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SHAPE OPTIMIZATION PROBLEMS OF HIGHER CODIMENSION

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Shape Optimization Problems of Higher Codimension

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Contents

Foreword

The field of shape optimization problems has received a lot of attention in recent years, particularly in relation to a number of applications in physics and engineering that require a focus on shapes instead of parameters or functions. In general for applications the aim is to deform and modify the admissible shapes in order to optimize a given cost function. The fascinating feature is that the variables are shapes, i.e., domains of $\mathbb{R}^d$, instead of functions. This choice often produces additional difficulties for the existence of a classical solution (that is an optimizing domain) and the introduction of suitable relaxed formulation of the problem is needed in order to get a solution which is in this case a measure. However, we may obtain a classical solution by imposing some geometrical constraint on the class of competing domains or requiring the cost functional verifies some particular conditions. The shape optimization problem is in general an optimization problem of the form

$$\min\{F(\Omega) : \Omega \in \mathcal{O}\},$$

where F is a given cost functional and $\mathcal{O}$ a class of domains in $\mathbb{R}^d$. They are many books written on shape optimization problems; among them we may cite [4], [18], [21], [23], [31], [55], [63], [64], [67], [70], [73], [79] [80], [87], [92]. The thesis is organized as follows: the first chapter is dedicated to the brief introduction and presentation of some examples. In Academic examples, we present the isoperimetric problems, minimal and capillary surface problems and the spectral optimization problems while in applied examples the Newton's problem of optimal aerodynamical profile and optimal mixture of two conductors are considered. The second chapter is concerned with some basics elements of geometric measure theory that will be used in the sequel. After recalling some notions of abstract measure theory, we deal with the Hausdorff measures which are important for defining the notion of approximate tangent space. Finally we introduce the notion of approximate tangent space to a measure and to a set and also some differential operators like tangential differential, tangential gradient and tangential divergence. The third chapter is devoted to the topologies on the set of domains in $\mathbb{R}^d$. Three topologies induced by convergence of domains are presented namely the convergence of characteristic functions, the convergence in the sense of Hausdorff and the convergence in the sense of compacts as well as the relationship between those different topologies. In the fourth chapter we present a shape optimization problem governed by linear state equations. After dealing with the continuity of the solution of the Laplacian problem with respect to the domain variation (including counter-examples to the continuity and the introduction to a new topology: the γ- convergence), we analyze the existence of optimal shapes and the necessary condition of optimality in the case where an optimal shape exists. The shape optimization problems governed by nonlinear state equations are treated in chapter five. The plan of study is the same as in chapter four that is continuity with respect to the domain variation of the solution of the p-Laplacian problem (and more general operator in divergence form), the existence of optimal shapes and the necessary condition of optimality in the case where an optimal shape exists. The last chapter deals with asymptotical shapes. After recalling the notion of Γ-convergence, we study the asymptotic of the compliance functional in different situations. First we study the asymptotic of an optimal p-compliance-networks which is the compliance associated to p-Laplacian problem with control variables running in the class of one dimensional closed connected sets with assigned length. We provide also the connection with other asymptotic problems like the average distance problem. The asymptotic of the p-compliance-location which deal with the compliance associated to the p-Laplacian problem with control variables running in the class of sets of finite numbers of points, is deduced from the study of the asymptotic of p-compliance-networks. Secondly we study the asymptotic of an optimal compliance-location. In this case we deal with the compliance associated to the classical Laplacian problem and the class of control variables is the class of identics n balls with radius depending on n and with fixed capacity.

Main Notations

$\mathcal{L}^k$	k-dimensional Lebesgue measure
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\frac{d\mu}{d\nu}$	Radon-Nikodym derivative of μ w.r.t. ν
$\mu \ll \nu$	the measure μ is absolutely continuous w.r.t. ν
$\mu \perp \nu$	measures μ and ν are mutually singular
$\mu_n \rightharpoonup \nu$	the sequence of measures μ_n converges weakly to the measure μ
$\mu_n \overset{*}{\rightharpoonup} \nu$	the sequence of measures μ_n converges *weakly** to the measure μ
$\mu = f\nu$	measure μ absolutely continuous w.r.t. ν
$A \Subset B$	A has compact closure in B
χ_A	characteristic function of A e.i. $\chi_A(X) = 1$ if $x \in A$ and 0 otherwise
$\mathbb{S}_r^{k-1}$	sphere of radius r in $\mathbb{R}^k$
$B_r(x)$	ball of radius r centered at x
$d^H(K_1, K_2)$	Hausdorff distance between compact sets K_1 and K_2
$d_H(A, B)$	Hausdorff distance between open sets A and B
$\mathcal{O}$	class of domains in $\mathbb{R}^d$
$P(\Omega)$	perimeter of the set Ω
$P_D(\Omega)$	perimeter of the set Ω relative to D
$\partial\Omega$	boundary of the set Ω
$C^0(\Omega, B)$	space of continuous functions from Ω to B
$C_c(\Omega)$	space of continuous functions with compact support in Ω
$C_c^\infty(\Omega, B)$	space of smooth functions from Ω to B with compact support
$L^p(\Omega)$	space of p-sommable Lebesgue measurable functions
$W^{1,p}(\Omega)$	standard Sobolev space of L^p functions, with distributional gradient in L^p
$W_0^{1,p}(\Omega)$	closure of $C_c^\infty(\Omega)$ in the $W^{1,p}$ norm
$\mathcal{P}(\Omega)$	space of Borel probability measures over Ω
div	divergence operator, if $u : \mathbb{R}^d \to \mathbb{R}^d$ then $\mathrm{div} u = \sum_{j=1}^d \frac{\partial u_j}{\partial x_j}$
∇	gradient operator, if $u : \mathbb{R}^d \to \mathbb{R}$ then $\nabla u = \left(\frac{\partial u}{\partial x_1} \cdots \frac{\partial u}{\partial x_d} \right)$
Δ	Laplace operator, if $u : \mathbb{R}^d \to \mathbb{R}$ then $\Delta u = \sum_{j=1}^d \frac{\partial^2 u_j}{\partial x_j^2}$
Δ_p	p-Laplace operator, if $u : \mathbb{R}^d \to \mathbb{R}$ then $\Delta_p u = \sum_{j=1}^d \frac{\partial}{\partial x_j} \left(\lvert\nabla u\rvert^{p-2} \frac{\partial u}{\partial x_j} \right)$
$d^E f$	tangential differential of f on E
$\mathrm{div}^E f$	tangential divergence of f on E
$\nabla^E f$	tangential gradient of f on E

Chapter 1

Introduction and Examples

This first chapter is dedicated to the brief introduction and presentation of some examples. After a brief presentation of the shape optimization problems, we describe two categories of examples: academics and applied. In Academic examples, we present the isoperimetric problems, minimal and capillary surface problems and the spectral optimization problems while in applied examples the Newton's problem of optimal aerodynamical profile and optimal mixture of two conductors are considered.

1.1 Introduction

In this chapter, we present a shape optimization problem in a general setting and describe some practical shape optimization problems or some problems that can be seen as a shape optimization problems. A shape optimization problem is an optimization problem of the form:

$$\min\{F(\Omega) : \Omega \in \mathcal{O}\},$$

where F is a given cost functional that has to be optimized and $\mathcal{O}$ a class of domains in $\mathbb{R}^d$. In this kind of optimization problems, we do not have always the existence of an optimal solution and we may need some additional conditions under which the existence of an optimal shape occurs. These conditions are either restricting the set of competing domains or assume the cost functional to satisfy some particular form. We want to stress that, in several situations an optimal domain does not exist; this is mainly due to the fact that in these cases the minimizing sequences are highly oscillating and converge to a limit object which is a measure. Then the solutions in these cases are measures (in general not domains). One class of optimal shape problems is the following: minimizing the integral functional of the form

$$\int_D F(x, u(x), \nabla u(x))dx$$

where u is the solution of some partial differential equation solved on Ω subset of D. In this thesis, we will deal mostly with this kind of shape optimization problems where u is a solution of an elliptic equation in divergence form with Dirichlet boundary conditions. A variant of this formulation is that of control problems; in this formulation the shape is in general the control. In this case one minimizes an energy or work functional with respect to the design parameters. The nature of these parameters can vary. They may reflect material properties of the structure. In this case, the control variables enter into coefficients of differential equation (as in the example of an optimal mixture of two conductors given below). If one optimizes the distribution of loads applied to the structure, then the control variables appear in the right hand side of the equation.

If an optimal shape exists, in general we do not know it. In order to give a qualitative description of the optimal solutions of a shape optimization problem, it is important to derive the so-called necessary conditions of optimality. These conditions, as it usually happens in all optimization problems, have to be derived from the comparison of the cost of an optimal solution Ω to the cost of other suitable admissible domains, close enough to Ω. This procedure is what is usually called a variation near the solution. The difficulty in obtaining necessary conditions of optimality for shape optimization problems consists in the fact that, being the unknown domain, the notion of neighborhood is not a priori clear; the possibility of choosing a domain variation could then be rather wide. The same method can be applied, when no classical solution exists, to relaxed solutions, and this will provide qualitative information about the behavior of minimizing sequences of the original problem. One may be interested also in other questions such as the geometric or topological properties of an optimal shape (symmetry, convexity, connectedness, open sets...) and other regularity properties. In general we may distinguish three branches of shape optimization.

1. Sizing optimization: a typical size of a structure is optimized (for example, a thickness distribution of a beam or a plate);

2. shape optimization itself: the shape of a structure is optimized without changing the topology;

3. topology optimization: the topology of a structure, as well as the shape is optimized.

In the following section, some examples of shapes are described.

1.2 Some academic examples

We present three academic examples. The isoperimetric problems which have many variants and are seen as one of the classical example of shape optimization problems, the minimal and capillary surfaces and the spectral optimization problems are discussed.

1.2.1 Isoperimetric Problems

The Isoperimetric problems go back to the antiquity. One classical example is as follows: We have in possession the cloture of given length and we want to find a shape of a camp that can be enclosed by this cloture and has maximal area. It is well known since Greek mathematicians that the solution of this problem is a disc. The mathematical formulation is the so called isoperimetric inequality which is: if Ω is a planar domain with finite area ($|\Omega| < \infty$) and of perimeter $P(\Omega)$, then

$$|\Omega| \leq \frac{1}{4\pi} P(\Omega)^2$$

and the equality occurs when Ω is a disc. A variance of this problem is sometimes associated to the name of queen Dido. In fact Dido was daughter of the phenician king Tiro. Her brother Pygmalion, after the dead of their father, killed Dido's husband which was a rich and powerful priest of the God Melkart. So she decided to leave with her husband's treasures and some followers and docked at the African north coast. There she bought from the king of Messitania, Jarbas as much land as it can be contained in ox-skin. Dido cut that skin in many thin strips and them she stringed them together getting the longest possible strip of skin. The queen Dido chosen to draw her Cartage (new kingdom or new city) in a such a way that one border is the African coast (fix border) and the other free border (materialized by strip of skin) is an arc of circle which provided effectively a domain with the greatest possible area. This gives a solution to the isoperimetric problem.

An analogous isoperimetric inequality in any dimension d can be written as follows:

$$|\Omega|^{d-1} \leq \frac{1}{d^d w_d} P(\Omega)^d$$

where $P(\Omega)$ stands for the area of the boundary of the d- dimensional domain Ω, $|\Omega|$ its volume and ω_d is the d-dimensional volume of the unit ball. One interesting problem is to exchange the role of the volume and the perimeter, that is looking for a domain which has a minimal perimeter among all domains with given volume. Mathematically, this is given by the following shape optimization problem:

$$\min\{P(\Omega), \quad \Omega \text{ bounded domain of } \mathbb{R}^d, |\Omega| = m_0\} \tag{1.1}$$

where m_0 is a given positive real number. Here, once again the solution is a ball. An other version is to consider the problem of minimization of the perimeter of all domains with given volume and contained in a given domain T. The solution of this problem is either a ball if the domain T can contain a ball of volume m_0 or a domain whose one boundary is part of a sphere and the other is part of the boundary of T (Dido's problem). To conclude this part we give the general formulation of the isoperimetric problem. Given a closed D subset of $\mathbb{R}^d$ and f a given $L^1_{loc}(\mathbb{R}^d)$ function. We minimize the perimeter among all Borel subsets $\Omega \subset D$ once the quantity $\int_\Omega f(x)dx$ is prescribed. The perimeter of a Borel set Ω is defined by

$$P(\Omega) = \int_{\mathbb{R}^d} |\nabla \chi_\Omega| dx = \mathcal{H}^{d-1}(\partial^* \Omega)$$

where $\nabla \chi_\Omega$ is the distributional derivative of the characteristic function of Ω and $\partial^*\Omega$ is the reduced boundary of Ω in the sense of geometric measure theory. By using the property of the BV spaces, when D is bounded we obtain the lower semicontinuity and the coercivity of the perimeter for the L^1 convergence. This allows us to apply the Direct method of the calculus of variations and to obtain the existence of an optimal solution for the problem

$$\min\{P(\Omega) : \Omega \subset D, \quad \int_\Omega f(x)dx = m_0\}. \tag{1.2}$$

It is easy to see that in general the problem (1.2) may have no solution if we drop the assumption D is bounded (see for instance [20], [31]).

1.2.2 Minimal and Capillary Surfaces

The minimal surface called also Plateau's problem is considered to be the wellspring of questions in geometric measure theory. Named in honor of the nineteenth century Belgian physicist Joseph Plateau who studied surface tension phenomena in general, and soap films and soap bubbles in particular, the question (in its original formulation) was to show that a fixed, simple closed curve in three-space will bound a surface of the type of a disc and having minimal area. Further, one wishes to study uniqueness for this minimal surface, and also to determine its other properties. This problem may be seen as a shape optimization problem. Given a simple closed curve γ in $\mathbb{R}^3$ we have the following minimization problem

$$\min\{\text{area}(\Omega) : \Omega \ \text{ surface with boundary } \ \gamma\} \tag{1.3}$$

Jesse Douglas solved the original Plateau's problem by considering the minimal surface to be a harmonic mapping (which one sees by studying the Dirichlet integral). Unfortunately, Douglas methods do not adapt well to higher dimensions, so it is desirable to find other techniques with broader applicability. A capillary surface is one of the generalization of the minimal surface. Consider an interact container containing a liquid. This liquid acts to the internal wall of the container by capillarity and the question is to find a form taken by the interface liquid-air (free boundary of the liquid). More details may be found in [61]. The mathematical formulation is: assume we have an open bounded set with smooth boundary D in $\mathbb{R}^3$ which represent the interior of the container and the volume of the liquid v, denoting by Ω the space occupied by the liquid, the total energy of the system (container plus liquid) is the sum of surface tension

$$E_1(\Omega) := \text{area}(\partial\Omega \cap D) + (\cos\gamma)\text{area}(\partial\Omega \cap \partial D) \tag{1.4}$$

and of the potential energy of gravity

$$E_2(\Omega) := -\int_\Omega K(x)dx \tag{1.5}$$

where γ is a given angle and K a bounded function both from the characteristic of the liquid. The shape optimization considered is of the form

$$\min\{E_1(\Omega) + E_2(\Omega) : \quad \Omega \subset D, \quad |\Omega| = v\} \tag{1.6}$$

The solution of this problem that is the optimal domain has a mean curvature equals to K everywhere on the free boundary $\partial\Omega \cap D$ and the free boundary make an angle θ with the wall of the container.

1.2.3 Spectral optimization problems

For every admissible domain Ω we consider the Dirichlet Laplacian $-\Delta$ which, under mild conditions on Ω, admits a compact resolvent and so a discrete spectrum $\lambda(\Omega)$. The cost functional is of the form

$$F(\Omega) = \Phi(\lambda(\Omega))$$

for a suitable function Φ. For instance, taking $\Phi(\lambda) = \lambda_k$ we may consider the optimization problem for the k-th eigenvalue of $-\Delta$

$$\min\{\lambda_k(\Omega) : \Omega \subset \mathcal{O}\}. \tag{1.7}$$

The volume constraint implies the existence of the classical solution of the minimization problem (1.7). More generally the volume constraint implies also the existence of the classical solution of minimization problem (1.7) where the cost functional is $\Phi(\lambda(\Omega))$ for some increasing and lower semicontinuous function Φ. A detailed presentation of the spectral optimization problem may be found in [36]

1.3 Some applied examples

We present two applied examples namely the Newton's problem of optimal aerodinamical profiles and the optimal mixture of two conductors. Let us mention also that they are other applied examples like image segmentation, identification of cracks or default, magnetic shaping and so on.

1.3.1 Newton's problem of optimal aerodynamical profiles

The problem of finding the best aerodynamical profile for a body in a fluid stream under some constraints on its size can be seen as a shape optimization problem. This problem was first considered by Newton, who gave a rather simple variational expression for the aerodynamical resistance of a convex body in a fluid stream, assuming that the competing bodies are radially symmetric, which makes the problem one dimensional. Here are his words:

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the given direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder. ... I reckon that this proposition will be not without application in the building of ships.

Under the assumption that the resistance is due to the impact of fluid particles against the body surface, that the particles are supposed all independent, and the tangential friction is negligible, by simple geometric considerations we may obtain the following expression of the resistance along the direction of the fluid stream, where we normalize all the physical constant to one:

$$F(u) = \int_\Omega \frac{1}{1+|\nabla u|^2}dx. \tag{1.8}$$

In the expression above Ω stands for the cross section of the body at the basis level, and $u(x)$ a function whose graph is the body boundary. The geometrical constraint in the problem consists in requiring that the admissible competing bodies be convex; this is also consistent with the physical assumption that all the fluid particles hit the body at most once. In problem (1.8) this turns out to be equivalent to assume that Ω is convex and $u : \Omega \to [0, +\infty)$ is concave. We consider the minimization problem

$$\min\{F(u) : u \ \ \text{concave}, \quad 0 \le u \le M\}, \tag{1.9}$$

where F is the functional in (1.8). Notice that the integral functional F is neither convex nor coercive, therefore we cannot apply the direct method of the calculus of variations for getting the existence of an optimal solution. However thanks to the concavity constraint, the existence of a minimizer u has been proved in [40]. A complete discussion on the problem may be found in [31].

1.3.2 Optimal mixtures of two conductors

This is another problem which can be seen as a shape optimization problem. It consists of the determination of the optimal distribution of two given conductors (for instance in the thermostatic model, where the state function is the temperature of the system) into a given region. If Ω denotes a given bounded open subset of $\mathbb{R}^d$ (the prescribed container), denoting by α and β the conductivities of the two materials, the problem consists in filling Ω with the two materials in the most performant way according to some given cost functional. The volume of each material can be prescribed. We denote by A the domain where the conductivity is α and by $a_A(x)$ the conductivity coefficient

$$a_A(x) = \alpha\chi_A(x) + \beta\chi_{\Omega\setminus A}(x).$$

Then the state equation which associates the control A to the state u (the temperature of the system, once the conductor α fills the domain A) becomes

$$\begin{cases} \operatorname{div}(a_A(x)\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{1.10}$$

where f is the given source density. We denote by u_A the unique solution of (1.10). If we take as a cost functional an integral of the form

$$\int_\Omega F(x, \chi_A, u_A, \nabla u_A)dx$$

in general an optimal configuration does not exist (see [71], [82]). However the addition of a perimeter penalization is enough to have the existence of optimal classical optimizers. More precisely, we take as a cost functional

$$J(u, A) = \int_\Omega F(x, \chi_A, u_A, \nabla u_A)dx + \lambda P_\Omega(A)$$

where $\lambda > 0$, and the minimization problem take the form

$$\min\{J(u, A) : A \subset \Omega, \;\; u \text{ solves } (1.10)\} \tag{1.11}$$

Notice that the existence result above still holds if we replace the perimeter constraint by the volume constraint of the form $|A| = m$. Let us mention also a similar problem considered in [8] which is

$$\min\{E(u, A) + \lambda P_\Omega(A) : u \in H_0^1(\Omega), \quad A \subset \Omega\}$$

where $\lambda > 0$ and

$$E(u, A) = \int_\Omega \big(a_A(x)|\nabla u|^2 + \chi_A(x)g_1(x, u) + \chi_{\Omega\setminus A}(x)g_2(x, u)\big)\, dx.$$

It has been proved that this optimization problem has classical solutions and every solution A is an open set provided g_1 and g_2 are Borel measurable functions and satisfy the inequalities

$$g_i(x, s) \geq b(x) - k|s|^2, \quad i = 1, 2$$

where $b \in L^1(\Omega)$ and $k < \alpha\lambda_1$, being λ_1 the first eigenvalue of $-\Delta$ on Ω.

Chapter 2

Elements of geometric measure theory

This chapter is concerned with some basics elements of geometric measure theory that will be used in the sequel. After recalling some notions of abstract measure theory, we deal with the Hausdorff measures which are important for defining the notion of approximate tangent space to measures and sets. Finally we introduce the notion of approximate tangent space to a measure and to a set and also some differential operators like tangential differential, tangential gradient and tangential divergence.

2.1 Measure theory

We recall briefly some results on abstract measure theory. For proofs and more details see [10], [57], [88], [90]. Let X be a topological space, denote by $\mathcal{B}(X)$ the σ-algebra of all Borel subsets of X that is, the smallest σ-algebra containing all open subsets of X and by $\mathcal{P}(X)$ the collection of all subsets of X.

Definition 2.1.1. *A function $\mu : \mathcal{P}(X) \to [0, +\infty]$ is called an outer measure if $\mu(\emptyset) = 0$ and μ is countably subadditive, i.e.*

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i), \quad \textit{whenever } A \subset \bigcup_{i=1}^{\infty} A_i.$$

Definition 2.1.2. *If μ is an outer measure on $\mathcal{P}(X)$ and $\mathcal{C}$ is a σ-algebra, μ is said to be countably additive (or σ-additive) on $\mathcal{C}$ if*

$$\mu\left(\bigsqcup_i A_i\right) = \sum_i \mu(A_i)$$

for every countable family $(A_i)_i \subset \mathcal{C}$ and $\bigsqcup$ stands for disjoint union.

Definition 2.1.3. *(Carathéodory measurability) If μ is an outer measure, a set $A \subset X$ is said to be μ-measurable if*

$$\mu(F) = \mu(F \cup A) + \mu(F \setminus A), \quad \textit{for every } F \subset X.$$

Proposition 2.1.4. *If μ is an outer measure, then the collection of μ-measurable sets is a σ-algebra.*

If we restrict the outer measure μ to the σ-algebra of all μ-measurable sets, we get a nonnegative and countably additive set function that we will call measure.

Definition 2.1.5. *(regular outer measure) Let μ be an outer measure on X. We say that μ is regular if for any set $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$.*

Now we will introduce the notion of signed and vector measures.

Definition 2.1.6. *Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$ be a σ-algebra. A function $\mu : \mathcal{C} \to \mathbb{R}^d$ is called a vector-valued measure if μ is additive in the sense that*

$$\mu\left(\bigsqcup_{i}^{\infty} A_i\right) = \sum_{i}^{\infty} \mu(A_i) \tag{2.1}$$

for every countable family $(A_i)_i$ of pairwise disjoint subsets of $\mathcal{C}$ and the right hand side of (2.1) is assumed to be finite. Moreover, given μ as above we define the function $|\mu| : \mathcal{C} \to [0, +\infty)$ as

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{\infty} ||\mu(A_i)|| : A = \bigsqcup_{i=1}^{\infty} A_i, \quad A_i \in \mathcal{C}\right\}. \tag{2.2}$$

The function $|\mu|$ is called the variation of μ and the quantity $|\mu|(X)$ the total variation of μ.

Theorem 2.1.7. *Let $X, \mathcal{C}$ and μ be as in Definition 2.1.6, then the following hold:*

1. *Every infinite sum as in (2.1) is absolutely convergent;*
2. *the total variation $|\mu|$ is countably additive on $\mathcal{C}$, hence it is a measure;*
3. *the quantity $|\mu|(X)$ is finite, therefore $|\mu|$ is a finite measure.*

They are several topologies on the set of measures. We give here the *weak** convergence. Let $(\mu_n)_n$ be a sequence of measures, we say that μ_n converges *weakly** to the measure μ and we write $\mu_n \overset{*}{\rightharpoonup} \mu$ if

$$\lim_{n\to+\infty} \int_X f d\mu_n = \int_X f d\mu \quad \forall f \in C^0(X, \mathbb{R}^d).$$

Proposition 2.1.8. *Let $(\mu_n)_n$ be a sequence of Radon measure on the locally compact, separable metric space X such that $\mu_n \rightharpoonup \mu$, then*

1. *if the measures μ_n are positive, then for every lower semicontinuous function $f : X \to [0, +\infty]$*

$$\liminf_{n\to+\infty} \int_X f d\mu_n \geq \int_X f d\mu$$

and for every upper semicontinuous function $g : X \to [0, +\infty]$ with compact support

$$\limsup_{n\to+\infty} \int_X g d\mu_n \leq \int_X g d\mu :$$

2. *if $|\mu_n|$ is locally weakly* convergent to λ, then $\lambda \geq |\mu|$. Moreover if X is a relatively compact set such that $\lambda(\partial X) = 0$, then $\mu_n(X) \to \mu(X)$ as $n \to +\infty$. More generally*

$$\int_X f d\mu = \lim_{n\to+\infty} \int_X f d\mu_n$$

for any bounded Borel function $f : X \to \mathbb{R}$ with compact support such that the set of its discontinuity points is λ-negligible.

We may apply the part 1 of the statement to the characteristic functions of open and compact sets and obtaining some particular interesting cases. Assume μ_n is locally *weakly** convergent to μ. Then for every compact set K, we have

$$\mu(K) \geq \limsup_{n\to+\infty} \mu_n(K) \tag{2.3}$$

and for every open set A it holds

$$\mu(A) \leq \liminf_{n\to+\infty} \mu_n(A) \tag{2.4}$$

2.2 Hausdorff Measure

In this section, we introduce the Hausdorff measures $\mathcal{H}^k$. This class of measures provide a general extension of the classical notion of length, surface area and volume.

Definition 2.2.1. *For $k \geq 0$, we set*

$$\omega_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(1+\frac{k}{2})}, \quad \textit{where} \quad \Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx$$

(Γ is the well-known Euler function). If $\delta \in (0, +\infty]$ and $A \subset \mathbb{R}^d$, we define

$$\mathcal{H}^h_\delta := \frac{\omega_k}{2^k} \left\{ \sum_{i\in I} (diam(A_i))^k : diam(A_i) < \delta, \;\; A \subset \bigcup_{i\in I} A_i \right\}. \tag{2.5}$$

and

$$\mathcal{H}^k(A) := \sup_{\delta>0} \mathcal{H}^k_\delta(A) = \lim_{\delta\to 0} \mathcal{H}^k_\delta(A),$$

where the second equality is obtained from the fact that $\mathcal{H}_\delta^k$ is nonincreasing with respect to δ. The quantity $\mathcal{H}^k(A)$ is called the k-dimensional Hausdorff measure of the set A.

Proposition 2.2.2. *(properties of Hausdorff measures)*

1. *For every $k \geq 0$, $\mathcal{H}^k$ is a Borel measure;*

2. *For $0 \leq s < m < +\infty$ and $A \subset \mathbb{R}^d$.*

$$\mathcal{H}^s(A) < +\infty \Longrightarrow \mathcal{H}^m(A) = 0$$

$$\mathcal{H}^m(A) > 0 \Longrightarrow \mathcal{H}^s(A) = +\infty;$$

3. *For every Borel set $A \subset \mathbb{R}^d$ and every $\delta \in (0, +\infty]$*

$$\mathcal{H}^d(A) = \mathcal{H}_\delta^d(A) = \mathcal{L}^d(A),$$

   where $\mathcal{L}^d$ stands for the d-dimensional Lebesgue measure;

4. *every Hausdorff measure $\mathcal{H}^k$ is Borel regular that is for every set A there exists a Borel set B such that $A \subset B$ and $\mathcal{H}^k(A) = \mathcal{H}^k(B)$;*

5. *for every $x \in \mathbb{R}^d$ and every positive real number λ*

$$\mathcal{H}^k(x + A) = \mathcal{H}^k(A), \quad \mathcal{H}^k(\lambda A) = \lambda^k \mathcal{H}^k(A)$$

We introduce also a notation for the restriction $\mathcal{H}^k \llcorner B$ of the Hausdorff measure to a set $B \subset \mathbb{R}^d$, defined by

$$\mathcal{H}^k \llcorner B(A) := \mathcal{H}^k(B \cap A) \quad \forall A \subset \mathbb{R}^d.$$

Definition 2.2.3. *(definition of Hausdorff dimension) Let $A \subset \mathbb{R}^d$ be a given subset. we define the Hausdorff dimension of the set A as*

$$\mathcal{H} - dim(A) := \inf\{k \geq 0 : \mathcal{H}^k(A) = 0\}.$$

If $k > \mathcal{H} - \dim(A)$ then $\mathcal{H}^k(A) = 0$, and if $k < \mathcal{H}\text{-}\dim(A)$ then $\mathcal{H}^k(A) = +\infty$. In the case $k = \mathcal{H}\text{-}\dim(A)$ nothing can be say a priori about the value of $\mathcal{H}^k(A)$.

2.3 Approximate tangent space

We start by the countably $\mathcal{H}^k$-rectifiable set.

Definition 2.3.1. *Let E be a $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ and $k = 0, \cdots, d$. We say that the set E is countably $\mathcal{H}^k$- rectifiable if $E = \bigcup_{j=0}^{\infty} E_j$ so that:*

1. $\mathcal{H}^k(E_0) = 0$*;*

2. for $j > 0$*,* $E_j \subset f_j(\mathbb{R}^k)$ *where* $f_j : \mathbb{R}^k \longrightarrow \mathbb{R}^d$ *are Lipschitz maps.*

The following proposition whose proof can be found in [59] [10] gives the different characterizations of countably $\mathcal{H}^k$-rectifiable set.

Proposition 2.3.2. *The following statement are equivalent*
a) E *is countably* $\mathcal{H}^k$*-rectifiable set;*
b) $E = \bigcup_{j=0}^{\infty} E_j$ *with* $\mathcal{H}^k(E_0) = 0$ *and* $E_j \subset S_j$ *where* S_j *are k-dimensional Lipschitz surfaces in* $\mathbb{R}^d$ *for* $j > 0$*;*
b′) $E = \bigcup_{j=0}^{\infty} E_j$ *with* $\mathcal{H}^k(E_0) = 0$ *and for* $j > 0, E_j \subset S_j$ *where* S_j *are k-dimensional surfaces of class* C^1 *in* $\mathbb{R}^d$*;*
c) $E = \bigcup_{j=0}^{\infty} E_j$ *with* $\mathcal{H}^k(E_0) = 0$ *and for* $j > 0, E_j \subset \Gamma_j$ *where* Γ_j *are graphs of Lipschitz maps* $g_j : \mathbb{R}^k \to \mathbb{R}^{d-k}$ *(up to identification of* $\mathbb{R}^k \times \mathbb{R}^{d-k}$ *with* $\mathbb{R}^d$*, and rotation);*
c′) $E = \bigcup_{j=0}^{\infty} E_j$ *with* $\mathcal{H}^k(E_0) = 0$ *and for* $j > 0, E_j \subset \Gamma_j$ *where* Γ_j *are graphs of* C^1 *maps* $g_j : \mathbb{R}^k \to \mathbb{R}^{d-k}$*.*

Now, we are interested in defining the tangent bundle of a countably $\mathcal{H}^k$-rectifiable set. We start by the definition of the approximated tangent space to a vector valued Radon measure. In the sequel $G(k, d)$ will denote the Grassmaniann manifold of k-dimensional non-oriented planes in $\mathbb{R}^d$.

Definition 2.3.3. *(Approximate tangent space to a measure) Let* μ *be an* $\mathbb{R}^m$*-valued Radon measure in an open set* $\Omega \subset \mathbb{R}^d$ *and* $x \in \Omega$*. We say that* μ *has an approximate tangent* $\pi \in G(k, d)$ *with multiplicity* $\theta \in \mathbb{R}^m$ *at* x*, and denoted by*

$$Tan^k(\mu, x) = \theta \mathcal{H}^k \llcorner \pi$$

if $\rho^{-k}\mu_{x,\rho}$ *locally weakly** *converge to* $\theta \mathcal{H}^k \llcorner \pi$ *in* $\mathbb{R}^d$ *as* $\rho \downarrow 0$ *where*

$$\mu_{x,\rho}(B) = \mu(x + \rho B) \;\; \text{for } B \in \mathcal{B}(\mathbb{R}^d), \;\; B \subset \frac{\Omega - x}{\rho}$$

According to the definition of $\mu_{x,\rho}$, the existence of the approximate tangent space with multiplicity θ can be rewrite as follows:

$$\lim_{\rho \downarrow 0} \rho^{-k} \int_{\Omega} \phi\left(\frac{y - x}{\rho}\right) d\mu(y) = \theta \int_{\pi} \phi(y) d\mathcal{H}^k(y), \quad \forall \phi \in C_c(\mathbb{R}^d)$$

For $\rho > 0$ small enough the support of the function $y \mapsto \phi((y - x)/\rho)$ is contained in Ω, then the formula does make sense.

Remark 2.3.4 Let E be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ with locally finite $\mathcal{H}^k$-measure and let $\mu = \mathcal{H}^k \llcorner E$. By the behavior of the Hausdorff measure under translations and homotheties, one can easily check that $\rho^{-k}\mu_{x,\rho} = \mathcal{H}^k \llcorner E_{x,\rho}$ where $E_{x,\rho} = (E - x)/\rho$.

Therefore $\pi \in G(k,d)$ is the approximate tangent space to $\mathcal{H}^k \llcorner E$ at x with multiplicity 1 if and only

$$\lim_{\rho \downarrow 0} \int_{E_{x,\rho}} \phi(y) d\mathcal{H}^k(y) = \int_{\pi} \phi(y) d\mathcal{H}^k(y), \quad \forall \phi \in C_c(\mathbb{R}^d).$$

In the following proposition we give a local property of the approximate tangent space. This result will be useful to the definition of the tangent space to a set. Proof can be found in [10].

Proposition 2.3.5. *Let $\mu_j = \theta_j \mathcal{H}^k \llcorner S_j, \ j = 1, 2$ be positive k-rectifiable measures and let π_j be the approximate tangent space to μ_j, defined for $\mathcal{H}^k - a.e \ x \in S_j$. Then*

$$\pi_1(x) = \pi_2(x) \ \ \textit{for} \ \ \mathcal{H}^k - a.e. \ x \in S_1 \cap S_2 \tag{2.6}$$

By positive k-rectifiable measure, we mean the measure $\theta \mathcal{H}^k \llcorner S$ where S is a countably $\mathcal{H}^k$-rectifiable set and θ positive function

If we assume $S_1 = S_2$ in proposition 2.3.5, we realize that the approximate tangent space to $\theta \mathcal{H}^k \llcorner S$ does not depend on θ but only on S; the equation (2.6) suggests the possibility to define the approximate tangent space $Tan^k(S,x)$ to a countably $\mathcal{H}^k$-rectifiable set S in the following way.

Definition 2.3.6. *Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^k$-rectifiable set and let S_j be a partition of $\mathcal{H}^k$-almost all S into $\mathcal{H}^k$-rectifiable sets; we define $Tan^k(S,x)$ to be the approximate tangent space to $\mathcal{H}^k \llcorner S_j$ at x for any $x \in S_j$ where the latter is defined.*

Remark 2.3.7 Notice that the measure $Tan(\mu, x)$ is univocally defined at any point x where it exists, and from this measure both the approximate tangent space and the multiplicity at x can be recovered. In contrast, the definition 2.3.6 is well posed (i.e. independent of the partition (S_j) chosen) only if we understand $Tan^k(S,x)$ as an equivalent class of $\mathcal{H}^k$-measurable maps from S to $G(k,d)$. In fact by a simple application of (2.6), two different partition produce tangent space maps coinciding $\mathcal{H}^k$-a.e. on S and satisfying the locality property

$$Tan^k(S,x) = Tan^k(S',x) \ \ \text{for} \ \ \mathcal{H}^k - a.e. \, x \in S \cap S'$$

for any pair of countably $\mathcal{H}^k$-rectifiable sets S, S' and the consistency property

$$supp \left[Tan^k(\theta \mathcal{H}^k \llcorner S, x) \right] = Tan^k(S,x) \ \ \text{for} \ \ \mathcal{H}^k - a.e. \, x \in S$$

for any Borel function $\theta : S \to (0; \infty)$ locally summable with respect to $\mathcal{H}^k \llcorner S$.

The following remark stress the particular case of the approximate tangent space to C^1 and Lipschitz k-graphs.

Remark 2.3.8 Let $\Gamma = \{x : f(\pi x) = \pi^{\perp} x\}$ be a k-dimensional graph of class C^1, and consider $P(x) = \{v + df_{\pi x}(v) : v \in \pi\}$. Then

$$Tan^k(\mathcal{H}^k \llcorner \Gamma, x) = \mathcal{H}^k \llcorner P(x) \quad \forall x \in \Gamma.$$

Now we want to define differentiability and some other differential operator on the countably $\mathcal{H}^k$-rectifiable set. We start by the following definition.

Definition 2.3.9. *Let E be a countably $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^m$ a Lipschitz function. We say that f is tangentially differentiable at $x \in E$ if the restriction of f to the affine space $x + Tan^k(E, x)$ is differentiable at x. The tangential differential $d^E f_x : Tan^k(E, x) \mapsto \mathbb{R}^m$ is a linear map.*

It is clear that if f is differentiable at x, then $d^E f_x$ is the restriction of the differential df_x to $Tan^k(E, x)$ provided that the approximate tangent space exists. Since $x \mapsto Tan^k(E, x)$ is understood as an equivalent class of map from E to $G(k, d)$, the same is true for tangential differential d^E. Hence, even for smooth functions in the ambient space, the tangential differential $d^E f_x$ is not well defined at a specific point x, but quantities such as $\int_F J_k d^E f_x dx$ are well defined for every Borel set $F \in \mathcal{B}(E)$. The tangential differential inherits from approximate tangent spaces a very useful locality property:

$$\exists d^E f_x \ \ \mathcal{H}^k - a.e. \text{on } \ E \Longrightarrow \exists d^F f_x = d^E f_x \ \ \mathcal{H}^k - a.e. \text{on } \ E \cap F$$

for E and F countably $\mathcal{H}^k$-rectifiable. The following result whose proof can found in [10] is a natural extension of Rademacher's differentiability theorem.

Proposition 2.3.10. *Under the same notation of definition (2.3.9) $d^E f_x$ exists for $\mathcal{H}^k$-a.e. $x \in E$.*

Definition 2.3.11. *(tangential gradient) Let E be a countably $\mathcal{H}^k$-rectifiable set and $\phi \in C^1(\Omega)$. If $x \in E \cap \Omega$ and h is any vector belonging to the approximate tangent space $Tan^k(E, x)$ to E at x , the directional derivative $\nabla_h \phi(x)$ is defined as*

$$\nabla_h \phi(x) = \langle \nabla \phi(x), h \rangle$$

where $\nabla \phi(x)$ is the restriction of the gradient of ϕ as function of Ω to the set E and the tangential gradient $\nabla^E \phi(x)$ of ϕ at x is defined as

$$\nabla^E \phi(x) = \sum_{j=1}^{k} \nabla_{\tau_j} \phi(x) \tau_j,$$

where $\tau_1, \cdots, \tau_k$ is an orthonormal basis of the approximate tangent space $Tan^k(E,x)$. Thus the tangential gradient is just the orthogonal projection of $\nabla\phi(x)$ on the approximate tangent plane $Tan^k(E,x)$. The linear map $d^E f_x : Tan^k(E,x) \mapsto \mathbb{R}$ is related to the directional derivative as follows:

$$d^E f_x(h) = \nabla_h \phi(x) \quad h \in Tan^k(E,x)$$

provided the approximate tangent space exists.

Definition 2.3.12. *Let E be a countably $\mathcal{H}^k$-rectifiable subset of a set Ω and $\phi \in [C^1(\Omega)]^d$. The tangential divergence of ϕ on E is defined by*

$$\mathrm{div}^E \phi(x) = \sum_{m=1}^{d} \langle \nabla^E \phi_m(x), e_m \rangle \quad \textit{for} \quad \mathcal{H}^k - a.e.\ x \in E$$

where $\phi = (\phi_1, \cdots, \phi_d)$ and $(e_1, \cdots, e_d)$ is the standard orthonormal basis of $\mathbb{R}^d$.

We can rewrite $\mathrm{div}^E \phi(x)$ in this way

$$\mathrm{div}^E \phi(x) = \sum_{m=1}^{d} \sum_{j=1}^{k} \langle \nabla \phi_m(x), \tau_j \rangle \langle \tau_j, e_m \rangle = \sum_{j=1}^{k} \langle \nabla_{\tau_j} \phi(x), \tau_j \rangle.$$

This expression shows that $\mathrm{div}^E \phi(x)$ is the orthogonal projection of the $\mathrm{div}\phi$ to the approximate tangent space $\mathrm{Tan}^k(E,x)$

Chapter 3

Topology on Domains of $\mathbb{R}^d$

A shape optimization problem is an optimization of the form

$$\min_{\Omega \in \mathcal{O}} F(\Omega), \tag{3.1}$$

where $\mathcal{O}$ is a collection of subsets of $\mathbb{R}^d$. To prove the existence of such a minimum, we use the so called Directs methods of the calculus of variations which is as follows: first ensure that $m = \{\inf F(\Omega), \Omega \in \mathcal{O}\}$ is finite; second take a minimizing sequence that is a sequence $\{\Omega_n\}_n$ of elements of $\mathcal{O}$ such that $\lim_{n\to+\infty} F(\Omega_n) = m$ which converges in some sense to a set $\Omega \in \mathcal{O}$ and $F(\Omega) \leq m$. Therefore we need some topology on the set of domains.

3.1 Different Topologies on Domains

In the set of domains there is not a canonical topology. This fact allows us to consider many topologies on the set of domains. We will consider here three topologies which are topologies induced by the convergence of characteristics functions, convergence in the sense of Hausdorff and convergence in the sense of compacts.

3.1.1 The Convergence of Characteristic functions

Let $\Omega \subset \mathbb{R}^d$ be a measurable set then we call the characteristic function of Ω the function χ_Ω which takes the value 1 on Ω and 0 outside. Let $(\Omega_n)_n$ be any sequence of measurable sets; then the sequence of characteristic functions χ_{Ω_n} is weakly* compact in L^∞ that is there exists $\chi \in L^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \varphi \chi_{\Omega_n} dx = \int_{\mathbb{R}^d} \varphi \chi dx, \quad \forall \varphi \in L^1(\mathbb{R}^d).$$

Notice that the function χ is not in general a characteristic function unless the convergence is strong in L^p_{loc} for some $p \in [1, +\infty)$. More precisely the *weak** limit is a characteristic function only if the convergence is strong.

Proposition 3.1.1. *If $(\Omega_n)_n$ and Ω are measurable subsets of $\mathbb{R}^d$ such that χ_{Ω_n} weakly* converges in $L^\infty(\mathbb{R}^d)$ to χ_Ω, then $\chi_{\Omega_n} \to \chi_\Omega$ in $L^p_{loc}(\mathbb{R}^d)$ for all $p \in [1, +\infty)$ and almost everywhere.*

Proof: By hypothesis we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} (\chi_{\Omega_n} - \chi_\Omega)\varphi dx = 0, \quad \forall \varphi \in L^1(\mathbb{R}^d). \tag{3.2}$$

Let B_r be a ball of center 0 and radius r and Ω^c the complement of Ω. Taking $\varphi = \chi_{B_r}\chi_{\Omega^c}$ in (3.2) we have

$$0 = \lim_{n\to+\infty} \int_{\mathbb{R}^d} \chi_{\Omega_n}\chi_{B_r}\chi_{\Omega^c}(x)dx = \lim_{n\to+\infty} |B_r \cap (\Omega_n \setminus \Omega)|.$$

Now taking $\varphi = \chi_{B_r}$ in (3.2) we get

$$0 = \lim_{n\to+\infty} \int_{B_r} (\chi_{\Omega_n} - \chi_\Omega)(x)dx = \lim_{n\to+\infty} \{|B_r \cap (\Omega_n \setminus \Omega)| - |B_r \cap (\Omega \setminus \Omega_n)|\}.$$

Hence getting also $|B_r \cap (\Omega \setminus \Omega_n)| \to 0$ as $n \to +\infty$. But

$$\int_{B_r} |(\chi_{\Omega_n} - \chi_\Omega)(x)|^p dx = |B_r \cap (\Omega_n \setminus \Omega)| + |B_r \cap (\Omega \setminus \Omega_n)|,$$

and the proof is over. □

Definition 3.1.2. *Let $(\Omega_n)_n$ and Ω be measurable subsets of $\mathbb{R}^d$. We say that Ω_n converges to Ω in the sense of characteristic functions as $n \to +\infty$ if*

$$\chi_{\Omega_n} \to \chi_\Omega \ \text{ in } L^p_{loc}(\mathbb{R}^d), \quad \forall p \in [1, +\infty).$$

3.1.2 The Convergence in the sense of Hausdorff

Let D be a compact set in $\mathbb{R}^d$, $\mathcal{K}_D$ the set of all compact non empty subsets of D and d the Euclidean distance on $\mathbb{R}^d$.

Definition 3.1.3. *Given $K_1, K_2 \in \mathcal{K}_D$ we define*

$$\begin{aligned} \forall x \in D, d(x, K_1) &:= \inf_{y\in K_1} d(x, y) \\ \rho(K_1, K_2) &:= \sup_{x\in K_1} d(x, K_2) \\ d^H(K_1, K_2) &:= \max\{\rho(K_1, K_2), \rho(K_2, K_1)\}. \end{aligned} \tag{3.3}$$

It is easy to check that d^H is a distance on $\mathcal{K}_D$ and it is called Hausdorff distance. One may show that $(\mathcal{K}_D, d^H)$ is a compact complete metric space. Now we give the definition of convergence of compact and open sets in the sense of Hausdorff.

Definition 3.1.4. *Let $(K_n)_{n\in\mathbb{N}}$ and K be compact sets contained in D. We say that K_n converges to K as $n \to +\infty$ in the Hausdorff sense if*

$$d^H(K_n, K) \to 0 \text{ as } n \to +\infty.$$

We denote this convergence by $K_n \xrightarrow{H} K$.

Definition 3.1.5. *Let $(\Omega_n)_{n\in\mathbb{N}}$ and Ω be open sets contained in D. We say that Ω_n converges to Ω as $n \to +\infty$ in the sense of Hausdorff if*

$$d_H(\Omega_n, \Omega) := d^H(D \setminus \Omega_n, D \setminus \Omega) \to 0 \text{ as } n \to +\infty. \tag{3.4}$$

We denote this convergence by $\Omega_n \xrightarrow{H} \Omega$.

We should mention that the distance $d_H(\Omega_1, \Omega_2)$ of two open sets contained in D is independent of the compact D. In fact if we take another compact $\tilde{D}$ containing both open sets, it holds

$$d_H(\Omega_1, \Omega_2) := d^H(D \setminus \Omega_1, D \setminus \Omega_2) = d^H(\tilde{D} \setminus \Omega_1, \tilde{D} \setminus \Omega_2).$$

We should mention also that here and after on we use the same terminology (convergence in the sens of Hausdorff , Hausdorff topology) for compact and open sets. We summarize some properties of Hausdorff distance. For some details see [56], [65] and [87]. Convergence of compact sets in the sense of Hausdorff.

1. A nonincreasing sequence of non empty compact sets converges to their intersection;
2. a nondecreasing sequence of non empty compact sets contained in D converges to their union;
3. if K_n converges to K in the Hausdorff sense then $K = \cap_n(\overline{\cup_{p\geq n} K_p})$;
4. the inclusion is stable under Hausdorff convergence.

Convergence of open sets in the sense of Hausdorff.

1. a nondecreasing sequence of open sets contained in D converges in Hausdorff sense to the union;
2. a nonincreasing sequence of open sets converges to the interior of the intersection;
3. inclusion is stable under Hausdorff convergence;
4. intersection is stable under Hausdorff convergence;

5. the union is not stable under Hausdorff convergence. More precisely we have

$$\left.\begin{array}{r}\Omega_n^1 \xrightarrow{H} \Omega^1 \\ \Omega_n^2 \xrightarrow{H} \Omega^2 \\ \Omega_n^1 \cup \Omega_n^2 \xrightarrow{H} \Omega\end{array}\right\} \Rightarrow \Omega^1 \cup \Omega^2 \subset \Omega,$$

   and the inclusion may be strict;

6. if $(\Omega_n)_n$ is a sequence of open sets converging to Ω and K a compact set contained in Ω then K is contained in Ω_n for n large enough;

7. the convexity is preserved by Hausdorff convergence;

8. the connectedness is not preserved by Hausdorff convergence;

9. the volume is not preserved by Hausdorff convergence. In fact the volume is lower semicontinuous for convergence of open sets;

3.1.3 The Convergence in the sense of Compacts

Definition 3.1.6. *Let $(\Omega_n)_{n\in\mathbb{N}}$ and Ω be open sets in $\mathbb{R}^d$. We say that Ω_n converges to Ω in the sense of compacts if*

$$\begin{array}{l}\forall K \, compact \subset \Omega, \;\; we\ have\ K \subset \Omega_n \;\; for\ n\ large\ enough \\ \forall L compact \;\; \subset \overline{\Omega}^c, \;\; we\ have\ L \subset \overline{\Omega}_n^c \;\; for\ n\ large\ enough.\end{array} \tag{3.5}$$

The big inconvenient of this topology is that the limit is not unique. We may check that a sequence of open sets $(\Omega_n)_n$ which converges to Ω in the sense of compact converges also to any open set ω such that $\overline{\omega} = \overline{\Omega}$. In fact this topology is not separable. To have the uniqueness, instead of working with open sets one may work with the class of open sets given by the following equivalent relation

$$\Omega_1 \simeq \Omega_2 \Leftrightarrow \overline{\Omega}_1 = \overline{\Omega}_2.$$

3.2 Link between those different topologies

The aim of this part is to show by some examples that the three topologies defined previously doest not imply each other. We give three examples where we have convergence in one topology and not in the others or we have convergence to another open set.

Example 3.2.1 Let Ω^1 be an open set obtained by removing from a unit disc in $\mathbb{R}^2$ the segment $[0,1]\times\{0\}$ and $\Omega^2 = B(0,1)\cap\bigcup_{k=1}^{\infty} B(x_k, r_k)$ where x_k is a dense sequence of $B(0,1)$ and (r_k) sequence of positive real numbers such that $\sum_{k\geq 1} r_k^d < 1$. Let

$\Omega_n := B(0, 1 + 1/n)$ then we easily have that Ω_n converges to Ω^1 and Ω^2 in the sense of compacts. The sequence Ω_n converges in the Hausdorff topology to $B(0,1) \neq \Omega^1$ therefore it cannot converges to Ω^1 in the sense of Hausdorff. Since Ω^2 does not have the same measure as $B(0,1)$ it cannot be a limit of Ω_n in the sense of characteristics functions.

Example 3.2.2 ([87]) In $\mathbb{R}^2$, set $F = [0,3]^2$ and

$$\begin{aligned} \Omega_n &= \{(x,y) \in F : 0 < x < 3, \quad 0 < y < 2 + \sin(nx)\}, \quad K_n = \Omega^c, \\ \Omega &= (0,3) \times (0,1), \quad K = \Omega^c. \end{aligned}$$

Then Ω_n converges to Ω in the sense of Hausdorff since

- $\rho(K_n, K) = 0$ since $K_n \subset K$
- $\forall x \in K$, we have $d(x, K_n) \leq \pi/n$ and then $\rho(K, K_n) \to 0$ as $n \to +\infty$.

Any compact of type $[a,b] \times [c,d]$ where $0 < a < b < 3$ and $1 < c < d < 3$ is contained in $\overline{\Omega}^c$ but never contained in $\overline{\Omega}_n^c$ for n large enough. It is also clear that it does not converge to Ω in the sense of characteristic functions since

$$\int_F |\chi_\Omega - \chi_\Omega| dx = \int_F \chi_{\Omega_n \setminus \Omega} dx = \int_0^3 \int_1^{2+\sin(nx)} dy dx = 3 + \frac{1 - \cos(3n)}{n}$$

which converges to 3 as $n \to +\infty$.

Example 3.2.3 In $\mathbb{R}$ let us set

$$\Omega_n = \bigcup_{k=0}^{2^n - 1} (\frac{k}{2^n}, \frac{k+1}{2^n}) = [0,1] \setminus \bigcup_{k=0}^{2^n} \{\frac{k}{2^n}\}.$$

Then Ω_n converges in the sense of Hausdorff to the empty set, in the sense of characteristics functions to $(0,1)$ (since $\chi_{\Omega_n} = \chi_{(0,1)}$ a.e.) does not converge in the sense of compact to some open set.

In the sequel, we will be concerned only with the Hausdorff convergence. In relation with this restriction, we give the following compactness result.

Theorem 3.2.4. *Let K_n be a sequence of compact sets contained in a fixed compact set D. Then there exists a compact set K contained in D and a subsequence K_{n_j} which converges in Hausdorff sense to K as $j \to +\infty$.*

Corollary 3.2.5. *Let Ω_n be a sequence of open sets contained in a fixed compact set D. Then there exists an open set Ω contained in D and a subsequence Ω_{n_j} which converges in Hausdorff sense to Ω as $j \to +\infty$.*

For the proof of the Theorem and Corollary one may consult [67].

Chapter 4

Shape optimization problems governed by linear state equations

This chapter deals with the shape optimization problems governed by linear state equations. After studying the continuity of the solution of the Laplacian problem with respect to the domain variation (including counter-examples to the continuity and the introduction to a new topology: the γ- convergence), we analyze the existence of optimal shapes and the necessary condition of optimality in the case where an optimal shape exists.

4.1 Continuity with respect to a domain

The existence of the optimal shape needs the continuity, or at least the lower semi-continuity of the functional associated to the problem, that is also the continuity of the solution of the partial differential equation associated to the functional, for some topology of domains variation. In this section, we analyze the continuity of the map $\Omega \to u_\Omega \in H_0^1(D)$ where u_Ω is the solution of the Dirichlet problem on an open variable Ω domain contained in a fixed open set D. The class of admissible domains Ω will be endowed with the Hausdorff topology.

4.1.1 Dirichlet problem for the Laplacian

Let Ω be a bounded open subset in $\mathbb{R}^d$ and $f \in H^{-1}(\Omega)$. The Dirichlet problem is to find u solution of the equation

$$u \in H_0^1(\Omega), \quad -\Delta u = f$$

in distributional sense that means

$$u \in H_0^1(\Omega), \quad \int_\Omega \nabla u \nabla v dx = \langle f, v \rangle_{H^{-1}(D)\times H_0^1(D)} \quad \forall v \in H_0^1(\Omega), \tag{4.1}$$

It is well known that this problem has a unique solution. Moreover the solution u minimizes the functional $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \langle f, v\rangle_{H^{-1}(D)\times H_0^1(D)} \quad v \in H_0^1(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^2 dx = \langle f, v\rangle_{H^{-1}(D)\times H_0^1(D)}. \tag{4.2}$$

The continuity problem is as follows: given a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ contained in a fixed open set D and converging in some sense to an open set Ω. Can we say that $u^f_{\Omega_n}$ "converges" to u^f_Ω where $u^f_{\Omega_n}$ and u^f_Ω are respectively the solution of the Dirichlet problem associated to Ω_n and Ω. Quite often, the "classical" topologies defined in the third chapter do not ensure this convergence. Therefore the question is reformulated as follows: under which additional conditions the continuity takes place? Or can we define a convergence on Ω_n which ensure that of $u^f_{\Omega_n}$? It has been proved (see [94]) that this continuity property does not depend on f; in fact, if the continuity holds for $f \equiv 1$ then it holds for any f. Let us start by this fundamental estimate.

Proposition 4.1.1. *There exists a constant $C = C(D)$ such that for any open $\Omega \subset D$, the solution of (4.1) extended by zero on $D \setminus \Omega$ satisfies*

$$||u^f_\Omega||_{H_0^1(D)} \leq C||f||_{H^{-1}(D)} \ \forall f \in H^{-1}(D). \tag{4.3}$$

Proof: setting $u = u^f_\Omega$ it follows from (4.2) that

$$\int_\Omega |\nabla u|^2 dx \leq ||f||_{H^{-1}(D)}||u||_{H_0^1(\Omega)}.$$

By Poincaré inequality, there exists a constant C_1 depending only on D such that

$$C_1||u||^2_{H_0^1(D)} \leq \int_D |\nabla u|^2 dx = \int_\Omega |\nabla u|^2 dx \leq ||f||_{H^{-1}(D)}||u||_{H_0^1(D)}$$

and the result follows. □

Corollary 4.1.2. *Let Ω_n be a sequence of open sets in D. Then up to a subsequence $u^f_{\Omega_n}$ converges weakly to $u^* \in H_0^1(D)$. Moreover, if there exists an open set $\Omega \subset D$ such that $u^* = u^f_\Omega$, then the convergence is strong in $H_0^1(D)$.*

Proof: The first part is just a consequence of the uniform boundedness in (4.3) and of the weakly sequentially compactness of the closed unit ball in Hilbert's space $H_0^1(D)$. The second part follows from the passage to the limit (weak) in the equality:

$$\int_D |\nabla u^f_{\Omega_n}|^2 dx = \int_D f u^f_{\Omega_n} dx$$

Using the fact that u^* coincide with u^f_Ω, we have

$$\lim_{n\to\infty}\int_D |\nabla u^f_{\Omega_n}|^2 dx = \int_D f u^f_\Omega dx = \int_D |\nabla u^f_\Omega|^2 dx$$

which proves a strong convergence of $\nabla u^f_{\Omega_n}$ in $(L^2(D))^d$. $\square$

We would like that the limit u^* coincides with u^f_Ω, where Ω is the limit in some sense of Ω_n. This is not always the case. We give below some counter-examples.

4.1.2 Counter-examples to the continuity

Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in the unit disc D, let Γ_n be the n first points of the sequence $(x_n)_{n\in\mathbb{N}}$ and $\Omega_n = D\setminus\Gamma_n$. Then Ω_n converges in the Hausdorff sense to the empty set, but $u^f_{\Omega_n}$ does not converge to $u^f_\emptyset = 0$. In fact $H^1_0(\Omega_n) = H^1_0(D)$, hence $u^f_{\Omega_n} = u^f_D$. This result comes from the fact that point has capacity zero in dimension greater or equal to 2 (as we will see later). More precisely for fixed n, we can find a sequence $v_s \in C^\infty_c(D)$ such that

$$v_s \to 0 \text{ in } H^1_0(D), v_s = 1 \text{ in a neighborhood of } \Gamma_n, \quad 0 \le v_s \le 1.$$

Therefore every $v \in C^\infty_c(D)$ is the limit in $H^1_0(D)$ of $v(1-v_s) \in C^\infty_c(\Omega_n)$ and the announced equality $u^f_{\Omega_n} = u^f_D$ follows. We can even make a hole around each point x_n without getting convergence of the sequence $u^f_{\Omega_n}$ to 0. More precisely, let $\Omega_n = D\setminus\bigcup_{1\le k\le n}\overline{B}(x_k,r_k)$ where r_k is a sequence of positive real numbers satisfying

$$\sum_{n\ge1} -1/\log(r_n) < \eta, \quad \sum_{n\ge1} r_n^2 < 1. \tag{4.4}$$

The sequence of open sets Ω_n decreases to $E = D\setminus\bigcup_{n\ge1}\overline{B}(x_n,r_n)$ which has non zero measure thanks to the second condition on the sequence r_n. On the other hand, Ω_n converges to the empty set in Hausdorff sense. But again $u^1_{\Omega_n}$ does not converge to $u^1_\emptyset = 0$. In fact let $\psi \in C^\infty_c(D)^+$ and consider the function

$$\forall x\in D, \phi(x) := \psi(x)[1+\sum_{n\ge1}\alpha_n\log(|x-x_n|)]^+,$$

where $\alpha_n = -1/\log(r_n)$ and a^+ stands for the positive part of a. This function is well defined thanks to the first condition on the sequence r_n: in fact, since each function $[x\to\log|x-x_n|]$ belongs to $L^1(D)$, the series $\sum_{n\ge1}\alpha_n\log(|x-x_n|)$ is convergent in $L^1(D)$ and of norm bounded by $k\eta$ where $k = ||\log(\cdot)||_{L^1(2D)}$. The function vanishes on the union of balls $\overline{B}(x_n,r_n)$, that is vanishes outside E and has compact support in each Ω_n. It is not identically zero if we chose η small enough and ψ of support large enough. Moreover, it is in $H_0^1(D)$ (we can compute easily its H^1 norm). It is then in $H_0^1(\Omega_n)$. In particular we have

$$\int_{\Omega_n} \nabla u^1_{\Omega_n} \nabla \phi dx = \int_{\Omega_n} \phi dx.$$

Assuming u^* as the weak limit of a subsequence of $u^1_{\Omega_n}$ and passing to the limit we get:

$$\int_D \nabla u^* \nabla \phi dx = \int_E \phi dx.$$

Since E has non zero measure and ϕ is nonnegative and not identically zero on E, we deduce that u^* is not identically zero. In general, when Ω_n has holes whose number tends to infinity with n, the continuity may fail. Moreover the sequence $u^1_{\Omega_n}$ may converge to a solution of a problem which is not associated to the Laplace operator itself. In fact let us consider the classical situation in the setting of the homogenization where the open sets Ω_n are obtained from an open set D by removing a big number of small holes uniformly distributed. The sequence Ω_n tends to the empty set in Hausdorff sense. But, the limit of the sequence $u^f_{\Omega_n}$ depends on the size of holes. We have the following intuitive idea.

- If the holes are "small", then $u^f_{\Omega_n}$ converges to u^f_D;
- if the holes are "big", then $u^f_{\Omega_n}$ converges to 0;
- there exists a critical size for the holes such that u^* is the solution of another partial differential equation on D.

Let us consider the following bidimensional classical example due to F. Murat and D. Cioranescu [47]. Let $D = (0,1)^2$ and, for $0 < i, j < n, x_{ij} = (\frac{i}{n}, \frac{j}{n})$. Let us consider $\Omega_n = D \setminus \bigcup_{1<i,j<n} \overline{B}(x_{ij}, r_n)$. By Proposition 4.1.1, up to a subsequence, $u^f_{\Omega_n}$ converges weakly to u^* in $H_0^1(D)$ and u^* is characterized according to the size of the holes:

Proposition 4.1.3. *1. If* $\lim_{n\to\infty} \frac{\log r_n}{n^2} = -\infty$, *then* $u^* = u^f_D$*;*

2. if $\lim_{n\to\infty} \frac{\log r_n}{n^2} = 0$, *then* $u^* = 0$ *;*

3. if $\lim_{n\to\infty} \frac{\log r_n}{n^2} = -c < 0$, *then* u^* *is the solution of the problem*

$$u^* \in H_0^1(D), \quad -\Delta u^* + \frac{2\pi}{c} u^* = f.$$

Proof: Let us denote by u_n the solution of the equation

$$\left\{ \begin{array}{rcl} -\Delta u &=& f \text{ in } \Omega_n \\ u &=& 0 \text{ in } \partial\Omega_n. \end{array} \right. \tag{4.5}$$

we start by the last assertion. Let us set $B_n = \bigcup_{0<i,j<n} B^n_{i,j}$ and the $C_n = \bigcup_{0<i,j<n} C^n_{i,j}$ where $B^n_{i,j}$ is the open ball with center $x_{i,j}$ and radius $1/2n$, and $C^n_{i,j}$ the closed ball with

center $x_{i,j}$ and radius $r_n = e^{-cn^2}$ for c a fixed constant (the constant appearing in the last statement of the proposition). For every n and $0 < i, j < n$ let $w_{i,j}^n \in H^1(B_{i,j}^n \setminus C_{i,j}^n)$ be the solution of the equation $\Delta w_{i,j}^n = 0$ on $B_{i,j}^n \setminus C_{i,j}^n$ which satisfies the boundary conditions $w_{i,j}^n = 0$ on $\partial C_{i,j}^n$ and $w_{i,j}^n = 1$ on $\partial B_{i,j}^n$. An explicit computation of the solution gives

$$w_{i,j}^n(x) = \frac{\ln|x - x_{i,j}| + cn^2}{cn^2 - \ln(2n)} \quad \text{for} \quad x \in B_{i,j}^n \setminus C_{i,j}^n.$$

We define w_n as the function which is equal to $w_{i,j}^n$ on $B_{i,j}^n \setminus C_{i,j}^n$, extended by 0 on C_n and by 1 on $D \setminus B_n$. We may observe that

- $0 \leq w_n \leq 1$;
- $\nabla w_n \rightharpoonup 0$ in $(L^2(D))^2$ as $n \to +\infty$, hence w_n converges weakly in $H^1(D)$ to a constant function. The computation of the limit of the integral $\int_D w_n dx$ shows that the constant is equal to 1.

Let $\varphi \in C_c^\infty(D)$. Then $\varphi w_n \in H_0^1(\Omega_n)$, hence φw_n may be chosen as a test function for the equation (4.5):

$$\int_D \nabla u_n \nabla w_n \varphi dx + \int_D \nabla u_n \nabla \varphi w_n dx = \int_D f \varphi w_n dx.$$

The second and the third terms of this equality converge respectively to $\int_D \nabla u^* \nabla \varphi dx$ and $\int_D f \varphi dx$. For the first term the Green formulas gives

$$\int_D \nabla u_n \nabla w_n \varphi dx = \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} u_n \frac{\partial w_n}{\partial \nu} \varphi d\sigma - \int_D u_n \nabla w_n \nabla \varphi dx.$$

The boundary term on $\partial C_{i,j}^n$ does not appear since u_n vanishes on it. The last term of the identity converges to 0 as $n \to \infty$. We get

$$\begin{aligned} \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} u_n \frac{\partial w_n}{\partial \nu} \varphi d\sigma &= \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} \frac{2n}{cn^2 - \ln(2n)} u_n \varphi d\sigma \\ &= \frac{2n^2}{cn^2 - \ln(2n)} \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} \frac{1}{n} u_n \varphi d\sigma. \end{aligned}$$

Let us denote by $\mu_n \in H^{-1}(D)$ the distribution defined

$$\langle \mu_n, \psi \rangle_{H^{-1}(D) \times H_0^1(D)} = \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} \frac{1}{n} \psi d\sigma.$$

We will prove that this distribution converges strongly in $H^{-1}(D)$ to πdx. Let v_n be the solution of the equation

$$\begin{cases} -\Delta v_n &= 4 \text{ in } \cup_{0<i,j<n} B_{i,j}^n \\ v_n &= 0 \text{ on } D \setminus \cup_{0<i,j<n} B_{i,j}^n. \end{cases}$$

then we have

$$\frac{\partial v_n}{\partial \nu} = \frac{1}{n} \quad \text{on} \quad \bigcup \partial B_{i,j}^n.$$

We notice that $v_n \to 0$ strongly in $H^1(D)$, therefore $\Delta v_n \to 0$ strongly in $H^{-1}(D)$. One may observe also that

$$\begin{aligned}
\langle -\Delta v_n, \psi \rangle_{H^{-1}(D)\times H_0^1(D)} &= \sum_{0<i,j<n} \int_{B_{i,j}^n} \nabla v_n \nabla \psi dx \\
&= \sum_{0<i,j<n} \int_{\partial B_{i,j}^n} \frac{1}{n} \psi d\sigma - \sum_{0<i,j<n} \int_{B_{i,j}^n} 4\psi dx.
\end{aligned}$$

Passing to the limit as $n \to \infty$ and using the fact that $1_{\cup B_{i,j}^n} \rightharpoonup \frac{\pi}{4} 1_D$ weakly in L^2 we get that

$$\mu_n \to \pi dx \quad \text{strongly in} \quad H^{-1}(D).$$

Consequently $u^* \in H_0^1(D)$ satisfies the equation

$$\forall \varphi \in C_c^\infty(D) \quad \int_D \nabla u^* \nabla \varphi dx + \frac{2\pi}{c} \int_D u^* \varphi dx = \int_D f \varphi dx,$$

that is

$$\begin{cases} -\Delta u^* + \frac{2\pi}{c} u^* &= f \text{ in } D \\ u^* &= 0 \text{ on } \partial D. \end{cases}$$

One may adapt this proof for the first two points. □

4.1.3 The γ-Convergence

Here we introduce a new topology on the class of sets in $\mathbb{R}^d$. The γ-convergence is nothing else than the topology on the set of open sets which expresses the continuity with respect to a domain of the solution of Dirichlet's problem.

Definition 4.1.4. *We say that a sequence of open sets Ω_n contained in D γ-converges to the open set $\Omega \subset D$, and we denote $\Omega_n \xrightarrow{\gamma} \Omega$, if for all $f \in H^{-1}(D)$ we have $u_{\Omega_n}^f \to u_\Omega^f$ in $H_0^1(D)$.*

The deal here is to find a class of subset of $\mathbb{R}^d$ for which the γ-convergence is compact. We start by the set that has the ε-cone property (domains satisfying a uniform exterior cone property).

Definition 4.1.5. *Let y be a point in $\mathbb{R}^d$, ζ a unitary vector and ε a positive real number. We call cone of summit y of direction ζ and opening ε, the cone defined by*

$$C(y, \zeta, \varepsilon) = \{z \in \mathbb{R}^d, (z - y, \zeta) \geq cos(\varepsilon)|z - y| \text{ and } 0 < |z - y| < \varepsilon\}.$$

We say that an open set Ω has the ε-cone property if

$$\forall x \in \partial\Omega, \;\; \exists \zeta_x \;\; \textit{unitary vector such that } \forall y \in \overline{\Omega} \cap B(x,\varepsilon) \;\; C(y, \zeta_x, \varepsilon) \subset \Omega$$

As example an Euclidean ball in $\mathbb{R}^d$ has ε-cone property. The following open sets do not have the ε-cone property: $\mathbb{R}^d \setminus \{0\}$, $\{(x,y) \in \mathbb{R}^d; xy > 0\}$.

Remark 4.1.6 It is easily shown that open convex sets have ε-cone property.

Now we introduce the notion of capacity associated to the H^1 norm. We define first the capacity of compact sets, then of open sets and finally the capacity of any sets.

Definition 4.1.7. *Let K be a compact set of $\mathbb{R}^d$, we define*

$$cap(K) := \inf\{\int_{\mathbb{R}^d} |\nabla u|^2 dx; \quad u \in C_c^\infty(\mathbb{R}^d), u \geq 1 \;\; on \;\; K\}$$

and for ω open set in $\mathbb{R}^d$, we define

$$cap(\omega) := \sup\{cap(K), \quad K \, compact \;\; K \subset \omega\}$$

We check that

Lemma 4.1.8. *For all compact K,*

$$cap(K) = \inf\{cap(\omega); \;\; \omega \;\; open, \;\; K \subset \omega\}$$

Proof: By definition $cap(K) \leq \inf\{cap(\omega); \;\; \omega$ open, $\;\; K \subset \omega\}$. Since the map $K \to cap(K)$ is nondecreasing for the inclusion of compact sets, if $\omega \subset K_1$, ω open set and K_1 compact set, we have $cap(\omega) \leq cap(K_1)$. Let now $\varepsilon > 0$ and $u \in C_c^\infty(\mathbb{R}^d)$ with $u \geq \chi_K$, $||u||_{H^1} \leq (1+\varepsilon) cap(K_1)$- Let us consider the open set $\omega = [(1+\varepsilon)u > 1]$ and the compact set $K_1 = [(1+\varepsilon)u \geq 1]$. We get

$$K \subset \omega, \quad cap(\omega) \leq (1+\varepsilon)^2 ||u||_{H^1}^2 \leq (1+\varepsilon)^2 [cap(K) + \varepsilon],$$

and the desired result follows. □

Definition 4.1.9. *If E is a subset of $\mathbb{R}^d$, we define*

$$cap(E) := \inf\{cap(\omega); \;\; \omega \;\; open, \;\; E \subset \omega\}.$$

We have the following property which provides another definition of capacity.

Proposition 4.1.10. *For all $E \subset \mathbb{R}^d$,*

$$cap(E) = \inf\{\int_{\mathbb{R}^d} |\nabla u|^2 dx : \;\; u \geq 1 \quad a.e \; in \; a \; neighborhood \; of \;\; E\}.$$

It follows that $cap(E) = 0$ is equivalent to the existence of a sequence u_n converging to 0 in $H^1(\mathbb{R}^d)$ and greater than 1 in a neighborhood of E. Therefore a set of capacity zero has zero Lebesgue measure.

In the same way, for an open bounded set D in $\mathbb{R}^d$, we define the relative capacity to D denoted by $cap_D(\cdot)$ or $cap(\cdot, D)$.

Definition 4.1.11. *For all compact $K \subset D$, we define*

$$cap_D(K) := \inf\{\int_D |\nabla u|^2 dx;\ \ u \in C_c^\infty(D),\ \ u \geq 1 \ \ on \ \ K\}.$$

For all open sets ω of D, we define

$$cap_D(\omega) := \sup\{cap_D(K);\ \ K \ \ compact,\ \ K \subset \omega\}.$$

If E is any subset of D, we define

$$cap_D(E) := \inf\{cap_D(\omega);\ \ \omega \ \ open,\ \ E \subset \omega\}.$$

Remark 4.1.12 The capacity of a point in $\mathbb{R}^d$ for $d \geq 2$ is zero. See [67] for the proof. In general we have:
if E is a subset of $\mathbb{R}^d$ contained in the manifold of dimension $d-2$, then $cap(E) = 0$,
if E is a subset of $\mathbb{R}^d$ which contains a piece of smooth hypersurface, then $cap(E) > 0$,
if E is smooth manifold of dimension d_E, then $cap(E) = 0$ is equivalent to $d_E \leq d-2$.
See for example [2] for the proof and more general calculus.

Here we give a quick overview of quasi-continuity and quasi-open sets

Definition 4.1.13. *We say that a property holds quasi-everywhere (q.e) if it holds outside a set of capacity zero.*

Definition 4.1.14. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be quasi-continuous if there exists a nonincreasing sequence of open sets ω_n of $\mathbb{R}^d$ satisfying $\lim_{n\to\infty} cap(\omega_n) = 0$ and the restriction of f to the complement ω_n^c of ω_n is continuous.*

Proposition 4.1.15. *All $f \in H^1(\mathbb{R}^d)$ have unique quasi-continuous representative that is for every $f \in H^1(\mathbb{R}^d)$ there exists a unique quasi-continuous function u such that $f = u$ q.e..*

For proof see [67] Theorem 3.3.29

Definition 4.1.16. *A subset Ω of D is called quasi-open if there exists a nonincreasing sequence of open sets ω_n such that: $\lim_{n\to\infty} cap(\omega_n) = 0$ and $\forall n,\ \ \Omega \cup \omega_n$ is open.*

Proposition 4.1.17. *A countable union of quasi-open sets is quasi-open set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a quasi-continuous function and α a real number. Then $[f > \alpha]$ is quasi-open. In particular, if $u \in H^1(\mathbb{R}^d)$, then $[\tilde{u} > \alpha]$ is quasi-open where $\tilde{u}$ is a quasi-continuous representation of u.*

For proof see [67] page 102.

Definition 4.1.18. *Let α and r be two positive real numbers. We say that an open set $\Omega \subset D$ has the capacity density (α, r) if*

$$\forall x \in \partial\Omega, \quad \frac{cap(\Omega^c \cap B_r(x), B_{2r}(x))}{cap(B_r(x), B_r(x))} \geq \alpha > 0.$$

For $\alpha < 1$ fixed, denote

$$\mathcal{O}_{\alpha,r_0} = \{\Omega \subset D, \ \forall r, \ 0 < r < r_0, \ \Omega \ \textit{has capacity density} \ (\alpha, r)\}.$$

We see that this condition is weaker than the ε-cone property. In fact it holds if for all point x on the boundary such that there exists a cone of size independent of x contained in Ω^c. This boundary regularity implies Hölderian regularity up to the boundary of solution of Dirichlet's problem provided f is sufficiently regular.

Now we introduce the Wiener's condition which is weaker than the capacity density. A set Ω is regular in the Wiener's sense if it belongs to the following set

$$\begin{aligned} \mathcal{W}_w(D) = \{ &\Omega \subset D : \forall x \in \partial\Omega, \ \forall 0 < r < R < 1; \\ &\int_r^R \left(\frac{cap(\Omega^c \cap \overline{B}_t(x), B_{2t}(x))}{cap(\overline{B}_t(x), B_{2t}(x))} \right) \frac{dt}{t} \geq w(r, R, x) \Big\} \end{aligned}$$

where $B_t(x)$ is the ball with center x and radius t, and

$$w : (0,1) \times (0,1) \times D \to [0, +\infty)$$

is such that

1. $\lim_{r \to 0} w(r, R, x) = +\infty$, locally uniformly on x;
2. w is lower semicontinuous in the third variable.

Let us introduce those notations that we will use in the sequel.

- The class $\mathcal{O}_{\text{convex}} \subset \mathcal{O}(D)$ of convex sets contained in D;
- the class $\mathcal{O}_{\text{unif cone}} \subset \mathcal{O}(D)$ of domains satisfying a uniform exterior cone property;
- the class $\mathcal{O}_{\text{unif flat cone}} \subset \mathcal{O}(D)$ of domains satisfying a uniform flat cone condition, i.e., as above, but with the weaker requirement that the cone may be flat, that is of dimension $d - 1$;
- $\mathcal{O}_{\text{cap density}} \subset \mathcal{O}(D)$ of domains satisfying a uniform capacity density condition for some α, r;
- $\mathcal{O}_{\text{unif Wiener}} \subset \mathcal{O}(D)$ of domains satisfying a uniform Wiener condition i.e. $\mathcal{O}_{\text{unif Wiener}} = \mathcal{W}_w(D)$.

Roughly speaking, we may establish the following inclusions :

$$\mathcal{O}_{\text{convex}} \subset \mathcal{O}_{\text{unif cone}} \subset \mathcal{O}_{\text{unif flat cone}} \subset \mathcal{O}_{\text{cap density}} \subset \mathcal{O}_{\text{unif Wiener}}. \tag{4.6}$$

Proposition 4.1.19. *Each of previous classes of domains is compact with respect to the Hausdorff convergence.*

Taking into account the Proposition 4.1.19 and the inclusions (4.6), it suffices to prove the following theorem for having γ-compactness in each of the previous classes.

Theorem 4.1.20. *(Uniform Wiener's condition) Let $(\Omega_n)_n \subset \mathcal{W}_w(D)$ which converges to Ω in the Hausdorff topology. Then $\Omega \in \mathcal{W}_w(D)$ and Ω_n γ-converges to Ω*

We conclude the part of continuity of the Dirichlet's problem by this result due to Šverák which turns out to be a consequence of Theorem 4.1.20. Let $l \geq 1$ be an integer, for all $\Omega \subset D$ open sets, we denote $\#\Omega^c$ the number of connected components of the complement of Ω. We define the class
$\mathcal{O}_l(D) = \{\Omega \subset D, \;\; \Omega, \;\; open, \;\; \#\Omega^c \leq l\}$.

Theorem 4.1.21. *Let Ω_n be a sequence of open sets in the class $\mathcal{O}_l(D)$ which converge to an open set Ω in the sense of Hausdorff and assume $d = 2$. Then for all $f \in H^{-1}(D)$, $u^f_{\Omega_n}$ converges to u^f_{Ω}.*

We will prove these two theorems in a general setting in Section 5.2, Theorem 5.2.10 and Theorem 5.2.13.

4.2 Existence of Optimal Shapes

Let $\mathcal{O}$ be a class of admissible open subsets of D and $J : \mathcal{O} \to [0, +\infty]$ be a γ-lower semicontinuous functional. We consider the following minimization problem.

$$\min\{J(\Omega) : |\Omega| \leq m, \;\; \Omega \subset \mathcal{O}\}. \tag{4.7}$$

The γ-convergence on the class of all open subsets of D is not compact if the dimension d is greater than 1; In fact several shape optimization problems of the form (4.7) do not admit any solution, and the introduction of a relaxed formulation is needed in order to describe the behavior of minimizing sequences. We describe some particular cases where the problem (4.7) admits a solution and give some examples where the existence of optimal domain fails.

4.2.1 Existence of optimal domains under some constraints

The direct method of the calculus of variation and the Proposition 4.1.19 Theorem 4.1.20 and Theorem 4.1.21 give the following result.

Theorem 4.2.1. *Let $F : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function. Then the shape optimization problem*

$$\min \left\{ \int_\Omega F(x, u^f_\Omega, \nabla u^f_\Omega) dx : \quad \Omega \in \mathcal{O}_{ad} \right\}$$

has at least one solution for

$$\mathcal{O}_{ad} = \mathcal{O}_{convex}, \mathcal{O}_{unif\ cone}, \mathcal{O}_{unif\ flat\ cone}, \mathcal{O}_{cap\ density}, \mathcal{O}_{unif\ wiener}, \mathcal{O}_l (d = 2)$$

respectively. Here u^f_Ω stands for the solution of the Dirichlet's problem associated to f and Ω.

Another interesting result of the existence of the optimal domain is the following theorem due to Buttazzo and Dal Maso. For the proof see [38].

Theorem 4.2.2. *(Buttazzo - Dal Maso) Let $J : \mathcal{O} \to \overline{\mathbb{R}}$ be a function which is γ-lower semicontinuous and monotone decreasing with respect to the set inclusion where $\mathcal{O}$ is the class quasi-open sets. Then the optimization problem*

$$\min \{ J(\Omega) : |\Omega| = m, \ \Omega \in \mathcal{O} \}$$

admits at least one solution in $\mathcal{O}$.

4.2.2 Example of non existence of an optimal shape

We give three examples of non-existence of an optimal domain.

Example 4.2.3 Let $D = (0, 1)^2, f \in L^2(D),\ f > 0$ a.e. on D, m a positive real number and w the solution of

$$\begin{cases} w \in H^1_0(D) \cap H^2_0(D) \\ -\Delta w + \frac{2\pi}{m} w = f \text{ in } D \end{cases}$$

Let us consider the functional J defined by

$$J(\Omega) = \int_\Omega (u^f_\Omega - w)^2 dx$$

this is a functional of least square type useful in applications. Fix a real number a, $0 < a < 1$ and let us consider

$$\mathcal{O}_a = \{ \Omega \subset D, \ \Omega \text{ open}, \ |\Omega| \geq a \}.$$

Then the problem $\min_{\Omega \in \mathcal{O}_a} J(\Omega)$ does not admit a solution. In fact, we have seen in the counterexample to the continuity that if we set

$$\Omega_n = D \setminus \bigcup_{i,j} \overline{B}(x_{ij}, e^{-mn^2}),$$

then, $u^f_{\Omega_n}$ converges to w weakly in $H^1_0(D)$ and strongly in $L^2(D)$. Therefore $J(\Omega_n) = ||u^f_{\Omega_n} - w||_{L^2(D)}$ converges to 0. Since it is clear that the measure of Ω_n tends to $1 > a$, the open sets Ω_n are in the class $\mathcal{O}_a$ for n large enough and thus the infimum of J is equal to 0. If this infimum is achieved for some open set Ω we should get $u^f_\Omega = w$ in $L^2(\Omega)$ and since $-\Delta u^f_\Omega = f$ in $\mathcal{D}'(\Omega)$, we should have $-\Delta w = f$ in $\mathcal{D}'(\Omega)$. But, by definition of w, $-\Delta w + \frac{2\pi}{m} w = f a.e.$ in D. This should implies that $w = 0$ on Ω, which is contradictory with the fact that $f > 0, a.e.$ on D and $|\Omega| > 0$. This argument can easily be adapted for proving that there exists not even a quasi-open solution.

Example 4.2.4 We bring this second example from [31]. Let D be a bounded open set and $f \in L^2(D)$, we want to minimize the functional

$$J(\Omega) = \int_D (u^f_\Omega - u_0)^2 dx,$$

where u_0 is a given function in $L^2(D)$. A physical interpretation of this problem can be the following: D is a box or a room heated from a source of heat f and $D \setminus \Omega$ represents a place where we put a cooling device (say ice). The purpose of the problem is to determine the ice position for which the temperature of the box is close as much as possible to the known ideal temperature u_0. We set the problem in a simple configuration and prove the non existence of optimal domains. Let us choose $f \equiv 1,\ u_0 \equiv c \equiv$ constant and D the unit ball in $\mathbb{R}^2$. By the maximum principle, for all $\Omega \subset D$,we have

$$0 \leq u^1_\Omega \leq u^1_D = \frac{1-r^2}{4} \leq \frac{1}{4}.$$

If $c \geq \frac{1}{4}$ we have

$$u^1_\Omega - c \leq u^1_D - c \leq 0,$$

therefore

$$J(\Omega) = \int_D (u^1_\Omega - c)^2 dx \geq \int_D (u^1_D - c)^2 dx = J(D),$$

this proves that $\Omega = D$ realize the minimum of J.
If $0 < c < \frac{1}{8}$ it is easy to see that D is not the minimum of J. In fact, denoting by B_R the disc of center O and radius $R < 1$, we have $u^1_{B_R} = \frac{R^2-r^2}{4}$ for $r = |x| < R$ and $J(B_R)$ is given by

$$2\pi \int_0^R (\frac{R^2-r^2}{4} - c)^2 r dr + 2\pi \int_R^1 (0-c)^2 = \frac{\pi}{48}(R^6 - 12cR^4 + 48c^2).$$

A simple calculation shows that $J(B_R) < J(D)$ for $R = \sqrt{8c} < 1$. Let us prove that J cannot admit a minimum (at least regular) in this case. Assume that there exists a regular minimum Ω, It is different from D, and $|\Omega| < |D|$. Suppose that its closure is different from D (this happen when Ω regular). Let B_ε be a ball of radius ε contained in $D \setminus \overline{\Omega}$. Set $\Omega_\varepsilon = \Omega \cup B_\varepsilon$ and let us show that for ε small enough, Ω_ε is a better open set than Ω. Since Ω_ε has two disjoints connected components, we may compute separately the solution on each connected component. But on Ω, $u^1_{\Omega_\varepsilon}$ coincides with u^1_Ω and on B_ε it may be computed explicitly. It is easy to see that, for ε small enough, we have $0 < u^1_\Omega < c$ on B_ε. Let us compare $J(\Omega_\varepsilon), J(\Omega)$.

$$\begin{aligned} J(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} (u^1_{\Omega_\varepsilon} - c)^2 dx + \int_{D\setminus\Omega_\varepsilon} c^2 dx \\ &= \int_\Omega (u^1_\Omega - c)^2 dx + \int_{B_\varepsilon} (u^1_{B_\varepsilon} - c)^2 dx + \int_{D\setminus\Omega} c^2 dx - \int_{B_\varepsilon} c^2 dx \\ &= J(\Omega) + \int_{B_\varepsilon} (u^1_{B_\varepsilon} - c)^2 - c^2 dx. \end{aligned}$$

For ε small enough, $0 < u^1_{\Omega_\varepsilon} < c$, and thus $(u^1_{\Omega_\varepsilon} - c)^2 < c^2$, this implies that $J(\Omega_\varepsilon) < J(\Omega)$. Therefore J cannot have a regular minimum. Moreover we may prove that there exists not even a non regular minimum as it has been done in [46]

Example 4.2.5 ([84]) Let D_1 and D_2 be two open bounded subsets of $\mathbb{R}^d$ such that the closure of D_1 is contained in D_2. Let $\mathcal{O}$ be the set of all open sets Ω such that $D_1 \subset \Omega \subset D_2$ that is

$$\mathcal{O} = \left\{ \Omega \subset \mathbb{R}^d \ \text{ open } \ D_1 \subset \Omega \subset D_2 \right\}.$$

Let $z \in L^2(D_1)$ be a given function. Consider the functional defined on $\mathcal{O}$ by

$$J(\Omega) = \int_{D_1} (u^f_\Omega - z)^2 dx;$$

where u^f_Ω stands for the solution of the Dirichlet problem

$$\begin{cases} -\Delta u + u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases}$$

We are interested in the problem

$$\min \{J(\Omega) : \Omega \in \mathcal{O}\} \tag{4.8}$$

To prove that the problem (4.8) does not admit in general a solution we restrict ourselves to the very particular 2-dimensional setting. Put $D_1 = B_1(0) \cup B_4(0) \setminus \overline{B}_3(0)$, $D_2 = B_5(0)$, $f = 1$ on D_2, $z = u_1$ on $B_1(0)$ and $z = u_2$ on $B_4(0) \setminus \overline{B}_3(0)$. where u_1 is the solution of

$$\begin{cases} -\Delta u + u &= 1 \text{ in } B_2(0) \\ u &= \delta \text{ on } \partial B_2(0) \end{cases}$$

and u_2 the solution of

$$\begin{cases} -\Delta u + u &= 1 \text{ in } B_5(0) \setminus \overline{B}_2(0) \\ u &= \delta \text{ on } \partial B_2(0) \\ u &= 0 \text{ on } \partial B_5(0) \end{cases}$$

δ is a positive number less than the value of $u^1_{D_2}$ on $\partial B_2(0)$. Let us consider the sequence $(\Omega_n)_n, n \geq 6$, of domains

$$\Omega_n = D_2 \setminus \bigcup_{i=1}^{n} B_{\delta_n}(x_i^n),$$

where $x_i^n \in \partial B_2(0), i = 1, \cdots, n$, are the vertices of a regular n-polygons, and the number $\delta_n > 0$ is chosen so that

$$\int_{\partial B_2(0)} u^1_{\Omega_n} d\mathcal{H}^1 = 4\pi\delta,$$

(for $\Omega \subset \mathcal{O}$, we assume u^1_Ω extended by zero to D_2). The sequence $u^1_{\Omega_n}$ converges weakly in $H^1(D_2)$ to

$$u_\infty(x) = \begin{cases} u_1(x), & |x| < 2 \\ u_2(x), & |x| \geq 2 \end{cases}$$

Therefore $\inf\{J(\Omega) : \Omega \in \mathcal{O}\} = 0$. At the same time, for all $\Omega \in \mathcal{O}$ we have $u^1_\Omega \neq z$ in D_1.

The previous examples show that some shape optimization problems do not admit a solution. It is then, useful to search the solution outside of the setting of domains in $\mathbb{R}^d$, introducing the relaxed form of the Dirichet problem. The relaxed form of a shape optimization problem with Dirichlet condition on the free boundary involve relaxed controls which are measures. See for Example [31] for more details and [37] for complete discussion. It is known that the relaxed control depends only on the state equation. If we take the equation of the form

$$-\Delta u = f \text{ in } \Omega, \quad u \in H^1_0(\Omega)$$

where the control variable runs in the class of opens subsets of a given bounded domain $D \subset \mathbb{R}^d$ and f is a given function in $L^2(D)$. For a sequence $(\Omega_n)_n$ of open subsets of D we denote by u_n the solution of the equation

$$-\Delta u = f \text{ in } \Omega_n, \quad u \in H^1_0(\Omega_n). \tag{4.9}$$

The relaxation consists of studying the limit behavior of the sequence u_n as $n \to +\infty$. the relaxed problem has been shown to be equal

$$-\Delta u + \mu u = f \text{ in } \Omega, \quad u \in H^1_0(\Omega)$$

where μ is the Borel measure defined by

$$\mu(B) = \begin{cases} +\infty \text{ if } cap(B \cap \{w = 0\}) > 0 \\ \int_B \frac{1}{w} d\nu \text{ if } cap(B \cap \{w = 0\}) = 0. \end{cases} \tag{4.10}$$

Here $\nu = \Delta w + 1 \geq 0$ in $\mathcal{D}'(D)$ is a nonnegative Radon measure belonging to $H^{-1}(D)$, and w is the limit of the solution of the equation (4.9) with $f = 1$

4.3 Necessary Condition of Optimality

We recall some derivation formulas of integral on moving domains. We consider the function of the form

$$\varepsilon \mapsto I(\varepsilon) = \int_{\Omega_\varepsilon} f(\varepsilon, \Phi_\varepsilon(x))dx,$$

$\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$ is the image of a measurable set $\Omega \subset R^d$ by a family of diffeomorphisms $\Phi_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ defined for all $\varepsilon \in [0, T)$ and for all $y \in \mathbb{R}^d, \Phi_0(y) = y$. By a change of variable of the type $x = \Phi_\varepsilon(y)$ we get

$$I(\varepsilon) = \int_\Omega f(\varepsilon, \Phi_\varepsilon(y))J_\varepsilon(y)dy,$$

where $J_\varepsilon(y) = \det(D_y\Phi_\varepsilon(y))$ stands for the Jacobian of Φ_ε (we will use the notation Φ_ε or $\Phi(\varepsilon)$).Assume

$$\Phi : \varepsilon \in [0, T) \to W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ differentiable at } 0 \text{ with } \Phi(0) = I, \frac{\partial \Phi}{\partial \varepsilon}(0) = V, \tag{4.11}$$

where I is the identity matrix. In other word Φ is the family of diffeomorphisms generated by the vector field V. We have the following differentiability formulas.

Theorem 4.3.1. *Let Φ satisfying (4.11) be given. Assume that*

$$\varepsilon \in [0, T) \in L^1(\mathbb{R}^d) \quad \textit{is differentiable at } 0, \tag{4.12}$$

$$f(0, \cdot) \in W^{1,1}(\mathbb{R}^d). \tag{4.13}$$

Then the function $\varepsilon \to I(\varepsilon) = \int_{\Omega_\varepsilon} f(\varepsilon, x)dx$ is differentiable at 0 and we have

$$I'(0) = \int_\Omega [\frac{\partial f}{\partial \varepsilon}(0, y) + \operatorname{div}_y(fV)(0, y)]dx. \tag{4.14}$$

Moreover if Ω is an open set with Lipschitz boundary, then

$$I'(0) = \int_\Omega \frac{\partial f}{\partial \varepsilon}(0, y)dx + \int_{\partial\Omega} (fV)(0, y)\nu(y)d\mathcal{H}^{d-1}(y), \tag{4.15}$$

where ν denotes the unit normal of $\partial\Omega$.

It may happens that the function $f(\varepsilon,\cdot)$ is defined only on the moving domain Ω_ε but not on all $\mathbb{R}^d$. In this case the previous theorem still works provided f admits an extension. We have

Corollary 4.3.2. *Let Φ satisfying (4.11) and $\varepsilon \in [0,T) \to f(\varepsilon,\cdot) \in L^1(\Omega_\varepsilon)$. Let assume that*

$$\varepsilon \in [0,T) \to F(\varepsilon) = f(\varepsilon, \Phi(\varepsilon,\cdot)) \text{ is differentiable at } 0, \tag{4.16}$$

and there exits a linear continuous extension operator $P : L^1(\Omega) \to L^1(\mathbb{R}^d)$ such that $P(f(0,\cdot)) \in W^{1,1}(\mathbb{R}^d)$. Then there exists an extension $\varepsilon \in [0,T) \to \tilde{f}(\varepsilon,) \in L^1(\mathbb{R}^d)$ which is differentiable at 0 and

$$\frac{\partial \tilde{f}}{\partial \varepsilon}(0,\cdot) = \frac{\partial F}{\partial \varepsilon}(0,\cdot) - \nabla P(f(0,\cdot))V.$$

Moreover the function $\varepsilon \to \int_{\Omega_\varepsilon} f(\varepsilon,x)dx$ is differentiable at 0 and the formulas (4.14) holds by setting for a.e.$x \in \Omega$, $\frac{\partial f}{\partial \varepsilon}(0,x) = \frac{\partial \tilde{f}}{\partial \varepsilon}(0,x)$.

We give a consequence of the theorem concerning the derivation on an interval

Corollary 4.3.3. *Let us assume*

$$\Phi \in C^1([0,T);W^{1,\infty}(\mathbb{R}^d)), f \in C^1([0,T);L^1(\mathbb{R}^d)) \cap C^1([0,T);W^{1,\infty}(\mathbb{R}^d)),$$

We set $V(\varepsilon,x) = \frac{\partial \Phi}{\partial \varepsilon}(\varepsilon, \Phi(\varepsilon)^{-1}(x))$. Then the function $\varepsilon \in [0,T) \to I(\varepsilon)$ is continuously differentiable on $[0,T)$ and we have

$$I'(\varepsilon) = \int_{\Omega_\varepsilon} [\frac{\partial f}{\partial \varepsilon}(\varepsilon,x) + \mathrm{div}(fV)(\varepsilon,x)]dx. \tag{4.17}$$

We consider also the surface integral of the form

$$G(\theta) = \int_{\Gamma_\theta} g(\theta)d\sigma(\theta) = \int_{\Gamma_\theta} g(\theta) \circ (I+\theta)J^\theta d\sigma(\theta)$$

where J^θ is the tangential Jacobian of the map $x \mapsto x+\theta(x)$ that is $J^\theta = Jac^{\Gamma_\theta}(I+\theta) = \det(I+\nabla\theta)||^t(I+\nabla\theta)^{-1}\nu||$ and $\Gamma = \partial\Omega$, $\Gamma_\theta = (I+\theta)(\Gamma)$ and θ is an element of the space $C^{1,\infty} := C^1 \cap W^{1,\infty}$ endowed with the $W^{1,\infty}$ norm.

Theorem 4.3.4. *Assume that Ω is a bounded open set of class C^1. Let $\theta \in C^{1,\infty} \mapsto g(\theta) \in W^{1,1}(\Omega_\theta)$ be a map such that the map $\theta \in C^{1,\infty} \mapsto h(\theta) \circ (I+\theta) \in W^{1,1}(\Omega)$ is differentiable at 0. Then the map $\theta \mapsto \mathcal{G}(\theta) = \int_{\Gamma_\theta} g(\theta)d\sigma(\theta)$ is differentiable at 0 and we have*

$$\forall \xi \in C^{1,\infty},\ \mathcal{G}'(0)\xi = \int_\Gamma h'(0)\xi + g(0)\mathrm{div}^\Gamma \xi.$$

For every compact $K \subset \Omega$, $\theta \in C^{1,\infty} \mapsto g(\theta)|_K \in L^1(K)$ is differentiable at 0 *and we have*

$$\forall \xi \in C^{1,\infty}, \;\; g'(0)\xi = h'(0)\xi - \nabla g(0)\xi \in L^1(\Omega).$$

Moreover, if Ω is of class C^2 and $g(0) \in W^{2,1}(\Omega)$, then

$$\begin{aligned}\mathcal{G}'(0)\xi &= \int_\Gamma g'(0)\xi + \nabla g(0)\xi + g(0)\mathrm{div}^\Gamma \xi \\ &= \int_\Gamma g'(0)\xi + [\frac{\partial g(0)}{\partial \nu} + Hg(0)](\xi \cdot \nu).\end{aligned}$$

where ν denotes the unit normal and H the mean curvature of Γ.

Proposition 4.3.5. *Let Ω be a bounded open set of class C^2 and $\Phi : \varepsilon \in [0,T) \to C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ differentiable at* 0 *with $\Phi(0) = I, \frac{\partial \Phi}{\partial \varepsilon}(0) = V$. Assume the map $\varepsilon \mapsto g(\varepsilon) \circ \Phi(\varepsilon) \in W^{1,1}(\Omega)$ is differentiable at* 0 *with $g(0) \in W^{2,1}(\Omega)$. Then the map $\varepsilon \mapsto G(\varepsilon) = \int_{\Gamma_\varepsilon} g(\varepsilon) d\sigma$ is differentiable at* 0*, $\varepsilon \mapsto g(\varepsilon)|_\omega \in W^{1,1}(\omega)$ is differentiable at* 0 *for every open set ω compactly supported in Ω; the derivative $g'(0)$ is in $W^{1,1}(\Omega)$ and we have*

$$G'(0) = \int_\Gamma g'(0) + [\frac{\partial g(0)}{\partial \nu} + Hg(0)](V \cdot \nu).$$

The goal is to apply those differentiability formulas to shape optimization problem in order to derive the shape derivative. The minimization problem we consider is the following

$$\min\{J(\Omega) : \Omega \in \mathcal{O}_{ad}\}$$

where $J(\Omega) = \int_\Omega F(x, u(x), \nabla u(x))dx + \int_\Gamma G(x, u(x), \nabla u(x))d\sigma$, $\mathcal{O}_{ad}$ is a class of domains for which the minimization problem has a solution and u the solution of the linear equation

$$A(x, \nabla u) = f \;\text{ in }\; \Omega, \quad B(x, \nabla u) = g \;\text{ on }\; \Gamma.$$

The functions $F = F(x,u,z), G = G(x,u,z)$ are assumed to be smooth. We assume also that the domain Ω is of class C^k, $k \geq 1$. Let Φ_ε be a smooth diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$ (which is a flow generated by a smooth vector field V) and denote by $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$ the transported domain and by u_ε the state function corresponding to the transformed domain that is the solution of the equation

$$A(x, \nabla u_\varepsilon) = f \;\text{ in }\; \Omega_\varepsilon, \quad B(x, \nabla u_\varepsilon) = g \;\text{ on }\; \Gamma_\varepsilon.$$

The new functional is the following

$$J(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} F(x, u_\varepsilon(x), \nabla u_\varepsilon(x))dx + \int_{\Gamma_\varepsilon} G(x, u_\varepsilon(x), \nabla u_\varepsilon(x))d\sigma.$$

We will not give details but only main steps. We have

$$\begin{aligned}
\frac{d}{d\varepsilon}|_{\varepsilon=0} J(\Omega_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{J(\Omega_\varepsilon) - J(\Omega))}{\varepsilon} \\
&= \int_\Omega F_u(x, u(x), \nabla u(x))u'(x)dx + \int_\Omega F_z(x, u(x), \nabla u(x)) \cdot \nabla u' dx \\
&+ \int_\Gamma F(x, u(x), \nabla u(x))V \cdot \nu d\sigma + \int_\Gamma G(x, u(x), \nabla u(x))z' d\sigma \\
&+ \int_\Gamma G_z(x, u(x), \nabla u(x)) \cdot \nabla u' d\sigma + \int_\Gamma [G_z(x, u(x), \nabla u(x)) \cdot (\nabla^2 u)\nu)]V \cdot \nu d\sigma \\
&+ \int_\Gamma H(x)G(x, u(x), \nabla u(x))V \cdot \nu d\sigma,
\end{aligned} ,$$

where $z = u|_\Gamma$, ν the unit normal vector of Γ and H the mean curvature of Γ. Moreover we have

$$z' = u' + \frac{\partial u}{\partial \nu} V \cdot \nu \ \text{ on } \ \Gamma. \tag{4.18}$$

Let us assume that the shape derivative $u' \in W^{s,l}(\Omega)$ is determined as the unique solution of the following linear equation

$$\langle Au', \varphi \rangle_{W^{-s,l'}(\Omega)\times W^{s,l}(\Omega)} = L(\varphi) \quad \forall \varphi \in W^{s,l}(\Omega), \tag{4.19}$$

where $A \in \mathcal{L}(W^{s,l}(\Omega), W^{-s,l'}(\Omega))$ and $L(\cdot) \in W^{-s,l'}(\Omega)$ are given elements. Let us denote by $q \in W^{-s,l'}(\Omega)$ the adjoint state that is the solution of the equation

$$\begin{aligned}
\langle \psi, A^* q \rangle = &\int_\Omega F_u(x, u(x), \nabla u(x))\psi(x)dx + \int_\Omega F_z(x, u(x), \nabla u(x)) \cdot \nabla \psi(x)dx \\
&+ \int_\Gamma G(x, u(x), \nabla u(x))\psi(x)d\sigma + \int_\Gamma G_z(x, u(x), \nabla u(x)) \cdot \nabla \psi(x)d\sigma
\end{aligned} \tag{4.20}$$

$\forall \psi \in W^{s,l}(\Omega)$, where A^* is the adjoint operator of A. Using the fact that

$$\langle u', A^* q \rangle = \langle Au', q \rangle = L(q)$$

we get

$$
\begin{aligned}
\frac{d}{d\varepsilon}|_{\varepsilon=0}J(\Omega_\varepsilon) = L(q) &+ \int_\Gamma F(x,u(x),\nabla u(x))V\cdot\nu d\sigma \\
&+ \int_\Gamma G(x,u(x),\nabla u(x))\frac{\partial u}{\partial\nu}V\cdot\nu d\sigma \\
&+ \int_\Gamma [G_z(x,u(x),\nabla u(x))\cdot(\nabla^2 u)\nu)]V\cdot\nu d\sigma \\
&+ \int_\Gamma H(x)G(x,u(x),\nabla u(x))V\cdot\nu d\sigma.
\end{aligned}
\tag{4.21}
$$

If Ω is the optimal domain then

$$
\frac{d}{d\varepsilon}|_{\varepsilon=0}J(\Omega_\varepsilon) = 0.
$$

This gives the necessary condition of optimality. In the previous computation the existence of u' and the equation satisfied by it are crucial. We give below the general way to find them. Assume formally that we have the following linear or nonlinear boundary value problem (equation on transported domain)

$$
A(\varepsilon,u_\varepsilon) = f \text{ in } \Omega_\varepsilon, \quad B(\varepsilon,u_\varepsilon) = g \text{ on } \Gamma_\varepsilon, \tag{4.22}
$$

where $A(\varepsilon,\cdot)$ and $B(\varepsilon,\cdot)$ are operators on the spaces functions defined on Ω_ε and Γ_ε.

The first step is to show the derivability of the function $\varepsilon\mapsto u_\varepsilon$. This may be done by applying the theorem of implicit functions to the transported operators on Ω and Γ. We get first the regularity of the function $\varepsilon\mapsto U_\varepsilon = u_\varepsilon\circ\Phi_\varepsilon$ and then the derivability of $\varepsilon\mapsto u_\varepsilon$ at least inside Ω. This allows to define $u' = u'(0)$ on the entire Ω and the regularity up to the boundary is deduced from the expression $u' = U' - \nabla u\cdot\Phi'$ (at $\varepsilon = 0$).

The second step is the computation of the derivative u'. To this aim we differentiate the equation (4.22) under appropriate regularity assumption. It follows that u' satisfies the new boundary value problem

$$
\partial_\varepsilon A(0,u) + \partial_u A(0,u)u' = 0 \text{ in } \Omega, \tag{4.23}
$$

$$
\partial_\varepsilon B(0,u) + \partial_u B(0,u)u' = \frac{\partial}{\partial\nu}(g - B(0,u))(V\cdot\nu) \text{ on } \Gamma. \tag{4.24}
$$

The equation (4.23) is easily obtained by differentiating the equation in distributional sense. For the equation (4.24), let us set $Z_\varepsilon = B(\varepsilon,u_\varepsilon) - g$ and differentiate the equation $Z_\varepsilon\circ\Phi_\varepsilon = 0$ on Γ at $\varepsilon = 0$. Then we get

$$
Z' + \nabla Z\cdot V = 0, \quad \nabla^\Gamma Z = 0,
$$

which implies

$$\nabla Z = \frac{\partial Z}{\partial \nu}\nu, \quad Z' = -\frac{\partial Z}{\partial \nu}(V \cdot \nu).$$

Here $Z' = \partial_\varepsilon B(0,u) + \partial_u B(0,u)u'$ and the expression (4.24) follows. For rigorous statement and more details see [80], [81], [89], [92].

Chapter 5

Shape optimization problems governed by nonlinear state equations

This chapter treats the shape optimization problems governed by nonlinear state equations. It is a nonlinear version of Chapter 4. We study first the continuity of the solution of the p-Laplacian problem (and also of the more general monotone operator in divergence form) with respect to the domain variation. secondly we analyze the existence of optimal shapes and give the necessary condition of optimality in the case where an optimal shape exists.

5.1 p-Capacity of a set

In the remark 4.1.3 , we have seen that set of Hausdorff dimension less than $d-2$ has capacity zero. To measure a set of capacity zero, we need a more general capacity that will be called p-capacity (the capacity is a particular case of the p-capacity when $p=2$).

Definition 5.1.1. *(p-capacity of a set)*
For a set K contained $\mathbb{R}^d$,

$$cap_p(K) := \inf\left\{\int_{\mathbb{R}^d} |\nabla\varphi|^p dx,\ \ \varphi \in C_c^\infty(\mathbb{R}^d), \varphi \geq 1 \ \ on \ \ K\right\}.$$

For the general elliptic operator in divergence form $\operatorname{div}(A(\nabla u))$, the capacity is defined in the same way by replacing the integrand $|\nabla\varphi|^p$ by $A(\nabla\varphi)\nabla\varphi$. More details may be found in [54]. The cap_p is also called the p-capacity or the capacity associated to the norm of $W^{1,p}$. In the same way we may define also the p-capacity of a set K relative to a bounded set B containing K.

$$cap_p(K,B) := \inf\left\{\int_B |\nabla\varphi|^p dx,\ \ \varphi \in C_c^\infty(B), \varphi \geq 1 \ \text{ on } \ K\right\}.$$

It is said that a property holds p-quasi everywhere (in short $p-q.e.$) if it holds outside a set of p-capacity zero. It said that a property holds almost everywhere (in short $a.e.$) if it holds outside a set of Lebesgue measure zero.

A function u is said p-quasi-continuous if for any ε there exists an open set A_ε such that $cap_p(A_\varepsilon, B) < \varepsilon$ and u is continuous in $\Omega \setminus A_\varepsilon$. Let us recall that any function $u \in W^{1,p}(\Omega)$ has a unique (up to a set of p-capacity zero) p-quasi continuous representative. Let us also recall the following results from [16] and [66] respectively.

Theorem 5.1.2. *If $u \in W^{1,p}(\mathbb{R}^d)$, then $u|_\Omega \in W^{1,p}(\Omega)$ if and only if $u = 0$ $p-q.e$ on Ω^c for a p-quasi-continuous representative.*

Theorem 5.1.3. *Let Ω be a bounded open subset of $\mathbb{R}^d$, and $u \in W^{1,p}(\Omega)$. If $u = 0$ $a.e.$ in Ω, then $u = 0$ $p-q.e.$ in Ω.*

We summarize here some properties of the p-capacity. For the proof see for example [57].

Proposition 5.1.4. *$A, B \subset \mathbb{R}^d$*

- $A \subset B \Longrightarrow cap_p(A) \leq cap_p(B)$;
- $cap_p(A) = \inf\{cap_p(U) : U \text{ open } A \subset U\}$;
- $cap_p(\lambda A) = \lambda^{d-p} cap_p(A)$ *for* $\lambda > 0$;
- $cap_p(L(A)) = cap_p(A)$ *for any affine isometry* $L : \mathbb{R}^d \to \mathbb{R}^d$;
- $cap_p(A) \leq C\mathcal{H}^{d-p}(A)$ *for some constant* C *depending only on* d *and* p;
- $\mathcal{L}^d(A) \leq C cap_p(A)^{d/d-p}$ *for some constant* C *depending only on* d *and* p;
- $cap_p(A \cup B) + cap_p(A \cap B) \leq cap_p(A) + cap_p(B)$;
- *if* $A_1 \subset \cdots \subset A_k \subset A_{k+1} \cdots$ *then*

$$\lim_{k\to\infty} cap_p(A_k) = cap_p\left(\bigcup_{k=1}^{\infty} A_k\right)$$

- *if* $A_1 \supset \cdots \supset A_k \supset A_{k+1} \cdots$ *are compact, then*

$$\lim_{k\to\infty} cap_p(A_k) = cap_p\left(\bigcap_{k=1}^{\infty} A_k\right).$$

5.2 Continuity with respect to a domain

The aims of this paragraph is to prove some continuity result of the solution of the p-Laplacian equation with respect to a domain. We follow [33]. For Ω open subset of a fixed ball D in $\mathbb{R}^d$, $d \geq 2$ and $h \in W_0^{1,p}(D)$, $f \in W^{-1,q}(D)$ given, we are interested in the study of the continuity of the map: $\Omega \mapsto u_{\Omega,f,h}$ where $u_{\Omega,f,h}$ is the weak solution of the Dirichlet problem

$$\begin{cases} \Delta_p u &= f \quad in \quad \Omega \\ \quad u &= h \quad on \quad \partial\Omega \end{cases} \tag{5.1}$$

In Chapter 4 section 1.3, we have characterized the continuity in terms of the γ-convergence ($p = 2$) of domains. Here we will do it for general p. We prove first the continuity result for the p-Laplacian (γ_p-convergence) and in a second step, we prove that γ_p-convergence implies the weak continuity for the solution of a more general Dirichlet problem associated with a monotone operator. For this reason we introduce this large class of operators and, in the next section, we prove for them some preliminary results necessaries for the continuity , which are obviously true for the p-Laplacian. Assume that $a : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $d \geq 2$, satisfies:

$$\text{for every} \quad \zeta \in \mathbb{R}^d \quad \text{the function} \quad a(\cdot, \zeta) \quad \text{is measurable} \tag{5.2}$$

$$\text{for } \; a.e.\, x \in \mathbb{R}^d \quad \text{the function} \quad a(x, \cdot) \quad \text{is continuous} \tag{5.3}$$

$$a(x, t\zeta) = |t|^{p-2} t a(x, \zeta), \quad t \in \mathbb{R}^d, \;\; t \neq 0. \tag{5.4}$$

The monotonicity assumption on $a(x, \zeta)$ are as usual that is there exist two positive constant c_0, c_1 with $0 < c_0 \leq c_1 < \infty$ such that, for $a.e.x \in \mathbb{R}^d$ and for every $\zeta_1, \zeta_2 \in \mathbb{R}^d$ we have:
in the case $2 \leq p < +\infty$

$$\langle a(x, \zeta_1) - a(x, \zeta_2), \zeta_1 - \zeta_2 \rangle \geq c_0 |\zeta_1 - \zeta_2|^p \tag{5.5}$$

$$|a(x, \zeta_1) - a(x, \zeta_2)| \leq c_1 (|\zeta_1| + |\zeta_2|)^{p-2} |\zeta_1 - \zeta_2| \tag{5.6}$$

in the case $1 < p \leq 2$

$$\langle a(x, \zeta_1) - a(x, \zeta_2), \zeta_1 - \zeta_2 \rangle \geq c_0 (|\zeta_1| + |\zeta_2|)^{p-2} |\zeta_1 - \zeta_2|^2 \tag{5.7}$$

$$|a(x, \zeta_1) - a(x, \zeta_2)| \leq c_1 |\zeta_1 - \zeta_2|^{p-1} \tag{5.8}$$

In particular equation (5.4)-(5.8) imply that for $a.e.$ $x \in \mathbb{R}^d$ and for any $\zeta \in \mathbb{R}^d$:

$$\langle a(x,\zeta),\zeta\rangle \geq c_0|\zeta|^p \tag{5.9}$$

$$|a(x,\zeta)| \leq c_1|\zeta|^{p-1} \tag{5.10}$$

Let us fix a ball D in $\mathbb{R}^d$. By the assumption made on $a(x,\zeta)$ the operator $Au = -\text{div}(a(x,Du))$ turns out to be continuous and strongly monotone from $W_0^{1,p}(D)$ into its dual $W^{-1,q}(D)$ via the pairing:

$$\langle Au, v\rangle = \int_B a(x,\nabla u)\nabla v dx \quad \forall u,v \in W_0^{1,p}(D) \tag{5.11}$$

The particular case where $a(x,\zeta) = |\zeta|^{p-2}\zeta$, A is the p-Laplacian operator

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$$

in this case (5.5)-(5.8) are satisfied with $c_0 = 2^{2-p}$ and $c_1 = p-1$ for $p \geq 2$ and $c_0 = 1, c_1 = 2^{2-p}$ for $p \leq 2$.

The scheme of the proof of the continuity is as follows. we are interested in equation (5.1) and consider a sequence of open sets $\{\Omega_n\}_{n\in\mathbb{N}}$ converging in the Hausdorff topology to some set Ω.

1. The sequence of the solution of equation (5.1) on Ω_n (with a general monotone operator) is bounded in $W_0^{1,p}(D)$ and any weak limit term of the subsequence solve equation (5.1) on Ω.

2. For the p-Laplacian it is sufficient to study continuity for equation (5.12) with $f = 0$.

3. Using the capacity density conditions, we prove that the limit term satisfies the boundary condition on Ω and hence it is solution on Ω.

4. The passage from the p-Laplacian to the general monotone operator in divergence form is obtained via Mosco convergence.

5.2.1 Some estimate for the solution and passage to the limit for the moving domain

Here we will focus on the general operator. Let us consider a mapping $a(x,\zeta)$ satisfying (5.2)-(5.10) and A be the operator defined by (5.11). Fix $f \in W^{-1,q}(D)$ and $h \in W_0^{1,p}(D)$. Then for any open subset Ω of D, we consider the following Dirichlet problem: find u such that

$$\begin{cases} Au &= f \;\; in \;\; \Omega \\ u &= h \;\; \text{on} \;\; \partial\Omega \end{cases} \tag{5.12}$$

in the weak sense that means

$$\int_\Omega a(x,\nabla u)\nabla\varphi dx = \int_\Omega f\varphi dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

From [53] we have the existence and the uniqueness of the equation (5.12) and some estimate for the solution in terms of the data f and h. From these estimates it follows a boundedness result for the solution of (5.12) which is uniform with respect to Ω, and a result on the continuous dependence of the solution of (5.12) on f and h. Let us denote the solution of (5.12) by $u_{\Omega,f,h}$. Then $u_{\Omega,f,h} \in W^{1,p}(\Omega)$. We can extend the function $u_{\Omega,f,h}$ by h outside Ω to an element of $W_0^{1,p}(D)$ since $u_{\Omega,f,h} - h \in W_0^{1,p}$. $\tilde{u}_{\Omega,f,h}$ will stand for this extension. It follows that

$$||\tilde{u}_{\Omega,f,h}||^p_{W_0^{p}(D)} = \int_\Omega |\nabla u_{\Omega,f,h}|^p dx + \int_{D\setminus\Omega} |\nabla h|^p dx.$$

Proposition 5.2.1. *For every $f \in W^{-1,q}(D)$, and $h \in W_0^{1,p}(D)$ the problem (5.12) has a unique solution which satisfies:*

$$\int_\Omega |\nabla u_{\Omega,f,h}|^p \le C(||f||^q_{W^{-1,q}(D)} + \int_D |\nabla h|^p dx)$$

where C is a constant depends only on p, c_0, c_1.

This proposition helps to prove a boundedness result, uniformly with respect to Ω, for the function $\tilde{u}_{\Omega,f,h}$. The proof can be found in [53], theorem 2.1

Corollary 5.2.2. *For any $\Omega \subseteq B$, let $u_{\Omega,f,h}$ be the solution of (5.12). Then $||\tilde{u}_{\Omega,f,h}||_{W_0^{1,p}(D)} \le C$ where C is a constant depending only on f, h, p, c_0, c_1.*

Proof: Setting $u = u_{\Omega,f,h}$, from proposition 5.2.1 we have:

$$\int_\Omega |\nabla u|^p \le C(||f||^q_{W^{-1,q}(D)} + \int_D |\nabla h|^p dx)$$

By adding $\int_{D\setminus\Omega} |\nabla h|^p dx$ to both sides, we get

$$||\tilde{u}_{\Omega,f,h}||^p_{W_0^{1,p}} \le \tilde{C}(||f||^q_{W^{-1,q}(D)} + \int_D |\nabla h|^p dx)$$

which gives a desired conclusion. □

The following lemma will be used to prove the continuous dependence of the solution of (5.12) on f and h. The Lemma is true for any pairs of functions u_1 and $u_2 \in W^{1,p}(\Omega)$. We are going to use this result in the case of two solutions of (5.12) on Ω with respect to different pairs of data (f,h). For proof see [53], lemma 2.2

Lemma 5.2.3. *Let $u_1, u_2 \in W^{1,p}(\Omega)$. If $2 \le p < \infty$, then*

$$c_0 \int_\Omega |\nabla u_1 - \nabla u_2|^p dx \le \int_\Omega \langle a(x, \nabla u_1) - a(x, \nabla u_2), \nabla u_1 - \nabla u_2 \rangle dx. \tag{5.13}$$

If $1 < p \le 2$, then

$$c_0 (\int_\Omega |\nabla u_1 - \nabla u_2|^p dx)^{\frac{2}{p}} \le K(u_1, u_2) \int_\Omega \langle a(x, \nabla u_1) - a(x, \nabla u_2), \nabla u_1 - \nabla u_2 \rangle dx \tag{5.14}$$

where $K(u_1, u_2) = 2(\int_\Omega |\nabla u_1|^p dx + \int_\Omega |\nabla u_2|^p dx)^{\frac{2-p}{p}}$.

Now fix $f_1, f_2 \in W^{-1,q}(D)$ and $h_1, h_2 \in W_0^{1,p}(D)$. By $u_i = u_{\Omega, f_i, h_i}$ for $i = 1, 2$ we denote the solution of (5.12) on the same domain Ω associated to the data $(f_i, h_i), i = 1, 2$. We get:

Lemma 5.2.4. *Fix f_1, f_2, h_1, h_2 as above. Let u_1, u_2 be the corresponding solution of (5.12). If $2 \le p < +\infty$, then:*

$$c_0 \int_\Omega |\nabla u_1 - \nabla u_2|^p dx \le K(||\nabla h_1 - \nabla h_2||_{L^p(D)} + ||f_1 - f_2||_{W^{-1,q}(D)}). \tag{5.15}$$

If $1 < p \le 2$, then:

$$c_0 (\int_\Omega |\nabla u_1 - \nabla u_2|^p dx)^{\frac{2}{p}} \le K K(u_1, u_2)(||\nabla h_1 - \nabla h_2||_{L^p(D)} + ||f_1 - f_2||_{W^{-1,q}(D)}) \tag{5.16}$$

where K is a constant depending only on $||f_1||_{W^{-1,q}(D)}, ||f_2||_{W^{-1,q}(D)}$, $||h_1||_{W_0^{1,p}(D)}, ||h_2||_{W_0^{1,p}(D)}, c_0, c_1, p$ and $K(u_1, u_2)$ is the constant in Lemma 5.2.3.

Proof: Writing (5.12) for u_1 and u_2 and the test function $\varphi = u_1 - h_1$ and $\varphi = u_2 - h_2$, we have

$$\begin{aligned} &\int_\Omega \langle a(x, \nabla u_1) - a(x, \nabla u_2), \nabla u_1 - \nabla u_2 \rangle dx = \\ &\int_\Omega \langle a(x, \nabla u_1) - a(x, \nabla u_2), \nabla h_1 - \nabla h_2 \rangle dx + \\ &\langle f_1 - f_2, u_1 - h_1 \rangle - \langle f_1 - f_2, u_2 - h_2 \rangle = A + B + C \end{aligned} \tag{5.17}$$

We will consider the three term separately. Let us start by the first one. Using (5.10) and the Hölder's inequality we obtain:

$$
\begin{aligned}
A &= \int_\Omega \langle a(x,\nabla u_1) - a(x,\nabla u_2), \nabla h_1 - \nabla h_2\rangle dx \\
&\le \int_\Omega |a(x,\nabla u_1)||\nabla h_1 - \nabla h_2| dx + \int_\Omega |a(x,\nabla u_2)||\nabla h_1 - \nabla h_2| dx \\
&\le c_1 \left(\int_\Omega |\nabla u_1|^{p-1} |\nabla h_1 - \nabla h_2| dx + \int_\Omega |Du_2|^{p-1}|\nabla h_1 - \nabla h_2| dx \right) \\
&\le c_1 \sum_{i=1}^{2} \left(\int_\Omega |\nabla u_i|^p dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla h_1 - \nabla h_2|^p dx \right)^{\frac{1}{p}}
\end{aligned}
$$

Now it follows from Proposition 5.2.1 that for $i = 1, 2$

$$
(\int_\Omega |\nabla u_i|^p dx)^{\frac{1}{q}} \le c(||f_i||^q_{W^{-1,q}(D)} + \int_D |\nabla h_i|^p dx)^{\frac{1}{q}}.
$$

Therefore there exists a constant α_0 such that

$$
A \le \alpha_0 ||\nabla h_1 - \nabla h_2||_{L^p(\Omega)} \le \alpha_0 ||\nabla h_1 - \nabla h_2||_{L^p(D)} \tag{5.18}
$$

where α_0 depends on $f_1, f_2, h_1, h_2, c_0, c_1, p$. Let us consider B. The Cauchy-Schwarz's inequality yields

$$
\begin{aligned}
B &\le ||f_1 - f_2||_{W^{-1,q}(D)} ||u_1 - h_1||_{W_0^{1,p}(B)} \\
&\le ||f_1 - f_2||_{W^{-1,q}(D)} (||\tilde{u}_1||_{W_0^{1,p}(D)} + ||h_1||_{W_0^{1,p}(D)}),
\end{aligned}
$$

and from corollary 5.2.2 we get:

$$
B \le \alpha_1 ||f_1 - f_2||_{W^{-1,q}(D)} \tag{5.19}
$$

where α_1 depends on f_1, h_1, c_0, c_1, p. From the part C again the Cauchy-Schawrz's inequality gives

$$
C \le \alpha_2 ||f_1 - f_2||_{W^{-1,q}(D)} \tag{5.20}
$$

where α_2 depends on f_2, h_2, c_0, c_1, p. Now combining (5.17)-(5.20) we get

$$
\int_\Omega \langle a(x,\nabla u_1) - a(x,\nabla u_2), \nabla u_1 - \nabla u_2\rangle \le K(||\nabla h_1 - \nabla h_2||_{L^p(D)} + ||f_1 - f_2||_{W^{-1,q}(D)}), \tag{5.21}
$$

where $K = \max(\alpha_0, \alpha_1, \alpha_2)$.
If $p \ge 2$, from (5.13) and (5.21) we have:

$$
c_0 \int_\Omega |\nabla u_1 - \nabla u_2|^p dx \le K(||\nabla h_1 - \nabla h_2||_{L^p(D)} + ||f_1 - f_2||_{W^{-1,q}(D)}),
$$

while for $1 < p \le 2$, from (5.14) and (5.21) we get

$$
c_0 (\int_\Omega |\nabla u_1 - \nabla u_2|^p dx)^{\frac{2}{p}} \le K(u_1, u_2) K(||\nabla h_1 - \nabla h_2||_{L^p(D)} + ||f_1 - f_2||_{W^{-1,q}(D)})
$$

as desired. $\square$

Remark 5.2.5 If $h_n \to h$ in $W_0^{1.p}(D)$ and $f_n \to f$ in $W^{-1,q}(D)$, then $\alpha_0, \alpha_1, \alpha_2$ can be chosen independently of n and bounded. So the constant K in Lemma 5.2.4 is uniformly bounded, and using Proposition 5.2.1, the same is true for $K(u_n, u)$.

From Lemma 5.2.4 and this remark we can prove the continuous dependence on f and h of the solution $u_{\Omega,f,h}$ which is uniform with respect to Ω.

Theorem 5.2.6. *If $h_n \to h$ strongly in $W_0^{1,p}(D)$ and $f_n \to f$ in $W^{-1,q}(D)$, then: $\tilde{u}_{\Omega,f_n,h_n} \to \tilde{u}_{\Omega,f,h}$ uniformly with respect to Ω.*

Proof: We set $\tilde{u}_n = \tilde{u}_{\Omega,f_n,h_n}$ and $\tilde{u} = \tilde{u}_{\Omega,f,h}$. We recall that

$$||\tilde{u} - \tilde{u}_n||^p_{W_0^{1,p}(D)} = ||\nabla u - \nabla u_n||^p_{L^p(\Omega)} + ||\nabla h - \nabla h_n||^p_{L^p(D\backslash\Omega)}.$$

If $p \geq 2$, from (5.14) we have

$$||\tilde{u} - \tilde{u}_n||^p_{W_0^{1,p}(D)} \leq \tilde{K}(||Dh - Dh_n||_{L^p(D)} + ||\nabla h - \nabla h_n||^p_{L^p(D\backslash\Omega)} + ||f - f_n||_{W^{-1,q}(D)}),$$

and since $h_n \to h$ strongly in $W_0^{1,p}(D)$ and $f_n \to f$ strongly in $W^{-1,q}(D)$, we have

$$||\tilde{u} - \tilde{u}_n||^p_{W^{1,p}(D)} \leq \varepsilon \quad \text{for n large enough.}$$

If $< p \leq 2$ from (5.15) we get

$$||\tilde{u} - \tilde{u}_n||^2_{W_0^{1,p}(D)} \leq M(||\nabla h - \nabla h_n||_{L^p(D)} + ||\nabla h - \nabla h_n||^2_{L^p(D\backslash\Omega)} + ||f - f_n||_{W^{-1,q}(D)}),$$

where M is a constant independent of n (see Remark 5.2.1). It follows as previously that

$$||\tilde{u} - \tilde{u}_n||^2_{W^{1,p}(D)} \leq \varepsilon \quad \text{for n large enough.}$$

These estimates for the solution, allow us to consider the moving domain case. Classically, if Ω_n converge to Ω in the Hausdorff topology, the continuity is achieved by proving that any weak convergent subsequence of solution of the equation in Ω_n tends to the solution of the equation on Ω.

A given function is a solution of the equation in Ω if it satisfies the equation in distributional sense and satisfies the boundary condition. Let us consider a sequence of domains , which is convergent in Hausdorff topology. We start by the passage to the limit in the equation. $\square$

Proposition 5.2.7. *Let $\{\Omega_n\}_{n\in\mathbb{N}}, \Omega \subset D$ and $\Omega_n \xrightarrow{H} \Omega$. Then $||\tilde{u}_{\Omega_n,f,h}||_{W_0^p(D)}$ is uniformly bounded, and if $\tilde{u}_{\Omega,f,h} \rightharpoonup u$, then*

$$\int_\Omega a(x, \nabla u)\nabla\varphi dx = \int_\Omega f\varphi dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Proof: Set $u_n = u_{\Omega_n,f,h}$. To prove the above equality, it is enough to prove that

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \text{ in } L^p(\Omega).$$

Let us consider a test function $\varphi \in C_c^\infty(\Omega)$, set $K = spt\varphi \Subset \Omega$, and since $\Omega_n \xrightarrow{H} \Omega$, we have that $K \Subset \Omega_n$ for n large enough. Then $\varphi \in C_c^\infty(\Omega_n)$ and since u_n is a solution in Ω_n, we can write

$$\int_{\Omega_n} a(x, \nabla u_n)\nabla\varphi dx = \int_{\Omega_n} f\varphi dx.$$

Consider the set $K^\varepsilon = \{x \in \mathbb{R}^d : d(x, K) < \varepsilon\}$ with ε small enough so that $K^\varepsilon \Subset \Omega$; if n is large enough, we still have $K^\varepsilon \Subset \Omega_n$. Let us prove that:

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \text{ in } L^p(K^{\frac{\varepsilon}{2}}).$$

Fix $g \in C_c^\infty(\Omega)$ such that $g = 1$ on $K^{\frac{\varepsilon}{2}}$, $g = 0$ on $\text{ext}(K^{\frac{\varepsilon}{2}})$, and $0 \leq g \leq 1$ on K^ε. Consider the test function

$$\phi_n = g(u_n - u) \in W_0^{1,p}(K^\varepsilon).$$

If we write the equation for u_n in Ω_n with the test function ϕ_n we have

$$\int_{K^\varepsilon} a(x, \nabla u_n)\nabla(g(u_n - u))dx = \int_{K^\varepsilon} fg(u_n - u)dx.$$

Performing the derivative of test function in the left hand side we get

$$\begin{aligned}
\int_{K^\varepsilon} a(x, \nabla u_n)g\nabla(u_n - u)dx &= \int_{K^\varepsilon} fg(u_n - u)dx - \int_{K^\varepsilon} a(x, \nabla u_n)(u_n - u)\nabla g dx \\
&\leq \int_{K^\varepsilon} fg(u_n - u)dx \\
&\quad + (\int_D |a(x, \nabla\tilde{u}_n)|^q dx)^{\frac{1}{q}} (\int_D |\nabla g(\tilde{u}_n - u)|^p dx)^{\frac{1}{p}}.
\end{aligned}$$

Remark that up to a subsequence the first term of right hand side tends to zero for $n \to \infty$ since $gu_n \rightharpoonup gu$ in $W^{1,p}(K^\varepsilon)$. The second term is a product of a uniformly bounded sequence (by the assumption on $a(x, \zeta)$) and of a vanishing term since $\tilde{u}_n \to u$ in $L^p(D)$. Finally we conclude with the following inequality:

$$\limsup_{n\to\infty} \int_{K^\varepsilon} a(x, \nabla u_n)g\nabla(u_n - u)dx \leq 0.$$

On the other hand, we get

$$\int_{K^\varepsilon} a(x, \nabla u)g\nabla(u_n - u)dx \to 0, \quad \text{as} \quad n \to \infty$$

since $\nabla u_n \rightharpoonup \nabla u$ in $L^p(K^\varepsilon)$; therefore, by subtracting, we have

$$\limsup_{n\to\infty} \int_{K^\varepsilon} g[a(x, \nabla u_n) - a(x, \nabla u)]\nabla(u_n - u)dx \leq 0.$$

By the monotonicity assumption on $a(x,\zeta)$ and the positivity of g we have

$$g[a(x,\nabla u_n) - a(x,\nabla u)]\nabla(u_n - u) \geq 0 \quad \forall x \in K^\varepsilon,$$

and hence

$$\lim_{n\to\infty}\int_{K^\varepsilon} g[a(x,\nabla u_n) - a(x,\nabla u)]\nabla(u_n - u)dx = 0.$$

By the positivity of the integrand and by the equality $g = 1$ on K^ε we have

$$\lim_{n\to\infty}\int_{K^{\frac{\varepsilon}{2}}} [a(x,\nabla u_n) - a(x,\nabla u)]\nabla(u_n - u)dx = 0.$$

From [66], Lemma 3.73, we get that $a(x,\nabla u_n) \rightharpoonup a(x,\nabla u)$ in $L^p(K^{\frac{\varepsilon}{2}})$, so that we can write

$$\int_{K^{\frac{\varepsilon}{2}}} a(x,\nabla u_n)\nabla\varphi dx = \int_{K^{\frac{\varepsilon}{2}}} f\varphi dx,$$

where φ is the chosen test function. If we compute the limit for $n \to \infty$ we have

$$\int_{K^{\frac{\varepsilon}{2}}} a(x,\nabla u)\nabla\varphi dx = \int_{K^{\frac{\varepsilon}{2}}} f\varphi dx,$$

since $spt\varphi = K \Subset K^{\frac{\varepsilon}{2}}$ the previous equality holds in Ω . As φ was arbitrarily chosen, u satisfies the equation on Ω in the sense of distributions. $\square$

5.2.2 Behavior on the boundary of the limit term for the p-Laplacian

This section is devoted to the p-Laplacian $1 < p < +\infty$. We want to prove the γ_p-convergence result, so in order to obtain the continuity of the map $\Omega \mapsto u_{\Omega,f,0}$, from Proposition 5.2.7, we have just to prove that $u|_\Omega \in W_0^{1,p}(\Omega)$, and hence u will be the unique solution of the equation on Ω.

Lemma 5.2.8. *Let $\Omega_n, \Omega \subset D$ be open subsets of D. If*

$$\tilde{u}_{\Omega_n,0,h} \to \tilde{u}_{\Omega,0,h} \quad \forall h \in W_0^{1,p}(D),$$

then

$$\tilde{u}_{\Omega_n,f,0} \to \tilde{u}_{\Omega,f,0} \quad \forall f \in W^{-1,q}(D).$$

Proof: Set $v_n = u_{\Omega_n,0,h}, v = u_{\Omega,0,h}$ and $u_n = u_{\Omega_n,f,0}, u = u_{\Omega,f,0}$. We have $u \in W_0^{1,p}(\Omega) \subset W_0^{1,p}(D)$. Since $\tilde{u}_n$ is bounded in $W_0^{1,p}(D)$, up to a subsequence

$$\tilde{u}_n \rightharpoonup w \text{ in } W_0^{1,p}(D).$$

We prove that $u = w$. We consider the new problem:

$$\begin{cases} \Delta_p \varphi_n &= 0 \quad in \quad \Omega_n \\ \varphi_n &= -w \quad on \quad D \setminus \Omega_n. \end{cases} \tag{5.22}$$

Taking into (5.22) as test function $\tilde{\varphi}_n + w - \tilde{u}_n \in W_0^{1,p}(\Omega_n)$ we have

$$\int_D |\nabla \tilde{\varphi}_n|^{p-2} \nabla \tilde{\varphi}_n \nabla (\tilde{\varphi}_n + w - \tilde{u}_n) dx = 0.$$

Now denoting by φ the solution of (5.22) on Ω, we have $\nabla(\tilde{\varphi}_n + w - \tilde{u}_n) \rightharpoonup \nabla \varphi$ in $L^p(D)$ and $|\nabla \tilde{\varphi}_n|^{p-2} \nabla \tilde{\varphi}_n \to |\nabla \tilde{\varphi}|^{p-2} \nabla \tilde{\varphi}$ in $L^q(D)$ (this comes from the convergence of the norm in $L^p(D)$ and a.e. pointwise convergence which derives from $\nabla \tilde{\varphi}_n \to \nabla \tilde{\varphi}$ in $L^p(D)$), so passing to the limit we get

$$\int_D |\nabla \tilde{\varphi}|^{p-2} \nabla \tilde{\varphi} \nabla \tilde{\varphi} dx = 0$$

which means $\tilde{\varphi} = 0$. But we know that $\tilde{\varphi} = -w$ in $D \setminus \Omega$, so $w = 0$ in $D \setminus \Omega$. We have just proved that the weak limit w satisfy the boundary condition, and from Proposition 5.2.7, the equation on the limit set Ω, hence $w = u$ from the uniqueness of the solution. We have just proved that $\tilde{u}_n \rightharpoonup \tilde{u}$. To conclude, it is sufficient to prove the convergence of the norm $||\tilde{u}_n||_{W_0^{1,p}(D)} \to ||\tilde{u}||_{W_0^{1,p}(D)}$ which implies the strong convergence. Using the weak formulation of the equation on Ω_n and Ω we obtain

$$\begin{aligned} \lim_{n \to +\infty} ||\tilde{u}_n||^p_{W_0^{1,p}(D)} &= \lim_{n \to +\infty} \langle -\Delta_p \tilde{u}_n, \tilde{u}_n \rangle = \lim_{n \to +\infty} \langle f, \tilde{u}_n \rangle \\ &= \langle f, \tilde{u} \rangle = \langle -\Delta_p \tilde{u}, \tilde{u} \rangle = ||\tilde{u}||^p_{W_0^{1,p}(D)}. \end{aligned} \tag{5.23}$$

which concludes the proof □

The previous Lemma suggests us the study of the oscillation of the solution of the problem

$$\begin{cases} \Delta_p u &= 0 \quad \text{in} \quad \Omega \\ u &= h \quad on \quad D \setminus \Omega \end{cases} \tag{5.24}$$

near the boundary, for smooth function h.

Let us first introduce this class of domains which is the generalization of the one introduced in Section 4.1.3 (regular Wiener sets)

$$\begin{aligned} \mathcal{W}_w(D) = \{ &\Omega \subset D : \forall x \in \partial\Omega, \ \forall 0 < r < R < 1; \\ &\int_r^R \left(\frac{cap_p(\Omega^c \cap \overline{B}_t(x), B_{2t}(x))}{cap_p(\overline{B}_t(x), B_{2t}(x))} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq w(r, R, x) \Big\} \end{aligned}$$

where $B_t(x)$ is the ball with center x and radius t, and

$$w : (0,1) \times (0,1) \times D \to [0, +\infty)$$

is such that

1. $\lim_{r\to 0} w(r,R,x) = +\infty$, locally uniformly on x;
2. w is lower semicontinuous in the third variable.

We recall the following theorem from [66].

Theorem 5.2.9. *Suppose that Ω is bounded. Let $h \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$ and let u be the solution of the equation (5.24). If $y \in \partial\Omega$, then for all $0 < r \le R$ it holds*

$$\begin{aligned} osc(u, \Omega \cap B_r(y)) \le osc(h, \partial\Omega \cap \overline{B}_{2R}(y)) + \\ + osc(h, \partial\Omega) \exp\left(-c \int_r^R \left(\frac{cap_p(\Omega^c \cap \overline{B}_t(y); B_{2t}(y))}{cap_p(\overline{B}_t(y); B_{2t}(y))} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right), \end{aligned}$$

where c is a fixed constant and

$$osc(u, A) = \sup_{x\in A} u(x) - \inf_{x\in A} u(x).$$

If $h \in C_c^\infty(D)$ and $\Omega \in \mathcal{W}_w(D)$ there exists some constant M such that $\forall x, z \in \Omega\cap B_r(y)$

$$|u(x) - u(z)| \le MR + M\exp(-cw(r,R,y)).$$

Since y is a regular point (by hypothesis $\Omega \in \mathcal{W}_w(D)$ and $\lim_{r\to 0} w(r,R,y) = +\infty$) we have

$$\lim_{z\to y, z\in\Omega\cap B_r(y)} u(z) = h(y).$$

Then we can write

$$|u(x) - h(y)| \le MR + M\exp(-cw(r,R,y)),$$

hence

$$|u(x) - u(y)| \le |h(x) - h(y)| + MR + M\exp(-cw(r,R,y)),$$

or

$$|u(x) - u(y)| \le 2MR + M\exp(-cw(r,R,y)) \tag{5.25}$$

We have all the necessary ingredients for proving the following principal result of this section.

Theorem 5.2.10. *Let $\Omega_n \in \mathcal{W}_w(D)$, and assume Ω_n converges in the Hausdorff topology to Ω. Then $\Omega \in \mathcal{W}_w(D)$ and Ω_n γ_p-converges to Ω.*

Proof: The family $\mathcal{W}_w(D)$ is compact in the Hausdorff topology. The nonlinear case may be proved as the linear one therefore $\Omega \in \mathcal{W}_w(D)$. From Lemma 5.2.8 we have only to prove that $\tilde{u}_{\Omega_n,0,h} \to \tilde{u}_{\Omega,0,h}$ for every $h \in W_0^{1,p}(D)$.

Step 1: Let $h \in C_c^\infty(D)$; we start by proving that $\tilde{u}_{\Omega_n,0,h} \to \tilde{u}_{\Omega,0,h}$. For simplicity set $u_n = u_{\Omega_n,0,h}$, then up to extraction of a subsequence $\tilde{u}_n \to u$ in $W_0^{1,p}(D)$. from Proposition 5.2.7 we know that u satisfies the equation on Ω, so it is sufficient to prove that $u = h$ p-q.e. on $D \setminus \Omega$. From the Banach-Saks theorem, we can construct a sequence of convex function combinations

$$\psi_n = \sum_{k=n}^{N_n} \alpha_k^n \tilde{u}_k \text{ with } \sum_{k=n}^{N_n} \alpha_k^n = 1 \tag{5.26}$$

such that

$$\psi_n(x) \to u(x) \quad p - q.e.onD. \tag{5.27}$$

So let us consider a point $x \in D \setminus \Omega$ were (5.27) holds. For any $\varepsilon > 0$ we will prove that $|u(x) - h(x)| \leq \varepsilon$ which implies that $u(x) = h(x)$ p-q.e. on $D \setminus \Omega$. From (5.26) we get

$$|u(x) - h(x)| \leq \sum_{k=n}^{N_n} \alpha_k^n |\tilde{u}_k(x) - h(x)|,$$

so it is sufficient to prove that there exists k_ε such that, $\forall k \geq k_\varepsilon$ we have

$$|\tilde{u}_k(x) - h(x)| \leq \varepsilon. \tag{5.28}$$

If $x \in D \setminus \Omega_k$ then (5.28) is true. In order to apply inequality (5.25) for the open set Ω_k we fix R_0 small enough such that

$$2MR_0 \leq \frac{\varepsilon}{4}.$$

using the fact $\lim_{r \to 0} w(r, R, x) = \infty$, locally uniformly on x we find a neighborhood U of x and r_0 small enough such that $B_{r_0}(x) \subset U$ and

$$M \exp(-cw(r, R, y)) \leq \frac{\varepsilon}{4}$$

for all $r \leq r_0$, and for all $y \in U$. Using the Hausdorff topology properties, there exists some k_0 large enough, such that for all $k \geq k_0$ we have $B_{r_0}(y) \cap \Omega_k^c \neq \emptyset$. If $x \in \Omega_k$, there exists some sequence $z_k \in B_{r_0}(x) \cap \partial\Omega_k$ and one can apply inequality (5.25) for u_k, r_0, R_0, z_k and get

$$|u_k(x) - h(x)| \leq \varepsilon \quad \forall k \geq k_0$$

which implies (5.28).

Step 2: we have $u_n \in W_0^{1,p}(\Omega_n)$ and

$$\int_{\Omega_n} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega_n).$$

We choose $\varphi = u_n - h$ as a test function then it follows

$$\int_{\Omega_n} |\nabla u_n|^{p-2} \nabla u_n \nabla u_n dx = \int_{\Omega_n} |\nabla u_n|^{p-2} \nabla u_n \nabla h dx. \tag{5.29}$$

From step 1 we have $\tilde{u}_n \rightharpoonup u$ in $W_0^{1,p}(D)$ and from [24] theorem 2.1, we have (passing to a subsequence if necessary)

$$\nabla \tilde{u}_n \to \nabla \tilde{u}, \;\; a.e. \text{ in } \; D.$$

For getting the strong convergence of the solution we prove convergence of norms. Using (5.29) we have

$$\begin{aligned}
||\tilde{u}_n||^p_{W_0^{1,p}(D)} &:= \int_{\Omega_n} |\nabla u_n|^p dx + \int_{D\setminus\Omega_n} |\nabla h|^p dx \\
&= \int_{\Omega_n} |\nabla u_n|^{p-2} \nabla u_n \nabla u_n dx + \int_{D\setminus\Omega_n} |\nabla h|^p dx \\
&= \int_{\Omega_n} |\nabla u_n|^{p-2} \nabla u_n \nabla h dx + \int_{D\setminus\Omega_n} |\nabla h|^p dx \\
&= \int_{D} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla h dx + \int_{D\setminus\Omega_n} |\nabla h|^p dx - \int_{D\setminus\Omega_n} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla h dx \\
&= \int_{D} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla h dx.
\end{aligned}$$

Therefore passing to the limit and using (5.29)

$$\begin{aligned}
\lim_{n\to+\infty} ||\tilde{u}_n||^p_{W_0^{1,p}(D)} &= \lim_{n\to+\infty} \int_{D} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla h dx \\
&= \int_{D} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla h dx \\
&= \int_{D\setminus\Omega} |\nabla h|^{p-2} \nabla h \nabla h dx + \int_{\Omega} |\nabla h|^{p-2} \nabla h \nabla h dx \\
&= \int_{D\setminus\Omega} |\nabla h|^p dx + \int_{\Omega} |\nabla u|^p dx = ||\tilde{u}||^p_{W_0^{1,p}(D)}
\end{aligned}$$

where we have used the weak L^p convergence of $\nabla \tilde{u}_n$ to $\nabla \tilde{u}$ and the almost everywhere pointwise convergence via the Mazur's lemma to obtain the weak L^q convergence of $|\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n$ to $|\nabla \tilde{u}|^{p-2} \nabla \tilde{u}$.

Step 3: Fix $h \in W_0^{1,p}(D)$. By density we have the existence of a sequence $(h_k)_k \subset C_c^\infty(D)$ such that $h_k \to h$ strongly in $W_0^{1,p}(D)$. Then $||h_k||_{W_0^{1,p}(D)}$ is uniformly bounded and we get

$$\begin{aligned}
||\tilde{u}_{\Omega_n,0,h} - \tilde{u}_{\Omega,0,h}||_{W_0^{1,p}(D)} &\leq ||\tilde{u}_{\Omega_n,0,h} - \tilde{u}_{\Omega,0,h_k}||_{W_0^{1,p}(D)} \\
&\leq ||\tilde{u}_{\Omega_n,0,h_k} - \tilde{u}_{\Omega,0,h_k}||_{W_0^{1,p}(D)} \\
&\leq ||\tilde{u}_{\Omega,0,h_k} - \tilde{u}_{\Omega,0,h}||_{W_0^{1,p}(D)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},
\end{aligned}$$

where we have used the continuity result of the Step 2 in the smooth boundary data for the second term and from the continuity dependence of the solution $\tilde{u}_{\Omega_n,0,h}$ with respect to the variation of the boundary data which is uniform in Ω_n. $\square$

A particular class contained in the family $\mathcal{W}_w(D)$ is the following (for the linear case section) For $c, r > 0$, we said that Ω satisfies the $p-(r,c)$ capacity condition if:

$$\forall x \in \partial\Omega, \forall 0 < \delta < r \qquad \frac{cap_p(\Omega^c \cap \overline{B}_\delta(x), B_{2\delta}(x))}{cap_p(\overline{B}_\delta(x), B_{2\delta}(x))} \geq c.$$

$\mathcal{O}_{p,c,r}(D)$ will denote the set of all open subsets of D which satisfy the $p-(r,c)$ capacity density condition. This family is compact in the Hausdorff topology.

5.2.3 Generalized Šverák result for $d-1 < p \leq d$

In this section we extend the result of Šverák Theorem 4.1.21 to the arbitrary dimension d and for $p \in (d-1, d]$. The reason of this choice is that in $\mathbb{R}^d$ every piece of curve has positive p-capacity for $p > d-1$. The case $p > d$ is trivial since all the functions in $W^{1,p}(\mathbb{R}^d)$ are continuous. To prove the Šverák type result we give some preliminary lemmas.

Lemma 5.2.11. *Let $\gamma_{[x,\xi]} \subset B_r(x)$ be a curve joining x to ξ such that $\xi \in \partial B_r(x)$. Then*

$$cap_p(\gamma_{[x,\xi]}; B_{2r}(x)) \geq cap_p([x,\xi]; B_{2r}(x)),$$

where $[x,\xi]$ denotes the segment with extrema x and ξ.

Proof: For some $\varepsilon > 0$ we consider a nonnegative function $\varphi \in C_c^\infty(B_{2R}(x))$ such that

$$\int_{B_{2r}(x)} |\nabla\varphi|^p dx \leq cap_p(\gamma_{[x,\xi]}; B_{2r}(x)) + \varepsilon$$

and $\varphi \geq 1$ on a neighborhood of $\gamma_{[x,\xi]}$. Then we denote by φ^* the Steiner symmetrization of φ with respect to the line $x\xi$. Then $\varphi^* \in W_0^{1,p}(B_{2r}(x))$, $\varphi^* \geq 1$ on $U^* \supset [x,\xi]$ and

$$\int_{B_{2r}(x)} |\nabla\varphi^*|^p dx \leq \int_{B_{2r}(x)} |\nabla\varphi|^p dx.$$

Since $cap_p([x,\xi]; B_{2r}(x)) \leq \int_{B_{2r}(x)} |\nabla\varphi^*|^p dx$, letting $\varepsilon \to 0$ concludes the proof. $\square$

Lemma 5.2.12. *Let $K \subset \mathbb{R}^d$ be a compact connected set. Then for all $x \in K$ and $r < \frac{1}{2} diam K$ we have*

$$\frac{cap_p(K \cap \overline{B}_r(x), B_{2r}(x))}{cap_p(\overline{B}_r(x), B_{2r}(x))} \geq \frac{cap_p([0,1] \times \{0\}^{d-1}, B_2(0))}{cap_p(\overline{B}_1(0), B_2(0))}$$

Proof: Let us take $x \in K$ and consider the set $K^\delta = \{x \in \mathbb{R}^d : d(x,K) < \delta\}$. Then $\overline{K^\delta} \subset K^{\delta+\varepsilon}$ for all $\varepsilon > 0$ and the property of the capacity on decreasing sequences of compact sets together with its monotonicity gives

$$cap_p(K \cap \overline{B}_r(x), B_{2r}(x)) = \lim_{\delta \to 0} cap_p(K^\delta \cap \overline{B}_r(x), B_{2r}(x)). \tag{5.30}$$

The set K^δ is open, contains K and so it is connected by curves. Since K^δ is not contained in $\overline{B}_r(x)$ (since $r < \frac{1}{2} diam K$), there exists $\xi \in \partial B_r(x) \cap K^\delta$ and a continuous curve $\gamma_{[x,\xi]}$ which connect x to ξ and lies in $\overline{B}_r(x) \cap K^\delta$. To conclude the proof it is sufficient to use Lemma 5.2.11 and the behavior of the capacity on homothetic sets. In fact we have

$$\begin{aligned} cap_p(K^\delta \cap \overline{B}_r(x), B_{2r}(x)) &\geq cap_p(\gamma_{[x,\xi]} \cap \overline{B}_r(x), B_{2r}(x)) \\ &\geq cap_p([x,\xi] \cap \overline{B}_r(x), B_{2r}(x)) \end{aligned}$$

The using (5.30) and letting $\delta \to 0$, we obtain

$$cap_p(K \cap \overline{B}_r(x), B_{2r}(x)) \geq cap_p([x,\xi] \cap \overline{B}_r(x), B_{2r}(x)).$$

Since

$$\frac{cap_p([x,\xi] \cap \overline{B}_r(x), B_{2r}(x))}{cap_p(\overline{B}_r(x), B_{2r}(x))} = \frac{cap_p([0,1] \times \{0\}^{d-1}, B_2(0))}{cap_p(\overline{B}_1(0), B_2(0))}$$

we conclude the proof. □

Now, we give a generalized Šverák result

Theorem 5.2.13. *Let $d-1 < p \leq d$. Consider the sequence $(\Omega_n)_n \subset \mathcal{O}_l(D)$ which converges to the Hausdorff topology to Ω. Then $\Omega \in \mathcal{O}_l(D)$ and Ω_n γ_p-converges to Ω.*

Proof: The set $\mathcal{O}_l(D)$ is compact in the Hausdorff topology, so $\Omega \in \mathcal{O}_l(D)$. we decompose $\overline{D} \setminus \Omega_n$ into its connected components

$$\overline{D} \setminus \Omega_n = K_1^n \cup \cdot \cup K_l^n.$$

Some of these components may be empty. Using the compactness of the Hausdorff topology, we have (up to extracting subsequences)

$$K_i^n \to K_i \quad \forall i = 1, \cdots, k.$$

Then K_i are still compact, connected and

$$\Omega = D \setminus K_1(\cup \cdots \cup K_l).$$

There are three types of connected components for the limit set: $K_i = \emptyset, K_i$ is a point, K_i contains at least two points. Consider the following family of indices which corresponds to the first two cases: $I = \{i : diam K_i = 0\}$ and consider the new sets

$$\Omega_n^+ = D \setminus \bigcup_{i \in \{1,\cdots,l\} \setminus I} K_i^n.$$

Then $\Omega_n^+ \supset \Omega_n$ and $\Omega_n^+ \to \Omega^+$ in Hausdorff topology where $\Omega^+ = D \setminus \bigcup_{i\in\{1,\cdots,l\}\setminus I} K_i$. For n large enough there exists some $r > 0$ such that diam $K_i^n > r, \forall i \in \{1, \cdots, l\} \setminus I$ and using Lemma 5.2.12 and Theorem 5.2.10 we have

$$u_{\Omega_n^+,f,0} \to u_{\Omega^+,f,0}.$$

Let us fix $f = 1$. We have

$$u_{\Omega_n^+,1,0} \to u_{\Omega^+,1,0} \tag{5.31}$$

and from the boundedness of $u_{\Omega_n,1,0}$ (possibly passing to a subsequence)

$$u_{\Omega_n,1,0} \rightharpoonup u \tag{5.32}$$

from which

$$u_{\Omega_n^+,1,0} - u_{\Omega_n,1,0} \rightharpoonup u_{\Omega^+,1,0} - u. \tag{5.33}$$

From the maximum principle we have

$$u_{\Omega_n^+,1,0} \geq u_{\Omega_n,1,0} \geq 0. \tag{5.34}$$

From (5.31) and (5.34) we get $u \geq 0$, and from (5.33) and (5.34) we get $u_{\Omega^+,1,0} \geq u$. So

$$0 \leq u \leq u_{\Omega^+,1,0} \tag{5.35}$$

which implies that $u \in W_0^{1,p}(\Omega^+)$. Since $cap_p(\Omega^+ \setminus \Omega) = 0$ then $u \in W_0^{1,p}(\Omega)$, i.e. $u = u_{\Omega,1,0}$ thanks to Proposition 5.2.7. We have proved that $u_{\Omega_n,1,0} \rightharpoonup u_{\Omega,1,0}$ and from theorem 6.3 of [53] we have

$$u_{\Omega_n,f,0} \rightharpoonup u_{\Omega,f,0}, \quad \forall W^{-1,q}(D). \tag{5.36}$$

Now, using the equation on Ω_n and Ω, we have:

$$\begin{aligned} \lim_{n\to+\infty} ||u_{\Omega_n,f,0}||^p_{W_0^{1,p}(\Omega_n)} &= \lim_{n\to+\infty} \langle -\Delta_p u_n, u_n\rangle = \lim_{n\to+\infty} \langle f, u_n\rangle \\ &= \langle f, u\rangle = \langle -\Delta_p u, u\rangle = ||u_{\Omega,f,0}||^p_{W_0^{1,p}(\Omega)}. \end{aligned} \tag{5.37}$$

(5.36) and (5.37) give

$$u_{\Omega_n,f,0} \to u_{\Omega,f,0} \text{ strongly in } W_0^{1,p}(D) \quad \forall f \in W^{-1,q}(D),$$

i.e. the γ_p-convergence. □

Remark 5.2.14 The γ_p-convergence is equivalent to the convergence in the sense of Mosco of the associated Sobolev spaces. A sequence of Sobolev spaces $W_0^{1,p}(\Omega_n)$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$ if the following two conditions hold:

1. $\forall u \in W_0^{1,p}(\Omega)$ $\exists u_n \in W_0^{1,p}(\Omega_n)$ such that $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^d)$;

2. $\forall u_{n_k} \in W_0^{1,p}(\Omega_{n_k})$ $u_{n_k \to u}$ strongly in $W^{1,p}(\mathbb{R}^d)$ we have $u \in W_0^{1,p}(\Omega)$.

If $\Omega_n \to \Omega$ in the Hausdorff topology the condition 1 follows from the fact that any compact contained in Ω is also contained in Ω_n for n large enough. Condition 2 generally fails, but it is sufficient to prove only for a particular sequence of functions $u_{n_k} \in W_0^{1,p}(\Omega_{n_k})$, namely the solution of the equation (5.1) with $f = 1$ and $h = 0$ (see [53]). As an immediate consequence, the γ_p-convergence implies the stability of the solution of a more general equation (5.2)-(5.10). In this case, we have only the weak continuity of the solution in $W_0^{1,p}(D)$. Let $Au = -\text{div}(a(x, \nabla u))$, where $a(x, \zeta)$ satisfies the assumption (5.2)-(5.10).

Theorem 5.2.15. *Let $\Omega_n \in \mathcal{W}_w(D)$ be a sequence which converges to Ω in Hausdorff topology. Then*

$$\tilde{u}_{\Omega_n,f,h} \rightharpoonup \tilde{u}_{\Omega,f,h}.$$

Proof: Let $\tilde{u}_{\Omega_n,f,h}$ be the solution of the general equation (5.12). From Proposition 5.2.7 we have $\tilde{u}_{\Omega_n,f,h} \rightharpoonup u$, where $u|_\Omega$ satisfies the equation on Ω. The capacitary constraint implies the Mosco convergence of $W_0^{1,p}(\Omega_n)$ to $W_0^{1,p}(\Omega)$. Using the definition of the limit in the sense of Mosco we have $(u - h)|_\Omega \in W_0^{1,p}(\Omega)$, and from the uniqueness of the solution of the equation (5.12) we get $u = u_{\Omega,f,h}$. $\square$

5.3 Existence of Optimal Shapes

Proposition 5.3.1. *The following classes of domains are γ_p-compact:*
$\mathcal{O}_{convex}, \mathcal{O}_{p-unif\ cone}, \mathcal{O}_{p-unif\ flat\ cone}\mathcal{O}_{p-cap\ density}, \mathcal{O}_{p-unif\ wiener}, \mathcal{O}_l(d-1 < p \leq d)$.

Proof: We know that all those classes are compact in the Hausdorff topology. The inclusions (4.6) remains valid then the γ_p compactness of those classes follows from Theorem 5.2.10 and Theorem 5.2.13 $\square$

The direct method of the calculus of variation and the Proposition 5.3.1 give the following result.

Theorem 5.3.2. *Let $F : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function. Then the shape optimization problem*

$$\min \left\{ \int_\Omega F(x, u_\Omega^f, \nabla u_\Omega^f) dx : \quad \Omega \in \mathcal{O}_{ad} \right\}$$

has at least one solution for

$$\mathcal{O}_{ad} = \mathcal{O}_{convex}, \mathcal{O}_{p-unif\ cone}, \mathcal{O}_{p-unif\ flat\ cone}, \mathcal{O}_{p-cap\ density}, \mathcal{O}_{p-unif\ wiener}, \mathcal{O}_l(d-1 < p \leq d)$$

respectively. Here u_Ω^f stands for the solution of the Dirichlet's problem associated to f and Ω.

As in section 4.2 of the previous chapter, we may construct the relaxed form of the Dirichlet problem for some nonlinear elliptic equations of monotone type. Let D be a bounded open subset of $\mathbb{R}^d$, and let $A : W_0^{1,p}(D) \to W^{-1,q}(D)$ be a monotone operator of the form

$$Au = -\mathrm{div}(a(x, \nabla u)),$$

where a satisfies the conditions (5.2)-(5.10) and $1 < p \leq +\infty$ and q is the conjugate exponent of p. We denote by $\mathcal{M}_0^p(D)$ the set of all nonnegative Borel measure μ on D ($+\infty$ valued is allowed), such that

- $\mu(B) = 0$ for every Borel set $B \subset D$ with $cap_p(B) = 0$;
- $\mu(B) = \inf\{\mu(U),\ U\text{quasi-open } B \subset U\}$ for every Borel set $B \subset D$.

If $\mu \in \mathcal{M}_0^p(D)$, then space $W_0^{1,p}(D) \cap L_\mu^p(D)$ is well defined, since all functions in $W_0^{1,p}(D)$ are defined μ-almost everywhere in D. It is easy to see that $W_0^{1,p}(D) \cap L_\mu^p(D)$ is Banach space with the norm $||u||^p_{W_0^{1,p}(D)\cap L_\mu^p(D)} = ||u||^p_{W_0^{1,p}(D)} + ||u||^p_{L_\mu^p(D)}$. Here we will give just the relaxed form. All the details and proofs can be found in [53]. Given $f \in W^{-1,q}(D)$ and a sequence Ω_n of open subsets of D, we denote by u_n the solution of the following equation

$$\left\{\begin{array}{rcl} Au_n & = & f \ \ in \ \ \Omega_n \\ u_n & = & 0 \ \ on \ \ \partial\Omega_n. \end{array}\right. \tag{5.38}$$

We extend u_n to all D by setting the value zero outside Ω. This equation has to be understood in the weak sense that is

$$\int_D a(x, \nabla u_n)\nabla v dx = \langle f, v\rangle, \quad \forall v \in W_0^{1,p}(\Omega_n).$$

Following [53], there exists a subsequence (not relabeled) of $(\Omega_n)_n$ such that for every $f \in W^{-1,q}(D)$, the sequence $(u_n)_n$ weakly converges to the solution of the equation

$$\left\{\begin{array}{l} Au + \mu|u|^{p-2} = f \\ u \in W_0^{1,p}(D) \cap L_\mu^p(D), \end{array}\right. \tag{5.39}$$

where μ is the Radon measure defined by

$$\mu(B) = \left\{\begin{array}{c} +\infty \ \text{ if } \ cap_p(B \cap \{w = 0\}) > 0 \\ \displaystyle\int_B \frac{1}{w^{p-1}} d\nu \ \text{ if } \ cap_p(B \cap \{w = 0\}) = 0. \end{array}\right. \tag{5.40}$$

Here $\nu = 1 - Aw \geq 0$ in $\mathcal{D}'(D)$ is a nonnegative Radon measure belonging to $W^{-1,q}(D)$ and w the weak limit in $W_0^{1,p}(D)$ of the equation (5.38) with $f = 1$. In the case where the operator A is the p-Laplacian operator, the equation (5.39) takes the form

$$\begin{cases} -\Delta_p u + \mu|u|^{p-2} = f \\ u \in W_0^{1,p}(D) \cap L^p_\mu(D), \end{cases} \tag{5.41}$$

and the expression of the Radon measure μ remains unchanged except that $\nu = \Delta_p w + 1$.

5.4 Necessary Condition of Optimality

This section is dedicated to the necessary conditions of optimality. We try to give the stationary configuration when the optimal shape exists. In this case we consider the following classes of control variables Σ: the class of closed connected subsets of $\mathbb{R}^2$, the class of finite numbers of points of $\mathbb{R}^d$ for $d > 1$ and the class of closed connected subsets in $\mathbb{R}^d$ for $d > 2$. In the two last cases, some extra difficulty arrive because of the codimension of Σ which is greater than 1. For simplicity, we assume Ω has Lipschitz boundary and $u = 0$ on $\partial\Omega \cup \Sigma$. We assume as much as needed the regularity on the data. Before looking for the necessary conditions of optimality, we recall some definitions and results which will be helpful; we refer to [28] for more details

For a measure μ we denote for μ a.e. x by $P_\mu(x, \cdot) : \mathbb{R}^d \to \mathrm{Tan}(\mu, x)$ the orthogonal projection of $\mathbb{R}^d$ on $\mathrm{Tan}(\mu, x)$.

Definition 5.4.1. *The curvature of μ is defined as the vector valued distribution*

$$H_\mu := \mathrm{div}(P_\mu \mu).$$

In other words H_μ is defined by

$$\langle H_\mu, X \rangle = -\int_{\mathbb{R}^d} \mathrm{div}^\mu X d\mu \quad \forall X \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d),$$

where $\mathrm{div}^\mu X = \sum_{j=1}^d (P_\mu(\nabla X^j))_j$.

We denote by $\mathcal{M}_{BC}$ the set of all positive and finite Borel regular measures of $\mathbb{R}^d$ whose curvature is a Borel regular measure with finite total mass. Since the curvature H_μ of a measure $\mu \in \mathcal{M}_{BC}$ is not necessary absolutely continuous with respect to μ, by Radon-Nikodym theorem, we can write

$$H_\mu = h(\mu)\mu + \partial\mu,$$

where $h(\mu) \in L^1_\mu(\mathbb{R}^d, \mathbb{R}^d)$ is the density of H_μ with respect to μ (also called the pointwise curvature) and $\partial\mu$ is the singular part of H_μ with respect to μ (also called the boundary of μ).

Remark 5.4.2 If $\mu = \mathcal{H}^k \llcorner \Sigma$, with Σ a C^2 k-manifold with boundary in $\mathbb{R}^d$, then by classical divergence theorem we have

$$H_\mu = \nu \mathcal{H}^{k-1} \llcorner \partial\Sigma + h \mathcal{H}^k \llcorner \Sigma,$$

where h stands for the mean curvature vector of Σ and ν the co-normal unit vector of $\partial\Sigma$.
When the tangent space to μ is reduced to zero μ a.e., H_μ is zero. This is for instance the case where μ is a finite sum of Dirac masses, or μ is concentrated on α-dimensional Cantor subset C of $[0,1]$ with $\mathcal{H}^\alpha(C) \in (0,+\infty)$.

Definition 5.4.3. *Let Σ be a countable $\mathcal{H}^k$ rectifiable set and $\mu = \theta\mathcal{H}^k \llcorner \Sigma$ be the associated rectifiable measure. A function $h \in L^1_\mu(\Sigma, \mathbb{R}^d)$ is said to be generalized mean curvature of Σ if*

$$\int_{\mathbb{R}^d} \operatorname{div}^\Sigma X d\mu = -\int_{\mathbb{R}^d} X \cdot h d\mu \quad \forall X \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d).$$

In this case we denote the generalized mean curvature of Σ by H_Σ.

Theorem 5.4.4. *Let $(\mu_r)_r$ be a bounded sequence in $\mathcal{M}_{BC}$ such that μ_r weakly converges to μ and* $\operatorname{dimTan}(\mu_r)\mu_r$ *weakly converges to $g\mu$. Then the condition*

$$\operatorname{dimTan}(\mu, x) \leq g(x) \quad \mu - a.e. \tag{5.42}$$

is necessary and sufficient to have

$$P_{\mu_r}\mu_r \rightharpoonup P_\mu \mu. \tag{5.43}$$

In this case we have

$$H_{\mu_r} \rightharpoonup H_\mu. \tag{5.44}$$

Proof: see [28] $\square$

Let Ω be an open subset of $\mathbb{R}^d$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0,+\infty]$ be a positive Carathédory function. We assume F smooth and satisfying the condition

$$F(x,u,z) \leq a(x) + u^p + |z|^p,$$

where a is an $L^1(\Omega)$ function. We consider the functional

$$J(\Sigma) := \int_\Omega F(x, u(x), \nabla u(x)) dx$$

where u is the solution of the equation

$$\left\{ \begin{array}{rcl} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &=& f \text{ in } \Omega \setminus \Sigma \\ u &=& 0 \text{ on } \partial\Omega \cup \Sigma. \end{array} \right. \tag{5.45}$$

Let us consider the following classes of control variables:

$$\mathcal{A}(\Omega) := \left\{ \Sigma \subset \Omega : \Sigma \text{ closed connected}, \quad \mathcal{H}^1(\Sigma) < +\infty \right\}$$

$$\mathcal{B}(\Omega) := \left\{\Sigma \subset \Omega : \Sigma \ \text{ discrete}, \quad \mathcal{H}^0(\Sigma) = \#(\Sigma) < +\infty\right\}$$

and the shape optimization problems

$$\min\left\{J(\Sigma) + \lambda\mathcal{H}^1(\Sigma) : \Sigma \in \mathcal{A}(\Omega)\right\} \tag{5.46}$$

$$\min\left\{J(\Sigma) + \lambda\mathcal{H}^0(\Sigma) : \Sigma \in \mathcal{B}(\Omega)\right\}. \tag{5.47}$$

The penalization terms $\lambda\mathcal{H}^1(\Sigma)$ and $\lambda\mathcal{H}^0(\Sigma)$ with $\lambda > 0$ replace the constraint on $\mathcal{H}^1(\Sigma)$ and $\mathcal{H}^0(\Sigma)$ prevent the minimization sequence to spread over all the domain Ω and hence getting a trivial solution. The existence of minimizer in the two shape optimization problems (5.46) and (5.47) is just a consequence of the Šverák (see [94] for $p = 2$ and [33] for general p) and the Blaschke and *Goℓab* theorems. Our goal is to derive the first order necessary condition of optimality. We try to give the stationary configuration of the optimal shape by distinguish three cases according to technical computations.

5.4.1 Case of closed connected subsets in $\mathbb{R}^2$

Let u be the weak solution of the state equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f \ \text{ in } \Omega\setminus\Sigma \\ u &= 0 \ \text{ on } \partial\Omega\cup\Sigma \end{cases} \tag{5.48}$$

that means

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx = \int_\Omega f v dx \quad \forall v \in W_0^{1,p}(\Omega\setminus\Sigma).$$

As done in section 4.4.1 let us introduce the family of diffeomorphisms. $\varphi_\varepsilon(x) = x + \varepsilon X(x)$ where X is a smooth vector field from $\mathbb{R}^d$ to $\mathbb{R}^d$ supported in Ω. For ε small enough, φ_ε maps Ω into Ω. Set $A_\varepsilon = \varphi_\varepsilon(A)$ and consider the new state equation in the deformed domain

$$\begin{cases} -\operatorname{div}(|\nabla u_\varepsilon|^{p-2})\nabla u_\varepsilon) &= f \ \text{ in } \Omega\setminus\Sigma_\varepsilon \\ u_\varepsilon &= 0 \ \text{ on } \partial\Omega\cup\Sigma_\varepsilon. \end{cases} \tag{5.49}$$

The corresponding functional is

$$\mathcal{F}(\Sigma_\varepsilon) = \int_\Omega F(x, u_\varepsilon(x), \nabla u_\varepsilon(x))dx + \lambda\mathcal{H}^1(\Sigma_\varepsilon). \tag{5.50}$$

To differentiate the above function, we use Theorem 4.3.1 since φ satisfies (4.11) and F smooth. By taking the derivative of the functional (5.50) at $\varepsilon = 0$ we get, thanks to (4.14) the following result

$$\frac{\partial}{\partial\varepsilon}\,|_{\varepsilon=0}\,\mathcal{F}(\Sigma_\varepsilon) = \int_\Omega (F_u u' + F_z\cdot\nabla u' + \operatorname{div}(FX)))dx + \lambda\frac{\partial}{\partial\varepsilon}\,|_{\varepsilon=0}\,\mathcal{H}^1(\Sigma_\varepsilon)$$

where u' is the solution of the equation

$$
\begin{cases} -\mathrm{div}(G_u(\nabla u')) &= 0 \text{ in } \Omega \setminus \Sigma \\ u' &= 0 \text{ on } \partial\Omega \\ u' &= -\nabla u \cdot X \text{ on } \Sigma \end{cases} \tag{5.51}
$$

and

$$
G_u(Z) = |\nabla u|^{p-2} Z + (p-2)|\nabla u|^{p-4}(\nabla u \cdot Z)\nabla u.
$$

The functional G_u is computed according to the procedure introduced in Chapter 4 for computing u' and the equation (5.51) is the precise version of equations (4.23), (4.24) associated to the p-Laplacian operator. The derivative $\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \mathcal{H}^1(\Sigma_\varepsilon)$ that appears in the above variation, according to theorem 7.31 of [10], gives the following

$$
\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \mathcal{H}^1(\Sigma_\varepsilon) = \int_\Sigma \mathrm{div}^\Sigma X d\mathcal{H}^1 = -\langle H_\Sigma, X \rangle
$$

As Σ is countably $\mathcal{H}^1$-rectifiable, div^Σ should be the projection of the divergence to the approximate tangent of Σ at $\mathcal{H}^1$-a.e point of Σ. Unfortunately, the quantity $\int_\Omega (F_u u' + F_z \cdot \nabla u')dx$ is not exploitable. To overcome this problem we introduce the adjoint state equation.

$$
\begin{cases} -\mathrm{div}(G_u(\nabla q)) &= F_u - \mathrm{div}(F_z) \text{ in } \Omega \setminus \Sigma \\ q &= 0 \text{ on } \partial\Omega \cup \Sigma \end{cases} \tag{5.52}
$$

which has to be understood in the distributional sense

$$
\int_\Omega (F_u v + \mathrm{div}(F_z) v)dx - \int_{\Omega \setminus \Sigma} (\mathrm{div}(G_u(\nabla q))v)dx = 0 \;\; \forall v \in \mathcal{D}'(\Omega \setminus \Sigma).
$$

We are not interested in the regularity of the functions u and q in the whole domain Ω but only near the optimal set. Close to the optimal set, q is H^1_{loc}. In the variational formulation of the equation (5.52) if we take u' as a test function, we have

$$
\int_\Omega (F_u u' - \mathrm{div}(F_z) u')dx + \int_{\Omega \setminus \Sigma} (\mathrm{div}(G_u(\nabla q))u')dx = 0. \tag{5.53}
$$

Let Ω^+ and Ω^- be two sets such that $\Omega = \Omega^+ \cup \Omega^-$ and $\Sigma \subset \partial\Omega^+ \cap \partial\Omega^-$. Assume that Σ, $\partial\Omega$ and f provide sufficient regularity for u, u' and q so that the Green formula can be applied to (5.53). For the sequel, we use the following notation. ∇u^+ stands for the trace on Σ of ∇u restricted to Ω^+, $\frac{\partial u^+}{\partial \nu}$ for the trace of the respective normal derivative, $F_z^+ \cdot \nu = F_z(x, 0, \nabla u^+) \cdot \nu$. Similarly ∇u^- stands for the trace on Σ of ∇u restricted to Ω^-, $\frac{\partial u^-}{\partial \nu}$ for the trace of the respective normal derivative , $F_z^- \cdot \nu = F_z(x, 0, \nabla u^-) \cdot \nu$. Recall also that

$$
\nabla u^\pm = \frac{\partial u^\pm}{\partial \nu} \nu
$$

because $u^\pm = u = 0$ on Σ (i.e the tangential derivative of $u^\pm$ over Σ vanishes). Let us compute separately the term of the equation (5.53) and starting by the first part, we get

$$A^+ = \int_{\Omega^+} (F_u u' - \operatorname{div}(F_z)u')dx = \int_{\Omega^+} (F_u u' + F_z \cdot \nabla u')dx - \int_{\partial\Omega^+} u' F_z \cdot \nu d\mathcal{H}^1$$

where ν is the outer normal of Ω^+. It is easy to observe that $u' = 0$ on $\partial\Omega \cap \partial\Omega^+$ and $\partial\Omega^+ = \Sigma \cup (\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)) \cup (\partial\Omega \cap \partial\Omega^+)$ so

$$A^+ = \int_{\Omega^+} (F_u u' + F_z \cdot \nabla u')dx + \int_\Sigma \frac{\partial u^+}{\partial \nu}(F_z^+ \cdot \nu) X\nu d\mathcal{H}^1 - \int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} u' F_z \cdot \nu d\mathcal{H}^1.$$

Likewise, taking into account the fact the outer normal of Ω^- restricted to $\Omega^+ \cap \Omega^-$ is $-\nu$, one gets

$$A^- = \int_{\Omega^-} (F_u u' + F_z \cdot \nabla u')dx - \int_\Sigma \frac{\partial u^-}{\partial n}(F_z^- \cdot \nu) X\nu d\mathcal{H}^1 + \int_{\partial\Omega^- \setminus (\partial\Omega \cup \Sigma)} u' F_z \cdot \nu d\mathcal{H}^1.$$

Then, combining both of them and using the fact that the two sets $\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)$ and $\partial\Omega^- \setminus (\partial\Omega \cup \Sigma)$ coincides, give

$$A = A^+ + A^- = \int_\Omega (F_u u' + F_z \cdot \nabla u')dx + \int_\Sigma \left(\frac{\partial u^+}{\partial \nu} F_z^+ \cdot \nu - \frac{\partial u^-}{\partial \nu} F_z^- \cdot \nu \right) X\nu d\mathcal{H}^1.$$

For the second term, the integration by part leads to

$$B^+ = \int_{\Omega^+ \setminus \Sigma} \operatorname{div} G_u(\nabla q) u' dx = -\int_{\Omega^+ \setminus \Sigma} G_u(\nabla q) \cdot \nabla u' dx + \int_{\partial(\Omega^+ \setminus \Sigma)} u' G_u(\nabla q) \cdot \nu d\mathcal{H}^1$$

where ν is the outer normal of Ω^+ as in the previous case. It is easily seen that

$$B^+ = -\int_{\Omega^+ \setminus \Sigma} G_u(\nabla q) \cdot \nabla u' dx - \int_\Sigma \frac{\partial u^+}{\partial \nu}(G_u(\nabla q)^+ \cdot \nu) X\nu d\mathcal{H}^1 + \int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} u' G_u(\nabla q) \cdot \nu d\mathcal{H}^1.$$

Similarly, under the same observation as above, we have

$$B^- = -\int_{\Omega^- \setminus \Sigma} G_u(\nabla q) \nabla u' dx + \int_\Sigma \frac{\partial u^-}{\partial \nu}(G_u(\nabla q)^- \cdot \nu) X\nu d\mathcal{H}^1 - \int_{\partial\Omega^- \setminus (\partial\Omega \cup \Sigma)} u' G_u(\nabla q) \cdot \nu d\mathcal{H}^1.$$

Therefore, summing up one obtains ($B = B^+ + B^-$)

$$B = -\int_{\Omega \setminus \Sigma} G_u(\nabla q) \cdot \nabla u' dx + \int_\Sigma \left(\frac{\partial u^-}{\partial \nu} G_u(\nabla q)^- \cdot \nu - \frac{\partial u^+}{\partial \nu} G_u(\nabla q)^+ \cdot \nu \right) X\nu d\mathcal{H}^1.$$

By the linearity of the function G_u, we get

$$\int_{\Omega \setminus \Sigma} G_u(\nabla q) \cdot \nabla u' dx = \int_{\Omega \setminus \Sigma} G_u(\nabla u') \cdot \nabla q dx$$

but, by integration by parts, it follows that

$$\int_{\Omega\setminus\Sigma} G_u(\nabla u')\cdot\nabla q dx = -\int_{\Omega\setminus\Sigma}\mathrm{div}(G_u(\nabla u'))q dx + \int_{\partial\Omega\cup\Sigma} qG_u(\nabla u')\cdot\nu d\mathcal{H}^1 = 0$$

because u' is the weak solution of equation (5.51) and q vanishes on $\partial\Omega\cup\Sigma$. Finally we obtain

$$\begin{aligned}\int_\Omega F_u u' + F_z\cdot\nabla u' dx = -\int_\Sigma\Big(&\frac{\partial u^+}{\partial\nu}F_z^+\cdot\nu - \frac{\partial u^-}{\partial\nu}F_z^-\cdot\nu + \frac{\partial u^-}{\partial n}G_u(\nabla q)^-\cdot\nu \\ &- \frac{\partial u^+}{\partial\nu}G_u(\nabla q)^+\cdot\nu\Big)X\nu d\mathcal{H}^1.\end{aligned}$$

To compute the term $\int_\Omega \mathrm{div}(FX)dx$, we assume for simplicity that Ω has a Lipschitz boundary. Then

$$\int_\Omega \mathrm{div}(FX)dx = \int_{\partial\Omega} FX\nu d\mathcal{H}^{d-1} = 0$$

since X is supported in Ω. It follows that

$$\begin{aligned}&\int_\Omega (F_u u' + F_z\cdot\nabla u' + \mathrm{div}(FX))dx - \lambda\langle H_\Sigma, X\rangle = -\lambda\langle H_\Sigma, X\rangle \\ &-\int_\Sigma\left(\frac{\partial u^+}{\partial\nu}F_z^+\cdot\nu - \frac{\partial u^-}{\partial\nu}F_z^-\cdot\nu - \frac{\partial u^-}{\partial\nu}G_u(\nabla q)^-\cdot\nu + \frac{\partial u^+}{\partial\nu}G_u(\nabla q)^+\cdot\nu\right)X\nu d\mathcal{H}^1\end{aligned}$$

but, by simple computation, we have

$$\begin{aligned}G_u(\nabla q)^+\cdot\nu &= \left|\nabla u^+\right|^{p-2}\nabla q^+\cdot\nu + (p-2)\left|\nabla u^+\right|^{p-4}(\nabla u^+\cdot\nabla q^+)\nabla u^+\cdot\nu + t\nabla q^+\cdot\nu \\ &= \left|\frac{\partial u^+}{\partial\nu}\right|^{p-2}\frac{\partial q^+}{\partial\nu} + (p-2)\left|\frac{\partial u^+}{\partial\nu}\right|^{p-4}\left(\frac{\partial u^+}{\partial\nu}\frac{\partial q^+}{\partial\nu}\right)\frac{\partial u^+}{\partial\nu} \\ &= (p-1)\left|\frac{\partial u^+}{\partial\nu}\right|^{p-2}\frac{\partial q^+}{\partial\nu}\end{aligned}$$

and also similarly

$$G_u(\nabla q)^-\cdot\nu = (p-1)\left|\frac{\partial u^-}{\partial\nu}\right|^{p-2}\frac{\partial q^-}{\partial\nu},$$

therefore combining all the computations together we have

$$\begin{aligned}\int_\Omega (F_u u' + F_z\nabla u' + &\mathrm{div}(FX))dx - \lambda\langle H_\Sigma, X\rangle = -\lambda\langle H_\Sigma, X\rangle \\ &-\int_\Sigma\left(\frac{\partial u^+}{\partial\nu}F_z^+\cdot\nu - \frac{\partial u^-}{\partial\nu}F_z^-\cdot\nu\right)X\nu d\mathcal{H}^1 \\ &+(p-1)\int_\Sigma\left(\left|\frac{\partial u^-}{\partial\nu}\right|^{p-2}\frac{\partial u^-}{\partial\nu}\frac{\partial q^-}{\partial\nu} - \left|\frac{\partial u^+}{\partial\nu}\right|^{p-2}\frac{\partial u^+}{\partial\nu}\frac{\partial q^+}{\partial\nu}\right)X\nu d\mathcal{H}^1\end{aligned}$$

This equality holds for every vector field X, then we derive the following optimality condition:

$$\lambda\langle H_\Sigma, \nu\rangle - \left(\frac{\partial u^+}{\partial \nu}F_z^+ \cdot \nu - \frac{\partial u^-}{\partial \nu}F_z^- \cdot \nu\right)$$
$$+ (p-1)\left(\left|\frac{\partial u^-}{\partial \nu}\right|^{p-2}\frac{\partial u^-}{\partial \nu}\frac{\partial q^-}{\partial \nu} - \left|\frac{\partial u^+}{\partial \nu}\right|^{p-2}\frac{\partial u^+}{\partial \nu}\frac{\partial q^+}{\partial \nu}\right) = 0.$$

We can rewrite this optimality condition in this form:

$$\lambda\langle H_\Sigma, \nu\rangle + \left(\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu}\right)^{\pm} = 0.$$

We have proved the following result.

Theorem 5.4.5. *Let Σ be an optimal set in the minimization problem (5.46) and u the corresponding solution of the state equation. Assume $d = 2$, then u satisfies the following necessary condition of optimality:*

$$\lambda\langle H_\Sigma, \nu\rangle + \left(\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu}\right)^{\pm} = 0,$$

where ν is the unit normal vector of Σ, H_Σ the generalized mean curvature of Σ and q the solution of the adjoint state equation (5.52).

5.4.2 Case of points in $\mathbb{R}^d, d > 1$

In the case of points, some extra difficulties arise. For a similar equation as in (5.51) we need to define a gradient on the point. But the gradient and the normal are not defined on points (in fact there are infinite many choices which depend on the direction). The strategy is to study configurations which are close to the optimal one and obtain the optimal configuration as a limit of the studied configurations. Let x_0 be the optimal point. We consider, for r small and positive real number, the set $\Sigma_r = \psi(\overline{B_r(x_0)})$ where $B_r(x_0)$ is the ball centered at x_0 and ψ is a smooth diffeomorphism from Ω to Ω such that x_0 is invariant by ψ. The associated state equation is

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= f \text{ in } \Omega \setminus \Sigma_r \\ u &= 0 \text{ on } \partial\Omega \cup \Sigma_r. \end{cases} \tag{5.54}$$

For the functional, we consider

$$\mathcal{F}(\Sigma_r) = \frac{1}{r^d}\int_\Omega F(x, u(x), \nabla u(x))dx + \lambda\frac{1}{r^d}\mathcal{H}^d(\Sigma_r).$$

The factors $\frac{1}{r^{d-1}}$ and $\frac{1}{r^d}$ are in order to avoid the functional to degenerate to the trivial limit functional which vanishes everywhere. Notice that as $r \to 0$ the solution of the equation (5.54) converges strongly in $W_0^{1,p}(\Omega)$ to the solution of the same equation defined on $\Omega \setminus \{x_0\}$. Moreover since $p > d$ thanks to Sobolev embedding theorem, the solution is Hölder continuous. Using the same trick as above that is transforming the domain by φ_ε, find the new state equation and new functional, by taking the derivative of the functional at $\varepsilon = 0$, one gets

$$\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \mathcal{F}((\Sigma_r)_\varepsilon) = \frac{1}{r^{d-1}} \int_\Omega (F_u u' + F_z \cdot \nabla u' + \operatorname{div}(FX))\, dx - \lambda \frac{1}{r^d} \langle H_{\Sigma_r}, X \rangle$$

where u' is solution of the equation

$$\begin{cases} -\operatorname{div}(G_u(\nabla u')) &= 0 \text{ in } \Omega \setminus \Sigma_r \\ u' &= 0 \text{ on } \partial\Omega \\ u' &= -\nabla u \cdot X \text{ on } \Sigma_r, \end{cases} \tag{5.55}$$

$G_u(\nabla u')$ is as before and H_{Σ_r} is the mean curvature of Σ_r. Using the fact that x_0 is optimal and r is small enough (we are in a small neighborhood of the optimal point), we obtain $\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \mathcal{F}((\Sigma_r)_\varepsilon) = o(1)$

To overcome the problem of $\nabla u'$ as in the previous case, we introduce the adjoint state equation.

$$\begin{cases} -\operatorname{div}(G_u(\nabla q)) &= F_u - \operatorname{div}(F_z) \text{ in } \Omega \setminus \Sigma_r \\ q &= 0 \text{ on } \partial\Omega \cup \Sigma_r. \end{cases} \tag{5.56}$$

This equation has to be understood in the distributional sense

$$\int_{\Omega \setminus \Sigma_r} (F_u v - \operatorname{div}(F_z) v) dx + \int_{\Omega \setminus \Sigma_r} \operatorname{div}(G_u(\nabla q)) v dx = 0 \quad \forall v \in \mathcal{D}'(\Omega \setminus \Sigma_r).$$

In particular

$$\int_{\Omega \setminus \Sigma_r} (F_u u' - \operatorname{div}(F_z) u') dx + \int_{\Omega \setminus \Sigma_r} \operatorname{div}(G_u(\nabla q)) u' dx = 0.$$

By integration by parts, the first term of the equation yields

$$\int_{\Omega \setminus \Sigma_r} (F_u u' - \operatorname{div}(F_z) u') dx = \int_{\Omega \setminus \Sigma_r} (F_u u' + F_z \cdot \nabla u') dx - \int_{\partial \Sigma_r} u' F_z \cdot \nu_r d\mathcal{H}^{d-1}$$

where ν_r is the inward normal of Σ_r. The computation is quite similar to the case of closed connected subset of $\mathbb{R}^2$ and one gets

$$\int_{\Omega \setminus \Sigma_r} \operatorname{div}(G_u(\nabla q)) u' dx = - \int_{\partial \Sigma_r} \left((p-1) \left| \frac{\partial u}{\partial \nu} \right|^{p-2} \frac{\partial u}{\partial \nu} \frac{\partial q}{\partial \nu} \right) X \nu d\mathcal{H}^{d-1}.$$

Here $\partial \Sigma_r$ plays the role of Σ in the two dimensional case. Moreover all the quantities vanish in the interior side of Σ then we are interested only on the other side that is the exterior side of Σ_r

$$\int_{\Omega \setminus \Sigma_r} (F_u u' + F_z \cdot \nabla u') dx = - \int_{\partial \Sigma_r} \left(\frac{\partial u}{\partial \nu} F_z \cdot \nu - (p-1) \left| \frac{\partial u}{\partial \nu} \right|^{p-2} \frac{\partial u}{\partial \nu} \frac{\partial q}{\partial \nu} \right) X \nu d\mathcal{H}^{d-1}.$$

Using the above calculation, one can rewrite the derivative of the functional as follows:

$$\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \mathcal{F}((S_r)_\varepsilon) = -\frac{\lambda}{r^d}\langle H_{\Sigma_r}, X\rangle + \\ -\frac{\lambda}{r^{d-1}}\int_{S_r}\left(\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu}\right) X\nu d\mathcal{H}^{d-1}.$$

By the change of variables of type $x = \psi(r,\theta)$, $\theta \in \mathbb{S}^{d-1}$ we get

$$-\int_{\mathbb{S}^{d-1}}\int_0^r \left(\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu}\right) X\nu J(\theta) dr d\theta \\ -\frac{\lambda}{r^d}\langle H_{S_r}, X\rangle = o(1).$$

In this notation $u = u(\psi(r,\theta))$, $q = q(\psi(r,\theta))$, $\nu = \nu(\psi(r,\theta))$, $F_z = F_z(\psi(r,\theta), u(\psi(r,\theta)), \nabla u(\psi(r,\theta)))$ and $J(\theta)$ is the Jacobian determinant of the function: $\theta \mapsto \psi(\theta)$. It remains to study the limit as r tends to 0. We do it in the particular way by letting $\psi(r,\theta)$ goes to x_0 in a fixed direction as r goes to 0. To express the dependence of the limit on the direction $\psi(\theta)$, we use the following notation: $\nu(\psi(r,\theta)) \to \nu(\psi(\theta))$ as $r \longrightarrow 0$; the same notation will be also used for other functions in the integrand. This gives:

$$\int_{\mathbb{S}^{d-1}} \left(\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu}\right) X\nu J d\theta = 0. \tag{5.57}$$

All the terms in the integrand are evaluated at $\psi(\theta)$. The quantity $r^{-d}\langle H_{\Sigma_r}, X\rangle$ goes to zero as r goes to zero. In fact if we set $\mu_r = r^{-d}\mathcal{H}^d \llcorner \Sigma_r$ this measure belongs to $\mathcal{M}_{BC}$ and weakly converges to the Dirac mass $\omega_d \delta_{x_0}$ concentrated at x_0. Since $\dim\mathrm{Tan}(\mu_r) = d$ for all $r > 0$ it follows that $(\dim\mathrm{Tan}(\mu_r))\mu_r = d\mu_r$ weakly converges to $d\omega_d\delta_{x_0}$. Therefore since $\dim\mathrm{Tan}(\delta_0) = 0 < d$ we may apply Theorem 5.4.4 with f equals to the constant function d to have the weak convergence of the mean curvature H_{μ_r} to the mean curvature $H_{\delta_{x_0}}$ which is identically zero. As a consequence the generalized mean curvature H_{Σ_r} of Σ_r weakly converges to the generalized mean curvature $H_{\delta_{x_0}}$ of the point x_0. The equality in (5.57) holds for every $X \in C_c^\infty(\Omega)$ and every ψ diffeomorphism. Again it holds true for X constant in the neighborhood of the optimal point and for all ψ diffeomorphism satisfying the condition

$$\int_{\mathbb{S}^{d-1}} \nu(\psi(\theta))J(\theta)d\theta = 0.$$

This allows us to write

$$\frac{\partial u}{\partial \nu}F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2}\frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu} = const.$$

This expression is constant for all ψ and $\theta \in \mathbb{S}^{d-1}$. This means that it is constant in any direction. Then we have following optimality condition:

$$\frac{\partial u}{\partial \nu} F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2} \frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu} = const.$$

Let us consider a particular case of this problem. We assume $d = 2$ and $F = f(x)u$ where u is the solution of the p-Laplacian equation. To express the dependence of u on p we denote it by u_p instead of u and the same for q. Since $p > 2$ we want to study the limit as $p \to 2^+$ of the problem. The sequence u_p are bounded in $H_0^1(\Omega \setminus \Sigma)$ then up to extracting subsequence, it converges weakly to some function u. It is easy to see that u coincides with the solution of the classical Laplacian that is solution of the p-Laplacian equation when $p = 2$. From the adjoint state equation, we may deduce also that the limit of q_p as $p \to 2^+$ coincides with the solution of the classical Laplacian equation. We may then rewrite the necessary condition of optimality in the following form:

$$\left|\frac{\partial u}{\partial \nu}\right| = const.$$

The result proved is summarized below.

Theorem 5.4.6. *Let Σ be an optimal set in the minimization problem (5.47) and u the corresponding solution of the state equation. Assume $d > 1$, then u satisfies the following necessary condition of optimality:*

$$\frac{\partial u}{\partial \nu} F_z \cdot \nu - (p-1)\left|\frac{\partial u}{\partial \nu}\right|^{p-2} \frac{\partial u}{\partial \nu}\frac{\partial q}{\partial \nu} = const,$$

where ν and q are respectively the limit as r goes to zero in a given direction of the unit normal vector of Σ_r and the solution of the adjoint state equation (5.56)

Remark 5.4.7 The case of points in $\mathbb{R}$ is similar to the case of closed connected subset in $\mathbb{R}^2$.

5.4.3 Case of closed connected subsets in $\mathbb{R}^d$ with $d > 2$

Here the strategy is the same. Let Σ be the optimal configuration. We study the configuration which is close to the optimal one and pass to the limit. As in the case of points, we consider a tube $\Sigma_r = \{x \in \mathbb{R}^x : d(x, \Sigma) \leq r\}$. The associated state equation is the following

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= f \text{ in } \Omega \setminus \Sigma_r \\ u &= 0 \text{ on } \partial\Omega \cup \Sigma_r. \end{cases} \tag{5.58}$$

The procedure is quite similar to the previous case. The corresponding general functional is set as follows.

$$\mathcal{F}(\Sigma_r) = \frac{1}{\mathcal{H}^{d-2}(\mathbb{S}_r^{d-2})}\int_\Omega F(x,u(x),\nabla u(x)) + \frac{\lambda}{|\Sigma_r|}\mathcal{H}^{d-1}(\Sigma_r),$$

where $\mathbb{S}_r^{d-2}$ is a $(d-2)$- dimensional sphere of radius r and centered on points of Σ.

From the previous computation, we deduce the derivative of the functional:

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\,|_{\varepsilon=0}\mathcal{F}((\Sigma_r)_\varepsilon) &= -\frac{\lambda}{|\Sigma_r|}\langle H_{\Sigma_r}, X\rangle,\\ &-\frac{1}{\mathcal{H}^{d-2}(\mathbb{S}_r^{d-2})}\int_{\partial\Sigma_r}\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}+t\left|\frac{\partial u}{\partial\nu}\right|^2\right)X\nu d\mathcal{H}^{d-1}\end{aligned}$$

where H_{Σ_r} is the generalized mean curvature of Σ_r. Remark that all the equations are the same as in the case of points in $\mathbb{R}^d$. To pass to the limit, we use the same trick as in the case of points in $\mathbb{R}^d$. First we disintegrate the measure $\mathcal{H}^{d-1}$ and get

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\,|_{\varepsilon=0}\,\mathcal{F}((\Sigma_r)_\varepsilon) &= -\frac{\lambda}{|\Sigma_r|}\langle H_{\Sigma_r}, X\rangle\\ &-\frac{1}{\mathcal{H}^{d-2}(\mathbb{S}_r^{d-2})}\int_\Sigma\int_{\mathbb{S}_r^{d-2}}\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right)X\nu d\mathcal{H}^{d-2}d\mathcal{H}^1.\end{aligned}$$

The measure $\mathcal{H}^{d-2}(\mathbb{S}_r^{d-2})^{-1}\mathcal{H}^{d-2}\llcorner\mathbb{S}_r^{d-2}$ converges weakly to $(d-1)\omega_{d-1}\delta_x$ where x is the center of the sphere $\mathbb{S}_r^{d-2}$ and ω_{d-1} is the volume of the unit $d-1$-dimensional ball. Due to the hypothesis made on data of the problem the measure

$$\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right)\nu\mathcal{H}^{d-2}(\mathbb{S}_r^{d-2})^{-1}\mathcal{H}^{d-2}\llcorner\mathbb{S}_r^{d-2}$$

weakly converges to the measure

$$\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right)(d-1)\omega_{d-1}\nu\delta_x$$

for $\mathcal{H}^1$ a.e. $x\in\Sigma$. The limit here is computed in a fixed direction $\theta\in\mathbb{S}^{d-2}$. For the curvature part, notice first that the measure $\mu_r = |\Sigma_r|^{-1}\mathcal{H}^d\llcorner\Sigma_r\in\mathcal{M}_{BC}$ weakly converges to the measure $\omega_{d-1}\mathcal{H}^1\llcorner\Sigma$ and $(\dim\mathrm{Tan}(\mu_r))\mu_r = d\mu_r$ weakly converges to the measure $dd\omega_{d-1}\mathcal{H}^1\llcorner\Sigma$. The fact that $\dim\mathrm{Tan}\mathcal{H}^1\llcorner\Sigma = 1 < d$ allows to apply again the Theorem 5.4.4 to have weak convergence of mean curvature of μ_r to that of $\mathcal{H}^1\llcorner\Sigma$ and consequently the weak convergence of H_{Σ_r} to H_Σ. Summarizing all computed results we get

$$\frac{\partial}{\partial\varepsilon}\,|_{\varepsilon=0}\,\mathcal{F}(\Sigma_\varepsilon) = -\omega_{d-1}\lambda\langle H_\Sigma, X\rangle$$
$$-\,(d-1)\omega_{d-1}\int_\Sigma\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right)X\nu d\mathcal{H}^1 = 0.$$

This relation is true for every vector field X therefore we get the following necessary of optimality:

$$\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right) = \frac{\lambda}{d-1}\langle H_\Sigma,\nu\rangle.$$

The result proved is summarized below.

Theorem 5.4.8. *Let Σ be an optimal set in the minimization problem (5.46) and u the solution of the associated state equation. Assume $d > 2$, then u satisfies the following necessary condition of optimality:*

$$\left(\frac{\partial u}{\partial\nu}F_z\cdot\nu-(p-1)\left|\frac{\partial u}{\partial\nu}\right|^{p-2}\frac{\partial u}{\partial\nu}\frac{\partial q}{\partial\nu}\right) + \frac{\lambda}{d-1}\langle H_\Sigma,\nu\rangle = 0,$$

where ν is the unit normal vector of Σ in a given direction, H_Σ the generalized mean curvature of Σ and q the limit as r goes to zero of the solution of the adjoint state

Remark that this necessary condition of optimality depends on the direction $\theta \in \mathbb{S}^{d-2}$. Those directions are contained in the $d-1$ plane which is orthogonal to the approximate tangent line to Σ.

Chapter 6

Asymptotic shapes

The study of asymptotical problems in the optimal location of an increasing amount of resources has been developed intensively, even in recent time, mainly using an approach based on Γ-convergence. In [29] the so-called location problem (choosing a set Σ composed by n points in a domain Ω in order to minimize the average distance of the points of Ω to Σ) is studied when $n \to +\infty$, finding a Γ-limit of a suitable sequence of functionals defined on the space of probability measures on $\overline{\Omega}$. In [76] the same analysis has been performed for the so-called irrigation problem, where points are replaced by closed connected one dimensional sets of finite length, and the constraint $\#\Sigma \leq n$ by $\mathcal{H}^1(\Sigma) \leq l$. Both problems are linked to the Monge-Kantorovich optimal transportation theory (see e.g. [7] [29]). However these asymptotical problems are not completely understood since explicit minimizing sequences are not known in general, apart some simple cases, usually in dimension two. For instance, for the location problem it is known that placing the points on a regular triangular grid, so that each one is in the middle of a cell shaped like a regular hexagon, gives an asymptotically minimizing sequence (see [60] or [25] for stronger results). For the irrigation problem, the asymptotically minimizing sequence is obtained by considering the set made by n segments of length 1 equi-spaced and parallel to two faces of the unit square union the boundary of unit square (see [76] or Lemma 6.2.26).

On the other hand, many researches have been carried out on shape optimization problems involving PDEs, more precisely optimizing the shape of a domain where to solve a PDE (in general of elliptic type with prescribed boundary conditions), in order to minimize the value of an objective functional depending on the solution of the PDE. There is a wide literature on shape optimization problems, both from theoretical and numerical point of view. The reader may find a lot of examples and details in the following books: [4], [23], [31], [67], [92]. One of the simplest shape optimization problem, which is also one of the most important in application, is the compliance minimization problem. It consists in finding a domain Ω which minimize the integral $\int_\Omega fu dx$ where u is the solution of elliptic equation $-\Delta u = f$ (or more general elliptic equation) with Dirichlet boundary condition on $\partial\Omega$. Here we consider three compliance minimization problem. First we deal with the p-compliance-Network which is the compliance minimization problem where the unknown domain where to solve the PDE with Dirichlet boundary conditions is the complement of an one dimension set with assigned length. The second is the p-compliance-location where we replace the one dimensional closed connected set with length l by a set of n points. The last one is compliance-location and the domain is searched among the complement of a finite union of balls with given radius. The aim of this chapter is to study the asymptotic behavior of the optimal sets.

6.1 Γ-Convergence

This section is devoted to the main properties of Γ-convergence, in particular to those that are useful in the actual computation of Γ-limits. For more details, one may consult [51], [30].

6.1.1 The definition of Γ-convergence

Let us consider the family $F_n : X \to [-\infty, +\infty]$ defined on topological space X. We say that F_n Γ-converges to $F : X \to [-\infty, +\infty]$ at $x \in X$ as $n \to +\infty$ if we have

$$F(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{n \to +\infty} \inf_{y \in U} F_n(y) = \sup_{U \in \mathcal{N}(x)} \limsup_{n \to +\infty} \inf_{y \in U} F_n(y) \tag{6.1}$$

where $\mathcal{N}(x)$ denotes the family of all neighborhoods of x in X. In this case we say that $F(x)$ is the Γ-limit of F_n at x and we write

$$F(x) = \Gamma - \lim_{n \to +\infty} F_n(x). \tag{6.2}$$

If (6.2) holds for every $x \in X$ we say that F_n Γ-converges to F. Sometime one may consider family of functionals $F_n : X_n \to [-\infty, +\infty]$, where the domain depend on n. In this case it is understood that we identify such functionals with

$$\tilde{F}_n(x) = \begin{cases} F_n(x) & \text{if} \quad x \in X_n \\ +\infty & \text{if} \quad x \in X \setminus X_n, \end{cases}$$

where X is a space containing all X_n where the convergence take place. In applications we will deal with metric spaces (as L^p spaces) or metrizable spaces (as bounded subsets of Sobolev spaces or of spaces of measures, equipped with weak topology), that in addition are also separable. For such spaces the definitions above are simplified as follows.

Theorem 6.1.1. *(equivalent definition of* Γ*-convergence) Let X be a metric space and $F_n, F :\to [-\infty, +\infty]$. Then the* Γ*-convergence of F_n to F at x is equivalent to any of*

the following conditions
(a) we have

$$F(x) = \inf\{\liminf_{n\to+\infty} F_n(x_n) : x_n \to x\} = \inf\{\limsup_{n\to+\infty} F_n(x_n) : x_n \to x\}; \tag{6.3}$$

(b) we have

$$F(x) = \min\{\liminf_{n\to+\infty} F_n(x_n) : x_n \to x\} = \min\{\limsup_{n\to+\infty} F_n(x_n) : x_n \to x\}; \tag{6.4}$$

(c)(sequential Γ*-convergence) we have*

(i)($\liminf$ *inequality) for every sequence* $(x_n)_n$ *converging to* x

$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n); \tag{6.5}$$

(ii)($\limsup$ *inequality) there exists a sequence* (x_n) *converging to* x *such that*

$$F(x) \geq \limsup_{n\to+\infty} F_n(x_n); \tag{6.6}$$

(d) the $\liminf$ *inequality (c)(i) holds and*
(ii)' (existence of recovery sequence) there exists a sequence x_n *converging to* x *such that*

$$F(x) = \lim_{n\to+\infty} F_n(x_n); \tag{6.7}$$

(e) the $\liminf$ *inequality (c)(i) holds and*
(ii)' (approximate $\limsup$ *inequality) for all* $\eta > 0$ *there exists a sequence* x_n *converging to* x *such that*

$$F(x) \geq \limsup_{n\to+\infty} F_n(x_n) - \eta; \tag{6.8}$$

Moreover, the Γ*-convergence of* F_n *to* F *on the whole* X *is equivalent to*
(f)(limit of minimum problems) inequality

$$\inf_U F \geq \limsup_{n\to+\infty} \inf_U F_n \tag{6.9}$$

holds for all open sets U *and inequality*

$$\inf_K F \leq \sup\{\liminf_{n\to+\infty} \inf_U F_n : U \supset K, U \text{ open}\} \tag{6.10}$$

holds for all compact sets K*.*

Remark 6.1.2 From the definition above we can make some observation

1. Stability under continuous perturbation: if F_n Γ-converges to F and $G : X \to [-\infty, +\infty]$ is d-continuous function then $F_n + G$ Γ-converges to $F + G$. This is an immediate consequence of the definition (e.g. from condition (d));

2. Γ-limit of a constant sequence: Γ-convergence does not enjoy the property that a constant family $F_n = F$ converges to F. In fact if this where true, then from the $\liminf$ inequality we would have $F(x) \leq \liminf_{n\to+\infty} F(x_n)$ for all $x_n \to x$; i.e., F is lower semicontinuous (which is not always true);

3. Comparison with uniform and pointwise convergence. the previous observation in particular shows that we cannot deduce the existence of the Γ-limit from the pointwise convergence. if F_n converges to G pointwise and $F = \Gamma - \lim_{n\to+\infty} F_n$ then $F \leq G$. However, if F_n converges uniformly to a continuous function on an open set U the we easily see that F_n Γ-converges to F;

Upper and lower Γ-limits

As for usual limits, it is convenient to define quantity that always exist (as upper and lower limits) and express the existence of the Γ-limits as an equality between those two quantities. From Theorem 6.1.1(a) we may define the upper and lower Γ-limits in the following way.

$$\Gamma - \liminf_{n\to+\infty} F_n(x) = \inf\{\liminf_{n\to+\infty} F_n(x_n) : x_n \to x\}, \tag{6.11}$$

$$\Gamma - \limsup_{n\to+\infty} F_n(x) = \inf\{\limsup_{n\to+\infty} F_n(x_n) : x_n \to x\}, \tag{6.12}$$

In this case the existence of the Γ-limits is formulated as

$$\Gamma - \liminf_{n\to+\infty} F_n(x) = \Gamma - \limsup_{n\to+\infty} F_n(x) = F(x). \tag{6.13}$$

Remark 6.1.3 If F_{n_k} is a subsequence of F_n then

$$\Gamma - \liminf_{n\to+\infty} F_n \leq \Gamma - \liminf_{k\to+\infty} F_{n_k}, \quad \Gamma - \limsup_{k\to+\infty} F_{n_k} \leq \Gamma - \limsup_{n\to+\infty} F_n.$$

In particular, if $F = \Gamma - \lim_{n\to+\infty} F_n$ exists then $F = \Gamma - \lim_{k\to+\infty} F_{n_k}$

6.1.2 Γ-convergence and lower semicontinuity

As remarked above the Γ-limit of a constant family $F_n = F$ does not converges to F. This is true, however, if F is d-lower semicontinuous. More, the class of lower semicontinuous functions provides a stable class for Γ-convergence. This is summarized in the following propositions.

Proposition 6.1.4. *(lower semicontinuous of Γ-limits) The Γ-upper and lower limits of a family F_n are d-lower semicontinuous functions.*

Proposition 6.1.5. *(Γ-limits and lower semicontinuous envelopes)*

1. *The Γ-limit of a constant sequence $F_n = F$ is equal to*

$$\overline{F}(x) = \liminf_{y \to x} F(y); \tag{6.14}$$

   that is, the lower semicontinuous envelope of F, defined as the largest lower semicontinuous function not greater that F.

2. *The Γ-limit is stable by substituting F_n by its lower semicontinuous envelope $\overline{F}_n$ i.e.*

$$\Gamma - \liminf_{n \to +\infty} F_n = \Gamma - \liminf_{n \to +\infty} \overline{F_n}, \quad \Gamma - \limsup_{n \to +\infty} F_n = \Gamma - \limsup_{n \to +\infty} \overline{F_n}. \tag{6.15}$$

Remark 6.1.6 If $F_n \to F$ pointwise then $\Gamma - \limsup_{n \to +\infty} F_n \leq F$, and hence, taking both lower semicontinuous envelopes, it holds $\Gamma - \limsup_{n \to +\infty} F_n \leq \overline{F}$.

6.1.3 Computation of Γ-limits

In general, the computation of the Γ-limits of a family F_n is divided into the computation of the lower and upper bound. A lower bound is a functional G such that $G \leq \Gamma - \liminf_{n \to +\infty} F_n$ i.e.

$$G(x) \leq \liminf_{n \to +\infty} F_{n_k}(x_{n_k}) \text{ for all } k \to +\infty \text{ and } x_{n_k} \to x. \tag{6.16}$$

The lower semicontinuity of the Γ-limit allows us to limit our research for lower bound to the class of lower semicontinuous G. If we can characterize a large enough family $\mathcal{G}$ of G satisfying (6.16) then the optimal lower bound obtained as $\overline{G}(x) =: \sup\{G(x) : G \in \mathcal{G}\}$. Since $\overline{G}$ is the supremum of the family of lower semicontinuous functions it is lower semicontinuous. The optimization of the lower bound suggests an anzatz to approximate a target element $x \in X$ by a family $\overline{x}_n \to x$, thus defining $H(x) := \lim_{n \to +\infty} F_n(\overline{x}_n)$. By definition $H \geq \limsup_{n \to +\infty} F_n$, so that H is an upper bound for the Γ-limit. If we use more ansatze, we obtain a family $\mathcal{H}$ and then a candidate optimal upper bound as $\overline{H}(x) = \inf\{H(x) : H \in \mathcal{H}\}$. The existence (and computation) of the Γ-limits is then expressed in the equality $\overline{G} = \overline{H}$.

Remark 6.1.7 (a density argument) The lower semicontinuity of the Γ-lim sup can be used to reduce its computation to a dense class. Let d' be a distance on X inducing a topology which is weaker than that induced by d i.e. $d'(x_n, x) \to 0$ implies $d(x_n, x) \to 0$, and suppose that
$(i)\mathcal{D}$ is a dense subset of X for d';
(ii) we have $\Gamma - \limsup_{n\to+\infty} F_n(x) \leq F(x)$ on $\mathcal{D}$, where F is a function which is continuous with respect to d; then we have $\Gamma - \limsup_{n\to+\infty} F_n \leq F$ on X. To check this, it suffices to note that if $d'(x_k, x) \to 0$ and $x_k \in \mathcal{D}$ then

$$\begin{aligned}\Gamma - \limsup_{n\to+\infty} F_n(x) &\leq \liminf_k (\Gamma - \limsup_{n\to+\infty} F_n(x_k)) \\ &\leq \liminf_k F(x_k) = F(x).\end{aligned}$$

6.1.4 Properties of Γ-convergence

Definition 6.1.8. *We will say that a sequence $F_n : X \to \overline{\mathbb{R}}$ is equi-coercive if for all $t \in \mathbb{R}$ there exists a compact set K_t such that $\{F_n \leq t\} \subset K_t$.*

We can state now the main convergence result of Γ-convergence.

Theorem 6.1.9. *(fundamental theorem of Γ-convergence) Let (X, d) be a metric space, let F_n be a equi-coercive sequence of functions on X, and let $F = \Gamma - \limsup_{n\to+\infty} F_n$; then*

$$\exists \min_X F = \lim_{n\to+\infty} \inf_X F_n. \tag{6.17}$$

Moreover, if $(x_n)_n$ is a precompact sequence such that $\lim_{n\to+\infty} F_n(x_n) = \lim_{n\to+\infty} \inf_X F_n$, then every limit of a subsequence of $(x_n)_n$ is a minimum for F.

Γ-limits of monotone sequences

We give some simple examples when the Γ-limit does exist and is easily computed.
(i) $F_{n+1} \leq F_n$ for all $n \in \mathbb{N}$, then

$$\Gamma - \lim_n F_n = \overline{(\inf_n F_n)} = \overline{(\lim_n F_n)}. \tag{6.18}$$

In fact as $F_n \to \inf_k F_k$ pointwise, by Remark 6.1.2 we have $\Gamma - \limsup_n F_n \leq \overline{(\inf_k F_k)}$ wile the other inequality comes from the inequality $\overline{(\inf_k F_k)} \leq \inf_k F_k \leq F_k$;
(ii) if $F_n \leq F_{n+1}$ for all $n \in \mathbb{N}$, then

$$\Gamma - \lim_n F_n = \sup_n \overline{F}_n = \lim_n \overline{F}_n; \tag{6.19}$$

in particular if F_n is lower semicontinuous for every $n \in \mathbb{N}$, then

$$\Gamma - \lim_n F_n = \lim_n F_n.$$

In fact, since $\overline{F}_n \to \sup_k \overline{F}_k$ pointwise,

$$\Gamma - \limsup_n F_n = \Gamma - \limsup_n \overline{F}_n \leq \sup_k \overline{F}_k$$

by Remark 6.1.2. On the other hand $\overline{F}_k \leq F_n$ for all $n \geq k$ so that the convex inequality easily follows.

Γ-limits and pointwise properties

Proposition 6.1.10. *If each element of the family (F_n) is positively homogeneous of degree d (respectively convex, a quadratic form) then their Γ-limit F is positively homogeneous of degree d (respectively convex, a quadratic form).*

Topological properties of Γ-convergence

Proposition 6.1.11. *(compactness) Let (X, d) be a separable metric space, and for all $n \in \mathbb{N}$ let $F_n : X \to \overline{\mathbb{R}}$ be a function. Then there exists an increasing sequence of integers n_k such that the $\Gamma - \lim_k F_{n_k}(x)$ exists for all $x \in X$.*

Proposition 6.1.12. *(Urysohn property) We have $\Gamma - \lim_n F_n = F$ if and only if for every subsequence F_{n_k} there exists a further subsequence which Γ-converges to F.*

In the following section we will apply this general theory of Γ-convergence for studying the limits behavior of some precise functional namely the compliance functional.

6.2 Asymptotics of an optimal p-compliance-networks

We consider the problem of the optimal location of a Dirichlet region in a d-dimensional domain Ω subjected to a given force f in order to minimize the p-compliance of the configuration. We look for the optimal region among the class all closed connected sets of assigned length l. Then we let l tends to infinity and we look for the Γ-limit of a suitable rescaled functional, in order to get information of the asymptotical distribution of the optimal set. We highlight as well the case where the Dirichlet region is searched among discrete sets of finite cardinality. We consider the problem of finding the best location of the Dirichlet region Σ in a d-dimensional domain Ω associated to an elliptic equation in divergence form, namely

$$\begin{cases} \Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

where the right hand side f is a nonnegative element of $L^q(\Omega)$, q being the conjugate exponent of p. We are interested in the minimization of the p-compliance functional

$$C_p(\Sigma) = \int_\Omega f u_{f,\Sigma,\Omega} dx,$$

where $u_{f,\Sigma,\Omega}$ stands for the unique solution of the above equation. The admissible class for control variables Σ we consider here is the class of all closed connected sets with given one dimensional Hausdorff measure. We assume that $p > d-1$ otherwise one dimensional set will have zero p-capacity and the problem will be meaningless. It is easy to obtain the optimal configuration Σ_l of the above optimization problem (see Theorem 6.2.1) as a consequence of Šverák result (see Theorem 4.1.21 for $p=2$ and Theorem 5.2.13 for general p). We are interested in the asymptotic behavior of Σ_l as $l \to +\infty$; more precisely we want to obtain the limit distribution of Σ_l as a limit probability measure that minimize the Γ-limit functional of the suitable rescaled p-compliance functional. In the last section, we deal with the case where the Dirichlet region is searched among the class of discrete sets on finite number of elements under the assumption that $p > d$.

6.2.1 p-compliance under length constraint

Let $p > d-1$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and l a positive given real number, we define

$$\mathcal{A}_l(\Omega) = \{\Sigma \subset \overline{\Omega}, \quad \text{closed and connected}, \quad 0 < \mathcal{H}^1(\Sigma) \leq l\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity, we denote by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{6.20}$$

By the maximum principle, the nonnegativity of the function f implies that of u. For $f \geq 0$, we define the p-compliance functional as follows:

$$\begin{aligned} C_p(\Sigma) = F_p(\Sigma, f, \Omega) &= \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p dx \\ &= q \max\left\{ \int_\Omega (v - \frac{1}{p}|\nabla v|^p) dx : v \in W_0^{1,p}(\Omega \setminus \Sigma) \right\}, \end{aligned}$$

where q stands for the conjugate exponent of p. The existence of the minimal p-compliance configuration is just a consequence of a generalized Šverák compactness-continuity result (see Theorem 5.2.13).

Theorem 6.2.1. *For any real number $l > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_l(\Omega)\} \tag{6.21}$$

admits at least one solution.

Here we are interested to the asymptotic behavior of the optimal set Σ_l of the problem (6.21) as $l \to +\infty$. Let us associate to every $\Sigma \in \mathcal{A}_l(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and define a functional $F_l : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$F_l(\mu) = \begin{cases} l^{\frac{q}{d-1}} C_p(\Sigma) & \text{if} \quad \mu = \mu_\Sigma, \ \Sigma \in \mathcal{A}_l(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{6.22}$$

The scaling factor $l^{\frac{q}{d-1}}$ is needed in order to avoid the functional to degenerate to the trivial limit functional which vanishes everywhere. Our main result deal with the behavior as $l \to +\infty$ of the functional F_l, and we state it in terms of Γ-convergence.

Theorem 6.2.2. *The functional F_l defined in (6.22) Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional F defined on $\mathcal{P}(\overline{\Omega})$ by*

$$F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx, \tag{6.23}$$

where μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ is a positive constant depending only on d and p and is defined by

$$\theta = \inf\{\liminf_{l \to +\infty} l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) : \Sigma_l \in \mathcal{A}_l(I^d)\} \tag{6.24}$$

$I^d = (0,1)^d$ being the unit cube in $\mathbb{R}^d$. When the dependence of θ on p will be necessary, we will use the notation $\theta(p)$.

According to the general theory of Γ-convergence (see Theorem 6.1.9), we deduce the following consequence of Theorem 6.2.2:

- if Σ_l is a solution of the minimization problem (6.21), then up to a subsequence $\mu_{\Sigma_l} \rightharpoonup \mu$ as $l \to +\infty$, where μ is a minimizer of F;
- since F has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_l} converges to the unique minimizer μ of F given by $\mu = c f^{\frac{q(d-1)}{q+d-1}} \mathcal{L}^d$ where c is such that μ is a probability measure that is $c = 1/\left(\int_\Omega f^{\frac{q(d-1)}{q+d-1}} dx\right)$

- the minimal value of F is equal to $\theta c^{\frac{q+d-1}{d-1}}$, and the sequence of the values

$$\inf \{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_l(\Omega)\}$$

is asymptotically equivalent to $l^{\frac{q}{d-1}} \inf \left\{F(\mu) : \mu \in \mathcal{P}(\overline{\Omega})\right\}$.

6.2.2 Proof of Γ-limit in $\mathbb{R}^2$

We will prove Theorem 6.2.2 in several steps, the most important two correspond to Γ-lim inf and Γ-lim sup inequalities.

The Γ-lim inf inequality

In the following proposition we prove that the Γ-lim inf functional is bounded below by the candidate limit F.

Proposition 6.2.3. *Under the same hypotheses of Theorem 6.2.2, denoting by F^- the functional $\Gamma\text{-}\liminf_l F_l$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any sequence $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that μ_{Σ_l} weakly* converges to μ, we have*

$$\liminf_{l \to +\infty} l^q \int_\Omega f u_{f,\Sigma_l,\Omega} dx \geq F(\mu).$$

Proof: Let us fix ε and define a set $G_{\varepsilon,l}$ in the following way: for a positive number a, denote by $I_a^2 = (-a,a)^2$ a square large enough to contain Ω, the set $G_{\varepsilon,l}$ is a regular grid composed by n horizontal lines and n vertical lines with $n = \lfloor \frac{\varepsilon l}{4a} \rfloor$, so that the total length is approximatively εl; then we intersect the grid with Ω. Let $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ and set $u_l' = u_{f,\Sigma_l',\Omega}$. Since $u_l \geq u_l'$, it is enough to estimate the integral $l^q \int_\Omega f u_l'$. It is obvious that $0 \leq u_l' \leq u_{f,G_{\varepsilon,l},\Omega}$ and Lemma 6.2.4 gives

$$||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d, \varepsilon_0, \varepsilon, f) l^{-q}.$$

It follows that $l^q u_l'$ is L^p bounded, so up to a subsequence $l^q u_l' \rightharpoonup w$ weakly in $L^p(\Omega)$. Thus

$$\lim_{l \to +\infty} l^q \int_\Omega g u_l' dx = \int_\Omega g w dx, \quad \forall g \in L^q(\Omega).$$

So it is enough to estimate w from below. We will show that, for almost any $x_0 \in \Omega$, it holds

$$w(x_0) \geq \theta \frac{f(x_0)^{1/(p-1)}}{(\mu_a + \varepsilon)^q}. \tag{6.25}$$

To this aim, we first estimate w on a square Q centered at the point $x_0 \in \Omega$. We assume that x_0 is a Lebesgue point for f and $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as Q shrinks around x_0. Assume also $f(x_0) > 0$ otherwise (6.25) would be trivial. We have

$$\lim_{l \to +\infty} l^q \int_\Omega u_l' dx = \int_\Omega w dx,$$

we use

$$u_l' \geq u_{f,\Sigma_l',Q} \geq u_{f(x_0),\Sigma_l',Q} - |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| \text{ in } Q,$$

where the first inequality comes from the fact that we add Dirichlet boundary condition on Q. The second part of Lemma 6.2.5 or Lemma 6.2.6 give

$$\int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}|dx \leq l^{-q}|Q|r(Q).$$

It remains to estimate the second term. First of all let us define the number $L(l,Q) = \mathcal{H}^1(\Sigma_l' \cap Q)$ and observe that

$$u_{f(x_0),\Sigma_l',Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma_l',Q}.$$

For simplicity of the notation, we denote $u_{1,\Sigma_l',Q}$ by v_l. By a change of variables, if we assume the side of square Q to be λ and we define $v_{l,\lambda} = \lambda^q v_l(\lambda x)$ (thinking for instance that both squares are centered at the origin), we get $v_{l,\lambda} = u_{1,\lambda^{-1}\Sigma_l',I^d}$. It is easy to see that

$$\lambda^{-1}\Sigma_l' \in \mathcal{A}_{L(l,Q)/\lambda}(I^d);$$

moreover, it holds $L(l,Q) \to +\infty$ as $l \to +\infty$, since

$$L(l,Q) \geq \mathcal{H}^1(G_{\varepsilon,l} \cap Q) \approx \varepsilon l|Q|. \tag{6.26}$$

Using (6.26) and the fact that $\mu_l = l^{-1}\mathcal{H}^1(\Sigma_l)$, we may estimate the ratio between $L(l,Q)$ and l. It follows from the *weak** convergence of μ_l to μ (and using 2.3)that $\limsup_{l\to+\infty} \mu_l(Q) \leq \mu(\overline{Q})$. So we have

$$\limsup_{l\to+\infty} \frac{L(l,Q)}{l} \leq \mu(\overline{Q}) + \varepsilon|Q|. \tag{6.27}$$

Using the definition of θ and the change of variables $y = \lambda x$ we have,

$$\begin{aligned}
\liminf_{l\to+\infty} L(l,Q)^q \int_Q v_l(y)dy &= \liminf_{l\to+\infty} L(l,Q)^q \lambda^2 \int_{I^d} v_{l,\lambda}(x)dx \\
&= \liminf_{l\to+\infty} \left(\lambda^{-1}L(l,Q)\right)^q \lambda^{2+2q} \int_{I^d} v_{l,\lambda}(x)dx \\
&\geq \lambda^{2+q}\theta,
\end{aligned}$$

hence using the fact that $\lambda^2 = |Q|$ we get

$$\begin{aligned}
\liminf_{l\to+\infty} l^q \int_Q v_l(y)dy &\geq \liminf_{l\to+\infty} \left(\frac{l}{L(l,Q)}\right)^q \liminf_{l\to+\infty} L(l,Q)^q \int_Q v_l(y)dy \\
&\geq \lambda^{2+q}\theta \left(\frac{1}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^q \\
&= \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^q |Q|\theta.
\end{aligned}$$

This implies that

$$|Q|^{-1}\int_Q w dx \geq -r(Q) + \left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^q \theta f(x_0)^{1/(p-1)}.$$

We know that $r(Q)$ tends to 0 when the square Q shrinks to x_0, whenever x_0 is a Lebesgue point for f. Now we let the square Q shrinks toward x_0 with x_0 satisfying the previous assumption, then we get

$$w(x_0) \geq \frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a(x_0)+\varepsilon)^q}.$$

It follows that

$$\liminf_{l\to+\infty} l^q \int_\Omega f u_l dx \geq \int_\Omega f w dx \geq \theta \int_\Omega \frac{f^q}{(\mu_a+\varepsilon)^q} dx,$$

and the desired inequality holds by letting ε tend to 0 that is

$$\liminf_{l\to+\infty} l^q \int_\Omega f u_l dx \geq \theta \int_\Omega \frac{f^q}{\mu_a^q} dx.$$

□

Lemma 6.2.4. *The following facts hold*

1. *There exists a constant C such that, for all functions $v \in W_0^{1,p}(I^2)$ we have*

$$\int_{I^2} |v|^p dx \leq C \int_{I^2} |\nabla v|^p dx.$$

2. *If we replace I^2 by a square Q of side λ, the inequality remains valid with the constant $\lambda^p C$ instead of C.*

3. *As a consequence, for any $\varepsilon > 0$, any $0 < l < \infty$, any domain Ω and any function $v \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l}) \subset W_0^{1,p}(\Omega)$ (where $G_{\varepsilon,l}$ is the grid introduced in the proof of Proposition 6.2.3) we have $||v||_{L^p(\Omega)} \leq C(\varepsilon)l^{-1}||u||_{W_0^{1,p}(\Omega)}$ for a suitable constant $C(\varepsilon)$.*

4. *As a further consequence, if $f \in L^q(\Omega)$ with $f \geq 0$ the function $u_{f,G_{\varepsilon,l},\Omega}$ satisfies $||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq l^{-q}||f||_{L^q(\Omega)}^{1/(p-1)}$.*

Proof: The first assertion is the well-known Poincaré inequality. The second is obtained just by a scaling of the first. To prove the third, let us extend the function v to the large square $I_a^2 \supset \Omega$ by setting the value zero outside Ω. Such an extension is in $W_0^{1,p}(I_a^2)$ due to the Dirichlet boundary condition on Ω. Then we consider the squares Q_j which come from the subdivision of I_a^2 into the squares given by the grid $G_{\varepsilon,l}$. Their side is of order l^{-1}. Notice that the extended function vanishes on the boundary of each square Q_j. By applying the second statement of this Lemma, we get

$$\int_{Q_j} |v|^p dx \leq C(\varepsilon) l^{-p} \int_{Q_j} |\nabla v|^p dx,$$

and by summing over j, we get

$$\int_{I_a^2} |v|^p dx \leq C(\varepsilon) l^{-p} \int_{I_a^2} |\nabla v|^p dx.$$

Since v vanishes outside Ω we may restrict the integrals to Ω and raise to the power $1/p$, thus getting the desired result. In the sequel of this section, the norm $||v||_{W_0^{1,p}(\Omega)}$ will stands for the L^p norm of the gradient $||\nabla v||_{L^p(\Omega)}$. It remains to prove the last assertion. By using the weak formulation of PDE defined by $u_{f,G_{\varepsilon,l},\Omega}$ we have

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,l},\Omega}|^p dx = \int_\Omega f u_{f,G_{\varepsilon,l},\Omega} dx \leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)}.$$

Recalling the fact that $u_{f,G_{\varepsilon,l},\Omega} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l})$, we get

$$\begin{aligned} ||u_{f,G_{\varepsilon,l},\Omega}||^p_{W_0^{1,p}(\Omega)} &\leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)} \\ &\leq C(\varepsilon) l^{-1} ||u_{f,G_{\varepsilon,l},\Omega}||_{W_0^{1,p}(\Omega)} ||f||_{L^q(\Omega)}, \end{aligned}$$

and the result follows. □

Lemma 6.2.5. *Assume $p \geq 2$. If $f, g \in L^q(\Omega)$ and u_f and u_g denote the respective solution of the p-Laplacian Equation with Dirichlet boundary conditions on Σ_l', then*

$$l^q ||u_f - u_g||_{L^1(\Omega)} \leq C ||f - g||^{1/(p-1)}_{L^q(\Omega)} |\Omega|^{1/q},$$

where the constant C depends only on p. If $\Omega = Q$ (a square centered at x_0), $g = f(x_0)$ and x_0 is a Lebesgue point for f, we have

$$l^q ||u_f - u_g||_{L^1(Q)} \leq C|Q| \left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|} \right)^{1/p} = |Q| r(Q).$$

Proof: The starting point is the inequality

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq C ||u_f - u_g||_{L^p(\Omega)} ||f - g||_{L^q(\Omega)},$$

that comes from the monotonicity inequality

$$|z - w|^p \leq C(|z|^{p-2} z - |w|^{p-2} w) \cdot (z - w),$$

which is valid for any $p \geq 2$ and pair of vectors (z, w) (see equation (5.5)). Thanks to Lemma 6.2.4, we know that the inequality $||v||_{L^p(\Omega)} \leq Cl^{-1}||v||_{W_0^{1,p}(\Omega)}$ is valid for any function vanishing on Σ_l'. Since $u_f - u_g$ vanishes on Σ_l', we have

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq Cl^{-1}||u_f - u_g||_{W_0^{1,p}(\Omega)}||f - g||_{L^q(\Omega)},$$

which implies that

$$||u_f - u_g||_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-p)}||f - g||^{1/(p-1)}_{L^q(\Omega)},$$

and then

$$\begin{aligned} ||u_f - u_g||_{L^1(\Omega)} &\leq C|Q|^{1/q}||u_f - u_g||_{L^p(\Omega)} \\ &\leq C|Q|^{1/q}l^{-1}||u_f - u_g||_{W_0^{1,p}(\Omega)} \\ &\leq C|Q|l^{-q}||f - g||^{1/(p-1)}_{L^q(\Omega)}. \end{aligned}$$

This proves the first assertion; the second one is just a simple consequence. □

Lemma 6.2.6. *Assume $p \leq 2$. If $f, g \in L^q(\Omega)$ and u_f and u_g denote the respective solution of the p-Laplacian Equation with Dirichlet boundary conditions on Σ_l', then*

$$l^q||u_f - u_g||_{L^1(\Omega)} \leq C||f - g||_{L^q(\Omega)}|\Omega|^{1/q}\left(||f||^q_{L^q(\Omega)} + ||||^q_{L^q(\Omega)}\right)^{(2-p)/p},$$

where the constant C depends only on p. If $\Omega = Q$ (a square centered at x_0), $g = f(x_0)$ and x_0 is a Lebesgue point for f, with $f(x_0) \neq 0$, we have

$$l^q||u_f - u_g||_{L^1(Q)} \leq C|Q||f(x_0)|^{(2-p)/(p-1)}\left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|}\right)^{1/q} = |Q|r(Q).$$

Proof: Here the starting point is the inequality

$$||u_f - u_g||^{2p}_{W_0^{1,p}(\Omega)} \leq C||u_f - u_g||^p_{L^p(\Omega)}||f - g||^p_{L^q(\Omega)}\left(||u_f||^p_{W_0^{1,p}(\Omega)} + ||u_g||^p_{W_0^{1,p}(\Omega)}\right)^{(2-p)}, \tag{6.28}$$

that follows from the monotonicity inequality which is valid for any $p \leq 2$ and any pair of vectors (z, w) (see equation (5.7)). Choosing $z = \nabla u_f,\ w = \nabla u_g$ using the weak formulation of the p-Laplacian equation and integrating, we get

$$\int_\Omega |\nabla u_f - \nabla u_g|^p(|\nabla u_f| + |\nabla u_g|)^{p-2}dx \leq \int_\Omega (u_f - u_g)(f - g)dx.$$

The inequality (6.28) is a consequence of a suitable Hölder inequality. We estimate the term $||u_f||^p_{W_0^{1,p}(\Omega)}$. Since $\int_\Omega |\nabla u_f|^p dx = \int_\Omega f u_f dx$ we get

$$||u_f||^p_{W_0^{1,p}(\Omega)} \leq ||u_f||_{L^p(\Omega)}||f||_{L^q(\Omega)} \leq Cl^{-1}||u_f||_{W_0^{1,p}(\Omega)}||f||_{L^q(\Omega)}$$

(we have used the fact that u_f vanishes on Σ_l') and we deduce

$$||u_f||_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-p)}||f||_{L^q(\Omega)}^{1/(p-1)}.$$

A similar estimate holds for $||u_g||_{W_0^{1,p}(\Omega)}$. Using the inequalities just proved and estimate $||u_f - u_g||_{L^p(\Omega)}$ by $||u_f - u_g||_{W_0^{1,p}(\Omega)}$, the inequality (6.28) becomes

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq Cl^{-p/2}||u_f - u_g||^{p/2}_{W_0^{1,p}(\Omega)}||f-g||^{p/2}_{L^q(\Omega)}l^{q(p/2-1)}\left(||f||^p_{L^q(\Omega)} + ||g||_{L^q(\Omega)}\right)^{1-p/2}.$$

This implies, by simplifying and raising to the power $2/p$:

$$||u_f - u_g||_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-p)}||f-g||_{L^q(\Omega)}l^{q(p/2-1)}\left(||f||^p_{L^q(\Omega)} + ||g||_{L^q(\Omega)}\right)^{(2-p)/p}.$$

The estimate on the $L^1(\Omega)$ norm is obtained as usual by passing first to the $L^p(\Omega)$ norm (up to a factor $|\Omega|^{1/q}$) and then to $W_0^{1,p}(\Omega)$ norm (up to a factor l^{-1}):

$$||u_f - u_g||_{L^1}(\Omega) \leq C|\Omega|^{1/q}l^{-q}||f-g||_{L^q(\Omega)}\left(||f||^p_{L^q(\Omega)} + ||g||_{L^q(\Omega)}\right)^{(2-p)/p},$$

which gives the first part of the thesis. For the second part it is sufficient to notice that, if x_0 is a Lebesgue point for f and $g = f(x_0)$, one gets (assuming $f(x_0) \neq 0$)

$$\frac{||f||_{L^q(Q)}}{||g||_{L^q(Q)}} = 1 + r(Q).$$

This allows us to write $|Q|^{1/q}f(x_0)$ instead of $||f||_{L^q(Q)}$, making an error which is negligeable (and of the from $|Q|r(Q)$). The inequality in the second statement follows and the proof is over. □

The Γ-lim sup inequality

To get the Γ-lim sup inequality, we need this crucial lemma.

Lemma 6.2.7. *Given $\Sigma_0 \in \mathcal{A}_{l_0}(I^2)$, a domain $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$, we consider the sequence of sets*

$$\Sigma^k = \bigcup_{y \in k^{-1}\mathbb{Z}^2} (y + k^{-1}\Sigma_0) \cap \overline{\Omega}.$$

We have $\Sigma^k \in \mathcal{A}_{l(k,\Omega)}(\Omega)$, where $l(k,\Omega) \approx |\Omega|kl_0$, Then we consider the sequence $(u_k)_k$ given by

$$u_k = k^q u_{f,\Sigma^k,\Omega}.$$

If we assume $\partial I^2 \subset \Sigma_0$, then we have $u_k \rightharpoonup c(\Sigma_0)f^{1/(p-1)}$ as $k \to \infty$, where the weak convergence is in the $L^p(\Omega)$ sense and $c(\Sigma_0)$ is the constant given by $\int_{I^2} u_{1,\Sigma_0,I^2}dx$.

Proof: First notice that the sequence $(u_k)_k$ is bounded in $L^p(\Omega)$, thanks to Lemma 6.2.4. So up to a subsequence it converges weakly in $L^p(\Omega)$ to some function. Let us consider the subsequence (denoted by the same indices) $(u_k)_k$ and its weak limit $w_{f,\Sigma_0,\Omega}$. It is obvious that the pointwise value of this limit function depends only on the local behavior of f. In fact, we may produce small cubes around each point $x \in \Omega$ which do not affect each other and if $f = \sum_j f_j 1_{A_j}$ is piecewise constant (the pieces A_j being disjoint open sets, for instance), then for k large enough the value of u_k at $x \in A_j$ depends only of f_j (u_k vanishes on $k^{-1}\partial I^2$). From the rescaling property of the p-Laplacian operator Δ_p, if f is a piecewise constant function, it holds $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} w_{1,\Sigma_0,\Omega}$. It is clear that in the case $f = 1$, since we are simply homogenizing the function u_{1,Σ_0,I^2}, the limit of the whole sequence $(u_k)_k$ exists and does not depend on the global geometry of Ω, but it is a constant and it is the same constant if we have I^2 instead of Ω. An easy computation shows that the constant is $c(\Sigma_0)$. It remains to extend the equality for non piecewise constant function belonging to $L^q(\Omega)$. Let $f \in L^q(\Omega)$ be a generic function and $(f_n)_n$ a sequence of piecewise constant functions approaching f in $L^q(\Omega)$. Up to a subsequence it holds $k^q u_{f,\Sigma_k,\Omega} \rightharpoonup w_{f,\Sigma_0,\Omega}$ and $k^q u_{f_n,\Sigma^k,\Omega} \rightharpoonup f_n^{1/(p-1)} c(\Sigma_0)$ as $k \to +\infty$. By Lemma 6.2.4 or Lemma 6.2.5 depending on p it holds also

$$||k^q u_{f,\Sigma^k,\Omega} - k^q u_{f_n,\Sigma^k,\Omega}||_{L^1(\Omega)} \leq R(||f - f_n||_{L^q(\Omega)}),$$

where $R(t) \approx t^{1/(p-1)}$ or $R(t) \approx t$ depending on p. Taking into account the lower semicontinuity of the $L^1(\Omega)$-norm with respect to the $L^p(\Omega)$-weak topology, we get, passing to the limit as $k \to +\infty$,

$$||w_{f,\Sigma_0,\Omega} - f_n^{1/(p-1)} c(\Sigma_0)||_{L^1(\Omega)} \leq R(||f - f_n||_{L^q(\Omega)}).$$

We now pass to the limit as $n \to +\infty$ and using Fatou's Lemma (up to a subsequence f_n converges pointwise a.e. to f), we get $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} c(\Sigma_0)$ and the proof is over.

□

Now we want to build efficient sets Σ_0 satisfying the key assumption of our previous Lemma, that is $\partial I^2 \subset \Sigma_0$ (we will call boundary-covering sets those sets for which such an inclusion holds).

Remark 6.2.8 This is a point where we strongly use the two-dimensional setting we have chosen in this section. In higher dimension, it is not possible to cover all the boundary by the means of a finite length. A strategy to overcome this difficulty is "almost-covering" the boundary of $[0,1]^d$ by means of grid of finite length and then estimating the difference between the solution with the Dirichlet boundary conditions on this grid and on the faces of the cubes. This strategy will be developed in the next section for proving the Γ-limit result in the higher dimension.

Lemma 6.2.9. *For any $\varepsilon > 0$ there exists $l_0 > 0$ such that for any $l > l_0$ we find a set $\Sigma \in \mathcal{A}_l(I^2)$ which is boundary-covering, with*

$$l^q \int_{I^2} u_{1,\Sigma,I^2} dx < (1+\varepsilon)\theta.$$

Proof: Given a small positive number δ , by definition of θ, we may find a set $\Sigma_1 \in \mathcal{A}_{l_1}(I^2)$ such that

$$l_1^q \int_{I^2} u_{1,\Sigma_1,I^2} dx < (1+\delta)\theta$$

and moreover the number l_1 may be chosen as large as we want. Now, we want to enlarge the set Σ_1 to get a set Σ_2 which is boundary-covering. We add to Σ_1 the boundary of I^2 and some segments to connect it to the original set. The new length $l_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed $l_1 + 5$. It is possible to choose l_1 so that

$$\left(\frac{l_1+5}{l_1}\right)^q \leq 1+\delta.$$

This implies that

$$l_2^q \int_{I^2} u_{1,\Sigma_2,I^2} dx < (1+\delta)^2\theta.$$

Now, if we are given a large number l, we homogenize the set Σ_2 into I^2 of order $k = \lfloor l/l_2 \rfloor$ and obtain the set $\Sigma \in \mathcal{A}_{kl_2}(I^2)$. Thanks to the rescaling property of the p-Laplacian operator, we have $(kl_2)^q \int_{I^2} u_{1,\Sigma,I^2} dx = l_2^q \int_{I^2} u_{1,\Sigma_2,I^2} dx$. Therefore

$$l^q \int_{I^2} u_{1,\Sigma,I^2} dx \leq \left(\frac{k+1}{k}\right)^q (1+\delta)^2\theta.$$

If $l > l_2\delta^{-1}$, then $k > \delta^{-1}$ and $1 + 1/k < 1+\delta$, so that we get

$$l^q \int_{I^2} u_{1,\Sigma,I^2} dx \leq (1+\delta)^{2+q}\theta.$$

It is now sufficient to choose δ sufficiently small so that $(1+\delta)^{2+q} < 1+\varepsilon$ and set $l_0 = l_2\delta^{-1}$. $\square$

We have all the ingredients for proving the Γ-$\limsup$ inequality. We will start from a particular class of measures. Let us call piecewise constant probability measures those probability measures $\mu \in \mathcal{P}(\overline{\Omega})$ which are of the form

$$\mu = \rho dx, \text{ with, } \rho \in L^1(\Omega), \int_\Omega \rho dx = 1, \rho > 0,$$

for a piecewise constant function $\rho = \sum_{j=1}^m \rho_j I_{\Omega_j}$, the pieces Ω_j being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \cup_{j=1}^m \Omega_j$.

Proposition 6.2.10. *Under the same hypotheses of Theorem 6.2.2, we have*

$$F^+(\mu) \leq F(\mu), \quad \textit{where} \quad F^+ = \Gamma - \limsup_{l\to+\infty} F_l,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any such a measure μ and $\varepsilon > 0$, there exists a family of sets $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that the measure μ_{Σ_l} weakly* *converges to the measure μ and moreover*

$$\limsup_{l\to+\infty} l^q \int_\Omega f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^q} dx.$$

Proof: Apply Lemma 6.2.9 and take a boundary-covering set $\Sigma_0 \in \mathcal{A}_{l_0}(I^2)$ such that

$$l_0^q \int_{I^2} u_{1,\Sigma_0,I^2} dx < (1+\varepsilon)\theta.$$

Now, we define the set Σ_l^j by homogenizing into Ω_j the set Σ_0 of order $k(l,j)$ that is

$$\Sigma_l^j = \overline{\Omega_j} \cap k(l,j)^{-1}(\mathbb{Z}^d + \Sigma_0).$$

Then we choose $\Sigma_l = \cup_j \Sigma_l^j \cup \cup_j \partial\Omega_j$ and $l_1 = \mathcal{H}^1(\cup_j \partial\Omega)$. The family Σ_l is admissible (i.e. $\Sigma_l \in \mathcal{A}_l(\Omega)$ and $\mu_{\Sigma_l} \rightharpoonup \mu$) if we have, as $l \to \infty$,

$$\sum_{j=0}^{m} |\Omega_j| k(l,j) l_0 + l_1 \le l \quad \text{and is asymptotic to} \quad l;$$

$$\frac{k(l,j)l_0}{l} \to \rho_j \quad \text{for} \quad j = 0, \cdots, m.$$

All theses conditions are satisfied if we set

$$k(l,j) = \left\lfloor \frac{(l-l_1)\rho_j}{l_0} \right\rfloor.$$

We have covered the internal boundary of the sets Ω_j in order to get the local behavior in which different zones Ω_j are independent on each other. We are interested in the estimate of the functional $F_l(\Sigma_l)$ that is

$$l^q \int_\Omega f u_{f,\Sigma_l,\Omega} dx = \sum_{j=0}^{m} \left(\frac{l}{k(l,j)}\right)^q \int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l^j,\Omega_j} dx.$$

The disintegration of the integral performed here allows to apply on each Ω_j Lemma 6.2.7, which gives the weak convergence in L^p

$$k(l,j)^q u_{f,\Sigma_l^j,\Omega_j} \rightharpoonup c(\Sigma_0) f^{1/(p-1)}.$$

The factors $(l/k(l,j))^q$ converge to $(l_0/\rho_j)^q$ as $l \to +\infty$. The choice of the set Σ_0 gives $l_0^q c(\Sigma_0) \le (1+\varepsilon)\theta$, so that we obtain

$$\limsup_{l\to+\infty} l^q \int_{\Omega_j} f u_{f,\Sigma_l^j,\Omega_j} dx \le (1+\varepsilon)\theta\rho_j^{-q} \int_{\Omega_j} f^q dx,$$

and summing up over j, we get

$$\limsup_{l\to+\infty} l^q \int_\Omega f u_{f,\Sigma_l,\Omega} dx \le (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^q} dx.$$

□

We have to extend the result to non piecewise constant measures. By the general theory of Γ-convergence (see Remark 6.1.3), we know that it is enough to prove the Γ-lim sup inequality on a class which is dense in energy. Hence, due to the lower semicontinuity of the functional F, it is sufficient to prove the following

Proposition 6.2.11. *For any measure $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_n)_n$ of piecewise constant measures such that $\mu_n \rightharpoonup \mu$ and*

$$\limsup_n F(\mu_n) \leq F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^q} dx.$$

Proof: First observe that the inequality is trivial whenever $F(\mu) = +\infty$. Assume now that $F(\mu) < +\infty$ and start proving the inequality for measures which are absolutely continuous with respect to the Lebesgue measure and have positive densities bounded away from zero. Given a measure $\mu = \rho dx$, with $\rho \geq c > 0$, it is possible to find a sequence of measures $\mu_n = \rho_n dx$ such $\rho_n \to \rho$ strongly in L^1 and μ_n are piecewise constant with $\rho_n \geq c$. The pointwise $a.e$ convergence of ρ_n to ρ may be assumed and the inequality $F(\mu) \geq \limsup_n F(\mu_n)$ follows easily (we have even an equality). So we have extended the result to any absolutely continuous measure with density bounded below away from zero. To get the result for any measure $\mu \in \mathcal{P}(\overline{\Omega})$, it is sufficient to prove that any measure μ may be approximated weakly* by absolutely continuous measure μ_n with densities bounded below away from zero and $\limsup_n F(\mu_n) \leq F(\mu)$. Let us take $\mu = \rho dx + \mu^s$, where μ^s is the singular part of the measure μ with respect to the Lebesgue measure and ρ the density of the absolutely continuous part. We construct the sequence of absolutely continuous measure μ_n by setting $\mu_n = ((1-1/n)\rho + a_n + \phi_n)dx$, where $a_n = n^{-1}\int_\Omega \rho dx$ and $\phi_n dx \rightharpoonup \mu^s$ with $\int_\Omega \phi_n dx = \int_{\overline{\Omega}} d\mu^s$. The fact that $F(\mu) < +\infty$ implies that ρ cannot vanish, hence $a_n > 0$ and $\rho_n = (1-1/n)\rho + a_n + \phi_n$ is bounded below by the positive constant a_n. We have as well that μ_n weakly* converges to μ and

$$\begin{aligned} F(\mu_n) &= \theta \int_\Omega \frac{f^q}{((1-1/n)\rho + a_n + \phi_n)^q} \leq \theta \int_\Omega \frac{f^q}{((1-1/n)\rho)^q} dx \\ &= (1-\frac{1}{n})^{-q} F(\mu) \end{aligned}.$$

Passing to the lim sup on the inequality, we get the desired result. □

6.2.3 Proof of Γ-limit in $\mathbb{R}^d, d \geq 3$

We will prove the Γ-convergence result in two steps corresponding to Γ-lim inf and Γ-lim sup.

Γ-lim inf **inequality**

Before proving the Γ-lim inf inequality, we need some results and constructions. We start by a construction of a set $G_{\varepsilon,l}$ which will be useful later. Let Ω be a domain, I^d be a unit cube in $\mathbb{R}^d$ and a be a positive real number such that the cube $(-a,a)^d$, that we will denote by I_a^d contains Ω. Let M be a union of d segments of length 1 joining at the center of the unit cube I^d and connecting two parallel faces of the unit cube in the given direction. The segments are made in such a way that their endpoints coincide with the middle points of the faces of I^d. We consider the set $G_{\varepsilon,l}$ to be the homogenization of the set M of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ into I_a^d. It is clear that due to the particularity of the set M, the set $G_{\varepsilon,l}$ is connected and $\mathcal{H}^1(G_{\varepsilon,l}) \approx \varepsilon l$.

Lemma 6.2.12. *1. Let $Q_R \subset \mathbb{R}^d$ be a cube of side R and $A \subset \overline{Q}_R$ a closed subset of Q_R of positive p-capacity, then there exists a constant $C = C(d,p)$ such that, for all functions $v \in C^\infty(\overline{Q}_R)$ with nonnegative mean value and vanishing on A, we have*

$$\int_{Q_R} |v|^p dx \leq \frac{CR^d}{Cap_p(A, Q_{2R})} \int_{Q_R} |\nabla v|^p dx,$$

where $cap_p(A, Q_{2R})$ stands for the relative p-capacity of the set A inside Q_{2R}.

2. *For any $\varepsilon > 0$, any $0 < l < +\infty$, any domain Ω and any function $v \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l}) \subset W_0^{1,p}(\Omega)$ ($G_{\varepsilon,l}$ is the network constructed above) it holds $||v||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{1}{1-d}} ||v||_{W_0^{1,p}(\Omega)}$, where $\varepsilon_0 = cap_p(M, 2I^d)$.*

3. *As a consequence, if we have a nonnegative function $f \in L^q(\Omega)$, then the function $u_{f,G_{\varepsilon,l},\Omega}$ satisfies $||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0) l^{\frac{q}{1-d}} ||f||^{q/(d-1)}_{L^q(\Omega)}$*

Proof: The first assertion is a variant of the well-known Poincaré inequality. See [74] for more comment. For proving the second one, we first choose the function v to be a nonnegative smooth function on a large cube I_a^d which vanish outside $\Omega \setminus G_{\varepsilon,l}$. We consider the subdivision of cube I_a^d into subcubes as done above and consider the associated network $G_{\varepsilon,l}$. The side of subcubes is of order $l^{1/(1-d)}$. Let us denote the subcubes by Q_j. The set $I_a^d \setminus G_{\varepsilon,l}$ can be seen as the homogenized of order $k = \lfloor(\frac{\varepsilon l}{ad})^{1/(d-1)}\rfloor$ of $I^d \setminus M$ into I_a^d (M is the set constructed above). Let us set $\varepsilon_0 = cap_p(M, 2I^d)$ and notice that v vanishes on $G_{\varepsilon,l}$. By applying the first statement of this Lemma, it follows that

$$\int_{Q_j} |v|^p dx \leq \frac{Ck^{-d}}{Cap_p(k^{-1}M, 2Q_j)} \int_{Q_j} |\nabla v|^p dx \leq \frac{Cl^{p/(1-d)}}{Cap_p(M, 2I^d)} \int_{Q_j} |\nabla v|^p dx$$

and by summing up over j we get

$$\int_{I_a^d} |v|^p dx \leq \frac{C}{\varepsilon_0} l^{p/(1-d)} \int_{I_a^d} |\nabla v|^p dx.$$

Using the fact that v vanishes outside Ω, we may restrict the integrand to Ω, raise each term of the inequality to the power $1/p$ and thus getting the result by noticing that the L^p norm of the gradient $||\nabla v||_{L^p(\Omega)}$ stands for the norm $||v||_{W_0^{1,p}(\Omega)}$. The general case follows by density. For the last inequality, we use the weak version of the PDE which gives

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,l},\Omega}|^p dx = \int_\Omega f u_{f,G_{\varepsilon,l},\Omega} dx \leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)}.$$

Since $u_{f,G_{\varepsilon,l},\Omega} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l})$ we get

$$\begin{aligned}||u_{f,G_{\varepsilon,l},\Omega}||^p_{W_0^{1,p}(\Omega)} &\le ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)}||f||_{L^q(\Omega)} \\ &\le C(d,\varepsilon_0,\varepsilon)l^{1/(1-d)}||u_{f,G_{\varepsilon,l},\Omega}||_{W_0^{1,p}(\Omega)}||f||_{L^q(\Omega)},\end{aligned}$$

and the desired result follows. □

Before proving the Γ-lim inf inequality, we need the following estimate which will be helpful. It is the equivalent of Lemma 6.2.5

Lemma 6.2.13. *Let $f, g \in L^q(\Omega)$ be given and u_f and u_g denote the solution of p-Laplacian equation with respective right hand side f, g and with Dirichlet boundary condition on $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ (where Σ_l is an element of $\mathcal{A}_l(\Omega)$ and $G_{\varepsilon,l}$ the above constructed network), then*

$$l^{q/(d-1)}||u_f - u_g||_{L^1(\Omega)} \le C|\Omega|^{1/q}||f-g||^{1/(d-1)}_{L^q(\Omega)},$$

where $C = C(d,p,\varepsilon_0,\varepsilon)$. In particular, if $\Omega = Q$ a cube centered at x_0, $g = f(x_0)$ and x_0 is a Lebesgue point for f, then

$$l^{q/(d-1)}||u_f - u_g||_{L^1(Q)} \le C|Q|\left(\frac{\int_Q |f(x)-f(x_0)|^q dx}{|Q|}\right)^{1/p} = |Q|r(Q).$$

Proof: $p > d-1 \ge 2$ and from monotonicity formulas (see equation (5.5)), it follows that

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \le C||u_f - u_g||_{L^p(\Omega)}||f-g||_{L^q(\Omega)},$$

where we have used $z = \nabla u_f$ and $w = \nabla u_g$. From Lemma 6.2.12, we have the inequality $||v||_{L^p(\Omega)} \le Cl^{1/(1-d)}||v||_{W_0^{1,p}(\Omega)}$ which holds for every function v vanishing on Σ_l'. Since the function $u_f - u_g$ vanishes on Σ_l', we have

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \le Cl^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)}||f-g||_{L^q(\Omega)},$$

which gives

$$||u_f - u_g||_{W_0^{1,p}(\Omega)} \le Cl^{1/(1-d)(p-1)}||f-g||^{1/(p-1)}_{L^q(\Omega)},$$

and using Hölder inequality, we get

$$\begin{aligned}||u_f - u_g||_{L^1(\Omega)} &\le |\Omega|^{1/q}||u_f - u_g||_{L^p(\Omega)} \\ &\le C|\Omega|^{1/q}l^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)} \\ &\le C|\Omega|^{1/q}l^{q/(1-d)}||f-g||^{1/(p-1)}_{L^q(\Omega)},\end{aligned}$$

and the first part of the statement follows. The second part is an obvious consequence of the first part.

□

In the following proposition, we prove that the Γ-lim inf functional is bounded below by the candidate limit functional F in (6.23).

Proposition 6.2.14. *Under the same hypotheses of Theorem 6.2.2, denoting by F^- the functional Γ-$\liminf_l F_l$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any sequence $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that μ_{Σ_l} weakly* converges to μ, we have*

$$\liminf_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \geq F(\mu).$$

Proof: Let $\Sigma'_l = \Sigma_l \cup G_{\varepsilon,l}$ and set $u'_l = u_{f,\Sigma'_l,\Omega}$. Since $u_l \geq u'_l$, it is enough to estimate the integral $l^{\frac{q}{d-1}} \int_\Omega f u'_l$. It is obvious that $0 \leq u'_l \leq u_{f,G_{\varepsilon,l},\Omega}$ and Lemma 6.2.12 gives

$$||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d, \varepsilon_0, \varepsilon, f) l^{\frac{q}{1-d}}.$$

It follows that $l^{\frac{q}{d-1}} u'_l$ is L^p bounded, so up to a subsequence $l^{\frac{q}{d-1}} u'_l \rightharpoonup w$ weakly in $L^p(\Omega)$. Thus

$$\lim_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega g u'_l dx = \int_\Omega g w dx, \quad \forall g \in L^q(\Omega).$$

So it is enough to estimate w from below. We will show that, for almost any $x_0 \in \Omega$, it holds

$$w(x_0) \geq \frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a + \varepsilon)^{\frac{q}{d-1}}}. \tag{6.29}$$

As in the two-dimensional case, we first estimate w on a cube Q centered at the point $x_0 \in \Omega$. We assume that x_0 is a Lebesgue point for f and $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as Q shrinks around x_0. Assume also $f(x_0) > 0$ otherwise (6.29) would be trivial. We have

$$\lim_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega u'_l dx = \int_\Omega w dx,$$

we use

$$u'_l \geq u_{f,\Sigma'_l,Q} \geq u_{f(x_0),\Sigma'_l,Q} - |u_{f,\Sigma'_l,Q} - u_{f(x_0),\Sigma'_l,Q}| \text{ in } Q,$$

where the first inequality comes from the fact that we add Dirichlet boundary condition on Q. The second part of Lemma 6.2.13 gives

$$\int_Q |u_{f,\Sigma'_l,Q} - u_{f(x_0),\Sigma'_l,Q}| dx \leq l^{\frac{q}{1-d}} |Q| r(Q).$$

It remains to estimate the second term. First of all let us define the number $L(l,Q) = \mathcal{H}^1(\Sigma'_l \cap Q)$ and observe that

$$u_{f(x_0),\Sigma'_l,Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma'_l,Q}.$$

For simplicity of the notation, we denote $u_{1,\Sigma'_l,Q}$ by v_l. By a change of variables, if we assume the side of cube Q to be λ and we define $v_{l,\lambda} = \lambda^q v_l(\lambda x)$ (thinking for instance that both cubes are centered at the origin), we get $v_{l,\lambda} = u_{1,\lambda^{-1}\Sigma'_l,I^d}$. It is easy to see that

$$\lambda^{-1}\Sigma'_l \in \mathcal{A}_{L(l,Q)/\lambda}(I^d);$$

moreover, it holds $L(l,Q)\to+\infty$ as $l\to+\infty$, since

$$L(l,Q)\geq\mathcal{H}^1(G_{\varepsilon,l}\cap Q)\approx\varepsilon l|Q|. \tag{6.30}$$

Using (6.30) and the fact that $\mu_l=l^{-1}\mathcal{H}^1(\Sigma_l)$, we may estimate the ratio between $L(l,Q)$ and l. It follows from the weak* convergence of μ_l to μ that $\limsup_{l\to+\infty}\mu_l(Q)\leq\mu(\overline{Q})$. So we have

$$\limsup_{l\to+\infty}\frac{L(l,Q)}{l}\leq\mu(\overline{Q})+\varepsilon|Q|. \tag{6.31}$$

Using the definition of θ and the change of variables $y=\lambda x$ we have,

$$\begin{aligned}\liminf_{l\to+\infty}L(l,Q)^{\frac{q}{d-1}}\int_Q v_l(y)dy&=\liminf_{l\to+\infty}L(l,Q)^{\frac{q}{d-1}}\lambda^d\int_{I^d}v_{l,\lambda}(x)dx\\&=\liminf_{l\to+\infty}\left(\lambda^{-1}L(l,Q)\right)^{\frac{q}{d-1}}\lambda^{d+q+\frac{q}{d-1}}\int_{I^d}v_{l,\lambda}(x)dx^{\cdot}\\&\geq\lambda^{d+q+\frac{q}{d-1}}\theta\end{aligned}$$

hence using the fact that $\lambda^d=|Q|$ we get

$$\begin{aligned}\liminf_{l\to+\infty}l^{\frac{q}{d-1}}\int_Q v_l(y)dy&\geq\liminf_{l\to+\infty}\left(\frac{l}{L(l,Q)}\right)^{\frac{q}{d-1}}\liminf_{l\to+\infty}L(l,Q)^{\frac{q}{d-1}}\int_Q v_l(y)dy\\&\geq\lambda^{d+q+\frac{q}{d-1}}\theta\left(\frac{1}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}}\\&=\left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}}|Q|\theta.\end{aligned}$$

This implies that

$$|Q|^{-1}\int_Q wdx\geq-r(Q)+\left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}}\theta f(x_0)^{1/(p-1)}.$$

We know that $r(Q)$ tends to 0 when the cube Q shrinks to x_0, whenever x_0 is a Lebesgue point for f. Now we let the cube Q shrinks toward x_0 with x_0 satisfying the previous assumption, then we get

$$w(x_0)\geq\frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a(x_0)+\varepsilon)^{\frac{q}{d-1}}}.$$

It follows that

$$\liminf_{l\to+\infty}l^{\frac{q}{d-1}}\int_\Omega fu_l dx\geq\int_\Omega fwdx\geq\theta\int_\Omega\frac{f^q}{(\mu_a+\varepsilon)^{\frac{q}{d-1}}}dx,$$

and the desired inequality holds by letting ε tend to 0 that is

$$\liminf_{l\to+\infty}l^{\frac{q}{d-1}}\int_\Omega fu_l dx\geq\theta\int_\Omega\frac{f^q}{\mu_a^{\frac{q}{d-1}}}dx.$$

□

Γ-lim sup **inequality**

Before proving the Γ-lim sup inequality we introduce one definition and prove some preliminaries results. We start by the definition of tiling set.

Definition 6.2.15. *A set $\Sigma \in \mathcal{A}_l(I^d)$ is called tiling set if $\Sigma \cap \partial I^d$ coincides with the 2^d vertices of I^d.*

Remark 6.2.16 If $\Sigma \in \mathcal{A}_l(I^d)$ is tiling set and Σ_k is the homogenization of order k of Σ into I^d, then Σ_k remains connected and

$$\mathcal{H}^1(\Sigma_k) = k^{d-1}\mathcal{H}^1(\Sigma).$$

Lemma 6.2.17. *Given $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ a tiling set, a domain $\Omega \subset \mathbb{R}^d$ and $f \in L^q(\Omega)$, we consider the sequence of sets*

$$\Sigma^k = \bigcup_{y \in k^{-1}\mathbb{Z}^d} (y + k^{-1}\Sigma_0 \cup \partial I^d) \cap \overline{\Omega}$$

and consider the sequence of functions $(u_k)_k$ given by

$$u_k = k^q u_{f,\Sigma^k,\Omega},$$

then $u_k \rightharpoonup c(\Sigma_0) f^{1/(p-1)}$ in $L^p(\Omega)$ as $k \to +\infty$, where $c(\Sigma_0)$ is a constant given by $\int_\Omega u_{1,\Sigma_0,I^d} dx$.

Proof: The proof is the same as the proof of Lemma 6.2.7 $\square$

Remark 6.2.18 This result remains true even if Σ_0 is not tiling. In fact in the proof we do not need to use the fact that Σ_0 is tiling . We keep it for the up coming construction. One problem in the previous Lemma is that we have used the whole boundary of the unit cube which is not an one dimensional set (since $d \geq 3$) and consequently the set Σ^k is not an one dimensional set. In the following Lemma, we prove an estimate on an unit cube which will be useful for proving that $u_{f,\Sigma^k,\Omega}$ may be approximated by $u_{f,\Sigma^k_l,\Omega}$ where Σ^k_l is an one dimensional closed and connected set.

Lemma 6.2.19. *Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set, then for every $\alpha > 0$ there exists $T^\alpha_l \in \mathcal{A}_l(I^d)$ such that if we denote by $u_l = u_{f,\Sigma \cup T^\alpha_l, I^d}$ and v_l the solution of the equation*

$$\begin{cases} -\Delta_p u &= f \ \text{in } I^d \setminus \Sigma \cup T^\alpha_l \\ u &= 0 \ \text{in } \Sigma \cup T^\alpha_l, \end{cases}$$

then $v_l \leq u_l + \alpha^{-q} C l^{\frac{q}{1-d}}$ on I^d where C is a constant independent of l and α.

Proof: The boundary of the d-dimensional unit cube is a union of $d-1$-dimensional unit cubes. On each $d-1$-dimensional unit cube, let us consider a type of set $G_{\alpha,l}$ previously constructed by choosing $k = \lfloor \frac{\alpha l^{1/d-1}}{d} \rfloor$ (in the this case the set M is constructed in the same way but as subset of I^{d-1} instead of I^d) and then take the union of all these sets. It is clear that it is a closed connected one dimensional subset of ∂I^d and his length is approximatively $(\alpha l^{\frac{1}{d-1}})^{d-2}$. To this set, we add another one which is the homogenization of order $k = \lfloor \frac{\alpha l^{1/(d-1)^2}}{d} \rfloor$ of the unit segment passing by the center and joining two opposite face of the unit cube into the unit cube. This set whose length is approximatively $(\alpha l^{\frac{1}{d-1}})^{d-2}$ is not connected but his union with the $G_{\alpha,l}$ is connected. Denoting this union by T_l^α, we observe that it is a closed connected one dimensional set and $\mathcal{H}^1(T_l^\alpha) \approx (\alpha l^{\frac{1}{d-1}})^{d-2}$. We may assume $\Sigma_l \cup T_l^\alpha$ connected since it suffice to add some segment to connect them. Adapting the proof of Lemma 6.2.12 by writing more precisely inequalities and replacing ε by α, we get $||v_l||_{L^p(I^d)} \leq C\alpha^{-q} l^{\frac{q}{1-d}}$ and $||u_l||_{L^p(I^d)} \leq C\alpha^{-q} l^{\frac{q}{1-d}}$ where $C = C(p, f, d)$ and v_l and u_l are functions of the statement of the Lemma. From the maximum principle we get $v_l - u_l \geq 0$ and from the above boundedness and Hölder inequality it holds

$$0 \leq \int_{I^d} (v_l - u_l) dx \leq C\alpha^{-q} l^{\frac{q}{1-d}}.$$

We obtain easily the existence of some constant $C = C(p, f, d)$ (it may be different from the above constant C) such that the inequality

$$v_l - u_l \leq C\alpha^{-q} l^{\frac{q}{1-d}}$$

holds in I^d and the proof is over. $\square$

The sets satisfying the hypothesis of the Lemma 6.2.19 will be called almost boundary-covering sets. Now we built an almost boundary-covering set that will be used for the construction of the recovering sequence for the Γ-lim sup inequality.

Lemma 6.2.20. *For any $\varepsilon > 0$, there exists $l_0 > 0$ such that for all $l > l_0$ we find a set $\Sigma \in \mathcal{A}_l(I^d)$ which is almost boundary-covering, with*

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx < (1+\varepsilon)\theta$$

and consequently if we denote by $u_{1,\Sigma}$ the solution of the same equation which vanish only on Σ and not on whole the boundary of I^d we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma} dx < (1+\varepsilon)\theta + C\alpha^{-q}.$$

Proof: Given a small positive number δ $(0 < \delta \ll 1)$, by definition of θ, we may find a set $\Sigma_1 \in \mathcal{A}_{l_1}(I^d)$ such that

$$l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx < (1+\delta)\theta$$

and moreover the number l_1 may be chosen as large as we want. Now, we want to enlarge the set Σ_1 to get a set Σ_2 which is almost boundary-covering. Let $\gamma = \bigcup_{j=1}^{2^d} S_j$ where S_j is the shortest segment joining Σ_1 to the j^{th} vertex of I^d cube. We set $\Sigma_2 = \Sigma_1 \cup T^{\alpha}_{l_1} \cup \gamma$ where $T^{\alpha}_{l_1}$ is the set T^{α}_{l} previously constructed with l replaced by l_1. Up to adding one segment, we may assume Σ_2 connected. The length $l_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed the number $l_1 + (\alpha l_1^{\frac{1}{d-1}})^{d-2} + (2^d+1)\sqrt{d}$. It is possible to chose l_1 so that

$$\left(\frac{l_1 + (\alpha l_1^{\frac{1}{d-1}})^{d-2} + (2^d+1)\sqrt{d}}{l_1}\right)^{\frac{q}{d-1}} \leq 1+\delta.$$

This implies

$$l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d}dx \leq \left(\frac{l_2}{l_1}\right)^{\frac{q}{d-1}} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d}dx \leq (1+\delta)^2\theta.$$

Now if we are given a large number l, we homogenize the set Σ_2 of order $k = \lfloor \left(\frac{l}{l_2}\right)^{\frac{1}{d-1}} \rfloor$ into I^d and the homogenized set Σ belongs to $\mathcal{A}_{k^{d-1}l_2}(I^d)$ and is still almost boundary-covering. For this set Σ it holds (using the rescaling property)

$$(k^{d-1}l_2)^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d}dx = l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d}dx.$$

Noticing that $l^{\frac{q}{d-1}} \leq \left(\frac{k+1}{k}\right)^q \left(k^{d-1}l_2\right)^{\frac{q}{d-1}}$, we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d}dx \leq \left(\frac{k+1}{k}\right)^q (1+\delta)^2\theta.$$

If $l > l_2\delta^{-1}$, using the fact that $\delta \ll 1$, an easy computation shows that $1+1/k < 1+\delta$ so that we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d}dx \leq (1+\delta)^{2+q}\theta.$$

Now it is sufficient to choose δ so small that $(1+\delta)^{2+q} < 1+\varepsilon$, choose $l_0 = l_2\delta^{-1}$ and the result follows. □

We have all the ingredients for proving the Γ-lim sup inequality. We will start from a class of piecewise constant measures (see the definition of piecewise constant measures just before Proposition 6.2.10 and we will keep the same notation) $\mu = \rho dx$ with $\rho = \sum_{j=1}^m \rho_j I_{\Omega_j}$.

Proposition 6.2.21. *Under the same hypotheses of Theorem 6.2.2, we have*

$$F^+(\mu) \leq F(\mu), \quad \textit{where} \quad F^+ = \Gamma - \limsup_{l\to+\infty} F_l,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any such a measure μ and $\varepsilon > 0$, there exists a family of sets $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that the measure μ_{Σ_l} weakly converges to the measure μ and moreover*

$$\limsup_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

Proof: Apply Lemma 6.2.20 and take an almost boundary-covering set $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ such that

$$l_0^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_0,I^d} dx < (1+\varepsilon)\theta.$$

Now, we define the set Σ_l^j by homogenizing into Ω_j the set Σ_0 of order $k(l,j)$ that is

$$\Sigma_l^j = \overline{\Omega_j} \cap k(l,j)^{-1}(\mathbb{Z}^d + \Sigma_0).$$

Since Σ_0 is tiling , for $k(l,j)$ large enough Σ_l^j remains connected and

$$\mathcal{H}^1(\Sigma_l^j) = |\Omega_j| K(l,j)^{d-1} \mathcal{H}^1(\Sigma_0) \leq |\Omega_j| K(l,j)^{d-1} l_0.$$

Let $\Sigma_{l_1} \in \mathcal{A}_{l_1}(\Omega)$ be a set contained in the internal boundary of the union of Ω_j and converges to it in the Hausdorff topology as $l_1 \to +\infty$ (Σ_{l_1} may obtained by homogenizing some kind of grid contained in ∂I^d of some order into $\cup_{j=0}^m \partial\Omega_j$). Due to the connectedness of Σ_{l_1}, the corresponding solution converges to the solution associated to the internal boundary of $\cup_{j=0}^m \Omega_j$ as well. Then we choose $\Sigma_l = \cup_{j=0}^m \Sigma_l^j \cup \Sigma_{l_1}$. We may assume Σ_l connected otherwise we add some segments to connect all the pieces. The family of sets Σ_l is admissible (i.e. $\Sigma_l \in \mathcal{A}_l(\Omega)$ and $\mu_{\Sigma_l} \rightharpoonup \mu$) if we have, as $l \to +\infty$,

$$\sum_{j=0}^m |\Omega_j| k(l,j)^{d-1} l_0 + l_1 \leq l \quad \text{and is asymptotic to} \quad l;$$

$$\frac{k(l,j)^{d-1} l_0}{l} \to \rho_j \quad \text{for} \quad j = 0, \cdots, m.$$

It is easy to see that all theses conditions are satisfied if we set

$$k(l,j) = \left\lfloor \left(\frac{l - l_1}{l_0} \rho_j \right)^{\frac{1}{d-1}} \right\rfloor.$$

Let us introduce the following sets

$$\Gamma_l^j = \overline{\Omega}_j \cap k(l,j)^{-1}(\mathbb{Z}^d + \partial I^d), \quad \Gamma_l = \bigcup_j \Gamma_l^j.$$

Thanks to Lemma 6.2.19 we have

$$\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l^j,\Omega_j} dx \leq \int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l \cup \Gamma_l^j,\Omega_j} dx + C\alpha^{-q} l_0^{\frac{q}{1-d}}.$$

In fact, we consider subcubes $Q_{k(l,j)}$ which are obtained by the partition of Ω_j made by Γ_l^j, then in each subcube $Q_{k(l,j)}$, the Lemma 6.2.19 gives

$$u_{f,\Sigma_l^j} \le u_{f,\Sigma_l^j,Q_{k(l,j)}} + C\alpha^{-q}(k(l,j)l_0^{\frac{1}{d-1}})^{-q}.$$

By multiplying this inequality by f (notice that $f \ge 0$), Integrating over $Q_{k(l,j)}$ and summing up, we get

$$\int_{\Omega_j} f u_{f,\Sigma_l^j,\Omega_j} dx \le \int_{\Omega_j} f u_{f,\Sigma_l^j} dx \le \int_{\Omega_j} f u_{f,\Sigma_l^j \cup \Gamma_l^j,\Omega_j} dx + C\alpha^{-q}(k(l,j)l_0^{\frac{1}{d-1}})^{-q}$$

where the first inequality comes from the maximum principle and the second is obtained by observing that on each cube $Q_{k(l,j)}$ it holds $u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} = u_{f,\Sigma_l^j,Q_{k(l,j)}}$

We choose l_1 to be a function of l (for example $l_1 = l^{\frac{d-1}{d}}$) in such a way that l_1 goes to $+\infty$ whenever l goes to $+\infty$. We are interested in the estimate of the value of $F_l(\Sigma_l)$

$$\begin{aligned}
l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx &= \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l,\Omega} dx \\
&\le \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l,\Omega_j} dx + c(l_1)\right) \\
&\le \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} f k(l,j)^q u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} dx + c(l_1) + C\alpha^{-q} l_0^{\frac{q}{1-d}}\right)
\end{aligned}$$

where $c(l_1)$ goes to zero as l_1 tend to infinity. By applying Lemma 6.2.17 to each Ω_j we get the following weak convergence in L^p.

$$k(l,j)^q u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} \rightharpoonup c(\Sigma_0) f^{1/(p-1)} \quad \text{as} \quad l \to +\infty$$

and the term $\left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}}$ converges to $\left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}}$ as $l \to +\infty$ for $j = 0, \cdots, m$. The choice of the set Σ_0 implies that $l_0^{\frac{q}{d-1}} c(\Sigma_0) < (1+\varepsilon)\theta$, so we have

$$\limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_{\Omega_j} f u_{f,\Sigma_l,\Omega} dx \le (1+\varepsilon)\theta \rho_j^{\frac{q}{d-1}} \int_{\Omega_j} f^q dx + C\alpha^{-q}, \text{ for } j = 0, \cdots, m$$

and summing up and using the fact that $\alpha^{-q} \to 0$ as $\alpha \to +\infty$, we get

$$\limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \le (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

□

Extending this result to a non piecewise constant measures is just a consequence of the general theory of Γ-convergence stating that it is sufficient to verify the lim sup inequality in a class which is dense in energy. Hence, to conclude, we need only this result that may be proved in the same way as the Proposition 6.2.11

Proposition 6.2.22. *For any measure $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_n)_n$ of piecewise constant measures such that $\mu_n \rightharpoonup \mu$ and*

$$\lim_n F(\mu_n) = F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx.$$

6.2.4 Some estimate on θ

In this section we will prove some estimate on the constant $\theta(p)$ and in particular we will show that $\theta(p)$ is neither 0 nor $+\infty$ so that our limit functional is not trivial.

Proposition 6.2.23. *We have*

$$\theta(p) < +\infty \quad \forall p > d-1.$$

Proof: Let $\Sigma_l \in \mathcal{A}_l(I^d)$ be a tiling set. For any positive integer number n, let us denote by Σ_l^n the homogenization of the set Σ_l of order n into I^d. Clearly, Σ_l^n is connected and $\mathcal{H}^1(\Sigma_l^n) = n^{d-1}l$. Using the rescaling property of the p-Laplacian operator, it follows that

$$\theta(p) \leq \liminf_n (n^{d-1}l)^{\frac{q}{d-1}} F_p(\Sigma_l^n, 1, I^d) = l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) < +\infty$$

which concludes the proof. □

For proving the lower bound, we need some preliminary results. We start by this lemma which is a consequence of the proof of Theorem 4.4.5 in [11].

Lemma 6.2.24. *Given a closed connected set $\Sigma \in \mathbb{R}^d$ with $\mathcal{H}^1(\Sigma) < +\infty$, there exists a sequence of connected sets Σ_j such that each Σ_j is a union of finite number of segments, $\mathcal{H}^1(\Sigma_j) \leq \mathcal{H}^1(\Sigma)$ and $\Sigma_j \to \Sigma$ in the Hausdorff distance.*

Lemma 6.2.25. *Let Σ be a closed connected subset of $\mathbb{R}^d$ with $\mathcal{H}^1(\Sigma) < +\infty$. Then*

$$|\{x \in \mathbb{R}^d : h_\Sigma(x) \leq t\}| \leq \mathcal{H}^1(\Sigma)\omega_{d-1}t^{d-1} + \omega_d t^d, \tag{6.32}$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$ and $h_\Sigma(x) = d(x, \Sigma)$ is the distance from x to the compact Σ.

Proof: For every $E \subset \mathbb{R}^d$, set $A_t(E) = \{x \in \mathbb{R}^d : h_\Sigma(x) < t\}$. We first suppose that $\Sigma = \bigcup_{i=1}^m s_i$ where each s_i is a segment. Let $\Sigma^j = \bigcup_{i=1}^j s_i$. Since Σ is connected, we may suppose that $s_{j+1} \cap \Sigma^j \neq \emptyset$ for $j < m$. For a single segment s,

$$|A_t(s)| = \mathcal{H}^1(s)\omega_{d-1}t^{d-1} + \omega_d t^d, \tag{6.33}$$

and hence the claim of the lemma is true for $m = 1$. Now suppose that

$$|A_t(\Sigma^j)| \le \mathcal{H}^1(\Sigma^j)\omega_{d-1}t^{d-1} + \omega_d t^d, \tag{6.34}$$

for some $j < m$, and let us prove the same estimate with $j+1$ in the place of j. Using (6.33) and (6.34) we have

$$\begin{aligned}
|A_t(\Sigma^{j+1})| &= |A_t(\Sigma^j \cup s_{j+1})| = |A_t(\Sigma^j) \cup A_t(s_{j+1})| \\
&= |A_t(\Sigma^j)| + |A_t(s_{j+1})| - |A_t(\Sigma^j) \cap A_t(s_{j+1})| \\
&\le (\mathcal{H}^1(\Sigma^j) + \mathcal{H}^1(s_{j+1}))\omega_{d-1}t^{d-1} + 2\omega_d t^d - |A_t(\Sigma^j) \cap A_t(s_{j+1})|.
\end{aligned}$$

Now it suffice to observe that, since $\Sigma^j \cap s_{j+1} \neq \emptyset$, then $A_t(\Sigma^j) \cap A_t(s_{j+1})$ contains a ball of radius t. Therefore the claim follows by induction on m. The general case follows from Lemma 6.2.24, approximating Σ by union of segments in the Hausdorff distance (which implies the uniform convergence of the corresponding distance functions), and observing that the functional $|A_t(\Sigma)|$ is lower semicontinuous in this topology (see [9], Prop. 2.1.). $\square$

Lemma 6.2.26. *Let $\Sigma \in \mathcal{A}_l(I^d)$ and $r > 0$, then*

$$\liminf_{l\to+\infty} l^{\frac{r}{d-1}} \int_{I^d} (h_\Sigma(x))^r dx \ge \frac{d-1}{(r+d-1)\omega_{d-1}^{\frac{r}{d-1}}}.$$

Moreover as far as $d = 2$ it holds

$$\inf\left\{\liminf_{l\to+\infty} l^r \int_{I^d} (h_\Sigma(x))^r dx : \Sigma \in \mathcal{A}_l(I^2)\right\} = \frac{1}{2^r(r+1)}.$$

Proof: Let A_t denotes the set of points $x \in \mathbb{R}^d$ such that $h_\Sigma(x) < t$. By Lemma 6.2.25,

$$|A_t \cap I^d| \le l\omega_{d-1}t^{d-1}\left(1 + \frac{t\omega_d}{l\omega_{d-1}}\right) \le l\omega_{d-1}t^{d-1}\left(1 + \frac{\sqrt{d}\omega_d}{l\omega_{d-1}}\right), \ \ t \in (0, \sqrt{d})$$

and hence raising to the power $r/(d-1)$, we get

$$|A_t \cap I^d|^{\frac{r}{d-1}} \le (l\omega_{d-1})^{\frac{r}{d-1}} t^r \left(1 + \frac{K}{l}\right)^{\frac{r}{d-1}}, \ \ t \in (0, \sqrt{d}) \tag{6.35}$$

where K is a constant depending only on p, d. Now using $|\nabla h_\Sigma| = 1$ a.e. on I^d and the coarea formulas, we have

$$|A_t \cap I^d| = \int_0^t P_s ds, \ \ \int_{A_t \cap I^d} (h_\Sigma(x))^r dx = \int_0^t s^r P_s ds, \ \ t > 0$$

where P_s is the perimeter of A_s in I^d, hence

$$\frac{d}{dt}|A_t \cap I^d| = P_t, \quad \frac{d}{dt}\int_{A_t\cap I^d}(h_\Sigma(x))^r dx = t^r P_t, \;\; t > 0.$$

Therefore, multiplying (6.35) by P_t we obtain that

$$\frac{d}{dt}|A_t \cap I^d|^{\frac{r+d-1}{d-1}} \leq (l\omega_{d-1})^{\frac{r}{d-1}}\left(1+\frac{K}{l}\right)^{\frac{r}{d-1}}\frac{d}{dt}\int_{A_t\cap I^d}(h_\Sigma(x))^r dx,$$

for every $t \in (0, \sqrt{d})$. Since $\sup_{I^d} h_\Sigma \leq \mathrm{diam} I^d = \sqrt{d}$, by integrating the last inequality over $(0, \sqrt{d})$, we get

$$1 = |I^d| \leq \frac{r+d-1}{d-1}(l\omega_{d-1})^{\frac{r}{d-1}}\left(1+\frac{K}{l}\right)^{\frac{r}{d-1}}\int_{I^d}(h_\Sigma(x))^r dx,$$

and passing to the lim inf as $l \to +\infty$ in the inequality, the desired result follows. For the two dimensional situation, since the above result holds for every $\Sigma \in \mathcal{A}_l(I^d)$, it follows that

$$\inf\left\{\liminf_{l\to+\infty} l^r \int_{I^d}(h_\Sigma(x))^r dx : \Sigma \in \mathcal{A}_l(I^2)\right\} \geq \frac{1}{(r+1)\omega_1^r} = \frac{1}{2^r(r+1)}.$$

it remains to prove the reverse inequality. Let S_n be the subset of the closed unit square in $\mathbb{R}^2$ made of $n+1$ equi-spaced vertical segments of unit length, and let $\Sigma_n = S_n \cup B$ where B is the base of the square. Clearly, Σ_n is connected and $\mathcal{H}^1(\Sigma_n) = n+2$. Moreover

$$\int_{I^d}(h_{\Sigma_n}(x))^r dx \leq \int_{I^d}(h_{S_n}(x))^r dx = 2n\int_0^{\frac{1}{2n}} t^r dt = \frac{1}{(r+1)(2n)^r}.$$

Therefore

$$\liminf_{n\to+\infty}\mathcal{H}^1(\Sigma_n)^r\int_{I^d}(h_{\Sigma_n}(x))^r dx \leq \liminf_{n\to+\infty}\frac{(n+2)^r}{(r+1)(2n)^r} = \frac{1}{2^r(r+1)}.$$

This proves the opposite inequality. $\square$

Proposition 6.2.27.

$$\theta(p) \geq \frac{(d-1)q^{-q}}{(q+d-1)\omega_{d-1}^{\frac{q}{d-1}}},$$

where ω_r stands for the volume of unit ball in $\mathbb{R}^r$.

Proof: First, we prove that

$$F_p(\Sigma_l, 1, I^d) \geq q^{-q} D_q(\Sigma_l \cup \partial I^d),$$

where $D_r(\Sigma) = \int_{I^d}(h_\Sigma(x))^r dx$. For every real number A and for every real number $r > 1$, we have

$$F_p(\Sigma_l, 1, I^d) = q \max\left\{\int_{I^d}(v - \frac{1}{p}|\nabla v|^p)dx : v \in W_0^{1,p}(I^d \setminus \Sigma_l)\right\}$$
$$\geq q\int_{I^d}(A(h(x))^r - \frac{1}{p}|\nabla(A(h(x))^r)|^p)dx,$$

where h is a distance function given by $h(x) = d(x, \Sigma_l \cup \partial I^d)$ and satisfied $|\nabla h| = 1$ (and consequently $|\nabla h^r| = rh^{r-1}$). Choosing $r = q$ the conjugate exponent of p, we get

$$F_p(\Sigma_l, 1, I^d) \geq q(A - A^q\left(\frac{q^p}{p}\right))\int_{I^d} h^q dx.$$

The result follows by optimizing on A (the optimal choice is $A = q^{-q}$). In the Lemma 6.2.26 we have proved that for any set $\Sigma_l \in \mathcal{A}_l(I^d)$ it holds

$$\liminf_l l^{\frac{q}{d-1}}\int_{I^d}(h_\Sigma(x))^q dx \geq \frac{d-1}{(q+d-1)\omega_{d-1}^{\frac{q}{d-1}}}.$$

Here, in the case $d = 2$, ∂I^2 is a one dimensional set then we apply Lemma 6.2.26 to $\Sigma \cup \partial I^2$ (adding one segment to make it connected if necessary) and getting the lower bound. For the case where $d \geq 3$ the same proof may be adapted by doing some modification and obtaining the same result even if $\Sigma_l \cup \partial I^d$ is not an one dimensional set i.e.

$$\liminf_l l^{\frac{q}{d-1}}\int_{I^d} h(x)^q dx \geq \frac{d-1}{(q+d-1)\omega_{d-1}^{\frac{q}{d-1}}},$$

and the desired result holds. □

6.2.5 Average distance as limit as $p \to \infty$

In this section our general presentation of the problem for any $p > d-1$ is exploited to let $p \to +\infty$: this allows us to compare it to some average distance problems. In some sense, the limit of these problems as $p \to +\infty$ correspond to the minimization of the functional D_1 introduced in the previous section. The goal of this section is to complete the previous results by showing a commutative Γ-convergence diagram: if we fix p and let the length constraint tends to $+\infty$ we get a limit depending p, given by (6.23). We want to show that both at finite level of fixed l and at the asymptotic level of the limit functional, we have Γ-convergence as $p \to$ to the corresponding functional arising in the average distance theory. The following Lemma is well-known.

Lemma 6.2.28. *Let Ω be a fixed domain, $p_0 < +\infty$ a fixed exponent with the conjugate $q_0 = p_0/(p_0 - 1)$ and $f \in L^{q_0}(\Omega)$ a nonnegative function. Then the sequence of* *functionals* $K_p : W_0^{1,p_0}(\Omega) \to [0, +\infty]$ *defined by*

$$K_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega f v dx$$

Γ-*converges as* $p \to +\infty$, *with respect to the weak convergence in* $W_0^{1,p_0}(\Omega)$, *to the functional* K_∞ *given by*

$$K_\infty(v) = \begin{cases} -\int_\Omega f v dx & \text{if} \quad ||v||_{W^{1,\infty}} \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In particular we have

$$\lim_{p \to +\infty} \min\{K_p(v) : v \in W_0^{1,p_0}(\Omega)\} = -\int_\Omega f(x) h_{\partial\Omega}(x) dx,$$

where $h_{\partial\Omega}(x) = d(x, \partial\Omega)$

Theorem 6.2.29. *Fix* $l > 0$, *an exponent* $p_0 > d$ *with conjugate* q_0, *and a nonnegative function* $f \in L^{q_0}(\Omega)$. *Consider the functionals*

$$C_p(\Sigma) = F(\Sigma, f, \Omega) \quad \text{for all } \Sigma \in \mathcal{A}_l(\Omega),$$

where $\mathcal{A}_l(\Omega)$ *is endowed with the Hausdorff convergence. As* $p \to +\infty$ *we have* Γ-*convergence of* $(C_p)_p$ *to the average distance functional* D *given by*

$$D(\Sigma) = \int_\Omega h(x) f(x) dx,$$

where $h(x) = d(x, \Sigma \cup \partial\Omega)$.

Proof: To prove the Γ-lim sup inequality we will prove pointwise convergence. This is to be done by fixing Σ, regarding the compliance as a maximum, and considering Γ-convergence on these problems, which would give us as byproduct the convergence of the optimal values. This Γ-convergence follows from Lemma 6.2.28, changing the signs in the functionals and applying it to the domain $\Omega \setminus \Sigma$. For the Γ-lim inf inequality take $\Sigma_p \to \Sigma$ and the corresponding potentials u_p. It is easy to see that this sequence is bounded in $W^{1,p_0}(\Omega)$ and, since $p_0 > d$, thanks to the compact embedding in C^0 of the Sobolev space $W^{1,p_0}(\Omega)$, we may also suppose $u_p \to u$ uniformly. If we prove, for almost any $x_0 \in \Omega$ such that $f(x_0) > 0$, the inequality $u(x_0) \geq d(x_0, \Sigma \cup \partial\Omega)$, the goal is achieved. To prove the inequality take $x_0 \in \Sigma \cup \partial\Omega$ and a radius $r < d(x_0, \Sigma \cup \partial\Omega)$. Since Σ_p converges in the Hausdorff topology to Σ it will eventually hold $r < d(x_0, \Sigma_p \cup \partial\Omega)$ as well. Hence, if we take the solution u_p of the p-Laplacian equation

$$\begin{cases} \Delta_p v_p &= f \quad \text{in} \quad B_r(x_0) \\ v_p &= 0 \quad \text{on} \quad \partial B_r(x_0), \end{cases}$$

we have the inequality $v_p \leq u_p$. hence it is sufficient to estimate the uniform limit of v_p. Since v_p is bounded in $W^{1,p_0}(B_r(x_0))$ we may suppose weak (and hence uniform) convergence to a function. By a Γ-convergence result of Lemma 6.2.28, we know that such a limit must optimize the limit problem, i.e. it must realize the maximum of $\int_{B_r(x_0)} fvdx$ among all 1-Lipschitz function v vanishing on $\partial B_r(x_0)$. The maximum is realized by the function $x \mapsto d(x, \partial B_r(x_0))$, which is the highest among these functions, but it could be realized by other functions as well. Those maximizing functions v should satisfy $v(x) = d(x, \partial B_r(x_0)) a.e.$ on $\{f > 0\}$. Yet, if $f(x_0) > 0$ and x_0 is a Lebesgue point for f, using the continuity of v and of the distance function (which are both Lipschitz continuous) we obtain $v(x_0) = d(x_0, \partial B_r(x_0)) = r$. Actually, by using again the 1-Lipschitz behavior of v, this proves the equality $v(x) = d(x, \partial B_r(x_0))$ for any $x \in B_r(x_0)$. This easily proves that the uniform limit u of the function u_p must satisfy $u(x_0) \geq r$ and letting r tend to $d(x_0, \Sigma \cup \partial\Omega)$, we get the desired inequality and the Γ-$\liminf$ inequality we were looking for. $\square$

Theorem 6.2.30. *Fix a nonnegative function $f \in L^{q_0}(\Omega)$ and consider the sequence of functionals $C_{p,\infty}$ on $\mathcal{P}(\overline{\Omega})$ (endowed with the weak topology) given, for $p > 1$, by*

$$C_{p,\infty}(\mu) := \int_\Omega \left(\frac{f}{\mu_a^{1/(d-1)}} \right)^q dx, \quad where\ q = \frac{p}{p-1}.$$

Then as $p \to +\infty$ we have the Γ-convergence of the sequence $(C_{p,\infty})_p$ to the functional $C_{\infty,\infty}$ defined by

$$C_{\infty,\infty}(\mu) := \int_\Omega \frac{f}{\mu_a^{1/(d-1)}} dx.$$

Proof: The result is straightforward, since we are considering the L^q of the same function $f/\mu_a^{1/(d-1)}$. The inequality

$$||v||_{L^q(\Omega)}^q \geq ||v||_{L^1(\Omega)} |\Omega|^{-1/(p-1)}$$

is sufficient to deal with the Γ-$\liminf$ inequality: let $\mu_p \rightharpoonup \mu$ then

$$\begin{aligned} \liminf_p ||f/(\mu_p)_a^{1/(d-1)}||_{L^q(\Omega)}^q &\geq \liminf_p ||f/(\mu_p)_a^{1/(d-1)}||_{L^1(\Omega)} |\Omega|^{-1/(p-1)} \\ &\geq ||f/\mu_a^{1/(d-1)}||_{L^1(\Omega)} \end{aligned},$$

where the last inequality comes from the semicontinuity of the limit functional and from the fact that $|\Omega|^{-1/(p-1)} \to |\Omega|^0 = 1$. The Γ-$\limsup$ inequality follows from the convergence of the L^q norm to the L^1 norm. $\square$

Theorem 6.2.31. *Let F_l be the functional defined on $\mathcal{P}(\overline{\Omega})$ by*

$$F_l(\mu) := \begin{cases} l^{\frac{1}{d-1}} \int_\Omega fh(x)dx & if \quad \mu = \mathcal{H}^1(\Sigma)^{-1} \mathcal{H}^1 \llcorner \Sigma, \ \Sigma \in \mathcal{A}_l(\Omega) \\ +\infty & otherwise, \end{cases}$$

where $h(x) = d(x, \Sigma \cup \partial\Omega)$*. Then the functional* F_l Γ*-converges with respect to the weak** *topology of* $\mathcal{P}(\overline{\Omega})$*, to the functional* F_∞ *defined on* $\mathcal{P}(\overline{\Omega})$ *by*

$$F_\infty(\mu) := \theta_\infty \int_\Omega \frac{f}{\mu_a^{\frac{1}{d-1}}} dx,$$

where μ_a *stands for the absolutely continuous part of* μ *with respect to the Lebesgue measure and* θ_∞ *is given by*

$$\theta_\infty = \inf\{\liminf_{l\to\infty} l^{\frac{1}{d-1}} \int_{I^d} h_1(x)dx, \Sigma \in \mathcal{A}_l(I^d)\},$$

with $h_1(x) = d(x, \Sigma \cup \partial I^d)$

Proof: See [76]. □

To complete the framework of the convergence as $p \to +\infty$, we have just to control the constants $\theta(p)$. From the proof of Proposition 6.2.27 we have

$$F_p(\Sigma_l, 1, I^d) \geq q^{-q} \int_{I^d} h_1(x)^q dx,$$

hence

$$\begin{aligned}
\liminf_{l\to+\infty} l^{\frac{1}{d-1}} F_p(\Sigma_l, 1, I^d) &\geq q^{-q} \liminf_{l\to+\infty} l^{\frac{1}{d-1}} \int_{I^d} h_1(x)^q dx \\
&\geq q^{-q} \liminf_{l\to+\infty} \left(l^{\frac{1}{d-1}} \int_{I^d} h_1(x)dx \right)^q \\
&\geq q^{-q} \left(\liminf_{l\to+\infty} l^{\frac{1}{d-1}} \int_{I^d} h_1(x)dx \right)^q \\
&\geq q^{-q} (\theta_\infty)^q
\end{aligned}.$$

We obtain

$$\theta(p) \geq q^{-q}(\theta_\infty)^q,$$

therefore

$$\liminf_{p\to+\infty} \theta(p) \geq \liminf_{p\to+\infty} q^{-q}(\theta_\infty)^q = \theta_\infty.$$

For the upper bound, using Lemma 6.2.28 we have

$$\limsup_{p\to+\infty} \theta(p) \leq \limsup_{p\to+\infty} l^{\frac{q}{d-1}} F_p(\Sigma, 1, I^d) = l^{\frac{1}{d-1}} \int_{I^d} h_1(x)dx,$$

for all $\Sigma \in \mathcal{A}_l(I^d)$. Passing to the limit as $l \to +\infty$ and optimization over $\Sigma \in \mathcal{A}_l(I^d)$ we get

$$\limsup_{p\to+\infty} \leq \theta_\infty.$$

According to Proposition 6.2.27 we have $\theta_\infty = \frac{1}{4}$ for $d = 2$ and $\theta_\infty \geq \frac{d-1}{d\omega_{d-1}^{\frac{1}{d-1}}}$ for higher dimension. The commutative diagram we highlighted in this section is summarized in the following diagram:

$$\begin{array}{ccc}
\mathcal{A}_l(\Omega) \ni \Sigma \mapsto l^{\frac{q}{d-1}} \int_\Omega f u^{(p)}_{f,\Sigma,\Omega} dx & \xrightarrow{l \to +\infty} & \mathcal{P}(\overline{\Omega}) \ni \mu \mapsto \theta(p) \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx \\
\Big\downarrow{\scriptstyle p \to +\infty} & & \Big\downarrow{\scriptstyle p \to +\infty} \\
\mathcal{A}_l(\Omega) \ni \Sigma \mapsto l^{\frac{1}{d-1}} \int_\Omega f(x) h(x) dx & \xrightarrow{l \to +\infty} & \mathcal{P}(\overline{\Omega}) \ni \mu \mapsto \theta_\infty \int_\Omega \frac{f}{\mu_a^{\frac{1}{d-1}}} dx
\end{array}$$

6.2.6 Asymptotics of an optimal p-compliance-location

In this section we consider the case where the control variables are searched among discrete sets of finite elements. Let $p > d$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and n a positive given integer number, we define

$$\mathcal{A}_n(\Omega) = \{\Sigma \subset \overline{\Omega} : 0 < \mathcal{H}^0(\Sigma) \leq n\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity (since $p > d$, every point has positive p-capacity), we denote as before by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W^{1,p}_0(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W^{1,p}_0(\Omega \setminus \Sigma). \tag{6.36}$$

For $f \geq 0$, we define the p-compliance functional as before and again the existence of the minimal p-compliance configuration is just a consequence of the Hölderianity of $u_{f,\Sigma_n,\Omega}$.

Theorem 6.2.32. *For any integer number $n > 0$, Ω a bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_n(\Omega)\} \tag{6.37}$$

admits at least one solution.

As before, we are interested to the asymptotic behavior of the optimal set Σ_n of the problem (6.37) as $n \to +\infty$. Let us associate to every $\Sigma \in \mathcal{A}_n(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = n^{-1} \delta_\Sigma$$

and define a functional $G_n : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$G_n(\mu) = \begin{cases} n^{\frac{q}{d}} C_p(\Sigma) & \text{if } \mu = \mu_\Sigma, \Sigma \in \mathcal{A}_n(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{6.38}$$

The scaling factor $n^{\frac{q}{d}}$ is needed in order to avoid the functionals to degenerate to the trivial limit functional which vanishes everywhere. Again the main result deal with the behavior as $n \to +\infty$ of the functional G_n, and is stated in terms of Γ-convergence.

Theorem 6.2.33. *The functional G_n defined in (6.38) Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional G defined on $\mathcal{P}(\overline{\Omega})$ by*

$$G(\mu) = \theta_1 \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d}}} dx, \tag{6.39}$$

where μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ is a positive constant depending only on d and p and is defined by

$$\theta_1 = \inf\{\liminf_{n\to+\infty} n^{\frac{q}{d}} F_p(\Sigma_n, 1, I^d) : \Sigma_n \in \mathcal{A}_n(I^d)\} \tag{6.40}$$

$I^d = (0,1)^d$ being the unit cube in $\mathbb{R}^d$.

We deduce the following consequence of Theorem 6.2.33:

- if Σ_n is a solution of the minimization problem (6.37), then up to a subsequence $\mu_{\Sigma_n} \rightharpoonup \mu$ as $n \to +\infty$, where μ is a minimizer of G;
- since G has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_n} converges to the unique minimizer μ of G given by $\mu = cf^{\frac{qd}{q+d}}\mathcal{L}^d$ where c is such that μ is a probability measure that is $c = 1/\left(\int_\Omega f^{\frac{qd}{q+d}} dx\right)$
- the minimal value of G is equal to $\theta_1 c^{\frac{q+d}{d}}$, and the sequence of the values $\inf\{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_n(\Omega)\}$ is asymptotically equivalent to $n^{\frac{q}{d}}\theta_1 c^{\frac{q+d}{d}}$.

We will not prove Theorem 6.2.33 since the proof follows the same line as the proof of Theorem 6.2.2 but we will point out some necessary modifications. The Lemma 6.2.12 is crucial for the proof of the Γ-$\liminf$ inequality. This Lemma remains valid in the case of discrete set provided that the power $d-1$ is replaced by d. In this case it suffices that v vanishes on one point since point has positive p-capacity (remember that $p > d$). An other important element in the proof of the Γ-$\liminf$ inequality is the set $G_{\varepsilon,l}$. Here, we will call it $G_{\varepsilon,n}$ and its construction is obtained by the homogenization of order $\lfloor \left(\frac{\varepsilon n}{2ad}\right)^{1/d} \rfloor$ of the center of the unit cube into the cube $I_a^d = (-a,a)^d$ which contains Ω. For the Γ-$\limsup$ inequality, proofs are essentially the same except the fact that we do not need tiling set and replace l by n. We conclude this section with the estimate of the constant θ_1. To prove the finiteness it suffice to use the set Σ_n which the homogenization of order n of the center of the unit cube into the unit cube. For the lower bound, the proof follows that of Proposition 6.2.27 and gives

$$\theta_1 \geq \frac{d}{(q+d)w_d^{\frac{q}{d}}}.$$

Also in this case we may have a commutative diagram that we summarize below

$$\begin{array}{ccc}
\mathcal{A}_n(\Omega) \ni \Sigma \mapsto n^{\frac{q}{d}} \int_\Omega f u^{(p)}_{f,\Sigma,\Omega} dx & \xrightarrow{n \to +\infty} & \mathcal{P}(\overline{\Omega}) \ni \mu \mapsto \theta(p) \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d}}} dx \\
\Big\downarrow {\scriptstyle p \to +\infty} & & \Big\downarrow {\scriptstyle p \to +\infty} \\
\mathcal{A}_n(\Omega) \ni \Sigma \mapsto n^{\frac{1}{d}} \int_\Omega f(x) h(x) dx & \xrightarrow{n \to +\infty} & \mathcal{P}(\overline{\Omega}) \ni \mu \mapsto \theta_\infty \int_\Omega \frac{f}{\mu_a^{\frac{1}{d}}} dx
\end{array}$$

6.3 Asymptotics of an optimal compliance-location in $\mathbb{R}^d$

We consider the problem of finding the best location of the Dirichlet region Σ for a d-dimensional membrane Ω subjected to a given vertical force f. The vertical displacement of the membrane satisfies the elliptic equation

$$\begin{cases} -\Delta u &= f \ \text{ in } \Omega \setminus \Sigma \\ u &= 0 \ \text{ in} \Sigma \cup \partial\Omega, \end{cases}$$

and the rigidity of the membrane is measured through the compliance functional

$$C(\Sigma) = \int_\Omega f u_{f,\Sigma,\Omega} dx,$$

where $u_{f,\Sigma,\Omega}$ stands for the unique solution of the above equation. The maximal rigidity of the membrane is obtained by minimizing the compliance functional $C(\Sigma)$ in a class of admissible regions Σ. The admissible class for control variables Σ we consider is the class of all n identical balls with prescribed capacity β. It is easy to obtain the optimal configuration Σ_n of the above optimization problem (see Theorem 6.3.1).As before we are interested in the asymptotic behavior of Σ_n as $n \to +\infty$; more precisely we want to obtain the limit distribution of Σ_n as a limit probability measure that minimize the Γ-limit functional of the suitable rescaled compliance functional.

6.3.1 Compliance under capacity constraint

For any bounded open set $\Omega \subset \mathbb{R}^d$, $\beta > 0$ and $R > 0$ both real numbers and $n \in \mathbb{N}$ we define:

$$\mathcal{A}(\beta, n)(\Omega) = \left\{ \Sigma \subset \overline{\Omega} : \Sigma = \overline{\Omega} \cap \bigcup_{i=1}^n \overline{B(x_i, r)} \ \text{ for } \ x_i \in \Omega_r, r = r_n \right\},$$

where $r_n = \beta^{\frac{1}{d-2}} n^{-1/(d-2)}$ if $d \geq 3$ and $r_n = e^{-n/\beta}$ if $d = 2$ and Ω_r stands for the r-neighborhood of Ω. When the dependence of the radius on the β will be necessary, we will write explicitly $r_n = r_n(\beta)$

Given $\Omega \subset \mathbb{R}^d$ and $f \in L^2(\Omega)$, for any compact set $\Sigma \subset \overline{\Omega}$ with positive Lebesgue measure, we define the function $u_{f,\Sigma,\Omega}$ as the weak solution of the problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

which means precisely $u \in H_0^1(\Omega \setminus \Sigma)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \text{ for any } \varphi \in H_0^1(\Omega \setminus \Sigma). \tag{6.41}$$

Notice that by the maximum principle, $f \geq 0$ implies $u_{f,\Sigma,\Omega} \geq 0$. For $f \geq 0$, we define the compliance over a subset of a given domain Ω as

$$F(\Sigma, f, \Omega) = \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^2 dx.$$

Now we restrict such a compliance functional to the sets in $\mathcal{A}(\beta, n)(\Omega)$, that is adding both the capacity constraint $cap(\Sigma) = \beta$ and a geometrical constraint that is force Σ to be composed by an assigned number of identical balls. This is our n^{th} compliance minimization problem. The severe geometric constraint of the elements of $\mathcal{A}(\beta, n)(\Omega)$ gives a necessary compactness to get the following existence result.

Theorem 6.3.1. *For any $n \in \mathbb{N}$ and $R > 0, \beta > 0$ fixed, if Ω is any bounded open subset of $\mathbb{R}^d$ and $f \geq 0$ belongs to $L^2(\Omega)$, the problem*

$$\min \{F(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}(\beta, n)(\Omega)\}, \tag{6.42}$$

admits a solution.

Then we would like to let n tends to infinity and look at the asymptotic of the problem, mainly at the distribution of the centers of balls. To this aim let us associate to each $\Sigma \in \mathcal{A}(\beta, n)(\Omega)$ a probability measure on $\overline{\Omega}$, given by $\mu_\Sigma = n^{-1} \sum_{i=1}^n \delta_{p(x_i)}$ where $p : \mathbb{R}^d \to \overline{\Omega}$ is a fixed projection of the whole space $\mathbb{R}^d$ to $\overline{\Omega}$ and $(x_i)_{i=1,\cdots,n}$ are the centers of the balls composing Σ. The role of the projection p is simply to hand the case where the center of the ball $\overline{B(x_i, r)}$ lies outside $\overline{\Omega}$. Such an measure is an atomic measure uniformly distributed on the centers of ball (or on their projection). Now define functionals $F_n : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$F_n(\mu) = \begin{cases} F(\Sigma, f, \Omega) & \text{if } \mu = \mu_\Sigma, \ \Sigma \in \mathcal{A}(\beta, n)(\Omega); \\ +\infty & \textit{otherwise.} \end{cases}$$

We will prove a Γ-convergence for the sequence F_n when the space $\mathcal{P}(\overline{\Omega})$ is endowed with the *weak** topology of probability measures. To introduce the limit functional F we need to define the quantity:

$$\theta(\beta) := \inf\left\{\liminf_{n\to+\infty} F(\Sigma_n,1,I^d) : \Sigma_n \in \mathcal{A}(\beta,n)(I^d)\right\}, \tag{6.43}$$

where $I^d = (0,1)^d$ is the unit cube in $\mathbb{R}^d$. It is easy to see that θ is a decreasing function on $\mathbb{R}^+$ which vanishes after some point. In fact for $\beta \geq (\sqrt{d}/2)^{(d-2)}$ for $d \geq 3$ and $\beta \geq (\ln\sqrt{2})^{-1}$, it is possible to use n balls of radius $r = r_n$ with β large enough to build a set $\Sigma \in \mathcal{A}(\beta,n)(I^d)$ covering the whole cube I^d, thus getting a vanishing solution $u_{f,\Sigma,I^d} = 0$ and $F(\Sigma,f,I^d) = 0$. Let us call t_1 the first vanishing point, i.e.

$$t_1 := \inf\{t \in \mathbb{R} : \theta(t) = 0\}.$$

We denote by θ^- and θ^+ the lower and upper semicontinuous envelopes of θ, respectively. They are given by

$$\theta^-(\beta) = \sup\{\theta(\alpha) : \alpha > \beta\};$$

$$\theta^+(\beta) = \inf\{\theta(\alpha) : \alpha < \beta\}.$$

It is easy to check that the following formula holds:

$$\theta^-(\beta) = \inf\left\{\liminf_{n} F(\Sigma_n,1,I^d) : \Sigma_n \in \mathcal{A}(\beta_n,n)(I^d),\ \ \beta_n \to \beta\right\}. \tag{6.44}$$

Remark 6.3.2 Due to the monotonicity of the function θ it is easy to see that for any $\beta_1 < \beta_2$ we have $\theta^+(\beta_2) \leq \theta^-(\beta_1)$ and in particular $\theta^-(0) \geq \theta^+(\beta)$ for any $\beta > 0$.

We may now define the candidate limit functional F by setting, for $\mu \in \mathcal{P}(\overline{\Omega})$

$$F(\mu) = \theta^-(0)\int_\Omega \frac{f^2}{\mu_a^{2/d}}dx, \tag{6.45}$$

where μ_a denotes the density of the absolutely continuous part of μ with respect to the Lebesgue measure. It is clear from (6.45) that the behavior of the function θ does not affect the minimization problem for F. The result we will prove is the following.

Theorem 6.3.3. *Given any bounded open set $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, a nonnegative function $f \in L^2(\Omega)$ and $R, \beta > 0$, the sequence of functional $(F_n)_n$ previously defined Γ-converges to F as $n \to +\infty$ with respect to the weak$*$ topology on $\mathcal{P}(\overline{\Omega})$.*

We deduce the following consequence of the Theorem 6.3.3

- if Σ_n is a solution of the minimization problem (6.42), it holds up to a subsequence, $\mu_{\Sigma_n} \rightharpoonup \mu$ as $n \to +\infty$, where μ is a minimizer of F;
- since F has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, we have the convergence of the whole sequence μ_{Σ_n} to the unique minimizer μ, which is given by $\mu = cf^{2d/(d+2)}dx$ (and c is computed so that μ is a probability measure, i.e. $c = 1/\int_\Omega f^{2d/(d+2)}dx$);
- the minimal value of F is equal to $\theta^-(0)c^{-\frac{d+2}{d}}$ and the sequence of the values $\inf\{F(\Sigma,f,\Omega) : \Sigma \in \mathcal{A}(\beta,n)(\Omega)\}$ is asymptotical to $\inf\left\{F(\mu) : \mu \in \mathcal{P}(\overline{\Omega})\right\}$.

6.3.2 Proof of Γ-limit

Also here we will prove Theorem 6.3.3 in several steps, the most important two corresponding to the Γ-lim inf and Γ-lim sup. Before that we need some preliminaries results. Let us fix $\varepsilon > 0$, in analogy to the constructed set $G_{\varepsilon,l}$, define the set $G_{\varepsilon,n}$ as follows

$$G_{\varepsilon,n} = \bigcup_{y \in k^{-1}\mathbb{Z}^d \cap [-a,a]^d} \overline{B(y,r)}, \quad r = r_n, \quad k = \lfloor (\varepsilon n)^{1/d} \rfloor.$$

Now we define $\Sigma_n' = \Sigma_n \cup G_{\varepsilon,n}$ and we set $u_n' = u_{f,\Sigma_n',\Omega}$. Let us introduce the number

$$k(n,Q) = \# \left(\left\{ i : \overline{B(x_i,r)} \cap Q \neq \emptyset \right\} \cup \left\{ j : \overline{B(y_j,r)} \cap Q \neq \emptyset \right\} \right).$$

It holds $k(n,Q) \to +\infty$ as $n \to +\infty$, since

$$k(n,Q) \geq \# \left\{ j : \overline{B(y_j,r)} \cap Q \neq \emptyset \right\} \approx \varepsilon n |Q|. \tag{6.46}$$

We may also estimate the ratio $k(n,Q)$ by n, by using (6.46) and the fact that $\# \left\{ i : \overline{B(x_i,r)} \cap Q \neq \emptyset \right\} = n\mu_n(Q_r)$, where $r = r_n \to 0$ as $n \to +\infty$ and Q_r denotes the r-neighborhood of Q. From $\mu_n \rightharpoonup \mu$ it follows that $\limsup_n \mu_n(Q_{r_n}) \leq \mu(\overline{Q})$, so that

$$\limsup_n \frac{k(n,Q)}{n} \leq \mu(\overline{Q}) + \varepsilon |Q|. \tag{6.47}$$

Lemma 6.3.4. *The following facts hold.*

1. *For any $\varepsilon > 0$, any $0 < n < +\infty$, any domain Ω and any function $v \in H_0^1(\Omega \setminus G_{\varepsilon,l}) \subset H_0^1(\Omega)$ ($G_{\varepsilon,n}$ is the set constructed above) it holds $||v||_{L^2(\Omega)} \leq C(d,\varepsilon)||v||_{H_0^1(\Omega)}$.*

2. *As a consequence, if we have a nonnegative function $f \in L^2(\Omega)$, then the function $u_{f,G_{\varepsilon,n},\Omega}$ satisfies $||u_{f,G_{\varepsilon,n},\Omega}||_{L^2(\Omega)} \leq C(d)||f||_{L^2(\Omega)}$.*

3. *For any $\Sigma_n \in \mathcal{A}(\beta,n)(\Omega)$, $f \in L^2(\Omega)$ a nonnegative function and $Q \subset \Omega$ a cube it holds*

$$u_{f,\Sigma_n,\Omega} \geq \left(\frac{n}{k(n,Q)} \right)^{\frac{2}{d}} u_{f,\Sigma_n',Q} + b_n \text{ on } Q,$$

where $b_n \rightharpoonup 0$ in $H_0^1(Q)$ and

$$||u_{f,G_{\varepsilon,n},\Omega}||_{L^2(Q)} \leq C(d) \left(\frac{n}{k(n,Q)} \right)^{\frac{-2}{d}} ||f||_{L^2(Q)}$$

Proof: The proof is along the line of the proof of Lemma 6.2.12. We choose A to be balls composing $G_{\varepsilon,n}$ and $p = 2$. For proving the first part, we first choose the function v to be a nonnegative smooth function on the large cube I_a^d which vanish outside $\Omega \setminus G_{\varepsilon,n}$. We consider the subdivision of cube I_a^d into subcubes as done above and consider the associated set $G_{\varepsilon,n}$. The side of subcubes is of order $n^{-1/d}$. Let us denote the subcubes by Q_j. The set $I_a^d \setminus G_{\varepsilon,n}$ can be seen as the homogenized of order $k = \lfloor (\frac{\varepsilon n}{ad})^{1/d} \rfloor$ of $I^d \setminus M$ into I_a^d (M is the a ball of radius $n^{-2/(d(d-2))}$ for $d \geq 3$ and of radius $\sqrt{n}e^{-n}$ for $d = 2$). Let us set $\varepsilon_0 = cap(M, 2I^d)$ and notice that v vanishes on $G_{\varepsilon,n}$. By applying the first statement of the Lemma 6.2.12, it follows that

$$\int_{Q_j} |v|^2 dx \leq \frac{Ck^{-d}}{cap(k^{-1}M, 2Q_j)} \int_{Q_j} |\nabla v|^2 dx \leq \frac{Cn^{-2/d}}{cap(M, 2I^d)} \int_{Q_j} |\nabla v|^2 dx.$$

Observing that $cap(M, 2I^d)$ is bounded below by $Cn^{-2/d}$ with C independent of n we have

$$\int_{Q_j} |v|^2 dx \leq C \int_{Q_j} |\nabla v|^2 dx$$

and by summing up over j we get

$$\int_{I_a^d} |v|^2 dx \leq C \int_{I_a^d} |\nabla v|^2 dx.$$

Using the fact that v vanishes outside Ω, we may restrict the integrand to Ω, raise each term of the inequality to the power $1/2$ and thus getting the result by noticing that the L^2 norm of the gradient $||\nabla v||_{L^2(\Omega)}$ stands for the norm $||v||_{H_0^1(\Omega)}$. The general case follows by density. For the second inequality, we use the weak version of the PDE which gives

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,n},\Omega}|^2 dx = \int_\Omega f u_{f,G_{\varepsilon,n},\Omega} dx \leq ||u_{f,G_{\varepsilon,n},\Omega}||_{L^2(\Omega)} ||f||_{L^2(\Omega)}.$$

Since $u_{f,G_{\varepsilon,n},\Omega} \in H_0^1(\Omega \setminus G_{\varepsilon,n})$ we get

$$||u_{f,G_{\varepsilon,n},\Omega}||^2_{H_0^1(\Omega)} \leq ||u_{f,G_{\varepsilon,n},\Omega}||_{L^2(\Omega)} ||f||_{L^2(\Omega)} \leq C(d,\varepsilon) ||u_{f,G_{\varepsilon,n},\Omega}||_{H_0^1(\Omega)} ||f||_{L^2(\Omega)},$$

and the desired result follows. For the first of the third point take

$$b_n = \min\{u_{f,\Sigma_n,\Omega} - \left(\frac{n}{k(n,Q)}\right)^{\frac{2}{d}} u_{f,\Sigma'_n,Q}, 0\}$$

and for the second part, repeat the proof of the second point and observing that $|Q|n \leq k(n,Q)$. $\square$

Γ-$\liminf$ **inequality**

In the following proposition we prove that the Γ-$\liminf$ is bounded below by the candidate limit functional F introduced in (6.45).

Proposition 6.3.5. *Under the same hypotheses of Theorem 6.3.3, denoting by F^- the Γ-$\liminf_n F_n$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that, for a sequence $(\Sigma_n)_n$ such that μ_{Σ_n} weakly* converges to μ and $\Sigma_n \in \mathcal{A}(\beta, n)(\Omega)$, it holds $\liminf_n \int_\Omega f u_n dx \geq F(\mu)$, where u_n stands for $u_{f,\Sigma_n,\Omega}$.*

Proof: We set $u'_n = u_{f,\Sigma'_n,Q}$ and we have $u_n \geq \left(\frac{n}{k(n,Q)}\right)^{\frac{2}{d}} u'_n + b_n$. The sequence $(\left(\frac{n}{k(n,Q)}\right)^{\frac{2}{d}} u'_n)_n$ is L^2- bounded thanks to Lemma 6.3.4. Due to the particular setting we have to study the local behavior of the energy. Let $x_0 \in \Omega$ and take a cube Q centered at the point $x_0 \in \Omega$. We will assume that x_0 is a Lebesgue point for f and it satisfies the condition $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as Q shrinks around x_0. These assumptions are verified for almost any point $x_0 \in \Omega$. The sequence $\left(\frac{n}{k(n,Q)}\right)^{\frac{2}{d}} u'_n$ is L^2 bounded then it converges weakly to a function $w \in L^2(Q)$. we have $\liminf_n \left(\frac{n}{k(n,Q)}\right)^{2/d} \int_Q f u'_n dx = \int_Q f w dx$. This shows that it is enough to estimate w from below. We have

$$\int_Q w dx = \lim_n \left(\frac{n}{k(n,Q)}\right)^{2/d} \int_Q u'_n dx.$$

We use

$$u'_n = u_{f,\Sigma'_n,Q} = u_{f(x_0),\Sigma'_n,Q} + u_{f-f(x_0),\Sigma'_n,Q} \geq u_{f(x_0),\Sigma'_n,Q} - u_{|f-f(x_0)|,\Sigma'_n,Q} \text{ in } Q,$$

where the inequality is obtained by the maximum principle. By applying Lemma 6.3.4 to the cube Q it follows that

$$||u_{|f-f(x_0)|,\Sigma'_n,Q}||_{L^2(Q)} \leq \left(\frac{n}{k(n,Q)}\right)^{-2/d} C(d)||f - f(x_0)||_{L^2(Q)}.$$

By Hölder inequality, we have the following estimate

$$\begin{aligned} \left(\frac{n}{k(n,Q)}\right)^{2/d} \int_Q u_{|f-f(x_0)|,\Sigma'_n,Q} dx &\leq |Q|^{1/2}||u_{|f-f(x_0)|,\Sigma'_n,Q}||_{L^2(Q)} \\ &\leq C(d)|Q|^{1/2}||f - f(x_0)||_{L^2(Q)} \end{aligned} \tag{6.48}$$

Now we have to evaluate the remaining term. Remark that $u_{f(x_0),\Sigma'_n,Q} = f(x_0)u_{1,\Sigma'_n,Q}$. For simplicity of the notation, we set $v_n = u_{1,\Sigma'_n,Q}$. By a change of variable, if we assume λ to be the side of cube Q, we get $v_{n,\lambda} = u_{1,\lambda^{-1}\Sigma'_n,Q}$ where $v_{n,\lambda}$ is defined by $v_{n,\lambda}(x) = \lambda^{-2}v_n(\lambda x)$. A simple computation shows that

$$\lambda^{-1}\Sigma'_n \in \mathcal{A}\left(\zeta_n, k(n,Q)\right)(I^d),$$

where $\zeta_n = \frac{\beta k(n,Q)}{\lambda^{d-2} n}$ if $d \geq 3$ and $\zeta_n = \frac{\beta k(n,Q)}{n}$ if $d = 2$.

Taking into account the equality (6.44), we easily check that for $d \geq 3$ we have

$$\begin{aligned}
\liminf_n \int_Q v_n dx &= \liminf_n \lambda^{d+2} \int_{I^d} v_{n,\lambda} dx \\
&\geq \lambda^{d+2} \theta^- \left(\frac{\beta}{\lambda^{d-2}} (\mu(\overline{Q}) + \varepsilon |Q|) \right)
\end{aligned}$$

So noticing that $\lambda^d = |Q|$ we get

$$\begin{aligned}
\liminf_n \left(\frac{n}{k(n,Q)} \right)^{2/d} \int_Q v_n dx &\geq \liminf_n \left(\frac{n}{k(n,Q)} \right)^{2/d} \liminf_n \int_Q v_n dx \\
&\geq \lambda^{d+2} \theta^- \left(\frac{\beta}{\lambda^{d-2}} (\mu(\overline{Q}) + \varepsilon |Q|) \right) \left(\frac{1}{\mu(\overline{Q}) + \varepsilon |Q|} \right)^{2/d} \\
&= \theta^- \left(\beta |Q|^{2/d} (\frac{\mu(\overline{Q}) + \varepsilon |Q|}{|Q|}) \right) |Q| \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon |Q|} \right)^{2/d}.
\end{aligned}$$

This implies, recalling (6.48),

$$\begin{aligned}
|Q|^{-1} \int_Q w dx &\geq -C(d) |Q|^{-1/2} ||f - f(x_0)||_{L^2(Q)} \\
&\quad + f(x_0) \theta^- \left(\beta |Q|^{2/d} (\frac{\mu(\overline{Q}) + \varepsilon |Q|}{|Q|}) \right) \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon |Q|} \right)^{2/d}.
\end{aligned}$$

For the case of dimension $d = 2$, the computations are quite similar and we obtain exactly the above inequality with $d = 2$.

Now we let Q shrink towards x_0, thus getting, using the lower semicontinuity of the function θ^-, for $a.e. x_0 \in \Omega$ (x_0 satisfies the previous assumption)

$$w(x_0) \geq f(x_0) \theta^-(0) \left(\frac{1}{\mu_a(x_0) + \varepsilon} \right)^{2/d}.$$

It follows that

$$\liminf_n \int_\Omega f u_n dx \geq \int_\Omega f w dx \geq \theta^-(0) \int_\Omega \frac{f^2 dx}{(\mu_a(x) + \varepsilon)^{2/d}}.$$

We let ε tends to zero and hence getting the desired result that is

$$\liminf_n \int_\Omega f u_n dx \geq \theta^-(0) \int_\Omega \frac{f^2 dx}{\mu_a(x)^{2/d}}.$$

□

Γ-lim sup **inequality**

Now, we have to prove the Γ-lim sup that is the reverse inequality. To this aim, we need this crucial Lemma

Lemma 6.3.6. *Given $\Sigma_0 \in \mathcal{A}(\beta_0, n_0)(I^d)$, a domain $\Omega \subset \mathbb{R}^d$ and $f \in L^2(\Omega)$, we consider this sequence of sets*

$$\Sigma^k = \bigcup_{y \in k^{-1}\mathbb{Z}^d} (y + k^{-1}\Sigma_0 \cup \partial I^d) \cap \overline{\Omega}$$

and the sequence functions $(u_k)_k$, defined by

$$u_k = k^2 u_{f,\Sigma^k,\Omega}.$$

We have $u_k \rightharpoonup c(\Sigma_0)f$ as $k \to \infty$, where the weak convergence is in the L^2 sense and $c(\Sigma_0)$ is the constant given by $\int_{I^d} u_{1,\Sigma_0,I^d} dx$.

Proof: The proof is the same as the proof of Lemma 6.2.7 □

Remark 6.3.7 The problem in the previous Lemma is that we have used the whole boundary of the unit cube which is not an union of balls of radius r_n and consequently the set Σ^k is not an element of the set $\mathcal{A}(\beta, n)$. In the following Lemma, we will prove that $u_{f,\Sigma^k,\Omega}$ may be approximate by $u_{f,\Sigma^k_l,\Omega}$ where Σ^k_n is an union of n_0 identical balls of radius r_n and capacity β (we will call those sets for which this condition is satisfied, almost boundary-covering sets).

We will use the technics developed for perforated domains in general or of sieves (see for example [12], [13],[47],[78], [86] for details). In those situations one proves that there exists a critical radius (perforated domains are obtained by removing balls from fixed domain) such that the Dirichlet problem converges to a limit problem which is not associated to the Laplacian operator (it appears a strange term) and when the radius is large or small enough, we are in a trivial case where the limit problem is associated to the Laplacian operator.

Let T_n be a set of $n^{1-1/d}$ balls of radius $r = r_n$ (depending on the dimension d) such that the centers are uniformly distributed on the boundary of the unit cube I^d. Let Σ_n and Σ be the respective homogenized of T_n and ∂I^d of order 1 into Ω that is

$$\Sigma_n = \overline{\Omega} \cap (\mathbb{Z}^d + T_n), \quad \Sigma = \overline{\Omega} \cap (\mathbb{Z}^d + \partial I^d).$$

Moreover we choose T_n to be such that Σ_n is a set of balls whose centers are $n^{\frac{-1}{d}}$-periodically distributed on Σ and Σ is a plane of symmetric of balls composing Σ_n. Let u_n and u be two sequences of functions defined by

$$u_n = u_{f,\Sigma_n,\Omega}, \quad u = u_{f,\Sigma,\Omega}.$$

Our goal is to prove that we may approximate u by the function u_n solution of the equation

$$\begin{cases} -\Delta u_n &= f \text{ in } \Omega \setminus \Sigma_n \\ u_n &= 0 \text{ in } \Sigma_n \cup \partial\Omega. \end{cases} \tag{6.49}$$

Let us denote by $\{x_j^n\}_{j\in J}$ the set of centers of the balls in Σ_n, and consider the open ball B_j^n with center x_j^n and radius $\frac{n^{\frac{-1}{d}}}{2}$ and the closed ball C_j^n with center x_j^n and radius $r_n = \delta_n = n^{-\frac{d-1}{d(d-2)}}$ if $d \geq 3$ and $\delta_n = e^{-\sqrt{n}}$ if $d = 2$. The sets $(B_j^n)_j$ are pairwise disjoint and $C_j^n \subset B_j^n$ for every j. In this part, operations as sum, union, intersection are made for j running in J (J depend on n). We begin with the case $d \geq 3$

Lemma 6.3.8. *Let w_n be the function which is equal on B_j^n to the solution of the equation*

$$\begin{cases} -\Delta w_n &= 0 \text{ in } (B_j^n \setminus C_j^n) \setminus (\Omega \setminus \Sigma) \\ w_n &= 1 \text{ on } C_j^n \\ w_n &= 0 \text{ on } \partial B_j^n \\ \frac{\partial w_n}{\partial \nu} &= 0 \text{ on } (B_j^n \cap \Sigma) \setminus C_j^n, \end{cases}$$

and $w_n = 0$ on $\Omega \setminus \cup B_j^n$. Then w_n converges weakly to zero in $H_0^1(\Omega)$ and the measure $|\nabla w_n|^2 dx$ weakly converges to the measure $C\mathcal{H}^{d-1}\llcorner\Sigma$ where C is the capacity of the unit ball in $\mathbb{R}^d$.

Proof: The weak convergence of w_n to zero is straightforward. For the convergence of the measure we have:

$$\int_\Omega |\nabla w_n|^2 dx = \sum_j \int_{B_j^n} |\nabla w_n|^2 dx = \frac{\mathcal{H}^{d-1}(\Sigma)}{n^{1-1/d}} \int_{B_{n^{-1/d}/2}(0)} |\nabla w_n|^2 dx$$

$$= \frac{\mathcal{H}^{d-1}(\Sigma)}{n^{1-1/d}} \min\left\{ \int_{B_{\frac{n^{-1/d}}{2}}(0)} |\nabla u|^2 dx : u \in H_0^1(B_{\frac{n^{-1/d}}{2}}(0)), u = 1 \text{ on } C_j^n \right\}$$

$$= \frac{\mathcal{H}^{d-1}(\Sigma)}{n^{1-1/d}} \min\left\{ \int_{B_{\frac{n^{-1/d}}{2\delta_n}}(0)} |\nabla u|^2 \delta_n^{d-2} dy : u \in H_0^1(B_{\frac{n^{-1/d}}{2\delta_n}}(0)), u = 1 \text{ on } B_1(0) \right\}$$

$$= \mathcal{H}^{d-1}(\Sigma) \min\left\{ \int_{B_{\frac{1}{2}n^{\frac{1}{d(d-2)}}}(0)} |\nabla u|^2 dy : u \in H_0^1(B_{\frac{1}{2}n^{\frac{1}{d(d-2)}}}(0)), u = 1 \text{ on } B_1(0) \right\}$$

where in the third equality we have used the change of variable of type $x = \delta_n y$. Passing to the limit as n tends to $+\infty$ the min in the right hand side converges to the capacity of the unit ball relative to $\mathbb{R}^d$. Since there is convergence of mass we get the weak convergence of measure. $\square$

Lemma 6.3.9. *Set $C_n = \cup_j C_j^n$ and define the functionals*

$$F_n(v) := \begin{cases} \int_\Omega |\nabla v|^2 dx & \text{if } v \in H_0^1(\Omega \setminus C_n), \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$F(v) := \int_\Omega |\nabla v|^2 dx + C \int_\Sigma v^2 d\mathcal{H}^{d-1},$$

where C is capacity of the unit ball in $\mathbb{R}^d$. Then F_n Γ-converges to F in the weak topology of $H_0^1(\Omega)$. Moreover if we replace the radius δ_n of balls of C_n by the radius r_n then F_n Γ-converges to G in the weak topology of $H_0^1(\Omega)$ where G is defined by

$$G(v) := \int_\Omega |\nabla v|^2 dx, \;\; v \in H_0^1(\Omega \setminus \Sigma).$$

Proof: For the Γ-$\limsup$ we will prove pointwise convergence. Let $\varphi \in C_c^\infty(\Omega)$ and take $v_n = v - w_n v$ where w_n is the function defined in the Lemma 6.3.8 then $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ since $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$. We have also $v_n = 0$ on C_n (since $w_n = 1$ on C_n) and

$$\lim_{n\to+\infty} \int_\Omega |\nabla v_n|^2 dx = \int_\Omega |\nabla v|^2 dx + C \int_\Sigma |v|^2 d\mathcal{H}^{d-1}$$

and by the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ the required result follows. To prove the Γ-$\liminf$ inequality let $v_n, v \in H_0^1(\Omega)$ be such that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ and $v_n = 0$ on C_n. Let $\varphi \in C_c^\infty(\Omega)$ be given and set $\varphi_n = \varphi - w_n\varphi$ then it holds

$$\int_\Omega |\nabla v_n|^2 dx \geq 2 \int_\Omega \nabla v_n \nabla \varphi_n dx - \int_\Omega |\nabla \varphi_n|^2 dx.$$

From the proof of Γ-$\limsup$ we have

$$\int_\Omega |\nabla \varphi_n|^2 dx \to \int_\Omega |\nabla \varphi|^2 dx + C \int_\Sigma \varphi^2 d\mathcal{H}^{d-1}.$$

$$\begin{aligned}
\liminf_{n\to+\infty} \int_\Omega \nabla v_n \nabla \varphi_n &= \liminf_{n\to+\infty} \int_\Omega \nabla v_n (\nabla\varphi - \nabla(w_n\varphi)) dx \\
&\geq \liminf_{n\to+\infty} \int_\Omega \nabla v_n \nabla\varphi dx - \limsup_{n\to+\infty} \int_\Omega \nabla v_n \nabla(w_n\varphi) dx. \\
&= \int_\Omega \nabla v \nabla \varphi dx - \limsup_{n\to+\infty} \int_\Omega \nabla(v_n\varphi) \nabla w_n dx
\end{aligned}$$

Using the Green formulas, we get

$$\begin{aligned}
\int_\Omega \nabla(v_n\varphi)\nabla w_n dx &= \sum_j \int_{B_j^n \cap \Omega} \nabla(v_n\varphi)\nabla w_n dx \\
&= \sum_j \int_{\partial B_j^n \cap \Omega} \varphi v_n \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1} + \sum_j \int_{\partial C_j^n \cap \Omega} \varphi v_n \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1} \\
&+ \sum_j \int_{(B_j^n \cap \Sigma)\setminus C_j^n} \varphi v_n \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1} + \sum_j \int_{C_j^n \cap \Sigma} \varphi v_n \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1}.
\end{aligned}$$

On the right hand, the two last terms vanish since $\frac{\partial w_n}{\partial \nu} = 0$ on $(B_j^n \cap \Sigma) \setminus C_j^n$ (due to last condition of the equation satisfied by w_n) and on $C_j^n \cap \Sigma$ (since $w_n = 1$ on C_j^n).

$$\sum_j \chi_{\partial B_j^n} \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1} \to -C\mathcal{H}^{d-1} \llcorner \Sigma$$

in $H_{loc}^{-1}(\mathbb{R}^d)$ strongly and as consequence $\sum_j \chi_{\partial C_j^n} \frac{\partial w_n}{\partial \nu} d\mathcal{H}^{d-1}$ converges strongly to zero in $H_{loc}^{-1}(\mathbb{R}^d)$. Using the fact that $\varphi v_n \rightharpoonup \varphi v$ in $H_0^1(\Omega)$ we obtain

$$\liminf_{n\to+\infty} \int_\Omega \nabla v_n \nabla \varphi_n dx \geq \int_\Omega \nabla v \nabla \varphi dx + C \int_\Sigma v\varphi d\mathcal{H}^{d-1},$$

therefore we have

$$\liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^2 dx \geq 2\int_\Omega \nabla v \nabla \varphi dx + 2C \int_\Sigma v\varphi d\mathcal{H}^{d-1} - \int_\Omega |\nabla \varphi|^2 dx - C\int_\Sigma \varphi^2 d\mathcal{H}^{d-1}.$$

Now, we let φ tend to v strongly in $H_0^1(\Omega)$ and thus getting

$$\liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^2 dx \geq \int_\Omega |\nabla v|^2 dx + C\int_\Sigma v^2 d\mathcal{H}^{d-1}$$

and the first part of the Lemma is proved. For the second point, let us denote by C_n' the union of balls C_j^n with radius r_n instead of δ_n. For all real number $\xi > 1$, we have $r_n \geq \xi\delta_n$ for n large enough. Since balls $\overline{B}_{\xi\delta_n}$ are contained in balls $\overline{B}_{r_n}$ with same centers (that is $\xi C_n \subset C_n'$) then

$$\left\{v \in H_0^1(\Omega \setminus C_n')\right\} \supset \left\{v \in H_0^1(\Omega \setminus \xi C_n)\right\}.$$

Therefore

$$\begin{aligned} S^- &= \Gamma - \liminf_{n\to+\infty} \begin{cases} \int_\Omega |\nabla v|^2 dx & \text{if } v \in H_0^1(\Omega \setminus C_n'), \\ +\infty & \text{otherwise,} \end{cases} \\ &\geq \Gamma - \lim \begin{cases} \int_\Omega |\nabla v|^2 dx & \text{if } v \in H_0^1(\Omega \setminus \xi C_n), \\ +\infty & \text{otherwise,} \end{cases} \\ &= \int_\Omega |\nabla v|^2 dx + \xi^{d-2}\int_\Sigma v^2 d\mathcal{H}^{d-1},\ \ v \in H_0^1(\Omega) \end{aligned}.$$

The coefficient ξ^{d-2} is due to the property of the capacity. This inequality holds for every $\xi > 1$, then we deduce that $S^- \geq G(v)$. The Γ-$\limsup$ holds by pointwise limit as before then $S^+ \leq G(v)$ and the proof is over. As consequence, the solution u_n of the equation (6.49) up to extraction of subsequence converges weakly in $H_0^1(\Omega)$ to the minimizer u of $v \mapsto G(v) + \int_\Omega fvdx$ which is the solution of the equation

$$\begin{cases} -\Delta u &= f \text{ on } \Omega \setminus \Sigma \\ u &= 0 \text{ on } \partial\Omega \cup \Sigma \end{cases}$$

notice that we have used the continuity of the map $v \mapsto \int_\Omega fvdx$ in the weak topology of $H_0^1(\Omega)$ for getting the Γ-convergence of $v \mapsto F_n(v) + \int_\Omega fvdx$ toward the map

$v \mapsto G(v) + \int_\Omega f v dx.$ $\square$

For the case $d = 2$ we cannot use the same result as in the case $d \geq 3$ since Lemma 6.3.8 fails. we adapt the counter example to the continuity of Proposition 4.1.3. The notations are those of Lemma 6.3.9 and we choose δ_n the radius of C_j^n to be equal $e^{-\sqrt{n}}$ and the radius of B_j^n is $1/(2\sqrt{n})$. For every n and j let $w_j^n \in H^1(B_j^n \setminus C_j^n)$ be the solution of the equation $\Delta w_j^n = 0$ on $B_j^n \setminus C_j^n$ which satisfies the boundary conditions $w_j^n = 0$ on ∂C_j^n and $w_j^n = 1$ on ∂B_j^n. An explicit computation of the solution gives

$$w_j^n(x) = \frac{\ln|x - x_j| + \sqrt{n}}{\sqrt{n} - \ln(2\sqrt{n})} \quad \text{for} \quad x \in B_j^n \setminus C_j^n.$$

We define w_n as the function which is equal to w_j^n on $B_j^n \setminus C_j^n$, extended by 0 on C_n and by 1 on $D \setminus B_n$. We may observe that

- $0 \leq w_n \leq 1$;
- $\nabla w_n \rightharpoonup 0$ in $L^2(\Omega)^2$ as $n \to +\infty$, hence w_n converges weakly in $H^1(\Omega)$ to a constant function. The computation of the limit of the integral $\int_\Omega w_n dx$ shows that the constant is equal to 1.

Let $\varphi \in C_c^\infty(\Omega)$. Then $\varphi w_n \in H_0^1(\Omega_n)$, hence φw_n may be chosen as a test function for the equation (6.49):

$$\int_\Omega \nabla u_n \nabla w_n \varphi dx + \int_\Omega \nabla u_n \nabla \varphi w_n dx = \int_\Omega f \varphi w_n dx.$$

The second and the third terms of this equality converge respectively to $\int_\Omega \nabla u \nabla \varphi dx$ and $\int_\Omega f \varphi dx$. For the first term the Green formulas gives

$$\int_\Omega \nabla u_n \nabla w_n \varphi dx = \sum_j \int_{\partial B_j^n} u_n \frac{\partial w_n}{\partial \nu} \varphi d\sigma - \int_\Omega u_n \nabla w_n \nabla \varphi dx.$$

The boundary term on ∂C_j^n does not appear since u_n vanishes on it. The last term of the identity converges to 0 as $n \to \infty$. We get

$$\begin{aligned} \sum_j \int_{\partial B_j^n} u_n \frac{\partial w_n}{\partial \nu} \varphi d\sigma &= \sum_j \int_{\partial B_j^n} \frac{2\sqrt{n}}{\sqrt{n} - \ln(2\sqrt{n})} u_n \varphi d\sigma \\ &= \frac{2\sqrt{n}}{\sqrt{n} - \ln(2\sqrt{n})} \sum_j \int_{\partial B_j^n} u_n \varphi d\sigma. \end{aligned}$$

Let us denote by $\mu_n \in H^{-1}(\Omega)$ the distribution defined

$$\langle \mu_n, \psi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \sum_j \int_{\partial B_j^n} \psi d\sigma.$$

We will prove that this distribution converges strongly in $H^{-1}(\Omega)$ to $\mathcal{H}\llcorner\Sigma$. Let v_n be the solution of the equation

$$\left\{\begin{array}{rcl} -\Delta v_n & = & 4 \text{ in } \cup_j B_j^n \\ v_n & = & 0 \text{ on } \Omega\setminus\cup_j B_j^n. \end{array}\right.$$

then we have

$$\frac{\partial v_n}{\partial \nu} = \frac{1}{\sqrt{n}} \quad \text{on} \quad \bigcup \partial B_j^n.$$

An easy computation shows that $\sqrt{n}v_n \to 0$ strongly in $H^1(\Omega)$, therefore $\sqrt{n}\Delta v_n \to 0$ strongly in $H^{-1}(\Omega)$. One may check also that

$$\begin{aligned} \langle -\Delta v_n, \psi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} &= \sum_j \int_{B_j^n} \nabla v_n \nabla\psi dx \\ &= \sum_j \int_{\partial B_j^n} \frac{1}{\sqrt{n}}\psi d\sigma - \sum_j \int_{B_j^n} 4\psi dx. \end{aligned}$$

Let us introduce the measure $\lambda_n = \sum_j 4\sqrt{n}\mathcal{L}^2\llcorner B_j^n$. This is a sequence of positive finite Radon measure uniformly bounded (one may check easily that $|\lambda_n|(\mathbb{R}^2) \leq C$ for all n and some C independent of n). We have

$$\lambda_n(\mathbb{R}^2) = \int_{\mathbb{R}^2} d\lambda_n = \sum_j 4\sqrt{n}\int_{B_j^n} dx \to \mathcal{H}^1\llcorner\Sigma(\mathbb{R}^2) \text{ as } n \to +\infty.$$

Let A be an open set in $\mathbb{R}^2$ then

$$\mathcal{H}^1(\Sigma\cap A) = \mathcal{H}^1(\Sigma\cap A\cap(\cup_j B_j^n)) = \sum_j \mathcal{H}^1(\Sigma\cap A\cap B_j^n) \leq \sum_j 4\sqrt{n}\mathcal{L}^2(A\cap B_j^n) = \lambda_n(A)$$

where for the two first equalities we have used the fact $(B_j^n)_j$ are pairwise disjoint and cover Σ up to a set of $\mathcal{H}^1$ measure zero ($\Sigma\setminus\cup_j B_j^n$ is a discrete set) and for the inequality we have used $\mathcal{H}^1(\Sigma\cap A\cap B_j^n) \leq 4\sqrt{n}\mathcal{L}^2(B_j^n\cap A)$. Then passing to the lim inf as $n \to +\infty$ in the inequality we get $\liminf \lambda_n(A) \geq \mathcal{H}^1\llcorner\Sigma(A) = \mathcal{H}^1(\Sigma\cap A)$. Since λ_n is a positive finite Radon measure such that $\lambda_n(\mathbb{R}^2)$ converges to $\mathcal{H}^1\llcorner\Sigma(\mathbb{R}^2)$ and for any open set A of $\mathbb{R}^2$ it holds $\liminf \lambda_n(A) \geq \mathcal{H}^1\llcorner\Sigma(A) = \mathcal{H}^1(\Sigma\cap A)$ then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2}\psi d\lambda_n = \int_{\mathbb{R}^2}\psi d\mathcal{H}^1\llcorner\Sigma = \int_\Sigma \psi d\mathcal{H}^1$$

for any continuous and bounded function ψ. The measure λ_n weakly converges to $\mathcal{H}^1\llcorner\Sigma$ and since $\sum_j 4\sqrt{n}\chi_{B_j^n}$ is L^∞ bounded the convergence is strong in $H^{-1}(\Omega)$. From this we deduce

$$\mu_n \to \mathcal{H}^1\llcorner\Sigma \text{ strongly in } H^{-1}(\Omega).$$

Consequently $u \in H_0^1(\Omega)$ satisfies the equation

$$\forall \varphi \in C_c^\infty(\Omega) \quad \int_\Omega \nabla u \nabla \varphi dx + 2 \int_\Sigma u \varphi d\mathcal{H}^1 = \int_\Omega f \varphi dx.$$

If we change δ_n into the radius of balls composing Σ_n that is $r_n = e^{-n/\beta}$ then the solution of the equation (6.49) converges strongly to the solution of equation

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases} \tag{6.50}$$

Lemma 6.3.10. *For any $\beta > 0$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, we find $\beta" < \beta$ and a set $\Sigma \in \mathcal{A}_R(\beta", n)(I^d)$ which is almost boundary-covering, with $\int_{I^d} u_{1,\Sigma,I^d} dx < (1+\varepsilon)\theta^+(\beta)$.*

Proof: let us fix $\delta > 0$ and $\beta' < \beta$ such that $\theta(\beta') < (1+\delta)\theta^+(\beta)$. From the definition $\theta(\beta')$ there exists a set $\Sigma_1 \in \mathcal{A}_R(\beta', n_1)(I^d)$ such that

$$\int_{I^d} u_{1,\Sigma_1,I^d} dx < (1+\delta)\theta(\beta')$$

and, moreover the number n_1 may be chosen as large as we want. Now we enlarge the set Σ_1 to get a set Σ_2 which is almost boundary-covering: we add to Σ_1 some m balls of radius $r = r_{n_1}(\beta')$ where $r_{n_1}(\beta') = (\frac{\beta}{n_1})^{1/(d-2)}$ if $d \geq 3$ and $r_{n_1}(\beta') = e^{\frac{-n_1}{\beta'}}$ if $d = 2$ (the same radius of balls composing Σ_1). In order to almost cover ∂I^d in the sense of our definition, we chose m so that $n_1\delta > m \gg n_1^{1-\varepsilon}$ where $0 < \varepsilon \leq 1/d$. We distribute uniformly the centers of the m balls on the boundary of the unit cube. It follows from the previous Lemma that Σ_2 is almost boundary-covering set. With our setting it is possible to chose n_1 so that $m \leq \delta n_1$ and $\beta'(\frac{n_1+m}{n_1}) = \beta" < \beta$. Notice that

$$\Sigma_2 \in \mathcal{A}_R\left(\beta'(\frac{n_1+m}{n_1}), n_1+m\right)(I^d).$$

If we set $n_2 = n_1 + m$ then $\Sigma_2 \in \mathcal{A}(\beta", n_2)(I^d)$. Moreover

$$\int_{I^d} u_{1,\Sigma_2,I^d} dx \leq \int_{I^d} u_{1,\Sigma_1,I^d} dx < (1+\delta)^2\theta^+(\beta).$$

Let us choose n_0 to be the smallest value of n_2. Now, if we are giving a large number n, by the flexibility of the choice of n_1, we may choose n_1 (and $m \leq \delta n_1$) so that $n_2 = n$ and moreover the new set Σ_{n_2} is still almost boundary-covering. It is sufficient to choose δ sufficiently small so that $(1+\delta)^2 < 1+\varepsilon$ to get the result. $\square$

Lemma 6.3.11. *For any $\beta > 0$, it holds $\theta(\beta) \leq \beta^{\frac{-2}{d}}\theta^-(0)$*

Proof: If $\beta \leq 1$, the inequality follows easily from the definition of θ^- and the fact that $\beta^{\frac{-2}{d}} \geq 1$. To prove the inequality in the case where $\beta > 1$, we will prove for every integer k and $\alpha > 0$ the inequality $\theta(k^2\alpha) \leq k^{-2}\theta(\alpha)$. For any $\Sigma \in \mathcal{A}(\alpha, n)(I^d)$ we take Σ_k as its homogenization of order k into I^d. The capacity of Σ_k is $k^2\alpha$. Due to the rescaling property of the Laplacian operator we have

$$k^2 \int_{I^d} u_{1,\Sigma_k,I^d} dx = \int_{I^d} u_{1,\Sigma,I^d} dx.$$

Then passing to the lim inf as $n \to +\infty$ and minimizing over $\Sigma \in \mathcal{A}(\alpha, n)(I^d)$, the right hand side of the equality is equal to $\theta(\alpha)$ wile the left hand side is bigger than $k^2\theta(k^2\alpha)$ since the set $\{\Sigma_k : \Sigma \in \mathcal{A}(\alpha, n)(I^d)\}$ is a subset of $\mathcal{A}(k^2\alpha, k^d n)(I^d)$. The inequality $\theta(k\alpha) \leq k^{-2}\theta(\alpha)$ follows. Now let δ be a positive number such that $\lfloor(\beta+\delta)^{\frac{1}{d}}\rfloor^{-2} \leq \beta^{\frac{-2}{d}}$ and set $k = \lfloor(\beta+\delta)^{\frac{1}{d}}\rfloor$

$$\theta(\beta) = \theta(k^2 \times k^{-2}\beta) \leq k^{-2}\theta(k^{-2}\beta) \leq k^{-2}\theta^-(0) \leq \beta^{\frac{-2}{d}}\theta^-(0),$$

where we have used the previous inequality with $\alpha = k^{-2}\beta$. With this we conclude the proof. □

As before we will prove first the Γ-lim sup inequality for a class of piecewise constant probability measures. Let $\mu \in \mathcal{P}(\overline{\Omega})$ be of the form

$$\mu = \rho dx, \quad \text{with } \rho \in L^1(\Omega), \quad \int_\Omega \rho dx = 1, \ \ \rho > 0,$$

for a piecewise constant function $\rho = \sum_{j=0}^m \rho_j I_{\Omega_j}$, the pieces Ω_j being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \cup_{j=1}^m \Omega_j$. For the simplicity of the notation, we set

$$\tilde{F}(\mu) = \theta^-(0) \int_\Omega \frac{f^2}{\mu_a^{2/d}} dx.$$

Proposition 6.3.12. *Under the same hypotheses of the Theorem 6.3.3, we have*

$$F^+(\mu) \leq \tilde{F}(\mu), \text{where } F^+ = \Gamma - \limsup_n F_n,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$. This means that, for any such a measure μ and any $\varepsilon > 0$, there exists a sequence of sets $(\Sigma_n)_n$ such that μ_{Σ_n} weakly converges to μ, $\Sigma_n \in \mathcal{A}(\beta, n)(\Omega)$ and moreover*

$$\limsup_n \int_\Omega f u_{f,\Sigma_n,\Omega} dx \leq (1+\varepsilon)\theta^-(0) \int_\Omega \frac{f^2}{\rho^{2/d}} dx.$$

Proof: By applying the Lemma 6.3.10 to all of the numbers ρ_j, we may find some numbers n_j and some sets $\Sigma^j \in \mathcal{A}(\rho_j", n_j)(I^d)$ which are all almost boundary-covering and such that

$$\int_{I^d} u_{1,\Sigma^j,I^d} dx < (1+\varepsilon)\theta^+(\rho_j).$$

Since n_j may be chosen as large as we want, we choose them in such a way that all balls have approximatively the same radius. Now we define for every integer k and every $j = 0, \cdots, m$ the set

$$A_{k,j} = \left\{ y \in k^{-1}\mathbb{Z}^d : (y + k^{-1}\Sigma^j) \cap \Omega_j \neq \emptyset \right\}.$$

and the following sets.

$$\Sigma_1^j := \bigcup_{y \in A_{1,j}} y + \Sigma^j, \quad \Sigma_k^j := \bigcup_{y \in A_{n,j}} y + k^{-1}\Sigma^j, \quad \Sigma_{k,1}^j := \bigcup_{y \in A_{n,j}} y + k^{-1}\Sigma_1^j.$$

We may notice that

$$\Sigma_1^j \in \mathcal{A}(\rho_j'|\Omega_j|, |\Omega_j| n_j) \text{ and } \Sigma_k^j \subset \Sigma_{k,1}^j, \text{ for } j = 0, \cdots, m \text{ and } \forall k,$$

and the total number of balls in the union is $n_1 = \sum_{j=0}^m \rho_j' n_j$ and the capacity is $\sum_{j=0}^m \rho_j'|\Omega_j| < 1$ since $\rho_j' < \rho_j$ for every j. Let Σ_{m_1} be a set of m_1 balls for almost covering the internal boundary of the union of Ω_j inside Ω. Thanks to Lemma 6.3.10, it is sufficient to choose m_1 so that $n_1^{1-\varepsilon} \leq m_1 < n_1$ for $1/d \leq \varepsilon < 1$. We chose m_1 so that $m_1/n_1 \to 0$ as $n \to +\infty$. Define the set $\Sigma_n = \Sigma_{m_1} \cup \bigcup_{j=0}^m \Sigma_1^j$ then $\Sigma_n \in \mathcal{A}(\beta_n, n)(\Omega)$ where $n = n_1 + m_1$ and $\beta_n = cap(\Sigma_n) \approx \frac{m_1 \rho_j'}{n_j} + \sum_{j=0}^m \rho_j'|\Omega|$. An easy computation shows that

$$n = \frac{1}{\rho_j}\left(1 + \frac{m_1}{n_1}\right) n_j.$$

As consequence $\frac{n_j}{n} \to \rho_j$ as $n \to +\infty$ and $\beta_n \to \sum_{j=0}^m \rho_j'|\Omega|$ as $n \to +\infty$ because $n_j \approx \rho_j n$ and $m_1/n \to 0$ as $n \to +\infty$. We have constructed a set $\Sigma_n \in \mathcal{A}(\beta_n, n)(\Omega)$ and the measure $\mu_n = \mu_{\Sigma_n}$ *weakly** converges to the measure $\mu = \rho dx$. Now it remains the estimate of the lim sup of the quantity $\int_\Omega f u_{f,\Sigma_n,\Omega} dx$ as n tends to the infinity. We will do estimate first on Ω_j. It holds

$$\int_{\Omega_j} f u_{f,\Sigma_1^j,\Omega_j} dx = k^2 \int_{\Omega_j} f u_{f,\Sigma_{k,1}^j,\Omega_j} dx \leq k^2 \int_{\Omega_j} f u_{f,\Sigma_k^j,\Omega_j} dx \quad \forall k \in \mathbb{N}^*,$$

where the equality follows from the rescaling property of the Laplacian operator and the inequality from $u_{f,\Sigma_{k,1}^j,\Omega_j} \leq u_{f,\Sigma_k^j,\Omega_j}$ which is a consequence of the fact that $\Sigma_k^j \subset \Sigma_{k,1}^j$ and the maximum principle. Passing to the limit as $k \to +\infty$ and using Lemma 6.3.6 we get

$$\int_{\Omega_j} f u_{f,\Sigma_1^j,\Omega_j} dx \leq \lim_{k \to +\infty} k^2 \int_{\Omega_j} f u_{f,\Sigma_k^j,\Omega_j} dx = C(\Sigma^j) \int_{\Omega_j} f^2 dx.$$

The Lemma 6.3.10, Lemma 6.3.11 and the definition of θ^+ give for $\varepsilon < \rho_j$

$$C(\Sigma^j) < (1+\varepsilon)\theta^+(\rho_j) \leq (1+\varepsilon)\theta(\rho_j - \varepsilon) \leq (1+\varepsilon)(\rho_j - \varepsilon)^{\frac{-2}{d}}\theta^-(0).$$

Therefore

$$\int_{\Omega_j} f u_{f,\Sigma_1^j,\Omega_j} dx \leq (1+\varepsilon)(\rho_j - \varepsilon)^{\frac{-2}{d}}\theta^-(0) \int_{\Omega_j} f^2 dx.$$

The Lemma 6.3.9 allows us to write

$$\int_{\Omega_j} f u_{f,\Sigma_n,\Omega} dx = \int_{\Omega_j} f u_{f,\Sigma_n,\Omega_j} dx + o(n_j) = \int_{\Omega_j} f u_{f,\Sigma_1^j,\Omega_j} dx + o(n_j)$$

and then

$$\int_{\Omega} f u_{f,\Sigma_n,\Omega} dx = \sum_{j=0}^{m} \int_{\Omega_j} f u_{f,\Sigma_n,\Omega} dx = \sum_{j=0}^{m} \int_{\Omega_j} f u_{f,\Sigma_1^j,\Omega_j} dx + o(n).$$

The previous result implies that

$$\limsup_{n\to+\infty} \int_{\Omega} f u_{f,\Sigma_n,\Omega} dx \leq (1+\varepsilon)\theta^{-}(0) \sum_{j=0}^{m} (\rho_j - \varepsilon)^{\frac{-2}{d}} \int_{\Omega_j} f^2 dx,$$

and the desired result follows by letting ε goes to zero that is

$$\limsup_{n\to+\infty} \int_{\Omega} f u_{f,\Sigma_n,\Omega} dx \leq \theta^{-}(0) \sum_{j=0}^{m} \rho_j^{\frac{-2}{d}} \int_{\Omega_j} f^2 dx = \theta^{-}(0) \int_{\Omega} \frac{f^2}{\rho^{\frac{2}{d}}} dx.$$

□

The extension of this result to the case of a generic probability measure is the same as those in Proposition 6.2.22 and Proposition 6.2.11 and the proof is the same.

6.3.3 Some estimate on θ

This section is dedicated to the study of the function θ. We already stressed that θ is non increasing function and vanishes from a point t_1 on. In the following, we prove that θ is piecewise constant.

Proposition 6.3.13.

$$\theta(\beta) = \begin{cases} \theta^{-}(0) & \text{if} \quad \beta < t_1 \\ 0 & \text{if} \quad \beta \geq t_1 \end{cases}$$

Proof: From the definition of the function θ we have

$$\begin{aligned} \theta(\beta) &= \inf \left\{ \liminf_n F(\Sigma_n, 1, I^d) : \Sigma_n \in \mathcal{A}(\beta, n)(I^d) \right\} \\ &= \liminf_n \min \left\{ F(\Sigma_n, 1, I^d) : \Sigma_n \in \mathcal{A}(\beta, n)(I^d) \right\}. \end{aligned}$$

By the general theory of Γ-convergence such a $\liminf$ is in fact a limit equal to the minimum of the limit problem in I^d with $f = 1$. Therefore

$$\begin{aligned} \theta(\beta) &= \min \left\{ \theta^{-}(0) \int_{I^d} \rho^{-2/d} dx : \rho \geq 0, \rho \in L^1(I^d), \int_{I^d} \rho dx = 1 \right\} \\ &= \theta^{-}(0) \min \left\{ \int_{I^d} \rho^{-2/d} dx : \rho \geq 0, \rho \in L^1(I^d), \int_{I^d} \rho dx = 1 \right\}. \end{aligned}$$

It is clear that the minimum is achieved for $\rho = 1$ and consequently $\theta(\beta) = \theta^{-}(0)$ for $\beta < t_1$. $\square$

Remark 6.3.14 From Lemma 6.3.11 and Proposition 6.3.13, it follows that t_1 (the first vanishing point of the function θ) is less or equal to 1.

It remains now to prove that $\theta^{-}(0)$ is neither 0 nor $+\infty$ so that our limit functional is not trivial. First we will prove that $\theta(\beta) < +\infty$ for any $\beta > 0$.

Proposition 6.3.15. *For any $\beta > 0$, $\theta(\beta) < +\infty$.*

Proof: To prove that $\theta(\beta)$ is finite for every $\beta > 0$ it is sufficient to consider a particular sequence of sets $\Sigma_n \in \mathcal{A}(\beta, n)(I^d)$ and then compute the lim inf in the definition of $\theta(\beta)$. Let us consider the number n of the form $n = k^d$ where $k \in \mathbb{N}$ and build for each k, a set Σ_n which is composed by $n = k^d$ balls of radius $r = (\frac{\beta}{n})^{1/(d-2)}$ if $d \geq 3$ and $r = e^{-\frac{n}{\beta}}$ if $d = 2$ with their centers placed at the middle points of a k^d cubes of side $1/k$ of a regular lattice partitioning the cube I^d. The set $\Sigma_n \in \mathcal{A}(\beta, n)(I^d)$ and the Lemma 6.3.4 gives

$$||u_{1,\Sigma_n,I^d}||_{L^2(I^d)} \leq C(d).$$

Moreover by the maximum principle $u_{1,\Sigma_n,I^d} \leq v$ where v is the solution of the equation

$$\begin{cases} -\Delta v &= 1 \text{ in } B_{r_0}(x_0) \\ v &= 0 \text{ on } \partial B_{r_o}(x_o) \end{cases}$$

$r_0 = \frac{\sqrt{d}}{2}$ is the radius of the smallest ball containing the cube I^d and centered at its same center and x_0 the center of the unit cube. The function v may be explicitly computed and an easy calculation shows that

$$\int_{I^d} u_{1,\Sigma_n,I^d} dx \leq \int_{I^d} v dx \leq \frac{d}{4}.$$

As consequence for all $\beta > 0$ we have $\theta(\beta) \leq \frac{d}{4}$. $\square$

We complete the part of the property of the function θ by proving that $\theta^{-}(0) > 0$.

Proposition 6.3.16. *For any $0 < \beta < t_1$, we have $\theta(\beta) > 0$*

Proof: For a fixed $n \in \mathbb{N}$, let us take n fixed points $(x_j)_{j=1}^n \in I^d$, consider the set $\Sigma_\beta = \bigcup_{j=1}^n \overline{B(x_j, r)} \in \mathcal{A}(\beta, n)(I^d)$ and set $\Omega_\beta = I^d \setminus \Sigma_\beta$. By the Holder inequality, we get the following estimate:

$$\mathcal{H}^{d-1}(\partial\Omega_\beta) \left(\int_{\partial\Omega_\beta} |\frac{\partial}{\partial\nu} u_{1,\Sigma_\beta,I^d}|^2 d\mathcal{H}^{d-1} \right) \geq \left(\int_{\partial\Omega_\beta} \frac{\partial}{\partial\nu} u_{1,\Sigma_\beta,I^d} d\mathcal{H}^{d-1} \right)^2 = |\Omega_\beta|^2, \quad (6.51)$$

where the last equality follows by integrating by part $\int_{\Omega_\beta} -\Delta u_{1,\Sigma_\beta,I^d} dx$. We may estimate as well the surface measure of the boundary of Ω_β. More precisely

$$\mathcal{H}^{d-1}(\partial\Omega_\beta) \le nd\omega_d(r_n(\beta))^{d-1}, \tag{6.52}$$

where $r_n(\beta)$ is the radius of the balls and depends on the dimension d. We use now shape derivative (see Corollary 4.3.3) by perturbing the domain Ω_β by a vector field which is normal to the boundary of the balls and proportional to n_d and get

$$-\frac{d}{d\beta}F(\Sigma_\beta, 1, I^d) = n_d \int_{\partial\Omega_\beta} |\frac{\partial}{\partial\nu}u_{1,\Sigma_\beta,I^d}|^2 d\mathcal{H}^{d-1}, \tag{6.53}$$

where

$$n_d = \begin{cases} n^{\frac{-1}{d-2}} & \text{if} \quad d \ge 3 \\ e^{-n} & \text{if} \quad d = 2 \end{cases}$$

Let us consider first the case $d \ge 3$. By combining equations (6.51), (6.52) and (6.53) it holds

$$\begin{aligned} -\frac{d}{d\beta}F(\Sigma_\beta, 1, I^d) &\ge \frac{n_d|\Omega_\beta|^2}{nd\omega_d(r_n(\beta))^{d-1}} \ge \frac{n_d(1-\omega_d r_n^d(\beta))^2}{nd\omega_d(r_n(\beta))^{d-1}} \\ &\ge \frac{1}{d\omega_d}\beta^{-\frac{d-1}{d-2}} - \frac{2}{\omega_d}\beta^{-1}n^{-\frac{d}{d-2}}, \end{aligned}$$

where we have used the inequalities $|\Omega_\beta|^2 \ge (1-\omega_d r_n^d(\beta))^2 \ge (1-2w_d r_n^d(\beta))$ to get the two last inequalities. For any $\beta \in (0, t_1)$, we integrate the inequality over the interval (β, t_1) and obtain

$$\begin{aligned} F(\Sigma_\beta, 1, I^d) \ge F(\Sigma_{t_1}, 1, I^d) &+ \frac{d-2}{d\omega_d}\left(\beta^{\frac{-1}{d-2}} - t_1^{\frac{-1}{d-2}}\right) \\ &- \frac{2}{\omega_d}n^{-d/(d-2)}(ln(\frac{t_1}{\beta})). \end{aligned}$$

Passing to the inf over $(x_j)_j$ and to the lim inf over n, we get

$$\theta(\beta) \ge \theta(t_1) + \frac{d-2}{d\omega_d}\left(\beta^{\frac{-1}{d-2}} - t_1^{\frac{-1}{d-2}}\right).$$

Using the fact that $\theta(t_1) = 0$ and $\beta < t_1$, we get

$$\theta(\beta) \ge \frac{d-2}{d\omega_d}\left(\beta^{\frac{-1}{d-2}} - t_1^{\frac{-1}{d-2}}\right) > 0.$$

For the case $d = 2$ we have

$$-\frac{d}{d\beta}F(\Sigma_\beta, 1, I^d) \ge \frac{1}{2\omega_2} - e^{-\frac{2n}{\beta}}.$$

For any $\beta \in (0, t_1)$, we integrate the inequality as before over (β, t_1) and get

$$F(\Sigma_\beta, 1, I^d) \ge F(\Sigma_{t_1}, 1, I^d) + \left(\frac{t_1 - \beta}{2\omega_2} - \int_\beta^{t_1} e^{-\frac{2n}{\beta}}d\beta\right).$$

Following the same argument as in the case $d \geq 3$, and using the fact that $\int_\beta^{t_1} e^{-\frac{2n}{\beta}} d\beta$ converges to zero as $n \to +\infty$ and $\omega_2 = \pi$ we have

$$\theta(\beta) \geq \frac{t_1 - \beta}{2\pi} > 0$$

which concludes the proof. □

6.3.4 One dimensional case

In the case of dimension 1 we are able to compute explicitly the function θ. Everything is simpler in dimension 1 since the balls we removed are intervals which disconnect the domain of the differential equation which is in this case an ODE, and so we can compute explicitly the solution. We point out that in dimension 1 the compliance is well-posed also for finite union of points and not only for small intervals. We can consider also the minimization problem with n points instead of n balls with fixed capacity. In this case, for an open interval $J = (a, b)$ we have the following functional

$$F_n(\mu) = \begin{cases} n^2 F(\Sigma, f, J) & \text{if } \mu = \mu_\Sigma \text{ and } \#\Sigma \leq n \\ +\infty & \text{otherwise} \end{cases}$$

Theorem 6.3.17. *Let $[a, b]$ be an interval of $\mathbb{R}$ and $f \in L^2([a, b])$ be a given nonnegative function. Then the functional F_n Γ-converges with respect to the weak start topology of $\mathcal{P}([a, b])$ to the functional*

$$F(\mu) = \frac{1}{12} \int_a^b \frac{f^2}{\mu_a^2} dx$$

Proof: If we set $\theta = \inf\{n^2 F(\Sigma, 1, (0, 1)), \quad \#\Sigma \leq n\}$ then from the section of asymptotic of an optimal p-compliance-location we have the Γ convergence of F_n with respect to the *weak** topology of $\mathcal{P}([a, b])$ to the functional $\theta \int_a^b \frac{f^2}{\mu_a^2} dx$ (proofs are essentially the same). It remains to show that $\theta = \frac{1}{12}$. Let Σ be the set of n distinct points of $(0, 1)$ then Σ partition $(0, 1)$ in $n + 1$ intervals. Let

$$x_0 = 0 < x_1 <, \cdots, < x_n = 1$$

be this partition and $l_j = x_{j+1} - x_j$ for $j = 0, \cdots, n$ the length of each subinterval (x_j, x_{j+1}). On (x_j, x_{j+1}) we have $u(x) = -\frac{(x - x_{j+1})(x - x_j)}{2}$. The energy is

$$I(u) = \frac{1}{12} \sum_{j=1}^{n} l_j^3.$$

We minimize $I(u)$ under the constraint $\sum_{j=1}^n l_j = 1$. By the convexity of the function $l \mapsto l^3$, the minimum is achieved for $l_j = n^{-1}$, and so $I(u) = (1/12)n^{-2}$ and hence $\theta = \liminf n^2 I(u) = 1/12$ and the proof is over. □

Bibliography

[1] R.S. Adams: *Sobolev Spaces*, Academic Press, 1975

[2] D.R. Adams, L.I. Hedberg: *Functions spaces and potential theory*, Springer 1996

[3] G. Allaire: *An introduction to homogenization and bound of effective properties applied to optimal shape design*, School on Homogenization, ICTP Trieste, 1993.

[4] G. Allaire: *Shape optimization by the homogenization method*. Springer Verlag, new york (2002).

[5] G. Allaire: *Conception optimale de structure*, cours Ecole Polytechnique, (2004).

[6] G. Allaire, A. Henrot: *On some recent advances in shape optimization*, C.R. Acad. Sci. Paris Ser. IIb Mecanique (le point sur concise review paper), 329 (2001), 383-396.

[7] L. Ambrosio: *Lecture note on mass transportation problems. Proceeding of the CIME school held in Martina Franca (Italy)*, Lecture note in Mathematics, Springer-Verlag (2003).

[8] L. Ambrosio, G. Buttazzo: *An optimal design problem with perimeter penalization*, Calc. Var., 1 (1993), 55-69.

[9] L. Ambrosio, I. Fonseca, P. Marcellini, L. Tartar: *On a volume constrained variational problem*, Arch. Rat. Mech. Anal., 149(1)21-47 (1999).

[10] L. Ambrosio, N. Fusco, and D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford mathematical monographs. Oxford University Press, Oxford, (2000).

[11] L. Ambrosio, P. Tilli: *Selected topics on "Analysis in metric spaces"*, Springer-Verlag

[12] N. Ansini: *The nonlinear sieve problem and applications to thin films*, Asymptotic Analysis 39 (2004) 113-145.

[13] N. Ansini, A. Braides: *Asymptotic analysis of periodically-perforated nonlinear media*, J. Math. Pures Appl. 81 (2002) 439-451.

[14] H. Attouch: *Variational convergence for functions and operators*, Applicable Math. serie, Pitman, London (1984).

[15] H. Attouch, C. Picard: *Variational inequalities with varying obstacles: the general form of the limit problem*, J. Funct. Anal.,50 no3 (1983), 329-386.

[16] T. Bagby, *Quasi topologies and rational approximation*, J. Funct. Anal., 19 (6),581-597, 1992.

[17] C. Bandle: *Isoperimetric inequalities and applications*, Pitman, London (1980).

[18] N. Banichuk: *Introduction to optimization of structures*, Springer-Verlag, New York, (1990).

[19] M. Barkoutou, M. Pierre: *Un résultat d'existence en optimisation de forme en utilisant une propriété géométrique de la normale*, ESAIM COCV, vol. 2, (1997), 105-123.

[20] E. Barozzi, E. H. A. Gonzalez: *Least area problems with a volume constraint*, In Variational methods for equilibrium problems of fluids, Astérisque 118, (1984), 33-53.

[21] M. Bendsoe: *Methods for optimization of structural topology, shape and material*, Springer Verlag (1995).

[22] M. Bendsoe, C. Mota Soares: *Topology optimization of structures*, Kluwer Acadmic Press, Dordrechts (1993).

[23] M.Bendsoe, O. Sigmund: *Topology Optimization, Theory, Methods and Application*, Springer Verlag (2003).

[24] L. Boccardo and F.Murat. *Almost everywhere convergence of the gradients of solutions of Elliptic and Parabolic equations*, Nonlinear Analysis, Theory, Methods and Applications, 10 (1972), 259-268.

[25] R. Bolton, F. Morgan: *Hexagonal economic regions solve the location problem*, Amer. Math. Monthly 109(2) (2002), 165-172.

[26] G. Bouchitté, G. Buttazzo: *New lower semicontinuity results for non convex functionals defined on measures*, Nonlinear Anal. 15(7)(1990) 679-692.

[27] G. Bouchitté, G. Buttazzo: *Integral representation of nonconvex functionals defined on measures*, Ann. Inst. H. Poincaré Anal. Aon Linéaire. 9, no. 1 (1992) 101-117.

[28] G. Bouchitté G. Buttazzo, I. Fragala: *Mean curvature of a measure and related variational problems*, Ann. Scuola Norm. Sup. Cl. Sci., 25(4):179-196, (1997).

[29] G. Bouchitté, C. Jimenez, M. Rajesh: *Asymptotique d'un probléme de positionement optimal*, C. R. Acad. Sci. Paris Ser. I, 335 (2002) 1-6.

[30] A. Braides: *Γ-convergence for Beginners*, Oxford University Press, Oxford (2002).

[31] D. Bucur, G. Buttazzo: *Variational Methods in Shapes Optimization Problems*. Progress in Nonlinear Differential Equation and Their Application, Vol. 65, Birkhauser, Basel, (2005).

[32] D. Bucur, G. Buttazzo, A. Henrot: *Existence results for some optimal partition problems*, Adv. Math. Sci. Appl., 8 no 2 (1998) 571-579.

[33] D. Bucur, P. Trebeschi: *Shape optimization governed by nonlinear state equations*, Proc. Roy. Soc. Edinburgh Sec: A, 128 (1998), 945-963.

[34] D. Bucur and J.P. Zolesio: *Wiener Criterion and shape continuity for the Dirichlet problem*, Bollettino U.M.I

[35] D. Bucur and J.P. Zolesio: *n-dimensional Shape Optimization under Capacitary constraints*, J. of Diff. Eq.,123, No2 (1995), 504-522.

[36] G. Buttazzo: *Spectral optimization problems*, Preprint, available at http://cvgmt.sns.it/cgi/get.cgi/papers/but10/santalo.pdf.

[37] G. Buttazzo and G. Dal Maso: *Shape Optimization for Dirichlet problems: Relaxed formulation and the Optimality Conditions*, Appl. Math. Optim.,23 (1991), 17-49.

[38] G. Buttazzo and G. Dal Maso: *An Existence Result for a class of Shape Optimization Problems*, Arch. Rational Mech. Anal. 122 (1993), 183-195.

[39] G. Buttazzo, G. Dal Maso, A. Garroni, A. Malusa: *On the relaxed formulation of some shape optomization problems*, Adv. Math. Sci. Appl 7 no 1 (1997) 1-24.

[40] G. Buttazzo, V. Ferone, B. Kawohl: *Minimum problems over sets of concave functions and related questions*,Math. Nachr.,173 (1995), 71-89.

[41] G.Buttazzo, E. Oudet, E. Stepanov: *Optimal transportation problems with free Dirichlet regions*, Progress in nonlinear Diff. Equation and their Apllication, 51, 41-65.

[42] G. Buttazzo, F. Santabrogio: *Asymptotical compliance optimization for connected network*, Networks and Heterogeneous Media, Vol II, no4,(2007) 761-777.

[43] G. Buttazzo, E. Stepanov: *Optimal transportation networks as free Dirichlet regions in the Monge-Kantorovich problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., II(4) (2003), 631-678.

[44] G. Buttazzo, E. Stepanov: *Minimization problems for average distance functionals.* In Calculus of variations: topics from the mathematical heritage of E. De Giorgi, volume 14 of Quad. Mat., pages 48-83. Dept. Math., Seconda Univ. Napoli, Caserta, (2004).

[45] D. Chenais, *On the existence of a solution in a domain identification problem*, J.of Math. Anal. and Appl.,52 (1975), 189-289.

[46] M. Chipot, G. Dal Maso: *Relaxed shape optimization: the case of nonnegative data for the Dirichlet problem*, Adv. Math. Sci. Appl., 1 (1992), 47-81.

[47] D. Cioranescu, F. Murat: *Un terme étrange venu d'ailleurs*, Nonlinear partial differential equations and their applications, College de France Seminar, Vol. II (1982), 98-138 and Vol. III (1982), 154-178.

[48] R. Courant, D. Hilbert: *Methods of mathematical physics*, Vol. I. Interscience Publishers, Inc., New York, N.Y. (1953).

[49] G. Dal Maso: *On the integral representation of certain local functionals*, Ricerche Mat., 32 (1983), 85-113.

[50] G. Dal Maso: *Γ-convergence and μ-capacities*, Ann. Scuola Norm. Sup. Pisa, 14 (1988), 423-464.

[51] G. Dal Maso: *An introduction to Γ-convergence*, Birkhauser, Basel, (1992).

[52] G. Dal Maso and A. Defranceschi, *Limits of Nonlinear Dirichlet problems in varying Domains*, Manuscripta Math.,61 (1988), 251-278.

[53] G. Dal Maso, F. Murat: *Asymptotic behavior and corrector for the Dirichlet problem in perforated domains with homogeneous monotone operators*, Ann.Sc Norm. Sup. Pisa Cl. Sci Ser. 4, 24 (1997), 239-290.

[54] G. Dal Maso, I.V. Skrypnik: *Capacity theory for monotone operators*, Potential Analysis 7: 765-803, (1997).

[55] M. Delfour, J.P. Zolesio: *Shapes and geometry. Analysis, Differential calculus, and Optimization*, Advances in Design and Control SIAM, Philadelphia, PA, (2001).

[56] C. Dellacherie: *Ensemble analytiques, capacités, mésure de Hausdorff*, Lecture Notes in Mathematics, Vol. 295. Springer-Verlag, Berlin-New York (1972).

[57] L.C. Evan, R.F. Gariepy: *Lecture note on measure theory and fine properties of functions*, CRC Press, (1992).

[58] K.J. Falconer: *The geometry of fractal sets*, Cambridge University Press, (1995).

[59] H. Federer: *Geometric measure theory*, Springer, (1969).

[60] L. Fejes Tóth: *Lagerungen in der Ebene auf der Kugel und im Raum*, Die Grundlehren der Math. Wiss., Vol. 65, Springer-Verlag, Berlin, (1953).

[61] R. Finn: *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften, Band 284, Springer-Verlag, New York, (1986).

[62] B. Fuglede: *Finely harmonic functions*, Lecture notes in Math., 289, Springer, Berlin, Heidelberg, New York (1972).

[63] J. Haslinger, R.A.E. Mäkinen: *Introduction to shape optimization. Theory, approximation and computation*, Advances in Design and Control SIAM, Philadelphia, (2003).

[64] J. Haslinger, P. Neitaanmäki: *Finite element approximation for optimal shape, material and topology design*, Second edition, John Wiley and Sons, Ltd.,Chichestet, (1996).

[65] F. Hausdorff: *Set theory*, Chelsea, (1962).

[66] J. Heinonen, T. Kilpela:inen, O. Martio: *Nonlinear potential theory of Degenerated Elliptics Equations*, Oxford science Pub.,(1975).

[67] A. Henrot, M. Pierre: *Variation et Optimisation de formes, une analyse géométrique*, Springer (2005).

[68] E. J. Hruslov: *The method of orthogonal projections and the Dirichlet boundary value problem in domain with "fine-grained" boundary*, (Russian) Mat. Sb. (N.S.) 88 (130) (1972), 38-60.

[69] D. Gilbarg , N.S. Trundinger: *Elliptics Partial Differential Equations of second order*, Reprint of the 1998 edition, Classic in Mathematics, Springer-verlag, Berlin (2001).

[70] B. Kawohl, O. Pironneau, L. Tartar, J.P. Zolesio: *Optimal shape design*, Lectures given at the joint C.I.M./C.I.M.E. Summer School held in Tróia, June 1-6, 1998. Edited by A. Cellina and A. Ornelas. Lecture notes in Mathematics, 1740, (2000).

[71] R. V. Kohn, G. Strang: *Optimal design and variational relaxation of variational problems*, I, II, III, Comm. Pure Appl. Math., 39 (1986), 113-137, 139-182, 353-377.

[72] N.S. Landkof: *Foundations of modern potential theory*, Springer Berlin (1972).

[73] E. Laporte, P. Le Tallec: *numerical methods in sensitivity analysis and shape optimization*, Birkhäuser, (2003).

[74] V.G. Maz'ja: *Sobolev spaces*, Springer, (1985).

[75] F. Morgan: *Geometric measure theory, a beginer's guide*, Academic Press, (1995).

[76] S. Mosconi, P. Tilli: Γ-*convergence for the irrigation problem*, J. of conv. Anal. 12 no.1 (2005) 145-158.

[77] J. Mossino: *Inégalités isopérimt́riques et apllications en physique*, Travaux en cours, Hermann, Paris (1984).

[78] F. Murat: *the neumann sieve*, Nonlinear variational problems (Isola d'Elba, 1983), 24-32, Res. Notes in Math., Pitman, Boston Mass., London, (1985).

[79] F. Murat, J. Simon: *Quelques résultats sur sur le contrôle par un domain gémétrique*, Publ. du Labo. d'Anal. Num., Paris VI, (1974), 1-46.

[80] F. Murat, J. Simon: *Sur le contrôle par un domaine géometrique*, Preprint 76015, Univ. Paris VI, (1976).

[81] F. Murat, J. Simon: *Sur le controle par un domain géométrique*, Publication du Laboratoire d'Analyse numérique de l'Université de Paris 6, 189, (1976).

[82] F. Murat, L. Tartar: *Calcul des variations et homogénéisation*, Proceedings of "Les Méthodes de l'homogénisation: Théorie et applications en physique", Ecole d'Eté d'Analyse Numérique C.E.A.-E.D.F.-INRIA, Bréau-sans-Nappe 1983, Collection de la direction des études et recherches d'electricité de France 57, Eyrolles, Paris, (1985), 319-369.

[83] F. Murat L. Tartar: *Optimality conditions and homogenization. Proceedings of Nonlinear variational problems*, Isola d'Elba 1983, Res. Notes in Math. 127, Pitman, London, (1985), 1-8.

[84] Yu. S. Osipov, A. P. Suetov: *On a problem of J.-L. Lions* Soviet Math. Dokl. Vol. 29 no 3 (1984) 487-491.

[85] C. Picard: *Analyse limite d'équation variationnelle dans un domain contenant une grille*, These d'État, Université Paris-Sud (Orsay), (1984).

[86] C. Picard: *Analyse limite d'équation variationnelle dans un domain contenant une grille*, RAIRO-Modélisation mathématique et Analyse Numérique tome 21, no2 (1987), 293-326.

[87] O. Pironneau: *Optimal shape disign for elliptic systems*, Springer Series in Computational Physics, Springer, New York (1984).

[88] W. Rudin: *Real and complex analysis*, McGraw-Hill, New York, (1974).

[89] J. Simon: *Differentiation with respect to a domain in boundary value problems*, Num. Func. Anal. Optimiz.,2 (1980), 649-687.

[90] L. Simon: *Lecture on geometric measure theory*, Proc. Center for Math. Anal., Australian Nat. Univ., (1983).

[91] J. Sokolowsky, A Zochowsky: *On the topological derivative in shape optimization*, SIAM J. Control Optim., 37(1999) 1251-1272.

[92] J. Sokolowski, J.P. Zolosio: *Introduction to Shape optimization, shape sensitivity Analysis*, Springer Series in Computational Mathematics, vol.16, Springer Berlin (1992).

[93] E. Stepanov: *Partial geometric regularity of some optimal connected transportation networks*, J. Math. Sciences (N.Y.), 132(4), (2006), 522-552.

[94] V. Šverák, *On optimal shape design*, J. Math. Pures Appl.,72-6 (1993), 537-551.

[95] W.P. Ziemer: *Weakly differentiable functions*, Springer, (1989).