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The Polyhedral Geometry of Partially Ordered Sets

A Common Aspect of Order Theory, Combinatorics, Representation Theory and Finite Frame Theory

September 2017

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Date of Defense

December 14, 2017

Abstract

Pairs of polyhedra connected by a piecewise-linear bijection appear in different fields of mathematics. The model example of this situation are the order and chain polytopes introduced by Stanley in [Sta86], whose defining inequalities are given by a finite partially ordered set. The two polytopes have different face lattices, but admit a volume and lattice point preserving piecewise-linear bijection called the transfer map. Other areas like representation theory and enumerative combinatorics provide more examples of pairs of polyhedra that are similar to order and chain polytopes.

The goal of this thesis is to analyze this phenomenon and move towards a common theoretical framework describing these polyhedra and their piecewise-linear bijections. A first step in this direction was done by Ardila, Bliem and Salazar in [ABS11], where the authors generalize order and chain polytopes by replacing the defining data with a marked poset. These marked order and chain polytopes still admit a piecewise-linear transfer map and include the Gelfand–Tsetlin and Feigin–Fourier–Littelmann–Vinberg polytopes from representation theory among other examples. We consider more polyhedra associated to marked posets and obtain new results on their face structure and combinatorial interplay. Other examples found in the literature bear resemblance to these marked poset polyhedra but do not admit a description as such. This is our motivation to consider distributive polyhedra, which are characterized by describing networks in [FK11] analogous to the description of order polytopes by Hasse diagrams. For a subclass of distributive polyhedra we are able to construct a piecewise-linear bijection to another polyhedron related to chain polytopes. We give a description of this transfer map and the defining inequalities of the image in terms of the underlying network.

Zusammenfassung

In verschiedenen Bereichen der Mathematik tauchen Paare von Polyedern auf, die durch eine stückweise lineare Transferabbildung in Bijektion stehen. Das Vorzeigebeispiel für diese Situation sind die von Stanley in [Sta86] eingeführten Ordnungs- und Kettenpolytope, deren beschreibende Ungleichungen durch endliche Halbordnungen gegeben sind. Die beiden Polytope unterscheiden sich in ihren Seitenverbänden, stehen jedoch durch eine stückweise lineare Transferabbildung in volumen- und gitterpunkttreuer Bijektion. Aber auch in anderen Bereichen wie der Darstellungstheorie und enumerativen Kombinatorik findet man solche Paare von Polyedern, deren beschreibende Ungleichungen stark an die von Ordnungs- und Kettenpolytopen erinnern.

Das Ziel dieser Arbeit ist es, dieses Phänomen strukturell zu analysieren und ein Theoriewerk zu schaffen, dass es erlaubt, diese Polyeder und ihre stückweise linearen Bijektionen von einem gemeinsamen Blickpunkt aus zu betrachten. Ein erster Schritt in diese Richtung wurde von Ardila, Bliem und Salazar bereits in [ABS11] vollzogen. Hier werden die Polytope aus der Ordnungstheorie verallgemeinert, indem die zugrundeliegenden Halbordnungen durch Markierungen ergänzt werden. Die so erhaltenen markierten Ordnungs- und Kettenpolytope stehen ebenfalls in stückweise linearer Bijektion und erlauben unter anderem die Beschreibung von Gelfand–Tsetlin- und Feigin–Fourier–Littelmann–Vinberg-Polytopen aus der Darstellungstheorie. Wir betrachten weitere markierten Halbordnungen zugeordnete Polyeder und erhalten neue Resultate über deren Seitenstruktur und kombinatorisches Zusammenspiel. Andere Beispiele aus der Literatur weisen zwar Ähnlichkeiten zu diesen Polyedern auf, lassen sich aber nicht als solche beschreiben. Über markierte Halbordnungen hinaus betrachten wir daher distributive Polyeder, die nach einer Charakterisierung in [FK11] durch gewisse Netzwerke beschrieben werden, ganz analog zur Beschreibung von Ordnungspolytopen durch Hasse-Diagramme. Für eine große Teilklasse dieser Polyeder lässt sich wieder eine stückweise lineare Bijektion zu einem mit Kettenpolytopen verwandten Polyeder herstellen. Wir konstruieren eine solche Transferabbildung und erhalten eine Beschreibung des Bildpolyeders durch lineare Ungleichung, die sich aus dem zugrundeliegenden Netzwerk ablesen lassen.

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Preface

The thesis at hand is the outcome of three years of research I conducted at the University of Bremen during my time as a PhD student. I was a research member of the *Explorationsprojekt* “Hilbert Space Frames and Algebraic Geometry” that was started by Eva-Maria Feichtner and Emily King to bring the fields of finite frame theory and algebraic as well as discrete geometry closer together. The project was funded by the *Zentrale Forschungsförderung* of the University of Bremen.

Together with my colleague Tim Haga, I studied so called polytopes of eigensteps that are used in frame theory to parametrize certain algebraic varieties of finite frames. Some discussions with participants of the workshop “Frames & Algebraic and Combinatorial Geometry” in Bremen showed that polytopes of eigensteps are closely related to Gelfand–Tsetlin polytopes from representation theory as well as order polytopes. This is when I got interested in order polytopes and their generalization to marked order polytopes. I had already obtained some results about marked order polytopes by myself when I realized that these contradict propositions in [Fou16] and [JS14]. Hence, I contacted the authors of both articles and in both cases we agreed that these results needed corrections. I wrote up my own results and published them in [Peg17], which is the basis of Chapter 6. Ghislain Fourier, the author of [Fou16], invited me to Cologne and together with Xin Fang we started a project that resulted in Chapter 7. Together with Jan-Philipp Litza, this project led to some results that I presented in a poster session of the “Einstein Workshop on Lattice Polytopes” in Berlin, where Raman Sanyal, one of the authors of [JS14], approached me to propose a joint project that is the basis of Chapter 8.

I am very happy about the fact that contacting authors about mistakes in published articles is welcome in this community and even resulted in joint projects in both occasions. For this I want to thank Ghislain Fourier and Raman Sanyal sincerely. I thank my advisor Eva-Maria Feichtner for giving me the opportunity to conduct my PhD research in Bremen and allowing me the freedom to follow my interests, even if that meant to diverge from the original direction of the project. I also want to thank my collaborators Xin Fang, Ghislain Fourier, Tim Haga, Jan-Philipp Litza and Raman Sanyal for the inspiring joint work. Thanks to all my colleagues for many fruitful and interesting discussions—also those over lunch at the cafeteria and over cappuccino at the coffee bar, some of them more funny than fruitful. For the tedious task of proofreading various parts of this thesis I thank Tim Lindemann, Jan-Philipp Litza, Viktoriya Ozornova, Ingolf Schäfer and Kirsten Schmitz—in alphabetic order. Finally, I want to thank my family and all my friends for their support and encouragement, but also for distraction when I needed a break from mathematics.

Bremen, September 2017
Christoph Pegel

Introduction

Polyhedra whose defining inequalities are given by the data of a partially ordered set historically emerged from two separate branches that just recently merged.

The first branch, started by Geissinger and Stanley in the 1980s, comes from order theory and combinatorial convex geometry. Given a finite poset P with a global maximum and a global minimum, Geissinger studied the polytope $\mathcal{O}(P)$ in $\mathbb{R}^P$ consisting of order-preserving maps $P \to \mathbb{R}$ sending the minimum to 0 and the maximum to 1 in [Gei81]. He found that vertices of this polytope correspond to non-trivial order ideals of P and describes how the volume of $\mathcal{O}(P)$ is given by the number of linear extensions of P. These results reappear in [Sta86], where Stanley called $\mathcal{O}(P)$ the *order polytope* associated to P and introduced a second polytope, the *chain polytope* $\mathcal{C}(P)$ with inequalities given by saturated chains in P. He introduced a piecewise-linear *transfer map* $\mathcal{O}(P) \to \mathcal{C}(P)$ that yields an Ehrhart equivalence of these polytopes. In particular, since the chain polytope $\mathcal{C}(P)$ only depends on the comparability graph of the poset P and has the same volume as $\mathcal{O}(P)$, this setting provides a geometric proof that the number of linear extensions of a poset only depends on the comparability graph. In the same spirit of comparing these two polytopes associated to a finite poset, a group around Hibi and Li characterized the posets such that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodular equivalent and constructed a bijection between the edge sets of both polytope in [HL16] and [HLSS17], respectively.

A second branch begins in the 1950s in representation theory, when Gelfand and Tsetlin introduced number patterns—now attributed to them as *Gelfand–Tsetlin patterns*—to enumerate the elements in a basis of the irreducible representation $V(\lambda)$ of the general linear group $\mathrm{GL}_n(\mathbb{C})$ in [GT50]. The defining conditions of these patterns give rise to the *Gelfand–Tsetlin polytope*—or *GT polytope* for short—associated to the highest weight λ of the representation, so that the elements in the Gelfand–Tsetlin basis correspond to the lattice points in the Gelfand–Tsetlin polytope. A different basis of $V(\lambda)$—previously conjectured to exist by Vinberg—was described by Feigin, Fourier and Littelmann in [FFL11] and is enumerated by the lattice polytopes of another polytope. Due to their importance in representation theory, the geometry of Gelfand–Tsetlin and Feigin–Fourier–Littelmann-Vinberg polytopes—or *FFLV polytopes* for short—has attracted the attention of different researchers, see [DM04; KM05; Kir10; KST12; GKT13; Ale16; FM17; ACK18]. While the description of the Gelfand–Tsetlin polytope shows similarities to order polytopes, the description of the FFLV polytope resembles that of chain polytopes.

Indeed, the two branches started to merge in 2011, when Ardila, Bliem and Salazar generalized the two poset polytopes of Stanley to *marked poset polytopes* in [ABS11], allowing marking conditions other than just sending minima to 0 and maxima to 1. This generalization allowed to consider GT and FFLV polytopes as the marked order and marked chain polytopes associated to the same marked poset (P, λ). Again, these two polytopes come with a piecewise-linear transfer map $\mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ that yields an Ehrhart equivalence and hence a geometric explanation as to why the GT and FFLV polytopes have the same number of lattice points—the dimension of $V(\lambda)$. Further results in this direction were achieved by Fang and Fourier as well as Jochemko and Sanyal in [Fou16; FF16] and [JS14], respectively.

The aim of this thesis is to continue this line of research and extend the class of polyhedra that admit a piecewise-linear transfer map.

Outline

In the first part of this thesis, consisting of Chapters 1 to 4, we introduce various examples of polyhedra found in the literature, whose descriptions show similarities to poset polytopes and often come with a transfer map like the one introduced by Stanley. Not all of the examples are marked poset polytopes and hence ask for a more general theory that we present in the second part of this thesis.

The second part starts with a review of marked poset polytopes in Chapter 5. Motivated by the face structure description of order polytopes given by Stanley, we study marked order polyhedra—a potentially unbounded generalization of marked order polytopes—in detail in Chapter 6. We follow a categorical approach and describe a functor $\mathcal{O}$ from the category of marked posets to the category of polyhedra and affine maps. The main results of this chapter are a combinatorial description of the face structure of marked order polyhedra in terms of partitions of the underlying poset as well as a regularity condition assuring that the facets of the polyhedron are in bijection with the covering relations of the poset. The face structure was previously studied by Jochemko and Sanyal in [JS14] and regularity was introduced by Fourier in [Fou16]. However, both articles contain minor mistakes resulting in incorrect characterizations of face partitions and regular marked posets, respectively. We also introduce *conditional marked order polyhedra* which are marked order polyhedra with additional linear constraints. These appear in representation theory as Gelfand–Tsetlin polytopes for weight subspaces of irreducible representations and in finite frame theory as polytopes of eigensteps. We generalize a method to determine dimensions of faces given by De Loera and McAllister for Gelfand–Tsetlin polytopes in [DM04] and show that—up to affine isomorphism—every polyhedron is a conditional marked order polyhedron.

In Chapter 7 we modify the transfer map $\mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ for marked poset polytopes by introducing a parameter $t \in [0,1]^\ell$, where ℓ is the number of unmarked elements in P. The results of this chapter are joint work with Xin Fang, Ghislain Fourier and Jan-Philipp Litza and have also appeared in [FFLP17]. For $t \equiv 0$ and $t \equiv 1$ the resulting polytopes are marked order and marked chain polytopes, while for t in a subset of $\{0,1\}^\ell$ we obtain the marked chain-order polytopes introduced by Fang and Fourier in [FF16]. Surprisingly, the images under this modified piecewise-linear transfer map are polytopes for all $t \in [0,1]^\ell$ and their combinatorial types stay constant along relative interiors of faces of the parametrizing hypercube. We provide a common description of the polytopes in this continuous family by a system of linear equations and inequalities. Using a theory of continuous degenerations and a subdivision obtained from a tropical hyperplane arrangement, we are able to describe the vertices of the generic polytope obtained for t in the interior of the hypercube.

Since some of the examples in the first part are outside the realm of marked poset polytopes, we consider a generalization that includes these examples in Chapter 8, which is based on joint work in progress with Raman Sanyal that will also appear in [PS17]. Instead of marked order polyhedra, we consider *distributive polyhedra*—those that form a distributive lattice with respect to the dominance order on $\mathbb{R}^n$. These have been characterized using *network matrices* by Felsner and Knauer in [FK11] and are a promising substitute to generalize the transfer map of marked poset polytopes even further. Indeed, we are able to show that as long as the underlying network of a distributive polyhedron contains only lossy cycles, we obtain a piecewise-linear bijection to another polyhedron falling into the class of *anti-blocking polyhedra*. Their description is very similar to that of chain polytopes, where instead of chains in a poset we have to consider infinite walks in a cyclic network.

We conclude the second part with Chapter 9, where we review the obtained results, formulate open questions and point to further directions of research.

Separated from the rest of this work, Chapter 10 in the appendix is a case study of certain conditional marked order polyhedra that appear as polytopes of eigensteps for finite equal norm tight frames. The results of the chapter are joint work with Tim Haga and have also been published in [HP16]. We provide a non-redundant system of linear equations and inequalities describing polytopes of eigensteps for equal norm tight frames and from that deduce their dimension and number of facets. Furthermore, we identify two affine isomorphisms in this class of polytopes as convex geometrical counterparts of known operations in finite frame theory, namely frame reversal and Naimark complements.

Preliminaries

Before looking at various related polyhedra in Part I, we want to introduce the main concepts from polyhedral geometry here and fix some notation. The rest of this section is essentially a continuous stream of definitions and well-known facts stated for reference that may be skipped on first read and only be consulted for unclear definitions later on. All terms defined here (and later in this thesis) are listed in the index at the end of this work.

Starting with the very basics, we write $A \subseteq B$ for the inclusion of sets A and B, and $A \subsetneq B$ for proper inclusions, that is, $A \subseteq B$ and $A \neq B$. We indicate unions of disjoint sets by writing $A \sqcup B$ instead of $A \cup B$. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of *non-negative integers*, *integers*, *rational numbers*, *real numbers* and *complex numbers*, respectively. Sometimes we use expressions like $\mathbb{N}_{>0}$, $\mathbb{R}_{\geq 0}$ etc. to denote positive integers, non-negative real numbers and so on. Given any non-negative integer $n \in \mathbb{N}$ we denote by $[n]$ the set $\{1, 2, \ldots, n\}$ of all positive integers less than or equal to n, in particular $[0] = \varnothing$ is the empty set.

Throughout the thesis, we will work in vector spaces $\mathbb{R}^S$ for some finite set S. This is the set of all maps $S \to \mathbb{R}$ equipped with pointwise addition and multiplication by real scalars. A point $x \in \mathbb{R}^S$ has coordinates that we denote by x_s or $x(s)$ for $s \in S$. Equipped with the inner-product $\langle x, y\rangle = \sum_{s \in S} x_s y_s$ it becomes a Hilbert space isometric to $\mathbb{R}^n$, where $n = |S|$ is the cardinality of S. A *linear form* on $\mathbb{R}^S$ is a linear map $\mathbb{R}^S \to \mathbb{R}$ and an *affine linear form* is a map $\alpha\colon \mathbb{R}^S \to \mathbb{R}$ such that the map given by $x \mapsto \alpha(x) - \alpha(0)$ is linear.

As a reference for polyhedral geometry we refer to the book of Ziegler [Zie95] and only restate the most important definitions here, adding those not found there. Any non-constant affine linear form α on $\mathbb{R}^S$ defines a *hyperplane* H and a *half-space* H^+, consisting of all points $x \in \mathbb{R}^S$ satisfying $\alpha(x) = 0$ or $\alpha(x) \geq 0$, respectively. A *polyhedron* in $\mathbb{R}^S$ is any set $Q \subseteq \mathbb{R}^S$ that may be expressed as an intersection of finitely many half spaces $Q = H_1^+ \cap H_2^+ \cap \cdots \cap H_r^+$. Since hyperplanes are intersections of two opposing half-spaces, this is equivalent to being an intersection of finitely many half-spaces and hyperplanes. In other words, a polyhedron is any solution set of finitely many linear equations and inequalities.

Two important subclasses of polyhedra are *polytopes* and *polyhedral cones*. A polyhedron Q is a polytope if it is a bounded subset of $\mathbb{R}^S$ and a polyhedral cone if it admits a description as an intersection of hyperplanes and half-spaces given by *linear* forms, that is, the solution of a system of linear equations and inequalities without constant terms. One of the fundamental results in polyhedral geometry is that polyhedra, polytopes and polyhedral cones all admit a second definition using Minkowski sums of convex and conical hulls of finite sets. To define these terms and state these fundamental results precisely, we introduce some more terminology. Given finitely many points $x_1, x_2, \ldots, x_m \in \mathbb{R}^S$, a linear combination $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$ is said to be a *conical combination* if all $\lambda_i \geq 0$, an *affine combination* if $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$ and a *convex combination* if it is affine and conical. A set $X \subseteq \mathbb{R}^S$ is *convex* if it contains all convex combinations of its points and a *cone* if it contains all conical combinations of its points. From the definitions we see that all polyhedra are convex and polyhedral cones are indeed cones. Each polyhedron Q in $\mathbb{R}^S$ comes with an associated polyhedral cone $\operatorname{rec}(Q)$, the *recession cone* of Q. It consists of all $y \in \mathbb{R}^S$ such that for any $x \in Q$ and $t \geq 0$ we have $x + ty \in Q$. When Q is given by linear inequalities $\alpha_i(x) \geq 0$ for $i = 1, 2, \ldots, r$ with each α_i an affine linear form on $\mathbb{R}^S$, then $\operatorname{rec}(Q)$ is given by the linear inequalities $\alpha_i'(x) \geq 0$, where $\alpha_i'(x) = \alpha(x) - \alpha(0)$ is the corresponding linear form. Given any set $Y \subseteq \mathbb{R}^S$ we define its *convex hull* $\operatorname{conv}(Y)$, *affine hull* $\operatorname{aff}(Y)$ and *conical hull* $\operatorname{cone}(Y)$ as the sets of all convex, affine and conical combinations of points in Y, respectively. For finitely many sets $X_1, X_2, \ldots, X_m \in \mathbb{R}^S$ their *Minkowski sum* $X_1 + X_2 + \ldots + X_m$ is the set consisting of all sums $x_1 + x_2 + \cdots + x_m$ with $x_i \in X_i$ for $i = 1, \ldots, m$. When all summands are convex, the Minkowski sum is convex as well.

We are now ready to state the aforementioned characterizations of polyhedra, polytopes and polyhedral cones. A set $Q \subseteq \mathbb{R}^S$ is a polyhedral cone if and only if it is the conical hull of finitely many points, it is a polytope if and only if it is the convex hull of finitely many points and it is a polyhedron if and only if it is the Minkowski sum of a polytope and a polyhedral cone.

One of the most important notions in polyhedral geometry is that of a *face* of a polyhedron. Given any polyhedron $Q \subseteq \mathbb{R}^S$, a face of Q is a subset $F \subseteq Q$ that may be expressed as

$$F = Q \cap \left\{ x \in \mathbb{R}^S : \alpha(x) = 0 \right\},$$

where α is an affine linear form that is non-negative on Q. In particular, for $\alpha \equiv 0$ and $\alpha \equiv 1$, we obtain Q itself and the empty set $\varnothing$ as faces of Q. The faces different from Q are called *proper faces*. The *dimension* $\dim(Q)$ of a polyhedron Q is defined as the dimension of its affine hull $\operatorname{aff}(Q)$ as an affine subspace of $\mathbb{R}^S$ and faces of dimension 0, 1 and $\dim(Q) - 1$ are called *vertices*, *edges* and *facets*, respectively. For $i \in \mathbb{N}$ we denote by $f_i(Q)$ the number of i-dimensional faces of Q and refer to the tuple $(f_0, f_1, \dots, f_{\dim Q})$ as the f*-vector* of Q. The *relative interior* $\operatorname{relint}(Q)$ of a polyhedron Q is the interior with the respect to its affine hull or equivalently the set of all points not contained in any proper face of Q. The relative interiors of the faces of a polyhedron are pairwise disjoint and hence each point $x \in Q$ uniquely determines a face such that $x \in \operatorname{relint}(F)$. Ordered by inclusion, the set of faces $\mathcal{F}(Q)$ forms a lattices graded by dimension called the *face lattice* of Q. Not every polyhedron has vertices, for example a half-space only has three faces, the empty face, the bounding hyperplane and the half-space itself, none of which is a vertex if the dimension of the half-space is at least 2. A polyhedron is called *pointed* if it has at least one vertex. The importance of pointed polyhedra lies in the fact that they are determined by their vertices and recession cones, to be precise: if Q is a pointed polyhedron with set of vertices V, then $Q = \operatorname{conv}(V) + \operatorname{rec}(Q)$.

Polyhedra come with various notions of equivalence. Two polyhedra $Q \subseteq \mathbb{R}^S$ and $R \subseteq \mathbb{R}^T$ are said to be *affinely equivalent* if there is an affine map $\mathbb{R}^S \to \mathbb{R}^T$ that restricts to a bijection $Q \to R$. The bijection is called an *affine isomorphism* in this case. They are said to be *combinatorially equivalent* if their faces lattices $\mathcal{F}(Q)$ and $\mathcal{F}(R)$ admit an order-preserving bijection, i.e., they are isomorphic lattices.

Sometimes we are interested in polyhedra in $\mathbb{R}^S$ that are in a way compatible with the lattice $\mathbb{Z}^S$ contained in $\mathbb{R}^S$.[1] For general reference on lattice points in polyhedra, we refer to the textbooks [Bar08; BR15] as well as the draft of lecture notes [HNP12]. We say a polytope $Q \subseteq \mathbb{R}^S$ is a *lattice polytope* if all of its vertices are lattice points, that is, points in $\mathbb{Z}^S$. Since polytopes are bounded, they only contain finitely many lattice points and to each polytope $Q \subseteq \mathbb{R}^S$ we can associate the counting function $\operatorname{Ehr}_Q : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{Ehr}_Q(k)$ is the number of lattice points in $kQ = \{kx : x \in Q\}$, the k-th *dilate* of Q. When Q is a lattice polytope, Ehr_Q is a polynomial called the *Ehrhart polynomial* of Q, due to Eugène Ehrhart. This yields another notion of equivalence we will come across in this thesis: two lattice polytopes Q and R are called *Ehrhart equivalent* if they have the same Ehrhart polynomial. For polyhedral cones in $\mathbb{R}^S$, the appropriate notion of being compatible with the lattice $\mathbb{Z}^S$ is that of a *rational polyhedral cone*. A polyhedral cone $Q \subseteq \mathbb{R}^n$ is called *rational*, if it may be defined by affine linear forms with rational coefficients, or equivalently with integral coefficients. In the spirit of the characterization of cones as conical hulls of finite sets, we see that a polyhedral cone is rational if and

[1] Note that lattice in this paragraph refers to a finitely generated subgroup of $(\mathbb{R}^S, +)$ while in the previous paragraphs it refers to a partially ordered set having all finite joins and meets.

only if it is the conical hull of finitely many lattice points. Less common in the literature but relevant for our work is the notion of a *lattice polyhedron*, a common generalization of lattice polytopes and rational polyhedral cones to arbitrary polyhedra. We say that $Q \subseteq \mathbb{R}^S$ is a lattice polyhedron if it can be expressed as Minkowski sum of a lattice polytope and a rational polyhedral cone. It is called *integrally closed* if it satisfies the *integer decomposition property*

$$\mathbb{Z}^S \cap kQ = (\mathbb{Z}^S \cap Q) + (\mathbb{Z}^S \cap Q) + \cdots + (\mathbb{Z}^S \cap Q)$$

for every $k \in \mathbb{N}$, where the Minkowski sum on the right hand side has k summands. A simple fact we will use is that unimodular simplices and their integral dilates are integrally closed. A lattice simplex $\Delta \subseteq \mathbb{R}^S$ is *unimodular* if the vectors emanating from a fixed vertex form part of a $\mathbb{Z}$-basis of $\mathbb{Z}^S$.

Lattice polyhedra ask for a finer notion of equivalence that takes lattice points into account. The right notion here is *unimodular equivalence*. An affine isomorphism $Q \to R$ for lattice polyhedra $Q \subseteq \mathbb{R}^S$ and $R \subseteq \mathbb{R}^T$ is called a *unimodular isomorphism*, if the uniquely determined extension $\operatorname{aff}(Q) \to \operatorname{aff}(R)$ to the affine hulls of Q and R restricts to a bijection $\mathbb{Z}^S \cap \operatorname{aff}(Q) \to \mathbb{Z}^T \cap \operatorname{aff}(R)$. In this case the polytopes are said to be unimodular equivalent and it follows that they are affinely equivalent, combinatorially equivalent and Ehrhart equivalent.

We will also consider subdivisions of polyhedra into smaller polyhedral pieces. A *polyhedral subdivision* of a polyhedron Q in $\mathbb{R}^S$ is a finite set $\mathcal{S}$ of polyhedra in $\mathbb{R}^S$ such that $\bigcup \mathcal{S} = Q$, for each polyhedron in $\mathcal{S}$ all its faces are elements of $\mathcal{S}$ as well and for $R_1, R_2 \in \mathcal{S}$ the intersection $R_1 \cap R_2$ is a face of both R_1 and R_2. The inclusion-wise maximal elements in a polyhedral subdivision $\mathcal{S}$ are called the *facets* or *chambers* of the subdivision while arbitrary elements of $\mathcal{S}$ are referred to as *faces* or *cells* of the subdivision.

A central role in our discussion of the polyhedral geometry of partially ordered sets is played by *piecewise-linear maps*. A continuous map $f : P \to Q$ between polyhedra is called piecewise-linear, if P admits a polyhedral subdivision $\mathcal{S}$ such that f restricts to an affine linear map on each cell in $\mathcal{S}$. In particular f is piecewise-linear if it is given in each component by taking maxima and/or minima of affine linear forms.

Part I.

Related Polyhedra in Order Theory, Combinatorics, Representation Theory and Finite Frame Theory

1. Order Theory

We start our showcase of polyhedra in the theory of partially ordered sets themselves, since the underlying structure is most explicit and visible in this setting. The polyhedra we are looking at in this chapter are *poset polytopes*, introduced by Geissinger and Stanley in the 1980s [Gei81; Sta86].

A *partially ordered set* $(P, \leq)$ is a set P together with a reflexive, transitive and anti-symmetric relation $\leq$. We use the usual short term *poset* and omit the relation $\leq$ in notation when the considered partial order is clear from the context. A finite poset is determined by its *covering relations*: we say p *is covered by* q and write $p \prec q$, if $p < q$ and whenever $p \leq r \leq q$ it follows that $r = p$ or $r = q$. Hence, we usually describe a finite poset by its *Hasse diagram*, which is the finite directed graph with nodes the elements of P and edges given by covering relations. See Figure 1.1 for examples of Hasse diagrams of finite posets. Instead of directed edges we follow the convention to draw p below q whenever $p \prec q$.

For general reference on the theory of partially ordered sets we refer to [Sta11]. However, we want to mention some basic notions here to familiarize the reader with our notation and terminology. Two elements p and q of a poset P are *comparable* if at least one of $p \leq q$ or $q \leq p$ holds. The partial order is *linear* or *total* if any two elements are comparable. A *linear extension* of a poset $(P, \leq)$ is a poset $(P, \leq')$ on the same set, such that $\leq'$ is linear and $p \leq' q$ whenever $p \leq q$. A subset $I \subseteq P$ is an *order ideal*, if whenever $q \in I$ and $p \leq q$, we have $p \in I$ as well.[1] The dual notion is that of *filters*, i.e., subsets $F \subseteq P$ such that $q \in F$ whenever $q \geq p$ for some $p \in F$. Note that filters are exactly the complements of order ideals. Given any subset $Q \subseteq P$, we obtain a poset $(Q, \leq_Q)$ with $p \leq_Q q$ for $p, q \in Q$ if and only if $p \leq q$ in P. Any poset obtained this way from P is called an *induced subposet* of P. A *chain* in a poset P is an induced subposet that is linear, i.e., it is a list of elements $p_1 < p_2 < \cdots < p_k$. A chain is said to be *saturated* if it is of the

[1] Some authors refer to what we call order ideals (filters) as *lower sets* (*upper sets*) and require order ideals (filters) to be non-empty and closed under finite joins (meets).

(a) the star poset (b) a linear poset or chain (c) an anti-chain (d) the diamond poset

Figure 1.1.: The Hasse diagrams of some finite posets. Only the posets (a), (b) and (d) are connected and only the posets (b) and (d) have a $\hat{0}$ and $\hat{1}$.

form $p_1 \prec p_2 \prec \cdots \prec p_k$, that is, all the relations are covering relations. An *anti-chain* is a subset of P with elements being pairwise incomparable. We say a poset P has a $\hat{0}$ if there is a unique minimal element $\hat{0} \in P$. Similarly, P has a $\hat{1}$ if there is a unique maximal element $\hat{1} \in P$. We call a poset P *connected*, if its Hasse diagram is a connected graph.

Having familiarized ourselves with posets, we want to introduce certain polytopes associated to them. These are order polytopes and chain polytopes. Order polytopes have been studied by Geissinger in [Gei81] and then reappeared in [Sta86], where Stanley also introduced chain polytopes and refers to both of them as *poset polytopes*.

1.1. Order Polytopes

To a finite poset P with $\hat{0}$ and $\hat{1}$, associate the *order polytope* $\mathcal{O}(P)$ in $\mathbb{R}^P$, consisting of all order-preserving maps $f : P \to \mathbb{R}$ with $f(\hat{0}) = 0$ and $f(\hat{1}) = 1$.[2] Here f being order-preserving means that $f(p) \leq f(q)$ whenever $p \leq q$. This simple construction yields a beautiful interplay of polyhedral geometry and order theory that we want to elaborate on in this section. Instead of denoting elements of $\mathbb{R}^P$ as maps $f : P \to \mathbb{R}$ and their values by $f(p)$ we usually write $x \in \mathbb{R}^P$ and use x_p instead of $x(p)$ to stress the fact that x is a point in a euclidean space with coordinates indexed by P.

Equivalently, the order polytope may be described by its vertices, which are indicator functions of non-trivial filters of P as shown in [Gei81, p. 127]. Thus, order polytopes are always lattice polytopes. Regarding the combinatorial structure of the order polytope, we can observe that the inequalities $x_p \leq x_q$ given by covering relations $p \prec q$ define the facets of $\mathcal{O}(P)$. In fact, these are just the two extreme cases of a combinatorial description of the face structure of order polytopes. Since non-trivial faces are intersections of facets, every face F will be described by a partition π of P such that all $x \in F$ are constant on the blocks of π.

Definition 1.1.1. A partition π of a finite poset P is called a *face partition* of P if it satisfies the following two conditions:

i) π is *P-compatible*: the transitive closure of the relation on π defined by $B \leq C$ if $p \leq q$ for some $p \in B$ and $q \in C$ is anti-symmetric and hence makes π a poset,
ii) π is *connected*: the blocks of π are connected as induced subposets of P.

Note that face partitions may as well be characterized as surjective order-preserving maps $f : P \to P'$ into some poset P' such that the fibers $f^{-1}(q)$ are connected for every $q \in P'$. Following Geissinger we call these maps *contractions* of P. Given a face partition π the corresponding contraction is just the quotient map $P \to \pi$, where π carries the induced poset structure given by P-compatibility. Given a contraction $f : P \to P'$ we obtain a face partition of P by taking the fibers of f as blocks.

We say a partition π' *refines* π if every block of π' is contained in a block of π. When thinking about a face partition π as a contraction $f : P \to \pi$, another face partition π'

[2] In Stanley's original work P is not required to have $\hat{0}$ and $\hat{1}$. Our $\mathcal{O}(P)$ would be $\widehat{\mathcal{O}}(P \setminus \{\hat{0}, \hat{1}\})$ in [Sta86]. We chose a different convention to match the setting in later chapters of this work.

with contraction $f' : P \to \pi'$ refines π if and only if f factors through f' by contractions, that is, $f = g \circ f'$ for a contraction $g : \pi' \to \pi$.

Theorem 1.1.2 ([Gei81, p. 130], [Sta86, Thm. 1.2]). *The face lattice of $\mathcal{O}(P)$ is isomorphic to the lattice of face partitions of P ordered by reverse refinement. The isomorphism is given by associating to a face partition π the face F_π consisting of all $x \in \mathcal{O}(P)$ constant on the blocks of π.*

From the description of a face F_π as in the previous theorem, it is immediate that $\dim(F_\pi) = |\pi| - 2$. Hence, facets correspond to partitions with only one non-trivial block $\{p, q\}$ for a covering relation $p \prec q$ and vertices correspond to partitions with exactly two blocks or equivalently contractions $P \to \{0, 1\}$, which are exactly the indicator functions of non-trivial filters of P.

In addition to the combinatorial description of the face structure of $\mathcal{O}(P)$, Stanley gave a *unimodular triangulation*—a subdivision into unimodular simplices—with facets enumerated by linear extensions of P. To a chain of order ideals $\mathcal{I} : \varnothing = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = P$ associate the $(r-2)$-dimensional simplex $F_{\mathcal{I}}$ in $\mathbb{R}^P$ consisting of all $x \in \mathbb{R}^P$ taking constant values on the pairwise disjoint sets $B_k = I_k \setminus I_{k-1}$ such that $x(B_1) = 0$, $x(B_r) = 1$ and $x(B_k) \le x(B_{k+1})$. By construction these simplices will always be contained in $\mathcal{O}(P)$ and in fact form a unimodular triangulation when $\mathcal{I}$ ranges over all possible chains of order ideals. The facets of this triangulation correspond to saturated chains of order ideals where exactly one element is added in each step. These saturated chains of order ideals are of course nothing else than linear extensions of P.

From this unimodular triangulation it is immediate that the normalized volume of $\mathcal{O}(P)$ is given by the number of linear extensions of P.

1.2. Chain Polytopes and the Transfer Map

A second polytope associated to a finite poset P with $\hat{0}$ and $\hat{1}$ that will turn out to share the same volume as the order polytope is the *chain polytope* $\mathcal{C}(P)$. It consists of all $y \in \mathbb{R}^P$ with non-negative coordinates satisfying $y_{\hat{0}} = 0$, $y_{\hat{1}} = 1$ and for each chain $\hat{0} < p_1 < p_2 < \cdots < p_k < \hat{1}$ an inequality

$$y_{p_1} + y_{p_2} + \cdots + y_{p_k} \le 1.$$

It is immediate that of these inequalities it suffices to consider only those given by saturated chains. In fact, these and the inequalities $y_p \ge 0$ correspond to the facets of $\mathcal{C}(P)$. As shown in [Sta86, Thm. 2.2], the vertices of $\mathcal{C}(P)$ are exactly the indicator functions of anti-chains in $P \setminus \{\hat{0}, \hat{1}\}$. As the vertices of $\mathcal{O}(P)$ correspond to non-trivial filters of P, whose sets of minima are exactly the anti-chains in $P \setminus \{\hat{0}, \hat{1}\}$, we see that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same number of vertices.

In fact, there is a piecewise-linear bijection between $\mathcal{O}(P)$ and $\mathcal{C}(P)$ that preserves vertices.

Theorem 1.2.1 ([Sta86, Thm. 3.2]). *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. The two maps $\varphi, \psi\colon \mathbb{R}^P \to \mathbb{R}^P$ defined by*

$$\varphi(x)_p = \begin{cases} x_p & \text{if } p \in \{\hat{0}, \hat{1}\}, \\ x_p - \max\left\{ x_q : q \text{ is covered by } p \right\} & \text{otherwise,} \end{cases}$$
$$\psi(y)_p = \begin{cases} y_p & \text{if } p \in \{\hat{0}, \hat{1}\}, \\ \max\left\{ y_{q_1} + \cdots + y_{q_k} : q_1 < \cdots < q_k = p \right\} & \text{otherwise} \end{cases}$$

are mutually inverse piecewise-linear maps. Furthermore, they restrict to piecewise-linear bijections $\varphi\colon \mathcal{O}(P) \to \mathcal{C}(P)$ and $\psi\colon \mathcal{C}(P) \to \mathcal{O}(P)$.

Since this statement is slightly stronger than [Sta86, Thm. 3.2.(a)], the proof is not carried out there, and we will see a variety of generalizations of this theorem later on, we use the opportunity to give a detailed proof here.

Proof of Theorem 1.2.1. The first thing to notice is that the map ψ satisfies the recursion

$$\psi(y)_p = y_p + \max\left\{ \psi(y)_q : q \text{ is covered by } p \right\} \tag{1.1}$$

for all $p \notin \{\hat{0}, \hat{1}\}$. To see this, first note that in the definition of ψ it is sufficient to consider only saturated chains $q_1 \prec \cdots \prec q_k = p$. Now each of these saturated chains ending in p passes through some q covered by p. Hence, maximizing over all of them can be achieved by taking for each q covered by p the maximum over all chains ending in q—which is just $\psi(y)_q$—and then taking the maximum of these values and add y_p. This yields the recursion in (1.1).

From the definition of φ and the recursion property of ψ we clearly see that $\varphi \circ \psi$ is the identity on $\mathbb{R}^P$. To verify that $\psi \circ \varphi$ is the identity as well, we use an inductive argument working our way up from the minimum $\hat{0}$ through the poset P. Starting at the bottom, we have $\psi(\varphi(x))_{\hat{0}} = x_{\hat{0}}$. Now let $\hat{0} < p < \hat{1}$ and assume by induction that $\psi(\varphi(x))_q = x_q$ for all $q < p$. It follows that

$$\psi(\varphi(x))_p = \varphi(x)_p + \max\Big\{ \underbrace{\psi(\varphi(x))_q}_{x_q} : q \text{ is covered by } p \Big\} = x_p.$$

This concludes the first part of the theorem: the maps φ and ψ are mutually inverse piecewise-linear self-maps on $\mathbb{R}^P$. It remains to show that they restrict to bijections between $\mathcal{O}(P)$ and $\mathcal{C}(P)$.

Let $x \in \mathcal{O}(P)$ and $y = \varphi(x)$. By definition of φ, we see that all coordinates of y are non-negative, since $x_q \leq x_p$ whenever $q \leq p$. Now consider a saturated chain $\hat{0} \prec p_1 \prec p_2 \prec \cdots \prec p_k \prec \hat{1}$. We have

$$y_{p_1} + y_{p_2} + \cdots + y_{p_k} \leq x_{p_1} + (x_{p_2} - x_{p_1}) + \cdots + (x_{p_k} - x_{p_{k-1}}) = x_{p_k} \leq x_{\hat{1}} = 1$$

and hence $y \in \mathcal{C}(P)$. Now let $y \in \mathcal{C}(P)$ and $x = \psi(y)$. Consider any covering relation $p \prec q$. If $p = \hat{0}$ we have $0 \leq y_q = x_q$. If $q = \hat{1}$ we have $x_p \leq 1$, since the sums along chains appearing in the definition of $\psi(y)_p$ are all bounded by 1 for $y \in \mathcal{C}(P)$. For a covering relation not involving $\hat{0}$ or $\hat{1}$, we have

$$\begin{aligned} x_p &= \max \left\{ y_{q_1} + \cdots + y_{q_k} : q_1 < \cdots < q_k = p \right\} \\ &\leq \max \left\{ y_{q_1} + \cdots + y_{q_k} : q_1 < \cdots < q_k = p \right\} + y_q \\ &\leq \max \left\{ y_{q_1} + \cdots + y_{q_k} + y_{q_{k+1}} : q_1 < \cdots < q_k < q_{k+1} = q \right\} = x_q. \end{aligned}$$

Hence, in all cases $x_p \leq x_q$ and we conclude that $x \in \mathcal{O}(P)$, finishing the proof. □

The map φ in Theorem 1.2.1 is called the *transfer map*, since it allows to transfer some—but not all—properties from $\mathcal{O}(P)$ to $\mathcal{C}(P)$. As stated earlier φ sends vertices of $\mathcal{O}(P)$ (indicator functions of filters) to vertices of $\mathcal{C}(P)$ (indicator functions of anti-chains in $P \setminus \{\hat{0}, \hat{1}\}$), so $\mathcal{C}(P)$ is a lattice polytope as well. Even more so, the transfer map is piecewise-unimodular, so the two polytopes have the same Ehrhart polynomial and volume. Since the transfer map is unimodular and orientation-preserving on each of the simplices in the unimodular triangulation of $\mathcal{O}(P)$, the transferred simplices form a unimodular triangulation of $\mathcal{C}(P)$. Indeed, we may think of φ as rearranging the simplices in the unimodular triangulation.

The fact that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ share the same volume, which is determined by the number of linear extension of P, has a subtle consequence that allows an elegant proof in this setting (cf. [Sta86, Cor. 4.5]): the number of linear extensions of a poset P only depends on the *comparability graph* of P, i.e., the simple graph with nodes the elements of P and edges between comparable elements.

Unfortunately, since the transfer map is only piecewise-linear, it does not behave well with respect to face structures. In fact, little is known about the face structure of chain polytopes. To the best of the authors knowledge, the following recent results by Hibi et al. capture all that is known on the face structure of chain polytopes beyond the description of facets and vertices mentioned before.

Theorem 1.2.2 ([HL16]). *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. The following are equivalent:*

i) the polytopes $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodular equivalent,

ii) the polytopes $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same number of facets,

iii) the star poset (see Figure 1.1a) does not appear as an induced subposet of P.

Theorem 1.2.3 ([HLSS17]). *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. The number of edges of the order polytope $\mathcal{O}(P)$ is equal to the number of edges of the chain polytope $\mathcal{C}(P)$. Furthermore, the degree sequences of the 1-skeleta of the two polytopes are the same if and only if the two polytopes are unimodular equivalent.*

In this theorem, the 1-*skeleton* of a polytope is the simple graph given by the vertices and edges of the polytope. For each vertex of a graph, its *degree* is the number of adjacent vertices and the *degree sequence* of a graph is the non-increasing list of all vertex degrees.

We finish our discussion of poset polytopes with a conjecture stated by Hibi and Li that we will see generalizations of in later chapters of this thesis.

Conjecture 1.2.4 ([HL16]). *Denote by f_i the number of i-dimensional faces of $\mathcal{O}(P)$ and by f_i' the number of i-dimensional faces of $\mathcal{C}(P)$. It holds that $f_i \leq f_i'$ for all $i \in \mathbb{N}$.*

2. Combinatorics

We continue our journey in the realm of combinatorics, where we want to present three kinds of polytopes that are closely related to poset polytopes. The polytopes we are dealing with in this chapter are Stanley–Pitman polytopes, Cayley polytopes, as well as lecture hall cones and polytopes together with a recent generalization.

2.1. Stanley–Pitman Polytopes

In [SP02] Stanley and Pitman study a polytope that is now referred to as the *Stanley–Pitman polytope* $\Pi_n(\xi)$. Given a tuple of positive real numbers $\xi = (\xi_1, \ldots, \xi_n)$ the polytope $\Pi_n(\xi)$ is the defined to be set of all $y \in \mathbb{R}^n$ with non-negative coordinates that satisfy

$$y_1 + \cdots + y_i \leq \xi_1 + \cdots + \xi_i \qquad \text{for } i = 1, \ldots, n.$$

Presented in this context, we immediately notice the shared properties with chain polytopes: it is defined by non-negativity constraints and some sums of coordinates being bounded above by constants. Following the analogy, we would expect $\Pi_n(\xi)$ to be a piecewise-linear image of a polytope related to order polytopes. Indeed, the authors consider the unimodular map $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\varphi(x_1, x_2, \ldots, x_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

The preimage $\varphi^{-1}(\Pi_n(\xi))$ is described by the inequalities $0 \leq x_1 \leq \cdots \leq x_n$ as well as

$$x_i \leq \xi_1 + \cdots + \xi_i \qquad \text{for } i = 1, \ldots, n.$$

Stanley and Pitman realize that $\varphi^{-1}(\Pi_n(\xi))$ is related to order polytopes and identify it as a *section of an order cone*. This turns out to be a special case of *marked order polyhedra* that we will discuss in Chapter 6. We also identify the map φ as an instance of a transfer map in this setting.

2.2. Cayley Polytopes

Consider the following combinatorial identity of integer partitions.

Theorem 2.2.1 ([Cay57]). *For each non-negative integer $n \in \mathbb{N}$, the number of positive integer tuples $(a_1, a_2, \ldots, a_n)$ satisfying $a_1 \leq 2$ and $a_{i+1} \leq 2a_i$ for $i = 1, \ldots, n-1$ is equal to the total number of partitions of non-negative integers less than 2^n into powers of 2.*

The integer tuples and partitions appearing in Cayley's theorem are called *Cayley compositions* and *Cayley partitions*, respectively. As an example, consider the case where $n = 3$. The Cayley compositions of length 3 are the 26 triples

$$
\begin{gathered}
(1,1,1),(1,1,2),(1,2,1),(1,2,2),(1,2,3),(1,2,4),(2,1,1),(2,1,2),(2,2,1),\\
(2,2,2),(2,2,3),(2,2,4),(2,3,1),(2,3,2),(2,3,3),(2,3,4),(2,3,5),(2,3,6),\\
(2,4,1),(2,4,2),(2,4,3),(2,4,4),(2,4,5),(2,4,6),(2,4,7),\text{ and }(2,4,8).
\end{gathered}
$$

The 26 Cayley partitions of non-negative integers less than 2^3 are

$$
\begin{aligned}
&0 = 0,\\
&1 = 1\cdot 2^0,\\
&2 = 2\cdot 2^0 = 1\cdot 2^1,\\
&3 = 3\cdot 2^0 = 1\cdot 2^1 + 1\cdot 2^0,\\
&4 = 4\cdot 2^0 = 1\cdot 2^1 + 2\cdot 2^0 = 2\cdot 2^1 = 4\cdot 2^0,\\
&5 = 5\cdot 2^0 = 1\cdot 2^1 + 3\cdot 2^0 = 2\cdot 2^1 + 1\cdot 2^0 = 1\cdot 2^2 + 1\cdot 2^0,\\
&6 = 6\cdot 2^0 = 1\cdot 2^1 + 4\cdot 2^0 = 2\cdot 2^1 + 2\cdot 2^0 = 3\cdot 2^1 = 1\cdot 2^2 + 2\cdot 2^0 = 1\cdot 2^2 + 1\cdot 2^1,\\
&7 = 7\cdot 2^0 = 1\cdot 2^1 + 5\cdot 2^0 = 2\cdot 2^1 + 3\cdot 2^0 = 3\cdot 2^1 + 1\cdot 2^0 = 1\cdot 2^2 + 3\cdot 2^0\\
&\quad = 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0.
\end{aligned}
$$

A new proof of Cayley's theorem appeared in [KP14]. Konvalinka and Pak identify both Cayley compositions and Cayley partitions with lattice points in certain polytopes and construct a unimodular map between them. We give a brief summary of these constructions in the following.

The polytope describing Cayley compositions is the *Cayley polytope* C_n in $\mathbb{R}^n$ given by the inequalities $1 \le x_1 \le 2$ and $1 \le x_{i+1} \le 2x_i$ for $i = 1, \dots, n-1$. These inequalities do of course not describe an order polytope, but they share the property that each inequality either compares a coordinate to a constant or just compares two coordinates. To geometrically describe Cayley partitions, identify a partition

$$m_1 \cdot 2^{n-1} + m_2 \cdot 2^{n-2} + \dots + m_n \cdot 1$$

with the integer tuple $(m_1, m_2, \dots, m_n)$. Under this identification, Cayley partitions are exactly the integer tuples satisfying $0 \le m_i$ for $i = 1, \dots, n$ as well as

$$2^{n-1}m_1 + 2^{n-2}m_2 + \dots + m_n \le 2^n - 1. \tag{2.1}$$

Now consider the unimodular map $\widehat{\varphi} \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\widehat{\varphi}(x_1, x_2, \dots, x_n) = (2 - x_1, 2x_1 - x_2, \dots, 2x_{n-1} - x_n).[1]$$

[1]The map $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ in [KP14] is the inverse of the map $\widehat{\varphi}$.

Denote by $B_n = \widehat{\varphi}(C_n)$ the image of the Cayley polytope under this unimodular map. We immediately obtain the describing inequalities of B_n: the inequalities $x_1 \le 2$ and $x_{i+1} \le 2x_i$ translate to $0 \le y_i$ for $i = 1, \dots, n$, while the inequalities $1 \le x_i$ translate to

$$\sum_{i=1}^{k} 2^{k-i} y_i \le 2^k - 1 \qquad \text{for } k = 1, \dots, n. \tag{2.2}$$

We see that $k = n$ yields the defining inequality (2.1) for Cayley partitions and hence $\widehat{\varphi}$ does indeed map Cayley compositions to Cayley partitions. Furthermore, this assignment is surjective, since every Cayley partition also satisfies the inequalities in (2.2) for $k < n$: let $(m_1, m_2, \dots, m_n)$ describe a Cayley partition, i.e., it satisfies (2.1). It follows that

$$\sum_{i=1}^{k} 2^{k-i} m_i = 2^{k-n} \sum_{i=1}^{k} 2^{n-i} m_i \le 2^{k-n} \sum_{i=1}^{n} 2^{n-i} m_i \le 2^{k-n}(2^n - 1) = 2^k - 2^{k-n}.$$

Since the expression on the left hand side is integral, the stronger inequality (2.2) holds as well, for all $k = 1, \dots, n$.

Note the similarities to chain polytopes and their transfer maps: the polytope B_n is the image of C_n under a unimodular map with coordinates involving differences $Y - X$ coming from inequalities $X \le Y$. Its defining inequalities are non-negativity constraints $y_i \ge 0$ as well as some weighted sums of coordinates being bounded above by constants.

We will see in Chapter 8 that Cayley polytopes fall into a subclass of *distributive polyhedra* that allow a piecewise-linear bijection to *anti-blocking polyhedra* and identify the map $\widehat{\varphi}$ as a transfer map in this very general setting.

2.3. Lecture Hall Cones and Polytopes

The last examples of polyhedra related to poset polytopes we want to present in this chapter are *lecture hall cones*, *lecture hall polytopes* and recent generalizations of those.

Lecture hall cones have been introduced by Bousquet-Mélou and Eriksson in [BE97a] to study a variant of a theorem of Euler that says that the number of partitions of an integer N into odd parts is the same as the number of partitions of N into distinct parts. In their article, they consider partitions of N into *small odd parts*, which are the odd integers less than $2n$ for some fixed n. They find that the number of such partitions equals the number of *lecture hall partitions* of N that have length n.

A *lecture hall partition* of length n is an integral point in the *lecture hall cone* L_n, consisting of all points $x \in \mathbb{R}^n$ satisfying the inequalities

$$0 \le \frac{x_1}{1} \le \frac{x_2}{2} \le \cdots \le \frac{x_n}{n}. \tag{2.3}$$

The name is motivated by the following setting: consider a lecture hall with n rows of seats, where the i-th row is positioned i units of measure away from the speaker. If the i-th row of seats is raised to a height of x_i units of measure, the condition in (2.3) asserts

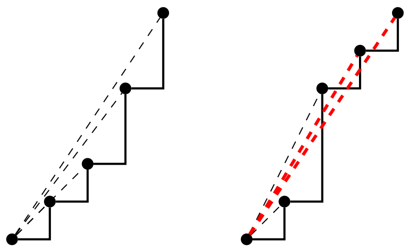


Figure 2.1.: The partitions $(1, 2, 4, 6)$ and $(1, 4, 5, 6)$ interpreted as row heights in a lecture hall. In the first case, the speaker is visible from all rows, while in the second case the view in the last two rows is obstructed.

that the view from each row is not obstructed by the rows in front. For example $(1, 2, 4, 6)$ is a lecture hall partition of length 4, while $(1, 4, 5, 6)$ is not, as illustrated in Figure 2.1.

A generalization of these partitions studied in the follow-up article [BE97b] is to place the i-th row of seats in distance s_i of the speaker. Thus, for an integer tuple $s = (s_1, s_2, \ldots, s_n)$, they defined the cone $L_n^{(s)}$ of *s-lecture hall partitions* by

$$0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n}.$$

A compact variant of this has been studied in [SS12], where the *s-lecture hall polytope* $P_n^{(s)}$ is defined by the inequalities

$$0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n} \leq 1.$$

We see that the inequalities are similar to those of an order polytope associated to a chain, just with added scaling factors for each coordinate. Indeed, this observation led to a recent generalization of lecture hall cones and polytopes to arbitrary finite posets in [BL16]. Given a finite poset P and an arbitrary map $s\colon P \longrightarrow \mathbb{N}_{>0}$, they define the *lecture hall order polytope $\mathcal{O}(P, s)$* as the set of all $x \in \mathbb{R}^P$ such that

$$\frac{x_p}{s_p} \leq \frac{x_q}{s_q} \quad \text{for } p \leq q$$

and

$$0 \leq \frac{x_p}{s_p} \leq 1 \quad \text{for all } p \in P.$$

Omitting the upper bound, they also define *lecture hall order cones.*

From the perspective taken in this thesis, we see that all variants of lecture hall cones and polytopes are similar to order polytopes in the sense that all inequalities either compare weighted coordinates or coordinates to a constant. We will see in Chapter 8 that all of these fall into the class of *distributive polyhedra* that may be described by an edge weighted digraph encoding the inequalities.

3. Representation Theory

In this chapter we look at polytopes from representation theory. The main examples here are *Gelfand–Tsetlin polytopes* and *Feigin–Fourier–Littelmann–Vinberg polytopes*, both of which enumerate bases for irreducible representations of the complex general linear group $\mathrm{GL}_n(\mathbb{C})$.

We refer to the books of Fulton, Harris [FH91], Hall [Hal15] and Procesi [Pro07] for detailed introductions to representation theory and only introduce the notions important for the following discussion here.

A *(complex) representation* of a group G can be described in various equivalent ways. The most elementary would be a (left) action of G on a (complex) vector space V by linear maps. From the description as a group action we immediately obtain the description as a homomorphism $G \longrightarrow \mathrm{GL}(V)$, sending $g \in G$ to the map $v \longmapsto gv$. Here $\mathrm{GL}(V)$ is the *general linear group* on V, consisting of all invertible linear maps $V \longrightarrow V$ with the group operation being composition of maps. Note that the group homomorphism $G \longrightarrow \mathrm{GL}(V)$ linearly extends to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \longrightarrow \mathrm{End}(V)$, where $\mathrm{End}(V)$ denotes the $\mathbb{C}$-algebra of all linear maps $V \longrightarrow V$ and $\mathbb{C}[G]$ is the group algebra of G, which is the complex vector space with basis G equipped with the product induced from G, i.e.,

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{j=1}^{m} \beta_i h_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_i\beta_i)(g_i h_j).$$

To summarize, a (complex) representation of G on the complex vector space V is one of the following equivalent structures:

- an action of G on V by invertible linear maps,
- a group homomorphism $G \longrightarrow \mathrm{GL}(V)$,
- a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \longrightarrow \mathrm{End}(V)$,
- a $\mathbb{C}[G]$-module structure on V.

Each of these different points of view has its advantages in different contexts. As usual in the literature, we will omit the action in notation and just call V the representation.

A subspace $W \subseteq V$ is called *invariant* or a *subrepresentation*, if $gw \in W$ for all $w \in W$, so W itself is a representation of G. If 0 and V are the only two invariant subspaces of V, the representation is said to be *irreducible*. Equivalently, a representation is irreducible if the $\mathbb{C}[G]$-module V is simple, i.e., the only two submodules of V are 0 and itself. Irreducible representations are of particular importance, since in good situations a group representation V will split into a direct sum of irreducible ones. If this is the case, we say that V is *completely reducible*.

If G is finite, Maschke's theorem says that every finite-dimensional representation of G is completely reducible, so the classification of finite-dimensional representations of G reduces to the study of irreducible representations of G.

3.1. Representations of the Complex General Linear Group

Let us now move to the representation theory of $\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^n)$, the group of all invertible complex $n \times n$ matrices. In this section, we mainly follow the combinatorial approach to the representation theory of $\mathrm{GL}_n(\mathbb{C})$ in [Ful97, Sec. 8], adding a discussion of the branching rule. A finite-dimensional representation V of $\mathrm{GL}_n(\mathbb{C})$ is called *holomorphic*, if after choosing a basis for $V \cong \mathbb{C}^m$, the group homomorphism $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ is holomorphic when $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}(V)$ are considered as subsets of $\mathbb{C}^{n^2}$ and $\mathbb{C}^{m^2}$, respectively. Using Weyl's unitarian trick (see for example [Ful97, Sec. 8.2]) it follows that holomorphic finite-dimensional representations of $\mathrm{GL}_n(\mathbb{C})$ are completely reducible.

3.1.1. Highest Weight Representations

Consider the chain of subgroups $H_n \subseteq B_n \subseteq \mathrm{GL}_n(\mathbb{C})$, where H_n consists of all invertible diagonal matrices and B_n consists of all invertible upper triangular matrices.[1] Given a representation V of $\mathrm{GL}_n(\mathbb{C})$, a vector $v \in V$ is called a *weight vector* with *weight* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ if

$$xv = x_1^{\mu_1} \cdots x_n^{\mu_n} v \quad \text{for all } x = \mathrm{diag}(x_1, \dots, x_n) \in H_n.$$

The weight vectors of weight μ together with 0 form the *weight subspace* V_μ of μ and similarly to the situation for eigenspaces of diagonalizable matrices, they decompose V as a direct sum of vector spaces

$$V = \bigoplus_\mu V_\mu,$$

where $\mu \in \mathbb{Z}^n$ ranges over all weights of V. Note that this is not a direct sum of representations, since the weight spaces are not invariant under the action of $\mathrm{GL}_n(\mathbb{C})$. The dimension of V_μ as a subspace of V is called the *multiplicity* of the weight μ. A weight vector is said to be of *highest weight*, if $B_n v = \mathbb{C}^* v$. That is, the invertible upper triangular matrices act on v by scalar multiplication and all scalars except zero appear. The importance of highest weights in the representation theory of $\mathrm{GL}_n(\mathbb{C})$ lies in the following theorem.

Theorem 3.1.1 ([Hum75, Sec. 31.3])**.** *i) A finite-dimensional holomorphic representation V of $\mathrm{GL}_n(\mathbb{C})$ is irreducible if and only if it has a highest weight vector v unique up to scaling. In this case, the weight of v is called the highest weight of V.*

ii) Two finite-dimensional holomorphic irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are isomorphic if and only if they have the same highest weight.

[1] The reader familiar with the theory of algebraic groups may recognize B_n as a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ and H_n as a maximal torus in B_n.

Hence, the classification of finite-dimensional holomorphic irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ is obtained by classifying the possible highest weights. It turns out that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ appears as a highest weight of some irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (cf. [FH91, Prop. 15.47]). Given any such $\lambda \in \mathbb{Z}^n$, we denote by $V(\lambda)$ the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ, it is determined up to isomorphism.

3.1.2. Young Diagrams and Schur Modules

We continue with a construction of the irreducible representations $V(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$. The construction we give here is detailed in [Ful97, Sec. 8].

We need a few operations to construct new representations from old ones. Namely, given representations V and W of a group G, all of the following vector spaces naturally carry the structure of a representation by G acting on each factor: any exterior power $\bigwedge^k V$, the tensor product $V \otimes W$, and the direct sum $V \oplus W$.

Back to the construction of representations for $\mathrm{GL}_n(\mathbb{C})$, our building blocks will be the *standard representation* $\mathbb{C}^n$ and the *determinantal representations* D^k for $k \in \mathbb{Z}$. The standard representation is just $\mathrm{GL}_n(\mathbb{C})$ acting on $\mathbb{C}^n$ by matrix multiplication and the determinantal representation D^k is given by $\mathrm{GL}_n(\mathbb{C})$ acting on $\mathbb{C}$ by $A \cdot z = \det(A)^k z$. Note that D^k is canonically isomorphic to $(\bigwedge^n \mathbb{C}^n)^{\otimes k}$ for $k \geq 0$, since $\bigwedge^n \mathbb{C}^n$ is spanned by $e_1 \wedge \cdots \wedge e_n$ and $Ae_1 \wedge \cdots \wedge Ae_n = \det(A)(e_1 \wedge \cdots \wedge e_n)$. Furthermore, $D^k \otimes D^l = D^{k+l}$ for all $k, l \in \mathbb{Z}$ under the canonical isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ via $z \otimes w \mapsto zw$.

We first consider the case where λ is a strictly positive weight, so $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Every such tuple is called an *integer partition*, since partitions of integers $N \geq 0$ into sums $N = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ not taking order of summands into account correspond to weakly decreasing tuples. Integer partitions are omnipresent in combinatorics and are often identified with certain diagrams. Given an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, its *Young diagram* is defined as

$$\mathrm{D}(\lambda) = \left\{ (i,j) \in \mathbb{N}^2 : 1 \leq i \leq k, 1 \leq j \leq \lambda_i \right\}.$$

The elements of $\mathrm{D}(\lambda)$ are called *boxes* as suggested by the usual way we depict Young diagrams. For example when $\lambda = (4,3,1)$, we have

$$\mathrm{D}(\lambda) = \mathrm{D}(4,3,1) = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), \\ (2,1), (2,2), (2,3), \\ (3,1) \end{array} \right\} = \ydiagram{4,3,1}\,.$$

Young diagrams as subsets $D \subseteq \mathbb{N}^2$ are characterized by the property that whenever $(i,j) \in D$ and $i' \leq i$, $j' \leq j$ we also have $(i',j) \in D$ and $(i,j') \in D$, i.e., the rows and columns have no gaps. Since this property is symmetric in the two coordinates, each Young diagram D comes with a *conjugate diagram* D' consisting of all (i,j) such that

$(j,i) \in D$. For example for $D = \mathrm{D}(4,3,1)$ as above, we have

$$D' = \begin{array}{|c|c|c|}\hline & & \multicolumn{1}{c|}{} \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \mathrm{D}(3,2,2,1).$$

For any integer partition λ there is a unique integer partition λ' such that $\mathrm{D}(\lambda') = \mathrm{D}(\lambda)'$. It is called the *conjugate partition* of λ and counts the numbers of boxes in each column of the Young diagram of λ. In the example, $(3,2,2,1)$ is the conjugate of $(4,3,1)$.

To each integer partition λ we associate a representation E^λ of $\mathrm{GL}_n(\mathbb{C})$ called a *Schur module*. Let $\mu = \lambda' = (\mu_1, \dots, \mu_l)$ be the conjugate partition of λ and consider the representation

$$A^\lambda = \bigwedge^{\mu_1} \mathbb{C}^n \otimes \bigwedge^{\mu_2} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\mu_l} \mathbb{C}^n. \tag{3.1}$$

That is, for each column in $\mathrm{D}(\lambda)$ we take the exterior power of $\mathbb{C}^n$ given by the length of the column and then take the tensor product of all these. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$, then A^λ is generated by monomials

$$e_T = \bigotimes_{j=1}^{l} \bigwedge_{i=1}^{\mu_j} e_{T(i,j)} \quad \text{for } T\colon \mathrm{D}(\lambda) \longrightarrow [n]. \tag{3.2}$$

Any map from a Young diagram D to some a set of number $M \subseteq \mathbb{N}$ is called a *Young tableau*[2] of shape D with entries in M and is usually denoted by drawing the actual diagram and putting numbers in the boxes. For example, a tableau of shape $\mathrm{D}(4,3,1)$ would be

$$T = \begin{array}{|c|c|c|c|}\hline 3 & 6 & 2 & 1 \\ \hline 4 & 8 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \,. \tag{3.3}$$

To construct E^λ we have to define an operation on Young tableaux called an *exchange*. Let $F\colon D \longrightarrow M$ be a tableau of shape D with column lengths $\mu = (\mu_1, \dots, \mu_l)$. An exchange of F yields another tableau $F'\colon D \longrightarrow M$ that is obtained by first picking two columns $j_1 < j_2$ and equally large sets of boxes B_1 in column j_1, B_2 in column j_2 and then interchanging the entries in B_1 and B_2, maintaining their vertical order. For example, if T is the tableau in (3.3) and we pick $j_1 = 1$, $j_2 = 2$ and $B_1 = \{(1,1),(3,1)\}$, $B_2 = \{(1,2),(2,2)\}$, the exchange produces the tableau

$$S = \begin{array}{|c|c|c|c|}\hline 6 & 3 & 2 & 1 \\ \hline 4 & 7 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \,.$$

Denote by T^{j_1,j_2,B_2} the family of all tableaux that can be obtained from T by an exchange given by j_1, j_2, B_2 and any choice of B_1. Note that different exchanges might produce

[2]What we call a Young tableau is called a filling in [Ful97]. We will refer to a Young tableau in the sense of [Ful97] as a semistandard Young tableau.

the same results, which is why we have to consider T^{j_1,j_2,B_2} as a family and not a set. For example for T as in (3.3) we have

$$T^{1,2,\{(1,2),(2,2)\}} = \left(\begin{array}{|c|c|c|c|}\hline 6&3&2&1\\\hline 8&4&4\\\cline{1-3} 7\\\cline{1-1}\end{array}\, , \begin{array}{|c|c|c|c|}\hline 6&3&2&1\\\hline 4&7&4\\\cline{1-3} 8\\\cline{1-1}\end{array}\, , \begin{array}{|c|c|c|c|}\hline 3&4&2&1\\\hline 6&7&4\\\cline{1-3} 8\\\cline{1-1}\end{array} \right).$$

Now let Q^λ be the subspace of A^λ spanned by the elements

$$e_T - \sum_{T' \in T^{j_1,j_2,B_2}} e_{T'},$$

for any tableau $T\colon \mathrm{D}(\lambda) \to [n]$ and choices of $j_1 < j_2$ and B_2. Using linearity in each factor we see that Q^λ is invariant under the action of $\mathrm{GL}_n(\mathbb{C})$ and we can define the Schur module E^λ as the quotient A^λ/Q^λ.

The elements $[e_T] \in E^\lambda$ for e_T the monomial in (3.2) still generate E^λ when T ranges over all tableaux with entries in $[n]$. Since we take exterior products in each column, it is enough to consider tableaux whose entries are strictly increasing from top to bottom in each column. Furthermore, the relations given by Q^λ yield that E^λ is generated by monomials $[e_T]$, where T is strictly increasing in the columns and weakly increasing in the rows. A tableau with these properties is called a *semistandard Young tableau* or *SSYT* for short. The monomials given by SSYTs of shape λ with entries in $[n]$ do in fact form a basis of the Schur module E^λ as a representation of $\mathrm{GL}_n(\mathbb{C})$, as is detailed in [Ful97, Sec. 8.1]. A consequence of this is that for $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k > n$ the representation E^λ is trivial, since there are no SSYTs with entries in $[n]$ when the first column of $\mathrm{D}(\lambda)$ has more than n boxes. This is of course easily verified from the definition of A^λ as in (3.1), since $\bigwedge^k \mathbb{C}^n$ is trivial when $k > n$.

Now assume λ has at most length n. If T is any tableau of shape λ with entries in $[n]$ and $x = \mathrm{diag}(x_1, \ldots, x_n) \in H_n$ is a diagonal matrix, we have

$$x \cdot [e_T] = \left[\bigotimes_{j=1}^{l} \bigwedge_{i=1}^{\mu_j} x e_{T(i,j)} \right] = \left[\bigotimes_{j=1}^{l} \bigwedge_{i=1}^{\mu_j} x_{T(i,j)} e_{T(i,j)} \right] = x_1^{\alpha_1} \cdots x_n^{\alpha_n} [e_T],$$

where α_k is the number of times k appears in T. Thus, every $[e_T]$ is a weight vector.

If we take the tableau T given by $(i,j) \mapsto i$, i.e., all entries in row i are equal to i, we obtain a vector of weight λ (padded with zeros if $k < n$). Acting on it with an upper triangular matrix $g \in B$, we can use the alternating property in each column and see that $g \cdot [e_T] = x \cdot [e_T]$ for $x \in H_n$ the diagonal matrix with the same diagonal as g. We conclude that this monomial is a highest weight vector and in fact it is the only one up to multiplication by a scalar (see [Ful97, Sec. 8.2, Lem. 4]). Hence, Theorem 3.1.1 implies that E^λ is an irreducible $\mathrm{GL}_n(\mathbb{C})$ module isomorphic to $V(\lambda)$. So we have a construction for all holomorphic finite-dimensional irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ with strictly positive highest weight.

Finally, we consider the general case where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an arbitrary highest weight for $\mathrm{GL}_n(\mathbb{C})$, so the λ_i are any integers satisfying $\lambda_1 \geq \cdots \geq \lambda_n$. Let k be an integer such that $\lambda_n + k > 0$ and define the integer partition λ_{+k} by $\lambda_{+k} = (\lambda_1 + k, \ldots, \lambda_n + k)$. Consider the tensor product $E^{\lambda_{+k}} \otimes D^{-k}$ of a Schur module as discussed above and the determinantal representation given by multiplication with $\det(A)^{-k}$ on $\mathbb{C}$. As a vector space $E^{\lambda_{+k}} \otimes D^{-k}$ is isomorphic to $E^{\lambda_{+k}}$ via $v \otimes z \mapsto zv$. However, as representations they are not isomorphic for $k \neq 0$. Let $w \in E^{\lambda_{+k}}$ be a weight vector of weight $(\alpha_1, \ldots, \alpha_n)$ and $x = \operatorname{diag}(x_1, \ldots, x_n) \in H_n$, then

$$x \cdot (w \otimes 1) = (x_1^{\alpha_1} \cdots x_n^{\alpha_n} w) \otimes ((x_1 \cdots x_n)^{-k} 1) = x_1^{\alpha_1 - k} \cdots x_n^{\alpha_n - k} (w \otimes 1),$$

so weights in $E^{\lambda_{+k}} \otimes D^{-k}$ are shifted by $-k$. We conclude that $E^{\lambda_{+k}} \otimes D^{-k}$ is an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ, as desired. Note that increasing k by one adds a tensor factor of $\bigwedge^n \mathbb{C}^n \cong D^1$ in the Schur module part $E^{\lambda_{+k}}$ that is canceled by an additional factor of D^{-1} in the determinantal part, so the construction does not depend on the choice of k up to canonical isomorphism.

3.2. Gelfand–Tsetlin Bases and Polytopes

In this section we construct a basis of $V(\lambda)$ known as the *Gelfand–Tsetlin basis* with elements naturally enumerated by the lattice points in a polytope. The main ingredient for the construction is the following *branching rule* that describes how an irreducible representation $V(\lambda)$ for $\mathrm{GL}_n(\mathbb{C})$ decomposes into irreducibles when restricted to $\mathrm{GL}_{n-1}(\mathbb{C})$. For a representation V of a group G with a subgroup $H \subseteq G$, we denote by $V|_H$ the *restriction* of V to a representation of H, i.e., the representation given by the restriction of $G \to \mathrm{GL}(V)$ to $H \to \mathrm{GL}(V)$. We consider $\mathrm{GL}_{n-1}(\mathbb{C})$ as the subgroup of $\mathrm{GL}_n(\mathbb{C})$ given by matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where A is an invertible $(n-1) \times (n-1)$ matrix.

Theorem 3.2.1 (Branching Rule, [Žel73, § 66 Thm. 2]). *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and $\mu = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{Z}^{n-1}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_{n-1}$ be highest weights of the irreducible representations $V(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ and $V(\mu)$ of $\mathrm{GL}_{n-1}(\mathbb{C})$, respectively. The restricted representation $V(\lambda)|_{\mathrm{GL}_{n-1}(\mathbb{C})}$ has a subrepresentation W_μ isomorphic to $V(\mu)$ if and only if the* interlacing condition

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

is satisfied. In this situation, W_μ is uniquely determined and furthermore $V(\lambda)|_{\mathrm{GL}_{n-1}(\mathbb{C})}$ decomposes as the direct sum

$$V(\lambda)\big|_{\mathrm{GL}_{n-1}(\mathbb{C})} = \bigoplus_\mu W_\mu \cong \bigoplus_\mu V(\mu),$$

where μ ranges over all integer tuples in $\mathbb{Z}^{n-1}$ satisfying the interlacing condition.

Now consider the chain of subgroups

$$\mathrm{GL}_1(\mathbb{C}) \subseteq \mathrm{GL}_2(\mathbb{C}) \subseteq \cdots \subseteq \mathrm{GL}_{n-1}(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C}) \tag{3.4}$$

and iteratively restrict $V(\lambda)$ to smaller $\mathrm{GL}_k(\mathbb{C})$, in each step applying the branching rule. This way we obtain a decomposition

$$V(\lambda)|_{\mathrm{GL}_1(\mathbb{C})} = \bigoplus_{\Lambda} W_\Lambda$$

into irreducible subrepresentations $W_\Lambda \subseteq V(\lambda)|_{\mathrm{GL}_1(\mathbb{C})}$ given by triangular patterns

$$\Lambda = \begin{pmatrix} \lambda_1^{(n)} & & \lambda_2^{(n)} & & \lambda_3^{(n)} & & \cdots & & \lambda_n^{(n)} \\ & \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & \cdots & & \lambda_{n-1}^{(n-1)} & \\ & & \ddots & & \ddots & & \iddots & & \\ & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\ & & & & \lambda_1^{(1)} & & & & \end{pmatrix} \in \mathbb{Z}^{n(n+1)/2}, \tag{3.5}$$

with first row $\lambda^{(n)} = \lambda$ and consecutive rows satisfying the interlacing condition from Theorem 3.2.1. Such triangular patterns of integers are called *Gelfand–Tsetlin patterns* or *GT patterns* for short, as they first appeared as enumerators for a basis of $V(\lambda)$ in [GT50]. Since $\mathrm{GL}_1(\mathbb{C})$ is abelian, all its irreducible representation are one dimensional (see [Hal15, Cor. 4.28]) and picking a non-zero vector $w_\Lambda \in W_\Lambda$ in each summand yields the Gelfand–Tsetlin basis for $V(\lambda)$. This basis is determined up to scalar multiplication and only depends on the choice of embeddings in the chain of subgroups in (3.4).

For any $\lambda \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ let $\mathrm{GT}(\lambda)$ be the *Gelfand–Tsetlin polytope* in $\mathbb{R}^{n(n+1)/2}$ given by patterns Λ as in (3.5) but with real entries, still satisfying $\lambda^{(n)} = \lambda$ in the first row as well as the interlacing conditions from Theorem 3.2.1 for consecutive rows. For an integral highest weight λ for $\mathrm{GL}_n(\mathbb{C})$, the GT patterns enumerating the Gelfand–Tsetlin basis of $V(\lambda)$ are exactly the lattice points in the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$.

It turns out that the vectors in $V(\lambda)$ given by GT patterns are always weight vectors. In fact, if w_Λ is a vector corresponding the GT pattern Λ as in (3.5), it has weight $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 = \lambda_1^{(1)}$ and

$$\mu_k = \sum_{i=1}^{k} \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)}.$$

In other words, the pattern has row sums $\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_n$ from bottom to top. This motivates the definition of the polytope $\mathrm{GT}(\lambda)_\mu$ whose integer points enumerate a basis of the weight subspace $V(\lambda)_\mu$. For general $\lambda, \mu \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ let $\mathrm{GT}(\lambda)_\mu$ be the *weighted Gelfand–Tsetlin polytope* in $\mathbb{R}^{n(n+1)/2}$ obtained by intersecting $\mathrm{GT}(\lambda)$ with the subspace described by the row sum conditions

$$\sum_{i=1}^{k} \lambda_i^{(k)} = \mu_1 + \cdots + \mu_k \quad \text{for } k = 1, \ldots, n.$$

In light of the previous sections, we see that the defining inequalities of the unweighted Gelfand–Tsetlin polytope $GT(\lambda)$ are similar to those of order polytopes.

Since Gelfand–Tsetlin polytopes appear not just in representation theory, but also in the geometry of Schubert varieties and flag varieties (e.g., [KM05; Kir10; KST12]), an increasing number of articles on their combinatoric and geometric properties appeared in the last decades. Regarding unweighted GT polytopes we want to mention [GKT13], where the generating function of the number of vertices of $GT(\lambda)$ is discussed, as well as [ACK18], where an approach using ladder diagrams is used to determine the exponential generating function for f-vectors of Gelfand–Tsetlin polytopes. On the side of weighted GT polytopes we point out [DM04], where a method using tiling matrices to determine dimensions of faces is used to obtain vertices of weighted Gelfand–Tsetlin polytopes. We will generalize this approach in Section 6.5.

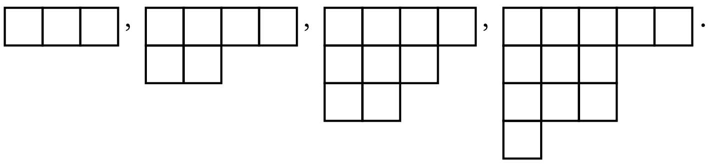
3.3. Gelfand–Tsetlin Patterns and SSYTs

Since we have described two bases of $V(\lambda)$—the GT basis obtained from iteratively applying the branching rule Theorem 3.2.1 and the basis for the Schur module $E^\lambda \cong V(\lambda)$ given by semistandard Young tableaux of shape $D(\lambda)$—it is natural to ask for a combinatorial bijection between GT patterns and SSYTs.

Assuming λ is a strictly positive weight—otherwise one can always consider λ_{+k} instead as done above—we can think of a GT pattern as a list of integer partitions and associate to each of them a Young diagram. For example, consider the Gelfand–Tsetlin pattern

$$\Lambda = \left(\begin{array}{ccccccc} 5 & & 3 & & 3 & & 1 \\ & 4 & & 3 & & 2 & \\ & & 4 & & 2 & & \\ & & & 3 & & & \end{array}\right).$$

From bottom to top, this yields the Young diagrams



Since GT patterns as in (3.5) satisfy $\lambda_i^{k+1} \le \lambda_i^k$, each diagram in the list will be contained in the next. Hence, we may encode the whole list as a Young tableau of shape given by the first row in the GT pattern: into each box put the number in which step the box appears when reading the GT pattern bottom to top. In the example we obtain the Young tableau

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 5 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array} \quad .$$

The interlacing conditions for GT patterns yield that this tableau will always be semistandard. In fact, this construction gives a bijection between GT patterns with top row $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ and SSYTs of shape $\mathrm{D}(\lambda)$ with entries in $[n]$ (cf. [Mac95, Ch. I, Sec. 1]).

However, this correspondence between GT patterns and SSYTs does not identify the two bases of $V(\lambda)$. Consider the case where $n = 3$ and $\lambda = (3, 2, 1)$ is the highest weight of the irreducible representation $V(\lambda)$ of $\mathrm{GL}_3(\mathbb{C})$. The branching rule yields that $\mu = (3, 1)$ gives a $\mathrm{GL}_2(\mathbb{C})$ invariant subspace W_μ of $V(\lambda)$ isomorphic to $V(\mu)$. This subspace is spanned by the GT basis vectors corresponding to patterns of the form

$$\begin{pmatrix} 3 & & 2 & & 1 \\ & 3 & & 1 & \\ & & * & & \end{pmatrix}$$

that correspond to the three SSYTs

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3&\multicolumn{1}{c}{}\\\cline{1-2} 3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&\multicolumn{1}{c}{}\\\cline{1-2} 3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,, \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3&\multicolumn{1}{c}{}\\\cline{1-2} 3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,. \tag{3.6}$$

The subspace of the Schur module E^λ spanned by the basis vectors $[e_T]$ for these three tableaux is *not* $\mathrm{GL}_2(\mathbb{C})$ invariant, as the following calculation shows. We use the notation

$$\begin{array}{|c|c|c|}\hline u&v&w\\\hline x&y&\multicolumn{1}{c}{}\\\cline{1-2} z&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} = [(u \wedge x \wedge z) \otimes (v \wedge y) \otimes w] \in E^\lambda \quad \text{for } u, v, w, x, y, z \in \mathbb{C}^3.$$

Using the properties of A^λ, i.e., linearity in each factor and alternating in the columns, we have

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \cdot \begin{array}{|c|c|c|}\hline e_1&e_1&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline e_1&e_1&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline e_1&e_2&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline e_1&e_1&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline e_1&e_2&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,.$$

The first three summands are basis vectors $[e_T]$ for T one of the tableaux in (3.6). The last one is $[e_T]$ for a non semi-standard tableau T and the relations of Q^λ given by the exchange for $j_1 = 2$, $j_2 = 3$ and $B_2 = \{(1, 3)\}$ allow to expand it as

$$\begin{array}{|c|c|c|}\hline e_1&e_2&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline e_1&e_1&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline e_1&e_2&e_3\\\hline e_2&e_1&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline e_1&e_1&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline e_1&e_1&e_3\\\hline e_2&e_2&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,.$$

Thus, in the SSYT basis we have

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \cdot \begin{array}{|c|c|c|}\hline e_1&e_1&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline e_1&e_1&e_1\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline e_1&e_2&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + 2\begin{array}{|c|c|c|}\hline e_1&e_1&e_2\\\hline e_2&e_3&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline e_1&e_1&e_3\\\hline e_2&e_2&\multicolumn{1}{c}{}\\\cline{1-2} e_3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\,,$$

where the last summand is not one of the three in (3.6). We conclude that the subspace spanned by the basis vectors $[e_T]$ for tableaux in (3.6) is not $\mathrm{GL}_2(\mathbb{C})$ invariant and hence the described correspondence of GT patterns and SSYTs does not identify the GT basis and SSYT basis of $V(\lambda)$.

3.4. Feigin–Fourier–Littelmann–Vinberg Polytopes

As we have seen in the previous sections, in many situations where a polytope with a description similar to an order polytope appears, there is a second polytope with a description similar to a chain polytope lurking. In fact, half a decade after Gelfand and Tsetlin constructed their basis of $V(\lambda)$, another basis was constructed by Feigin, Fourier and Littelmann in [FFL11] that has been conjectured to exist by Vinberg five years earlier.

The *Feigin–Fourier–Littelmann–Vinberg basis* or *FFLV basis* for short is also enumerated by certain patterns with integral entries. Given a highest weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ for $\mathrm{GL}_n(\mathbb{C})$, an *FFLV pattern* is a triangular pattern

$$\Lambda = \left(\begin{array}{ccccccccc} \lambda_1^{(n)} & & \lambda_2^{(n)} & & \lambda_3^{(n)} & & \cdots & & \lambda_n^{(n)} \\ & \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & \cdots & & \lambda_{n-1}^{(n-1)} & \\ & & \ddots & & \ddots & & \iddots & & \\ & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array} \right) \in \mathbb{Z}^{n(n+1)/2},$$

satisfying the following conditions:

i) the first row is given by $\lambda^{(n)} = \lambda$,
ii) all other entries are non-negative,
iii) for each *Dyck path* starting and ending in the first row, the sum of the entries along the path below the top row is less than or equal to the difference of the endpoints.

Here a Dyck path is a sequence of indices $((i_1, k_1), (i_2, k_2), \dots, (i_s, k_s))$ such that $k_1 = k_s = n$ and at every step (i, k) is either followed by $(i-1, k-1)$ or $(i, k+1)$. Condition iii) is then precisely

$$\lambda_{i_2}^{(k_2)} + \cdots + \lambda_{i_{s-1}}^{(k_{s-1})} \leq \lambda_{i_s} - \lambda_{i_1}.$$

An example for such a pattern with one Dyck path indicated is

$$\Lambda = \left(\begin{array}{ccccccc} 5 & & 3 & & 3 & & 1 \\ & 0 & & 0 & & 1 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{array} \right).$$

Condition iii) is satisfied for this path since

$$1 + 0 + 0 + 2 + 0 \leq 5 - 1.$$

Without requiring the entries to be integers, the conditions of FFLV patterns describe a polytope $\mathrm{FFLV}(\lambda)$ in $\mathbb{R}^{n(n+1)/2}$, whose lattice points correspond to the basis vectors in the FFLV basis of $V(\lambda)$ when λ is integral. The description of this polytope is similar to that of a chain polytope: all entries that are not fixed are required to be non-negative and each Dyck path gives an upper bound on the sum of the entries along the path. In [ABS11], Ardila, Bliem and Salazar generalized the concept of poset polytopes to *marked poset polytopes*, allowing $\mathrm{GT}(\lambda)$ and $\mathrm{FFLV}(\lambda)$ to be considered as the marked order polytope $\mathcal{O}(P,\lambda)$ and marked chain polytope $\mathcal{C}(P,\lambda)$ associated to a certain *marked poset* (P,λ). They also give a piecewise-unimodular transfer map $\mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ and hence provide a combinatorial explanation as to why the two polytopes have the same number of lattice points—the dimension of $V(\lambda)$.

We will come back to this generalization of poset polytopes in Chapter 5. It is worth noting at this point, that the representation theory literature offers many other examples of polytopes whose lattice points enumerate bases of irreducible representations of semisimple Lie algebras. Most of them are closely related to poset polytopes and some of them still fit in the more general framework we will discuss in Part II. See for example [Lit98], [ABS11] and [BD15].

4. Finite Frame Theory

In this chapter we discuss polytopes that arise in the theory of finite Hilbert space frames. Frame theory started in the theory of signal transmission as a generalization of Fourier analysis that allows for redundancy and adaptation. Generalizing an approach taken by Gabor [Gab46], the theory was thoroughly introduced by Duffin and Schaeffer in [DS52]. For a detailed account on the historical development, and an introduction to the finite dimensional theory up to recent research results we refer to the book of Casazza and Kutyniok [CK13].

Roughly speaking, frame theory addresses the following shortcomings of Fourier analysis: considering some signal, for example image data, audio data or similar, the aim of Fourier analysis is to express it in terms of oscillations, i.e., it describes a signal using frequencies and phases. In the discrete setting, this is merely a change from one orthonormal basis to another in a finite dimensional Hilbert space. Hence, losing any of the coefficients due to noise or a lossy transmission, the signal can not be recovered. Furthermore, the chosen orthonormal basis might not be well suited to capture the characteristics of the signals to be transmitted or processed. However, the advantage of an orthonormal basis is the very efficient calculation of coefficients: they are the inner products with the basis vectors. Frame theory aims to resolve these issues by allowing redundant spanning sets that might be adapted to the signals characteristic features, while still offering efficient calculations.

4.1. Basics of Finite Frame Theory

We start with a quick introduction to the theory of finite Hilbert space frames to then introduce sequences of eigensteps associated to frames as well as the polytopes they form.

Let $\mathcal{H}$ be any Hilbert space, i.e., a real or complex complete vector space with an inner product $\langle \cdot, \cdot \rangle$. A *Hilbert space frame* or just *frame* for $\mathcal{H}$ is a sequence $F = (f_1, f_2, \ldots)$ of vectors in $\mathcal{H}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\,\|x\| \leq \sum_{k=1}^{\infty} |\langle x, f_k \rangle|^2 \leq B\,\|x\| \qquad \text{for all } x \in \mathcal{H}. \tag{4.1}$$

The constants A and B are called *frame bounds*. If A and B are chosen as big, respectively small, as possible, they are called *optimal frame bounds*. One of the most important types of frames are *tight frames*, where the optimal frame bounds are equal, that is, one can choose $A = B$.

Since in any application both the dimension of the Hilbert space and the length of the frame are necessarily finite, the finite version of frame theory gained more and more interest in the recent years. In this setting, the Hilbert space $\mathcal{H}$ is of finite dimension d and a *finite frame* is a finite sequence $F = (f_1, f_2, \ldots, f_n)$ of vectors in $\mathcal{H}$ satisfying

$$A\,\|x\| \leq \sum_{k=1}^{n} |\langle x, f_k\rangle|^2 \leq B\,\|x\| \qquad \text{for all } x \in \mathcal{H}. \tag{4.2}$$

In finite dimensions, this definition might seem overly complicated: the upper frame bound will exist for any vector configuration and the existence of a lower frame bound is equivalent to F spanning all of $\mathcal{H}$. Hence, we could as well define a finite frame just as an ordered spanning set. However, the usual definition we gave here stresses the harmonic analysis flavored point of view the theory takes.

As stated above, frame theory wants to keep the efficient calculation method of just taking inner products for obtaining the coefficients that describe a signal. Hence, for any frame F, define the *analysis operator* T and *synthesis operator* T^* by

$$\begin{aligned} T\colon\ \mathcal{H} &\longrightarrow \mathbb{C}^n, & \qquad T^*\colon\ \mathbb{C}^n &\longrightarrow \mathcal{H}, \\ x &\longmapsto (\langle x, f_k\rangle)_{k=1}^{n}, & (a_k)_{k=1}^{n} &\longmapsto \sum_{k=1}^{n} a_k f_k. \end{aligned}$$

As the notation suggests and is easily verified, the analysis and synthesis operators of any frame form an adjoint pair of linear operator, i.e., $\langle Tx, a\rangle = \langle x, T^*a\rangle$ for all $x \in \mathcal{H}$, $a \in \mathbb{C}^n$. Here the inner product on $\mathbb{C}^n$ is the standard dot product

$$\langle a, b\rangle = \sum_{k=1}^{n} a_k \overline{b_k}.$$

The composition $S = T^*T$ given by $Sx = \sum_{k=1}^{n} \langle x, f_k\rangle f_k$ is called the *frame operator* of F. Ideally, this is the identity on $\mathcal{H}$, so that taking any signal, storing the inner products with the frame vectors and just using them as coefficients in a linear combination recovers the original signal. This is of course true, whenever the vectors in F form an orthonormal basis. However, there are frames different from orthonormal bases that still satisfy $S = \mathrm{id}_{\mathcal{H}}$. These are called *Parseval frames* and a frame is Parseval if and only if the optimal frame bounds are $A = B = 1$. For tight frames we still have $S = A\,\mathrm{id}_{\mathcal{H}}$ and in general the frame operator is a positive definite operator with smallest and largest eigenvalues being the frame bounds A and B, respectively. Since the eigenvalues of the frame operator play an essential role in frame theory, they are usually referred to as the eigenvalues of the frame itself.

4.2. The Frame Construction Problem

One of the difficulties in frame theory is to construct application specific frames with certain properties. A particular example is the need for signal encoding that is robust with respect to noise and loss of coefficients. One can show that unit norm tight frames—i.e., frames that are tight and consist solely of unit vectors—are optimally robust in a certain sense (cf. [CK03; HP04]). The general problem that was posed in this context is the *frame construction problem* (cf. [CFM+13a]): how to construct all frames $F = (f_1, \dots, f_n)$ for $\mathcal{H} = \mathbb{C}^d$ with prescribed norm squares $\|f_k\|^2 = \mu_k$ and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ of the frame operator?

Given the tuples $\mu = (\mu_1, \dots, \mu_n)$ and $\lambda = (\lambda_1, \dots, \lambda_d)$, the existence of a frame for $\mathbb{C}^d$ with these norm squares and eigenvalues can be answered using the *Schur–Horn theorem*, that characterizes the possible diagonals of Hermitian matrices with prescribed eigenvalues.

Identify the frame F with the $d \times n$ matrix

$$F = \begin{pmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_n \\ | & | & & | \end{pmatrix}.$$

Hence, F is the matrix of the synthesis operator with respect to the standard bases of $\mathbb{C}^d$ and $\mathbb{C}^n$, while its Hermitian conjugate F^* is the matrix of the analysis operator. The norm squares $\|f_k\|^2$ are the diagonal entries of the Hermitian $n \times n$ matrix F^*F called the *Gram matrix* of F. Note that we have $n \geq d$ since finite frames are spanning sets. The spectrum of F^*F is just a zero-padded version of the spectrum of the frame operator represented by the $d \times d$ matrix FF^*. To be precise, if $(\lambda_1, \dots, \lambda_d)$ is the spectrum of the frame operator in weakly decreasing order, then $(\lambda_1, \dots, \lambda_d, 0, \dots, 0)$ counting $n - d$ zeros is the spectrum of the Gram matrix. To see this, consider the singular value decomposition

$$F = U \begin{pmatrix} \alpha_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \alpha_d & 0 & \cdots & 0 \end{pmatrix} V^*, \tag{4.3}$$

where U and V are unitary matrices of size $d \times d$ and $n \times n$, respectively. Hence, we have

$$\begin{aligned} FF^* &= U \operatorname{diag}(|\alpha_1|^2, \dots, |\alpha_d|^2)\, U^*, \quad \text{and} \\ F^*F &= V \operatorname{diag}(|\alpha_1|^2, \dots, |\alpha_d|^2, 0, \dots, 0)\, V^*. \end{aligned} \tag{4.4}$$

From this decomposition we can immediately read of the eigenvalues and the spectra compare as claimed. Hence, we know that any frame F with the desired properties yields a Gram matrix F^*F with spectrum $(\lambda_1, \dots, \lambda_d, 0, \dots, 0)$. Conversely, given any positive-definite Hermitian $n \times n$ matrix M with spectrum $(\lambda_1, \dots, \lambda_d, 0, \dots, 0)$, we can construct a frame F such that $M = F^*F$: since M is Hermitian, it has an orthonormal eigenbasis, so there is a unitary $n \times n$ matrix U such that

$$M = U \operatorname{diag}(\lambda_1, \dots, \lambda_d, 0, \dots, 0)\, U^*. \tag{4.5}$$

Hence, we have $M = F^*F$, where F is the $d \times n$ matrix

$$\begin{pmatrix} \sqrt{\lambda_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \sqrt{\lambda_d} & 0 & \cdots & 0 \end{pmatrix} U^*. \tag{4.6}$$

We conclude that characterizing the possible norms of frames whose frame operator has eigenvalues $(\lambda_1, \ldots, \lambda_d)$ is equivalent to characterizing the possible diagonals of Hermitian matrices with eigenvalues $(\lambda_1, \ldots, \lambda_d, 0, \ldots, 0)$. We can now state the Schur–Horn theorem for finite frames.

Theorem 4.2.1 (Schur–Horn, [Hor54]). *There exists a frame $F = (f_1, \ldots, f_n)$ for $\mathbb{C}^d$ with norm squares $\|f_k\|^2 = \mu_k$ for $k = 1, \ldots, n$ and spectrum $(\lambda_1, \ldots, \lambda_d)$ if and only if $(\mu_1, \ldots, \mu_n)$ lies in the convex hull of all permutations of the vector $(\lambda_1, \ldots, \lambda_d, 0, \ldots, 0)$ in $\mathbb{R}^n$.*

Note that this formulation of the Schur–Horn theorem does not require the norm squares to be in any specific order. In the literature, the theorem is often stated in terms of *majorization*: given two vectors $a, b \in \mathbb{R}^n$, we say *a is majorized by b* and write $a \preceq b$, if after permuting the coordinates such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ we have

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for } 1 \leq k < n, \text{ and}$$
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

By a result of Rado that appeared in [Rad52], a is majorized by b if and only if a lies in the convex hull of all permutation of b, which justifies our formulation of the Schur–Horn theorem.

Unfortunately, the classical proof of the Schur–Horn theorem is non-constructive and other proofs may construct some, but not all frames with the desired properties. Only recently, a new approach to the frame construction problem using *sequences of eigensteps* led to a complete parametrization in [FMPS13; CFM+13a].

4.3. Polytopes of Eigensteps

In this section, we discuss the approach taken in [FMPS13; CFM+13a] to solve the frame construction problem. As we will see, one step in the parametrization of all frames for $\mathbb{C}^d$ with norm squares $\mu = (\mu_1, \ldots, \mu_n)$ and eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$ will be the parametrization of a certain polytope that coincides with a Gelfand–Tsetlin polytope.

Given any Hermitian $m \times m$ matrix M, we denote by $\sigma(M) \in \mathbb{R}^m$ the spectrum of m in weakly decreasing order. That is, $\sigma(M) = (\sigma_1(M), \ldots, \sigma_m(M))$, where $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_m(M)$ are the eigenvalues of M. To a frame F for $\mathbb{C}^d$ considered as a $n \times d$ matrix and a non-negative integer $k \leq n$ associate the *truncated frame* F_k which is just the matrix consisting of the first k columns of F.

Definition 4.3.1 (cf. [FMPS13]). To a frame $F = (f_1, \ldots, f_n)$ for $\mathbb{C}^d$ we associate the *sequence of outer eigensteps* and the *sequence of inner eigensteps* of F given by

$$\Lambda^{\text{out}}(F) = \left(\sigma(F_0 F_0^*), \ldots, \sigma(F_n F_n^*)\right), \quad \text{and}$$
$$\Lambda^{\text{in}}(F) = \left(\sigma(F_1^* F_1), \ldots, \sigma(F_n^* F_n)\right),$$

respectively. Note that $F_0F_0^*$ is the $d \times d$ zero matrix with all eigenvalues zero. We will usually consider $\Lambda^{\text{out}}(F)$ as the real $d \times (n+1)$ matrix

$$\Lambda^{\text{out}}(F) = \begin{pmatrix} \sigma_1(F_0F_0^*) & \cdots & \sigma_1(F_nF_n^*) \\ \vdots & & \vdots \\ \sigma_d(F_0F_0^*) & \cdots & \sigma_d(F_nF_n^*) \end{pmatrix} \in \mathbb{R}^{d\times(n+1)}$$

and refer to it as the *outer eigenstep tableau* or just *outer eigensteps* of F. We will usually consider $\Lambda^{\text{in}}(F)$ as the triangular pattern

$$\Lambda^{\text{in}}(F) = \left(\begin{array}{ccccc} & & & & \sigma_1(F_n^*F_n) \\ & & \sigma_1(F_3^*F_3) & \cdots & \sigma_2(F_n^*F_n) \\ & \sigma_1(F_2^*F_2) & & & \\ \sigma_1(F_1^*F_1) & & \sigma_2(F_3^*F_3) & \cdots & \vdots \\ & \sigma_2(F_2^*F_2) & & & \\ & & \sigma_3(F_3^*F_3) & \cdots & \sigma_{n-1}(F_n^*F_n) \\ & & & & \sigma_n(F_n^*F_n) \end{array}\right) \in \mathbb{R}^{n(n+1)/2}$$

and refer to it as the *inner eigenstep pattern* of F.

In the same fashion we compared the spectra of FF^* and F^*F in (4.4), we can do this for the truncated frames and see that outer and inner eigensteps encode the exact same information: for $i \leq d$, $\sigma(F_iF_i^*)$ is just a zero-padded version of $\sigma(F_i^*F_i)$ while for $i \geq d$, $\sigma(F_i^*F_i)$ is a zero-padded version of $\sigma(F_iF_i^*)$. In Figure 4.1 we illustrate how the outer eigenstep tableau $\Lambda^{\text{out}}(F)$ and the inner eigenstep pattern $\Lambda^{\text{in}}(F)$ fit together.

Note that the (outer and inner) eigensteps of a frame $F = (f_1, \ldots, f_n)$ also encode the norms of the frame vectors as we have

$$\sum_{i=1}^{d} \sigma_i(F_kF_k^*) = \operatorname{Tr}(F_kF_k^*) = \operatorname{Tr}(F_k^*F_k) = \sum_{j=1}^{k} \|f_j\|^2 \tag{4.7}$$

for outer eigensteps, and equivalently

$$\sum_{i=1}^{k} \sigma_i(F_k^*F_k) = \operatorname{Tr}(F_k^*F_k) = \sum_{j=1}^{k} \|f_j\|^2 \tag{4.8}$$

for inner eigensteps.

Now given a spectrum $\lambda = (\lambda_1, \ldots, \lambda_d)$ and norm squares $\mu = (\mu_1, \ldots, \mu_n)$, the approach in [FMPS13; CFM+13a] is to first construct all possible sequences of outer eigensteps $\Lambda \in \mathbb{R}^{d\times(n+1)}$ of frames with the desired norms and eigenvalues and then for each tableau Λ construct all frames F with $\Lambda^{\text{out}}(F) = \Lambda$. Denote by $\mathcal{F}_{\mu,\lambda}$ the set of all frames for $\mathbb{C}^d$ solving the frame construction problem for μ and λ, that is, all frames having norm squares μ and spectrum λ. As we will see, the sets $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda}) \subseteq \mathbb{R}^{d\times(n+1)}$ and $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda}) \subseteq \mathbb{R}^{n(n+1)/2}$ are both polytopes. Part of the description of these polytopes are the conditions on the

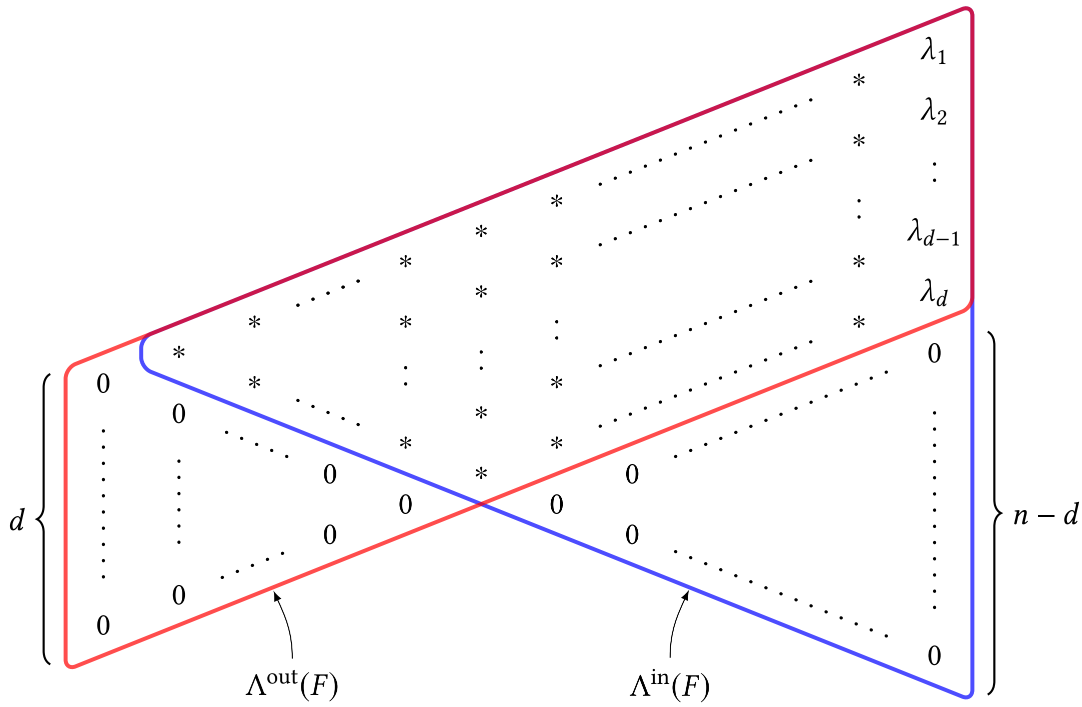


Figure 4.1.: The different zero-paddings of the spectral information of a frame F of length n for $\mathbb{C}^d$ as encoded in the outer eigenstep tableau $\Lambda^{\mathrm{out}}(F)$ and the inner eigenstep pattern $\Lambda^{\mathrm{in}}(F)$.

column sums imposed by the *trace condition* (4.7) or (4.8), respectively. In addition to these, there are inequalities that describe how the spectra of frame operators and Gram matrices can change when a frame vector is added. To obtain these, we have to apply the *min-max theorem* or *Courant–Fischer theorem* for eigenvalues of Hermitian matrices.

Theorem 4.3.2 (Min-Max Theorem, [HJ85, Thm. 4.2.11]). *Let M be a Hermitian $m \times m$ matrix. The eigenvalues of M satisfy the equations*

$$\sigma_i(M) = \max_{\substack{U \subseteq \mathbb{C}^m \\ \dim(U)=i}} \min_{\substack{u \in U \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle} = \min_{\substack{U \subseteq \mathbb{C}^m \\ \dim(U)=m-i+1}} \max_{\substack{u \in U \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle},$$

where U ranges over all subspaces of $\mathbb{C}^m$ with the given dimension.

Furthermore, if $(u_1, \ldots, u_m)$ is an orthonormal eigenbasis of M with $Mu_j = \sigma_j(M)u_j$ for all j, the maximum and minimum are attained for the subspaces $\operatorname{span}(u_1, \ldots, u_i)$ *and* $\operatorname{span}(u_i, \ldots, u_m)$*, respectively.* □

This description of the eigenvalues of Hermitian matrices implies the following lemma that yields interlacing conditions on sequences of inner eigensteps.

Lemma 4.3.3 (Cauchy's Interlace Theorem, [HJ85, Thm. 4.3.8]). *Let M be a Hermitian $m \times m$ matrix and M' be the $(m-1) \times (m-1)$ submatrix obtained from M by removing the last row and column. The spectra of M and M' satisfy the* interlacing condition

$$\sigma_1(M) \geq \sigma_1(M') \geq \sigma_2(M) \geq \sigma_2(M') \geq \cdots \geq \sigma_{m-1}(M) \geq \sigma_{m-1}(M') \geq \sigma_m(M).$$

Proof. We fix $i \in \{1, \ldots, m-1\}$ and show that

$$\sigma_i(M) \geq \sigma_i(M') \geq \sigma_{i+1}(M).$$

Let $u_1, \ldots, u_{m-1} \in \mathbb{C}^{m-1}$ be an orthonormal eigenbasis of M' such that $M'u_j = \sigma_j(M')u_j$ for all j and consider the subspace $S = \operatorname{span}(u_1, \ldots, u_i) \subseteq \mathbb{C}^{m-1}$. Denote by $\iota\colon \mathbb{C}^{m-1} \to \mathbb{C}^m$ the embedding into the first $m-1$ coordinates with image $\mathbb{C}^{m-1} \times 0$. Applying the min-max theorem 4.3.2, we obtain

$$\sigma_i(M') = \min_{\substack{u \in S \\ u \neq 0}} \frac{\langle M'u, u\rangle}{\langle u, u\rangle} = \min_{\substack{u \in \iota(S) \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle} \leq \max_{\substack{U \subseteq \mathbb{C}^m \\ \dim(U)=i}} \min_{\substack{u \in U \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle} = \sigma_i(M).$$

Now consider the subspace $S' = \operatorname{span}(u_i, \ldots, u_{m-1})$. Applying the min-max theorem again, we obtain

$$\sigma_i(M') = \max_{\substack{u \in S' \\ u \neq 0}} \frac{\langle M'u, u\rangle}{\langle u, u\rangle} = \max_{\substack{u \in \iota(S') \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle} \geq \min_{\substack{U \subseteq \mathbb{C}^m \\ \dim(U)=m-i}} \max_{\substack{u \in U \\ u \neq 0}} \frac{\langle Mu, u\rangle}{\langle u, u\rangle} = \sigma_{i+1}(M). \qquad \square$$

Corollary 4.3.4. *Let $F = (f_1, \ldots, f_n)$ be a frame for $\mathbb{C}^d$. The sequence of inner eigensteps $\Lambda^{in}(F) \in \mathbb{R}^{n(n+1)/2}$ satisfies the* interlacing condition

$$\sigma_1(F_{k+1}^* F_{k+1}) \geq \sigma_1(F_k^* F_k) \geq \cdots \geq \sigma_{m-1}(F_{k+1}^* F_{k+1}) \geq \sigma_{m-1}(F_k^* F_k) \geq \sigma_m(F_{k+1}^* F_{k+1}).$$

for $k = 1, \ldots, n-1$.

Proof. This is immediate from Lemma 4.3.3, since $F_k^* F_k$ is obtained from $F_{k+1}^* F_{k+1}$ by removing the last row and column. □

Hence, we conclude that $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$ is contained in the polytope described by the trace conditions in (4.8) given by μ, the final spectrum being equal to $(\lambda_1, \dots, \lambda_d, 0, \dots, 0)$, as well as the interlacing conditions in Corollary 4.3.4.

Interestingly, these are not only necessary but also sufficient conditions and hence fully characterize the sequences of eigensteps in $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$. In fact, there is a converse of Lemma 4.3.3 that follows from a lemma of Mirsky.

Lemma 4.3.5 ([Mir58, Lem. 2])**.** *Let $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_{m-1}$ be real numbers such that*

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{m-1} \geq \beta_{m-1} \geq \alpha_m,$$

then there exists a real (symmetric) matrix of the form

$$\begin{pmatrix} \beta_1 & & & p_1 \\ & \ddots & & \vdots \\ & & \beta_{m-1} & p_{m-1} \\ p_1 & \cdots & p_{m-1} & p_m \end{pmatrix}$$

with eigenvalues $\alpha_1, \dots, \alpha_m$. □

Corollary 4.3.6. *Let M' be a Hermitian $(m-1)\times(m-1)$ matrix. Given a weakly decreasing tuple $(\rho_1, \dots, \rho_m)$ of real numbers, there exists a vector $b \in \mathbb{C}^{m-1}$ and a real number c such that the $m \times m$ matrix*

$$M = \begin{pmatrix} M' & b \\ b^* & c \end{pmatrix}$$

has spectrum $(\rho_1, \dots, \rho_m)$ if and only if the interlacing condition

$$\rho_1 \geq \sigma_1(M') \geq \rho_2 \geq \sigma_2(M') \geq \cdots \geq \rho_{m-1} \geq \sigma_{m-1}(M') \geq \rho_m$$

is satisfied.

Proof. The *only if* part of the statement is exactly Cauchy's interlace theorem as stated in Lemma 4.3.3. Now given a Hermitian $(m-1) \times (m-1)$ matrix M' and a weakly decreasing tuple of real numbers $(\rho_1, \dots, \rho_m)$ satisfying the interlacing condition, we use an orthonormal eigenbasis to obtain

$$M' = U \operatorname{diag}(\sigma_1(M'), \dots, \sigma_{m-1}(M'))\, U^*$$

and apply Mirsky's Lemma 4.3.5 to obtain a matrix

$$P = \begin{pmatrix} \sigma_1(M') & & & p_1 \\ & \ddots & & \vdots \\ & & \sigma_{m-1}(M') & p_{m-1} \\ p_1 & \cdots & p_{m-1} & p_m \end{pmatrix}$$

with eigenvalues $\rho_1, \ldots, \rho_m$. Now let

$$M = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M' & b \\ b^* & c \end{pmatrix},$$

where

$$b = U \begin{pmatrix} p_1 \\ \vdots \\ p_{m-1} \end{pmatrix} \quad \text{and} \quad c = p_m. \qquad \square$$

Finally, we arrive at a complete characterization of the eigenstep patterns in $\Lambda^{\mathrm{in}}(\mathcal{F}_{\mu,\lambda})$.

Corollary 4.3.7 (cf. [FMPS13, Def. 2])**.** *Let $n \geq d$ be non-negative integers, $\mu = (\mu_1, \ldots, \mu_n)$ a tuple of non-negative real numbers and $\lambda = (\lambda_1, \ldots, \lambda_d)$ a weakly decreasing tuple of positive real numbers. A triangular pattern*

$$\Lambda = \left(\begin{array}{ccccc} & & & & \lambda_{1,n} \\ & & \lambda_{1,3} & \cdots & \lambda_{2,n} \\ & \lambda_{1,2} & & & \\ \lambda_{1,1} & & \lambda_{2,3} & \cdots & \vdots \\ & \lambda_{2,2} & & & \\ & & \lambda_{3,3} & \cdots & \lambda_{n-1,n} \\ & & & & \lambda_{n,n} \end{array} \right) \in \mathbb{R}^{n(n+1)/2}$$

appears as the sequence of inner eigensteps of a frame $F \in \mathcal{F}_{\mu,\lambda}$ if and only if the following conditions are satisfied:

i) the column sums are given by

$$\sum_{j=1}^{k} \lambda_{j,k} = \sum_{j=1}^{k} \mu_j, \quad \textit{for } k = 1, \ldots, n,$$

ii) the entries in the last column are

$$(\lambda_{1,n}, \ldots, \lambda_{n,n}) = (\lambda_1, \ldots, \lambda_d, 0, \ldots, 0), \quad \textit{and}$$

iii) the interlacing condition

$$\lambda_{1,k+1} \geq \lambda_{1,k} \geq \lambda_{2,k+1} \geq \lambda_{2,k} \geq \cdots \geq \lambda_{k,k+1} \geq \lambda_{k,k} \geq \lambda_{k+1,k+1}$$

is satisfied for $k = 1, \ldots, n-1$.

Proof. Given a frame $F \in \mathcal{F}_{\mu,\lambda}$ the pattern $\Lambda = \Lambda^{\mathrm{in}}(F)$ satisfies the column sum condition by (4.8). The last column is the spectrum of F^*F and is hence given by $(\lambda_1, \ldots, \lambda_d, 0, \ldots, 0)$ as obtained from the decomposition in (4.4). The interlacing conditions are satisfied by Corollary 4.3.4.

Conversely, let $\Lambda \in \mathbb{R}^{n(n+1)/2}$ be a triangular pattern satisfying the above conditions. We first construct a positive-definite Hermitian $n \times n$ matrix M such that for $k = 1, \dots, n$ the matrix M_k consisting of the first k rows and columns of M has spectrum $(\lambda_{1,k}, \dots, \lambda_{k,k})$. Start by letting $M_1 = (\lambda_{1,1})$ and note that $\lambda_{1,1}$ is non-negative since the interlacing conditions yield $\lambda_{1,1} \geq \lambda_{2,2} \geq \dots \geq \lambda_{n,n}$ and $\lambda_{n,n}$ is non-negative as an entry of the last column. We can now iteratively apply Corollary 4.3.6 to obtain the matrices $M_2, \dots, M_n = M$ and find a frame $F = (f_1, \dots, f_n)$ for $\mathbb{C}^d$ such that $M = F^*F$ as previously done in (4.5)–(4.6). The spectra of the truncated frames F_k are as desired since $F_k^* F_k = M_k$. Hence, $\Lambda^{\text{in}}(F) = \Lambda$. Furthermore, the frame vectors of F have norm squares $\mu_1, \dots, \mu_n$ since the norms are encoded in the eigenstep pattern as in (4.8) and fixed by the column sum condition. We conclude that $F \in \mathcal{F}_{\mu,\lambda}$, which finishes the proof. □

We conclude that $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$ is a polytope in $\mathbb{R}^{n(n+1)/2}$. Using the translation between outer and inner eigensteps as depicted in Figure 4.1 we may translate the characterization of inner eigensteps in Corollary 4.3.7 to outer eigensteps.

Corollary 4.3.8 (cf. [FMPS13, Def. 1], [CFM+13a, Def. 1])**.** *Let $n \geq d$ be non-negative integers, $\mu = (\mu_1, \dots, \mu_n)$ a tuple of non-negative real numbers and $\lambda = (\lambda_1, \dots, \lambda_d)$ a weakly decreasing tuple of positive real numbers. A tableau*

$$\Lambda = \begin{pmatrix} \lambda_{1,0} & \cdots & \lambda_{1,n} \\ \vdots & & \vdots \\ \lambda_{d,0} & \cdots & \lambda_{d,n} \end{pmatrix} \in \mathbb{R}^{d \times (n+1)}$$

appears as the sequence of outer eigensteps of a frame $F \in \mathcal{F}_{\mu,\lambda}$ if and only if the following conditions are satisfied:

i) the column sums are given by

$$\sum_{i=1}^{d} \lambda_{i,k} = \sum_{j=1}^{k} \mu_j, \quad \text{for } k = 1, \dots, n,$$

ii) the entries in the last column are

$$(\lambda_{1,n}, \dots, \lambda_{d,n}) = (\lambda_1, \dots, \lambda_d),$$

iii) the entries in the first column are zero

$$(\lambda_{1,0}, \dots, \lambda_{d,0}) = (0, \dots, 0), \quad \text{and}$$

iv) the interlacing condition

$$\lambda_{1,k+1} \geq \lambda_{1,k} \geq \lambda_{2,k+1} \geq \lambda_{2,k} \geq \dots \geq \lambda_{d,k+1} \geq \lambda_{d,k}$$

is satisfied for $k = 1, \dots, n-1$.

We will refer to both $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$ and $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ as the *polytope of eigensteps* as they are affinely isomorphic by the correspondence depicted in Figure 4.1.

Note that by the Schur–Horn theorem for finite frames (Theorem 4.2.1), the polytopes $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$ and $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ described in Corollaries 4.3.7 and 4.3.8 are non-empty if and only if μ lies in the convex hull of all permutations of $\tilde{\lambda} = (\lambda_1, \dots, \lambda_d, 0, \dots, 0)$, or equivalently if and only μ is majorized by $\tilde{\lambda}$.

Using sequences of eigensteps, the frame construction problem can now be split into two steps, as done in [FMPS13; CFM+13a]:

Step A. Parametrize the polytope of eigensteps $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ (or $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$).

Step B. Construct all frames with a given sequence of outer or inner eigensteps.

An algorithm called *Top Kill* solving step A, i.e., obtaining all points in $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$, is described in [FMPS13] and an algorithm solving step B is given in [CFM+13a]. We refer to the book chapter [FMP13] for a detailed account on both results.

4.3.1. Polytopes of Eigensteps and Gelfand–Tsetlin Polytopes

Comparing the description of the polytope $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda})$ with that of the weighted Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)_\mu$ in Section 3.2, we see that

$$\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda}) = \mathrm{GT}(\tilde{\lambda})_\mu, \quad \text{where } \tilde{\lambda} = (\lambda_1, \dots, \lambda_d, 0, \dots, 0).$$

Furthermore, denoting by $\mathcal{F}_\lambda$ the set of all frames for $\mathbb{C}^d$ of length n with spectrum λ, we obtain a description of the polytopes $\Lambda^{\text{in}}(\mathcal{F}_\lambda)$ and $\Lambda^{\text{out}}(\mathcal{F}_\lambda)$ by omitting the column sum conditions in Corollaries 4.3.7 and 4.3.8, respectively. The description obtained this way is identical to that of an unweighted Gelfand–Tsetlin polytope. To be precise,

$$\Lambda^{\text{in}}(\mathcal{F}_\lambda) = \mathrm{GT}(\tilde{\lambda}), \quad \text{where } \tilde{\lambda} = (\lambda_1, \dots, \lambda_d, 0, \dots, 0).$$

4.3.2. The Case of Equal Norm Tight Frames

When considering only tight frames where all frame vectors have the same norm—so μ and λ are constant—we can simplify the description of $\Lambda^{\text{in}}(\mathcal{F}_{\mu,\lambda}) = \mathrm{GT}(\tilde{\lambda})_\mu$ given in Corollary 4.3.7 to a non-redundant one. From this we can read off the dimension and number of facets. The details of this can be found in the appendix in Chapter 10. The results are joint work with Tim Haga and also appeared in [HP16].

Part II.

Towards a General Framework

Introduction

This part is devoted to the search for general framework capturing the phenomena observed in Part I. There we have seen various examples of a scenario that might be roughly described as the following:

In some mathematical context two polyhedra $P, Q \subseteq \mathbb{R}^n$ appear together with a piecewise-linear bijection $P \to Q$. The polyhedron P is given by linear equations and inequalities of some of the following types:

- equations $x_i = c$ fixing a coordinate to a constant $c \in \mathbb{R}$,
- inequalities $x_i \leq x_j$ comparing coordinates,
- inequalities $\alpha x_i \leq \beta x_j$ comparing coordinates weighted by positive constants $\alpha, \beta > 0$.

The polyhedron Q is given by linear equations and inequalities of the following type:

- equations $x_i = c$ fixing a coordinate to a constant $c \in \mathbb{R}$,
- inequalities $x_i \geq 0$ for all coordinates that are not fixed,
- inequalities $\alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \cdots + \alpha_k x_{i_k} \leq c$ bounding a positive combination of non-fixed coordinates by a constant $c \in \mathbb{R}$.

The first step in this direction was done by Ardila, Bliem and Salazar in [ABS11], where they extended the definition of poset polytopes and their transfer maps to posets with so called *markings*, that give fixed coordinates for the associated polytopes. This generalization allows to consider the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ and the Feigin–Fourier–Littelmann–Vinberg polytope $\mathrm{FFLV}(\lambda)$ discussed in Chapter 3 as the marked order polytope $\mathcal{O}(P, \lambda)$ and marked chain polytope $\mathcal{C}(P, \lambda)$ of some marked poset (P, λ), respectively. Since a lot of the results for ordinary poset polytopes do not immediately generalize to marked poset polytopes, this opened questions that had been answered for ordinary poset polytopes to be asked for marked poset polytopes as well. In particular, the elegant combinatorial description of the face structure of order polytopes $\mathcal{O}(P)$ given by Geissinger and Stanley that we have seen in Section 1.1 had to be adapted to marked order polytopes $\mathcal{O}(P, \lambda)$, since in the marked case even the correspondence between covering relations and facets breaks down in general. This lead to a notion of *regular* marked posets, that still produce marked order polytopes with facets in correspondence to covering relations in Fourier's work [Fou16]. However, this definition of regularity missed some redundant covering relations and we give a corrected definition in Section 6.2. Indeed, we provide a combinatorial description of the face structure of marked order polyhedra—a potentially unbounded generalization of marked order polytopes. This has previously been attempted by Jochemko and Sanyal in [JS14], but the same counterexample that shows Fourier's regularity condition was not strong enough also serves as a counterexample for the characterization of face partitions in [JS14, Prop. 2.3]. We discuss both problems in Remark 6.2.20.

Another result that is true for poset polytopes but not for marked poset polytopes is the transfer map preserving vertices. In the unmarked setting the correspondence of filters and anti-chains in a poset immediately yields a vertex description of the chain polytope. In the marked case, a vertex description of $C(P, \lambda)$ has yet to be found. Even in the unmarked case, the face structure of chain polytopes still lacks a combinatorial description. There are necessary and sufficient criteria for order and chain polytopes to be combinatorially (and unimodularly) equivalent—given by Hibi and Li for the unmarked case in [HL16] and by Fourier in [Fou16] for the marked case with minor imprecisions that were corrected in [FF16]—but when these do not hold, close to nothing is known about the face structure of both unmarked and marked chain polytopes. We take this as motivation to introduce a time parameter $t \in [0, 1]$ in the transfer map for marked poset polytopes to study how the marked order polytope continuously deforms into the marked chain polytope. It turns out that the image is a polytope for all t and the combinatorial type stays constant for intermediate $t \in (0, 1)$, so there is a *generic* marked poset polytope in between that degenerates to the marked order and chain polytope for $t = 0$ and $t = 1$, respectively. Fang and Fourier suggested to choose a different parameter t_p in each coordinate and this in fact still yields polytopes whose combinatorial types are now constant along the relative interiors of faces of the parametrizing hypercube $[0, 1]^\ell$. We call this the *continuous family of marked poset polytopes* and study it in Chapter 7. Their suggestion was inspired by a notion of *marked chain-order polytopes* they introduced in [FF16] motivated from representation theory, where the defining inequalities look like those for order polytopes for some fixed subset of poset elements and like those for chain polytopes for the other elements. Picking all $t_p \in \{0, 1\}$ in the parametrized transfer map, we recover Fang and Fourier's marked chain-order polytopes, which did not come with a transfer map a priori. This construction allows us to partly answer the question about vertices of marked chain polytopes and marked chain-order polytopes. To be precise, we give a combinatorial description of the vertices for the generic polytope obtained for $t_p \in (0, 1)$. For $t_p \in [0, 1]$ this still gives a set of points whose convex hull is the polytope, but the description may become redundant.

Considering the other examples from Part I, we recognize the Stanley–Pitman polytope $\Pi_n(\xi)$ discussed in Section 2.1 as a marked chain polytope and the map φ to be the transfer map from the corresponding marked order polytope in this setting. The weighted Gelfand–Tsetlin polytopes $\mathrm{GT}(\lambda)_\mu$ from Section 3.2 as well as the eigenstep polytopes $\Lambda^{\mathrm{in}}(\mathcal{F}_{\mu,\lambda})$ and $\Lambda^{\mathrm{out}}(\mathcal{F}_{\mu,\lambda})$ from Section 4.3 have additional sum conditions and hence are not marked order polytopes. We briefly discuss a generalization we call *conditional marked order polyhedra* in Section 6.5 that allows these additional linear constraints. However, this definition turns out to be too general to allow a lot of results. Indeed, every polyhedron is affinely isomorphic to a conditional marked order polyhedron.

The descriptions of the Cayley polytope from Section 2.2 and the lecture hall cones and polytopes from Section 2.3 have coefficients in the inequalities and hence do not fit into the theory of marked poset polyhedra. In this direction, we recently started a project with Raman Sanyal, vastly extending the class of polyhedra that allow a piecewise-linear deformation into a polyhedron with a description similar to marked chain polytopes. This is motivated by the work of Felsner and Knauer on *distributive polyhedra* in [FK11]. They describe distributive polyhedra by underlying directed graphs with edge weights, generalizing marked order polyhedra described by Hasse diagrams of posets in this context. We use this characterization in Chapter 8 to construct piecewise-linear bijective images of a large class of distributive polyhedra. These images have descriptions similar to marked chain polytopes. The polytopes and the transfer map appearing in the geometrical proof of Cayley's theorem in [KP14] as well as the objects in Section 2.3 on lecture hall cones and polytopes can be treated in this more general setting.

5. Marked Poset Polytopes

We start by reviewing the definitions and results obtained for marked poset polytopes as defined by Ardila, Bliem and Salazar in [ABS11].

5.1. Marked Order and Chain Polytopes and their Transfer Maps

Motivated by the similarity of Gelfand–Tsetlin and Feigin–Fourier–Littelmann–Vinberg polytopes to order and chain polytopes, respectively, the authors of [ABS11] give the following definition.

Definition 5.1.1 (cf. [ABS11, Def. 1.2][1]). Let P be a finite poset and $\lambda\colon P^* \to \mathbb{R}$ be a real valued order-preserving map on an induced subposet $P^* \subseteq P$. We say (P, λ) is a *marked poset* with *marking* λ, *marked elements* P^* and denote by $\tilde{P} = P \setminus P^*$ the set of all *unmarked elements*.

When P^* contains all minimal and maximal elements, we associate two *marked poset polytopes* to (P, λ). The *marked order polytope* $\mathcal{O}(P, \lambda) \subseteq \mathbb{R}^P$ is the set of all $x \in \mathbb{R}^P$ satisfying the equations $x_a = \lambda(a)$ for all marked elements $a \in P^*$ as well as the inequalities $x_p \leq x_q$ for all $p, q \in P$ with $p \leq q$. The *marked chain polytope* $\mathcal{C}(P, \lambda)$ is the set of all $y \in \mathbb{R}^P$ satisfying the equations $y_a = \lambda(a)$ for all $a \in P^*$, the inequalities $y_p \geq 0$ for all $p \in \tilde{P}$, as well as the inequalities

$$y_{p_1} + y_{p_1} + \cdots + y_{p_k} \leq \lambda(b) - \lambda(a)$$

for each chain $a < p_1 < p_2 < \cdots < p_k < b$ with $a, b \in P^*$ and all $p_i \in \tilde{P}$.

Note that both $\mathcal{O}(P, \lambda)$ and $\mathcal{C}(P, \lambda)$ are bounded since P^* is assumed to contain all extremal elements.

When P is a poset with $\hat{0}$ and $\hat{1}$, we recover the poset polytopes as discussed in Chapter 1 by choosing the marking $\lambda\colon \{\hat{0}, \hat{1}\} \to \mathbb{R}$ given by $\lambda(\hat{0}) = 0$ and $\lambda(\hat{1}) = 1$. When $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a highest weight for $\mathrm{GL}_n(\mathbb{C})$, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we obtain the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ and the Feigin–Fourier–Littelmann–Vinberg polytope $\mathrm{FFLV}(\lambda)$ as the marked order and marked chain polytope for the *Gelfand–Tsetlin poset* depicted in Figure 5.1. We usually describe a marked poset (P, λ) by its *marked Hasse diagram*, which is the Hasse diagram of P with the marked elements $a \in P^*$ drawn as boxes instead of dots and their marking $\lambda(a)$ written next to it in red.

[1] Our definitions slightly differ from those in [ABS11], where $\mathcal{O}(P, \lambda)$ and $\mathcal{C}(P, \lambda)$ are defined as subsets of $\mathbb{R}^{\tilde{P}}$. We use a different convention to match the rest of this work and avoid case distinctions in proofs.

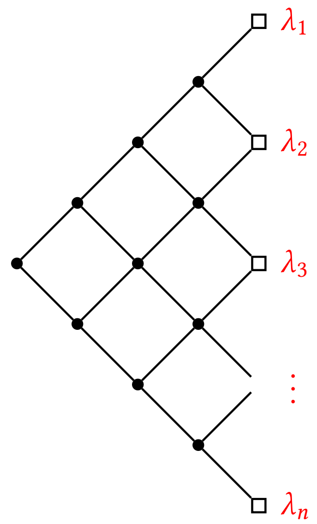


Figure 5.1.: The marked Hasse diagram of the Gelfand–Tsetlin poset for the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$.

As in the unmarked case, marked poset polytopes come with a transfer map $\mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$. To be precise, we have the following generalization of Theorem 1.2.1.

Theorem 5.1.2 (cf. [ABS11, Thm. 3.4])**.** *Let (P,λ) be a marked poset with all minimal and maximal elements marked. The two maps $\varphi, \psi \colon \mathbb{R}^P \to \mathbb{R}^P$ defined by*

$$\varphi(x)_p = \begin{cases} x_p & \text{if } p \in P^*, \\ x_p - \max\left\{ x_q : q \text{ is covered by } p \right\} & \text{otherwise,} \end{cases}$$

$$\psi(y)_p = \begin{cases} y_p & \text{if } p \in P^*, \\ \max\left\{ y_a + y_{q_1} + \cdots + y_{q_k} : a < q_1 < \cdots < q_k = p, a \in P^*, q_i \in \tilde{P} \right\} & \text{otherwise,} \end{cases}$$

are mutually inverse piecewise-linear maps. Furthermore, they restrict to piecewise-linear bijections $\varphi \colon \mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ and $\psi \colon \mathcal{C}(P,\lambda) \to \mathcal{O}(P,\lambda)$.

The statement is slightly stronger than the formulation in [ABS11], still we defer the proof to Chapter 7, where we give a proof for a larger class of transfer maps. Using this transfer map, the authors of [ABS11] show that $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$ are lattice polytopes with the same Ehrhart polynomial when λ maps to $\mathbb{Z}$.[2] In Chapter 7 we extend this to a larger family of Ehrhart equivalent polytopes associated to (P,λ), indexed by subsets of $\tilde{P}$.

Jochemko and Sanyal study marked poset polytopes from a combinatorial point of view in [JS14]. They describe the faces of marked order polytopes by partitions of P as

[2] We want to point out here that there is a slight mistake in the proof of [ABS11, Lem. 3.5], since the transfer map is not linear on the cells of the subdivision induced by the partial braid arrangement $x_p = x_q$ for all $p, q \in \tilde{P}$. However, if one takes the complete braid arrangement instead, i.e., $x_p = x_q$ for all $p, q \in P$, the argument is valid. In the terminology of [ABS11] this means to also consider hyperplanes $x_p = \lambda_a$ for $a \in A = P^*$ and $p \in P \setminus A = \tilde{P}$.

Stanley and Geissinger did for order polytopes. However, the conditions given in [JS14, Prop. 2.3] are not sufficient to guarantee that a partition does indeed define a face of the marked order polytope. We provide a corrected characterization of face partitions for marked order polytopes in Theorem 6.2.14. They also generalize Stanley's unimodular triangulation of $\mathcal{O}(P, \lambda)$ to a subdivision of $\mathcal{O}(P, \lambda)$ into products of simplices. As this subdivision will be relevant for our results as well, we quickly review the construction here.

5.2. A Subdivision into Products of Simplices

Recall from Chapter 1 that ordinary order and chain polytopes admit a unimodular triangulation with cells corresponding to chains of order ideals, the maximal ones corresponding to linear extensions of the given poset. For marked poset polytopes, there is still a subdivision with cells given by chains of order ideals that are compatible with the marking. To be precise, let $\mathcal{I} : \varnothing = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = P$ be a chain of order ideals in P. For each $p \in P$ denote by $i(\mathcal{I}, p)$ the smallest index k for which $p \in I_k$. The chain is said to be *compatible* with λ, if

$$i(\mathcal{I}, a) < i(\mathcal{I}, b) \quad \text{if and only if} \quad \lambda(a) < \lambda(b)$$

for all $a, b \in P^*$. Given a chain $\mathcal{I}$ of order ideals of P that is compatible with λ we obtain a partition of P by letting $B_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, r$. Let $F_{\mathcal{I}}$ be the polytope contained in $\mathcal{O}(P, \lambda)$ consisting of all points x constant on the blocks B_k with values weakly increasing along the linear order $B_1, \ldots, B_r$ of the blocks. This description implies that $F_{\mathcal{I}}$ is a product of simplices as shown in [JS14, Lemma 2.5]. We will see in Chapter 6 how $F_{\mathcal{I}}$ is naturally identified with a marked order polytope itself, with the inclusion into $\mathcal{O}(P, \lambda)$ induced by a surjective map of marked posets.

The collection of polytopes $F_{\mathcal{I}}$ with $\mathcal{I}$ ranging over all chains of order ideals compatible with λ forms a polyhedral subdivision of $\mathcal{O}(P, \lambda)$ into products of simplices. The chambers in this subdivision (cells of maximal dimension) correspond to chains, where in each step either exactly one unmarked element not in a constant interval, or a set containing all marked elements with some equal marking and the elements in the intervals between them is added. We call these *saturated compatible chains*.

Given any compatible chain of order ideals $\mathcal{I}$, the transfer map $\varphi \colon \mathcal{O}(P, \lambda) \to \mathcal{C}(P, \lambda)$ restricts to a linear map on the cell $F_{\mathcal{I}}$: the appearing maxima are determined by the fact that $x \in F_{\mathcal{I}}$ is constant on the blocks B_k and weakly increasing along their linear order. Hence, we can think of φ as linearly transforming each chamber in the subdivision to a polytope sitting in $\mathcal{C}(P, \lambda)$. In particular, we obtain a subdivision of $\mathcal{C}(P, \lambda)$ into products of simplices, by taking images of cells of the subdivision of $\mathcal{O}(P, \lambda)$.

6. Marked Order Polyhedra

We continue by studying marked order polyhedra, a potentially unbounded generalization of marked order polytopes associated to any marked poset (P, λ). We start by describing different ways to look at marked order polyhedra from an order theoretic, a convex geometric and a categorical point of view. We then study the face structure of marked order polyhedra in Section 6.2 and give a complete combinatorial characterization of partitions of the underlying poset corresponding to faces of the polyhedron. We specialize this characterization to facets and show that regular marked posets have facets in correspondence with the covering relations of the poset. In Section 6.4, we focus on convex geometrical properties of marked order polyhedra. We describe the recession cone of the polyhedra, how disjoint unions of posets correspond to products of polyhedra and give a Minkowski sum decomposition. Furthermore, we show that marked order polyhedra with integral markings are always lattice polyhedra. We close by adding linear conditions to marked order polyhedra in Section 6.5, generalizing a result on dimensions of faces obtained in [DM04] for weighted Gelfand–Tsetlin polytopes to these conditional marked order polyhedra.

The results of this chapter also appeared in [Peg17].

6.1. Half-Spaces, Extensions and a Functor

Recall from Definition 5.1.1 that a marked poset (P, λ) is a finite poset P together with a subset $P^* \subseteq P$ of marked elements and an order-preserving marking $\lambda \colon P^* \to \mathbb{R}$. As before, the set of unmarked elements $P \setminus P^*$ is denoted by $\tilde{P}$. We say the marking λ is *strict* if $\lambda(a) < \lambda(b)$ whenever $a < b$.

To study marked posets and the polyhedra we will associate to them, it is sometimes useful to take a more categorically minded point of view on marked posets. A map $f \colon (P, \lambda) \to (P', \lambda')$ between marked posets is an order-preserving map $f \colon P \to P'$ such that $f(P^*) \subseteq (P')^*$ and $\lambda'(f(a)) = \lambda(a)$ for all $a \in P^*$. With this definition of maps we see that marked posets form a category MPos. Letting Pos denote the category of posets and order-preserving maps, we can describe MPos as a category of certain diagrams in Pos. A marked poset (P, λ) is a diagram

$$P \hookleftarrow P^* \xrightarrow{\lambda} \mathbb{R}$$

in Pos, where $P^* \hookrightarrow P$ is the inclusion of an induced subposet P^* in a finite poset P. A map $f\colon (P,\lambda) \to (P',\lambda')$ is a commutative diagram

$$\begin{tikzcd} P & P^* \arrow[l, hook'] \arrow[r, "\lambda"] \arrow[d, "f|_{P^*}"] & \mathbb{R} \arrow[d, equal] \\ P' & P'^* \arrow[l, hook'] \arrow[r, "\lambda'"] & \mathbb{R}. \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[d, "f"] \end{tikzcd}$$

To each marked poset (P,λ) we associate a polyhedron $\mathcal{O}(P,\lambda)$ in $\mathbb{R}^P$.

Definition 6.1.1. Let (P,λ) be a marked poset. The *marked order polyhedron* $\mathcal{O}(P,\lambda)$ associated to (P,λ) is the set of all $x \in \mathbb{R}^P$ such that $x_p \le x_q$ for all $p,q \in P$ with $p \le q$ and $x_a = \lambda(a)$ for all $a \in P^*$.

Since the coordinates in P^* are fixed, we use the affinely isomorphic projection $\widetilde{\mathcal{O}}(P,\lambda)$ of $\mathcal{O}(P,\lambda)$ to $\mathbb{R}^{\tilde{P}}$ in examples.

When P^* contains all extremal elements of P, the polyhedron $\mathcal{O}(P,\lambda)$ is bounded. In this case $\mathcal{O}(P,\lambda)$ is the marked order polytope associated to (P,λ) as in Definition 5.1.1.

In more geometric terms, this definition is equivalent to

$$\mathcal{O}(P,\lambda) = \bigcap_{p<q} H^+_{p<q} \cap \bigcap_{a\in P^*} H_a,$$

where $H^+_{p<q}$ is the half-space in $\mathbb{R}^P$ defined by $x_p \le x_q$ and H_a is the hyperplane defined by $x_a = \lambda(a)$.

An interval $[a,b]$ in a marked poset (P,λ) is called *constant* if $a,b \in P^*$ and $\lambda(a) = \lambda(b)$. In this case $x_p = \lambda(a)$ for all $x \in \mathcal{O}(P,\lambda)$ and $p \in [a,b]$. With this terminology, a marking λ is strict if and only if (P,λ) contains no non-trivial constant intervals.

We can also think of the marked order polyhedron $\mathcal{O}(P,\lambda)$ as the set of all extensions of λ to order-preserving maps $x\colon P \to \mathbb{R}$ with $x|_{P^*} = \lambda$. That is, the set of all poset maps $x\colon P \to \mathbb{R}$ such that the diagram

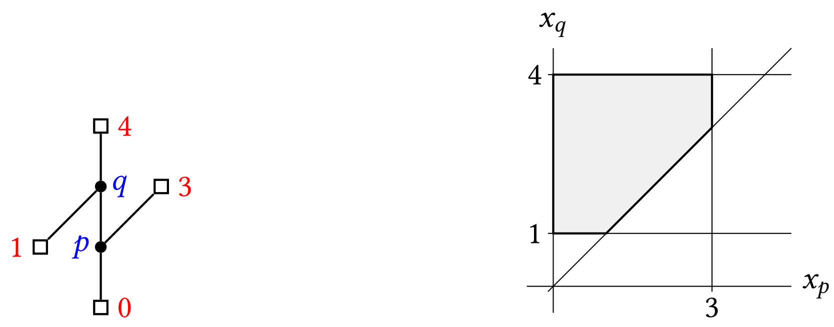
$$\begin{tikzcd} P \arrow[rr, bend right, "x"'] & P^* \arrow[l, hook'] \arrow[r, "\lambda"] & \mathbb{R} \end{tikzcd}$$

commutes. Putting together the diagram of a map $f\colon (P,\lambda) \to (P',\lambda')$ between marked posets and that of a point $x \in \mathcal{O}(P',\lambda')$, we see that we obtain a point $f^*(x)$ in $\mathcal{O}(P,\lambda)$ given by $f^*(x) = x \circ f$:

$$\begin{tikzcd} P \arrow[d, "f"] & P^* \arrow[l, hook'] \arrow[r, "\lambda"] \arrow[d, "f|_{P^*}"] & \mathbb{R} \arrow[d, equal] \\ P' \arrow[rr, bend right, "x"'] & P'^* \arrow[l, hook'] \arrow[r, "\lambda'"] & \mathbb{R}. \end{tikzcd}$$

Hence, letting Polyh denote the category of polyhedra and affine maps, we have a contravariant functor $\mathcal{O}\colon \mathsf{MPos} \to \mathsf{Polyh}$ sending a marked poset (P,λ) to the marked order polyhedron $\mathcal{O}(P,\lambda)$ and a map f between marked posets to the induced map f^* described above.

As we will see in the next proposition, any marking λ can be extended to P and any strict marking can be extended to a strictly order-preserving map $P \to \mathbb{R}$.



(a) a marked poset (P, λ)

(b) the polytope $\widetilde{O}(P, \lambda)$

Figure 6.1.: The marked poset (P, λ) from Example 6.1.3 and the associated marked order polytope $\widetilde{O}(P, \lambda)$.

Proposition 6.1.2. *Let (P, λ) be a marked poset. The associated marked order polyhedron is non-empty and if λ is strict, there is a point $x \in O(P, \lambda)$ such that $x_p < x_q$ whenever $p < q$.*

Proof. The order on $\mathbb{R}$ is dense and unbounded. Hence, whenever $a < c$ in $\mathbb{R}$ there is a $b \in \mathbb{R}$ such that $a < b < c$ and for any $b \in \mathbb{R}$ there are $a, c \in \mathbb{R}$ such that $a < b < c$. Since P is finite this allows us to successively extend λ to an order-preserving map on P. In fact, we can find an order-preserving extension x of λ such that for $p < q$ we have $x_p = x_q$ if and only if there are $a, b \in P^*$ such that $a \leq p < q \leq b$ and $\lambda(a) = \lambda(b)$. In particular, when λ was strict we can always find a strictly order-preserving extension. □

Example 6.1.3. We consider the marked order polytope given by the marked poset (P, λ) in Figure 6.1a. The blue labels name elements in P, while the red labels correspond to values of the elements of P^* under the marking λ. The (projected) associated marked order polytope $\widetilde{O}(P, \lambda)$ is shown in Figure 6.1b. ◊

6.2. Face Structure and Facets

In this section, we study the face structure of $O(P, \lambda)$. As it turns out, the faces of marked order polyhedra correspond to certain partitions of the underlying poset P. Our goal is to characterize those partitions combinatorially. We associate to each point x in $O(P, \lambda)$ a partition π_x of P, that will suffice to describe the minimal face of $O(P, \lambda)$ containing x. The partitions that are obtained in this way from points of the polyhedron will then—ordered by refinement—capture the polyhedrons face structure.

Definition 6.2.1. Let $Q = O(P, \lambda)$ be a marked order polyhedron. To each $x \in Q$ we associate a partition π_x of P induced by the transitive closure of the relation

$$p \sim_x q \quad \text{if} \quad x_p = x_q \text{ and } p, q \text{ are comparable.}$$

We may think of π as being obtained by first partitioning P into blocks of constant values under x and then splitting those blocks into connected components with respect to the Hasse diagram of P.

Given any partition π of P, we call a block $B \in \pi$ *free* if $P^* \cap B = \varnothing$ and denote by $\tilde{\pi}$ the set of all free blocks of π. Note that any $x \in \mathcal{O}(P, \lambda)$ is constant on the blocks of π_x and the values on the non-free blocks of π_x are determined by λ.

Let $x \in Q$ be a point of a polyhedron. We denote the minimal face of Q containing x by F_x. Hence, F_x is the unique face having x in its relative interior. Equivalently, F_x is the intersection of all faces of Q containing x.

Proposition 6.2.2. *Let $x \in Q = \mathcal{O}(P, \lambda)$ be a point of a marked order polyhedron with associated partition $\pi = \pi_x$. We have*

$$F_x = \{\, y \in Q : y \textit{ is constant on the blocks of } \pi \,\}$$

and $\dim F_x = |\tilde{\pi}|$.

Proof. For $p < q$ in P let $H_{p<q} = \partial H^+_{p<q}$ be the hyperplane defined by $x_p = x_q$ in $\mathbb{R}^P$. The minimal face of a point $x \in Q$ is then given by

$$F_x = Q \cap \bigcap_{\substack{p<q,\\ x_p = x_q}} H_{p<q}.$$

A point $y \in Q$ satisfies $y_p = y_q$ for all $p < q$ with $x_p = x_q$ if and only if y is constant on the blocks of π_x. Thus, F_x is indeed given by all $y \in Q$ constant on the blocks of π_x.

To determine the dimension of F_x, we consider its affine hull $\operatorname{aff}(F_x)$. It is obtained by intersecting the affine hull of Q with all $H_{p<q}$ such that $x_p = x_q$. The affine hull of Q itself is the intersection of all H_a for $a \in P^*$ and all $H_{p<q}$ such that $y_p = y_q$ for all $y \in Q$. Putting these facts together, we have

$$\operatorname{aff}(F_x) = \bigcap_{a \in P^*} H_a \;\cap \bigcap_{\substack{p<q,\\ y_p = y_q \forall y \in Q}} H_{p<q} \;\cap \bigcap_{\substack{p<q,\\ x_p = x_q}} H_{p<q} = \bigcap_{a \in P^*} H_a \;\cap \bigcap_{\substack{p<q,\\ x_p = x_q}} H_{p<q}.$$

This is exactly the set of all y constant on the blocks of π_x and satisfying $y_a = \lambda(a)$ for all $a \in P^*$. Such y are uniquely determined by values on the free blocks of π_x and thus $\dim(F_x) = |\tilde{\pi}_x|$ as desired. □

Corollary 6.2.3. *If λ is a strict marking on P, the dimension of $\mathcal{O}(P, \lambda)$ is equal to the number of unmarked elements in P.*

Proof. Since all coordinates in P^* are fixed by λ, we always have $\dim \mathcal{O}(P, \lambda) \le \left|\tilde{P}\right|$. If λ is strict, there is a point $x \in \mathcal{O}(P, \lambda)$ such that $x_p < x_q$ whenever $p < q$ by Proposition 6.1.2. Hence, π_x is the partition of P into singletons and $\dim F_x = |\tilde{\pi}_x| = \left|\tilde{P}\right|$. We conclude that $F_x = \mathcal{O}(P, \lambda)$, so x is a relative interior point and the marked order polyhedron has the desired dimension. □

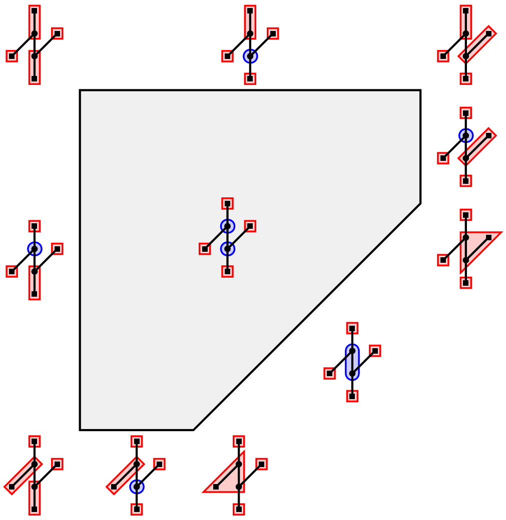


Figure 6.2.: The face partitions of the marked order polytope in Example 6.1.3.

Corollary 6.2.4. *Let $x \in Q = \mathcal{O}(P, \lambda)$ be a point of a marked order polyhedron. For $y \in Q$ we have $y \in F_x$ if and only if π_x is a refinement of π_y.*

Proof. By Proposition 6.2.2, $y \in F_x$ if and only if y is constant on the blocks of π_x. Let y be constant on the blocks of π_x. Any block B of π_x is connected with respect to the Hasse diagram of P and y takes constant values on B, hence B is contained in a block of π_y by construction and π_x is a refinement of π_y. Now let $y \in Q$ with π_x being a refinement of π_y. We conclude that y is constant on the blocks of π_x, since it is constant on the blocks of π_y and π_x is a refinement of π_y. □

Corollary 6.2.5. *Given any two points $x, y \in \mathcal{O}(P, \lambda)$, we have $F_y \subseteq F_x$ if and only if π_x is a refinement of π_y. In particular $F_y = F_x$ if and only if $\pi_y = \pi_x$.* □

Hence, the partition of $\mathcal{O}(P, \lambda)$ into relative interiors of its faces is the same as the partition given by $x \sim y$ if $\pi_x = \pi_y$ and we can associate to each non-empty face F a partition π_F with $\pi_F = \pi_x$ for any x in the relative interior of F. We call a partition π of P a *face partition* of (P, λ) if $\pi = \pi_F$ for some non-empty face of $\mathcal{O}(P, \lambda)$. We arrive at the following description of face lattices of marked order polyhedra.

Corollary 6.2.6. *Let $Q = \mathcal{O}(P, \lambda)$ be a marked order polyhedron. The poset $\mathcal{F}(Q) \setminus \{\varnothing\}$ of non-empty faces of Q is isomorphic to the induced subposet of the partition lattice on P given by all face partitions of (P, λ).* □

For the marked order polytope from Example 6.1.3, we illustrated the face partitions in Figure 6.2. The free blocks are highlighted in blue round shapes, non-free blocks in red angular shapes. We see that the dimensions of the faces are given by the numbers of free blocks in the associated face partitions and that face inclusions correspond to refinements of partitions.

In order to characterize the face partitions of a marked poset (P, λ) combinatorially, we introduce some properties of partitions of P.

Definition 6.2.7. Let (P, λ) be a marked poset. A partition π of P is *connected* if the blocks of π are connected as induced subposets of P. It is *P-compatible*, if the relation $\leq$ defined on π as the transitive closure of

$$B \leq C \quad \text{if} \quad p \leq q \text{ for some } p \in B, q \in C$$

is anti-symmetric. In this case $\leq$ is a partial order on π. A P-compatible partition π is called *(P, λ)-compatible*, if whenever $a \in B \cap P^*$ and $b \in C \cap P^*$ for some blocks $B \leq C$, we have $\lambda(a) \leq \lambda(b)$.

Remark 6.2.8. Whenever a partition π of a poset P is P-compatible, it is also *convex*. That is, for $a < b < c$ with a and c in the same block $B \in \pi$, we also have $b \in B$, since otherwise the blocks containing a and b would contradict the relation on the blocks being anti-symmetric. This implies that the blocks in a connected, P-compatible partition are not just connected as induced subposets of P but even connected as induced subgraphs of the Hasse diagram of P.

Proposition 6.2.9. *Let (P, λ) be a marked poset. A (P, λ)-compatible partition π of P gives rise to a marked poset $(P/\pi, \lambda/\pi)$ where P/π is the poset of blocks in π, $(P/\pi)^* = \pi \setminus \tilde{\pi}$ and $\lambda/\pi \colon (P/\pi)^* \to \mathbb{R}$ is defined by $(\lambda/\pi)(B) = \lambda(a)$ for any $a \in B \cap P^*$. Furthermore, the quotient map $P \to P/\pi$ defines a map $(P, \lambda) \to (P/\pi, \lambda/\pi)$ of marked posets.*

Proof. Since π is P-compatible, the blocks of π form a poset P/π as in Definition 6.2.7. Since π is (P, λ)-compatible, we have $\lambda(a) = \lambda(b)$ whenever $a, b \in B \cap P^*$ for some non-free block $B \in \pi$. Hence, the map λ/π is well-defined. It is order-preserving by the definition of (P, λ)-compatibility. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
P \arrow[d] & P^* \arrow[l, hook'] \arrow[r, "\lambda"] \arrow[d] & \mathbb{R} \arrow[d, equal] \\
P/\pi & (P/\pi)^* \arrow[l, hook'] \arrow[r, "\lambda/\pi"] & \mathbb{R}.
\end{tikzcd}$$

Thus, we have a quotient map $(P, \lambda) \to (P/\pi, \lambda/\pi)$. □

Proposition 6.2.10. *Every face partition π_F of (P, λ) is (P, λ)-compatible, connected and the induced marking on $(P/\pi_F, \lambda/\pi_F)$ is strict.*

Proof. Let F be a non-empty face of $\mathcal{O}(P, \lambda)$. It is obvious that π_F is connected by construction, since it is given by the transitive closure of a relation that only relates pairs of comparable elements. To verify that π_F is P-compatible, we need to check that the induced relation $\leq$ on the blocks of π_F is anti-symmetric. Assume we have blocks $B, C \in \pi_F$ such that $B \leq C$ and $C \leq B$. Since $B \leq C$, there is a finite sequence of blocks $B = X_1, X_2, \ldots, X_k, X_{k+1} = C$ such that for $i = 1, \ldots, k$ there are some $p_i \in X_i$, $q_i \in X_{i+1}$ with $p_i \leq q_i$. Take any x in the relative interior of F, then $x_{p_i} \leq x_{q_i}$ for $i = 1, \ldots, k$ and since x is constant on the blocks of π_F, we have $x_{q_i} = x_{p_{i+1}}$ for $i = 1, \ldots, k-1$. To summarize, we have

$$x_{p_1} \leq x_{q_1} = x_{p_2} \leq x_{q_2} = \cdots \leq \cdots = x_{p_k} \leq x_{q_k}. \tag{6.1}$$

Hence, the constant value x takes on B is less than or equal to the constant value x takes on C. Since we also have $C \le B$, we conclude that x takes equal values on the blocks B and C. From (6.1) we conclude that x takes equal values on all blocks X_i. From the definition of $\pi_x = \pi_F$ it follows that the blocks X_i are in fact all equal, in particular $B = C$ and the relation is anti-symmetric.

To see that π_F is (P,λ)-compatible, let $B, C \in \pi$ be non-free blocks with $B \le C$. By the same argument as above, we know that any $x \in F$ has constant value on B less than or equal to the constant value on C, so $\lambda(a) \le \lambda(b)$ for marked $a \in B$, $b \in C$. If $\lambda(a) = \lambda(b)$ we have $B = C$, by the same argument as above, so the induced marking is strict. □

Given any partition π of P, we can define a polyhedron F_π contained in $\mathcal{O}(P,\lambda)$ by

$$F_\pi = \{\, y \in Q : y \text{ is constant on the blocks of } \pi \,\}\,.$$

If $\pi = \pi_F$ is a face partition of (P,λ), we have $F_\pi = F$ by Proposition 6.2.2. However, F_π is not a face for all partitions π of P.

As long as π is (P,λ)-compatible, we can show that the polyhedron F_π is affinely isomorphic to the marked order polyhedron $\mathcal{O}(P/\pi, \lambda/\pi)$. The isomorphism will be induced by the quotient map $P \to P/\pi$. Our first step is to verify that this induced map is indeed an injection.

Lemma 6.2.11. *Let $f\colon (P,\lambda) \to (P',\lambda')$ be a map of marked posets. If f is surjective, the induced map $f^*\colon \mathcal{O}(P',\lambda') \to \mathcal{O}(P,\lambda)$ is injective.*

Proof. Let $x, y \in \mathcal{O}(P',\lambda')$ such that $f^*(x) = f^*(y)$. Given any $p \in P'$ we need to show $x_p = y_p$. Since f is surjective, $p = f(q)$ for some $q \in P$ and thus

$$x_p = x_{f(q)} = f^*(x)_q = f^*(y)_q = y_{f(q)} = y_p. \qquad \square$$

Proposition 6.2.12. *Let (P,λ) be a marked poset and π a (P,λ)-compatible partition. The quotient map $q\colon (P,\lambda) \to (P/\pi, \lambda/\pi)$ induces an injection*

$$q^*\colon \mathcal{O}(P/\pi, \lambda/\pi) \hookrightarrow \mathcal{O}(P,\lambda)$$

with image $q^*(\mathcal{O}(P/\pi, \lambda/\pi)) = F_\pi$.

Proof. By Lemma 6.2.11 we know that q^* is an injection. Hence, we only need to verify that F_π is the image of q^*. The image is contained in F_π, since whenever p and p' are in the same block $B \in \pi$, we have

$$q^*(x)_p = x_{q(p)} = x_B = x_{q(p')} = q^*(x)_{p'}.$$

Hence, all $q^*(x)$ are constant on the blocks of π. Conversely, given any point $y \in \mathcal{O}(P,\lambda)$ constant on the blocks of π, we obtain a well defined map $x\colon P/\pi \to \mathbb{R}$ sending each block to the constant value y_p for all p in the block. This map is a point $x \in \mathcal{O}(P/\pi, \lambda/\pi)$ mapped to y by q^*. □

The previous proposition tells us, that whenever we have a (P,λ)-compatible partition π, the marked order polyhedron $\mathcal{O}(P/\pi,\lambda/\pi)$ is affinely isomorphic to the polyhedron $F_\pi \subseteq \mathcal{O}(P,\lambda)$ via the embedding q^* induced by the quotient map. From now on, we refer to affine isomorphisms arising this way as the *canonical affine isomorphism* $\mathcal{O}(P/\pi,\lambda/\pi) \cong F_\pi$.

Corollary 6.2.13. *For every non-empty face F of a marked order polyhedron $\mathcal{O}(P,\lambda)$ we have a canonical affine isomorphism $\mathcal{O}(P/\pi_F,\lambda/\pi_F) \cong F$.* □

We are now ready to state and prove the characterization of face partitions of marked posets.

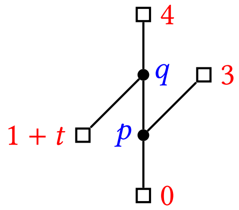
Theorem 6.2.14. *A partition π of a marked poset (P,λ) is a face partition if and only if it is (P,λ)-compatible, connected and the induced marking on $(P/\pi,\lambda/\pi)$ is strict.*

Proof. The fact that face partitions satisfy the above properties is the statement of Proposition 6.2.10. Now let π be a partition of P that is (P,λ)-compatible, connected and induces a strict marking λ/π. By Proposition 6.1.2, there is a point $z \in \mathcal{O}(P/\pi,\lambda/\pi)$ such that $z_B < z_C$ whenever $B < C$. Let $x \in \mathbb{R}^P$ be the point in the polyhedron $F_\pi \subseteq \mathcal{O}(P,\lambda)$ obtained as the image of z under the canonical affine isomorphism $\mathcal{O}(P/\pi,\lambda/\pi) \xrightarrow{\sim} F_\pi$. We claim that $\pi = \pi_x$, so π is a face partition. Since x is constant on the blocks of π and π is connected, we know that π is a refinement of π_x. Now assume that the equivalence relation $\sim_x$ defining π_x relates elements in different blocks of π. In this case, there are blocks $B \neq C$ of π with elements $p \in B$, $q \in C$ such that $x_p = x_q$ and $p < q$. This implies that $z_B = z_C$ and $B < C$, a contradiction to the choice of z. Hence, $\pi = \pi_x$ and π is a face partition of (P,λ). □

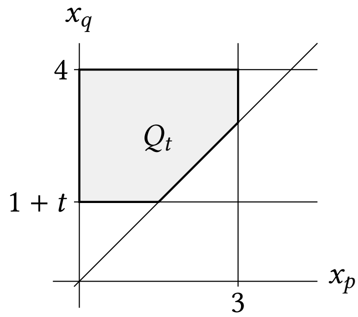
Remark 6.2.15. To decide whether a given partition π of a marked poset (P,λ) satisfies the conditions in Theorem 6.2.14, it is enough to know the linear order on $\lambda(P^*)$. The exact values of the marking are irrelevant. Hence, the face lattice of $\mathcal{O}(P,\lambda)$ is determined solely by discrete, combinatorial data. In fact, since the directions of facet normals do not depend on the values of λ, we can conclude that the normal fan $\mathcal{N}(\mathcal{O}(P,\lambda))$ is determined by this combinatorial data. However, the affine isomorphism type of $\mathcal{O}(P,\lambda)$ does depend on the exact values of λ.

Example 6.2.16. We construct a continuous family $(Q_t)_{t\in[0,1]}$ of marked order polytopes, whose underlying marked posets all yield the same combinatorial data in the sense of Remark 6.2.15, but Q_s and Q_t are affinely isomorphic if and only if $s = t$. Let (P,λ_t) be the marked poset shown in Figure 6.3a. Letting t vary in $[0,1]$, we obtain for each t a different affine isomorphism type, since two of the vertices of Q_t will move, while the other three stay fixed and are affinely independent as can be seen in Figure 6.3b. However, all Q_t share the same normal fan and are in particular combinatorially equivalent. ◊

We continue our study of the face structure of marked order polyhedra by having a closer look at facets. Since inequalities in the description of marked order polyhedra come from covering relations in the underlying poset, we expect a correspondence of



(a) the marked poset (P, λ_t)



(b) the polytope $Q_t = \widetilde{\mathcal{O}}(P, \lambda_t)$

Figure 6.3.: The marked poset (P, λ_t) from Example 6.2.16 and the associated marked order polytope $Q_t = \widetilde{\mathcal{O}}(P, \lambda_t)$.

facets to certain covering relations. If the marked poset satisfies a certain regularity condition, the facets are indeed in bijection with the covering relations. Hence, if we can change the underlying poset of a marked order polyhedron to a regular one, without changing the associated polyhedron, we obtain an enumeration of facets. We start by modifying an arbitrary marked poset to a strict one by contracting constant intervals.

Proposition 6.2.17. *Given any marked poset (P, λ), the partition π induced by the relations $a \sim p$ and $p \sim b$ whenever $[a, b]$ is a constant interval containing p yields a strictly marked poset $(P/\pi, \lambda/\pi)$ such that $\mathcal{O}(P/\pi, \lambda/\pi) \cong F_\pi = \mathcal{O}(P, \lambda)$ via the canonical affine isomorphism.*

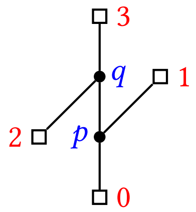
Proof. Let $x \in \mathcal{O}(P, \lambda)$ be a point constructed as in the proof of Proposition 6.1.2. By construction we have $x_p = x_q$ for $p < q$ if and only if there are $a, b \in P^*$ with $a \le p < q \le b$ with $\lambda(a) = \lambda(b)$. Thus, we conclude that $\pi_x = \pi$ and π is a face partition of $\mathcal{O}(P, \lambda)$. Since every point of $\mathcal{O}(P, \lambda)$ satisfies $x_a = x_p = x_b$ whenever $[a, b]$ is a constant interval containing p, we conclude that F_π is indeed the whole polyhedron. Hence, $\mathcal{O}(P/\pi, \lambda/\pi) \cong F_\pi = \mathcal{O}(P, \lambda)$, where λ/π is a strict marking by Proposition 6.2.10. □

Definition 6.2.18. Let (P, λ) be a marked poset. A covering relation $p \prec q$ is called *non-redundant* if for all marked elements a, b satisfying $a \le q$ and $p \le b$, we have $a = b$ or $\lambda(a) < \lambda(b)$. Otherwise the covering relation is called *redundant*. The marked poset (P, λ) is called *regular*, if all its covering relations are non-redundant.

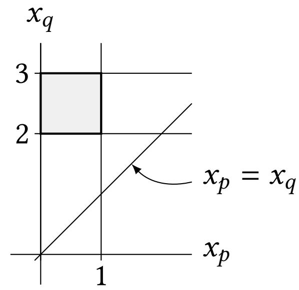
Apart from the desired correspondence of covering relations and facets, regularity of marked posets implies some useful properties of the marked poset itself.

Proposition 6.2.19. *Let (P, λ) be a regular marked poset. The following conditions are satisfied:*

i) the marking λ is strict,
ii) there are no covering relations between marked elements,
iii) every element in P covers and is covered by at most one marked element.



(a) a marked poset (P, λ)



(b) the polytope $\widetilde{O}(P, \lambda)$

Figure 6.4.: The marked poset (P, λ) from Remark 6.2.20 and the associated marked order polytope $\widetilde{O}(P, \lambda)$. The covering relation $p \prec q$ is redundant.

Proof. i) When $a < b$ are marked elements of P, there is some covering relation $p \prec q$ such that $a \leq p \prec q \leq b$. Since $a \leq q$ and $p \leq b$, we have $\lambda(a) < \lambda(b)$ by regularity.

ii) When $b \prec a$ is a covering relation between marked elements, we have $\lambda(a) < \lambda(b)$ by choosing $p = b$, $q = a$ in the regularity condition. This is a contradiction to λ being order-preserving.

iii) When $a, b \prec q$ for marked a, b, the regularity condition for $a \leq q$ and $b \leq b$ implies $a = b$ or $\lambda(a) < \lambda(b)$. By the same argument we get $a = b$ or $\lambda(b) < \lambda(a)$. We conclude that $a = b$. □

Remark 6.2.20. The conditions in Proposition 6.2.19 are necessary, but not sufficient for (P, λ) to be regular. The marked poset in Figure 6.4a satisfies all three conditions, but the covering relation $p \prec q$ is redundant.

In fact, this example shows that the process described by Fourier in [Fou16, Sec. 3] does not remove all redundant covering relations and hence leads to a notion of regularity that is not sufficient to have facets in correspondence with covering relations.

The same marked poset also serves as a counterexample to the characterization of face partitions in [JS14, Prop. 2.3]. Instead of partitions of P in terms of blocks, they use subposets of P that have all the elements of P but only some of the relations. The connected components of the Hasse diagram of such a subposet G give a connected partition π_G of P and conversely every connected partition π defines a subposet G_π on the elements of P by having $p \leq q$ in G_π if and only if $p \leq q$ in P and p and q are in the same block of π. When (P, λ) is the marked poset in Figure 6.4 and G is the subposet with $p \leq q$ as the only non-reflexive relation—i.e., $\{p, q\}$ is the only non-singleton block in π_G—the conditions in Proposition 2.3 of [JS14] are satisfied but G does not yield a face of $\mathcal{O}(P, \lambda)$ as can be seen in Figure 6.4b.

Theorem 6.2.21. *Let (P, λ) be a regular marked poset. The facets of $\mathcal{O}(P, \lambda)$ correspond to the covering relations in (P, λ).*

Proof. Since (P, λ) is strictly marked, the dimension of $\mathcal{O}(P, \lambda)$ is equal to the number of unmarked elements in P. Hence, a facet F corresponds to a (P, λ)-compatible, connected partition π of P such that λ/π is strict and π has exactly $|\tilde{P}| - 1$ free blocks. We claim that the number of non-free blocks of π is $|P^*|$. Assume there are marked elements $a \neq b$ in a common block B of π. Since π has $|\tilde{P}| - 1$ free blocks, at most one unmarked element can be in a non-free block. Since (P, λ) is regular, there are no covering relations between marked elements. Hence, since B is connected as an induced subgraph of the Hasse diagram of P and contains both a and b, it also contains the only unmarked element p in a non-free block, and we have one of the following four situations: $a \prec p \prec b$, $a \succ p \succ b$, $a \prec p \succ b$ or $a \succ p \prec b$. Since a and b are in the same block, they are identically marked and the first two possibilities contradict λ being strict. The other two possibilities contradict regularity, since p covers—or is covered by—more than one marked element. Hence, π has exactly $|P^*|$ non-free blocks and we conclude that π has $|P| - 1$ blocks overall. Therefore, π consists of $|P| - 2$ singletons and a single connected 2-element block corresponding to a covering relation of P.

Conversely, let $p \prec q$ be a covering relation of P. We claim that the partition π with the only non-singleton block $\{p, q\}$ is a face partition with $|\tilde{P}| - 1$ free blocks. Since (P, λ) is regular, it contains no covering relation between marked elements and π has exactly $|\tilde{P}| - 1$ free blocks. Since $\{p, q\}$ is the only non-singleton block and $p \prec q$, the partition π is connected and P-compatible. To verify that π is (P, λ)-compatible and λ/π is strict, let B, C be non-free blocks of π with $a \in B \cap P^*$ and $b \in C \cap P^*$ such that $B \leq C$. When $B = C$, we have $a = b$ and $\lambda(a) = \lambda(b)$. When $B < C$, we conclude $a < b$ or $a \leq q$, $p \leq b$, since $\{p, q\}$ is the only non-trivial block. In both cases, regularity implies $\lambda(a) < \lambda(b)$. □

Now that we established a regularity condition on marked posets that guarantees a bijection between covering relations in P and facets of the marked order polyhedron, we explain how to transform any given marked poset to a regular one.

Proposition 6.2.22. *Let (P, λ) be a strictly marked poset. Redundant covering relations in P can be removed successively to obtain a regular marked poset (P', λ) with the same associated marked order polyhedron $\mathcal{O}(P', \lambda) = \mathcal{O}(P, \lambda)$.*

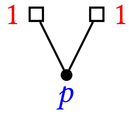
Proof. Let $p \prec q$ be a redundant covering relation in P. That is, there are marked elements $a \neq b$ satisfying $a \leq q$, $p \leq b$ and $\lambda(a) \geq \lambda(b)$. Let P' be obtained from P by removing the covering relation $p \prec q$ from P. Obviously $\mathcal{O}(P, \lambda)$ is contained in $\mathcal{O}(P', \lambda)$.

Now let $x \in \mathcal{O}(P', \lambda)$. To verify that x is a point of $\mathcal{O}(P, \lambda)$, we have to show $x_p \leq x_q$. Since λ is a strict marking on P, we can not have $a \leq p$. Otherwise $a \leq p \leq b$ implies $a < b$, in contradiction to $\lambda(a) \geq \lambda(b)$. Hence, removing the covering relation $p \prec q$ we still have $a \leq' q$ in P'. By the same argument $p \leq' b$. Thus, by the defining conditions of $\mathcal{O}(P', \lambda)$, we have

$$x_p \leq x_b = \lambda(b) \leq \lambda(a) = x_a \leq x_q.$$

Therefore, $x \in \mathcal{O}(P, \lambda)$ and we conclude $\mathcal{O}(P', \lambda) = \mathcal{O}(P, \lambda)$. This process can be repeated until all redundant covering relations have been removed, resulting in a regular marked poset defining the same marked order polyhedron. □

Remark 6.2.23. Note that Proposition 6.2.22 does not imply, that all covering relations that are redundant in (P, λ) can be removed simultaneously. Removing a single redundant covering relation can lead to other redundant covering relations becoming non-redundant. In the marked poset



both covering relations are redundant. However, removing any of the two covering relations renders the remaining covering relation non-redundant.

Given any marked poset (P, λ), we can apply the constructions of Proposition 6.2.17 and Proposition 6.2.22 to obtain a regular marked poset (P', λ') defining the same marked order polyhedron up to canonical affine isomorphism.

6.3. A Polyhedral Subdivision

In the previous section we have seen how surjective maps $f\colon (P, \lambda) \to (P', \lambda')$ give rise to inclusions $f^*\colon \mathcal{O}(P', \lambda') \hookrightarrow \mathcal{O}(P, \lambda)$. In particular, any (P, λ)-compatible partition π led to an inclusion of $\mathcal{O}(P/\pi, \lambda/\pi)$ into $\mathcal{O}(P, \lambda)$ whose image we refer to as F_π. We have seen that F_π is a face of $\mathcal{O}(P, \lambda)$ when π is connected and λ/π is strict.

In this section we come back to Jochemko and Sanyal's subdivision of marked order polytopes into products of simplices we discussed in Section 5.2. This subdivision immediately generalizes to marked order polyhedra and the above discussion of induced maps allows to consider the cells as inclusion of marked order polyhedra $\mathcal{O}(P_{\mathcal{I}}, \lambda_{\mathcal{I}})$ into $\mathcal{O}(P, \lambda)$.

Let $\mathcal{I}\colon \varnothing = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = P$ be a chain of order ideals in P compatible with λ, as defined in Section 5.2. Denote by $\pi_{\mathcal{I}}$ the partition of P with blocks $B_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, r$. Note that $\pi_{\mathcal{I}}$ is a (P, λ)-compatible partition since $\mathcal{I}$ is compatible with λ, so we have a marked poset $(P/\pi_{\mathcal{I}}, \lambda/\pi_I\mathcal{I})$. The poset $P/\pi_{\mathcal{I}}$ has a linear extension $P_{\mathcal{I}}$ given by $B_k \leq B_l$ if and only if $k \leq l$ and since $\mathcal{I}$ was compatible with λ, the marking $\lambda/\pi_{\mathcal{I}}$ yields a strict marking $\lambda_{\mathcal{I}}$ on $P_{\mathcal{I}}$. Hence, we have surjections

$$(P, \lambda) \longrightarrow (P/\pi_{\mathcal{I}}, \lambda/\pi_{\mathcal{I}}) \longrightarrow (P_{\mathcal{I}}, \lambda_{\mathcal{I}}),$$

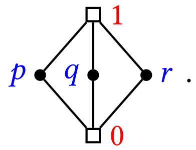
where the first map is a quotient map of marked posets as in Proposition 6.2.9 and the second map is the inclusion into a linear extension. These surjective maps of marked posets gives rise to inclusions of marked order polyhedra

$$\mathcal{O}(P_{\mathcal{I}}, \lambda_{\mathcal{I}}) \lhook\joinrel\longrightarrow \mathcal{O}(P/\pi_{\mathcal{I}}, \lambda/\pi_{\mathcal{I}}) \lhook\joinrel\longrightarrow \mathcal{O}(P, \lambda).$$

The image $F_{\mathcal{I}}$ of $\mathcal{O}(P_{\mathcal{I}}, \lambda_{\mathcal{I}})$ in $\mathcal{O}(P, \lambda)$ consists of all points $x \in \mathcal{O}(P, \lambda)$ constant on the blocks B_k with values weakly-increasing along their linear order $B_1, B_2, \ldots, B_r$.

Before stating and proving how these polyhedra form a subdivision of $\mathcal{O}(P, \lambda)$ we give an example of the above construction.

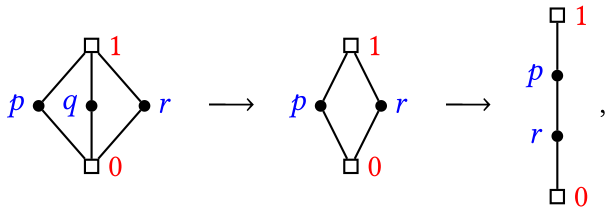
Example 6.3.1. Consider the marked poset

$(P, \lambda) \quad = $ 

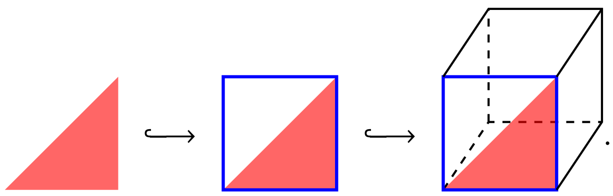
Its marked order polytope is a unit cube. For the chain of order ideals

$$\mathcal{I} : \varnothing \subsetneq \{0, q\} \subsetneq \{0, q, r\} \subsetneq \{0, p, q, r\} \subsetneq P$$

we obtain the surjective maps of marked posets



where the first map sends both 0 and q to 0. Applying the marked order polyhedron functor to this diagram yields the inclusions



◊

Using the above terminology, we are now ready to state the following proposition.

Proposition 6.3.2. *Let (P, λ) be a marked poset and denote by $\mathcal{S}$ the set of all $F_\mathcal{I}$ for $\mathcal{I}$ ranging over all chains of order ideals in P compatible with λ together with the empty set. Then $\mathcal{S}$ is a polyhedral subdivision of $\mathcal{O}(P, \lambda)$ in which each cell is a product of simplices and simplicial cones.*

Proof. The fact that $F_\mathcal{I}$ is always a product of simplices and simplicial cones is a consequence of $(P_\mathcal{I}, \lambda_\mathcal{I})$ being a linear marked poset by construction. It remains to be checked that $\mathcal{S}$ covers all of $\mathcal{O}(P, \lambda)$, is closed under taking faces and intersections of cells $F_\mathcal{I}$ and $F_\mathcal{J}$ result in a face of both.

Given any $x \in \mathcal{O}(P, \lambda)$ enumerate the appearing values $x(P) = \{a_1, a_2, \ldots, a_r\}$ such that $a_1 < a_2 < \ldots < a_r$. Define the chain $\mathcal{I}$ by letting I_k be the set of all $p \in P$ such that $x_p < a_{k+1}$ for $k = 0, \ldots, r-1$ and set $I_r = P$. This chain of order ideals is compatible with λ and yields a cell $F_\mathcal{I}$ having x in its relative interior. In fact, this is the only cell with x in its relative interior, since the relative interior of a cell $F_\mathcal{I}$ contains exactly those $x \in \mathcal{O}(P, \lambda)$ that are *strictly* increasing along the blocks $B_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, r$.

Given a cell $F_{\mathcal{I}}$ we can apply the face description in Theorem 6.2.14 to $(P_{\mathcal{I}}, \lambda_{\mathcal{I}})$ to see that faces of $F_{\mathcal{I}}$ are given by cells $F_{\mathcal{J}}$ with $\mathcal{J} \subseteq \mathcal{I}$ a coarsening, i.e., every ideal in $\mathcal{J}$ is also contained in $\mathcal{I}$.

Given two cells $F_{\mathcal{I}}$ and $F_{\mathcal{J}}$, their intersection $F_{\mathcal{I}} \cap F_{\mathcal{J}}$ consists of all $x \in \mathcal{O}(P, \lambda)$ such that x is constant and weakly increasing on both $B_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, r$ and $C_k = J_k, \setminus J_{k-1}$ for $k = 1, \ldots, s$. Let $\mathcal{K} = \mathcal{I} \cap \mathcal{J} \colon K_0 \subsetneq K_1 \subsetneq \ldots \subsetneq K_t$ be the chain consisting of all order ideals contained in both $\mathcal{I}$ and $\mathcal{J}$. We claim that $F_{\mathcal{I}} \cap F_{\mathcal{J}} = F_{\mathcal{K}}$ if $\mathcal{K}$ is compatible with λ and $F_{\mathcal{I}} \cap F_{\mathcal{J}} = \varnothing$ otherwise.

First assume that $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ is compatible with λ. We have $F_{\mathcal{K}} \subseteq F_{\mathcal{I}} \cap F_{\mathcal{J}}$, since $F_{\mathcal{K}}$ is a face of both $F_{\mathcal{I}}$ and $F_{\mathcal{J}}$ by the previous paragraph. Now let $x \in F_{\mathcal{I}} \cap F_{\mathcal{J}}$ and consider a block $D_j = K_j \setminus K_{j-1}$. The chains $\mathcal{I}$ and $\mathcal{J}$ both contain K_{j-1} and K_j, but in between the order ideals in $\mathcal{I}$ and $\mathcal{J}$ are pairwise distinct. For sake of contradiction assume that there are $p, p' \in D_j$ with $x_p < x_{p'}$, so x is not constant on D_j. Denote by L the set of all $q \in P$ such that $x_q < x_{p'}$. Note that L is an order ideal and has to be in both $\mathcal{I}$ and $\mathcal{J}$ since all values x takes on $P \setminus L$ are strictly larger than those on L. We conclude that $K_{j-1} \subsetneq L \subsetneq K_j$, a contradiction to $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$. Hence, x is constant on the blocks K_k. The fact that the values of x are weakly-increasing along the linear order $K_1, K_2, \ldots, K_t$ is a consequence of $\mathcal{K}$ being a coarsening of $\mathcal{I}$ and x being weakly increasing along the blocks $B_1, B_2, \ldots, B_r$ given by $\mathcal{I}$, since each block K_k is a union of consecutive blocks $B_i, B_{i+1}, \ldots, B_j$.

When $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ is not compatible with λ, the reason might be one of the following two situations:

a) there are $a, b \in P^*$ with $\lambda(a) < \lambda(b)$ but $i(\mathcal{K}, a) = i(\mathcal{K}, b)$,
b) there are $a, b \in P^*$ with $\lambda(a) \leq \lambda(b)$ but $i(\mathcal{K}, a) > i(\mathcal{K}, b)$.

In case a), assume $x \in F_{\mathcal{I}} \cap F_{\mathcal{J}}$ and conclude that $x_a = x_b$ by the above argument. This is a contradiction to $\lambda(a) < \lambda(b)$ and hence $F_{\mathcal{I}} \cap F_{\mathcal{J}} = \varnothing$. In case b), since $\mathcal{K}$ is a coarsening of $\mathcal{I}$, we can conclude that already $\mathcal{I}$ was not compatible with λ, so this case is excluded. □

6.4. Products, Minkowski Sums and Lattice Polyhedra

In this section we study some convex geometric properties of marked order polyhedra. We describe recession cones, a correspondence between disjoint unions of posets and products of polyhedra, characterize pointedness and use these results to obtain a Minkowski sum decomposition. At the end of the section we show that marked posets with integral markings always give rise to lattice polyhedra.

Proposition 6.4.1. *The recession cone of $\mathcal{O}(P, \lambda)$ is $\mathcal{O}(P, 0)$, where $0 \colon P^* \to \mathbb{R}$ is the zero marking on the same domain as λ.*

Proof. The recession cone of a polyhedron $Q \subseteq \mathbb{R}^n$ defined by a system of linear inequalities $Ax \geq b$ is given by $Ax \geq 0$. Hence, replacing all constant terms in the description of $\mathcal{O}(P, \lambda)$ by zeros we see that $\operatorname{rec}(\mathcal{O}(P, \lambda)) = \mathcal{O}(P, 0)$. □

Proposition 6.4.2. *Let (P_1, λ_1) and (P_2, λ_2) be marked posets on disjoint sets. Let the marking $\lambda_1 \sqcup \lambda_2 \colon P_1^* \sqcup P_2^* \to \mathbb{R}$ on $P_1 \sqcup P_2$ be given by λ_1 on P_1^* and λ_2 on P_2^*. The marked order polyhedron $\mathcal{O}(P_1 \sqcup P_2, \lambda_1 \sqcup \lambda_2)$ is equal to the product $\mathcal{O}(P_1, \lambda_1) \times \mathcal{O}(P_2, \lambda_2)$ under the canonical identification $\mathbb{R}^{P_1 \sqcup P_2} = \mathbb{R}^{P_1} \times \mathbb{R}^{P_2}$.*

Proof. The defining equations and inequalities of a product polyhedron $Q_1 \times Q_2$ in $\mathbb{R}^{P_1} \times \mathbb{R}^{P_2}$ are obtained by imposing both the defining conditions of Q_1 and Q_2. In case of $Q_1 = \mathcal{O}(P_1, \lambda_1)$ and $Q_2 = \mathcal{O}(P_2, \lambda_2)$ these are exactly the defining conditions of $\mathcal{O}(P_1 \sqcup P_2, \lambda_1 \sqcup \lambda_2)$. □

Note that this relation between disjoint unions of marked posets and products of the associated marked order polyhedra may be expressed as the contravariant functor $\mathcal{O} \colon \mathsf{MPos} \to \mathsf{Polyh}$ sending coproducts to products.

We now characterize marked posets whose associated polyhedra are *pointed*. A pointed polyhedron is one that has at least one vertex, or equivalently does not contain a line. The importance of pointedness lies in the fact that pointed polyhedra are determined by their vertices and recession cone. To be precise, a pointed polyhedron is the Minkowski sum of its recession cone and the polytope obtained as the convex hull of its vertices.

Proposition 6.4.3. *A marked order polyhedron $\mathcal{O}(P, \lambda)$ is pointed if and only if each connected component of P contains a marked element.*

Proof. Let $P_1, \dots, P_k$ be the connected components of P with $\lambda_i = \lambda|_{P_i}$ the restricted markings. By inductively applying Proposition 6.4.2, we have a decomposition

$$\mathcal{O}(P, \lambda) = \mathcal{O}(P_1, \lambda_1) \times \dots \times \mathcal{O}(P_k, \lambda_k).$$

Hence, $\mathcal{O}(P, \lambda)$ is pointed if and only if each $\mathcal{O}(P_i, \lambda_i)$ is pointed, reducing the statement to the case of P being connected.

Let (P, λ) be a connected marked poset and suppose $v \in \mathcal{O}(P, \lambda)$ is a vertex. By Proposition 6.2.2 the corresponding partition π has no free blocks. Hence, either P is empty or it has at least as many marked elements as the number of blocks in π.

Conversely, if P is connected and contains marked elements, the following procedure yields a vertex v of $\mathcal{O}(P, \lambda)$: start by setting $v_a = \lambda(a)$ for all $a \in P^*$. Pick any $p \in P$ such that v_p is not already determined and p is adjacent to some q in the Hasse-diagram of P with v_q already determined. Set v_p to be the maximum of all determined v_q with p covering q or the minimum of all determined v_q with p covered by q. Continue until all v_p are determined.

In each step, the defining conditions of $\mathcal{O}(P, \lambda)$ are respected and the procedure determines all v_p since P is connected and contains a marked element. By construction, each block of π_v will contain a marked element and thus v is a vertex by Proposition 6.2.2. □

Proposition 6.4.4. *Let $\lambda_1, \lambda_2 \colon P^* \to \mathbb{R}$ be markings on the same poset P. The Minkowski sum $\mathcal{O}(P, \lambda_1) + \mathcal{O}(P, \lambda_2)$ is contained in $\mathcal{O}(P, \lambda_1 + \lambda_2)$, where $\lambda_1 + \lambda_2$ is the marking sending $a \in P^*$ to $\lambda_1(a) + \lambda_2(a)$.*

Proof. Let $x \in \mathcal{O}(P, \lambda_1)$ and $y \in \mathcal{O}(P, \lambda_2)$. For any relation $p \leq q$ in P we have $x_p \leq x_q$ and $y_p \leq y_q$, hence $x_p + y_p \leq x_q + y_q$. For $a \in P^*$ we have $x_a + y_a = \lambda_1(a) + \lambda_2(a) = (\lambda_1 + \lambda_2)(a)$. Thus, $x + y \in \mathcal{O}(P, \lambda_1 + \lambda_2)$. □

We are now ready to give a Minkowski sum decomposition of marked order polyhedra, such that the marked posets associated to the summands have 0-1-markings. The decomposition is a generalization of [SP02, Theorem 4] and [JS14, Corollary 2.10], where the bounded case with P^* being a chain in P is considered.

Theorem 6.4.5. *Let (P, λ) be a marked poset with $P^* \neq \varnothing$ and $\lambda(P^*) = \{ c_0, c_1, \dots, c_k \}$ with $c_0 < c_1 < \cdots < c_k$. Let $c_{-1} = 0$ and define markings $\lambda_i \colon P^* \to \mathbb{R}$ for $i = 0, \dots, k$ by*

$$\lambda_i(a) = \begin{cases} 0 & \text{if } \lambda(a) < c_i, \\ 1 & \text{if } \lambda(a) \geq c_i. \end{cases}$$

Then $\mathcal{O}(P, \lambda)$ decomposes as the weighted Minkowski sum

$$\mathcal{O}(P, \lambda) = \sum_{i=0}^{k} (c_i - c_{i-1})\, \mathcal{O}(P, \lambda_i).$$

Proof. Since

$$\lambda = c_0\lambda_0 + (c_1 - c_0)\lambda_1 + \cdots + (c_k - c_{k-1})\lambda_k$$

and in general $\mathcal{O}(P, c\lambda) = c\, \mathcal{O}(P, \lambda)$, one inclusion follows immediately from Proposition 6.4.4. For the other inclusion, first assume that $\mathcal{O}(P, \lambda)$ is pointed. In this case, it is enough to consider vertices and the recession cone. Since the underlying posets and sets of marked elements agree for all polytopes in consideration, they all have the same recession cone $\mathcal{O}(P, 0)$ by Proposition 6.4.1. Let $v \in \mathcal{O}(P, \lambda)$ be a vertex. The associated face partition π has no free blocks and on each block v takes some constant value in $\lambda(P^*)$. For fixed $i \in \{0, \dots, k\}$ we enumerate the blocks of π where v takes constant value c_i by $B_{i,1}, \dots, B_{i,r_i}$. For a block $B \in \pi$ denote by $w_B = \sum_{p \in B} e_p \in \mathbb{R}^P$ the labeling of P with all entries in B equal to 1, all other entries equal to 0. This yields a description of v as

$$v = \sum_{i=0}^{k} c_i \sum_{j=1}^{r_i} w_{B_{i,j}}.$$

For $i = 0, \dots, k$ define points $v^{(i)} \in \mathbb{R}^P$ by

$$v^{(i)} = (c_i - c_{i-1}) \sum_{l=i}^{k} \sum_{j=1}^{r_l} w_{B_{l,j}}.$$

This gives a decomposition of v as $v^{(0)} + \cdots + v^{(k)}$. It remains to be checked that each $v^{(i)}$ is a point in the corresponding Minkowski summand. Since $v^{(0)}$ is just constant c_0 on the whole poset and λ_0 is the marking of all ones, we have $v^{(0)} \in c_0\, \mathcal{O}(P, \lambda_0)$. Fix $i \in \{1, \dots, k\}$. For $p \leq q$ we have $v_p \leq v_q$ and thus $p \in B_{i,j}$, $q \in B_{i',j'}$ for $i \leq i'$ by the chosen enumeration of blocks. Hence, by definition of $v^{(i)}$, the inequality $v_p^{(i)} \leq v_q^{(i)}$ is equivalent to one of the three inequalities $0 \leq 0$, $0 \leq c_i - c_{i-1}$ or $c_i - c_{i-1} \leq c_i - c_{i-1}$, all being true. The marking conditions of $\mathcal{O}(P, (c_i - c_{i-1})\lambda_i)$ are satisfied by $v^{(i)}$ as well, so $v^{(i)} \in (c_i - c_{i-1})\,\mathcal{O}(P, \lambda_i)$. We conclude that

$$v = \sum_{i=1}^{k} v^{(i)} \quad \in \quad \sum_{i=0}^{k} (c_i - c_{i-1})\,\mathcal{O}(P, \lambda_i)$$

for each vertex v of $\mathcal{O}(P, \lambda)$. Hence, the proof is finished for the case of $\mathcal{O}(P, \lambda)$ being pointed.

When $\mathcal{O}(P, \lambda)$ is not pointed, we can decompose $P = P' \sqcup P''$ where P' consists of all connected components without marked elements and P'' consists of all other components. Letting λ' and λ'' be the respective restrictions of λ, we have $\mathcal{O}(P, \lambda) = \mathcal{O}(P', \lambda') \times \mathcal{O}(P'', \lambda'')$ by Proposition 6.4.2, where $\mathcal{O}(P', \lambda')$ is not pointed while $\mathcal{O}(P'', \lambda'')$ is, by Proposition 6.4.3. Applying the previous result to $\mathcal{O}(P'', \lambda'')$ we obtain

$$\mathcal{O}(P, \lambda) = \mathcal{O}(P', \lambda') \times \left(\sum_{i=0}^{k} (c_i - c_{i-1})\,\mathcal{O}(P'', \lambda_i'') \right).$$

Since P' contains no marked elements, it is equal to its recession cone and we have

$$\mathcal{O}(P', \lambda') = \sum_{i=0}^{k} \mathcal{O}(P', \lambda').$$

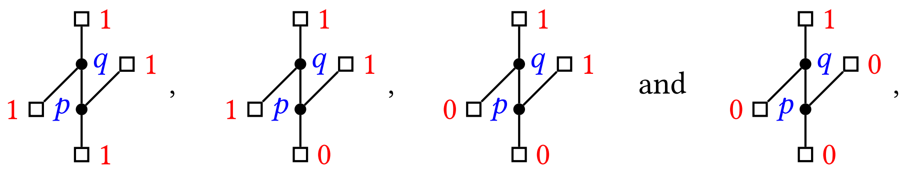
Therefore, using the identity $\sum_{i=0}^{k} P_i \times \sum_{i=0}^{k} Q_i = \sum_{i=0}^{k} (P_i \times Q_i)$ for products of Minkowski sums, we obtain

$$\begin{aligned}
\mathcal{O}(P, \lambda) &= \left(\sum_{i=0}^{k} \mathcal{O}(P', \lambda') \right) \times \left(\sum_{i=0}^{k} (c_i - c_{i-1})\,\mathcal{O}(P'', \lambda_i'') \right) \\
&= \sum_{i=0}^{k} \left(\mathcal{O}(P', \lambda') \times \mathcal{O}(P'', (c_i - c_{i-1})\lambda_i'') \right) \\
&= \sum_{i=0}^{k} \mathcal{O}(P' \sqcup P'', \lambda' \sqcup (c_i - c_{i-1})\lambda_i'').
\end{aligned}$$

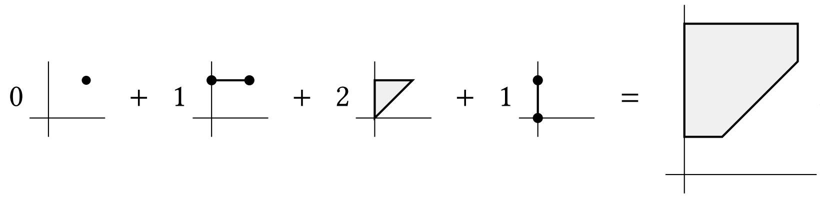
Since P' did non contain any markings that could be affected by scaling, the factors $(c_i - c_{i-1})$ can be put as dilation factors in front of the polyhedra. Again, since P' is unmarked, we have $\lambda' \sqcup \lambda_i'' = \lambda_i$ and $P' \sqcup P'' = P$, so we obtain the desired Minkowski sum decomposition. □

Remark 6.4.6. When $\mathcal{O}(P, \lambda)$ is a polytope, $\mathcal{O}(P, 1)$ is just a point and the marked poset polytopes $\mathcal{O}(P, \lambda_i)$ appearing in the Minkowski sum decomposition of Theorem 6.4.5 may all be expressed as ordinary poset polytopes as discussed by Stanley [Sta86] and Geissinger [Gei81] by contracting constant intervals and dropping redundant conditions.

Example 6.4.7. We apply the Minkowski sum decomposition of Theorem 6.4.5 to the marked order polytope $\mathcal{O}(P,\lambda)$ from Example 6.1.3. Since $\lambda(P^*) = \{0,1,3,4\}$ in this example, we obtain the four new markings λ_0, λ_1, λ_2 and λ_3 given by



respectively. The associated marked order polytopes and their weighted Minkowski sum are



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We finish this section by considering marked posets with integral markings. When all markings on a poset P are integral, we will see that $\mathcal{O}(P,\lambda)$ is a lattice polyhedron. A simple fact about lattice polyhedra we will need is that products of lattice polyhedra are lattice polyhedra. This is an immediate consequence of the Minkowski sum identity $(Q+R)\times(Q'+R') = (Q\times Q')+(R\times R')$ we already used, together with products of lattice polytopes being lattice polytopes and products of rational cones being rational cones.

Proposition 6.4.8. *Let (P,λ) be a marked poset such that $\lambda(P^*) \subseteq \mathbb{Z}$. Then the marked order polyhedron $\mathcal{O}(P,\lambda)$ is a lattice polyhedron.*

Proof. When $\mathcal{O}(P,\lambda)$ is pointed, it is enough to show that all vertices are lattice points and the recession cone is rational. By Proposition 6.2.2 the face partitions associated to vertices have no free blocks. Hence, all coordinates are contained in $\lambda(P^*)$, so vertices are lattice points. The recession cone is obtained as $\mathcal{O}(P,0)$ by Proposition 6.4.1, which is a rational polyhedral cone.

If $\mathcal{O}(P,\lambda)$ is not pointed, we use the decomposition $P = P' \sqcup P''$ of P into unmarked connected components in P' and the other components in P''. As in the previous proof, we obtain a product decomposition $\mathcal{O}(P,\lambda) = \mathcal{O}(P',\lambda')\times\mathcal{O}(P'',\lambda'')$ by Proposition 6.4.2. Since P'^* is empty we know that $\mathcal{O}(P',\lambda')$ is a rational polyhedral cone. Since all connected components of P'' contain marked elements, we know that $\mathcal{O}(P'',\lambda'')$ is pointed and hence a lattice polyhedron by the previous argument. We conclude that $\mathcal{O}(P,\lambda)$ is a lattice polyhedron. □

6.5. Conditional Marked Order Polyhedra

In this section we study intersections of marked order polyhedra with affine subspaces. We describe an affine subspace U of $\mathbb{R}^P$ by a linear map $s\colon \mathbb{R}^P \to \mathbb{R}^k$ and a vector $b \in \mathbb{R}^k$, such that $U = s^{-1}(b)$. Hence, U is the space of solutions to the linear system $s(x) = b$.

Definition 6.5.1. Given a marked poset (P,λ), a linear map $s\colon \mathbb{R}^P \to \mathbb{R}^k$ and $b \in \mathbb{R}^k$, we define the *conditional marked order polyhedron* $\mathcal{O}(P,\lambda,s,b)$ as the intersection $\mathcal{O}(P,\lambda) \cap s^{-1}(b)$.

The faces of $\mathcal{O}(P,\lambda,s,b)$ correspond to the faces of $\mathcal{O}(P,\lambda)$ whose relative interior meets $s^{-1}(b)$. Hence, they are also given by face partitions. However, given a face partition π of $\mathcal{O}(P,\lambda)$, deciding whether it is a face partition of $\mathcal{O}(P,\lambda,s,b)$ can not be done combinatorially in general. The problem is in determining whether the linear system $s(x) = b$ admits a solution in the relative interior of F_π. We come back to this issue later in the section. Still, given a point $x \in \mathcal{O}(P,\lambda,s,b)$, we obtain a face partition π_x and we can find the dimension of $F_x \subseteq \mathcal{O}(P,\lambda,s,b)$ by calculating a kernel of a linear map associated to π_x.

Given a partition π of P, we define the linear injection $r_\pi\colon \mathbb{R}^{\tilde{\pi}} \to \mathbb{R}^P$ by

$$r_\pi(z)_p = \begin{cases} z_B & \text{if } p \text{ is an element of the free block } B \in \tilde{\pi}, \\ 0 & \text{otherwise.} \end{cases}$$

We can describe r_π as taking a labeling z of the free blocks of π with real numbers and making it into a labeling of P with real numbers, by putting the values given by z on elements in free blocks, while labeling elements in non-free blocks with zero. If π is a face partition of $\mathcal{O}(P,\lambda)$, we have seen in the proof of Proposition 6.2.2 that the affine hull of $F_\pi \subseteq \mathcal{O}(P,\lambda)$ is a translation of $\operatorname{im}(r_\pi)$. The following proposition is a generalization of this observation to conditional marked order polyhedra.

Proposition 6.5.2. *Let x be a point of $\mathcal{O}(P,\lambda,s,b)$ with associated face partition $\pi = \pi_x$. Let U be the linear subspace of $\mathbb{R}^P$ parallel to the affine hull of the face $F_x \subseteq \mathcal{O}(P,\lambda,s,b)$. The map r_π restricts to an isomorphism $\ker(s \circ r_\pi) \xrightarrow{\sim} U$. In particular, the dimension of F_x is the same as the dimension of $\ker(s \circ r_\pi)$.*

Proof. Let F'_x be the minimal face of $\mathcal{O}(P,\lambda)$ containing x, so that $F_x = F'_x \cap s^{-1}(b)$. For the affine hulls we also have $\operatorname{aff}(F_x) = \operatorname{aff}(F'_x) \cap s^{-1}(b)$. Letting U' be the linear subspace parallel to $\operatorname{aff}(F'_x)$, just as U is the linear subspace parallel to $\operatorname{aff}(F_x)$, we obtain

$$U = U' \cap \ker(s) = \operatorname{im}(r_\pi) \cap \ker(s),$$

since $\ker(s)$ is the linear subspace parallel to $s^{-1}(b)$. This description implies that r_π restricts to an isomorphism $\ker(s \circ r_\pi) \xrightarrow{\sim} U$. $\square$

Remark 6.5.3. In the special case of Gelfand–Tsetlin polytopes with linear conditions given by a weight μ, this result appeared in [DM04] in terms of tiling matrices associated to points in the polytope. The tiling matrix is exactly the matrix associated to the linear map $s \circ r_\pi$.

Example 6.5.4. Let (P,λ) be the linear marked poset

$$0 \prec p \prec q \prec r \prec s \prec 5$$

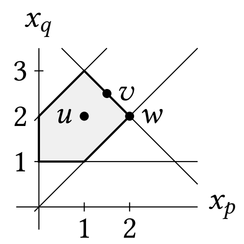


Figure 6.5.: The conditional marked order polytope $\mathcal{O}(P, \lambda, s, b)$ from Example 6.5.4 together with three points on faces of different dimensions.

and impose the linear conditions $x_p + x_r = 4$ and $x_q + x_s = 6$ on $\mathcal{O}(P, \lambda)$. We describe these conditions by intersecting with $s^{-1}(b)$ for the linear map $s\colon \mathbb{R}^P \to \mathbb{R}^2$ given by $s(x) = (x_p + x_r, x_q + x_s)$ and $b = (4, 6)$. Any point in $\mathcal{O}(P, \lambda, s, b)$ is determined by x_p and x_q, so we can picture the polytope in $\mathbb{R}^2$. Expressing the five inequalities in terms of x_p, x_q using the linear conditions, we obtain

$$0 \le x_p, \quad x_p \le x_q, \quad x_q \le 4 - x_p, \quad x_q \le 2 + x_p, \quad 1 \le x_q.$$

The resulting polytope in $\mathbb{R}^{\{p,q\}} \cong \mathbb{R}^2$ is illustrated in Figure 6.5.

We want to calculate the dimensions of the minimal faces of $\mathcal{O}(P, \lambda, s, b)$ containing the points $u = (1, 2)$, $v = (1.5, 2.5)$ and $w = (2, 2)$ in $\mathbb{R}^2$. In $\mathbb{R}^P$ these points and their associated partitions of P are

$$0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5, \quad 0 \mid 1.5 \mid 2.5 \;\; 2.5 \mid 3.5 \mid 5, \quad \text{and} \quad 0 \mid 2 \;\; 2 \;\; 2 \mid 4 \mid 5.$$

Hence, we have 4, 3 and 2 free blocks, respectively. The associated linear maps $s \circ r_\pi$ can be represented by the matrices

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. The kernels of these maps have dimension 2, 1 and 0 corresponding to the dimensions of the minimal faces containing u, v and w as one can see in Figure 6.5. $\Diamond$

Given any (P, λ)-compatible partition of P, we obtained a polyhedron F'_π contained in $\mathcal{O}(P, \lambda)$ in the previous section. Hence, we have a polyhedron F_π contained in $\mathcal{O}(P, \lambda, s, b)$ given by $F_\pi = F'_\pi \cap s^{-1}(b)$. As in the unconditional case, these polyhedra are canonically affine isomorphic to conditional marked order polyhedra given by the quotient $(P/\pi, \lambda/\pi)$.

Proposition 6.5.5. *Let (P, λ) be a marked poset, π a (P, λ)-compatible partition, $s\colon \mathbb{R}^P \to \mathbb{R}^k$ a linear map and $b \in \mathbb{R}^k$. Define s/π to be the composition $s \circ q^*$, where q^* is the inclusion $\mathbb{R}^{P/\pi} \hookrightarrow \mathbb{R}^P$ induced by the quotient map of marked posets. The polyhedron $F_\pi \subseteq \mathcal{O}(P, \lambda, s, b)$ is affinely isomorphic to the conditional marked order polyhedron $\mathcal{O}(P/\pi, \lambda/\pi, s/\pi, b)$ via the canonical isomorphism obtained by restricting q^*.*

Proof. By definition, F_π is the intersection of the face F'_π of $\mathcal{O}(P,\lambda)$ with $s^{-1}(b)$. We know that q^* restricts to an affine isomorphism $\mathcal{O}(P/\pi,\lambda/\pi) \xrightarrow{\sim} F'_\pi$. Hence, F_π is contained in the image of q^* as well and we have

$$F_\pi = F'_\pi \cap s^{-1}(b) = F'_\pi \cap \operatorname{im} q^* \cap s^{-1}(b) = F'_\pi \cap q^*((s \circ q^*)^{-1}(b)).$$

We may write F'_π as $q^*(\mathcal{O}(P/\pi,\lambda/\pi))$ and use injectivity of q^* to obtain

$$F_\pi = q^*(\mathcal{O}(P/\pi,\lambda/\pi)) \cap q^*((s/\pi)^{-1}(b)) = q^*(\mathcal{O}(P/\pi,\lambda/\pi) \cap (s/\pi)^{-1}(b)).$$

By definition of conditional marked order polyhedra, this is just the injective image of $\mathcal{O}(P/\pi,\lambda/\pi,s/\pi,b)$ under q^*, which finishes the proof. □

When F is a non-empty face of $\mathcal{O}(P/\pi,\lambda/\pi,s/\pi,b)$ we have an associated partition $\pi = \pi_F$, so that $F = F_\pi$. Thus, we obtain the same corollary on faces of conditional marked order polyhedra as in the unconditional case.

Corollary 6.5.6. *For every non-empty face F of a conditional marked order polyhedron $\mathcal{O}(P,\lambda,s,b)$ we have a canonical affine isomorphism*

$$\mathcal{O}(P/\pi_F,\lambda/\pi_F,s/\pi_F,b) \cong F.$$

□

The next proposition will allow us to consider *any* polyhedron as a conditional marked order polyhedron up to affine isomorphism. Thus, there is little hope to understand general conditional marked order polyhedra any better than we understand polyhedra in general.

Proposition 6.5.7. *Every polyhedron is affinely isomorphic to a conditional marked order polyhedron.*

Proof. Let $Q \subseteq \mathbb{R}^n$ be a polyhedron given by linear equations and inequalities

$$\sum_{i=1}^n a_{ki}x_i = c_k \qquad \text{for } k = 1,\dots,s,$$

$$\sum_{i=1}^n b_{li}x_i \le d_l \qquad \text{for } l = 1,\dots,t.$$

Define a poset P with ground set $\{p_1,\dots,p_n,q_1,\dots,q_t,r\}$ and covering relations $q_l \prec r$ for $l = 1,\dots,t$. Define a marking on $P^* = \{r\}$ by $\lambda(r) = 0$. The marked poset obtained this way is depicted in Figure 6.6. Let the linear system $s(x) = b$ for $x \in \mathbb{R}^P$ be given by

$$\sum_{i=1}^n a_{ki}x_{p_i} = c_k \qquad \text{for } k = 1,\dots,s,$$

$$\sum_{i=1}^n b_{li}x_{p_i} - x_{q_l} = d_l \qquad \text{for } l = 1,\dots,t.$$

The conditional marked order polyhedron $\mathcal{O}(P,\lambda,s,b)$ is affinely isomorphic to Q by the map $\mathcal{O}(P,\lambda,s,b) \to Q$ sending $x \in \mathbb{R}^P$ to $(x_{p_1},\dots,x_{p_n}) \in \mathbb{R}^n$. □

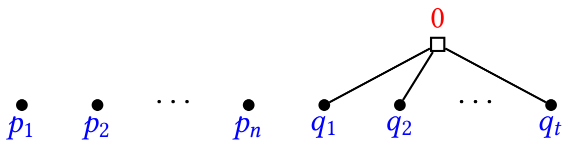


Figure 6.6.: The marked poset constructed in the proof of Proposition 6.5.7.

We may now come back to the question of when a face partition π of (P, λ) still corresponds to a face of $\mathcal{O}(P, \lambda, s, b)$. As discussed at the beginning of this section, we have to decide whether $s(x) = b$ admits a solution in the relative interior of the face F'_π of $\mathcal{O}(P, \lambda)$, that is, $\operatorname{relint}(F'_\pi) \cap s^{-1}(b) \neq \emptyset$. Using the affine isomorphism induced by the quotient map this is equivalent to

$$\operatorname{relint}\left(\mathcal{O}(P/\pi, \lambda/\pi)\right) \cap (s/\pi)^{-1}(b) \neq \emptyset.$$

Hence, we reduced the problem to deciding whether a linear system $s(x) = b$ admits a solution in the relative interior of a marked order polyhedron $\mathcal{O}(P, \lambda)$. However, even deciding whether $s(x) = b$ admits any solution in $\mathcal{O}(P, \lambda)$ is equivalent to deciding whether $\mathcal{O}(P, \lambda, s, b)$ is non-empty, which is in general just as hard as determining whether an arbitrary system of linear equations and linear inequalities admits a solution by Proposition 6.5.7.

We conclude that the concept of conditional marked order polyhedra is too general to obtain meaningful results. Still, in special cases the structure of an underlying poset and faces still corresponding to a subset of face partitions might be useful. An interesting class of conditional marked order polyhedra might consist of those, where P is connected and conditions are given by fixing sums along disjoint subsets of P, as is the case for Gelfand–Tsetlin polytopes with weight conditions.

In Chapter 10 we study a special case of such Gelfand–Tsetlin polytopes with weight conditions in the context of frame theory, namely polytopes of eigensteps as in Section 4.3 for the case of equal norm tight frames. We see that even determining the dimension and facets of such a conditional marked order polyhedron is a non-trivial task.

7. A Continuous Family of Marked Poset Polyhedra

Having studied marked order polyhedra in Chapter 6, we want to move towards marked chain polytopes in this chapter. As already mentioned in Chapter 1, little is known about the combinatorics even of ordinary unmarked chain polytopes. In the marked setting, the situation is even worse, since the transfer map $\mathcal{O}(P, \lambda) \to \mathcal{C}(P, \lambda)$ from the marked order polytope to the marked chain polytope does not even preserve vertices. In this chapter, we take the following approach to gain knowledge about marked chain polytopes: given a marked poset (P, λ) introduce a parameter $t \in [0, 1]$ to the transfer map to obtain a homotopy φ_t between the identity on $\mathcal{O}(P, \lambda)$ and the transfer map to $\mathcal{C}(P, \lambda)$. It turns out that $\varphi_t(\mathcal{O}(P, \lambda))$ is a polytope for all $t \in [0, 1]$ and the combinatorial type does not change when t varies in $(0, 1)$. Thus, we may think of the marked order polytope $\mathcal{O}(P, \lambda)$ at $t = 0$ and the marked chain polytope $\mathcal{C}(P, \lambda)$ at $t = 1$ as continuous degenerations of a *generic marked poset polytope* obtained for $t \in (0, 1)$. We will give a precise definition of what we mean by *continuous degeneration* and indicate how this point of view might help understanding the face structure of marked chain polytopes.

Inspired by previous representation theoretically motivated work of Fang and Fourier on *marked chain-order polytopes* in [FF16], we allow the parameter t to be different in each coordinate and obtain a family of polytopes $\mathcal{O}_t(P, \lambda)$ for $t \in [0, 1]^{\tilde{P}}$ that we refer to as the *continuous family of marked poset polytopes*. Analogous to the case of just one parameter, the combinatorial type stays constant along the relative interiors of the faces of the parametrizing hypercube. We recover the marked chain-order polytopes of Fang and Fourier at some of the cube's vertices, hence putting them in an elegant unified framework.

The results of this chapter are joint work with Xin Fang, Ghislain Fourier and Jan-Philipp Litza and have also appeared in [FFLP17].

7.1. Definition

Instead of defining $\mathcal{O}_t(P, \lambda)$ as an image of the marked order polyhedron under a modified transfer map, we give a description in terms of linear equations and inequalities, and provide the transfer map in the next section.

Definition 7.1.1. Let (P, λ) be a marked poset such that P^* contains at least all minimal elements of P. For $t \in [0, 1]^{\tilde{P}}$ define the *marked poset polyhedron* $\mathcal{O}_t(P, \lambda)$ as the set of all $x \in \mathbb{R}^P$ satisfying the following conditions:

i) for each $a \in P^*$ an equation $x_a = \lambda(a)$,

ii) for each saturated chain $p_0 \prec p_1 \prec p_2 \prec \cdots \prec p_r \prec p$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$, $p \in P$ and $r \geq 0$ an inequality

$$(1 - t_p)\left(t_{p_1} \cdots t_{p_r} x_{p_0} + t_{p_2} \cdots t_{p_r} x_{p_1} + \cdots + x_{p_r}\right) \leq x_p, \tag{7.1}$$

where $t_p = 0$ if $p \in P^*$. Note that the inequalities for $p \in P^*$ and $r = 0$ may be omitted, since they are consequences of λ being order-preserving.

Since the coordinates in P^* are fixed, we sometimes consider the projection $\widetilde{O}_t(P, \lambda)$ of $O_t(P, \lambda)$ in $\mathbb{R}^{\tilde{P}}$ instead.

In this chapter we assume (P, λ) to have at least all minimal elements marked throughout, so that Definition 7.1.1 always applies.

When not just the minimal but in fact all extremal elements of P are marked, the polyhedra $O_t(P, \lambda)$ will all be bounded and hence referred to as *marked poset polytopes*. In the rest of this chapter, whenever a terminology using the word "polyhedron" is introduced, the same term with "polyhedron" replaced by "polytope" is always implicitly defined for the case of all extremal elements of (P, λ) being marked.

We will refer to the family of all $O_t(P, \lambda)$ for $t \in [0, 1]^{\tilde{P}}$ as the *continuous family of marked poset polyhedra* associated to the marked poset (P, λ). When at least one parameter t_p is in $(0, 1)$, we call $O_t(P, \lambda)$ an *intermediate marked poset polyhedron* and when all t_p are in $(0, 1)$ a *generic marked poset polyhedron*.

7.2. Transfer Maps

We will continue by proving that the polyhedra defined in Definition 7.1.1 are in fact images of the marked order polyhedron under a parametrized transfer map.

Theorem 7.2.1. *The maps $\varphi_t, \psi_t \colon \mathbb{R}^P \to \mathbb{R}^P$ defined by*

$$\varphi_t(x)_p = \begin{cases} x_p & \text{if } p \in P^*, \\ x_p - t_p \max_{q \prec p} x_q & \text{otherwise,} \end{cases}$$

$$\psi_t(y)_p = \begin{cases} y_p & \text{if } p \in P^*, \\ y_p + t_p \max_{q \prec p} \psi_t(y)_q & \text{otherwise,} \end{cases}$$

are mutually inverse. Furthermore, φ_t restricts to a piecewise-linear bijection from $O(P, \lambda)$ to $O_t(P, \lambda)$.

Note that ψ_t is well-defined, since all minimal elements in P are marked. Given $t, t' \in [0, 1]^{\tilde{P}}$ the maps ψ_t and $\varphi_{t'}$ compose to a piecewise-linear bijection

$$\theta_{t,t'} = \varphi_{t'} \circ \psi_t \colon O_t(P, \lambda) \longrightarrow O_{t'}(P, \lambda),$$

such that $\varphi_t = \theta_{0,t}$ and $\psi_t = \theta_{t,0}$. We call the maps $\theta_{t,t'}$ *transfer maps*.

Proof. We start by showing that the maps are mutually inverse. For $p \in P^*$—so in particular for p minimal in P—we immediately obtain $\psi_t(\varphi_t(x))_p = x_p$ and $\varphi_t(\psi_t(y))_p = y_p$. Hence, let p be non-minimal, unmarked and assume by induction that $\psi_t(\varphi_t(x))_q = x_q$ and $\varphi_t(\psi_t(y))_q = y_q$ hold for all $q < p$. We have

$$\psi_t(\varphi_t(x))_p = \varphi_t(x)_p + t_p \max_{q \prec p} \psi_t(\varphi_t(x))_q = \varphi_t(x)_p + t_p \max_{q \prec p} x_q = x_p$$

and

$$\varphi_t(\psi_t(y))_p = \psi_t(y)_p - t_p \max_{q \prec p} \psi_t(y)_q = y_p.$$

Hence, the maps are mutually inverse.

We now show that φ_t maps $\mathcal{O}(P,\lambda)$ into $\mathcal{O}_t(P,\lambda)$. Let $x \in \mathcal{O}(P,\lambda)$ and $y = \varphi_t(x)$. Given any saturated chain $p_0 \prec p_1 \prec p_2 \prec \cdots \prec p_r \prec p$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$ and $p \in P$, we have $y_{p_i} \leq x_{p_i} - t_{p_i} x_{p_{i-1}}$ for $i \geq 1$ by definition of φ_t. Hence,

$$\begin{aligned}
&(1-t_p)\big(t_{p_1}\cdots t_{p_r} y_{p_0} + t_{p_2}\cdots t_{p_r} y_{p_1} \qquad\qquad + \cdots + y_{p_r}\big) \\
\leq\; &(1-t_p)\big(t_{p_1}\cdots t_{p_r} x_{p_0} + t_{p_2}\cdots t_{p_r}(x_{p_1} - t_{p_1}x_{p_0}) + \cdots + (x_{p_r} - t_{p_r}x_{p_{r-1}})\big) \\
=\; &(1-t_p)x_{p_r} \leq (1-t_p)\max\nolimits_{q \prec p} x_q \leq x_p - t_p \max\nolimits_{q \prec p} x_q = y_p.
\end{aligned} \tag{7.2}$$

Thus, we have shown that $y \in \mathcal{O}_t(P,\lambda)$ as it satisfies (7.1) for all chains.

Finally, we show that ψ_t maps $\mathcal{O}_t(P,\lambda)$ into $\mathcal{O}(P,\lambda)$. Let $y \in \mathcal{O}_t(P,\lambda)$ and $x = \psi_t(y)$. Now consider any covering relation $q \prec p$. If q is marked, the inequality (7.1) given by the chain $q \prec p$ yields

$$y_p \geq (1-t_p)x_q.$$

If q is not marked, set $p_r := q$ and inductively pick p_{i-1} such that $x_{p_{i-1}} = \max_{q' \prec p_i} x_{q'}$ until ending up at a marked element p_0. Inequality (7.1) given by the chain

$$p_0 \prec p_1 \prec \cdots \prec p_r = q \prec p.$$

still yields

$$\begin{aligned}
y_p &\geq (1-t_p)\left(t_{p_1}\cdots t_{p_r} y_{p_0} + t_{p_2}\cdots t_{p_r} y_{p_1} + \cdots + y_{p_r}\right) \\
&= (1-t_p)\left(t_{p_1}\cdots t_{p_r} x_{p_0} + t_{p_2}\cdots t_{p_r}(x_{p_1} - t_{p_1}x_{p_0}) + \cdots + t_{p_r}(x_{p_{r-1}} - t_{p_{r-1}}x_{p_{r-2}}) + y_q\right) \\
&= (1-t_p)\left(t_{p_r}x_{p_{r-1}} + y_q\right) = (1-t_p)\left(t_q \max_{q' \prec q} x_{q'} + y_q\right) = (1-t_p)x_q.
\end{aligned}$$

Hence, if p is not marked, we have

$$x_p = y_p + t_p \max_{q' \prec p} x_{q'} \geq y_p + t_p x_q \geq x_q.$$

If p is marked, $t_p = 0$ so $x_p = y_p \geq x_q$. Thus, all defining conditions of $\mathcal{O}(P,\lambda)$ are satisfied. $\square$

Remark 7.2.2. In contrast to the previous transfer maps in Theorems 1.2.1 and 5.1.2, the inverse transfer map ψ_t in Theorem 7.2.1 is given using a recursion. Unfolding the recursion, we might as well express the inverse transfer map for $p \in \tilde{P}$ in the closed form

$$\psi_t(y)_p = \max_{\mathfrak{c}} \left(t_{p_1} \cdots t_{p_r} y_{p_0} + t_{p_2} \cdots t_{p_r} y_{p_1} + \cdots + y_{p_r} \right),$$

where the maximum ranges over all saturated chains $\mathfrak{c}\colon p_0 \prec p_1 \prec \cdots \prec p_r$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$ and $r \geq 0$ ending in $p_r = p$.

In examples it is often convenient to consider the projected polyhedra $\widetilde{O}_t(P, \lambda)$ in $\mathbb{R}^{\tilde{P}}$. Accordingly we define projected transfer maps.

Definition 7.2.3. Denote by $\pi_{\tilde{P}}$ the projection $\mathbb{R}^P \to \mathbb{R}^{\tilde{P}}$ and by $\iota_\lambda\colon \mathbb{R}^{\tilde{P}} \to \mathbb{R}^P$ the inclusion given by $\iota(x)_a = \lambda(a)$ for all $a \in P^*$. Define the projected transfer maps $\widetilde{\varphi}_t, \widetilde{\psi}_t\colon \mathbb{R}^{\tilde{P}} \to \mathbb{R}^{\tilde{P}}$ by $\pi_{\tilde{P}} \circ \varphi_t \circ \iota_\lambda$ and $\pi_{\tilde{P}} \circ \psi_t \circ \iota_\lambda$, respectively.

7.3. Marked Chain-Order Polyhedra

Of particular interest are the marked poset polyhedra for $t \in \{0,1\}^{\tilde{P}}$. Each such t uniquely corresponds to a partition $\tilde{P} = C \sqcup O$ such that t is the characteristic function χ_C, i.e.,

$$t_p = \chi_C(p) = \begin{cases} 1 & \text{for } p \in C, \\ 0 & \text{for } p \in O. \end{cases}$$

In this case, we denote the marked poset polyhedron $O_t(P, \lambda)$ by $O_{C,O}(P, \lambda)$ and refer to it as a *marked chain-order polyhedron*. The elements of C will be called *chain elements* and the elements of O *order elements*. Specializing Definition 7.1.1 we obtain the following description:

Proposition 7.3.1. *Given any partition $\tilde{P} = C \sqcup O$, the marked chain-order polyhedron $O_{C,O}(P, \lambda)$ is given by the following linear equations and inequalities:*

i) for each $a \in P^*$ an equation $x_a = \lambda(a)$,

ii) for each chain element $p \in C$ an inequality $0 \leq x_p$,

iii) for each saturated chain $a \prec p_1 \prec p_2 \cdots \prec p_r \prec b$ between elements $a, b \in P^* \sqcup O$ with all $p_i \in C$ and $r \geq 0$ an inequality

$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_a.$$

As before, the case $a, b \in P^*$ and $r = 0$ can be omitted.

Proof. Let $t = \chi_C \in \{0,1\}^{\tilde{P}}$ and consider a chain $p_0 \prec p_1 \prec \cdots \prec p_r \prec p$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$, $p \in P$ and $r \geq 0$. This chain yields an inequality

$$(1 - t_p)\left(t_{p_1} \cdots t_{p_r} x_{p_0} + t_{p_2} \cdots t_{p_r} x_{p_1} + \cdots + x_{p_r}\right) \leq x_p, \tag{7.3}$$

where $t_p = 0$ if $p \in P^*$.

When $p \in C$ we have $t_p = 1$ and (7.3) becomes $0 \leq x_p$. Since all minimal elements are marked, there is such a chain ending in p for each $p \in C$ and hence we get $0 \leq x_p$ for all $p \in C$ this way.

When $p \in P^* \sqcup O$, we have $t_p = 0$ and (7.3) reads

$$t_{p_1} \cdots t_{p_r} x_{p_0} + t_{p_2} \cdots t_{p_r} x_{p_1} + \cdots + x_{p_r} \leq x_p.$$

Since $t_{p_i} = \chi_C(p_i)$, letting $k \geq 0$ be maximal such that $p_k \in P^* \sqcup O$, we obtain

$$x_{p_k} + x_{p_{k+1}} + \cdots + x_{p_r} \leq x_p,$$

which is equivalent to

$$x_{p_{k+1}} + \cdots + x_{p_r} \leq x_p - x_{p_k}.$$

Conversely, consider any chain $a \prec p_1 \prec p_2 \cdots \prec p_r \prec b$ between elements $a, b \in P^* \sqcup O$ with all $p_i \in C$. If $a \in P^*$, the chain is of the type to give a defining inequality as in Definition 7.1.1 and we immediately get

$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_a.$$

If $a \in O$, extend the chain downward to a marked element to obtain a chain

$$q_0 \prec q_1 \prec \cdots \prec q_l \prec a \prec p_1 \prec \cdots \prec p_r \prec n.$$

Since a is the last element in the chain contained in $P^* \sqcup O$, the above simplification for the inequality given by this chain yields

$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_a. \qquad \square$$

Remark 7.3.2. The term "marked chain-order polytope" is used differently in [FF16]. Their definition only allows partitions $\tilde{P} = C \sqcup O$ such that there is no pair $p \in O$, $q \in C$ with $p < q$, i.e., C is an order ideal in $\tilde{P}$. We call such a partition an *admissible partition* and refer to $\mathcal{O}_{C,O}(P, \lambda)$ as an *admissible marked chain-order polyhedron (polytope)*. In this thesis, we allow arbitrary partitions for marked chain-order polyhedra instead of referring to this more general construction as "layered marked chain-order polyhedra" as suggested in [FF16].

Note that in particular we obtain the marked order polyhedron when all $t_p = 0$ and the marked chain polytope when all $t_p = 1$ and all maximal elements marked.

7.4. Integrality, Integral Closure and Unimodular Equivalence

When (P,λ) comes with an integral marking, so $\lambda(a) \in \mathbb{Z}$ for all $a \in P^*$, the authors of [ABS11] already showed that $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$ are Ehrhart equivalent lattice polytopes in the bounded case. In [Fou16] Fourier gives a necessary and sufficient condition for $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$ to be unimodular equivalent and together with Fang this was generalized to admissible marked chain-order polytopes in [FF16]. They also show that all the admissible marked chain-order polytopes are integrally closed lattice polytopes.

In this section we assume integral markings containing all extremal elements throughout and generalize the results above to non-admissible partitions. Let us start by showing that under these assumptions all the marked chain-order polytopes are lattice polytopes.

Proposition 7.4.1. *For $t \in \{0,1\}^{\tilde{P}}$ the marked chain-order polytope $\mathcal{O}_t(P,\lambda)$ is a lattice polytope.*

Proof. When $t \in \{0,1\}^{\tilde{P}}$, the transfer map $\varphi_t\colon \mathcal{O}(P,\lambda) \to \mathcal{O}_t(P,\lambda)$ is piecewise-unimodular. In particular, it maps lattice points to lattice points. When $\operatorname{im}(\lambda) \subseteq \mathbb{Z}$, we know that the marked poset polytope $\mathcal{O}(P,\lambda)$ is a lattice polytope. In fact, considering the subdivision into products of simplices from Section 6.3, each cell $F_{\mathcal{I}}$ is a lattice polytope as the image of the lattice polytope $\mathcal{O}(P_{\mathcal{I}},\lambda_{\mathcal{I}})$ under the lattice-preserving maps $\mathcal{O}(P_{\mathcal{I}},\lambda_{\mathcal{I}}) \to \mathcal{O}(P/\pi_{\mathcal{I}},\lambda/\pi_{\mathcal{I}}) \to \mathcal{O}(P,\lambda)$. Hence, all vertices in the subdivision of $\mathcal{O}(P,\lambda)$ are lattice points. Applying φ_t we obtain a subdivision of $\mathcal{O}_t(P,\lambda)$ with still all vertices being lattice points. Since the vertices of $\mathcal{O}_t(P,\lambda)$ have to appear as vertices in the subdivision, we conclude that $\mathcal{O}_t(P,\lambda)$ is a lattice polytope. □

Corollary 7.4.2. *The polytopes $\mathcal{O}_t(P,\lambda)$ for $t \in \{0,1\}^{\tilde{P}}$ are all Ehrhart equivalent.*

Proof. This is an immediate consequence of the transfer map being piecewise-unimodular and bijective. □

Proposition 7.4.3. *The polytopes $\mathcal{O}_t(P,\lambda)$ for $t \in \{0,1\}^{\tilde{P}}$ are all integrally closed.*

Proof. We will reduce $\mathcal{O}_t(P,\lambda)$ being integrally closed to the fact that unimodular simplices are integrally closed. Since we have a polyhedral subdivision of $\mathcal{O}_t(P,\lambda)$ into cells $\varphi_t(F_{\mathcal{I}})$, it suffices to show that each cell is integrally closed. On the cell $F_{\mathcal{I}}$, the transfer map φ_t is the restriction of a unimodular map $\mathbb{R}^P \to \mathbb{R}^P$. Hence, it is enough to show that each cell $F_{\mathcal{I}}$ in the subdivision of $\mathcal{O}(P,\lambda)$ is integrally closed. In fact, since $F_{\mathcal{I}}$ is the image of $\mathcal{O}(P_{\mathcal{I}},\lambda_{\mathcal{I}})$ under a map that identifies the affine lattices spanned by the polytopes, it suffices to show that marked order polytopes associated to linear posets with integral markings are integrally closed. Since these are products of marked order polytopes associated to linear posets with integral markings only at the minimum and maximum, it is enough to show that these are integrally closed. However, these are just integral dilations of unimodular simplices. □

Having identified a family of Ehrhart equivalent integrally closed lattice polytopes, we now move on to the question of unimodular equivalences within this family.

Given a marked poset (P, λ), we call an element $p \in \tilde{P}$ a *star element* if p is covered by at least two elements and there are at least two different saturated chains from a marked element to p. This notion has been used in [FF16] to study unimodular equivalence of admissible marked chain-order polytopes.

A finer notion we will use in our discussion is that of a *chain-order star element* with respect to a partition $C \sqcup O$ of $\tilde{P}$.

Definition 7.4.4. Given a partition $\tilde{P} = C \sqcup O$, an element $q \in O$ is called a *chain-order star element* if there are at least two different saturated chains $s \prec q_1 \prec \cdots \prec q_k \prec q$ with $s \in P^* \sqcup O$ and all $q_i \in C$ and there are at least two different saturated chains $q \prec q_1 \prec \cdots \prec q_k \prec s$ with $s \in P^* \sqcup O$ and all $q_i \in C$.

Note that if $C \sqcup O$ and $(C \sqcup \{q\}) \sqcup (O \setminus \{q\})$ are admissible partitions for some $q \in O$, i.e., C is an order ideal in $\tilde{P}$ and q is minimal in O, then q is an (O, C)-star element if and only it is a star element in the sense of [FF16].

Proposition 7.4.5[1]**.** *Let $C \sqcup O$ be a partition of $\tilde{P}$ and $q \in O$ not a chain-order star element. Let $O' = O \setminus \{q\}$ and $C' = C \sqcup \{q\}$, then $\mathcal{O}_{C,O}(P, \lambda)$ and $\mathcal{O}_{C',O'}(P, \lambda)$ are unimodular equivalent.*

Proof. We have to consider the following two cases.

(1) There is exactly one saturated chain $s \prec q_1 \prec \cdots \prec q_k \prec q$ with $s \in P^* \sqcup O$ and all $q_i \in C$. Define the unimodular map $\Psi\colon \mathbb{R}^P \to \mathbb{R}^P$ by letting

$$\Psi(x)_p = \begin{cases} x_q - x_s - \cdots - x_{q_k} & \text{if } p = q, \\ x_p & \text{otherwise.} \end{cases}$$

We claim that $\Psi(\mathcal{O}_{C,O}(P, \lambda)) = \mathcal{O}_{C',O'}(P, \lambda)$. The defining inequalities of $\mathcal{O}_{C,O}(P, \lambda)$ involving x_q are the following:

i) for each saturated chain $q \prec p_1 \prec p_2 \cdots \prec p_r \prec b$ with $b \in P^* \sqcup O$, $p_i \in C$ and $r \geq 0$ an inequality
$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_q,$$

ii) the inequality
$$x_{q_1} + \cdots + x_{q_k} \leq x_q - x_s.$$

Applying Ψ, these translate to

i) for each saturated chain $q \prec p_1 \prec p_2 \cdots \prec p_r \prec b$ with $b \in P^* \sqcup O$, $p_i \in C$ and $r \geq 0$ an inequality
$$x_{q_1} + \cdots + x_{q_k} + x_q + x_{p_1} + \cdots + x_{p_r} \leq x_b - x_s, \tag{7.4}$$

[1] In the proof of Proposition 7.4.5 we do not use integrality of the marking or P^* containing all extremal elements. Hence, the statement still holds when P^* only contains all minimal elements and the marking is not integral.

ii) the inequality

$$0 \leq x_q. \tag{7.5}$$

These are exactly the defining properties of $\mathcal{O}_{C',O'}(P,\lambda)$ involving x_q: the saturated chains $a \prec p_1 \prec p_2 \cdots \prec p_r \prec b$ with $a, b \in P^* \sqcup O'$ and all $p_i \in C'$ involving q at index k, must have $a = s$ and $p_i = q_i$ for $i \leq k$, so they yield the inequalities in (7.4). The inequality in 7.5 is what we get from $q \in C'$.

(2) There is exactly one saturated chain $q \prec q_1 \prec \cdots \prec q_k \prec s$ with $s \in P^* \sqcup O$ and all $q_i \in C$. An analogous argument as above shows that in this case the map $\Psi\colon \mathbb{R}^P \to \mathbb{R}^P$ defined by

$$\Psi(x)_p = \begin{cases} x_s - x_q - \cdots - x_{q_k} & \text{if } p = q, \\ x_p & \text{otherwise} \end{cases}$$

yields an unimodular equivalence of $\mathcal{O}_{C,O}(P,\lambda)$ and $\mathcal{O}_{C',O'}(P,\lambda)$. In this case every chain involving q that is relevant for $\mathcal{O}_{C',O'}(P,\lambda)$ must end in $\cdots \prec q \prec q_1 \prec \cdots \prec q_k \prec s$. □

7.5. Combinatorial Types

Having studied the marked chain-order polyhedra obtained for $t \in \{0,1\}^{\tilde{P}}$, we will now consider intermediate and generic $t \in [0,1]^{\tilde{P}}$. In this section we show that the combinatorial type of $\mathcal{O}_t(P,\lambda)$ stays constant when t varies inside the relative interior of a face of the parametrizing hypercube $[0,1]^{\tilde{P}}$.

The idea is to translate whether a defining inequality of $\mathcal{O}_t(P,\lambda)$ is satisfied for some $\varphi_t(x)$ with equality into a condition on x depending only on the face of $[0,1]^{\tilde{P}}$ the parameter t is contained in.

The key ingredient will be a relation on P depending on $x \in \mathcal{O}(P,\lambda)$.

Definition 7.5.1. Given $x \in \mathcal{O}(P,\lambda)$ let $\dashv_x$ be the relation on P given by

$$q \dashv_x p \quad\Longleftrightarrow\quad q \prec p \quad\text{and}\quad x_q = \max_{q' \prec p} x_{q'}.$$

Proposition 7.5.2. *Let $x \in \mathcal{O}(P,\lambda)$. Given a saturated chain $p_0 \prec p_1 \prec \cdots \prec p_r \prec p$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$ and $p \in P$, the corresponding defining inequality (7.1) is satisfied with equality by $\varphi_t(x)$ if and only if one of the following is true:*

i) $t_p = 1$ and $x_p = \max_{q \prec p} x_q$,

ii) $t_p < 1$ and $x_p = x_{p_r}$ as well as

$$p_{k-1} \dashv_x p_k \dashv_x \cdots \dashv_x p_r,$$

where $k \geq 1$ is the smallest index such that $t_{p_i} > 0$ for all $i \geq k$.

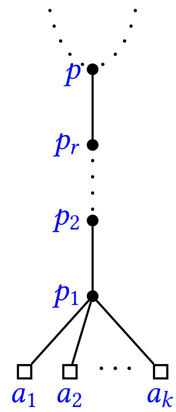


Figure 7.1.: The condition on $p \in P$ in Proposition 7.5.4.

Proof. Let $y = \varphi_t(x) \in \mathcal{O}_t(P,\lambda)$. When $t_p = 1$, the inequality (7.1) for y reads $0 \leq y_p$ which is equivalent to

$$\max_{q \prec p} x_q \leq x_p.$$

When $t_p < 1$ we may simplify (7.1) to

$$(1 - t_p)\left(t_{p_k} \cdots t_{p_r} y_{p_{k-1}} + \cdots + y_{p_r}\right) \leq y_p, \tag{7.6}$$

where $k \geq 1$ is the smallest index such that $t_{p_i} > 0$ for all $i \geq k$. The coefficients on the left hand side of (7.6) are all strictly positive and inspecting the estimation in (7.2) yields equality if and only if

$$\begin{aligned} x_{p_{i-1}} &= \max_{q \prec p_i} x_q \quad \text{for } i \geq k \quad \text{and} \\ x_{p_r} &= x_p. \end{aligned}$$

$\square$

Since the conditions of Proposition 7.5.2 only depend on each t_p being 0, 1 or in between, we obtain the following corollary.

Corollary 7.5.3. *The combinatorial type of $\mathcal{O}_t(P,\lambda)$ is constant along relative interiors of faces of the parametrizing hypercube $[0,1]^{\tilde{P}}$.*

Furthermore, some of the t_p do not affect the combinatorial type at all:

Proposition 7.5.4. *The combinatorial type of $\mathcal{O}_t(P,\lambda)$ does not depend on t_p for $p \in \tilde{P}$ such that there is a (unique) chain $p_1 \prec p_2 \prec \cdots \prec p_r \prec p_{r+1} = p$, where all $p_i \in \tilde{P}$, p_1 covers only marked elements and p_i is the only element covered by p_{i+1} for $i = 1,\ldots,r$.*

The condition on p in Proposition 7.5.4 is equivalent the subposet of all elements below p being of the form depicted in Figure 7.1.

Proof. Let $t, t' \in [0,1]^{\tilde{P}}$ such that $t_q = t'_q$ for $q \neq p$ and consider the transfer map $\theta_{t,t'}$. For $y \in \mathcal{O}_t(P,\lambda)$ and $q \neq p$ we have

$$\theta_{t,t'}(y)_q = \varphi_{t'}(\psi_t(y))_q = \psi_t(y)_q - t'_q \max_{q'<q} \psi_t(y)_{q'} = \psi_t(y)_q - t_q \max_{q'<q} \psi_t(y)_{q'} = y_q.$$

For the p-coordinate note that the given condition means walking down from p in the Hasse diagram of (P,λ) we are forced to walk along $p \succ p_r \cdots \succ p_1$ and p_1 covers only marked elements. Hence, we have

$$\begin{aligned}
\theta_{t,t'}(y)_p &= \psi_t(y)_p - t'_p \max_{q<p} \psi_t(y)_q = \psi_t(y)_p - t'_p \psi_t(y)_{p_r} \\
&= y_p + t_p y_{p_r} + t_p t_{p_r} y_{p_{r-1}} + t_p t_{p_{r-1}} t_{p_r} y_{p_{r-2}} \cdots + t_p t_{p_1} t_{p_2} \cdots t_{p_r} \max_{a<p_1} \lambda(a) \\
&\qquad - t'_p \left(y_{p_r} + t_{p_r} y_{p_{r-1}} + t_{p_{r-1}} t_{p_r} y_{p_{r-2}} \cdots + t_{p_1} t_{p_2} \cdots t_{p_r} \max_{a<p_1} \lambda(a) \right) \\
&= y_p + (t_p - t'_p) \left(y_{p_r} + t_{p_r} y_{p_{r-1}} + t_{p_{r-1}} t_{p_r} y_{p_{r-2}} \cdots + t_{p_1} t_{p_2} \cdots t_{p_r} \max_{a<p_1} \lambda(a) \right).
\end{aligned}$$

We conclude that $\theta_{t,t'}$ restricts to an affine isomorphism $\mathcal{O}_t(P,\lambda) \xrightarrow{\sim} \mathcal{O}_{t'}(P,\lambda)$. □

Corollary 7.5.5. *Letting k be the number of elements in $\tilde{P}$ not satisfying the condition in Proposition 7.5.4, there are at most 3^k different combinatorial types of marked poset polyhedra associated to a marked poset (P,λ).* □

7.6. Tropical Arrangements and Subdivisions

As discussed in Section 5.2 and Section 6.3, the marked order polyhedron $\mathcal{O}(P,\lambda)$ comes with a subdivision $\mathcal{S}$ into products of simplices and simplicial cones. Since the transfer map φ as well as the parametrized transfer map φ_t is linear on each cell of $\mathcal{S}$, we have a transferred subdivision $\mathcal{S}_t$ of $\mathcal{O}_t(P,\lambda)$ for all $t \in [0,1]^{\tilde{P}}$.

In this section we introduce a coarsening of $\mathcal{S}$ into linearity regions of φ, obtained by intersecting $\mathcal{O}(P,\lambda)$ with the cells in a tropical hyperplane arrangement determined by (P,λ). Our main reason to consider this subdivision is a result in Section 7.8, where we will show that the vertices of generic marked poset polyhedra are given by the vertices in this subdivision and hence can be obtained by first subdividing the marked order polyhedron to then transfer the vertices in the subdivision. The notation we use here is close to [FR15], where the combinatorics of tropical hyperplane arrangements are discussed in detail.

7.6.1. Tropical Hyperplane Arrangements

In tropical geometry, the usual ring structure $(\mathbb{R}, +, \cdot)$ we use for Euclidean geometry is replaced by the *tropical semiring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where $a \oplus b = \max(a,b)$, $a \odot b = a + b$

and $-\infty$ is the identity with respect to $\oplus$. Hence, a *tropical polynomial* is a convex piecewise-linear function in ordinary terms:

$$\bigoplus_{a \in \mathbb{N}^n} c_a \odot x_1^{\odot a_1} \odot \cdots \odot x_n^{\odot a_n} = \max \left\{ a_1 x_1 + \cdots + a_n x_n + c_a : a \in \mathbb{N}^n \right\}.$$

Given a *tropical linear form*

$$\alpha = \bigoplus_{i=1}^{n} c_i \odot x_i = \max \left\{ x_i + c_i : i = 1, \ldots, n \right\},$$

where some—but not all—coefficients are allowed to be $-\infty$, one defines a *tropical hyperplane* H_α consisting of all $x \in \mathbb{R}^n$ such that α is non-differentiable at x or equivalently, the maximum in $\alpha(x)$ is attained at least twice.

We may pick some of the coefficients c_i to be $-\infty$ to obtain tropical linear forms only involving some of the coordinates. For example when $n = 3$ we could have

$$\alpha = (1 \odot x_1) \oplus (3 \odot x_2) = \max\{1 + x_1, 3 + x_2\} = (1 \odot x_1) \oplus (3 \odot x_2) \oplus (-\infty \odot x_3)$$

and the tropical hyperplane H_α would just be the usual hyperplane $1 + x_1 = 3 + x_2$. Given a tropical hyperplane, one obtains a polyhedral subdivision of $\mathbb{R}^n$ with facets the linearity regions of α and the skeleton of codimension 1 being H_α as follows: for a tropical hyperplane $H = H_\alpha$ in $\mathbb{R}^n$ define the *support* $\operatorname{supp}(H)$ as the set of all $i \in [n]$ such that the coefficient c_i is different from $-\infty$ in α. For any non-empty subset $L \subseteq \operatorname{supp}(H)$ we have a cell

$$F_L(H) = \left\{ x \in \mathbb{R}^n : c_l + x_l = \max_{i \in \operatorname{supp}(H)} (x_i + c_i) \text{ for all } l \in L \right\}.$$

The facets $F_{\{l\}}$ for $l \in \operatorname{supp}(H)$ are the linearity regions of α and the cells F_L for $|L| \geq 2$ form a subdivision of H_α. Given any $x \in \mathbb{R}^n$, we define its *signature* $\operatorname{sig}_H(x)$ as the unique $L \subseteq \operatorname{supp}(H)$ such that x is in the relative interior of F_L. Equivalently, the signature of x is the set of indices achieving the maximum in $\alpha(x)$,

$$\operatorname{sig}_H(x) = \operatorname*{argmax}_{i \in \operatorname{supp}(H)} (x_i + c_i).$$

Using this terminology, we may also describe $F_L(H)$ as the set of all points $x \in \mathbb{R}^n$ with $L \subseteq \operatorname{sig}_H(x)$

Now let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a *tropical hyperplane arrangement*, that is, each H_i is a tropical hyperplane $H_{\alpha_i} \subseteq \mathbb{R}^n$ for a tropical linear form α_i. The common refinement $\mathcal{T}(\mathcal{H})$ of the polyhedral subdivision of $H_1, H_2, \ldots, H_m$ gives a polyhedral subdivision of $\mathbb{R}^n$ whose facets are the largest regions on which all α_i are linear and whose $(n-1)$-skeleton is a subdivision of $\bigcup \mathcal{H}$. To each $x \in \mathbb{R}^n$ we associate the *tropical covector* $\operatorname{tc}(x)\colon [m] \to 2^{[n]}$ recording the signatures with respect to all hyperplanes, that is

$$\operatorname{tc}(x) = \left(\operatorname{sig}_{H_1}(x), \operatorname{sig}_{H_2}(x), \ldots, \operatorname{sig}_{H_m}(x) \right).$$

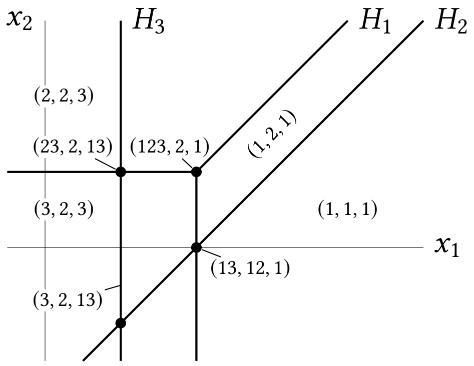


Figure 7.2.: The tropical hyperplane arrangement from Example 7.6.1 sliced at $x_3 = 0$ with some of the appearing tropical covectors listed.

Hence, the cells of $\mathcal{T}(\mathcal{H})$ are enumerated by the appearing tropical covectors when x varies over all points in $\mathbb{R}^n$. The set of all these tropical covectors is called the *combinatorial type* of $\mathcal{H}$ and denoted $\mathrm{TC}(\mathcal{H})$. For each $\tau \in \mathrm{TC}(\mathcal{H})$ the corresponding cell is given by

$$F_\tau = \bigcap_{i=1}^{m} F_{\tau_i}(H_i)$$

and its relative interior consists of all $x \in \mathbb{R}^n$ such that $\mathrm{tc}(x) = \tau$.

To digest all these definitions, let us look at a small example before using the introduced terminology to define a subdivision of marked poset polyhedra.

Example 7.6.1. Let $n = 3$ and consider the following tropical linear forms:

$$\begin{aligned}
\alpha_1 &= ((-2) \odot x_1) \oplus ((-1) \odot x_2) \oplus (0 \odot x_3) &&= \max\{x_1 - 2, x_2 - 1, x_3\},\\
\alpha_2 &= ((-2) \odot x_1) \oplus (0 \odot x_2) \oplus ((-\infty) \odot x_3) &&= \max\{x_1 - 2, x_2\},\\
\alpha_3 &= ((-1) \odot x_1) \oplus ((-\infty) \odot x_2) \oplus (0 \odot x_3) &&= \max\{x_1 - 1, x_3\}.
\end{aligned}$$

Let $\mathcal{H} = \{H_1, H_2, H_3\}$ be the tropical hyperplane arrangement with H_i given by α_i for $i = 1, 2, 3$. The supports of the three hyperplanes are $\mathrm{supp}(H_1) = \{1, 2, 3\}$, $\mathrm{supp}(H_2) = \{1, 2\}$ and $\mathrm{supp}(H_3) = \{1, 3\}$. Since tropical hyperplanes are invariant under translations along the all-one vector $(1, 1, \dots, 1) \in \mathbb{R}^n$, we obtain a faithful picture of the subdivision $\mathcal{T}(\mathcal{H})$ by just looking at the slice $x_n = 0$. This is done in Figure 7.2 for the example at hand with some of the appearing tropical covectors listed. ◊

7.6.2. The Tropical Subdivision

We are now ready to introduce the *tropical subdivision* of marked poset polyhedra. As before, let (P, λ) be a marked poset with at least all minimal elements marked. The transfer maps φ_t of Theorem 7.2.1 give rise to the tropical linear forms

$$\alpha_p = \max_{q \prec p} x_q = \bigoplus_{q \prec p} x_q \quad \text{for } p \in \tilde{P}.$$

When p is not covering at least two elements, the tropical linear form α_p has just one term and defines an empty tropical hyperplane since the maximum can never be achieved twice. Hence, let R denote the set of all $p \in \tilde{P}$ covering at least two elements and define a tropical hyperplane arrangement $\mathcal{H}(P, \lambda)$ in $\mathbb{R}^P$ with tropical hyperplanes $H_p = H_{\alpha_p}$ for all $p \in R$. By construction, the facets of $\mathcal{T}(\mathcal{H}(P, \lambda))$ are the linearity regions of φ_t for $t \in (0, 1]^{\tilde{P}}$.

The reason this subdivision will help study the combinatorics of marked poset polytopes is the following: by Proposition 7.5.2 the combinatorics of $\mathcal{O}_t(P, \lambda)$ can be determined by pulling points back to $\mathcal{O}(P, \lambda)$ and looking at the relation $\dashv_x$. But for $r \in R$ and $p \in P$ we have $p \dashv_x r$ if and only if $p \in \operatorname{tc}(x)_r$, so the information encoded in $\dashv_x$ is equivalent to knowing the minimal cell of $\mathcal{T}(\mathcal{H}(P, \lambda))$ containing x.

Using this tropical hyperplane arrangement, we can define a polyhedral subdivision of $\mathcal{O}(P, \lambda)$.

Definition 7.6.2. Let $\mathcal{T}(\mathcal{H}(P, \lambda))$ be the polyhedral subdivision of $\mathbb{R}^P$ associated to the marked poset (P, λ). The *tropical subdivision* $\mathcal{T}(P, \lambda)$of $\mathcal{O}(P, \lambda)$ is given by the intersection of faces of $\mathcal{O}(P, \lambda)$ with the faces of $\mathcal{T}(\mathcal{H}(P, \lambda))$:

$$\mathcal{T}(P, \lambda) = \{\, F \cap G \mid F \in \mathcal{F}(\mathcal{O}(P, \lambda)), G \in \mathcal{T}(\mathcal{H}(P, \lambda)) \,\} .$$

For $t \in [0, 1]^{\tilde{P}}$ define the tropical subdivision of $\mathcal{O}_t(P, \lambda)$ as

$$\mathcal{T}_t(P, \lambda) = \{\varphi_t(Q) \mid Q \in \mathcal{T}(P, \lambda)\} .$$

Note that $\mathcal{T}_t(P, \lambda)$ is polyhedral subdivision of $\mathcal{O}_t(P, \lambda)$ since φ_t is linear on each $G \in \mathcal{T}(P, \lambda)$ by construction. In particular, $\mathcal{T}_t(P, \lambda)$ is a coarsening of the subdivision $\mathcal{S}_t$ into products of simplices and simplicial cones.

7.7. Continuous Degenerations

By Corollary 7.5.3, the combinatorial type of $\mathcal{O}_t(P, \lambda)$ is constant along the relative interiors of the faces of the hypercube $[0, 1]^{\tilde{P}}$. Assume we are looking at some $\mathcal{O}_t(P, \lambda)$ with $t_p \in (0, 1)$ for a fixed p. Continuously changing t_p to 0 or 1, the combinatorial type of the polyhedron stays constant until it possibly jumps, when reaching 0 or 1, respectively. This motivates to think of the two polyhedra for $t_p = 0$ and $t_p = 1$ as continuous degenerations of the polyhedron for any $t_p \in (0, 1)$.

In this section we formally introduce a concept of continuous degenerations of polyhedra to then apply it to marked poset polyhedra.

7.7.1. Continuous Degenerations of Polyhedra

We start by defining continuous deformations of polyhedra, mimicking the situation in the continuous family.

Definition 7.7.1. Given two polyhedra Q_0 and Q_1 in $\mathbb{R}^n$, a *continuous deformation* from Q_0 to Q_1 consists of the following data:

i) A continuous map $\rho\colon Q_0 \times [0,1] \longrightarrow \mathbb{R}^n$, such that each $\rho_t = \rho(-,t)$ is an embedding, ρ_0 is the identical embedding of Q_0 and the image of ρ_1 is Q_1.

ii) Finitely many continuous functions $f^1, f^2, \dots, f^k\colon \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}$ such that for all i and t the maps $f^i_t = f^i(-,t)\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are affine linear forms and satisfy

$$\rho_t(Q_0) = \left\{ x \in \mathbb{R}^n \,\middle|\, f^i_t(x) \geq 0 \text{ for all } i \right\}.$$

Hence, the images $\rho_t(Q_0)$ are all polyhedra and we write Q_t for $\rho_t(Q_0)$ and say $(Q_t)_{t\in[0,1]}$ is a continuous deformation when the accompanying maps ρ and f^i are clear from the context.

Note that a continuous deformation of polyhedra as defined here consists of both a map moving the points around *and* a continuous description in terms of inequalities for all $t \in [0,1]$.

Definition 7.7.2. A continuous deformation $(Q_t)_{t\in[0,1]}$ as in Definition 7.7.1 is called a *continuous degeneration* if for all $x \in Q_0$, $t < 1$ and $i = 1, \dots, k$ we have $f^i_t(\rho_t(x)) = 0$ if and only if $f^i_0(x) = 0$.

From this definition we immediately obtain the following.

Proposition 7.7.3. *If $(Q_t)_{t\in[0,1]}$ is a continuous degeneration, the polyhedra Q_t for $t < 1$ are all combinatorially equivalent and ρ_t preserves faces and their incidence structure.*

Proof. Let the data of the continuous degeneration be given as in Definition 7.7.1. For $y \in Q_t$ denote by $\mathfrak{I}_t(y)$ the set of all $i \in [k]$ such that $f^i_t(y) = 0$. The set of all $\mathfrak{I}_t(y)$ for $y \in Q_t$ ordered by reverse inclusion is isomorphic to $\mathcal{F}(Q_t) \setminus \{\varnothing\}$ since relative interiors of faces of Q_t correspond to regions of constant $\mathfrak{I}_t$.

Since for all $x \in Q_0$, $t < 1$ and $i = 1, \dots, k$ we have $f^i_t(\rho_t(x)) = 0$ if and only if $f^i_0(x) = 0$, the sets $\mathfrak{I}_t(\rho_t(x))$ are fixed for $t < 1$ and hence ρ_t preserves the face structure. $\square$

We continue by illustrating the definition of continuous degenerations in an example before proceeding with the general theory.

Example 7.7.4. For $t \in [0,1]$ let $Q_t \subseteq \mathbb{R}^2$ be the polytope defined by the inequalities $0 \leq x_1 \leq 2$, $0 \leq x_2$ as well as

$$\begin{aligned} x_2 &\leq (1-t)x_1 + 1, \quad \text{and} \\ x_2 &\leq (1-t)(2-x_1) + 1. \end{aligned}$$

For $t = 0$, $t = \frac{1}{2}$ and $t = 1$ we have illustrated the polytope in Figure 7.3. Together with the map $\rho_t\colon Q_0 \longrightarrow \mathbb{R}^2$ given by $\rho_t(x)_1 = x_1$ for all t and

$$\rho_t(x)_2 = \begin{cases} x_2\, \frac{(1-t)x_1+1}{x_1+1} & \text{for } x_1 \leq 1, \\ x_2\, \frac{(1-t)(2-x_1)+1}{(2-x_1)+1} & \text{for } x_1 \geq 1 \end{cases}$$

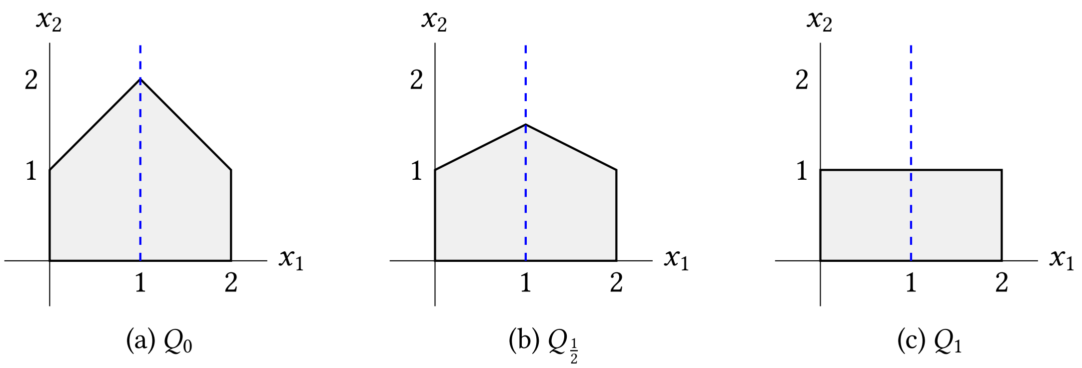


Figure 7.3.: The polytopes in the continuous degeneration from Example 7.7.4 for $t = 0$, $t = \frac{1}{2}$ and $t = 1$.

we obtain a continuous degeneration. Starting from the pentagon in Figure 7.3a at $t = 0$ we see increasingly compressed pentagons with the two top edges becoming more flat-angled until ending up with the rectangle in Figure 7.3c at $t = 1$. The map ρ_t just scales the x_2 coordinates accordingly, preserving the face-structure for $t < 1$. $\diamondsuit$

The key result on continuous degenerations that will allow conclusions on face structure of degenerations is that during a continuous degeneration, relative interiors of faces always map into relative interiors of faces. In other words, continuous degenerations can not "fold" faces of Q_0 so they split into different faces of Q_1, but only "straighten" some adjacent faces of Q_0 to become one face of Q_1.

Proposition 7.7.5. *Let $(Q_t)_{t\in[0,1]}$ be a continuous degeneration of polyhedra. Whenever F is a face of Q_0, there is a unique face G of Q_1 such that*

$$\rho_1(\operatorname{relint} F) \subseteq \operatorname{relint} G.$$

Proof. As in the previous proof, let $\mathfrak{I}_t(y)$ denote the set of indices $i \in [k]$ such that $f_t^i(y) = 0$. Using these incidence sets we may rephrase the proposition as follows: whenever $x, x' \in Q_0$ satisfy $\mathfrak{I}_0(x) = \mathfrak{I}_0(x)$, they also satisfy $\mathfrak{I}_1(\rho_1(x)) = \mathfrak{I}_1(\rho_1(x'))$.

Let F be the face of Q_0 having both x and x' in its relative interior and assume there exists a $j \in \mathfrak{I}_1(\rho_1(x)) \setminus \mathfrak{I}_1(\rho_1(x'))$ for sake of contradiction. Hence, we have $f_1^j(\rho_1(x)) = 0$ while $f_1^j(\rho_1(x')) > 0$. Let d denote the dimension of F then $\operatorname{relint} F$ is a manifold of dimension d. Since ρ_1 is an embedding, $\rho_1(\operatorname{relint} F)$ is a manifold of dimension d as well. Since the affine hull of $\rho_t(\operatorname{relint} F)$ is of dimension d for all $t < 1$, we conclude that the affine hull of $\rho_1(\operatorname{relint} F)$ is of dimension at most d. To see this, take any $d + 1$ points $y_0, \ldots, y_d$ in $\rho_1(\operatorname{relint} F)$. Their images $\rho_t(\rho_1^{-1}(y_0)), \ldots, \rho_t(\rho_1^{-1}(y_d))$ in $\rho_t(\operatorname{relint} F)$ are affinely dependent for $t < 1$, so they have to be affinely dependent for $t = 1$ as well by the continuity of ρ in t.

But as $\rho_1(\operatorname{relint} F)$ is a manifold of dimension d, we conclude that its affine hull has dimension exactly d and $\rho_1(\operatorname{relint} F)$ is an open subset of its affine hull. Given that both $\rho_1(x)$ and $\rho_2(x')$ are points in $\rho_1(\operatorname{relint} F)$, we conclude that there exists an $\varepsilon > 0$ such that the point

$$z = \rho_1(x) + \varepsilon(\rho_1(x) - \rho_1(x'))$$

is still contained in $\rho_1(\operatorname{relint} F)$. In particular, $z \in Q_1$. However, since f_1^j is an affine linear form, we have

$$f_1^j(z) = (1+\varepsilon) f_1^j(\rho_1(x)) - \varepsilon f_1^j(\rho_1(x')) < 0.$$

This contradicts $z \in Q_1$, which finishes the proof. $\square$

The consequence of Proposition 7.7.5 is that continuous degenerations induce maps between face lattices.

Corollary 7.7.6. *When $(Q_t)_{t\in[0,1]}$ is a continuous degeneration of polyhedra, we have a surjective order-preserving map of face lattices*

$$\operatorname{dg}\colon \mathcal{F}(Q_0) \longrightarrow \mathcal{F}(Q_1)$$

determined by the property

$$\rho_1(\operatorname{relint} F) \subseteq \operatorname{relint} \operatorname{dg}(F).$$

for non-empty F and $\operatorname{dg}(\varnothing) = \varnothing$. Furthermore, the map satisfies $\dim(\operatorname{dg}(F)) \geq \dim F$ for all $F \in \mathcal{F}(Q_0)$. $\square$

We will refer to the map in Corollary 7.7.6 as the *degeneration map*. Before coming back to marked poset polyhedra, we finish with a result on the f-vectors of continuous degenerations.

Proposition 7.7.7. *Let $(Q_t)_{t\in[0,1]}$ be a continuous degeneration of polyhedra. We have $f_i(Q_1) \leq f_i(Q_0)$ for all i.*

Proof. Let G be an i-dimensional face of Q_1. We claim that there is at least one i-dimensional face F of Q_0 such that $\operatorname{dg}(F) = G$. Since every polyhedron is the disjoint union of the relative interiors of its faces and ρ_1 is a bijection, we have

$$\operatorname{relint} G = \bigsqcup_{F\in\operatorname{dg}^{-1}(G)} \rho_1(\operatorname{relint} F).$$

Since $\operatorname{relint} G$ is a manifold of dimension $\dim G$ and each $\rho_1(\operatorname{relint} F)$ is a manifold of dimension $\dim F \leq \dim G$, there has to be at least one $F \in \operatorname{dg}^{-1}(G)$ of the same dimension as G. $\square$

7.7.2. Continuous Degenerations in the Continuous Family

We are now ready to apply the concept of continuous degenerations to the continuous family of marked poset polyhedra. Let us first identify for which pairs of parameters $u, u' \in [0,1]^{\tilde{P}}$ we expect to have a continuous degeneration from $\mathcal{O}_u(P,\lambda)$ to $\mathcal{O}_{u'}(P,\lambda)$ and then specify the deformation precisely.

Definition 7.7.8. Let $u \in [0,1]^{\tilde{P}}$ and let $I \subseteq \tilde{P}$ be the set of indices p, such that $u_p \in \{0,1\}$. Any $u' \in [0,1]^{\tilde{P}}$ such that $u'_p = u_p$ for $p \in I$ is called a *degeneration* of u.

Proposition 7.7.9. *Let u' be a degeneration of u. The map*

$$\begin{aligned} \rho\colon \mathcal{O}_u(P,\lambda) \times [0,1] &\longrightarrow \mathbb{R}^P, \\ (x,\xi) &\longmapsto \theta_{u,\xi u'+(1-\xi)u}(x) \end{aligned}$$

is a continuous degeneration with the accompanying affine linear forms given by the equations and inequalities in Definition 7.1.1 for $t = \xi u' + (1-\xi)u$.

Proof. The map ρ together with the affine linear forms given by Definition 7.1.1 is a continuous deformation by Theorem 7.2.1. The fact that ρ is a continuous degeneration follows from Proposition 7.5.2. □

Now the machinery of continuous degenerations immediately yields degeneration maps and results on the f-vectors of marked poset polyhedra.

Corollary 7.7.10. *Let $u, u' \in [0,1]^{\tilde{P}}$ such that u' is a degeneration of u. The continuous degeneration in Proposition 7.7.9 yields a degeneration map $\mathrm{dg}_{u,u'}\colon \mathcal{O}_u(P,\lambda) \to \mathcal{O}_{u'}(P,\lambda)$ in the sense of Corollary 7.7.6. In particular, the f-vectors satisfy*

$$f_i(\mathcal{O}_{u'}(P,\lambda)) \leq f_i(\mathcal{O}_u(P,\lambda)) \quad \textit{for all } i.$$

Furthermore, given a degeneration u'' of u', the degeneration maps satisfy

$$\mathrm{dg}_{u,u''} = \mathrm{dg}_{u',u''} \circ \mathrm{dg}_{u,u'}\,.$$

Proof. After applying Proposition 7.7.5, Corollary 7.7.6 and Proposition 7.7.7 to the situation at hand, all that remains to be proven is the statement about compositions of degeneration maps. This is an immediate consequence of $\theta_{u,u''} = \theta_{u',u''} \circ \theta_{u,u'}$. □

7.8. Vertices in the Generic Case

Using the tropical subdivision from Section 7.6 and the concept of continuous degenerations from Section 7.7, we are ready to to prove a theorem describing the vertices of generic marked poset polyhedra.

Theorem 7.8.1. *The vertices of a generic marked poset polyhedron $\mathcal{O}_t(P,\lambda)$ with $t \in (0,1)$ are exactly the vertices in its tropical subdivision $\mathcal{T}_t(P,\lambda)$.*

As a consequence, the vertices of the generic marked poset polyhedron can be obtained by subdividing the marked order polyhedron using the associated tropical subdivision and transferring the obtained vertices via the transfer map φ_t to $\mathcal{O}_t(P,\lambda)$. Furthermore, even for arbitrary $t \in [0,1]$, the set of points obtained this way will always contain the vertices of $\mathcal{O}_t(P,\lambda)$.

Before proceeding with the proof of Theorem 7.8.1 let us illustrate the situation with an example.

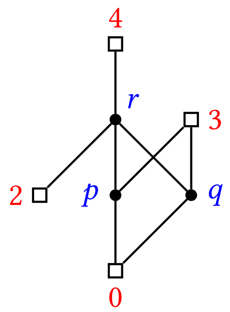


Figure 7.4.: The marked poset from Example 7.8.2

Example 7.8.2. Let (P,λ) be the marked poset given in Figure 7.4. The hyperplane arrangement $\mathcal{H}(P,\lambda)$ consists of just one tropical hyperplane given by the tropical linear form

$$\alpha_r = \max\{x_2, x_p, x_q\} = x_2 \oplus x_p \oplus x_q.$$

It divides the space $\mathbb{R}^P$ into three regions where either x_2, x_p or x_q is maximal among the three coordinates. Intersecting this subdivision with $\mathcal{O}(P,\lambda)$ we obtain the the tropical subdivision shown in Figure 7.5a, where the hyperplane itself is shaded in red. We see the 11 vertices of the polytope depicted in green and 3 additional vertices of the tropical subdivision that are not vertices of $\mathcal{O}(P,\lambda)$ in red. Since t_p and t_q are irrelevant for the affine type of $\mathcal{O}_t(P,\lambda)$ by Proposition 7.5.4—and in fact only get multiplied by 0 in the projected transfer map $\widetilde{\varphi}_t$—we only need to consider the parameter t_r. In Figure 7.5b we see the tropical subdivision of $\mathcal{O}_t(P,\lambda)$ for $t_r = \frac{1}{2}$. Now all vertices that appear in the subdivision are green, i.e., they are vertices of the polytope, as stated in Theorem 7.8.1. When $t_r = 1$, we obtain the tropic subdivision of the marked chain polytope $\mathcal{C}(P,\lambda)$ as shown in Figure 7.5c. Again, some of the vertices in the subdivision are not vertices of the polytope. ◊

To prove Theorem 7.8.1, we first need a lemma simplifying the description of vertices in $\mathcal{T}(P,\lambda)$. Recall that the tropical hyperplane arrangement introduced in Section 7.6 has tropical hyperplanes enumerated by R, the set of all unmarked elements in P covering at least two other elements.

Lemma 7.8.3. *Let v be a vertex in the tropical subdivision $\mathcal{T}(P,\lambda)$ of a marked order polyhedron $\mathcal{O}(P,\lambda)$ and denote by F and G the minimal faces of $\mathcal{O}(P,\lambda)$ and $\mathcal{T}(\mathcal{H}(P,\lambda))$ containing v, respectively, so that $\{v\} = F \cap G$. Denote by R_G the set of all $r \in R$ such that $|\operatorname{tc}(v)_r| \geq 2$ and let*

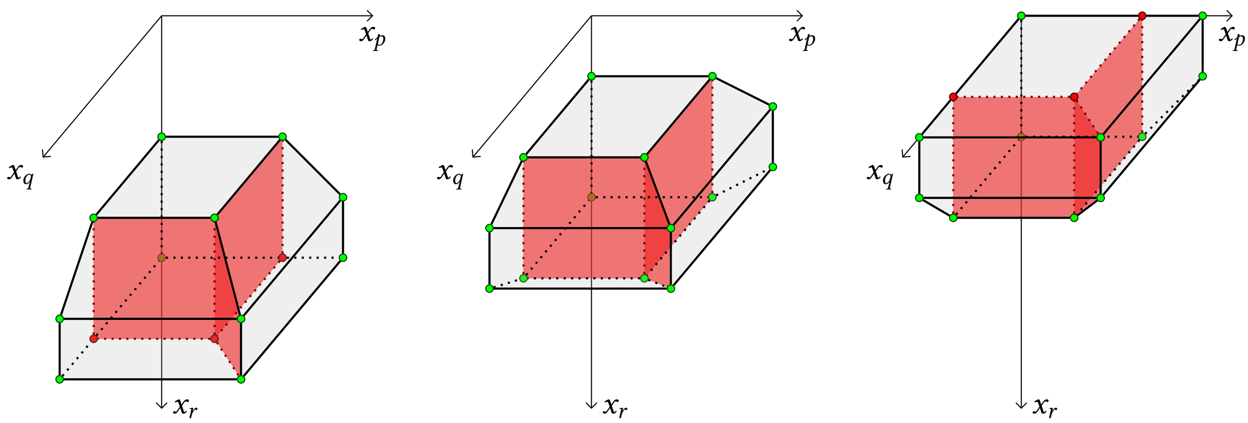
$$G' = \left\{ x \in \mathbb{R}^P \,\middle|\, x_q = x_{q'} \text{ for all } r \in R_G,\, q, q' \in \operatorname{tc}(v)_r \right\}$$

Then

$$\{v\} = F \cap G = F \cap G'.$$

Proof. By definition of the tropical subdivision $\mathcal{T}(\mathcal{H}(P,\lambda))$ of $\mathbb{R}^P$, we have

$$G = \left\{ x \in \mathbb{R}^P \,\middle|\, \operatorname{tc}(v)_r \subseteq \operatorname{tc}(x)_r \text{ for all } r \in R, \right\},$$



(a) $\widetilde{\mathcal{O}}(P,\lambda)$ (b) $\widetilde{\mathcal{O}}_{\frac{1}{2}}(P,\lambda)$ (c) $\widetilde{\mathcal{C}}(P,\lambda)$

Figure 7.5.: Tropical subdivision of the marked poset polytope from Example 7.8.2 for $t_r = 0$, $\frac{1}{2}$ and 1. For generic t the vertices of the subdivision coincide with the vertices of the polytope.

where $\operatorname{tc}(v)_r \subseteq \operatorname{tc}(x)_r$ is equivalent to $x_q = x_{q'}$ for $q, q' \in \operatorname{tc}(v)_r$ and $x_{q''} \leq x_q$ for $q'' \prec r$ with $q'' \notin \operatorname{tc}(v)_r$ and $q \in \operatorname{tc}(v)_r$. Hence, we may write

$$G = G' \cap H \cap L,$$

where

$$\begin{aligned} G' &= \left\{ x \in \mathbb{R}^P \,\middle|\, x_q = x_{q'} \text{ for all } r \in R_G,\, q, q' \in \operatorname{tc}(v)_r \right\}, \\ H &= \left\{ x \in \mathbb{R}^P \,\middle|\, x_{q''} \leq x_q \text{ for all } r \in R_G,\, q'' \prec r \text{ with } q'' \notin \operatorname{tc}(v)_r,\, q \in \operatorname{tc}(v)_r \right\}, \\ L &= \left\{ x \in \mathbb{R}^P \,\middle|\, \operatorname{tc}(v)_r \subseteq \operatorname{tc}(x)_r \text{ for all } r \in R \setminus R_G \right\}. \end{aligned}$$

Since $v_{q''} < v_q$ for $q'', q \prec r$ with $q'' \notin \operatorname{tc}(v)_r$ and $q \in \operatorname{tc}(v)_r$, we know that v is an interior point of H. Since for $r \notin R_G$ the set $\operatorname{tc}(v)_r$ has exactly one element, there are no conditions $x_q = x_{q'}$ for $q, q' \in \operatorname{tc}(v)_r$ and by the previous argument v is also an interior point of L. Hence, we have

$$\{v\} = (F \cap G') \cap (H \cap L),$$

where v is an interior point of $H \cap L$. Since $\mathbb{R}^P$ is Hausdorff and $F \cap G'$ is connected, this implies $\{v\} = F \cap G'$. □

We are now ready to prove Theorem 7.8.1.

Proof of Theorem 7.8.1. Let v be a vertex in the tropical subdivision $\mathcal{T}(P,\lambda)$ of $\mathcal{O}(P,\lambda)$, so that $\{v\} = F \cap G$, where F is the minimal face of $\mathcal{O}(P,\lambda)$ containing v and G is the minimal cell in $\mathcal{T}(\mathcal{H}(P,\lambda))$ containing v. Let $\operatorname{tc}(G)$ be the tropical covector corresponding to G and denote by R_G the set of all $r \in R$ such that $|\operatorname{tc}(G)_r| \geq 2$. In other words, R_G consists of all $p \in \tilde{P}$ such that at least two different $q \prec p$ maximize v_q. Fix $u \in [0,1]^{\tilde{P}}$ with $u_p \in (0,1)$ for $p \in R_G$ and $u_p = 0$ otherwise.

We claim that $\varphi_u(v)$ is a vertex of $\mathcal{O}_u(P,\lambda)$. Since u is a degeneration of any $t \in (0,1)^{\tilde{P}}$, we can conclude by Proposition 7.7.5 that $\varphi_t(v)$ is then also a vertex of $\mathcal{O}_t(P,\lambda)$ whenever $t \in (0,1)^{\tilde{P}}$.

By Lemma 7.8.3 we have $\{v\} = F \cap G'$, where G' is defined by the conditions $x_q = x_{q'}$ for $r \in R_G$, $q, q' \in \mathrm{tc}(v)_r$. Let Q be the minimal face of $\mathcal{O}_u(P,\lambda)$ containing $\varphi_u(v)$. If we can show $\psi_u(Q) \subseteq F$ and $\psi_u(Q) \subseteq G'$, we can conclude that Q is a single point and hence $\varphi_u(v)$ a vertex. Since $0 \in [0,1]^{\tilde{P}}$ is a degeneration of u, we have $\psi_u(\operatorname{relint} Q) \subseteq \operatorname{relint} F$ by Proposition 7.7.5 and conclude $\psi_u(Q) \subseteq F$ by taking closures.

To show that $\psi_u(Q) \subseteq G'$, let $r \in R_G$ and $q \in \mathrm{tc}(G)_r = \mathrm{tc}(v)_r$. For $y \in Q$ with image $z = \psi_u(y)$ in $\mathcal{O}(P,\lambda)$, we will show $q \dashv_z r$, so $q \in \mathrm{tc}(z)_r$. Hence, we obtain $\mathrm{tc}(v)_r \subseteq \mathrm{tc}(z)_r$ for $r \in R_G$ which implies $z \in G'$. Our strategy is as follows: construct a chain $\mathfrak{c}$ corresponding to a defining inequality of $\mathcal{O}_u(P,\lambda)$ satisfied by $\varphi_u(v)$ with equality, such that $q \dashv_v r$ is one of the corresponding conditions on v in Proposition 7.5.2. Since the inequality is satisfied by $\varphi_u(v)$ with equality, the same holds for $y \in Q$. Again, by Proposition 7.5.2, this implies that $q \dashv_z r$.

What remains to be done is constructing the chain $\mathfrak{c}$. In the following, we need a relation slightly stronger than $\dashv_x$. Let $a \mathrel{=\!\!|}_x b$ denote the relation on P defined by $b \in R_G$ and $a \dashv_x b$. That is, $a \mathrel{=\!\!|}_x b$ holds if and only if $b \in R_G$, $a \prec b$ and $x_a = \max_{q \prec b} x_q$.

First construct a chain from q downward to a marked element that is of the kind

$$a \prec \cdots \prec p' \prec p'_1 \mathrel{=\!\!|}_v \cdots \mathrel{=\!\!|}_v p'_l = q,$$

where $l \geq 1$ and $p'_1 \notin R_G$. That is, walk downwards in R_G along relations $\mathrel{=\!\!|}_v$ as long as possible, then arbitrarily extend the chain to some marked element $a \in P^*$. Let

$$\mathfrak{c}\colon a \prec \cdots \prec p' \prec p'_1 \prec \cdots \prec p'_{l-1}.$$

When q was marked, $\mathfrak{c}$ is just the empty chain. When $q \notin R_G$, we have $l = 1$, $p'_1 = q$ and $\mathfrak{c}$ ends in p'.

Now construct a maximal chain

$$q \mathrel{=\!\!|}_v r \mathrel{=\!\!|}_v p_1 \mathrel{=\!\!|}_v \cdots \mathrel{=\!\!|}_v p_k,$$

where $k \geq 0$. Let $p_{-1} = q$, $p_0 = r$, and let $B \in \pi_F$ be the block of the face partition of F containing p_k. We claim that B can not be a singleton: since $F \cap G'$ is a point, the conditions imposed by the face partition π_F together with the conditions given by G' determine all the coordinates, in particular x_{p_k}. However, p_k is neither marked, since it is an element of $R_G \subseteq \tilde{P}$, nor does it appear in one of the equations for G', since the chain was chosen maximal. Hence, the coordinate x_{p_k} must be determined by p_k sitting in a non-trivial block with some other coordinate already determined by the conditions imposed by λ, π_F and G'.

If there exists $p \in B$ with $p_k \prec p$, let

$$\mathfrak{d}\colon p_1 \prec \cdots \prec p_k \prec p.$$

The chain $\mathfrak{c} \prec q \prec r \prec \mathfrak{d}$ yields a defining inequality for $\mathcal{O}_u(P,\lambda)$. Since $u_{p'_1} = 0$, while $u_{p'_2}, \dots, u_{p'_l}, u_r, u_{p_1}, \dots, u_{p_k} > 0$ and $u_p \neq 1$ in case of $p \in \tilde{P}$, the describing inequality of $\mathcal{O}_u(P,\lambda)$ given by $\mathfrak{c} \prec q \prec r \prec \mathfrak{d}$ is satisfied with equality for some $\varphi_u(x)$ if and only if

$$p'_1 \dashv_x \cdots \dashv_x p'_l = q \dashv_x r \dashv_x p_1 \dashv_x \cdots \dashv_x p_k \quad \text{and} \quad x_{p_k} = x_p.$$

For $x = v$, all these conditions are satisfied. Hence, they are also satisfied by z. In particular $q \dashv_z r$ as desired.

If there exists no $p \in B$ with $p_k \prec p$, there must be some $p \in B$ with $p \prec p_k$, since B is not a singleton. In this case $v_p = v_{p_k}$ so in particular $p \dashv_v p_k$. Since $p_{k-1} \dashv_v p_k$ as well, we conclude $v_{p_{k-1}} = v_p = v_{p_k}$. Now let

$$\mathfrak{d}\colon p_1 \prec \cdots \prec p_k.$$

The inequality for $\mathcal{O}_u(P,\lambda)$ given by $\mathfrak{c} \prec q \prec r \prec \mathfrak{d}$ is satisfied with equality for $\varphi_u(x)$ if and only if

$$p'_1 \dashv_x \cdots \dashv_x p'_l = q \dashv_x r \dashv_x p_1 \dashv_x \cdots \dashv_x p_{k-1} \quad \text{and} \quad x_{p_{k-1}} = x_{p_k}.$$

Again, all these conditions hold for $x = v$, hence also for z and we can conclude $q \dashv_z r$ as before. □

7.9. Poset Transformations

Since having a strict or even regular marking already played an essential role in the theory of marked order polyhedra, it is a natural question to ask whether we can apply the poset transformation used in Section 6.2 and still obtain the same marked poset polyhedra up to affine equivalence for arbitrary $t \in [0,1]^{\tilde{P}}$. In this section we show that the answer is positive: modifying a marked poset to be strictly marked and modifying a strictly marked poset to be regular does not change the affine isomorphism type.

Proposition 7.9.1. *Contracting constant intervals in (P,λ) yields a strictly marked poset $(P/\pi, \lambda/\pi)$ such that $\mathcal{O}_{t'}(P/\pi, \lambda/\pi)$ is affinely isomorphic to $\mathcal{O}_t(P,\lambda)$ for all $t \in [0,1]^{\tilde{P}}$, where t' is the restriction of t to elements not contained in any non-trivial constant intervals.*

Proof. Let (P',λ') be the strictly marked poset obtained from (P,λ) by contracting constant intervals. Hence, P' is obtained from P by taking the quotient under the equivalence relation generated by $a \sim p$ and $p \sim b$ whenever $a \leq b$ are marked elements such that $\lambda(a) = \lambda(b)$ and $a \leq p \leq b$. The elements of P' are either singletons $\{p\}$ for p not contained in any non-trivial constant interval or non-trivial blocks B that are unions of non-trivial constant intervals. All non-trivial blocks B are marked and among the singletons $\{p\}$ only those with $p \in \tilde{P}$ are unmarked.

By Proposition 6.2.17, we have an affine isomorphism $q^*\colon \mathcal{O}(P',\lambda') \to \mathcal{O}(P,\lambda)$ induced by the quotient map $q\colon (P,\lambda) \to (P',\lambda')$. Now consider the two transfer maps

$\varphi_t\colon \mathcal{O}(P,\lambda) \to \mathcal{O}_t(P,\lambda)$ and $\varphi'_{t'}\colon \mathcal{O}(P',\lambda') \to \mathcal{O}_{t'}(P',\lambda')$. When B is a non-trivial block in P'—in other words an equivalence class with at least two elements—we have

$$\varphi_t(x)_p = (1-t_p)\lambda'(B)$$

for all unmarked $p \in B$ and $x \in \mathcal{O}(P,\lambda)$. When p is an unmarked element outside of constant intervals, we have $\varphi_t(q^*(x))_p = \varphi'_{t'}(x)_{\{p\}}$ for all $x \in \mathcal{O}(P',\lambda')$ by construction. Hence, the affine map $\gamma\colon \mathbb{R}^{\tilde{P}'} \to \mathbb{R}^{\tilde{P}}$ defined by

$$\gamma(x)_p = \begin{cases} (1-t_p)\lambda'(B) & \text{if } p \in \tilde{P} \cap B \text{ for a non-trivial block } B,\\ x_{\{p\}} & \text{otherwise} \end{cases}$$

restricts to an affine map $\widetilde{\mathcal{O}}_{t'}(P',\lambda') \to \widetilde{\mathcal{O}}_t(P,\lambda)$, such that the diagram

$$\begin{tikzcd} \widetilde{\mathcal{O}}(P',\lambda') \arrow[r, "q^*"] \arrow[d, "\tilde{\varphi}'_{t'}"] & \widetilde{\mathcal{O}}(P,\lambda) \arrow[d, "\tilde{\varphi}_t"] \\ \widetilde{\mathcal{O}}_{t'}(P',\lambda') \arrow[r, "\gamma"] & \widetilde{\mathcal{O}}_t(P,\lambda) \end{tikzcd}$$

commutes. Thus, it is an affine isomorphism. □

Proposition 7.9.2. *If (P,λ) is strictly marked, removing a redundant covering relation yields a marked poset (P',λ) such that $\mathcal{O}_t(P,\lambda) = \mathcal{O}_t(P',\lambda)$ for all $t \in [0,1]^{\tilde{P}}$.*

Proof. Let $p \prec q$ be a redundant covering relation in P. That is, there are marked elements $a \neq b$ satisfying $a \leq q$, $p \leq b$ and $\lambda(a) \geq \lambda(b)$. Let P' be obtained from P be removing the covering relation $p \prec q$.

Comparing the transfer maps φ_t and φ'_t associated to (P,λ) and (P',λ) defined on the same marked order polyhedron $\mathcal{O}(P,\lambda) = \mathcal{O}(P',\lambda)$ by Proposition 6.2.22, we see that they can only differ in the q-coordinate, which can only happen when q is unmarked. To be precise,

$$\varphi_t(x)_q = x_q - t_q \max_{q' \prec q} x_{q'} \qquad \text{and} \qquad \varphi'_t(x)_q = x_q - t_q \max_{\substack{q' \prec q\\ q' \neq p}} x_{q'}.$$

Since λ is strict, we can not have $a \leq p$. Otherwise we had $a < b$ in contradiction to $\lambda(a) \geq \lambda(b)$. Hence, when q is unmarked, there is a $p' \neq p$ such that $a \leq p' \prec q$. For all $x \in \mathcal{O}(P',\lambda) = \mathcal{O}(P,\lambda)$ we have

$$x_{p'} \geq \lambda(a) \geq \lambda(b) \geq x_p$$

and excluding p from the maximum does not change the transfer map at all. We conclude that

$$\mathcal{O}_t(P',\lambda) = \varphi'_t(\mathcal{O}(P',\lambda)) = \varphi_t(\mathcal{O}(P,\lambda)) = \mathcal{O}_t(P,\lambda). \qquad \square$$

Using the above transformations, we can always replace a marked poset (P,λ) by a regular marked poset (P',λ') yielding affinely equivalent marked poset polyhedra.

7.10. Facets and the Hibi–Li Conjecture

In Chapter 6 we have seen that regular marked posets yield a one-to-one correspondence of covering relations in (P,λ) and facets of $\mathcal{O}(P,\lambda)$. We strongly believe the same regularity condition implies that both the inequalities in Definition 7.1.1 for $t \in (0,1)^{\tilde{P}}$ and the inequalities in Proposition 7.3.1 for all partitions $\tilde{P} = C \sqcup O$—i.e., all $t \in \{0,1\}^{\tilde{P}}$—correspond to the facets of the described polyhedra. In fact, we can show that the latter implies the former and the conjecture is true for certain *ranked* marked posets.

Definition 7.10.1. A marked poset (P,λ) is called *tame* if the inequalities given in Proposition 7.3.1 correspond to the facets of $\mathcal{O}_{C,O}(P,\lambda)$ for all partitions $\tilde{P} = C \sqcup O$.

Conjecture 7.10.2[2]**.** *A marked poset (P,λ) is tame if and only if it is regular.*

We know that regularity is a necessary condition for being tame, since otherwise (P,λ) either contains non-trivial constant intervals and the covering relations in those do not correspond to facets of $\mathcal{O}(P,\lambda)$ or the marking is strict but there are redundant covering relations that do not correspond to facets of $\mathcal{O}(P,\lambda)$.

We start by considering marked chain polyhedra. We can show that any chain in (P,λ) that does not contain redundant covering relations defines a facet of $\mathcal{C}(P,\lambda)$.

Lemma 7.10.3. *Let (P,λ) be a marked poset and $\mathfrak{c}\colon a \prec p_1 \prec p_2 \prec \cdots \prec p_r \prec b$ be a saturated chain between elements $a,b \in P^*$ with all $p_i \in \tilde{P}$ and $r \geq 1$. If none of the covering relations in $\mathfrak{c}$ are redundant, the inequality*

$$x_{p_1} + \cdots + x_{p_r} \leq x_b - x_a \tag{7.7}$$

is not redundant in the description of $\mathcal{C}(P,\lambda) = \mathcal{O}_{\tilde{P},\varnothing}(P,\lambda)$ given in Proposition 7.3.1.

In particular we obtain the following result.

Corollary 7.10.4[3]**.** *Let (P,λ) be regular. The description of the marked chain polyhedron $\mathcal{C}(P,\lambda) = \mathcal{O}_{\tilde{P},\varnothing}(P,\lambda)$ given in Proposition 7.3.1 is non-redundant.*

Proof of Lemma 7.10.3. Our strategy is as follows. First show that (7.7) can be strictly satisfied by some point in $\mathcal{C}(P,\lambda)$, so the polyhedron is not contained in the corresponding hyperplane. Then construct a point $x \in \mathcal{C}(P,\lambda)$ such that (7.7) is satisfied with equality but all other inequalities that can be strictly satisfied by points in $\mathcal{C}(P,\lambda)$ are strictly satisfied by x. This shows that (7.7) is the only inequality describing a facet with x in its relative interior.

To see that (7.7) can be strictly satisfied, just take $x \in \mathbb{R}^P$ with $x_a = \lambda(a)$ for $a \in P^*$ and $x_p = 0$ for $p \in \tilde{P}$. Note that $\lambda(a) < \lambda(b)$ since otherwise all covering relations in $\mathfrak{c}$ would be redundant.

[2] In [FF16, Proposition 4.5], the statement of Conjecture 7.10.2 is given without proof for the case of admissible partitions and bounded polyhedra. At the time of writing this thesis, no proof of this statement is known.

[3] The result of Corollary 7.10.4 was previously stated without proof in [Fou16, Lemma 1] for bounded polyhedra.

For the second step, first linearly order the set of all markings in $[\lambda(a), \lambda(b)]$, so that

$$\lambda(P^*) \cap [\lambda(a), \lambda(b)] = \{\lambda_1, \ldots, \lambda_k\}$$

with $\lambda(a) = \lambda_1 < \cdots < \lambda_k = \lambda(b)$ and $k > 1$. For $i = 1, \ldots, k-1$ we define the following sets:

$$\begin{aligned} Z_i^{\uparrow} &= \{ p \in \mathfrak{c} : p \geq d \text{ for some } d \in P^* \text{ with } \lambda(d) \geq \lambda_{i+1} \}, \\ Z_i^{\downarrow} &= \{ p \in \mathfrak{c} : p \leq e \text{ for some } e \in P^* \text{ with } \lambda(e) \leq \lambda_i \}, \\ Z_i &= \mathfrak{c} \setminus \left(Z_i^{\uparrow} \sqcup Z_i^{\downarrow} \right). \end{aligned}$$

Note that $Z_i^{\uparrow}$ and $Z_i^{\downarrow}$ are disjoint, since any p in their intersection would give $d \leq p \leq e$ with $\lambda(d) \geq \lambda_{i+1} > \lambda_i \geq \lambda(e)$ contradicting λ being order-preserving.

For $p \in Z_i^{\uparrow}$ all elements of $\mathfrak{c}$ greater than p are also contained in $Z_i^{\uparrow}$ and for $p \in Z_i^{\downarrow}$ all elements of $\mathfrak{c}$ less than p are also contained in $Z_i^{\downarrow}$. Furthermore, we have $a \in Z_i^{\downarrow}$ and $b \in Z_i^{\uparrow}$ for all i. Thus, the chain $\mathfrak{c}$ decomposes into three connected subchains $Z_i^{\downarrow}, Z_i, Z_i^{\uparrow}$.

We claim that the middle part Z_i is always non-empty as well. Otherwise, the chain $\mathfrak{c}$ contains a covering relation $p \prec q$ with $p \in Z_i^{\downarrow}$ and $q \in Z_i^{\uparrow}$ and hence we had $d, e \in P^*$ with $e \geq p \prec q \geq d$ and $\lambda(e) \leq \lambda_i < \lambda_{i+1} \leq \lambda(d)$ so that $p \prec q$ is redundant.

We also claim that each $p_j \in \mathfrak{c}$ is contained in at least one of the Z_i. Since $a \leq p_j$, we can choose $i_0 \in [k]$ maximal such that $p_j \geq d$ for some d with $\lambda(d) \geq \lambda_{i_0}$. In the same fashion, choose $i_1 \in [k]$ minimal such that $p_j \leq e$ for some e with $\lambda(e) \leq \lambda_{i_1}$. We have $i_0 < i_1$, since otherwise there are $d \leq p_j \leq e$ with $\lambda(d) \geq \lambda(e)$, either rendering λ non order-preserving or any covering relation above or below p_j redundant. We conclude that $p_j \in Z_i$ for $i = i_0, \ldots, i_1 - 1$.

Define a point $x \in \mathbb{R}^P$ by letting $x_a = \lambda(a)$ for all $a \in P^*$ and for $p \in \tilde{P}$:

$$x_p = \begin{cases} \displaystyle\sum_{\substack{i=1,\ldots,k-1, \\ p \in Z_i}} \frac{\lambda_{i+1} - \lambda_i}{|Z_i|} & \text{for } p \in \mathfrak{c} \text{ and} \\ \varepsilon & \text{for } p \in \tilde{P} \setminus \mathfrak{c}, \end{cases}$$

where $\varepsilon > 0$ is small enough to satisfy the finitely many constraints in the rest of this proof. Note that all $|Z_i| > 0$ since the Z_i are non-empty and all $x_p > 0$ for $p \in \tilde{P}$ since each $p_j \in \mathfrak{c}$ is contained in at least one of the Z_i.

The inequality given by $\mathfrak{c}$ is satisfied with equality, since

$$\begin{aligned} \sum_{j=1}^{r} x_{p_j} &= \sum_{j=1}^{r} \sum_{\substack{i=1,\ldots,k-1, \\ p_j \in Z_i}} \frac{\lambda_{i+1} - \lambda_i}{|Z_i|} \\ &= \sum_{i=1}^{k-1} \sum_{\substack{j=1,\ldots,r \\ p_j \in Z_i}} \frac{\lambda_{i+1} - \lambda_i}{|Z_i|} = \sum_{i=1}^{k-1} (\lambda_{i+1} - \lambda_i) = \lambda_k - \lambda_1 = x_b - x_a. \end{aligned}$$

Now consider any chain $\mathfrak{d}\colon a' \prec q_1 \prec \cdots \prec q_s \prec b'$ different from $\mathfrak{c}$. We have to show that the inequality

$$x_{q_1} + \cdots + x_{q_s} \leq x_{b'} - x_{a'} \tag{7.8}$$

either can not be strictly satisfied by any point in $C(P, \lambda)$ or is strictly satisfied by x.

If $\lambda(a') = \lambda(b')$ the inequality can never be satisfied strictly by points in $C(P, \lambda)$. If $\lambda(a') < \lambda(b')$ we have

$$\sum_{j=1}^{s} x_{q_j} = \sum_{q\in\tilde{\mathfrak{d}}} x_q = \varepsilon\left|\tilde{\mathfrak{d}} \setminus \tilde{\mathfrak{c}}\right| + \sum_{q\in\tilde{\mathfrak{d}}\cap\tilde{\mathfrak{c}}} \sum_{\substack{i=1,\dots,k-1,\\ q\in Z_i}} \frac{\lambda_{i+1} - \lambda_i}{|Z_i|}, \tag{7.9}$$

where $\tilde{\mathfrak{c}}$ and $\tilde{\mathfrak{d}}$ denote the unmarked parts of $\mathfrak{c}$ and $\mathfrak{d}$, respectively.

Let S denote the double sum in (7.9) and consider the following cases:

i) We have $\lambda(a) < \lambda(a') < \lambda(b') < \lambda(b)$. Let $1 < i_0 < i_1 < k$ be the indices such that $\lambda_{i_0} = \lambda(a')$, $\lambda_{i_1} = \lambda(b')$. Note that all elements of $\tilde{\mathfrak{d}}$ are above a' with $\lambda(a') = \lambda_{i_0}$ so $\tilde{\mathfrak{d}} \cap \tilde{\mathfrak{c}} \subseteq Z_{i_0-1}^{\uparrow}$ and we have $\tilde{\mathfrak{d}} \cap Z_i = \varnothing$ for $i < i_0$. By the same reasoning $\tilde{\mathfrak{d}} \cap Z_i = \varnothing$ for $i \geq i_1$. Hence, we have

$$S = \sum_{q\in\tilde{\mathfrak{d}}\cap\tilde{\mathfrak{c}}} \sum_{\substack{i=i_0,\dots,i_1-1,\\ q\in Z_i}} \frac{\lambda_{i+1} - \lambda_i}{|Z_i|} \leq \lambda(b') - \lambda(a'),$$

with equality achieved if and only if $Z_i \subseteq \tilde{\mathfrak{d}}$ for $i = i_0, \dots, i_1 - 1$.

Let $p \in \tilde{\mathfrak{c}}$ be maximal such that $p \in Z_{i_1-1}$. Then there is a covering relation $p \prec q$ in $\mathfrak{c}$ with $q \in Z_{i_1-1}^{\uparrow}$. We have $p \notin \mathfrak{d}$, since otherwise $p < b'$ and $q > d$ for some d with $\lambda(d) \geq \lambda_{i_1} = \lambda(b)$, rendering $p \prec q$ redundant. Hence, $p \in Z_{i_1-1} \setminus \tilde{\mathfrak{d}}$ and $S < \lambda(b') - \lambda(a')$.

We conclude that (7.8) is strictly satisfied for small enough ε.

ii) We have $\lambda(a') < \lambda(b') \leq \lambda(a)$ or $\lambda(b) \leq \lambda(a') < \lambda(b')$. In this case we have $\mathfrak{d} \subseteq Z_i^{\downarrow}$ for all i or $\mathfrak{d} \subseteq Z_i^{\uparrow}$ for all i, respectively, so that $S = 0$. Choosing ε small enough yields strict inequality in (7.8).

iii) We have $\lambda(a) < \lambda(a') < \lambda(b) \leq \lambda(b')$ or $\lambda(a') \leq \lambda(a) < \lambda(b') < \lambda(b)$. By reasoning similar to item i) we have $S < \lambda(b) - \lambda(a')$ or $S < \lambda(b') - \lambda(a)$, respectively. In both cases $S < \lambda(b') - \lambda(a')$ and choosing ε small enough yields strict inequality in (7.8).

iv) We have $\lambda(a') \leq \lambda(a) < \lambda(b) \leq \lambda(b')$. In case $\tilde{\mathfrak{d}} \cap \tilde{\mathfrak{c}} = \tilde{\mathfrak{c}}$ we have $\lambda(a') < \lambda(a)$ and $\lambda(b) < \lambda(b')$ since otherwise the covering relation $a \prec p_1$ or $p_k \prec b$ would be redundant. Hence

$$\sum_{q\in\tilde{\mathfrak{d}}} x_q = \varepsilon\left|\tilde{\mathfrak{d}} \setminus \tilde{\mathfrak{c}}\right| + (\lambda(b) - \lambda(a)) < \lambda(b') - \lambda(a')$$

for ε small enough.

In case $\tilde{\mathfrak{d}} \cap \tilde{\mathfrak{c}} \neq \tilde{\mathfrak{c}}$, at least one summand is missing in S to achieve $\lambda(b) - \lambda(a)$ since each $p \in \tilde{\mathfrak{c}}$ is in at least one of the Z_i. Thus, $S < \lambda(b) - \lambda(a)$ and we may choose ε small enough to obtain

$$\sum_{q \in \tilde{\mathfrak{d}}} x_q < \lambda(b) - \lambda(a) \leq \lambda(b') - \lambda(a').$$

In all cases (7.8) is satisfied by x with strict inequality and we conclude that (7.7) is not redundant in the description of $C(P, \lambda)$ given in Proposition 7.3.1. □

For ranked marked posets, we can use Lemma 7.10.3 to show that Conjecture 7.10.2 holds.

Definition 7.10.5. A marked poset (P, λ) is called *ranked* if there exists a *rank function* $\operatorname{rk}\colon P \longrightarrow \mathbb{Z}$ satisfying

i) $\operatorname{rk} p + 1 = \operatorname{rk} q$ for all $p, q \in P$ with $p \prec q$,
ii) $\lambda(a) < \lambda(b)$ for all $a, b \in P^*$ with $\operatorname{rk} a < \operatorname{rk} b$.

Note that the rank function of a ranked marked poset is uniquely determined up to a constant on each connected component.

Proposition 7.10.6. *Let (P, λ) be regular and ranked, then (P, λ) is tame.*

Proof. Let $\operatorname{rk}\colon P \longrightarrow \mathbb{Z}$ be a rank function such that $\min\{\operatorname{rk} p : p \in P\} = 0$ and let $r = \max\{\operatorname{rk} p : p \in P\}$. Since $\lambda(a) < \lambda(b)$ for marked elements with $\operatorname{rk} a < \operatorname{rk} b$, we can choose real numbers $\xi_0 < \xi_1 < \cdots < \xi_{r+1}$ such that $\lambda(a) \in (\xi_i, \xi_{i+1})$ for $a \in P^*$ with $\operatorname{rk} a = i$.

Let $\tilde{P} = C \sqcup O$ be any partition. All inequalities $0 \leq x_p$ for $p \in C$ are non-redundant in the description of $\mathcal{O}_{C,O}(P, \lambda)$ given in Proposition 7.3.1. To see this, take any $x \in \mathcal{O}_{C,O}(P, \lambda)$ and let $x' \in \mathbb{R}^P$ be given by $x'_q = x_q$ for $q \neq p$ and $x_p = -1$.

Now consider any chain $\mathfrak{c}\colon a \prec p_1 \prec \cdots \prec p_r \prec b$ with $a, b \in P^* \sqcup O$ and all $p_i \in \tilde{P}$. If $r = 0$, we have to show that $x_a \leq x_b$ is a non-redundant inequality provided at least one of a and b is not marked. For this, define $x \in \mathcal{O}_{C,O}(P, \lambda)$ by

$$x_p = \begin{cases} \lambda(p) & \text{for } p \in P^*, \\ \xi_{\operatorname{rk} p} & \text{for } p \in O \setminus \{a, b\} \text{ with } \operatorname{rk} p \leq \operatorname{rk} a, \\ \xi_{\operatorname{rk} p + 1} & \text{for } p \in O \setminus \{a, b\} \text{ with } \operatorname{rk} p \geq \operatorname{rk} b, \\ \xi_{\operatorname{rk} b} & \text{for } p \in \{a, b\} \text{ if } a, b \in O, \\ \lambda(a) & \text{for } p \in \{a, b\} \text{ if } a \notin O, \\ \lambda(b) & \text{for } p \in \{a, b\} \text{ if } b \notin O, \\ \min_i\{\xi_{i+1} - \xi_i\} & \text{for } p \in C. \end{cases}$$

Using the fact that (P, λ) is ranked it is routine to check that x satisfies all inequalities of Proposition 7.3.1 strictly except for $x_a \leq x_b$.

Now consider the case where $r \geq 1$. The idea is to extend the marking λ to a marking λ' defined on $P^* \sqcup O$ such that (P, λ') has no redundant covering relations in $\mathfrak{c}$. We then have $\mathcal{O}_{C,O}(P,\lambda) \cap U = \mathcal{C}(P,\lambda')$ with U given by $x_p = \lambda'(p)$ for $p \in O$. Note that the description of $\mathcal{C}(P,\lambda')$ in Proposition 7.3.1 is exactly the description given for $\mathcal{O}_{C,O}(P,\lambda)$ in Proposition 7.3.1 with the additional equations $x_p = \lambda'(p)$ for $p \in O$. In the description of $\mathcal{C}(P,\lambda')$ the inequality given by $\mathfrak{c}$ is not redundant by Lemma 7.10.3 and hence the same inequality is not redundant in the description of $\mathcal{O}_{C,O}(P,\lambda) \cap U$. Thus, it can not be redundant in the description of $\mathcal{O}_{C,O}(P,\lambda)$ itself either.

It remains to construct the extended marking λ'. Let $\lambda'(p) = \lambda(p)$ for $p \in P^*$ and for $p \in O$ with $\operatorname{rk} p = i$ choose

$$\lambda'(p) \in \begin{cases} (\xi_i, \xi_{i+1}) & \text{for } i \notin \{\operatorname{rk} a, \operatorname{rk} b\}, \\ (\max\{\xi_i, \max\{\lambda(d) : \operatorname{rk} d = i\}\}, \xi_{i+1}) & \text{for } p = a \text{ if } a \in O, \\ (\xi_i, \min\{\xi_{i+1}, \min\{\lambda(d) : \operatorname{rk} d = i\}\}) & \text{for } p = b \text{ if } b \in O, \\ (\xi_i, \lambda'(a)) & \text{for } i = \operatorname{rk} a, p \neq a, \\ (\lambda'(b), \xi_{i+1}) & \text{for } i = \operatorname{rk} b, p \neq b. \end{cases}$$

The appearing open intervals are all non-empty so these choices are possible. Given any such λ', we still have $\lambda'(d) \in (\xi_i, \xi_{i+1})$ when $\operatorname{rk} d = i$, so (P, λ') is still a ranked marked poset. Let us verify that $\mathfrak{c}$ contains no redundant covering relation with respect to (P, λ').

i) The covering relation $a \prec p_1$ is non-redundant since $\lambda'(d) < \lambda'(a)$ for all marked elements $d \leq p_1$, $d \neq a$.
ii) The covering relation $p_r \prec b$ is non-redundant since $\lambda'(d) > \lambda'(b)$ for all marked elements $d \geq p_r$, $d \neq b$.
iii) All covering relations $p_j \prec p_{j+1}$ are non-redundant since (P, λ) is ranked.

Hence, we can apply Lemma 7.10.3 to $\mathcal{C}(P, \lambda')$ and obtain the desired result. □

Remark 7.10.7. In light of the proof of Proposition 7.10.6, a possible strategy to prove Conjecture 7.10.2 in general would be to extend markings such that along a given chain the covering relations stay non-redundant. However, we did not succeed in doing this for arbitrary (non-ranked) marked posets.

Remark 7.10.8. The marked posets relevant in representation theory appearing in [ABS11; BD15] are all ranked and regular after applying the transformations of Section 7.9 if necessary. Hence, they are tame and Proposition 7.3.1 gives non-redundant descriptions for all associated marked chain-order polyhedra.

At the beginning of this chapter, we mentioned that (P, λ) being tame also implies the description given for generic marked poset polyhedra $\mathcal{O}_t(P,\lambda)$ in Definition 7.1.1 is non-redundant.

Proposition 7.10.9. *Let (P, λ) be a tame marked poset. The description of any generic marked poset polyhedron $\mathcal{O}_t(P,\lambda)$ for $t \in (0,1)^{\tilde{P}}$ given in Definition 7.1.1 is non-redundant.*

Proof. The way we will prove non-redundance of the description in Definition 7.1.1 is to reconsider the proof of Proposition 7.3.1. We have seen that picking a parameter $u = \chi_C \in \{0,1\}^{\tilde{P}}$ for a partition $\tilde{P} = C \sqcup O$ we obtain the description in Proposition 7.3.1 but there might be multiple chains as in Definition 7.1.1 such that (7.1) degenerates to the same inequality listed in Proposition 7.3.1. Since we know the description in Proposition 7.3.1 is non-redundant for tame marked posets, we can do the following: take a chain $\mathfrak{c}$ giving an inequality for $\mathcal{O}_t(P,\lambda)$ as in Definition 7.1.1 and construct a partition $\tilde{P} = C \sqcup O$ such that no other chain yields the same inequality as $\mathfrak{c}$ for the marked chain-order polyhedron $\mathcal{O}_{C,O}(P,\lambda)$. Knowing that the description of $\mathcal{O}_{C,O}(P,\lambda)$ is non-redundant we conclude that $\mathfrak{c}$ can not be omitted in the description of $\mathcal{O}_t(P,\lambda)$ either whenever $u = \chi_C$ is a degeneration of t, in particular when $t \in (0,1)^{\tilde{P}}$.

Consider any chain $\mathfrak{c}\colon p_0 \prec p_1 \prec p_2 \prec \cdots \prec p_r \prec p$ with $p_0 \in P^*$, $p_i \in \tilde{P}$ for $i \geq 1$, $p \in P$ and $r \geq 0$. Let $C = \{p_1, \ldots, p_r\}$, $O = \tilde{P} \setminus (P^* \sqcup C)$ and note that $p \in P^* \sqcup O$. Since $p_0 \in P^*$ and $p \notin C$, no other chain gives the same inequality in the proof of Proposition 7.3.1. $\square$

We finish this chapter with a discussion of the Hibi–Li conjecture already mentioned in Chapter 1 as Conjecture 1.2.4 for ordinary poset polytopes. For marked order and chain polytopes as well as admissible marked chain-order polytopes an analogous conjecture was stated in [Fou16; FF16]. Let us state the conjecture in full generality here—for possibly unbounded marked chain-order polyhedra with arbitrary partitions $\tilde{P} = C \sqcup O$—and report on what can be said about the conjecture from the above discussion.

Conjecture 7.10.10. *Let (P,λ) be a marked poset with all minimal elements marked. Given partitions $\tilde{P} = C \sqcup O$ and $\tilde{P} = C' \sqcup O'$ such that $C \subseteq C'$, we have*

$$f_i\left(\mathcal{O}_{C,O}(P,\lambda)\right) \leq f_i\left(\mathcal{O}_{C',O'}(P,\lambda)\right) \quad \text{for all } i \in \mathbb{N}.$$

This refined version of the conjecture was stated in case of admissible partitions and bounded polyhedra by Fang and Fourier in [FF16]. It is clear, that it is enough to consider only the case $C' = C \sqcup \{q\}$ for some $q \in O$ and by the results of Section 7.9 we can assume (P,λ) is regular. When q is not a chain-order star element, we know that $\mathcal{O}_{C,O}(P,\lambda)$ and $\mathcal{O}_{C',O'}(P,\lambda)$ are unimodular equivalent by Proposition 7.4.5 and hence their f-vectors are identical. In fact, the statement of Proposition 7.4.5 is a necessary and sufficient condition for unimodular equivalence for tame marked posets and we can count facets to show Conjecture 7.10.10 holds for tame marked posets in codimension 1:

Proposition 7.10.11. *Let (P,λ) be a tame marked poset and $\tilde{P} = C \sqcup O$ any partition. Given $q \in O$ let $C' = C \sqcup \{q\}$ and $O' = O \setminus \{q\}$, then $\mathcal{O}_{C,O}(P,\lambda)$ and $\mathcal{O}_{C',O'}(P,\lambda)$ are unimodular equivalent if and only if q is not a chain-order star element. Otherwise, the number of facets increases by*

$$(k-1)(l-1),$$

where k is the number of saturated chains $s \prec q_1 \prec \cdots \prec q_k \prec q$ with $s \in P^* \sqcup O$ and all $q_i \in C$ and l is the number of saturated chains $q \prec q_1 \prec \cdots \prec q_k \prec s$ with $s \in P^* \sqcup O$ and all $q_i \in C$.

Proof. If q is not a chain-order star element, the polyhedra are unimodular equivalent by Proposition 7.4.5. For a tame marked poset, the number of facets of $\mathcal{O}_{C,O}(P,\lambda)$ is the number of inequalities in Proposition 7.3.1, and hence equal to

$$|C| + |\{\, a \prec p_1 \prec \cdots \prec p_r \prec b \mid r \geq 0, a, b \in P^* \sqcup O, p_i \in C \,\}|\,.$$

Changing an order element q to be a chain element, the first summand increases by 1, while in the second summand the $k + l$ chains ending or starting in q are replaced by the kl chains now going through q. Hence, the number of facets increases by

$$1 - (k + l) + kl = (k - 1)(l - 1). \qquad \square$$

If Conjecture 7.10.2 holds, we can conclude that the Hibi–Li conjecture as formulated in Conjecture 7.10.10 holds in codimension 1. For smaller dimensions, we have a common bound on all f-vectors of marked chain-order polyhedra associated to a marked poset (P, λ) by the f-vector of the generic marked poset polyhedron obtained from Corollary 7.7.10. Unfortunately, this does not help for obtaining a comparison as in the Hibi–Li conjecture.

8. Distributive and Anti-Blocking Polyhedra

In this chapter we aim to generalize the concept of piecewise-linear transfer maps to a larger class of polyhedra. The results of this chapter are joint work in progress with Raman Sanyal and will also appear in [PS17].

Motivated by the work of Felsner and Knauer in [FK11], we replace marked order polyhedra by so called *distributive polyhedra*. A polyhedron $Q \subseteq \mathbb{R}^n$ is called distributive if the set of points in Q forms a distributive lattice with respect to the *dominance order* given by $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \dots, n$. An equivalent definition is that whenever $x, y \in Q$, the component-wise minimum $\min(x, y)$ and maximum $\max(x, y)$ are contained in Q as well. The reason distributive polyhedra are good candidates to replace marked order polyhedra is that by the characterization given in [FK11], a polyhedron $Q \subseteq \mathbb{R}^n$ is distributive if and only if it can be defined using only inequalities $x_i \leq \alpha x_j + c$ for $\alpha, c \in \mathbb{R}$, $\alpha \geq 0$. Hence, their describing inequalities can be encoded in a directed graph with nodes $[n]$ and an edge from i to j with two weights α, c for each defining inequality $x_i \leq \alpha x_j + c$.

The order polytopes of Chapter 1 and the marked order polyhedra of Chapter 5 are distributive and the associated directed graph is essentially the Hasse diagram of P with markings replaced by weighted loops. To generalize the transfer map $\mathcal{O}(P, \lambda) \to \mathcal{C}(P, \lambda)$, note that the term $\max_{q \prec p} x_q$ for some $p \in \tilde{P}$ may as well be described as the maximum over all left hand sides of inequalities $\cdots \leq x_p$ describing $\mathcal{O}(P, \lambda)$. Hence, we can describe the transfer map geometrically by noting

$$\varphi(x)_p = x_p - \max_{q \prec p} x_q = \max\{\mu \geq 0 : x - \mu e_p \in \mathcal{O}(P, \lambda)\}.$$

Since this description does not depend on the poset structure at all, we may just do the same for any polyhedron $Q \subseteq \mathbb{R}^n$ and define a "transfer map" $\varphi \colon Q \to \mathbb{R}^n$ by

$$\varphi(x)_i = \max\{\mu \geq 0 : x - \mu e_i \in Q\},$$

assuming that each x_i appears on the right hand side of some inequality. In case of $P = \widetilde{\mathcal{O}}(P, \lambda)$—the marked order polyhedron projected to $\mathbb{R}^{\tilde{P}}$—this is exactly the transfer map to the projected marked chain polyhedron $\widetilde{\mathcal{C}}(P, \lambda)$. In general, it is not clear whether φ is injective or even whether the image is still a polyhedron.

We will show in this chapter that for a large class of distributive polyhedra, we do get an injective piecewise-linear map with image a polyhedron, hence generalizing the transfer maps of marked order polyhedra. The images $\varphi(P)$ will be *anti-blocking* polyhedra as introduced by Fulkerson in [Ful71]. A polyhedron $Q \subseteq \mathbb{R}^n$ is called anti-blocking, if its defining inequalities are $x_i \geq 0$ for $i = 1, \ldots, n$ together with inequalities of the form $a_1 x_1 + \cdots + a_n x_n \leq 1$ where all $a_i \geq 0$. Equivalently, Q is contained in the non-negative orthant $\mathbb{R}^n_{\geq 0}$ and whenever $x \in Q$ and $y \in \mathbb{R}^n_{\geq 0}$ with $y \leq x$ with respect to the dominance order, then $y \in Q$ as well.

8.1. Distributive Polyhedra and Marked Networks

We start by defining the underlying data of the distributive polyhedra we are considering.

Definition 8.1.1. A *marked network* $\Gamma = (V, E, \alpha, c, \lambda)$ consists of

i) a finite set of nodes V,
ii) a finite set of directed edges E with source $s(e) \in V$ and target $t(e) \in V$ such that $s(e) \neq t(e)$ for all $e \in E$,
iii) a positive edge weight $\alpha \colon E \to \mathbb{R}_{>0}$,
iv) a real edge weight $c \colon E \to \mathbb{R}$,
v) a marking $\lambda \colon V^* \to \mathbb{R}$ on a subset $V^* \subseteq V$ of *marked* nodes.

Note that different edges may have the same source and destination, so we are describing a loop-free directed multigraph equipped with two edge weights and a real marking of a subset of its nodes. We will denote an edge $e \in E$ such that $s(e) = v$ and $t(e) = w$ by $v \xrightarrow{e} w$. Analogous to marked posets, we denote the set of unmarked nodes by $\tilde{V} = V \setminus V^*$.

To each such network we associate a distributive polyhedron.

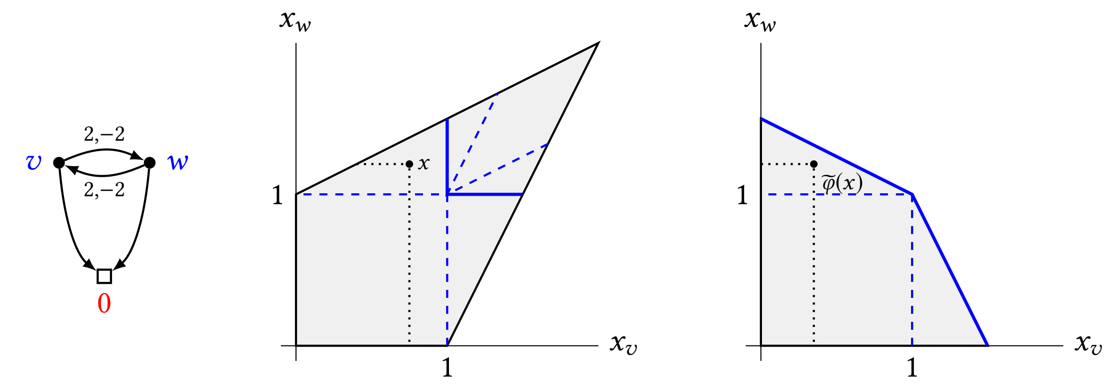
Definition 8.1.2. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network. The distributive polyhedron $\mathcal{D}(\Gamma)$ is the set of all $x \in \mathbb{R}^V$ such that $\alpha_e x_w + c_e \leq x_v$ for each edge $v \xrightarrow{e} w$ and $x_a = \lambda(a)$ for each marked node $a \in V^*$.

Since the coordinates in V^* are fixed, it is sometimes convenient to consider the projection $\widetilde{\mathcal{D}}(\Gamma)$ in $\mathbb{R}^{\tilde{V}}$ instead. We will denote by $\pi_{\tilde{V}}$ the projection $\mathbb{R}^V \to \mathbb{R}^{\tilde{V}}$ and by ι_λ the embedding $\mathbb{R}^{\tilde{V}} \to \mathbb{R}^V$ such that $\iota_\lambda(x)_a = \lambda(a)$ for all $a \in V^*$. Using this notation, we have $\widetilde{\mathcal{D}}(\Gamma) = \pi_{\tilde{V}}(\mathcal{D}(\Gamma))$ and $\mathcal{D}(\Gamma) = \iota_\lambda\big(\widetilde{\mathcal{D}}(\Gamma)\big)$.

Note that every distributive polyhedron $P \subseteq \mathbb{R}^n$ is given as $P = \widetilde{\mathcal{D}}(\Gamma)$ by some marked network Γ with $\tilde{V} = [n]$ by the Felsner–Knauer characterization. The difference between our marked networks and the networks used in [FK11] is that instead of allowing loops with $\alpha \in \{0, 2\}$ to obtain inequalities involving only one variable we have marked nodes, no loops and require $\alpha > 0$. Furthermore, the weights α are reciprocal to those in [FK11].[1]

For a marked poset (P, λ), let $\Gamma(P, \lambda)$ be given by the Hasse diagram of P with trivial weights and marking λ. To be precise, $\Gamma(P, \lambda)$ is given by $V = P$, an edge $e_{p,q}$ from p to q with weights $\alpha_{e_{p,q}} = 1$ and $c_{e_{p,q}} = 0$ for every covering relation $q \prec p$ and the marking is just λ. Note that in the usual way to draw Hasse diagrams, our edges are now pointing to the bottom. Comparing Definitions 6.1.1 and 8.1.2 we see that the marked order polyhedron $\mathcal{O}(P, \lambda)$ is exactly the distributive polyhedron $\mathcal{D}(\Gamma(P, \lambda))$.

[1]The reason for choosing a different convention of underlying networks in comparison to [FK11] is that loops would interfere with our definition of the transfer map and we want to have terminology closer to that of marked poset polyhedra.



(a) a network Γ (b) the polytope $\widetilde{\mathcal{D}}(\Gamma)$ (c) the image $\widetilde{\varphi}(\widetilde{\mathcal{D}}(\Gamma))$

Figure 8.1.: The marked network Γ of Example 8.2.2 with the associated distributive polytope and its "folded" image under the non-injective transfer map.

8.2. Transfer Maps and Failure of Injectivity

We are now ready to define transfer maps for distributive polyhedra whose underlying marked networks have at least all sinks marked. As usual, by a *sink* we mean a node $v \in V$ such that no edge in E has source v.

Definition 8.2.1. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network with at least all sinks marked. Define the *transfer map* $\varphi\colon \mathbb{R}^V \to \mathbb{R}^V$ by

$$\varphi(x)_v = \begin{cases} x_v & \text{if } v \in V^*, \\ x_v - \max_{v \xrightarrow{e} w}(\alpha_e x_w + c_e) & \text{otherwise.} \end{cases}$$

Furthermore, let $\widetilde{\varphi}\colon \mathbb{R}^{\tilde{V}} \to \mathbb{R}^{\tilde{V}}$ be given by $\widetilde{\varphi} = \pi_{\tilde{V}} \circ \varphi \circ \iota_\lambda$.

Unfortunately, transfer maps given by marked networks are not always injective. When Γ is acyclic, we can show injectivity similar to the injectivity of transfer maps in the continuous family of marked poset polyhedra. However, when Γ contains cycles, the behavior of the transfer map becomes more delicate.

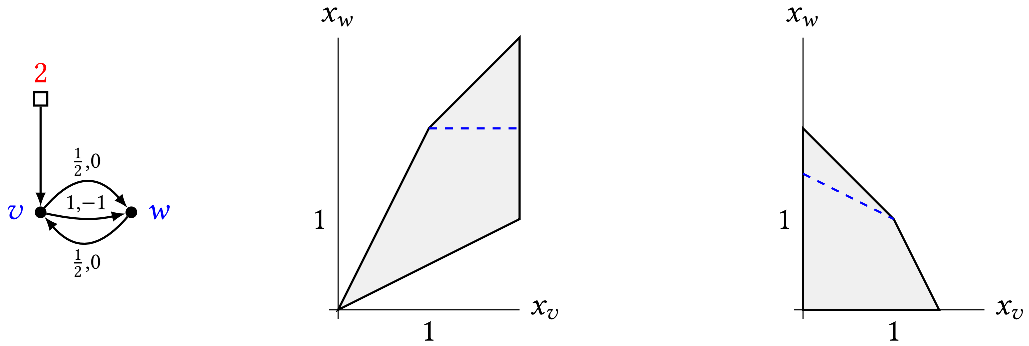
In the following we will depict an edge $v \xrightarrow{e} w$ with weights α_e and c_e as

$$v \xrightarrow{\alpha_e, c_e} w \ ,$$

where blue labels are node names. Marked nodes are drawn as squares with red labels and when edge weights are omitted, we always assume $\alpha_e = 1$ and $c_e = 0$.

Example 8.2.2. Let Γ be the marked network depicted in Figure 8.1a. The distributive polyhedron $\widetilde{\mathcal{D}}(\Gamma)$ is a kite given by the inequalities $0 \le x_v$, $0 \le x_w$, $2x_v - 2 \le x_w$ and $2x_w - 2 \le x_v$ as shown in Figure 8.1b. The transfer map for this network is given on $\mathbb{R}^{\tilde{V}}$ by

$$\widetilde{\varphi}\begin{pmatrix} x_v \\ x_w \end{pmatrix} = \begin{pmatrix} x_v - \max\{0, 2x_w - 2\} \\ x_w - \max\{0, 2x_v - 2\} \end{pmatrix}.$$



(a) a network Γ (b) the polytope $\widetilde{\mathcal{D}}(\Gamma)$ (c) the image $\widetilde{\varphi}(\widetilde{\mathcal{D}}(\Gamma))$

Figure 8.2.: The marked network Γ of Example 8.2.3 with the associated distributive polytope and its injective image under the transfer map.

It is not injective on $\widetilde{\mathcal{D}}(\Gamma)$ as for example the vertices $(0,0)$ and $(2,2)$ both get mapped to the origin. In fact, the map is 2-to-1 and "folds" the polytope along the thick blue line in Figure 8.1b. The dashed lines in the lower left half stay fixed under the transfer map and have the same image as the dashed lines in the upper right half. The geometric behavior of the transfer map mentioned in the introduction is shown for some $x \in \widetilde{\mathcal{D}}(\Gamma)$ using dotted lines. Given $x \in \widetilde{\mathcal{D}}(\Gamma)$ the transfer map measures how far we can shift x in each direction $-e_v$ for $v \in \tilde{V}$ while still staying inside $\widetilde{\mathcal{D}}(\Gamma)$. $\lozenge$

Example 8.2.3. Let Γ be the marked network depicted in Figure 8.2a. The distributive polyhedron $\widetilde{\mathcal{D}}(\Gamma)$ is a quadrilateral given by the inequalities $\frac{1}{2}x_v \leq x_w$, $\frac{1}{2}x_w \leq x_v$, $x_w - 1 \leq x_v$ and $x_v \leq 2$ as shown in Figure 8.2b. The transfer map for this network is given on $\mathbb{R}^{\tilde{V}}$ by

$$\widetilde{\varphi}\begin{pmatrix} x_v \\ x_w \end{pmatrix} = \begin{pmatrix} x_v - \max\{\frac{1}{2}x_w, x_w - 1\} \\ x_w - \frac{1}{2}x_v \end{pmatrix}.$$

In this example, the transfer map is injective and maps $\widetilde{\mathcal{D}}(\Gamma)$ to the anti-blocking polytope depicted in Figure 8.2c. The dashed line divides $\widetilde{\mathcal{D}}(\Gamma)$ into the two linearity regions of the transfer map. We will come back to this example in Section 8.3.2 after constructing inverse transfer maps and describing the inequalities for $\varphi(\mathcal{D}(\Gamma))$. $\lozenge$

As we have seen in Examples 8.2.2 and 8.2.3, some cyclic networks lead to injective transfer maps while others do not. The important difference in the two examples is the product of weights along the cycles.

Definition 8.2.4. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network. A *finite walk* in Γ is a finite alternating sequence of nodes and edges $(v_1, e_1, v_2, e_2, \cdots, v_r, e_r, v_{r+1})$ starting and ending with a node such that $v_i \xrightarrow{e_i} v_{i+1}$ for $i = 1, \ldots, r$. We usually denote a finite walk by

$$\gamma\colon v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_r} v_{r+1}.$$

The *length* of γ is given by $|\gamma| = r$ and its *weight* is defined as the product over the edge weights α_e, that is,

$$\alpha_\gamma = \prod_{i=1}^{r} \alpha_{e_i}.$$

If $v_{r+1} = v_1$, we call γ a *cycle*. Following [FK11] a cycle γ is called *lossy* if $\alpha_\gamma < 1$, *gainy* if $\alpha_\gamma > 1$ and *breakeven* if $\alpha_\gamma = 1$. We call a cycle *elementary* if apart from $v_1 = v_{r+1}$ no node is visited by γ twice.

In the following section, we will show that the observation made in Examples 8.2.2 and 8.2.3 is true in general: when Γ contains only lossy cycles, the transfer map is injective.

8.3. Lossy Cycles and Infinite Walks

Throughout this section we assume that $\Gamma = (V, E, \alpha, c, \lambda)$ is a marked network such that every sink is marked and every cycle is lossy. Our goal is to construct an inverse to the transfer map φ defined in Definition 8.2.1 and show that the projection of $\varphi(\mathcal{D}(\Gamma))$ to $\mathbb{R}^{\tilde{V}}$ is an anti-blocking polyhedron by giving explicit inequalities determined by walks in Γ.

If we want to mimic the recursive definition of the inverse transfer map in Theorem 7.2.1 or the closed form in Remark 7.2.2, a cyclic network brings new challenges: the recursion does not terminate for cyclic networks, while the closed form might have to involve infinite walks and hence infinite series. Inspecting the proof of Theorem 7.2.1, we see that the appearing chains are those obtained by starting at an unmarked element and walking down in the Hasse diagram of P along covering relations until ending up at some marked element. If we start at an unmarked node in a cyclic network and walk along edges we either end up at a marked element after finitely many steps or continue visiting unmarked elements indefinitely. Thus, we define a possibly infinite set of walks in Γ.

Definition 8.3.1. To Γ associate the set of walks $\mathcal{W}$ consisting of finite walks

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_r} v_{r+1} \quad \text{with } v_i \in \tilde{V} \text{ for } i \le r \text{ and } v_{r+1} \in V^*, \tag{8.1}$$

as well as infinite walks

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots \quad \text{with all } v_i \in \tilde{V}. \tag{8.2}$$

We also include finite walks with $r = 0$ in $\mathcal{W}$, i.e., for all marked $a \in V^*$ we have $(a) \in \mathcal{W}$.

Given a walk $\gamma \in \mathcal{W}$ starting in v and an edge $w \xrightarrow{e} v$ from an unmarked element $w \in \tilde{V}$, denote by $w \xrightarrow{e} \gamma$ the walk in $\mathcal{W}$ obtained from γ by prepending the edge e.

In order to define the inverse transfer map, we want to associate to each walk $\gamma \in \mathcal{W}$ an affine linear form $\Sigma(\gamma)$ on $\mathbb{R}^V$ such that the following properties are satisfied:

(I) $\Sigma(a)(x) = x_a$ for all $a \in V^*$,

(II) $\Sigma(v \xrightarrow{e} \gamma)(x) = \alpha_e \Sigma(\gamma)(x) + (x_v + c_e)$ for all $\gamma \in \mathcal{W}$ and $e \in E$ such that $v \xrightarrow{e} \gamma \in \mathcal{W}$.

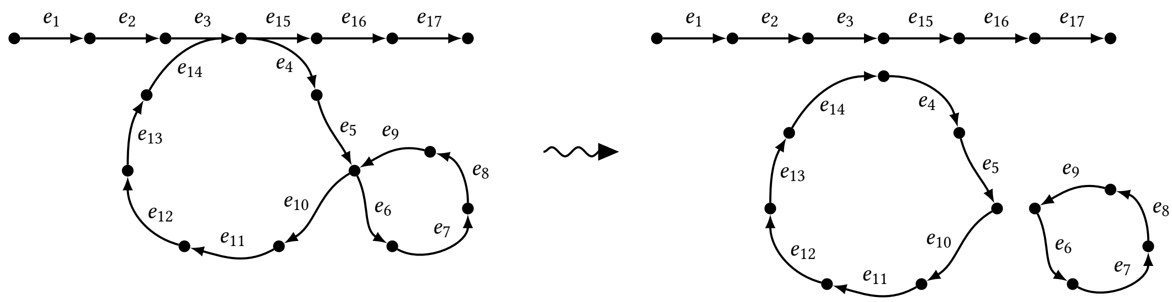


Figure 8.3.: The decomposition of a finite walk into an acyclic walk and elementary cycles used in the proof of Proposition 8.3.2.

To achieve this, we need the following statement on convergence of infinite series.

Proposition 8.3.2. *Let $\gamma \in \mathcal{W}$ be an infinite walk as in (8.2). The infinite series*

$$\sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1}\alpha_{e_j}\right)\left(x_{v_k}+c_{e_k}\right)$$

absolutely converges for all $x \in \mathbb{R}^V$.

Proof. Since Γ has only finitely many vertices and edges, we have $\left|x_{v_k}+c_{e_k}\right| \leq M$ for $M = \max_{v\in V, e\in E}(|x_v|+|c_e|)$. Therefore, it is enough to show absolute convergence of $\sum_{k=1}^{\infty}\prod_{j=1}^{k-1}\alpha_{e_j}$. Using the root test, it is sufficient to show that

$$\limsup_{k\to\infty}\left(\prod_{j=1}^{k}\alpha_{e_j}\right)^{\frac{1}{k}} < 1.$$

Since Γ is finite, there are only finitely many acyclic walks and elementary cycles and we may define

$$a = \max\left\{\alpha_\gamma^{1/|\gamma|} : \gamma \text{ is an elementary cycle}\right\}, \quad \text{and}$$
$$b = \max\left\{\alpha_\gamma^{1/|\gamma|} : \gamma \text{ is an acyclic walk}\right\}.$$

Now fix some $k \in \mathbb{N}$ and consider the finite walk

$$\gamma^{(k)} \colon v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_k} v_{k+1}$$

We may decompose $\gamma^{(k)}$ into an acyclic walk from v_1 to v_{k+1} and finitely many elementary cycles as depicted in Figure 8.3. If the acyclic component has $t \leq k$ edges there are $k-t$ edges in the elementary cycles in total and we obtain

$$\alpha_{\gamma^{(k)}} = \prod_{j=1}^{k}\alpha_{e_j} \leq a^{k-t}b^t = \left(\frac{b}{a}\right)^t a^k.$$

Let $\ell \in \mathbb{N}$ be the maximal length of an acyclic walk in Γ and set $c = \max\{b/a, 1\}$ to obtain

$$\left(\prod_{j=1}^{k} \alpha_{e_j}\right)^{\frac{1}{k}} \leq \left(c^\ell a^k\right)^{\frac{1}{k}} = c^{\frac{\ell}{k}} a \xrightarrow{k\to\infty} a.$$

Since all cycles in Γ are lossy by assumption, we have $a < 1$, finishing the proof. □

Using Proposition 8.3.2, we can define the desired linear forms.

Definition 8.3.3. For $\gamma \in \mathcal{W}$ define an affine linear form $\Sigma(\gamma)\colon \mathbb{R}^V \to \mathbb{R}$ as follows. If γ is a finite walk as in (8.1), let

$$\Sigma(\gamma)(x) = \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j}\right) \left(x_{v_k} + c_{e_k}\right) + \left(\prod_{j=1}^{r} \alpha_{e_j}\right) x_{v_{r+1}}.$$

If γ is an infinite walk as in (8.2), let

$$\Sigma(\gamma)(x) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j}\right) \left(x_{v_k} + c_{e_k}\right).$$

By construction, the defined linear forms satisfy the properties (I) and (II). Indeed, the properties (I) and (II) uniquely determine the linear forms $\Sigma(\gamma)$ given the convergence in Proposition 8.3.2.

Proposition 8.3.4. *For any $x \in \mathbb{R}^V$ we have $\sup_{\gamma\in\mathcal{W}} \Sigma(\gamma)(x) < \infty$.*

Proof. Let the constants $M, a, b, c \in \mathbb{R}$ and $\ell \in \mathbb{N}$ be given as in the proof of Proposition 8.3.2. For finite walks $\gamma \in \mathcal{W}$ as in (8.1), we have

$$\Sigma(\gamma)(x) = \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j}\right) \left(x_{v_k} + c_{e_k}\right) + \left(\prod_{j=1}^{r} \alpha_{e_j}\right) x_{v_{r+1}} \leq M \sum_{k=1}^{r+1} c^\ell a^{k-1} \leq \frac{Mc^\ell}{1-a}.$$

Likewise, for infinite walks as in (8.2), we have

$$\Sigma(\gamma)(x) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j}\right) \left(x_{v_k} + c_{e_k}\right) \leq M \sum_{k=1}^{\infty} c^\ell a^{k-1} = \frac{Mc^\ell}{1-a}.$$ □

8.3.1. The Inverse Transfer Map

We are now ready to obtain an inverse to the transfer map φ. For $v \in \tilde{V}$ denote by $\mathcal{W}_v$ the set of all walks $\gamma \in \mathcal{W}$ starting in v.

Theorem 8.3.5. *The transfer map $\varphi\colon \mathbb{R}^V \to \mathbb{R}^V$ is a piecewise-linear bijection with inverse $\psi\colon \mathbb{R}^V \to \mathbb{R}^V$ given by*

$$\psi(y)_v = \sup_{\gamma\in\mathcal{W}_v} \Sigma(\gamma)(y).$$

Furthermore, the inverse of $\widetilde{\varphi} = \pi_{\tilde{V}} \circ \varphi \circ \iota_\lambda$ is given by $\widetilde{\psi} = \pi_{\tilde{V}} \circ \psi \circ \iota_\lambda$.

Since part of the proof of Theorem 8.3.5 will be relevant when we give a description of $\varphi(\mathcal{D}(\Gamma))$ below, we provide the following lemma first.

Lemma 8.3.6. *For any $x \in \mathbb{R}^V$ and $v \in V$ we have*

$$\sup_{\gamma \in \mathcal{W}_v} \Sigma(\gamma)(\varphi(x)) \leq x_v.$$

Proof. Let $y = \varphi(x)$ for $x \in \mathbb{R}^V$. For a finite walk $\gamma \in \mathcal{W}$ as in (8.1) starting in $v_1 = v$, we have

$$\begin{aligned}
\Sigma(\gamma)(y) &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(y_{v_k} + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) y_{v_{r+1}} \\
&= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \max_{v_k \xrightarrow{e} w} (\alpha_e x_w + c_e) + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} \\
&\leq \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} \\
&= x_{v_1} = x_v.
\end{aligned}$$

For an infinite walk as in (8.2) starting in $v_1 = v$, we have

$$\begin{aligned}
\Sigma(\gamma)(y) &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(y_{v_k} + c_{e_k} \right) \\
&\leq \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) = x_{v_1} = x_v.
\end{aligned}$$

□

Proof of Theorem 8.3.5. For marked $a \in V^*$ the only walk in $\mathcal{W}_a$ is the trivial walk a with $\Sigma(a)(y) = y_a$ by property (I), so that $\psi(y)_a = y_a$. For unmarked $v \in \tilde{W}$, all walks $\gamma \in \mathcal{W}_v$ are of them form $v \xrightarrow{e} \gamma'$ for an edge $v \xrightarrow{e} w$ and $\gamma' \in \mathcal{W}_w$. Hence, by the recursive property (II) we have

$$\Sigma(\gamma)(y) = \alpha_e \Sigma(\gamma')(y) + (y_v + c_e).$$

We conclude that ψ satisfies the recursion

$$\psi(y)_v = y_v + \max_{v \xrightarrow{e} w} \left(\alpha_e \psi_y(w) + c_e \right) \quad \text{for all } v \in \tilde{V}.$$

Comparing this to the definition of φ, we see that $\varphi \circ \psi$ is the identity on $\mathbb{R}^V$.

Regarding the composition $\psi \circ \varphi$, first note that $\psi(\varphi(x))_a = x_a$ for all marked $a \in V^*$ and $\psi(\varphi(x))_v \leq x_v$ for all $v \in \tilde{V}$ by Lemma 8.3.6. Hence, to show that $\psi(\varphi(x))_v = x_v$ for $v \in \tilde{V}$, it is enough to construct a walk $\gamma \in \mathcal{W}_v$ such that $\Sigma(\gamma)(\varphi(x)) \geq x_v$. Let $v_1 = v$ and successively pick an edge $v_k \xrightarrow{e_k} v_{k+1}$ such that

$$\alpha_{e_k} x_{v_{k+1}} + c_{e_k} = \max_{v_k \xrightarrow{e} w} \left(\alpha_e x_w + c_e \right),$$

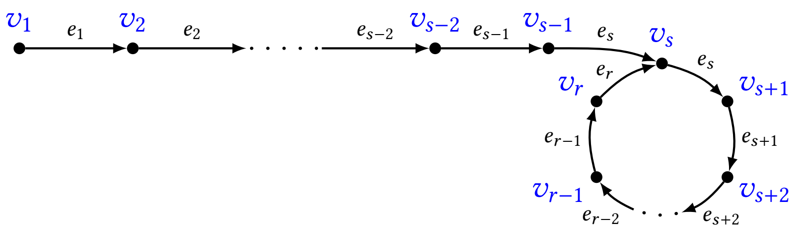


Figure 8.4.: A monocycle. Note that the visible nodes are pairwise distinct.

until either v_{k+1} is marked or v_{k+1} already appeared in $\{v_1, \ldots, v_k\}$.

In the first case we constructed a finite walk $\gamma \in \mathcal{W}_v$ as in (8.1) satisfying

$$\begin{aligned}\Sigma(\gamma)(\varphi(x)) &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(\varphi(x)_{v_k} + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) \varphi(x)_{v_{r+1}} \\ &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} = x_{v_1} = x_v.\end{aligned}$$

In the second case, we ended at an unmarked element $v_{r+1} = v_s$ for $s \leq r$. This yields an infinite walk $\gamma \in \mathcal{W}_v$ of the form

$$v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{s-1}} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots \tag{8.3}$$

That is, γ walks from v_1 to v_s and then infinitely often runs through the cycle

$$v_s \xrightarrow{e_s} v_{s+1} \xrightarrow{e_{s+1}} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s.$$

Treating indices $k > r$ accordingly, we obtain

$$\begin{aligned}\Sigma(\gamma)(\varphi(x)) &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(\varphi(x)_{v_k} + c_{e_k} \right) \\ &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) = x_{v_1} = x_v.\end{aligned}$$

In both cases $\Sigma(\gamma)(\varphi(x)) = x_v$ and we obtain $\psi(\varphi(x))_v = x_v$ as desired. We conclude that φ and ψ are mutually inverse piecewise-linear self-maps of $\mathbb{R}^V$. $\square$

Inspecting the proof of Theorem 8.3.5, we see that only a finite subset of $\mathcal{W}$ is necessary to define ψ. Namely, the acyclic finite walks with only the last node marked and the infinite walks that keep repeating an elementary cycle after a finite number of steps as in (8.3). We will refer to walks of the latter kind as *monocycles*. A more suggestive illustration of a monocycle can be found in Figure 8.4.

Definition 8.3.7. Let $\widehat{\mathcal{W}} \subseteq \mathcal{W}$ be the subset of walks $\gamma \in \mathcal{W}$ such that γ is either a finite acyclic walk or a monocycle as in (8.3) with pairwise distinct $v_1, \ldots, v_r$. For $v \in V$, Denote by $\widehat{\mathcal{W}}_v = \widehat{\mathcal{W}} \cap \mathcal{W}_v$ the set of walks in $\widehat{\mathcal{W}}$ starting in v.

Corollary 8.3.8. *The inverse transfer map $\psi \colon \mathbb{R}^V \to \mathbb{R}^V$ is given by*

$$\psi(y)_v = \max_{\gamma \in \widehat{\mathcal{W}}_v} \Sigma(\gamma)(y).$$

Since some of the $\gamma \in \widehat{\mathcal{W}}_v$ appearing in this description of the inverse transfer map might be monocycles, we want to give a finite expression for the linear form $\Sigma(\gamma)$ that is originally defined using an infinite series.

Proposition 8.3.9. *Let $\gamma \in \widehat{\mathcal{W}}$ be a monocycle in Γ, i.e.,*

$$\gamma \colon v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{s-1}} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots$$

with $v_i \in \tilde{V}$ for all $i \in \mathbb{N}$. Decompose γ into its acyclic beginning ρ and the repeating cycle δ, that is,

$$\begin{aligned} \rho &\colon v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{s-1}} v_s, \\ \delta &\colon v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s. \end{aligned}$$

Then for all $x \in \mathbb{R}^V$ we have

$$\Sigma(\gamma)(x) = \sum_{k=1}^{s-1} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) + \frac{\alpha_\rho}{1 - \alpha_\delta} \sum_{k=s}^{r} \left(\prod_{j=s}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right).$$

Proof. The infinite series in Definition 8.3.3 yields that $\Sigma(\gamma)(x)$ is equal to

$$\sum_{k=1}^{s-1} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) + \alpha_\rho \sum_{l=0}^{\infty} \left[\alpha_\delta^l \sum_{k=s}^{r} \left(\prod_{j=s}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) \right]. \tag{8.4}$$

Since all cycles in Γ are lossy, we have $\alpha_\delta < 1$ and the geometric series $\sum_{l=0}^{\infty} \alpha_\delta^l$ converges to $(1 - \alpha_\delta)^{-1}$. □

8.3.2. Anti-Blocking Images

In the previous section we showed that distributive polyhedra given by marked networks with only lossy cycles and at least all sinks marked admit a piecewise-linear injective transfer map $\varphi \colon \mathbb{R}^V \to \mathbb{R}^V$ analogous to the transfer map for marked order polyhedra. In this section we keep the same premise and focus on the image $\varphi(\mathcal{D}(\Gamma))$. We show that its projection to $\mathbb{R}^{\tilde{V}}$ is an anti-blocking polyhedron with describing inequalities given by the walks in $\widehat{\mathcal{W}}$.

Definition 8.3.10. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network with only lossy cycles and at least all sinks marked. The *walk polyhedron* $\mathcal{A}(\Gamma)$ is the set of all $y \in \mathbb{R}^V$ satisfying the following conditions:

i) for each $a \in V^*$ an equation $y_a = \lambda_a$,

ii) for each $v \in \tilde{V}$ an inequality $0 \leq y_v$, and

iii) for each walk $a \xrightarrow{e} \gamma$ with $a \in V^*$ and $\gamma \in \widehat{\mathcal{W}}$ an inequality

$$\alpha_e \Sigma(\gamma)(y) + c_e \leq \lambda(a).$$

Since the coordinates in V^* are fixed, we furthermore define $\tilde{\mathcal{A}}(\Gamma) = \pi_{\tilde{V}}(\mathcal{A}(\Gamma))$ to be the projection of $\mathcal{A}(\Gamma)$ to $\mathbb{R}^{\tilde{V}}$.

Theorem 8.3.11. *The transfer map φ defined in Definition 8.2.1 restricts to a piecewise-linear bijection $\mathcal{D}(\Gamma) \to \mathcal{A}(\Gamma)$.*

Proof. To show that $\varphi(\mathcal{D}(\Gamma)) \subseteq \mathcal{A}(\Gamma)$, let $y = \varphi(x)$ for $x \in \mathcal{D}(\Gamma)$. By definition of φ we have $y_a = a$ for $a \in V^*$ and $y_v \geq 0$ for $v \in \tilde{V}$. Now let $a \xrightarrow{e} \gamma$ be a walk with $a \in V^*$ and $\gamma \in \widehat{\mathcal{W}}$. By Lemma 8.3.6, we have $\Sigma(\gamma)(y) \leq x_v$ and hence

$$\alpha_e \Sigma(\gamma)(y) + c_e \leq \alpha_e x_v + c_e \leq x_a = \lambda(a).$$

Now let y be any point in $\mathcal{A}(\Gamma)$ and let $x = \psi(y)$. By definition of ψ we have $x_a = \lambda(a)$ for all $a \in V^*$. For any edge $v \xrightarrow{e} w$ we have to show that $\alpha_e x_w + c_e \leq x_v$. Letting $\gamma \in \widehat{\mathcal{W}}_w$ be a walk starting in w constructed as in the proof of Theorem 8.3.5 such that $\Sigma(\gamma)(y) = x_w$. If v is not marked we can use $y_v \geq 0$ and Lemma 8.3.6 to obtain

$$\alpha_e x_w + c_e = \alpha_e \Sigma(\gamma)(y) + c_e = \Sigma(v \xrightarrow{e} \gamma)(y) - x_v \leq \Sigma(v \xrightarrow{e} \gamma)(y) \leq x_v.$$

If v is marked, $v \xrightarrow{e} \gamma$ is a walk as in Definition 8.3.10, so that

$$\alpha_e x_w + c_e = \alpha_e \Sigma(\gamma)(y) + c_e \leq \lambda(v) = x_v.$$

We conclude that $x \in \mathcal{D}(\Gamma)$ and hence $\psi(\mathcal{A}(\Gamma)) = \mathcal{D}(\Gamma)$. □

Proposition 8.3.12. *The walk polyhedron $\tilde{\mathcal{A}}(\Gamma)$ is anti-blocking.*

Proof. By Definition 8.3.10 we have $0 \leq y_v$ for all all $y \in \tilde{\mathcal{A}}(\Gamma)$ and $v \in \tilde{V}$. Furthermore, the coefficients in an inequality $\alpha_e \Sigma(\gamma)(y) + c_e \leq \lambda(a)$ are all non-negative: for finite walks they are just finite products of edge weights $\alpha_{e'}$ while for monocycles some of them are multiplied by the positive factor $\alpha_\rho/(1 - \alpha_\delta)$ as described in Proposition 8.3.9. □

Example 8.3.13 (continuation of Example 8.2.3)**.** Recall the marked network Γ with two unmarked nodes from Example 8.2.3 that is depicted in Figure 8.2 together with the distributive polytope $\tilde{\mathcal{D}}(\Gamma)$ and its anti-blocking image now denoted by $\tilde{\mathcal{A}}(\Gamma)$.

Since Γ does not have any edges with marked target and both elementary cycles contain all unmarked elements, the elements of $\widehat{\mathcal{W}}$ are just the monocycles given by the elementary cycles with trivial acyclic beginning. Let e, f, g be the edges between v and w, where e and f are the edges $v \to w$ with weights $(\alpha_e, c_e) = (\frac{1}{2}, 0)$ and $(\alpha_f, c_f) = (1, -1)$, respectively, and g is the edge $w \to v$ with weights $(\alpha_g, c_g) = (\frac{1}{2}, 0)$. The elements of $\widehat{\mathcal{W}}$ are exactly

$$
\begin{aligned}
\gamma_1 &: v \xrightarrow{e} w \xrightarrow{g} v \xrightarrow{e} w \xrightarrow{g} \cdots, \\
\gamma_2 &: v \xrightarrow{f} w \xrightarrow{g} v \xrightarrow{f} w \xrightarrow{g} \cdots, \\
\gamma_3 &: w \xrightarrow{g} v \xrightarrow{e} w \xrightarrow{g} v \xrightarrow{e} \cdots, \quad \text{and} \\
\gamma_4 &: w \xrightarrow{g} v \xrightarrow{f} w \xrightarrow{g} v \xrightarrow{f} \cdots.
\end{aligned}
$$

From Proposition 8.3.9 with trivial acyclic beginning ($s = 1$) we obtain

$$
\begin{aligned}
\Sigma(\gamma_1)(y) &= \frac{4}{3}x_v + \frac{2}{3}x_w, \\
\Sigma(\gamma_2)(y) &= 2x_v + 2x_w - 2, \\
\Sigma(\gamma_3)(y) &= \frac{2}{3}x_v + \frac{4}{3}x_w, \quad \text{and} \\
\Sigma(\gamma_4)(y) &= x_v + 2x_w - 1.
\end{aligned}
$$

Hence, the inverse transfer map on $\mathbb{R}^{\tilde{V}}$ is given by

$$
\widetilde{\psi}\begin{pmatrix} y_v \\ y_w \end{pmatrix} = \begin{pmatrix} \max\{\frac{4}{3}x_v + \frac{2}{3}x_w, 2x_v + 2x_w - 2\} \\ \max\{\frac{2}{3}x_v + \frac{4}{3}x_w, x_v + 2x_w - 1\} \end{pmatrix}.
$$

Note that the linearity regions are the two half-spaces given by the hyperplane $\frac{1}{3}x_v + \frac{2}{3}x_w = 1$ containing the dashed line in Figure 8.2c.

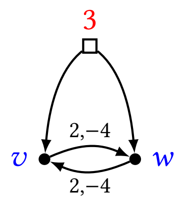
For the anti-blocking image $\widetilde{\mathcal{A}}(\Gamma)$ the only walks appearing in Definition 8.3.10 are $2 \to \gamma_1$ and $2 \to \gamma_2$ giving inequalities

$$
\begin{aligned}
\frac{4}{3}x_v + \frac{2}{3}x_w &\le 2, \quad \text{and} \\
2x_v + 2x_w &\le 4.
\end{aligned}
$$

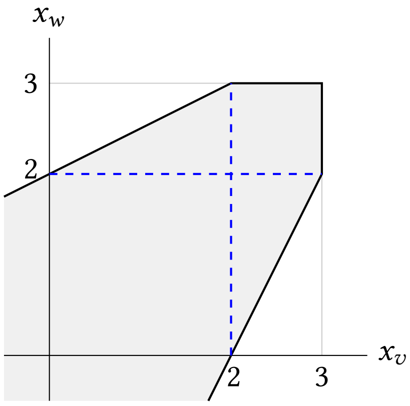
These correspond to the two non-trivial facets in Figure 8.2c. ◊

In Example 8.2.2, where we have a gainy cycle and the transfer map is not injective, the image was still an anti-blocking polytope. However, this is not true in general: in the following example we have a gainy cycle, an injective transfer map nevertheless, but the projected image $\pi_{\tilde{V}}(\mathcal{D}(\Gamma))$ is not anti-blocking.

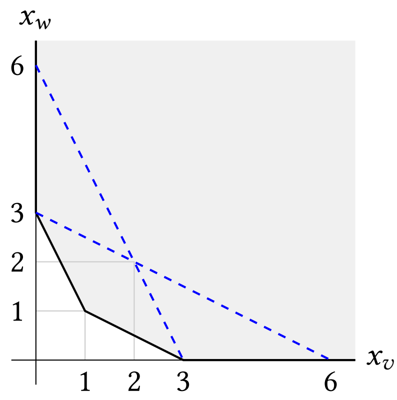
Example 8.3.14. Let Γ be the marked network shown in Figure 8.5a. The distributive



(a) a network Γ



(b) the polyhedron $\widetilde{\mathcal{D}}(\Gamma)$



(c) the image $\widetilde{\varphi}(\widetilde{\mathcal{D}}(\Gamma))$

Figure 8.5.: The marked network Γ of Example 8.3.14 with the associated distributive polyhedron and its non-anti-blocking image under the transfer map.

polyhedron $\widetilde{\mathcal{D}}(\Gamma)$ is the unbounded polyhedron in Figure 8.5b given by the inequalities $2x_v - 4 \leq x_w$, $2x_w - 4 \leq x_v$, $x_v \leq 3$ and $x_w \leq 3$. The transfer map is given on $\mathbb{R}^V$ by

$$\widetilde{\varphi}\begin{pmatrix} x_v \\ x_w \end{pmatrix} = \begin{pmatrix} x_v - 2x_w + 4 \\ x_w - 2x_v + 4 \end{pmatrix}.$$

Thus, the image $\widetilde{\varphi}(\widetilde{\mathcal{D}}(\Gamma))$ is the polyhedron given by inequalities $0 \leq y_v$, $0 \leq y_w$, $y_v + 2y_w \geq 3$ and $2y_v + y_w \geq 3$. It is depicted in Figure 8.5c and is not an anti-blocking polyhedron. In fact it is what is called a *blocking polyhedron* in [Ful71]: it is given given by inequalities $x_i \geq 0$ for all coordinates together with inequalities of the form $a_1x_1 + \cdots a_nx_n \geq 1$ where all $a_i \geq 0$. ◊

8.4. Duality

When $P \subseteq \mathbb{R}^n$ is a distributive polyhedron, the polyhedron $-P$ consisting of all $x \in \mathbb{R}^n$ such that $-x \in P$ is distributive as well. Considering P as a distributive lattice with the dominance order, we may think of $-P$ as its dual. To be precise, if we denote by P^{op} the set P equipped with the dual dominance order $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$ in the dominance order, then the map $P^{\mathrm{op}} \longrightarrow -P$ given by $x \mapsto -x$ is an isomorphism of distributive lattices: we have $x \leq y$ if and only if $-y \leq -x$ and $\max\{-x, -y\} = -\min\{x, y\}$.

When $P = \mathcal{D}(\Gamma)$ for some marked network Γ, there is a dual network Γ^{op} such that $-P = \mathcal{D}(\Gamma^{\mathrm{op}})$.

Definition 8.4.1. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network. The *dual marked network* $\Gamma^{\mathrm{op}} = (V, E^{\mathrm{op}}, \alpha^{\mathrm{op}}, c^{\mathrm{op}}, \lambda^{\mathrm{op}})$ is defined by

i) the same set of nodes V as Γ,
ii) an edge $w \xrightarrow{e^{\mathrm{op}}} v$ for each edge $v \xrightarrow{e} w$ in Γ,

iii) edge weights $\alpha_{e^{\mathrm{op}}} = \frac{1}{\alpha_e}$,
iv) edge weights $c_{e^{\mathrm{op}}} = \frac{c_e}{\alpha_e}$, and
v) the marking $\lambda^{\mathrm{op}} \colon V^* \to \mathbb{R}$ given by $\lambda^{\mathrm{op}}(a) = -\lambda(a)$.

Note that this really is a duality notion in the sense that $(\Gamma^{\mathrm{op}})^{\mathrm{op}} = \Gamma$.

Proposition 8.4.2. *For any marked network Γ we have $-\mathcal{D}(\Gamma) = \mathcal{D}(\Gamma^{\mathrm{op}})$.*

Proof. We have $x \in -\mathcal{D}(\Gamma)$ if and only if $-x \in \mathcal{D}(\Gamma)$, so x has to satisfy the conditions $-x_a = \lambda(a)$ for all $a \in V^*$ and $\alpha_e(-x_w) + c_e \leq (-x_v)$ for each edge $v \xrightarrow{e} w$ in Γ. These conditions are equivalent to $x_a = -\lambda(a) = \lambda^{\mathrm{op}}(a)$ for all $a \in V^*$ as well as

$$\frac{1}{\alpha_e} x_v + \frac{c_e}{\alpha_e} = \alpha_{e^{\mathrm{op}}} x_v + c_{e^{\mathrm{op}}} \leq x_w. \qquad \square$$

In our definition of the transfer map φ we considered the defining inequalities $\alpha_e x_w + c_e \leq x_v$ and for $v \in \tilde{V}$ set the v coordinate to the difference of x_v and the maximum left hand side in such an inequality. However, we could as well write the inequalities as $x_w \leq \frac{1}{\alpha_e} x_v - \frac{c_e}{\alpha_e}$ and then for $w \in \tilde{V}$ take the difference of x_w and the minimum over all right hand sides. This requires all sources to be marked and motivates the following definition.

Definition 8.4.3. Let $\Gamma = (V, E, \alpha, c, \lambda)$ be a marked network with at least all sources marked. Define the *dual transfer map* $\widehat{\varphi} \colon \mathbb{R}^V \to \mathbb{R}^V$ by

$$\widehat{\varphi}(x)_v = \begin{cases} -x_v & \text{if } v \in V^*, \\ -x_v + \min_{w \xrightarrow{e} v} \left(\frac{1}{\alpha_e} x_w - \frac{c_e}{\alpha_e} \right). & \end{cases}$$

As before, let $\widehat{\widehat{\varphi}} = \pi_{\tilde{V}} \circ \widehat{\varphi} \circ \iota_\lambda \colon \mathbb{R}^{\tilde{V}} \to \mathbb{R}^{\tilde{V}}$.

Now if Γ has all sources marked, then Γ^{op} has all sinks marked and we may compare the dual transfer map for Γ with the usual transfer map for the dual network Γ^{op}.

Proposition 8.4.4. *Let Γ be a marked network with at least all sources marked. The transfer maps satisfy the relation*

$$\widehat{\varphi}_\Gamma = \varphi_{\Gamma^{\mathrm{op}}} \circ -,$$

where $\widehat{\varphi}_\Gamma$ is the dual transfer map for Γ as in Definition 8.4.3, $\varphi_{\Gamma^{\mathrm{op}}}$ is the usual transfer map for Γ^{op} as in Definition 8.2.1 and $- \colon \mathbb{R}^V \to \mathbb{R}^V$ is given by $x \mapsto -x$.

Proof. For coordinates $v \in V^*$ the statement is trivial and for $v \in \tilde{V}$ we have

$$\begin{aligned} \varphi_{\Gamma^{\mathrm{op}}}(-x)_v &= -x_v - \max_{v \xrightarrow{e^{\mathrm{op}}} w} (-\alpha_{e^{\mathrm{op}}} x_w + c_{e^{\mathrm{op}}}) = -x_v - \max_{w \xrightarrow{e} v} \left(-\frac{1}{\alpha} x_w + \frac{c_e}{\alpha_e} \right) \\ &= -x_v + \min_{w \xrightarrow{e} v} \left(\frac{1}{\alpha} x_w - \frac{c_e}{\alpha_e} \right) = \widehat{\varphi}_\Gamma(x)_v. \end{aligned} \qquad \square$$

Applying the identity of Proposition 8.4.4 to the distributive polyhedron $\mathcal{D}(\Gamma)$, we obtain the following corollary.

Corollary 8.4.5. *For a marked network* Γ *with at least all sources marked and only gainy cycles, we have*

$$\widehat{\varphi}_\Gamma(\mathcal{D}(\Gamma)) = \varphi_{\Gamma^{\mathrm{op}}}(-\mathcal{D}(\Gamma)) = \varphi_{\Gamma^{\mathrm{op}}}(\mathcal{D}(\Gamma^{\mathrm{op}})) = \mathcal{A}(\Gamma^{\mathrm{op}}).$$

If we want to compare $\mathcal{A}(\Gamma^{\mathrm{op}})$ and $\mathcal{A}(\Gamma)$, we have to assume that Γ has all sinks and sources marked and all cycles are lossy and gainy, which is of course only possible if Γ is acyclic. We arrive at the following theorem.

Theorem 8.4.6. *Let* Γ *be an acyclic marked network with at least all sinks and sources marked. Denote by* $\pi_{\tilde{V}}$ *the projection* $\mathbb{R}^V \to \mathbb{R}^{\tilde{V}}$ *and let* $\overset{*}{-}\colon \mathbb{R}^V \to \mathbb{R}^V$ *be the map that takes negatives on coordinates in* V^* *and is the identity on coordinates in* $\tilde{V}$*. The following diagram of bijections commutes:*

$$\begin{tikzcd}
\mathcal{D}(\Gamma) \arrow[r, "-"] \arrow[d, "\widehat{\varphi}_\Gamma"] & \mathcal{D}(\Gamma^{\mathrm{op}}) \arrow[d, "\varphi_{\Gamma^{\mathrm{op}}}"] \\
\overset{*}{-}\mathcal{A}(\Gamma) \arrow[r, equal] \arrow[d, "\pi_{\tilde{V}}"] & \mathcal{A}(\Gamma^{\mathrm{op}}) \arrow[d, "\pi_{\tilde{V}}"] \\
\widetilde{\mathcal{A}}(\Gamma) \arrow[r, equal] & \widetilde{\mathcal{A}}(\Gamma^{\mathrm{op}}).
\end{tikzcd}$$

Proof. The only thing that remains to be shown is that $\mathcal{A}(\Gamma)$ maps to $\mathcal{A}(\Gamma^{\mathrm{op}})$ when taking negatives on marked coordinates. Since $\lambda^{\mathrm{op}} = -\lambda$ the equations $y_a = \lambda(a)$ are equivalent to $-y_a = \lambda^{\mathrm{op}}$ for all marked $a \in V^*$. The inequalities $0 \le y_v$ for $v \in \tilde{V}$ are the same for both polyhedra.

Now consider an inequality $\alpha_e \Sigma(\gamma)(y) + c_e \le \lambda_a$ for a walk $a \xrightarrow{e} \gamma$ in Γ with $a \in V^*$ and $\gamma \in \widehat{\mathcal{W}}$. Since Γ is acyclic, $a \xrightarrow{e} \gamma$ is of the form

$$a = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{r-1}} v_r \xrightarrow{e_r} v_{r+1} = b$$

with $a, b \in V^*$ and $v_i \in \tilde{V}$ for $i = 1, \ldots, r$. Hence, the inequality is given by

$$\sum_{k=1}^{r} \left(\prod_{j=0}^{k-1} \alpha_{e_j} \right) \left(y_{v_k} + c_{e_k} \right) + c_{e_0} \le \lambda_a - \left(\prod_{j=0}^{r} \alpha_{e_j} \right) \lambda(b).$$

Dividing both sides by $\prod_{j=0}^{r} \alpha_{e_j}$ and translating to the weights and marking of Γ^{op}, we obtain

$$\sum_{k=1}^{r} \left(\prod_{j=k}^{r} \alpha_{e_j^{\mathrm{op}}} \right) (y_{v_k} + c_{e_{k-1}^{\mathrm{op}}}) + c_{e_r^{\mathrm{op}}} \le \lambda^{\mathrm{op}}(b) - \left(\prod_{j=0}^{r} \alpha_{e_j^{\mathrm{op}}} \right) \lambda^{\mathrm{op}}(a),$$

which is exactly the inequality for $\mathcal{A}(\Gamma^{\mathrm{op}})$ given by the walk

$$b = v_{r+1} \xrightarrow{e_r^{\mathrm{op}}} v_r \xrightarrow{e_{r-1}^{\mathrm{op}}} \cdots \xrightarrow{e_1^{\mathrm{op}}} v_1 \xrightarrow{e_0^{\mathrm{op}}} v_0 = a. \qquad \square$$

Remark 8.4.7. In the setting of Theorem 8.4.6 the following diagram in general *does not* commute:

$$\begin{tikzcd}
\mathcal{D}(\Gamma) \arrow[r, equal] \arrow[d, "\widehat{\varphi}"] \arrow[dr, phantom, "!"] & \mathcal{D}(\Gamma) \arrow[d, "\varphi"] \\
\overset{*}{-}\mathcal{A}(\Gamma) \arrow[r, "\overset{*}{-}"] & \mathcal{A}(\Gamma).
\end{tikzcd}$$

In other words, the piecewise-linear self-map $\overset{*}{-} \circ \widehat{\varphi} \circ \psi$ on $\mathcal{A}(\Gamma)$ is non-trivial in general.

Example 8.4.8. We consider the order and chain polytopes associated to the linear poset $P\colon \widehat{0} \prec p \prec q \prec \widehat{1}$. Identifying $\mathbb{R}^{\tilde{P}}$ with $\mathbb{R}^2$ by writing points as $\left(\begin{smallmatrix} x_p \\ x_q \end{smallmatrix}\right)$, its projected order and chain polytopes are the triangles

$$\widetilde{\mathcal{O}}(P) = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}, \quad \text{and} \quad \widetilde{\mathcal{C}}(P) = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

The two transfer maps $\widetilde{\varphi}, \widehat{\widetilde{\varphi}}\colon \widetilde{\mathcal{O}}(P) \to \widetilde{\mathcal{C}}(P)$ are given by

$$\widetilde{\varphi}\begin{pmatrix}x_p\\x_q\end{pmatrix} = \begin{pmatrix}x_p\\x_q - x_p\end{pmatrix}, \quad \text{and} \quad \widehat{\widetilde{\varphi}}\begin{pmatrix}x_p\\x_q\end{pmatrix} = \begin{pmatrix}x_q - x_p\\1 - x_q\end{pmatrix}.$$

Hence, we obtain a non-trivial self-map $\sigma = \widehat{\widetilde{\varphi}} \circ \widetilde{\psi}$ on $\widetilde{\mathcal{C}}(P)$ with

$$\sigma\begin{pmatrix}y_p\\y_q\end{pmatrix} = \widehat{\widetilde{\varphi}}\begin{pmatrix}y_p\\y_p + y_q\end{pmatrix} = \begin{pmatrix}y_q\\1 - y_p - y_q\end{pmatrix}.$$

We see that σ is a cyclic permutation of the vertices of $\widetilde{\mathcal{C}}(P)$. ◊

9. Conclusion

Motivated by the examples of Part I, we have discussed various classes of polyhedra in the previous chapters. A diagram of the appearing classes, their inclusion relations as well as the discussed transfer maps is shown in Figure 9.1. Let us proceed by looking back to the examples of Part I and see how they fit into this picture.

9.1. Back to Part I

The order and chain polytopes from Chapter 1 have a known generalization to marked order and chain polytopes introduced by Ardila, Bliem and Salazar in [ABS11]. In Figure 9.1 we show their sleight generalization to the unbounded case, since requiring only minimal elements to be marked is enough to get a well-defined transfer map. Here we find the unweighted Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ and the Feigin–Fourier–Littelmann–Vinberg polytope $\mathrm{FFLV}(\lambda)$ from Chapter 3 as the marked order and marked chain polytope associated to the Gelfand–Tsetlin poset shown in Figure 5.1, respectively. Since the polytope of eigensteps $\Lambda(\mathcal{F}_\lambda)$ without norm conditions is an unweighted Gelfand–Tsetlin polytope as discussed in Section 4.3.1, it is a marked order polytope as well.

The weighted Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)_\mu$ as well as the polytope of eigensteps $\Lambda(\mathcal{F}_{\mu,\lambda})$ have additional linear constraints and hence fall into the class of conditional marked order polyhedra. However, as we have seen in Section 6.5, in fact every polyhedron is affinely isomorphic to a conditional marked order polyhedron and thus studying them in general is not very promising. Still, we were able to generalize a method introduced by De Loera and McAllister for Gelfand–Tsetlin polytopes in [DM04] to conditional marked order polyhedra, allowing the computation of the dimension of a face given a point in its relative interior.

The Stanley–Pitman polytope $\Pi_n(\xi)$ from [SP02], defined as the set of all $y \in \mathbb{R}^n_{\geq 0}$ with

$$y_1 + \cdots + y_i \leq \xi_1 + \cdots + \xi_i \quad \text{for } i = 1, \ldots, n$$

in Section 2.1, is the marked chain polytope $\widetilde{\mathcal{C}}(P,\lambda)$ associated to the marked poset (P,λ) in Figure 9.2, where the markings are given by $u_k = \sum_{i=1}^k \xi_i$. In fact, Stanley and Pitman already give a unimodular transformation of $\Pi_n(\xi)$ to $\widetilde{\mathcal{O}}(P,\lambda)$ and recognize it as a section of an order cone. The transformation they use is exactly the transfer map $\widetilde{\psi}\colon \widetilde{\mathcal{C}}(P,\lambda) \to \widetilde{\mathcal{O}}(P,\lambda)$.

Things get more interesting when considering the Cayley polytope C_n from Section 2.2. Recall that it is defined as the set of all $x \in \mathbb{R}^n$ such that $1 \leq x_1 \leq 2$ and $1 \leq x_{i+1} \leq 2x_i$ for $i = 1, \ldots, n-1$. The inequalities have coefficients different from 1 and hence C_n is not a marked order polytope anymore. However, it is a distributive polytope and equals

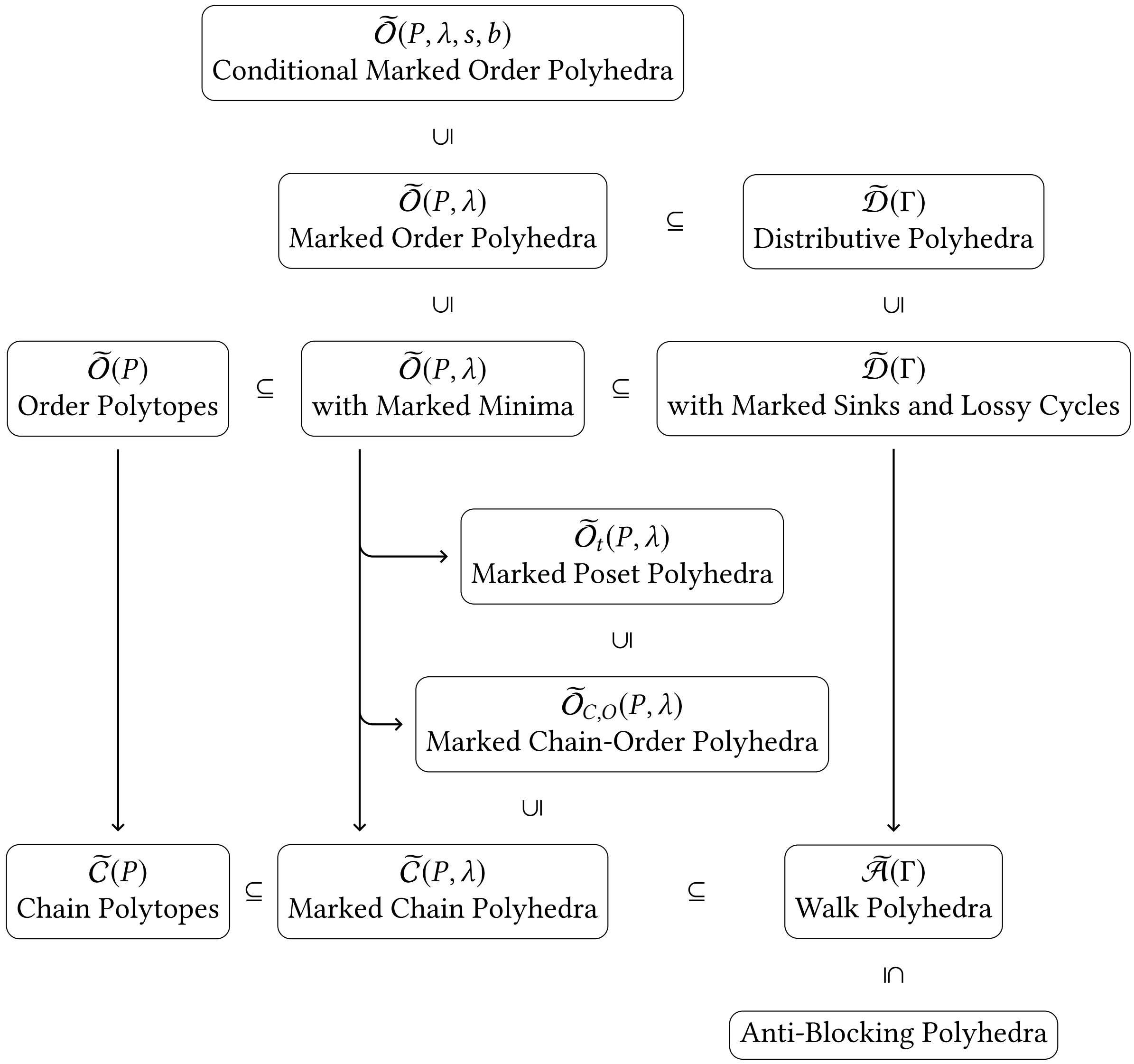


Figure 9.1.: A diagram of the different classes of polyhedra and their inclusions as discussed in this thesis. Arrows indicate the existence of piecewise-linear bijective transfer maps.

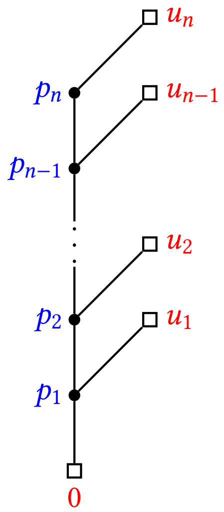


Figure 9.2.: The marked poset defining the Stanley–Pitman polytope.

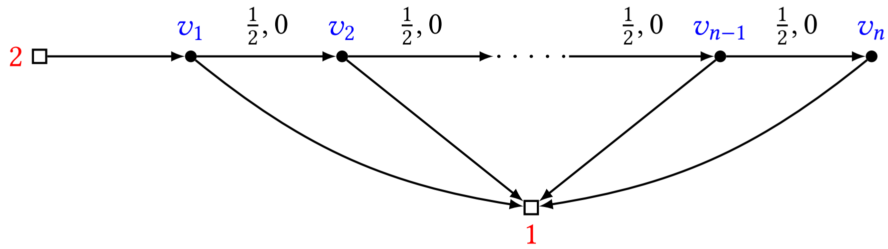


Figure 9.3.: The marked network defining the Cayley polytope.

$\widetilde{\mathcal{D}}(\Gamma)$ for the marked network in Figure 9.3. The geometric proof of Cayley's theorem (Theorem 2.2.1) in [KP14] uses a linear transformation to an anti-blocking polytope B_n that is exactly the walk polytope $\widetilde{\mathcal{A}}(\Gamma)$. Furthermore, the map $\widehat{\varphi}\colon C_n \to B_n$ is just the dual transfer map $\widetilde{\mathcal{D}}(\Gamma) \to \widetilde{\mathcal{A}}(\Gamma)$ as discussed in Section 8.4.

Let us now move to the lecture hall cones and polytopes discussed in Section 2.3. In the light of Chapter 6, we can define *marked lecture hall order polyhedra* as a common generalization of lecture hall cones and polytopes as in [BE97a], s-lecture hall cones and polytopes as in [BE97b], as well as lecture hall order cones and polytopes as in [BL16]. Let (P, λ) be any marked poset and $s\colon P \to \mathbb{N}_{>0}$ an arbitrary map. Define the marked lecture hall order polyhedron $\mathcal{O}(P, \lambda, s)$ as the set of all $x \in \mathbb{R}^n$ such that $x_a = s_a\lambda(a)$ for all marked $a \in P^*$ as well as

$$\frac{x_p}{s_p} \le \frac{x_q}{s_q} \quad \text{for } p \le q.$$

As usual, denote by $\widetilde{\mathcal{O}}(P, \lambda)$ the projection to $\mathbb{R}^{\tilde{P}}$. When $s(p) = 1$ for all p, we have $\mathcal{O}(P, \lambda, s) = \mathcal{O}(P, \lambda)$. When P is the linear poset $\hat{0} \prec p_1 \prec \cdots \prec p_n$ and $\lambda(\hat{0}) = 0$, we recover s-lecture hall cones and adding a maximal element with marking 1 we get the s-lecture hall polytope. For a poset with global minimum and maximum that are marked 0 and 1, respectively, we obtain the lecture hall order polytope and omitting the maximum we get the lecture hall order cone. Marked lecture hall order polyhedra are distributive and their defining network Γ is the marked Hasse diagram of (P,λ), where a covering relation $p \prec q$ gives an edge $q \xrightarrow{e} p$ with $\alpha_e = \frac{s_q}{s_p}$ and $c_e = 0$, and markings are multiplied by s_a. However, we may also describe $\mathcal{O}(P,\lambda,s)$ as a linear transformation of $\mathcal{O}(P,\lambda)$: let $T_s\colon \mathbb{R}^P \to \mathbb{R}^P$ be the linear map given by $T_s(x)_p = s_p x_p$, then $T_s(\mathcal{O}(P,\lambda)) = \mathcal{O}(P,\lambda,s) = \mathcal{D}(\Gamma)$. When all minima are marked, we may also define a *marked lecture hall chain polyhedron* $\mathcal{C}(P,\lambda,s) = \mathcal{A}(\Gamma)$. What we get is exactly the image of the marked chain polyhedron $\mathcal{C}(P,\lambda)$ under the linear map T_s. Even more so, the transfer maps fit in the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{O}(P,\lambda) & \xrightarrow{T_s} & \mathcal{O}(P,\lambda,s) & = & \mathcal{D}(\Gamma) \\
\downarrow{\scriptstyle \varphi_{(P,\lambda)}} & & & & \downarrow{\scriptstyle \varphi_\Gamma} \\
\mathcal{C}(P,\lambda) & \xrightarrow{T_s} & \mathcal{C}(P,\lambda,s) & = & \mathcal{A}(\Gamma).
\end{array}$$

In conclusion, all polyhedra appearing in Part I and the maps between them—except for weighted Gelfand–Tsetlin polytopes and polytopes of eigensteps with norm constraints—fit into the theory of distributive polyhedra with their transfer maps and anti-blocking images developed in Chapter 8.

9.2. Review, Open Questions and Further Directions

In this section we want to review the results of Chapters 6 to 8 and point out open questions as well as possible further directions of research.

Marked Order Polyhedra

In Chapter 6 we discussed marked order polyhedra, a potentially unbounded generalization of the marked order polytopes introduced by Ardila, Bliem and Salazar in [ABS11]. We were able to generalize the face structure description in terms of face partitions given by Stanley in [Sta86]. A similar study has previously been done by Jochemko and Sanyal in [JS14], but their requirements on face partitions missed certain edge cases as the example in Remark 6.2.20 shows. Furthermore, we gave a corrected definition of regularity—a condition on marked posets to obtain facets of the marked order polyhedron in correspondence with the covering relations of the underlying poset. This notion has been introduced in [Fou16] but also missed certain edge cases. Our approach to marked order polyhedra comes with a categorical description that was not discussed in the literature before. We described a contravariant functor $\mathcal{O}\colon \mathsf{MPos} \to \mathsf{Polyh}$ from the category of marked posets to the category of polyhedra and affine maps that associates to each marked poset its marked order polyhedron. This categorical approach allowed to identify face inclusions as being induced maps of certain quotient maps on marked posets and view identities like $\mathcal{O}(P_1 \sqcup P_2, \lambda_1 \sqcup \lambda_2) = \mathcal{O}(P_1,\lambda_1) \times \mathcal{O}(P_2,\lambda_2)$ as functorial properties of $\mathcal{O}$, in this case $\mathcal{O}$ sending coproducts to products.

To include the weighted Gelfand–Tsetlin polytopes $\mathrm{GT}(\lambda)_\mu$ and polytopes of eigensteps with norm conditions $\Lambda(\mathcal{F}_{\mu,\lambda})$, we introduced conditional marked order polyhedra. We were able to adopt the method of tiling matrices introduced by De Loera and McAllister in [DM04] to this larger class of polyhedra. However, we also argued that this class of polyhedra might be too big to obtain meaningful results, since *every* polyhedron is a conditional marked order polyhedron up to affine isomorphism, as shown in Proposition 6.5.7. In Chapter 10 we determine a non-redundant description of polytopes of eigensteps of equal norm tight frames—an instance of conditional marked order polyhedra. The methods used there do not have an obvious generalization even to arbitrary polytopes of eigensteps $\Lambda(\mathcal{F}_{\mu,\lambda})$ let alone to conditional marked order polyhedra in general. Hence, we state the following as the main open question of Chapter 6.

Question 9.2.1. Is there a subclass of conditional marked order polyhedra that allows a combinatorial face description and regularity condition similar to the theory of marked order polyhedra without additional constraints?

A Continuous Family of Marked Poset Polyhedra

In Chapter 7 we modified the transfer map $\mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ introduced for marked poset polytopes in [ABS11] as a generalization of the transfer map in [Sta86]. We required only minimal elements to be marked, which is enough to get a well-defined transfer map, and added a parameter $t_p \in [0,1]$ in the p-coordinate of the transfer map for each unmarked element $p \in \tilde{P}$. Surprisingly, the images $\mathcal{O}_t(P,\lambda) = \varphi_t(\mathcal{O}(P,\lambda))$ are still polyhedra for all $t \in [0,1]^{\tilde{P}}$ and their combinatorial type is constant along the relative interiors of the parametrizing hypercube. We gave a system of linear equations and inequalities depending on t that describes all of these polyhedra simultaneously. For $t \in \{0,1\}^{\tilde{P}}$ such that the set of all p with $t_p = 1$ is an order ideal in $\tilde{P}$, we recovered the marked chain-order polytopes introduced by Fang and Fourier in [FF16]. Using the transfer map, we found that the marked chain-order polytopes for arbitrary $t \in \{0,1\}^{\tilde{P}}$ still form an Ehrhart equivalent family of integrally closed lattice polytopes. Furthermore, the star element description of unimodular equivalences given by Fang and Fourier for admissible partitions has a direct generalization to arbitrary marked chain-order polyhedra.

The main new tools in Chapter 7 are the tropical subdivision discussed in Section 7.6 and the continuous degenerations of polyhedra from Section 7.7. Using both of them, we obtained a description of the vertices of generic marked poset polyhedra: in the tropical subdivision of $\mathcal{O}_t(P,\lambda)$, the vertices of the subdivision are exactly the vertices of the polytope when $t \in (0,1)^{\tilde{P}}$. Hence, they can be constructed by finding the vertices in the tropical subdivision of the marked order polyhedron $\mathcal{O}(P,\lambda)$—which can be done combinatorially—and transfer them to the generic marked poset polyhedron. When the parameter t moves to the boundary, the polyhedron degenerates and vertices can disappear. One of the open questions in this context is the following.

Question 9.2.2. Let v be any vertex of the generic marked poset polyhedron $\mathcal{O}_t(P,\lambda)$ for $t \in (0,1)^{\tilde{P}}$. Is there a combinatorial way to determine whether the image of v in

$\mathcal{O}_{C,O}(P, \lambda)$ is still a vertex for a given partition $\tilde{P} = C \sqcup O$? Furthermore, is there always some partition $\tilde{P} = C \sqcup O$ such that the image in $\mathcal{O}_{C,O}(P, \lambda)$ is still a vertex?

Having seen that we may modify a marked poset to a regular one without changing the associated marked order polyhedron, we were able to show in Section 7.9 that the same is true for marked poset polyhedra with an arbitrary parameter $t \in [0, 1]^{\tilde{P}}$. But what are the implications of regularity on the continuous family? We conjectured that our descriptions of marked chain-order polyhedra $\mathcal{O}_{C,O}(P, \lambda)$ and generic marked poset polyhedra $\mathcal{O}_t(P, \lambda)$ are non-redundant for regular marked posets in Conjecture 7.10.2 and showed that this is true for ranked marked posets in Proposition 7.10.6. Apart from this conjecture, the main open problem in this context is the Hibi–Li conjecture stated in Conjecture 7.10.10. In our refined formulation it states that that the f-vector of $\mathcal{O}_{C,O}(P, \lambda)$ is dominated by the f-vector of $\mathcal{O}_{C',O'}(P, \lambda)$ provided that C is contained in C'. We were able to show this for regular ranked marked posets in codimension 1 by counting facets.

Distributive and Anti-Blocking Polyhedra

In Chapter 8 we considered a larger class of polyhedra that still allows a piecewise-linear transfer map analogous to the one for marked order polyhedra. Motivated by the characterization of distributive polyhedra using network matrices done by Felsner and Knauer in [FK11], we defined a transfer map for every distributive polyhedron. Interestingly, this map is not always injective, but we were able to show that it is when all cycles in the underlying network are lossy, that is, have product of weights along the edges strictly less than one. This observation motivates the first question we want to state here.

Question 9.2.3. Which polyhedra $Q \subseteq \mathbb{R}^n$ admit a well-defined, injective, piecewise-linear transfer map $\varphi\colon Q \to \mathbb{R}^n$ given by

$$\varphi(x)_i = \max\{\, \lambda \in \mathbb{R}_{\geq 0} : x - \lambda e_i \in Q \,\}?$$

As Example 8.3.14 showed, even for some distributive polyhedra with non-lossy cycles the transfer map can still be injective.

In the case of only lossy cycles, we obtained a description of the image under the transfer map similar to the description of marked chain polyhedra. Instead of chains between marked elements, the inequalities are given by either finite walks between marked nodes, or infinite walks that start in a marked node and end in a repeating cycle. Both marked chain polyhedra and these walk polyhedra are anti-blocking: they are contained in $\mathbb{R}^n_{\geq 0}$ and satisfy the property that whenever $x \in Q$ and $0 \leq y \leq x$ with respect to dominance order, we have $y \in Q$ as well. From this perspective it is interesting to look back at the marked chain-order polyhedra $\mathcal{O}_{C,O}(P, \lambda)$ of Section 7.3. When the chain part is empty we obtain the marked order polyhedron $\widetilde{\mathcal{O}}(P, \lambda)$ which is distributive. When the order part is empty we obtain the marked chain polyhedron $\widetilde{\mathcal{C}}(P, \lambda)$ which is anti-blocking. For arbitrary partitions $\tilde{P} = C \sqcup O$ the marked chain-order polyhedron $\widetilde{\mathcal{O}}_{C,O}(P, \lambda)$ is neither distributive nor anti-blocking in general. However, we may define a mixed notion that specializes to being distributive and anti-blocking and captures the properties of marked chain-order polyhedra.

Definition 9.2.4. Let D and A be finite sets. A polyhedron Q in $\mathbb{R}^D \times \mathbb{R}^A$ is called *mixed distributive anti-blocking* if it satisfies the following properties:

i) given $(x, z) \in Q$ and $(y, z) \in Q$, we have $(\min(x, y), z) \in Q$ and $(\max(x, y), z) \in Q$,
ii) for all $(x, z) \in Q$ and $i \in A$ we have $z_i \geq 0$,
iii) when $(x, z) \in Q$ and $0 \leq y \leq x$, then $(x, y) \in Q$.

The minima, maxima and comparisons are with respect to the dominance orders on $\mathbb{R}^D$ and $\mathbb{R}^A$.

In the extreme cases where $A = \varnothing$ or $D = \varnothing$, we obtain the notions of distributive and anti-blocking polyhedra, respectively. It is straight forward to check that marked-chain order polyhedra are mixed distributive anti-blocking with respect to the decomposition $\mathbb{R}^{\tilde{P}} = \mathbb{R}^O \times \mathbb{R}^C$. Hence, it is natural to ask whether we can combine the approaches taken in Chapters 7 and 8 to obtain a "continuous family of network polyhedra".

Question 9.2.5. Does introducing a parameter $t \in [0, 1]^{\tilde{V}}$ in the transfer map of distributive polyhedra with marked sinks and lossy cycles yield a continuous family like the one for marked posets such that the combinatorial type of the images is constant along relative interiors of the parametrizing hypercube and the polyhedra at the vertices are mixed distributive anti-blocking?

Extending the idea that distributive polyhedra generalize marked order polyhedra, there are a number of questions we want to add here.

Question 9.2.6. Is there a description of the face structure of (some) distributive polyhedra like the one for marked order polyhedra?

Question 9.2.7. What is a "regular" marked network? That is, when are the edges of the network in correspondence with the facets of the associated distributive polyhedron?

Question 9.2.8. Do distributive polyhedra admit a natural subdivision into (products of) simplices on which the transfer map is linear?

Question 9.2.9. Is there a Minkowski sum decomposition of distributive polyhedra similar to the one for marked order polyhedra?

We believe that pursuing the questions raised in this concluding chapter will lead to a better understanding of the appearance of piecewise-linear maps in polyhedral geometry in general and hope for further applications of the theory developed in this thesis.

Appendix

10. Polytopes of Eigensteps of Finite Equal Norm Tight Frames

In this chapter, we come back to the polytope of eigensteps $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ as discussed in Chapter 4. We consider the special case of equal norm tight frames, that is, both the norm tuple $\mu = (\mu_1, \ldots, \mu_n)$ and the spectrum $\lambda = (\lambda_1, \ldots, \lambda_d)$ are constant. The results of this chapter are joint work with Tim Haga and have previously appeared in [HP16].

10.1. Introduction

By the Schur–Horn theorem for finite frames (Theorem 4.2.1) the frame variety $\mathcal{F}_{\mu,\lambda}$ for constant $\mu = (\mu_*, \ldots, \mu_*) \in \mathbb{R}^n$ and $\lambda = (\lambda_*, \ldots, \lambda_*) \in \mathbb{R}^d$ is non-empty if and only if $n\mu_* = d\lambda_*$. Hence, given any $\lambda_* > 0$ we must have norm squares $\mu_* = \frac{d}{n}\lambda_*$. Changing λ_* just dilates the polytope $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ and hence we can restrict ourselves to the case where $\lambda_* = n$ and $\mu_* = d$. For the rest of this chapter, we denote by $\Lambda_{n,d} = \Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ the polytope of eigensteps of finite equal norm tight frames, where $\mu = (d, \ldots, d) \in \mathbb{R}^n$ and $\lambda = (n, \ldots, n) \in \mathbb{R}^d$.

In this special case of equal norm tight frames, the describing equations and inequalities of the polytope of eigensteps can be drastically simplified. To be precise, we give a description of the polytope where the remaining inequalities are in one-to-one correspondence with the facets of the polytope and the remaining equations are linearly independent.

Using this description, we obtain formulae for the dimension of the polytope and its number of facets:

Theorem*. *Let $\Lambda_{n,d}$ be the polytope of eigensteps of equal norm tight frames of n vectors in a d-dimensional Hilbert space.*

1. *The dimension of $\Lambda_{n,d}$ is 0 for $d = 0$ and $d = n$, otherwise*

$$\dim(\Lambda_{n,d}) = (d-1)(n-d-1).$$

2. *For $2 \leq d \leq n-2$ the number of facets of $\Lambda_{n,d}$ is*

$$d(n-d-1) + (n-d)(d-1) - 2.$$

This theorem appears as Theorem 10.2.2 and Theorem 10.3.6, respectively. In Section 10.4 we return to frame theory and describe how the affine isomorphisms of polytopes

$$
\begin{array}{cccccccc}
0 = & \lambda_{d,0} & & \lambda_{d,1} & & \lambda_{d,2} & \cdots & \lambda_{d,n-1} & & \lambda_{d,n} & = n \\
0 = & \lambda_{d-1,0} & & \lambda_{d-1,1} & & \lambda_{d-1,2} & \cdots & \lambda_{d-1,n-1} & & \lambda_{d-1,n} & = n \\
0 = & \lambda_{d-2,0} & & \lambda_{d-2,1} & & \lambda_{d-2,2} & \cdots & \lambda_{d-2,n-1} & & \lambda_{d-2,n} & = n \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 = & \lambda_{1,0} & & \lambda_{1,1} & & \lambda_{1,2} & \cdots & \lambda_{1,n-1} & & \lambda_{1,n} & = n \\
\hline
\Sigma & 0 & & d & & 2d & \cdots & (n-1)d & & nd &
\end{array}
$$

Figure 10.1.: The conditions for a valid sequence of eigensteps for equal norm tight frames of length n and norms $\sqrt{d}$ in $\mathbb{C}^n$. A wedge $\lambda_{i,j}$ — $\lambda_{k,l}$ denotes an inequality $\lambda_{i,j} \leq \lambda_{k,l}$.

we obtained combinatorially are described by reversing the order of frame vectors and taking Naimark complements. We end with Section 10.5, where we discuss our results and some open questions.

Applying Corollary 4.3.8 to the case of equal norm tight frames, we arrive at the following description of $\Lambda_{n,d}$.

Corollary 10.1.1. *For integers $0 \leq d \leq n$, the polytope $\Lambda_{n,d}$ is the set of all matrices*

$$\Lambda = \left(\lambda_{i,k}\right)_{\substack{1\leq i\leq d,\\ 0\leq k\leq n}} \in \mathbb{R}^{d\times(n+1)}$$

satisfying the following conditions:

$$\lambda_{i,0} = 0 \qquad \text{for} \quad 1 \leq i \leq d, \tag{10.1}$$

$$\lambda_{i,n} = n \qquad \text{for} \quad 1 \leq i \leq d, \tag{10.2}$$

$$\sum_{i=1}^{d} \lambda_{i,k} = dk \qquad \text{for} \quad 0 \leq k \leq n, \tag{10.3}$$

$$\lambda_{i,k} \leq \lambda_{i,k+1} \qquad \text{for} \quad 1 \leq i \leq d, \quad 0 \leq k < n, \tag{10.4}$$

$$\lambda_{i,k} \leq \lambda_{i-1,k-1} \qquad \text{for} \quad 1 < i \leq d, \quad 0 < k \leq n. \tag{10.5}$$

We will refer to (10.1) *and* (10.2) *as the* first *and* last column conditions, *respectively. The equations in* (10.3) *are* column sum conditions, *while* (10.4) *and* (10.5) *will be referred to as the* horizontal *and* diagonal inequalities, *respectively, for reasons obvious from Figure 10.1.*

10.2. Dimension

In this section we determine the dimension of $\Lambda_{n,d}$. The dimension of the solution set of a system of linear equations and inequalities can be computed from the number of variables and the number of linearly independent equations, including those arising from inequalities that are always satisfied with equality. Thus, the first step is to remove redundant equations and recognize inequalities that are always satisfied with equality.

Proposition 10.2.1. *A matrix $(\lambda_{i,k}) \in \mathbb{R}^{d\times(n+1)}$ is a point of $\Lambda_{n,d}$ if and only if the following conditions are satisfied:*

$$
\begin{aligned}
\lambda_{i,k} &= 0 && \text{for} \quad i > k, && (10.6)\\
\lambda_{i,k} &= n && \text{for} \quad i < k + d - n + 1, && (10.7)\\
\sum_{i=1}^{d} \lambda_{i,k} &= dk && \text{for} \quad 0 < k < n, && (10.8)\\
\lambda_{i,k} &\le \lambda_{i,k+1} && \text{for} \quad 1 \le i \le d, \ \ i \le k < n - d + i - 1, && (10.9)\\
\lambda_{i,k} &\le \lambda_{i-1,k-1} && \text{for} \quad 1 < i \le d, \ \ i \le k < n - d + i, && (10.10)\\
\lambda_{d,d} &\ge 0, && && (10.11)\\
\lambda_{1,n-d} &\le n. && && (10.12)
\end{aligned}
$$

Proof. The idea behind the proof is to use the first and last column conditions together with the horizontal and diagonal inequalities to obtain triangles in the eigenstep tableaux that consist of fixed 0- or n-entries. Using those fixed triangles we can drop many of the now redundant inequalities from the system in Corollary 10.1.1. The remaining inequalities form a parallelogram with two legs as depicted in Figure 10.2.

We first prove the necessity of the modified conditions. The triangles described by (10.6) and (10.7)—from now on referred to as the two *triangle conditions*, see Figure 10.2 for reference—are an immediate consequence of the first and last column conditions together with the horizontal and diagonal inequalities. The remaining equations and inequalities already appear as part of the definition of $\Lambda_{n,d}$.

To prove sufficiency, we first see that the first and last column conditions are implied by the triangle conditions. The first and last column are always fixed, so the column sum conditions can be weakened to (10.8). Condition (10.11) together with the weakened horizontal and diagonal inequalities (10.9) and (10.10) is enough to guarantee that all $\lambda_{i,k}$ are non-negative. Thus, we will refer to (10.11) as the *lower bound condition*. Similarly (10.12) guarantees $\lambda_{i,k} \le n$ for all entries and will be referred to as the *upper bound condition*. Hence, from the original horizontal and diagonal inequalities (10.4) and (10.5) we only need those involving solely entries outside of the 0- and n-triangles. □

The remaining inequalities required by Proposition 10.2.1 are depicted in Figure 10.2. Note that Proposition 10.2.1 holds only for equal norm tight frames, in particular (10.7) is false for frames which are not tight.

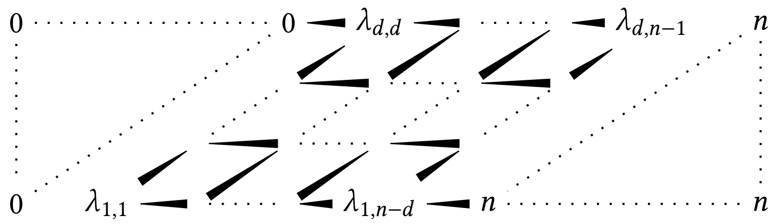


Figure 10.2.: The modified conditions for a valid sequence of eigensteps for equal norm tight frames with only the inequalities required by Proposition 10.2.1.

With the modified conditions from Proposition 10.2.1 we are now able to compute the dimension of $\Lambda_{n,d}$.

Theorem 10.2.2. *The dimension of the polytope $\Lambda_{n,d}$ is 0 for $d = 0$ and $d = n$, otherwise*

$$\dim(\Lambda_{n,d}) = (d-1)(n-d-1).$$

Proof. For $d = 0$ the only point in $\Lambda_{n,d}$ is the empty $0 \times (n+1)$ matrix, hence $\dim(\Lambda_{n,0}) = 0$. For $d = n$, the 0- and n-triangles fill up the whole matrix. Thus, $\Lambda_{n,n}$ also consists of a single point and $\dim(\Lambda_{n,n}) = 0$.

Otherwise, the triangle and sum conditions given by (10.6), (10.7) and (10.8) are linearly independent. Thus, by counting the equations, we obtain

$$\dim(\Lambda_{n,d}) \le d(n+1) - 2 \cdot \frac{d(d+1)}{2} - (n-1) = (d-1)(n-d-1).$$

To verify $\dim(\Lambda_{n,d}) \ge (d-1)(n-d-1)$, we show that $\Lambda_{n,d}$ contains a *special point* $\widehat{\Lambda}$ that satisfies all the inequalities (10.9) to (10.12) *strictly*, with the difference between the left and right hand sides of each inequality being equal to 1. The entries of $\widehat{\Lambda}$ not fixed by the triangle conditions are given by

$$\widehat{\lambda}_{i,k} := d + k - 2i + 1 \quad \text{for} \quad i \le k \le n - d + i - 1. \tag{10.13}$$

See Example 10.2.3 for reference. The smallest value in (10.13) is $\widehat{\lambda}_{d,d} = 1$, the largest is $\widehat{\lambda}_{1,n-d} = n - 1$, so the lower and upper bound conditions are strictly satisfied. The horizontal and diagonal inequalities hold strictly as well, since

$$\begin{aligned} \widehat{\lambda}_{i,k} &= d + k - 2i + 1 < d + (k+1) - 2i + 1 = \widehat{\lambda}_{i,k+1}, \\ \widehat{\lambda}_{i,k} &= d + k + 2i + 1 < d + (k-1) - 2(i-1) + 1 = \widehat{\lambda}_{i-1,k-1}. \end{aligned}$$

It remains to verify the column sum conditions (10.8). Letting $i_0 := \max\{0, k + d - n\}$ and

$i_1 := \min\{d, k\}$ we have

$$\begin{aligned}\sum_{i=1}^{d} \widehat{\lambda}_{i,k} &= \sum_{i=1}^{i_0} n + \sum_{i=i_0+1}^{i_1} \widehat{\lambda}_{i,k} + \sum_{i=i_1+1}^{d} 0 \\ &= i_0 n + \sum_{i=i_0+1}^{i_1} (d + k - 2i + 1) \\ &= i_0 n + (i_1 - i_0)(d + k - i_1 - i_0).\end{aligned}$$

In all four cases, this expression evaluates to dk. □

Example 10.2.3. For $n = 6$, $d = 4$ we obtain the special point of $\Lambda_{6,4}$ as

$$\widehat{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 2 & 3 & 6 & 6 \\ 0 & 0 & 3 & 4 & 6 & 6 & 6 \\ 0 & 4 & 5 & 6 & 6 & 6 & 6 \end{pmatrix}.$$

This tableau satisfies all inequalities in Proposition 10.2.1 strictly while also satisfying the column sum and triangle conditions. ◊

Note that the dimension of the polytope of eigensteps $\Lambda_{n,d}$ is related to the dimensions of certain frame varieties.

Remark 10.2.4. Let $\mathcal{F}^{\mathbb{R}}_{n,d} \subseteq \mathbb{R}^{d\times n}$ be the real algebraic variety of real unit norm tight frames, $\mathcal{F}^{\mathbb{C}}_{n,d} \subseteq \mathbb{C}^{d\times n} = \mathbb{R}^{2(d\times n)}$ the real algebraic variety of complex unit norm tight frames. The orthogonal group $O(d)$ and the unitary group $U(d)$ act on $\mathcal{F}^{\mathbb{R}}_{n,d}$ and $\mathcal{F}^{\mathbb{C}}_{n,d}$, respectively. The dimensions of $\mathcal{F}^{\mathbb{R}}_{n,d}$ and $\mathcal{F}^{\mathbb{C}}_{n,d}$ as determined in [CMS17, Prop. 5.5] are strictly greater than $\dim(\Lambda_{n,d})$ for $n, d > 0$. By Theorem 4.3 in [DS06], this is also true for the real dimension of $\mathcal{F}^{\mathbb{C}}_{n,d}/U(d)$, while the dimension of $\mathcal{F}^{\mathbb{R}}_{n,d}/O(d)$ is in fact equal to $\dim(\Lambda_{n,d})$.

10.3. Facets

In this section we investigate which of the remaining inequalities describing $\Lambda_{n,d}$ are necessary. In other words, we find the facet-describing inequalities of $\Lambda_{n,d}$. In particular, we obtain a formula for the number of facets.

To reduce the number of inequalities we need to consider separately, we use two kinds of dualities. One is an affine isomorphism between $\Lambda_{n,d}$ and $\Lambda_{n,n-d}$ that translates horizontal to diagonal inequalities and vice versa. The other is an affine involution on $\Lambda_{n,d}$, reversing the order of rows and columns of the eigenstep tableaux. We will see in Section 10.4 how these dualities correspond to certain operations on equal norm tight frames.

From the proof of Theorem 10.2.2 we know that the affine hull $\mathrm{aff}(\Lambda_{n,d})$ is the affine subspace of $\mathbb{R}^{d\times(n+1)}$ defined by the triangle and sum conditions ((10.6), (10.7) and (10.8)).

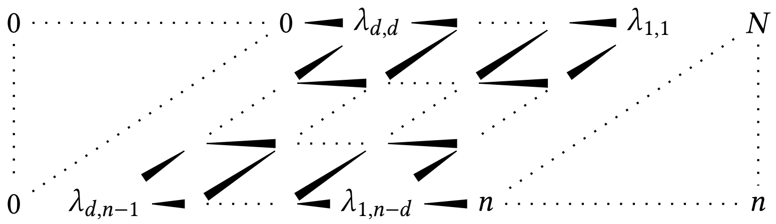


Figure 10.3.: The image of a sequence of eigensteps Λ as in Figure 10.2 under the affine isomorphism $\Psi_{n,d} \colon \Lambda_{n,d} \to \Lambda_{n,n-d}$ from Proposition 10.3.1.

Proposition 10.3.1. *There is an affine isomorphism*

$$\Psi_{n,d} \colon \operatorname{aff}(\Lambda_{n,d}) \longrightarrow \operatorname{aff}(\Lambda_{n,n-d})$$

given by

$$\big(\Psi_{n,d}(\Lambda)\big)_{i,k} = \begin{cases} \lambda_{d+i-k,n-k}, & \text{for } i \le k \le d+i-1, \\ 0, & \text{for } k < i, \\ n, & \text{for } k > d+i-1, \end{cases}$$

that restricts to an affine isomorphism $\Lambda_{n,d} \to \Lambda_{n,n-d}$.

As a map of eigenstep tableaux, $\Psi_{n,d}$ can be understood as interchanging rows and diagonals in the parallelogram of non-fixed entries while adjusting the sizes of 0- and n-triangles. For example, when $n = 5$, $d = 3$ the map $\Psi_{5,3} \colon \operatorname{aff}(\Lambda_{5,3}) \to \operatorname{aff}(\Lambda_{5,2})$ is given by

$$\Psi_{5,3}\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda_{3,3} & \lambda_{2,2} & \lambda_{1,1} & 5 \\ 0 & \lambda_{3,4} & \lambda_{2,3} & \lambda_{1,2} & 5 & 5 \end{pmatrix}.$$

In Figure 10.3 we illustrate the general structure of the image of an eigenstep tableau under $\Psi_{n,d}$.

Proof. We first need to verify that $\Lambda' := \Psi_{n,d}(\Lambda)$ is a point in $\operatorname{aff}(\Lambda_{n,n-d})$ for $\Lambda \in \operatorname{aff}(\Lambda_{n,d})$. The triangle conditions are satisfied by the definition of $\Psi_{n,d}$. To verify the sum conditions for $\Lambda_{n,n-d}$, let $1 \le k \le n-1$ and $l := n-k$, then

$$\begin{aligned} \sum_{i=1}^{n-d} \lambda'_{i,k} &= \sum_{i=1}^{\max\{0,k-d\}} n \quad + \sum_{i=\max\{0,k-d\}+1}^{\min\{n-d,k\}} \lambda_{d+i-k,n-k} \\ &= \max\{0,k-d\}n + \sum_{j=\max\{0,l+d-n\}+1}^{\min\{d,l\}} \lambda_{j,l} \\ &= \max\{0,k-d\}n + dl - \max\{0,l+d-n\}n \\ &= \max\{0,k-d\}n - \max\{0,d-k\}n + d(n-k) \\ &= (k-d)n + d(n-k) = (n-d)k. \end{aligned}$$

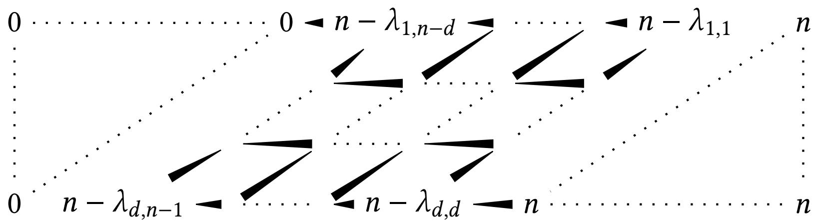


Figure 10.4.: The image of a sequence of eigensteps Λ as in Figure 10.2 under the involution $\Phi_{n,d}$ from Proposition 10.3.2.

To see that $\Psi_{n,d}$ restricts to an affine map $\Lambda_{n,d} \to \Lambda_{n,n-d}$ we need to consider all inequalities. Let $\Lambda \in \Lambda_{n,d}$ and $\Lambda' = \Psi_{n,d}(\Lambda)$. The lower and upper bound conditions are satisfied, since $\lambda'_{n-d,n-d} = \lambda_{d,d} \geq 0$ and $\lambda'_{1,d} = \lambda_{1,n-d} \leq n$. The remaining horizontal and diagonal inequalities (10.9) and (10.10) interchange under $\Psi_{n,d}$. Let $j := d + i - k$ and $l := n - k$, then we have

$$\begin{aligned} & \lambda'_{i,k} \leq \lambda'_{i,k+1} && \text{for} \quad 1 \leq i \leq n-d, \ \ i \leq k < d+i-1 \\ \Leftrightarrow \quad & \lambda_{j,l} \leq \lambda_{j-1,l-1} && \text{for} \quad 1 < j \leq d, \ \ j \leq l < n-d+j \end{aligned}$$

and

$$\begin{aligned} & \lambda'_{i,k} \leq \lambda'_{i-1,k-1} && \text{for} \quad 1 < i \leq n-d, \ \ i \leq k < d+i \\ \Leftrightarrow \quad & \lambda_{j,l} \leq \lambda_{j,l+1} && \text{for} \quad 1 \leq j \leq d, \ \ j \leq l < n-d+j-1. \end{aligned}$$

Hence, $\Psi_{n,d}$ restricts to an affine map $\Lambda_{n,d} \to \Lambda_{n,n-d}$. It is an isomorphism on both the affine hulls and the polytopes themselves, since $\Psi_{n,d}$ and $\Psi_{n,n-d}$ are mutually inverse. This needs to be checked only for the non-fixed entries:

$$\begin{aligned} \big(\Psi_{n,n-d}(\Psi_{n,d}(\Lambda))\big)_{i,k} &= \big(\Psi_{n,d}(\Lambda)\big)_{n-d+i-k,n-k} \\ &= \lambda_{d+n-d+i-k-n+k,n-n+k} = \lambda_{i,k}, \\ \big(\Psi_{n,d}(\Psi_{n,n-d}(\Lambda))\big)_{i,k} &= \big(\Psi_{n,n-d}(\Lambda)\big)_{d+i-k,n-k} \\ &= \lambda_{n-d+d+i-k-n+k,n-n+k} = \lambda_{i,k}. \end{aligned}$$ □

Proposition 10.3.2. *There is an affine involution $\Phi_{n,d} \colon \mathbb{R}^{d\times(n+1)} \longrightarrow \mathbb{R}^{d\times(n+1)}$ given by*

$$\Phi(\Lambda)_{i,k} = n - \lambda_{d-i+1,n-k},$$

that restricts to an affine involution $\Lambda_{n,d} \to \Lambda_{n,d}$.

The involution $\Phi_{n,d}$ can be described as rotating the whole eigenstep tableau by 180° and subtracting every entry from n, as depicted in Figure 10.4.

Proof. It is clear that $\Phi_{n,d}$ is an affine map $\mathbb{R}^{d\times(n+1)} \to \mathbb{R}^{d\times(n+1)}$. We use the original system of equations and inequalities given in Corollary 10.1.1 to verify $\Phi(\Lambda) \in \Lambda_{n,d}$ when

$\Lambda \in \Lambda_{n,d}$. For $k = 0, n$ we obtain

$$\Phi(\Lambda)_{i,0} = n - \lambda_{d-i+1,n} = n - n = 0,$$
$$\Phi(\Lambda)_{i,n} = n - \lambda_{d-i+1,0} = n - 0 = n.$$

Hence, (10.1) and (10.2) are satisfied by $\Lambda' := \Phi(\Lambda)$. The column sum conditions (10.3) are satisfied, since

$$\begin{aligned}\sum_{i=1}^{d} \Phi(\Lambda)_{i,k} &= \sum_{i=1}^{d} \left(n - \lambda_{d-i+1,n-k}\right) \\ &= \sum_{j=1}^{d} \left(n - \lambda_{j,n-k}\right) \\ &= dn - d(n-k) \\ &= dk.\end{aligned}$$

For the horizontal and diagonal inequalities, we observe that $\lambda_{i,k} \leq \lambda_{i',k'}$ is equivalent to $n - \lambda_{i',k'} \leq n - \lambda_{i,k}$.

Finally, $\Phi_{n,d}$ is an involution on both $\mathbb{R}^{d\times(n+1)}$ and $\Lambda_{n,d}$, since

$$\begin{aligned}\big((\Phi_{n,d} \circ \Phi_{n,d})(\Lambda)\big)_{i,k} &= n - \big(\Phi_{n,d}(\Lambda)\big)_{d-i+1,n-k} \\ &= n - (n - \lambda_{i,k}) \\ &= \lambda_{i,k}.\end{aligned}$$

□

The results noted in the following remark are easily verified by direct computation.

Remark 10.3.3. The special point $\widehat{\Lambda}$ of $\Lambda_{n,d}$ is fixed under $\Phi_{n,d}$ and mapped to the special point of $\Lambda_{n,n-d}$ by $\Psi_{n,d}$. Furthermore, Φ and Ψ commute. To be precise:

$$\Phi_{n,n-d} \circ \Psi_{n,d} = \Psi_{n,d} \circ \Phi_{n,d}.$$

Using the dualities given by Φ and Ψ, we now construct points that witness the necessity of most of the inequalities in Proposition 10.2.1.

Lemma 10.3.4. *Let $n \geq 5$, $2 \leq d \leq n-2$. Consider one of the inequalities in (10.9) to (10.12) which is not $\lambda_{2,2} \leq \lambda_{1,1}$, $\lambda_{1,1} \leq \lambda_{1,2}$, $\lambda_{d,n-2} \leq \lambda_{d,n-1}$ or $\lambda_{d,n-1} \leq \lambda_{d-1,n-2}$. Then there is a point in $\mathbb{R}^{d\times(n+1)}$ satisfying all conditions of Proposition 10.2.1 except the considered inequality.*

Proof. The idea behind the proof is to start with the special point $\widehat{\Lambda} \in \Lambda_{n,d}$ and locally change entries such that just one of the inequalities fails, while preserving all other conditions. Since $\Psi_{n,d}$ translates horizontal (10.9) to diagonal inequalities (10.10) and vice versa, it is enough to consider only horizontal inequalities. Also, since $\Phi_{n,d}$ maps the top row ($i = d$) to the bottom row ($i = 1$), the inequalities in the bottom row do not need to be considered either. Since $\Phi_{n,d}$ maps the first diagonal ($i = k$) to the last ($i = k + d - n + 1$)

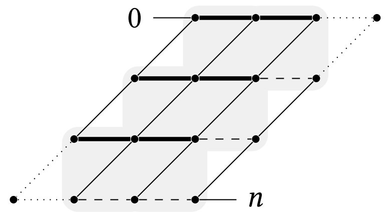


Figure 10.5.: The horizontal inequalities treated by the modification (10.14) shown in bold.

and vice versa, we do not need to consider the last horizontal inequality in each row. The remaining horizontal inequalities are treated with the following modification of $\widehat{\Lambda}$:

$$
\begin{array}{ccccccc}
 & & & & m-1 & & m \\
 & & & \swarrow & & \swarrow & \\
m-1 & \leftarrow & m & \leftarrow & m+1 & \leftarrow & m+2 \\
 & \swarrow & & \swarrow & & \swarrow & \\
m+1 & \leftarrow & m+2 & \leftarrow & m+3 & \leftarrow & m+4 \\
 & \swarrow & & \swarrow & & & \\
m+3 & & m+4 & & & &
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{ccccccc}
 & & & & m-1 & & m \\
 & & & \swarrow & & \swarrow & \\
m-1 & \leftarrow & m+1 & \nleftarrow & m & \leftarrow & m+2 \\
 & \swarrow & & \swarrow & & \swarrow & \\
m+1 & \leftarrow & m+1 & \leftarrow & m+4 & \leftarrow & m+4 \\
 & \swarrow & & \swarrow & & & \\
m+3 & & m+4 & & & &
\end{array}
\tag{10.14}
$$

Note that this modification of $\widehat{\Lambda}$ does not alter the column sums and causes only the slashed inequality in (10.14) to fail. If $2 \leq d < n-2$ the square of modified entries in (10.14) fits into the parallelogram of non-fixed entries. In Figure 10.5 we demonstrate how this modification can be used to obtain points that let each of the bold inequalities fail individually. The dashed inequalities are covered by the above argument using $\Phi_{n,d}$, while the dotted inequalities are the four exceptions mentioned in Lemma 10.3.4.

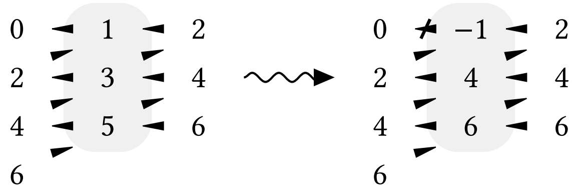
If $d = n-2$, the parallelogram of non-fixed entries becomes too thin to fit the squares of (10.14), so this case has to be treated separately. Instead of considering the horizontal inequalities for $d = n-2$, we can use the duality given by $\Psi_{n,d}$ and consider the diagonal inequalities for $d = 2$. We use the following modification of $\widehat{\Lambda}$:

$$
\begin{array}{ccccccc}
m-1 & \leftarrow & m & \leftarrow & m+1 & \leftarrow & m+2 \\
 & \swarrow & & \swarrow & & \swarrow & \\
m+1 & \leftarrow & m+2 & \leftarrow & m+3 & \leftarrow & m+4
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{ccccccc}
m-1 & \leftarrow & m+1 & \leftarrow & m+2 & \leftarrow & m+2 \\
 & \swarrow & & \not\swarrow & & \swarrow & \\
m+1 & \leftarrow & m+1 & \leftarrow & m+2 & \leftarrow & m+4
\end{array}
$$

The only inequality that remains to be treated is the lower bound condition $\lambda_{d,d} \geq 0$. The upper bound condition then follows from the duality given by $\Phi_{n,d}$. Here we use a modification of $\widehat{\Lambda}$ to construct a point that causes only the lower bound condition to fail. We first do this for $d = 2$:

$$
\begin{array}{ccccccc}
0 & \leftarrow & 1 & \leftarrow & 2 & \leftarrow & 3 \\
 & \swarrow & & \swarrow & & \swarrow & \\
2 & \leftarrow & 3 & \leftarrow & 4 & \leftarrow & 5
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{ccccccc}
0 & \nleftarrow & -1 & \leftarrow & 1 & \leftarrow & 3 \\
 & \swarrow & & \swarrow & & \swarrow & \\
2 & \leftarrow & 5 & \leftarrow & 5 & \leftarrow & 5
\end{array}
$$

This also covers the case $d = n-2$ by dualizing using $\Psi_{n,2}$. For $2 < d < n-2$ we use a different modification of $\widehat{\Lambda}$:



□

Example 10.3.5. For $n = 5$ and $d = 2$, we construct the points given by Lemma 10.3.4 explicitly. The special point of $\Lambda_{5,2}$ is

$$\widehat{\Lambda} = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 2 & 3 & 4 & 5 & 5 \end{pmatrix}.$$

The half-spaces described by the non-exceptional inequalities are $H_1\colon \lambda_{2,2} \geq 0$, $H_2\colon \lambda_{2,2} \leq \lambda_{2,3}$, $H_3\colon \lambda_{2,3} \leq \lambda_{1,2}$, $H_4\colon \lambda_{1,2} \leq \lambda_{1,3}$ and $H_5\colon \lambda_{1,3} \leq 5$. Applying Lemma 10.3.4 yields the following five points P_i, each satisfying all conditions except lying in the half-space H_i:

$$P_1 = \begin{pmatrix} 0 & 0 & -1 & 1 & 3 & 5 \\ 0 & 2 & 5 & 5 & 5 & 5 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} 0 & 0 & 2 & 1 & 3 & 5 \\ 0 & 2 & 2 & 5 & 5 & 5 \end{pmatrix},$$
$$P_3 = \begin{pmatrix} 0 & 0 & 2 & 3 & 3 & 5 \\ 0 & 2 & 2 & 3 & 5 & 5 \end{pmatrix},$$
$$P_4 = \Phi_{5,2}(P_2) = \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 5 \\ 0 & 2 & 4 & 3 & 5 & 5 \end{pmatrix},$$
$$P_5 = \Phi_{5,2}(P_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 2 & 4 & 6 & 5 & 5 \end{pmatrix}.$$

The two variables $\lambda_{2,2}$ and $\lambda_{2,3}$ completely parametrize the polytope, since $\lambda_{1,1} = 2$, $\lambda_{1,2} = 4 - \lambda_{2,2}$, $\lambda_{1,3} = 6 - \lambda_{2,3}$ and $\lambda_{2,4} = 3$ by the column sum conditions. Hence, we can illustrate the situation in the plane, as done in Figure 10.6. ◊

Using Lemma 10.3.4, we prove the following theorem, giving the number of facets of $\Lambda_{n,d}$.

Theorem 10.3.6. *For $2 \leq d \leq n-2$ the number of facets of $\Lambda_{n,d}$ is*

$$d(n-d-1) + (n-d)(d-1) - 2.$$

Proof. We first show for the case of $n \geq 5$ that $d(n-d-1)+(n-d)(d-1)-2$ inequalities are sufficient to describe $\Lambda_{n,d}$ in its affine hull.

Let $n \geq 5$, $2 \leq d \leq n-2$. Counting the horizontal and diagonal inequalities (10.9), (10.10) yields $d(n-d-1) + (d-1)(n-d)$ inequalities.

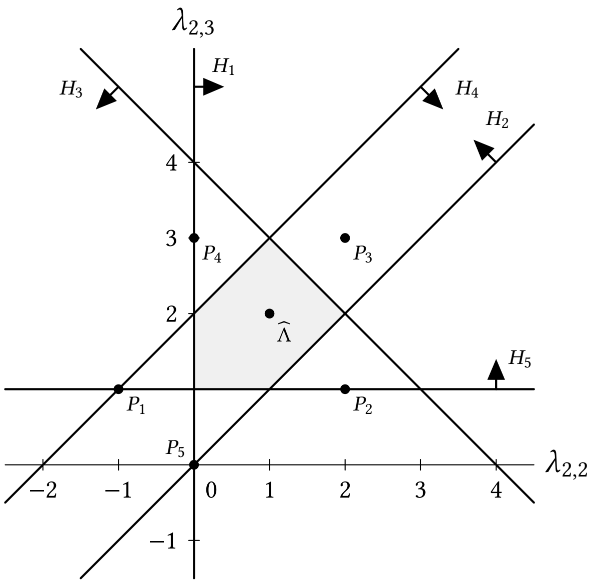


Figure 10.6.: For $\Lambda_{5,2}$ we have five necessary inequalities. The points P_i which satisfy all conditions but the defining inequality for the half-space H_i are constructed in Example 10.3.5.

We now show that the four inequalities between non-fixed entries that are already mentioned in Lemma 10.3.4 are in fact not necessary. Recall that these are

$$\lambda_{2,2} \leq \lambda_{1,1}, \tag{10.15}$$
$$\lambda_{1,1} \leq \lambda_{1,2}, \tag{10.16}$$
$$\lambda_{d,n-2} \leq \lambda_{d,n-1}, \tag{10.17}$$
$$\lambda_{d,n-1} \leq \lambda_{d-1,n-2}. \tag{10.18}$$

From the column sum and triangle conditions it follows that $\lambda_{1,1} = d$ and $\lambda_{1,2} + \lambda_{2,2} = 2d$. Thus, (10.15) and (10.16) are both equivalent to $\lambda_{2,2} \leq d$, which is already implied by $\lambda_{2,2} \leq \lambda_{2,3} \leq \lambda_{1,2} = 2d - \lambda_{2,2}$, when $d \leq n-2$. Therefore, (10.15) and (10.16) are superfluous.

Again, by the column sum and triangle conditions, we have $\lambda_{d,n-1} = n - d$ and $\lambda_{d,n-2} + \lambda_{d-1,n-2} = 2(n-d)$. Thus, (10.17) and (10.18) are both equivalent to $\lambda_{d-1,n-2} \geq n-d$, which is already implied by $\lambda_{d-1,n-2} \geq \lambda_{d-1,n-3} \geq \lambda_{d,n-2} = 2(n-d) - \lambda_{d-1,n-2}$, when $d \leq n-2$. However the two arguments are independent only when $n \geq 5$, since for $n = 4$, $d = 2$ we have $\lambda_{2,2} = \lambda_{d,n-2}$ and $\lambda_{1,2} = \lambda_{d-1,n-2}$.

Counting all inequalities, including the lower and upper bound conditions, excluding the four superfluous inequalities, we have

$$d(n-d-1) + (d-1)(n-d) + 2 - 4 = d(n-d-1) + (d-1)(n-d) - 2$$

inequalities that are sufficient to describe $\Lambda_{n,d}$. From Lemma 10.3.4 we know that all these inequalities are actually necessary, hence we obtain the desired number of facets.

For the case $n = 4$, $d = 2$, we have $\dim(\Lambda_{4,2}) = (2-1)(4-2-1) = 1$. The only polytope of dimension 1 is a line segment, the two endpoints being its facets. Thus, $\Lambda_{4,2}$ has two facets, as given by $d(n-d-1) + (d-1)(n-d) - 2$ for $n = 4$, $d = 2$. □

From Lemma 10.3.4 and Theorem 10.3.6 we conclude that removing the four exceptional inequalities from the description of $\Lambda_{n,d}$ in Proposition 10.2.1 yields a non-redundant system of equations and inequalities.

10.4. Connections Between Frame and Eigenstep Operations

Until now we focused on the combinatorics of sequences of eigensteps. In this section, we give descriptions of the affine isomorphisms $\Phi_{n,d}$ and $\Psi_{n,d}$ in terms of the underlying frames.

In the following, let $\tilde{F} := (f_{n-k+1})_{k=1}^{n}$ denote the frame with reversed order of frame vectors.

We obtain the following result:

Proposition 10.4.1. *Let $F = (f_k)_{k=1}^{n}$ be an equal norm tight frame in $\mathbb{C}^d$ with $\|f_n\|^2 = d$, then*

$$\Phi_{n,d}(\Lambda^{out} F) = \Lambda^{out}(\tilde{F}).$$

Proof. Decomposing the frame operator of F we have

$$n \cdot I_d = FF^* = \sum_{j=1}^{n} f_j f_j^* = \sum_{j=1}^{k} f_j f_j^* + \sum_{j=k+1}^{n} f_j f_j^* = F_k F_k^* + \tilde{F}_{n-k} \tilde{F}_{n-k}^*.$$

Thus, if $v \in \mathbb{C}^d$ is an eigenvector of $F_k F_k^*$ with eigenvalue γ, we obtain

$$\tilde{F}_{n-k} \tilde{F}_{n-k}^* v = (n \cdot I_d - F_k F_k^*) v = (n - \gamma) v.$$

So v is an eigenvector of $\tilde{F}_{n-k} \tilde{F}_{n-k}^*$ with eigenvalue $n - \gamma$ and $\Lambda^{\mathrm{out}}(\tilde{F}) = \Phi_{n,d}(\Lambda^{\mathrm{out}}(F))$. □

A well-known concept in finite frame theory is the notion of *Naimark complements*. In the case of Parseval frames, finding a Naimark complement of F amounts to finding a matrix G such that $\binom{F}{G}$ is unitary. By scaling, this definition can be extended to tight frames and in fact to all finite frames, as discussed in [CFM+13b]. In our context, we use the following definition:

Definition 10.4.2. Given an equal norm tight frame $F = (f_k)_{k=1}^{n}$ in $\mathbb{C}^d$ with $\|f_k\|^2 = d$, a frame $G = (g_k)_{k=1}^{n}$ in $\mathbb{C}^{n-d}$ satisfying $F^* F + G^* G = n \cdot I_n$ is called a *Naimark complement* of F.

Many properties of a frame F carry over to its Naimark complement G. In particular, a Naimark complement of an equal norm tight frame is again an equal norm tight frame, the norm being $\sqrt{n-d}$. The following proposition shows how the duality described by $\Psi_{n,d}$ corresponds to taking a Naimark complement and reversing the order of frame vectors.

Proposition 10.4.3. *Let $F = (f_k)_{k=1}^n$ be an equal norm tight frame in $\mathbb{C}^d$ with norms $\|f_k\|^2 = d$ and $G = (g_k)_{k=1}^n$ a Naimark complement of F, then*

$$\Psi_{n,d}(\Lambda^{out}(F)) = \Lambda^{out}(\tilde{G}).$$

Proof. Since $\Psi_{n,d}(\Lambda^{\mathrm{out}}(F)) = \Lambda^{\mathrm{out}}(\tilde{G})$ is equivalent to $\Lambda^{\mathrm{out}}(F) = \Psi_{n,n-d}(\Lambda^{\mathrm{out}}(\tilde{G}))$, we only need to consider the case $n \geq 2d$. We first consider the columns of F with indices $k < d$. Since F_k is a $d \times k$ matrix, $F_k F_k^*$ has at most k non-zero eigenvalues. To be precise, the spectrum of the frame operator of F_k is

$$\sigma(F_k F_k^*) = (\lambda_{1,k}, \dots, \lambda_{k,k}, \underbrace{0, \dots, 0}_{d-k}).$$

In order to obtain the eigensteps of G, we switch to Gram matrices. The Gram matrix of F_k is the $k \times k$ matrix $F_k^* F_k$, with spectrum

$$\sigma(F_k^* F_k) = (\lambda_{1,k}, \dots, \lambda_{k,k}),$$

which is obtained by considering the singular value decomposition of F_n.

Since G is a Naimark complement of F, we have $F^*F + G^*G = n \cdot I_n$. In particular,

$$\begin{aligned} n \cdot I_n = F^*F + G^*G &= \begin{pmatrix} F^* & G^* \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \\ &= \begin{pmatrix} F_k^* & G_k^* \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} F_k & \cdots \\ G_k & \cdots \end{pmatrix} = \begin{pmatrix} F_k^* F_k + G_k^* G_k & \cdots \\ \vdots & \ddots \end{pmatrix}. \end{aligned}$$

The first k rows and columns of this identity yield $F_k^* F_k + G_k^* G_k = n \cdot I_k$. Therefore

$$\sigma(G_k^* G_k) = (n - \lambda_{k,k}, \dots, n - \lambda_{1,k}).$$

Going back to the frame operator of G_k, which is the $(n-d) \times (n-d)$ matrix $G_k G_k^*$, we have

$$\sigma(G_k G_k^*) = (n - \lambda_{k,k}, \dots, n - \lambda_{1,k}, \underbrace{0, \dots, 0}_{n-d-k}).$$

Finally, using $\tilde{G}_{n-k} \tilde{G}_{n-k}^* + G_k G_k^* = GG^* = n \cdot I_{n-d}$, we obtain

$$\sigma(\tilde{G}_{n-k} \tilde{G}_{n-k}^*) = (\underbrace{n, \dots, n}_{n-d-k}, \lambda_{k,k}, \dots, \lambda_{1,k}),$$

which shows that the $(n-k)$-th column of $\Psi(\Lambda^{\mathrm{out}}(F))$ is equal to the $(n-k)$-th column of $\Lambda^{\mathrm{out}}(\tilde{G})$ for $n < d$.

For $k > n-d$, let $l := n-k$ so that $l < d$. Hence, the $(n-l)$-th column of $\Psi(\Lambda^{\mathrm{out}}(\tilde{F}))$ is the $(n-l)$-th column of $\Lambda^{\mathrm{out}}(G)$ by the previous argument. Since $\Lambda^{\mathrm{out}}(\tilde{F}) = \Phi_{n,d}(\Lambda^{\mathrm{out}}(F))$ and $\Lambda^{\mathrm{out}}(G) = \Phi_{n,n-d}(\Lambda^{\mathrm{out}}(\tilde{G}))$, we know that $\Psi_{n,d}(\Phi_{n,d}(\Lambda^{\mathrm{out}}(F)))$ and $\Phi_{n,n-d}(\Lambda^{\mathrm{out}}(\tilde{G}))$ agree in the k-th column. Using $\Psi_{n,d} \circ \Phi_{n,d} = \Phi_{n,n-d} \circ \Psi_{n,d}$ and the fact that $\Phi_{n,n-d}$ reverses the column order, we conclude that $\Psi_{n,d}(\Lambda^{\text{out}}(F))$ and $\Lambda^{\text{out}}(\tilde{G})$ agree in the $(n-k)$-th column as desired.

We now consider $d \leq k \leq n-d$. By the same arguments as before, we have

$$\sigma(F_k F_k^*) = (\lambda_{1,k}, \dots, \lambda_{d,k}),$$

$$\sigma(F_k^* F_k) = (\lambda_{1,k}, \dots, \lambda_{d,k}, \underbrace{0, \dots, 0}_{k-d}),$$

$$\sigma(G_k^* G_k) = (\underbrace{n, \dots, n}_{k-d}, n - \lambda_{d,k}, \dots, n - \lambda_{1,k}).$$

Since G_k is an $(n-d) \times k$ matrix, with $n-d \geq k$, the spectrum of the frame operator of G_k is

$$\sigma(G_k G_k^*) = (\underbrace{n, \dots, n}_{k-d}, n - \lambda_{d,k}, \dots, n - \lambda_{1,k}, \underbrace{0, \dots, 0}_{n-d-k}).$$

Thus

$$\sigma(\tilde{G}_{n-k} \tilde{G}_{n-k}^*) = (\underbrace{n, \dots, n}_{n-d-k}, \lambda_{1,k}, \dots, \lambda_{d,k}, \underbrace{0, \dots, 0}_{k-d}),$$

which shows that the $(n-k)$-th column of $\Psi(\Lambda^{\text{out}}(F))$ is equal to the $(n-k)$-th column of $\Lambda^{\text{out}}(\tilde{G})$ for $d \leq k \leq n-d$. □

10.5. Conclusion and Open Problems

As we have seen, in the special case of equal norm tight frames we are able to obtain a general non-redundant description of the polytope of eigensteps in terms of equations and inequalities. However, this description does not generalize to non-tight frames, where we lose the n-triangle in the eigenstep tableau. Hence, even the dimension of $\Lambda^{\text{out}}(\mathcal{F}_{\mu,\lambda})$ will depend on the multiplicities of eigenvalues in the spectrum that cause smaller triangles of fixed entries in the eigenstep tableaux.

From a discrete geometers point of view, it might be interesting to find a description of polytopes of eigensteps in terms of vertices. However, even restricting to equal norm tight frames, we were not able to calculate the number of vertices of $\Lambda_{n,d}$ in general, let alone find a description of the polytope as a convex hull of vertices. On the frame theoretic end, it might be interesting to study properties of frames F corresponding to certain points of the polytope. For example, interesting classes of equal norm tight frames might be the frames F such that $\Lambda^{\text{out}}(F)$ is the special point $\widehat{\Lambda}$, a boundary point of $\Lambda_{n,d}$ or a vertex of $\Lambda_{n,d}$.

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