are fit for a number of uses, one of which in particular we shall describe in this chapter.

The reader interested in a complete account should refer to [18]; the flag algebra application to Turán problems which we shall outline is the one used in [19] and subsequently employed by Baber and Talbot to prove that "hypergraphs do jump" after all. It involves a small fraction of the theoretic framework built by Razborov, and could be translated in purely combinatorial terms, with no reference to any Algebra or Measure Theory.

We have chosen, however, to give an account which tries to keep a perspective a little broader than strictly necessary, in hopes of giving an idea of the abstract and quite graceful side of flag algebras, as well as of the strongly present computational side, meanwhile providing some insight intended to help those who will attempt a reading of [18].

The basis of the flag algebra approach to Turán problems lies in the spirit of Lemma 1.3.

Consider a k-graph G with $|V(G)| \geq l$ and let $\mathcal{F}_l^0$ (this notation will be justified in due time) be the set of isomorphism classes of k-graphs on l

vertices. Then

$$d(G) = \sum_{H \in \mathcal{F}_l^0} p(H,G)d(H)$$

where $p(H,G)$ is the probability that a random l-set in $V(G)^{(l)}$ induces the graph H.

This is nothing but Lemma 1.3, since the sum from the right hand side is exactly the mean density of subgraphs of G induced by sets of l vertices.

Suppose now we were to consider a k-graph F, and required G to be F-free. We would have that, for all H in $\mathcal{F}_l^0$ such that $F \subset H$, $p(H,G) = 0$. Hence, if we denote by $\mathcal{H}_l^0$ the set $\{H \in \mathcal{F}_l^0 \,|\, H \text{ is } F\text{-free}\}$, we have

$$d(G) = \sum_{H \in \mathcal{H}_l^0} p(H,G)d(H).$$

Since l is fixed, we can (in theory) compute all members of $\mathcal{H}_l^0$ together with their edge densities. All this can tell us about the Turán density of F, however, is

$$\pi(F) \leq \max_{H \in \mathcal{H}_l^0} d(H)$$

which is hardly surprising (it is nothing but the inequality $\pi(F) \leq \pi(l,F)$) and not very helpful at all, since direct computation is likely to be possible only for small l.

On the other hand, what the systematic approach of flag algebras will render computationally feasible is to find appropriate real coefficients c_H such that

$$\liminf_{n\to\infty} \min_{G \in \mathcal{F}_n^0} \sum_{H \in \mathcal{F}_l^0} c_H p(H,G) \geq 0. \tag{5.1}$$

This would yield that for all $\epsilon > 0$, in the case of G being F-free, we get

$$d(G) + \epsilon \leq \sum_{H \in \mathcal{H}_l^0} p(H,G)(d(H) + c_H)$$

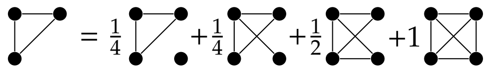
as long as $|V(G)|$ is greater than some $n(\epsilon)$. Thus we would be able to deduce $\pi(F) \leq \max_{H \in \mathcal{H}_l^0} d(H) + c_H$; if some of the coefficients c_H were negative this might be a much better bound than the trivial one from before.

5.2 Flag Algebras

The Algebra $\mathcal{A}^0$ and the Limit Flag Parameters

Define $\mathcal{F}^0 = \bigcup_{l \in \mathbb{N}} \mathcal{F}_l^0$ as the set of all isomorphism classes of k-graphs, and consider the vector space of all formal (finite) linear combinations of k-graphs with real coefficients, namely $\mathbb{R}\mathcal{F}^0$.

the 4 elements in $\mathcal{F}_3^0$ (namely $\overline{K_3}$, $\overline{P_2}$, P_2, K_3)



K_3 as a linear combination of elements in $\mathcal{F}_4^0$

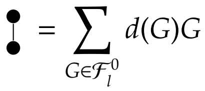


Figure 5.1: Some examples of elements and identities in $\mathcal{A}^0$ (in the case of 2-graphs).

We quotient out the subspace generated by all elements of the form

$$G - \sum_{H \in \mathcal{F}_l^0} p(G, H)H \tag{5.2}$$

where G is a k-graph on $m \leq l$ vertices (l is any positive integer), and obtain a real vector space which we call $\mathcal{A}^0$.

The reason for this choice, as we are about to see, lies in the chain rule

$$p(G, \hat{G}) = \sum_{H \in \mathcal{F}_l^0} p(G, H)p(H, \hat{G}) \tag{5.3}$$

(for $|V(\hat{G})| \geq l$, $|V(G)| \leq l$) which is nothing but a very straightforward generalization of Lemma 1.3, with a proof perfectly analogous to that from Chapter 1: notice that the hypergraph version of Lemma 1.3, using our new terminology, simply states that

$$p(K_k^{(k)}, \hat{G}) = \sum_{H \in \mathcal{F}_l^0} p(K_k^{(k)}, H)p(H, \hat{G})$$

where $K_k^{(k)}$ is the k-hyperedge.

Though we shall not need the full power of the upcoming constructions (the proof relating to jumps can be followed as a mere sequence of correct computations), we would like to present the formalism from [18] and sketch the ideas behind it, so as to introduce the reader to the world opened by Razborov's paper. The main idea (which also justifies our choices so far) is that we wish to build maps of the form

$$\varphi_{\hat{G}} : \sum_i \alpha_i H_i \longmapsto \sum_i \alpha_i p(H_i, \hat{G})$$

"for big $\hat{G}$" into our framework. Such maps, if $\hat{G}$ is fixed, will be zero on most of $\mathcal{A}^0$ ($p(H,G)$ will be 0 whenever $|V(G)| < |V(H)|$); the concept that we actually wish to formalize is that of *limit* maps of this form, for $|V(\hat{G})| \to \infty$.

We introduce a little terminology in order to succeed in such an endeavor.

An *increasing* sequence of k-graphs is a sequence $\mathcal{G} = (G_n)_{n\in\mathbb{N}}$ consisting of elements of $\mathcal{F}^0$ such that $|V(G_{n+1})| > |V(G_n)|$ for all n in $\mathbb{N}$.

We call an increasing sequence $(G_n)_{n\in\mathbb{N}}$ of k-graphs *convergent* if the limit $\lim_{n\to\infty} p(F, G_n)$ exists for all F in $\mathcal{F}^0$.

Remark 5.1. *We remark, since it will be of use later, that any increasing sequence of k-graphs contains a converging subsequence, which can be easily produced via an inductive diagonal argument.*

For each convergent sequence $\mathcal{G}$ of k-graphs we define the map $\varphi_{\mathcal{G}} : \mathcal{A}^0 \to \mathbb{R}$ as

$$\varphi_{\mathcal{G}} : f \longmapsto \lim_{n\to\infty} \varphi_{G_n}(f).$$

Elements of the form (5.1) now do become trivial in the sense that they have image 0 under all maps $\varphi_{\mathcal{G}}$: in fact, for all (big enough) $\hat{G}$

$$\varphi_{\hat{G}}\left(G - \sum_{H\in\mathcal{F}_l^0} p(G,H)H \right) = p(G,\hat{G}) - \sum_{H\in\mathcal{F}_l^0} p(G,H)p(H,\hat{G}) = 0$$

by the above mentioned chain rule, and such a relation is preserved by taking a limit.

Our objective will now be to prove that all maps $\varphi_{\mathcal{G}}$ are nonnegative when evaluated on some fixed element $\sum_{H\in\mathcal{F}_l^0} c_H H$. This yields exactly our requirement of (5.1), as we now prove.

In fact, set

$$\alpha = \liminf_{n\to\infty} \min_{G\in\mathcal{F}_n^0} \sum_H c_H p(H,G)$$

and suppose we have

$$\varphi_{\mathcal{G}}\left(\sum_H c_H H \right) \geq 0$$

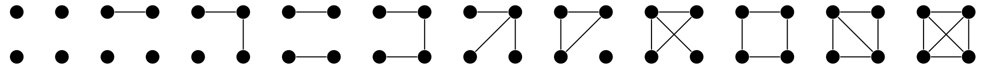
for all convergent sequences $\mathcal{G}$.

Let G_n be a k-graph on $[n]$ for which

$$\sum_H c_H p(H,G_n) = \min_{G\in\mathcal{F}_n^0} \sum_H c_H p(H,G);$$

then

$$\alpha = \liminf_{n\to\infty} \varphi_{G_n}\left(\sum_H c_H H \right)$$



All elements of $\mathcal{F}_4^0$ (in the 2-graph case)

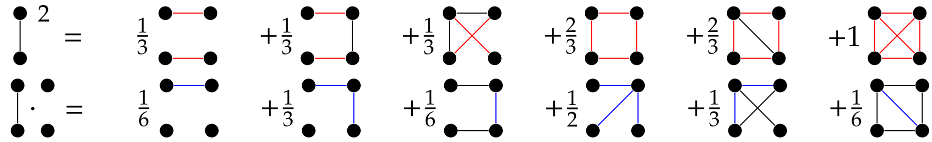


Figure 5.2: Two examples of the computation of products in the algebra $\mathcal{A}^0$.

so there is a subsequence G_{n_k} of G_n such that

$$\alpha = \lim_{n\to\infty} \varphi_{G_{n_k}}\left(\sum_H c_H H\right).$$

$(G_{n_k})_{k\geq 0}$ is an increasing sequence of k-graphs, and thus has a convergent subsequence which we call $\mathcal{H}$.

We get

$$\alpha = \varphi_{\mathcal{H}}\left(\sum_H c_H H\right)$$

which implies $\alpha \geq 0$ (namely, (5.1)).

This enables us to use the coefficients c_H in the argument from the preceding section.

We are ready for the next step in the construction: we endow $\mathcal{A}^0$ with the structure of a commutative associative algebra with 1 in the following way.

Given $H_1 \in \mathcal{F}_{l_1}^0$ and $H_2 \in \mathcal{F}_{l_2}^0$ take $H_1 \cdot H_2$ to be

$$\sum_{H\in\mathcal{F}_l^0} p(H_1, H_2; H)H$$

where $l \geq l_1 + l_2$ and $p(H_1, H_2; H)$ is the probability that, if we choose two disjoint sets W_1 and W_2 such that $|W_i| = l_i$ from the vertex set of H, $H[W_i]$ is (isomorphic to) the graph H_i.

It can be shown easily that, as soon as we quotient out the relations that define $\mathcal{A}^0$, $H_1 \cdot H_2$ does not depend on the choice of l. This is a straightforward application of the chain rule: for $l \geq m \geq l_1 + l_2$

$$\sum_{H\in\mathcal{F}_l^0} p(H_1, H_2; H)H = \sum_{H\in\mathcal{F}_l^0}\sum_{K\in\mathcal{F}_m^0} p(H_1, H_2; K)p(K, H)H =$$

$$= \sum_{K \in \mathcal{F}_m^0} p(H_1, H_2; K) \sum_{H \in \mathcal{F}_l^0} p(K, H)H = \sum_{K \in \mathcal{F}_m^0} p(H_1, H_2; K)K.$$

Extend the operation bilinearly on all of $\mathcal{A}^0$. Commutativity and associativity, as well as the fact that $\emptyset \in \mathcal{A}^0$ is the identity element, are very easy to prove.

The algebra operation is itself introduced to help bring our ideal maps smoothly into the framework, as we shall soon see.

The next key fact that we need is provided by the following lemma.

Lemma 5.1. *Take $F_1 \in \mathcal{F}_{l_1}^0$, $F_2 \in \mathcal{F}_{l_2}^0$; we have*

$$\limsup_{n\to\infty} \max_{G \in \mathcal{F}_n^0} |p(F_1, F_2; G) - p(F_1, G)p(F_2, G)| = 0.$$

Proof. Let $\alpha(n)$ be the probability that two random sets W_1 and W_2, each chosen independently in $[n]^{(l_1)}$ and $[n]^{(l_2)}$, have nonempty intersection. Then, since $p(F_1, F_2; G)$ represents the probability of the event that two sets chosen independently in $V(G)^{(l_1)}$ and $V(G)^{(l_2)}$ induce F_1 and F_2 respectively, conditioned by the event that the two sets have empty intersection (which has probability $1 - \alpha(|V(G)|)$), we have

$$|p(F_1, G)p(F_2, G) - p(F_1, F_2; G)| \leq \alpha(|V(G)|).$$

On the other hand,

$$\alpha(n) = 1 - \frac{\binom{n}{l_1}\binom{n-l_1}{l_2}}{\binom{n}{l_1}\binom{n}{l_2}}$$

hence

$$\lim_{n\to\infty} \alpha(n) = 0$$

which proves our statement. □

Remark 5.2. *Lemma 5.1 is stating nothing but a very intuitive fact: when the cardinality of a set goes to infinity, the effect of sampling becomes negligible.*

Let us go back to our maps $\varphi_{\hat{G}}$; we observe that their behavior on products of graphs is summarized by the expression

$$\varphi_{\hat{G}}(H_1 \cdot H_2) = \sum_{H \in \mathcal{F}_l^0} p(H_1, H_2; H)p(H, \hat{G}) = p(H_1, H_2; \hat{G});$$

noticing that, by Lemma 5.1,

$$\varphi_{\hat{G}}(H_1 \cdot H_2) = p(H_1, \hat{G})p(H_2, \hat{G}) + o(1)$$

(where $o(1)$ is a term which is infinitesimal when $|V(\hat{G})| \to \infty$) we may deduce $\varphi_{\hat{G}}(H_1 \cdot H_2) = \varphi_{\hat{G}}(H_1)\varphi_{\hat{G}}(H_2) + o(1)$.

What this is telling us is that our limit maps $\varphi_{\mathcal{G}}$ (where $\mathcal{G}$ is an increasing convergent sequence of k-graphs) are nothing but algebra homomorphisms from $\mathcal{A}^0$ to $\mathbb{R}$.

In fact, the merit of our framework is that such limit maps are now very easily captured: they are algebra homomorphisms ψ from $\mathcal{A}^0$ to $\mathbb{R}$, subject to the constraint that $\psi(H) \geq 0$ for all $H \in \mathcal{F}^0$ (which encodes the fact that $p(H, G) \geq 0$ for all G). The set of such homomorphisms is denoted by $Hom^+(\mathcal{A}^0, \mathbb{R})$. One containment we have just proved; the other may be proved as a consequence of Lemma 5.1 and of the Borel Cantelli Lemma. In fact it can then be shown that the set

$$C_{sem}(\mathcal{F}^0) = \{f \in \mathcal{A}^0 \mid \psi(f) \geq 0 \forall \psi \in Hom^+(\mathcal{A}^0, \mathbb{R})\}$$

which Razborov calls *semantic cone*, represents in a sense asymptotically true relations in Extremal Hypergraph Theory.

This concept is duly formalized, proved and discussed in Section 3 from [18], to which we refer the interested reader. The results we need for our computations are way more specific, but they fit very well into such a framework.

We remind the reader that our objective is to find an element of $\mathcal{A}^0$ of the form $f = \sum_{H \in \mathcal{H}_l^0} c_H H$ in the semantic cone $C_{sem}(\mathcal{F}^0)$. This we may achieve by taking $f = g^2$ for some $g \in \mathcal{A}^0$, or rather $f = Q(g)$, where Q is an appropriate positive semidefinite quadratic form.

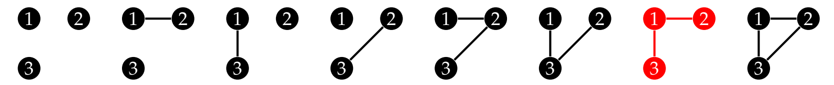
Before we get into the details of this idea, however, we shall generalize the construction we made for $\mathcal{A}^0$, this time in such a way that the algebra product incorporates information about the shape of graph intersections straight into our algebraic structure. The construction will be repeated in an almost identical way, and almost nothing will change in the corresponding statements and proofs (the chain rule and the Lemma corresponding to our Lemma 5.1 will work in a perfectly analogous way) so our exposition will this time be briefer.

Types and the Algebra $\mathcal{A}^\sigma$

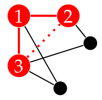
A *k-type* is a labelled k-graph: it is a k-graph of the form $\sigma = ([t], E)$, where this time we do not establish identity between types which are isomorphic as graphs. For example, there are 3 2-graphs on 3 vertices, but there are as many as 8 2-types on [3] (see Figure 5.3).

Given a type σ on $[t]$, a σ-flag is a couple $F = (F^0, \theta)$, where F^0 is a k-graph in $\mathcal{F}^0$ and $\theta : [t] \longrightarrow V(F^0)$ is a graph isomorphism from σ to $F^0[\theta([t])]$; a σ-flag is thus a partially labelled graph.

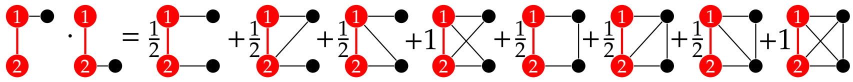
Consider $\mathcal{F}^\sigma$ to be the set of all σ-flags, $\mathcal{F}_l^\sigma$ the set of σ-flags of the form (F^0, θ) with $F^0 \in \mathcal{F}_l^0$.



All 2-types on 3 vertices
(cf. Figure 5.1 where all elements of $\mathcal{F}^0_3$ are drawn)



A σ-flag on 5 vertices, where $\sigma = ([3], \{12, 13\})$. The dotted line between the nodes labelled 2 and 3 represents the fact that a σ-flag is forced *not* to have an edge joining them.



The product of the two drawn flags in $\mathcal{A}^\sigma$, where σ is the labelled edge.

Figure 5.3: Types and flags.

Given two σ-flags $F_1 = (F^0_1, \theta_1)$ and $F = (F^0, \theta)$ on l_1 and l vertices respectively, with $l_1 \leq l$, we may define the value $p(F_1, F)$ as the probability that, if we pick a set W of $l_1 - t$ vertices uniformly at random from $V(F^0) \setminus \theta([t])$, the induced flag $(F^0[\theta([t]) \cup W], \theta)$ is isomorphic to the flag F_1.

We put $\mathcal{F}^\sigma$ through the same process as $\mathcal{F}^0$: consider the real vector space $\mathbb{R}\mathcal{F}^\sigma$ and quotient out the subspace generated by all elements of the form

$$G - \sum_{H \in \mathcal{F}^\sigma_l} p(G, H)H \tag{5.4}$$

to obtain the vector space $\mathcal{A}^\sigma$.

Notice $\mathcal{A}^0$ can be interpreted as the algebra built from the "empty" type.

Our purpose to incorporate information about intersections of the form σ into flags becomes even clearer once we describe $\mathcal{A}^\sigma$'s structure of a commutative algebra.

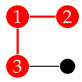
Consider the operation $\mathbb{R}\mathcal{F}^\sigma \times \mathbb{R}\mathcal{F}^\sigma \longrightarrow \mathbb{R}\mathcal{F}^\sigma$ defined as

$$F_1 \cdot F_2 = \sum_{H \in \mathcal{F}^\sigma_l} p(F_1, F_2; H)H$$

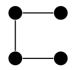
where, if we set $H = (H^0, \theta_H)$, $p(F_1, F_2; H)$ is the probability that, choosing uniformly at random a couple of disjoint subsets W_1 and W_2 of $V(H^0)$ such that $|W_i| = l_i - t$ and $W_i \cap \theta_H([t]) = \emptyset$, $(H^0[\theta_H([t]) \cup W_i], \theta_H)$ is the flag F_i.

The proof that the product (which we extend bilinearly) is well defined on $\mathcal{A}^\sigma$ is perfectly analogous to the one we wrote in the case of $\mathcal{A}^0$: notice that the chain rule is still true on $\mathcal{F}^\sigma$.

We now wish to send elements of $\mathcal{A}^\sigma$ back into $\mathcal{A}^0$ so that we can possibly use them for our argument of Section 5.1. This is done in a very natural way by defining an averaging operator from $\mathcal{A}^\sigma$ to $\mathcal{A}^0$.

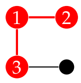


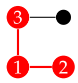
Consider the σ-flag F drawn on the left, where $\sigma = ([3], \{12, 13\})$.



Its corresponding graph in $\mathcal{F}^0$ is the path P_4.

There are $4 \cdot 3! = 24$ injections of $[3]$ into the vertex set of P_4. Of those, only the two that follow actually induce a σ-flag isomorphic to F:

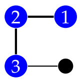




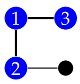
This implies that

$$[F]_\sigma = \frac{1}{12} P_4;$$

below are two examples of injections that do not induce σ-flags isomorphic to F, and therefore must not be counted.



The injection does not induce σ! There should be an edge between 1 and 3, and no edge between 2 and 3.



The injection does induce σ, but there should be an extra edge attached to 3, and 2 should have degree 1.

Figure 5.4: Computation of q_σ.

For a σ-flag $F = (F^0, \theta)$ in $\mathcal{A}^\sigma$, define

$$[F]_\sigma = q_\sigma(F) F^0$$

where $q_\sigma(F)$ is the cardinality of the set

$$\{\eta : [t] \to V(F^0) \mid \text{ such that } (F^0, \eta) \text{ is isomorphic to the flag } F\};$$

divided by the total number of injections of $[t]$ into $V(F^0)$; then extend linearly on all of $\mathcal{A}_\sigma$. Notice the averaging operator is well defined, since

$$[F]_\sigma = q_\sigma(F^0) F^0 = \sum_{H \in \mathcal{F}_l^0} p(F^0, H) q_\sigma(F^0) H = \left[\sum_{H \in \mathcal{F}_l^\sigma} p(F, H) H \right]_\sigma .$$

Positivity in General Flag Algebras

We may define functions φ_G from $\mathcal{A}^\sigma$ to $\mathbb{R}$ in exactly the same way as before, where G is now a σ-flag. This time we consider increasing (and convergent) sequences of σ-flags, and observe that maps $\varphi_{\mathcal{G}}$, where $\mathcal{G}$ is a convergent sequence of σ-flags, are algebra homomorphisms which are non-negative on single σ-flags.

For g in $\mathcal{A}^\sigma$, we write $g \geq 0$ to mean that for all convergent sequences $\mathcal{G}$ of σ-flags we have $\varphi_{\mathcal{G}}(g) \geq 0$ (we could have given the definition in terms of homomorphisms in Hom^+; the only reason we do not is that we have stated, but not completely proved, equivalence of the two concepts).

We have the following lemma:

Lemma 5.2. *Let $f = \sum_i \alpha_i F_i$ be an element of $\mathcal{A}^\sigma$ such that $f \geq 0$. Then $[f]_\sigma \geq 0$.*

Proof. One fist thing we remark is the fact that that, thanks to the defining relations (5.4), we may assume all σ-flags $F_i = (F_i^0, \eta_i)$ to have vertex sets of the same cardinality $|V(F_i^0)| = k$ for some k. This will be assumed throughout the proof.

Consider an increasing convergent sequence of k-graphs $\mathcal{G} = (G_n)_{n\in\mathbb{N}}$; we shall compute $\varphi_{\mathcal{G}}([f]_\sigma)$ and show it cannot be negative. We have

$$[f]_\sigma = \sum_i \alpha_i q_\sigma(F_i) F_i^0$$

therefore

$$\varphi_{G_n}([f]_\sigma) = \sum_i \alpha_i q_\sigma(F_i) p(F_i^0, G_n).$$

Write G for G_n; consider now for each i the value of

$$\sum_\theta p(F_i, G_\theta)$$

where the sum is taken over all injections θ of $[t]$ into $V(G)$ such that $G_\theta = (G, \theta)$ is a σ-flag. Without loss of generality assume $G = G_n$ has n vertices. We call $N(F_i, G_\theta)$ the number of k-sets $W \subseteq V(G)$ which contain $\theta([t])$ and are such that $(G[W], \theta)$ is the flag F_i, so that

$$p(F_i, G_\theta) = \frac{N(F_i, G_\theta)}{\binom{n-t}{k-t}}.$$

Notice that (by a simple double counting of spanned copies of F present in G) we have

$$\sum_\theta N(F_i, G_\theta) = q_\sigma(F_i)\binom{k}{t} t! N(F_i^0, G)$$

where the quantity $q_\sigma(F_i)\binom{k}{t}t!$ represents the number of injections of $[t]$ into $V(F_i^0)$ which yield a σ-flag isomorphic to F_i, and $N(F_i^0, G)$ is the number of k-sets in $V(G)^{(k)}$ which span a copy of F_i^0.

Hence

$$q_\sigma(F_i^0)p(F_i^0, G) = \frac{\binom{n-t}{k-t}}{\binom{n}{k}\binom{k}{t}t!}\sum_\theta p(F_i, G_\theta) = \frac{1}{\binom{n}{t}t!}\sum_\theta p(F_i, G_\theta).$$

Summing over i, we get that

$$\varphi_G([f]_\sigma) = \frac{1}{\binom{k}{t}t!}\sum_i \alpha_i \sum_\theta p(F_i, G_\theta) = \frac{1}{\binom{k}{t}t!}\sum_\theta \varphi_{G_\theta}(f).$$

Consider now an increasing sequence of σ-flags $\mathcal{H} = (H_n)$ built in such a way that $H_n = G_{n\eta}$, where

$$\varphi_{G_{n\eta}}(f) = \min_\theta \varphi_{G_{n\theta}}(f).$$

We have

$$\varphi_{G_n}([f]_\sigma) \geq p(G_n)\varphi_{H_n}(f),$$

where $p(G_n)$ is the probability that a random injection θ of $[t]$ into $V(G_n)$ makes (G_n, θ) a σ-flag.

Clearly, $p(G_n)$ is bounded (it is in $[0,1]$); and thus we must have

$$\lim_{n\to\infty} \varphi_{G_n}([f]_\sigma) \geq \limsup_{n\to\infty} p(G_n)\varphi_{H_n}(f).$$

Take an increasing sequence of indices $(n_k)_{k\in\mathbb{N}}$ such that $p(G_{n_k})$ converges, and also $(H_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $\mathcal{H}$. By our hypothesis (positivity of f as an element of $\mathcal{A}^\sigma$) we have that

$$\lim_{k\to\infty} p(G_{n_k})\varphi_{H_{n_k}}(f) \geq 0;$$

this finally yields

$$\lim_{n\to\infty} \varphi_{G_n}([f]_\sigma) \geq 0.$$

□

The Semidefinite Problem

We now have all the theoretic ingredients we need in order to describe and apply our method. What follows should now be viewed as just a sequence of perfectly trivial remarks.

For one thing, we have acquired a large family of elements of our algebra which must be trivially nonnegative in our sense: namely, as already anticipated, squares.

But we can go a little further with no effort at all: let $f = \sum_{i=1}^n \alpha_i F_i$ be an element of $\mathcal{A}^\sigma$ and $Q = (q_{ij})_{i,j\in[n]}$ a real positive semidefinite $n \times n$ matrix. Then clearly we also have

$$Q(f) = \sum_{i,j\in[n]} q_{ij}\alpha_i\alpha_j F_i \cdot F_j \geq 0.$$

Now remember our aim: we were looking for nonnegative elements of the form

$$\sum_{H \in \mathcal{F}_l^0} c_H H$$

where l we fix in advance; hopefully our coefficients will minimize

$$\max_{H \in \mathcal{H}_l^0} d(H) + c_H.$$

We have obtained a great number of candidates to this role: all those elements of the form $[Q(f)]_\sigma$, where σ is a type on $t \le l$ vertices, $f = \sum_i \alpha_i F_i$ is a linear combination of flags on no more than $\frac{l+t}{2}$ vertices, and Q is a positive semidefinite matrix.

In fact, choose a type σ (of cardinality, say, t); let $m \le \frac{l+t}{2}$ and take the element of $\mathcal{F}^\sigma$

$$f = \sum_{K \in \mathcal{H}_m^\sigma} K = \sum_i K_i$$

(where we choose to extend the sum over F-free σ-flags only, since graphs which are not F-free do not contribute in any way to the final part of the computation). Our problem is now to find Q such that

$$\max_{H \in \mathcal{H}_l^0} d(H) + \sum_{i,j \in [n]} q_{ij} \sum_{H_\theta = (H,\theta) \in \mathcal{H}_l^\sigma} q_\sigma(H_\theta) p(K_i, K_j; H_\theta)$$

is minimized.

We thus end up with a semidefinite problem which is tractable by computational methods: the field of semidefinite programming is well developed and offers suitable algorithms to help us in our quest. Of course difficulties arise unless our number l is quite small (since we need to count spanned subgraphs of a number of isomorphism classes, which is computationally difficult) but the striking thing about flag algebras is that even a small l may lead to drastic improvements in the bounds for a Turán density.

5.3 An Example

As an example we make a very easy computation in the case of 2-graphs which, in spite of being of no actual consequence in terms of results, will hopefully better acquaint the reader with the method and serve to clarify any doubt about the way it is applied. Suppose we were to try and compute the Turán density of the triangle K_3 (which, of course, we already know to be 1/4).

A first extremely rough estimate could be obtained by listing all triangle-free graphs on 3 vertices (thus all graphs on 3 vertices save the triangle, see

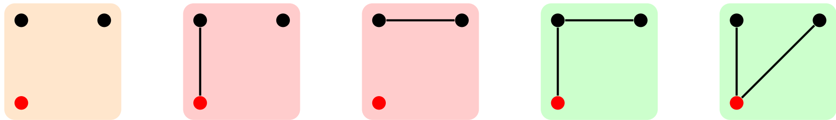
A	B	C	D	E
				
representatives of isomorphism class as graphs:				
$\overline{K_3}$	$\overline{P_2}$	$\overline{P_2}$	P_2	P_2
densities:				
0	1/3	1/3	2/3	2/3
corresponding coefficients $q_\sigma(\cdot)$:				
1	2/3	1/3	2/3	1/3
tables containing $\begin{array}{c\|c} p(F_0,F_0;\cdot) & p(F_0,F_1;\cdot) \\ \hline p(F_1,F_0;\cdot) & p(F_1,F_1;\cdot) \end{array}$:				
$\begin{array}{c\|c} 1 & 0 \\ \hline 0 & 0 \end{array}$	$\begin{array}{c\|c} 0 & 1/2 \\ \hline 1/2 & 0 \end{array}$	$\begin{array}{c\|c} 1 & 0 \\ \hline 0 & 0 \end{array}$	$\begin{array}{c\|c} 0 & 1/2 \\ \hline 1/2 & 0 \end{array}$	$\begin{array}{c\|c} 0 & 0 \\ \hline 0 & 1 \end{array}$

Table 5.1: Summaries of values relevant to the computation of $[Q(f)]_\sigma$.

below); clearly, the one with the highest density is the path P_2, which only gives us

$$\pi(K_3) \leq 2/3.$$

$d = 0$ $\quad$ $d = 1/3$ $\quad$ $d = 2/3$

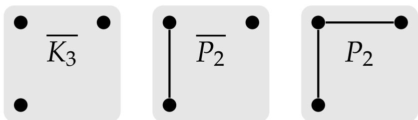


Figure 5.5: All elements of $\mathcal{H}_3^0$, where $F = K_3$.

Suppose we wanted to apply the method of flag algebras to obtain something better.

Since we chose $l = 3$, we can take the type σ to be $(\{1\}, \emptyset)$ (a single node) and consider the element of $\mathcal{A}^\sigma$

$$f = F_0 + F_1$$

where F_0 is the graph on 2 vertices with no edges (and one labelled vertex), and F_1 is the edge with one labelled vertex.

We need to consider

$$[Q(f)]_\sigma = \sum_{H \in \mathcal{H}_3^\sigma} \sum_{i,j} q_{ij} p(F_i, F_j; H) q_\sigma(H) H^0$$

for any positive semidefinite real 2×2 matrix $Q = (q_{ij})_{i,j\in\{0,1\}}$; hence we simply compute the probabilities $p(F_i, F_j; H)$ which are summarized in Table 5.1.

We get

$$[Q(f)]_\sigma = q_{00}\overline{K_3} + \left(\frac{2}{3}\frac{1}{2}(q_{01} + q_{10}) + \frac{1}{3}q_{00}\right)\overline{P_2} + \left(\frac{2}{3}\frac{1}{2}(q_{01} + q_{10}) + \frac{1}{3}q_{11}\right)P_2;$$

we need to choose Q in order to minimize

$$\max\left\{q_{00}, \frac{1}{3} + \frac{q_{01} + q_{10}}{3} + \frac{q_{00}}{3}, \frac{2}{3} + \frac{q_{01} + q_{10}}{3} + \frac{q_{11}}{3}\right\}.$$

The computation is easily done by hand, and yields the positive semidefinite matrix

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

which gives us

$$\pi(K_3) \leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2} < \frac{2}{3};$$

the bound, though we worked on extremely small graphs, is still far better than the trivial estimate given by $\pi(3, K_3)$.

5.4 Hypergraphs do Jump

Baber and Talbot used the method outlined in the preceding sections to show

Theorem 5.3. *Every α in $[0.2299, 0.2316)$ is a jump for 3-graphs.*

This is done by using Frankl and Rödl's Lemma 3.7: we show that there is a finite family $\mathcal{F}$ of 3-graphs such that $\pi(\mathcal{F}) \leq 0.2299$ and $6\lambda(F) > 0.2316$ for all F in $\mathcal{F}$.

In fact, the proof originates from an attempt by Erdős, who was trying to show that $2/9$ is a jump for 3-graphs: remember that $2/9 = 3!/3^3$ is the first value in $[0, 1)$ for which we do not know whether it is a jump for 3-graphs or not, see Section 3.2.

Erdős suggested that it might be possible to prove $\pi(\widehat{\mathcal{F}}) \leq 2/9$, where $\widehat{\mathcal{F}}$ consists of the three 3-graphs

$$F_1 = ([4], \{123, 134, 124\}),$$

$$F_2 = ([5], \{123, 124, 125, 345\}),$$

$$F_3 = ([5], \{123, 124, 235, 145, 345\}).$$

This is not true: in fact, it can be shown by taking appropriate blowups of some $\widehat{\mathcal{F}}$-free graphs that $\pi(\widehat{\mathcal{F}}) \geq 0.2319 > 2/9$.

On the other hand, we can compute Lagrangians for F_1, F_2 and F_3 without great difficulty (make use of Lemma 2.3). We get $6\lambda(F_1) = 8/27$, $6\lambda(F_2) = \frac{189+15\sqrt{5}}{961}$ and $6\lambda(F_3) = 6/25$, so $6\lambda(F_1) > 6\lambda(F_3) > 6\lambda(F_2) > 2/9$.

One possible way ahead is to try adding 3-graphs to the family, making sure their Lagrangian is more than 2/9, in such a way that the Turán density of the family is lowered to a value of 2/9 or less.

We add the two $\widehat{\mathcal{F}}$-free 3-graphs

$$F_4 = ([7], \{123, 135, 145, 245, 126, 246, 346, 356, 237, 147, 347, 257, 167\}),$$

$$F_5 = ([7], \{123, 124, 135, 145, 236, 346, 256, 456, 247, 347, 257, 357, 167\}).$$

Set $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\}$. It can be shown that $\lambda(F_4) > \lambda(F_2)$ and $\lambda(F_5) \geq \lambda(F_2)$.

It is still not true that $\pi(\mathcal{F}) \leq 2/9$. However, estimates with the previously explained flag algebra method lead us to

Lemma 5.4.

$$\pi(\mathcal{F}) \leq 0.2299$$

Lemma 5.4, together with the fact that $6\lambda(F_2) = 0.2316 = \min_{F\in\mathcal{F}} 3!\lambda(F)$, establishes Theorem 5.3.

Lemma 5.4 is proved by taking $l = 7$ and building an element of $\mathcal{A}_7^0$ of the form $f = \sum_{H\in\mathcal{H}_7^0} c_H H \geq 0$.

$\mathcal{H}_7^0$ has 4042 elements, which have been computed explicitly; Baber and Talbot use the types

- $\sigma_1 = ([1], \emptyset)$, $t_1 = 1$
- $\sigma_2 = ([3], \emptyset)$, $t_2 = 3$
- $\sigma_3 = ([3], \{123\})$, $t_3 = 3$
- $\sigma_4 = ([5], \{123, 124, 135\})$, $t_4 = 5$
- $\sigma_5 = ([5], \{123, 124, 345\})$, $t_5 = 5$
- $\sigma_6 = ([5], \{123, 124, 135, 245\})$, $t_6 = 5$

and compute the element

$$\sum_H c_H H = \sum_{i=1}^{6} \left[Q_i \left(\sum_{F\in\mathcal{H}^{\sigma_i}_{\frac{7+t_i}{2}}} F \right) \right]_{\sigma_i}$$

where each Q_i is chosen (tanks to a semidefinite programming library) in order to minimize

$$\max_{H\in\mathcal{H}_7^0} d(H) + c_H.$$

Of course it would have been ideal if all of the types on no more than $5 = 7-2$ vertices were used, but the problem turned out to be computationally intractable.

One last remark that we wish to make in reassurance of the worried reader is that there is no issue of rounding errors possibly introduced in the computation. The only part of the computation where floating point arithmetic is employed is where the semidefinite program solver [2] chosen by Baber and Talbot comes into play; its role is only the selection of appropriate positive semidefinite matrices: the fact that they are indeed positive semidefinite can be easily checked a posteriori using only integer operations, eliminating such an issue completely.

5.5 Some Results and Some Open Problems

Baber and Talbot, as well as Razborov, use the "flag algebra" method to give estimates for some more Turán densities.

In particular, Baber and Talbot give bounds for $\pi(K_4^-)$, where

$$K_4^- = ([4], \{123, 124, 134\})$$

is the 3-graph on [4] with one missing edge. They prove the upper bound in

$$\frac{2}{7} \leq \pi(K_4^-) \leq 0.2871.$$

This fact, combined with $\lambda(K_4^-) = \frac{4}{81}$ (which we proved in Example 2.3), enables us to also state

Theorem 5.5. *Every α in $[0.2871, \frac{8}{27})$ is a jump for 3-graphs.*

Going back to the original interval of jumps from Theorem 5.3, though its existence is in itself a very remarkable fact, we have made no apparent progress towards our original goal of proving whether or not 2/9 is a jump for 3-graphs. In fact, the problem seems to be very hard: we know that $\pi(\widehat{\mathcal{F}}) > 2/9$: proving that 2/9 is a jump (with the method used so far) would imply giving extremely precise estimates for the Turán density of some unknown family of hypergraphs.

On the other hand, we cannot hope to produce a proof that $k!/k^k$ is a jump for k-graphs following the lines of Frankl and Rödl's argument from Chapter 3: we would end up needing to prove $\lambda(H) \leq 1/k^k$, where H is any (small) subgraph of some big graph K. Since $1/k^k$ is the Lagrangian of the hyperedge, this we would not be able to do.

One last remark we want to make concerns the general method provided by flag algebras. It is very natural to wonder how far the process of choosing (a finite number of) appropriate k-types, elements of $\mathcal{A}^\sigma$ and quadratic forms can take us: in fact,

> ? *is it true that any nonnegative element f of $\mathcal{A}^0$ can be proved to be such by a finite computation of the kind discussed in the preceding sections?*

This question is formalized by Razborov (actually, a number of slightly different variants are proposed) by means of a “Cauchy-Schwartz” calculus whose statements are of the form $f \geq 0$, where $f \in \mathcal{A}^\sigma$, whose axioms are “trivial” inequalities (all those of the form $f^2 \geq 0$ and all instances of a Cauchy-Schwartz inequality which can be proved on flag algebras), with a number of very natural inference rules. (It should perhaps be noted that Razborov’s construction contains elements a little more general than the ones we considered within our account, though they are nothing really new to us: in particular, there are general averaging operators $[\cdot]_{\sigma,\eta}$ from $\mathcal{A}^\sigma$ to $\mathcal{A}^\eta$, defined analogously to our $[\cdot]_\sigma$.)

Razborov’s question, which is strongly related to our own from above, is whether or not his Cauchy-Schwartz calculus is complete.

This has been answered by Hatami and Norine in 2010: it is not.

In the paper [11] they consider the question

> ? *Given a positive integer r, r graphs $H_1, \ldots, H_r$ and r integers $a_1, \ldots, a_r$, does the inequality*
> $$\sum_{i\in[r]} a_i p(H_i, G) \geq 0$$
> *hold for all graphs G?*

They show, via a reduction to Hilbert’s 10th problem, that such a question is in general undecidable.

APPENDIX A

Table of symbols

$[n]$	$\{1, \ldots, n\}$
$X^{(k)}$	$\{Y \subseteq X \mid \lvert Y\rvert = k\}$
$X^{(\leq k)}$	$\{Y \subseteq X \mid \lvert Y\rvert \leq k\}$
$\mathcal{P}(X)$	$\{Y \subseteq X\}$
$K_t^{(k)}$	the complete k-graph on $[t]$
P_n	the path of length n
$d(G)$	the edge density of G
$G[W]$	subgraph induced by G on W
$\chi(G)$	the chromatic number of G
$\pi(F)$	the Turán density of F
$\pi(n, F)$	the edge density of a k-graph on $[n]$ with $ex(n, F)$ edges
$G(\mathbf{t})$	the $\mathbf{t}$-blowup of G
$p_G(\mathbf{x})$	the Lagrange polynomial for G
$\lambda(G)$	the Lagrangian of G
$\lfloor \cdot \rfloor$	floor
$\lceil \cdot \rceil$	ceiling
$\mathbf{e}_i$	the i-th vector in the standard basis of $\mathbb{R}^n$
$\mathbf{1}$	the vector $(1, \ldots, 1)$ in $\mathbb{R}^n$
$p^{(i)}(\mathbf{x})$	the derivative of the polynomial p with respect to the i-th variable
LHS	left hand side
RHS	right hand side
AM	Arithmetic Mean
GM	Geometric Mean
$\mid$	"divides"

Index of Definitions

Bibliography

[1] R. Baber and J. Talbot. Hypergraphs do jump. *Comb. Probab. Comput.*, 20:161–171, 2011.

[2] B. Borchers. CSDP, a C library for semidefinite programming. *Optim. Methods Software*, (11):613–623, 1999.

[3] W. G. Brown. On an open problem of Paul Turán concerning 3-graphs. In *In Studies in pure mathematics*, pages 91–93. Birkhauser, 1983.

[4] P. Erdős. On some extremal problems on r-graphs. *Discrete Mathematics*, 1(1):1 – 6, 1971.

[5] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc*, (52):1087–1091, 1946.

[6] P. Frankl and Z. Füredi. Extremal problems whose solutions are the blowups of the small Witt-designs. *J. Comb. Theory Ser. A*, 52:129–147, July 1989.

[7] P. Frankl and V. Rödl. Hypergraphs do not jump. *Combinatorica*, 4:149–159, 1984. 10.1007/BF02579215.

[8] Z. Füredi. Turán type problems. In *Surveys in Combinatorics*, pages 253–300. Cambridge Univ. Press, 1991.

[9] Z. Füredi and M. Simonovits. Triple systems not containing a fano configuration. *Combinatorics, Probability and Computing*, 14(04):467–484, 2005.

[10] R. L. Graham and B. L. Rothschild. *Ramsey theory (2nd ed.)*. Wiley-Interscience, New York, NY, USA, 1990.

[11] H. Hatami and S. Norine. Undecidability of linear inequalities in graph homomorphism densities. *J. Amer. Math. Soc.*, 24(02):547–565, 2011.

[12] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán. In *in the theory of graphs, (Hungarian, English summary), Mat. Lapok 15*, pages 228–238, 1964.

[13] P. Keevash. Hypergraph Turán problems. In *Surveys in Combinatorics 2011*. Cambridge University Press, 2011.

[14] P. Keevash and B. Sudakov. The Turán number of the Fano plane. *Combinatorica*, (25):561–574, 2005.

[15] A. V. Kostochka. A class of constructions for Turán's (3,4)-problem. *Combinatorica*, (2):187–192, 1982.

[16] W. Mantel. Problem 28, solution by. *Wiskundige Opgaven*, (10):60–61, 1907.

[17] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. *J. Math*, (17):533–540, 1965.

[18] A. Razborov. Flag algebras. *J. Symbolic Logic*, (72), 2007.

[19] A. Razborov. On 3-hypergraphs with forbidden 4-vertex configurations. *SIAM J. Discrete Math.*, pages 946–963, 2010.

[20] J. M. Talbot. Lagrangians of hypergraphs. *Comb. Probab. Comput.*, 11:199–216, 2002.

[21] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, (48):436–452, 1941.

Thanks

To Davide, who was the very first to listen to a confusing account of Lagrangians and blowups and jumps, on a sunny afternoon in a comfortably empty coffee room, because that was the moment when a shaky outline of this work actually materialized before my mind's eye; I dearly hope my blabbering did not dissuade him from any possible future contact with Combinatorics and Graph Theory.

To Bea, because I'm not sure I would have survived this August without the beauty and peace and adventure of that week in Sardinia.

To my aunt Beatrice and my uncle Clemente, because I'm entirely sure I would not have survived this August without their patient and utterly undeserved looking after: I hope they did not mind too much that I haunted their house as a thesis-writing, internet-connection-seeking ghost, muttering of mathematics under my breath, and conveniently materializing in the kitchen every day at lunch and dinner time.

To my little cousins: Ilaria, Federico, Mattia, who were a pleasant distraction and a delightful source of inspiration; because the curiosity of a mathematician is still quite the same as that of a child who's baffled by the mystery of adding and multiplying integers, and keeps repeating the same computation with a tinge of doubt and and a sea of underlying awe. I'm glad I was not given the chance to forget this.

To Simone, whose "well this is false isn't it?" I have learned to expect and to treasure: when I needed someone to listen and nod at random intervals, he was there.

To Dario, who did not only help me retain my sanity by playing frequent duets with me (he won't admit it, but he's an exceedingly good pianist): he was also an attentive and interested reader, and his comments have been very encouraging and constructive at a time when I most needed them to be.

A Master's thesis represents graduation, and graduation tends to sum up five long years of study, and people, and places, and life. Mathematics is not something you can do for anyone other than yourself, but I am strongly aware that one of the reasons I am still here is the chance I was given to share it, together with many other things, with a bunch of some of the most exciting, passionate, crazy, boring, lovable dorks that ever existed.

I cannot keep myself from thanking all of my friends: those who will walk the same path as me for some time – whatever that may be – and those who are leaving for new adventures. I know that every one of them will go on to become a most precious element of some maths or physics department, or whatever kind of workplace their future holds; and I hope they will bring to themselves and the people around them the same happiness that they have often brought me. I am giving them the last page of this thesis as a (very foolish) present, which I hope they may enjoy.

Have fun!