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Minimality of Hyperplane Arrangements and Configuration Spaces: a combinatorial approach

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Chapter 1

Introduction

The theory of Hyperplane Arrangements (more generally, Subspace Arrangements) is developing in the last (at least) three decades as an interesting part of Mathematics, which derives from and at the same time connects different classical branches. Among them we have: the theory of root systems (so, indirectly, Lie theory); Singularity theory, by the classical connection with simple singularities and braid groups and related groups (Artin groups); Combinatorics, through for example Matroid and Oriented Matroid theory; Algebraic Geometry, in connection with certain moduli spaces of genus zero curves and also through the classical study of the topology of Hypersurface complements; the theory of Generalized Hypergeometric Functions, and the connected development of the study of *local system* cohomologies; recently, the theory of box splines, partition functions, index theory.

Most of the theory is spread into a big number of papers, but there exists also (few) dedicated books, or parts of books, as [GMP88], [OT92], and the recent book [DCP09].

The subject of this thesis concerns some topological aspects of the theory which we are going to outline here.

So, consider an hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$. We assume here that $\mathcal{A}$ is finite, but most of the results hold with few modifications for any affine (locally finite) arrangement. It was known by general theories that the complement to the complexified arrangement $\mathcal{M}(\mathcal{A})$ has the homotopy type of an $n-$dimensional complex, and in [S87] an explicit construction of a combinatorial complex (denoted since then as the Salvetti complex, here denoted by $\mathbf{S}$) was made. In general, such complex has more $k-$cells than the k-th Betti number of $\mathcal{M}(\mathcal{A})$. It has been known for a long time that the cohomology of the latter space is free, and a combinatorial description of such cohomology was found (see [OT92] for references). The topological type of the complement is not combinatorial for general arrangements, but it is still unclear if this is the case for special classes of arrangements. Nevertheless, suspecting special properties for the topology of the complement, it was proven that the latter enjoys a strong *minimality* condition. In fact, in [DiP03],[R02] it was shown that $\mathcal{M}(\mathcal{A})$ has the homotopy type of a CW-complex having exactly β_k k-cells, where β_k is the k-th Betti number.

This was an *existence-type* result, with no explicit description of the minimal complex.

A more precise description of the minimal complex, in the case of real defined arrangements, was found in [Y05], using classical Morse theory. A better explicit description was found in [SS07], where the authors used Discrete Morse theory over **S** (as introduced in [F98, F02]). There they introduce a *total* ordering (denoted *polar ordering*) for the set of *facets* of the induced stratification of $\mathbb{R}^n$, and define an explicit discrete vector field over the face-poset of **S**. There are as many *k-critical cells* for this vector field as the $k-$th Betti number ($k \geq 0$). It follows from discrete Morse theory that such a discrete vector field produces:

i) a homotopy equivalence of **S** with a minimal complex;

ii) an explicit description (up to homotopy) of the boundary maps of the minimal complex, in terms of *alternating paths*, which can be computed explicitly from the field.

A different construction (which has more combinatorial flavor) was given in [De08] (see also [DeS10]).

In this thesis we consider this kind of topological problems around minimality. First, even if the above construction allows in theory to produce the minimal complex explicitly, the boundary maps that one obtains by using the alternating paths are not *themselves minimal,* in the sense that several pairs of the same critical cell can delete each other inside the attaching maps of the bigger dimensional critical cells. So, a problem is to produce a minimal complex with *minimal* attaching maps.

We are able to do that in the two-dimensional affine case (see chapter 5, [GMS09]).

Next, we generalize the construction of the vector field to the case of so called *d-complexified* arrangements.

First, consider classical Configuration Spaces in $\mathbb{R}^d$ (sometimes written as $F(n, \mathbb{R}^d)$) : they are defined as the set of ordered $n-$tuples of *pairwise different* points in $\mathbb{R}^d$. Taking coordinates in $(\mathbb{R}^d)^n = \mathbb{R}^{nd}$

$$x_{ij},\ i = 1, \dots, n,\ j = 1, \dots, d,$$

one has

$$F(n, \mathbb{R}^d) \ = \ \mathbb{R}^{nd} \setminus \cup_{i \neq j}\ H_{ij}^{(d)},$$

where $H_{ij}^{(d)}$ is the codimension d-subspace

$$\cap_{k=1,\dots,d}\ \{x_{ik} = x_{jk}\}.$$

So, the latter subspace is the intersection of d hyperplanes in $\mathbb{R}^{nd}$, each obtained by the hyperplane $H_{ij} = \{x \in \mathbb{R}^n\ :\ x_i = x_j\}$, considered on the $k-$th component in $(\mathbb{R}^n)^d = \mathbb{R}^{nd}$, $k = 1, \dots, d$.

By a *Generalized Configuration Space* (for brevity, simply a Configuration Space) we mean an analog construction, which starts from any *Hyperplane Arrangement* $\mathcal{A}$ in $\mathbb{R}^n$. For each $d > 0$, one has a $d-$*complexification* $\mathcal{A}^{(d)} \subset M^d$ of $\mathcal{A}$, which is given by the collection $\{H^{(d)},\ H \in \mathcal{A}\}$ of the *d-complexified* subspaces. The *configuration space* associated to $\mathcal{A}$ is the complement to the subspace arrangement

$$\mathcal{M}^{(d)} \ = \ \mathcal{M}(\mathcal{A})^{(d)} := \ (\mathbb{R}^n)^d \setminus \bigcup_{H \in \mathcal{A}} H^{(d)} \ .$$

For $d = 2$ one has the standard complexification of a real hyperplane arrangement. There is a natural inclusion $\mathcal{M}^{(d)} \hookrightarrow \mathcal{M}^{(d+1)}$ and the limit space is contractible (in case of an arrangement associated to a reflection group W, the limit of the orbit space with respect to the action of W gives the classifying space of W; see [DCS00]) .

In this thesis we give an explicit construction of a minimal CW-complex for the configuration space $\mathcal{M}(\mathcal{A})^{(d)}$, for all $d \geq 1$. That is, we explicitly produce a CW-complex having as many i-cells as the i-th Betti number β_i of $\mathcal{M}(\mathcal{A})^{(d)}$, $i \geq 0$.

For $d = 1$ the result is trivial, since $\mathcal{M}^{(1)}$ is a disjoint union of convex sets (the *chambers*). Case $d = 2$ was discussed above. For $d > 2$ the configuration spaces are simply-connected, so by general results they have the homotopy type of a minimal CW-complex. Nevertheless, having explicit "combinatorial" complexes is useful in order to produce geometric bases for the cohomology. In fact, we give explicit bases for the homology (and cohomology) of $\mathcal{M}^{(d+1)}$ which we call (d)-*polar bases*. As far as we know, there is no other precise description of a geometric $\mathbb{Z}$-basis in the literature, except for some particular arrangements, in spite of the fact that the $\mathbb{Z}$-module structure of the homology is well known: it derives from a well known formula in [GMP88] that such homology depends only on the intersection lattice of the d-complexification $\mathcal{A}^{(d)}$, and such lattice is the same for all $d \geq 1$. The tool we use here is still discrete Morse theory. Starting from the previous explicit construction in [DCS00] of a non-minimal CW-complex (see also [BZ92]) which we denote here by $\mathbf{S}^{(d)}$, which has the homotopy type of $\mathcal{M}^{(d+1)}$, we construct an explicit *combinatorial gradient vector field* on $\mathbf{S}^{(d)}$ and we give a precise description of the critical cells. One finds that critical cells live in dimension id, for $i = 1, \ldots, n'$, where n' is the *rank* of the arrangement $\mathcal{A}$ ($n' \leq n$).

Notice that the proof of minimality, in case $d > 2$, is straightforward from our construction because of the gap between the dimensions of the critical cells.

One can conjecture that *torsion-free subspace arrangements are minimal*: that is, when the complement of the arrangement has torsion-free cohomology, then it is a minimal space.

We pass now to a more precise description of the contents of the several parts of the thesis.

Chapters 2, 3 and 4 are introductive, the original part can be found at most in chapters 5 and 6.

Chapter 2 is an introductory collection of the main tools needed in the following parts. It includes: Orlik-Solomon algebra and related topics, as the so called *broken circuit bases*; the definition of Salvetti complex; the main definitions and results of the Discrete Morse Theory, following the original work by Forman ([F98, F02]).

In chapter 3 we deal with general subspace arrangements. In section 3.1 we recall Goresky-MacPherson formula. We consider here the explicit example given in [J94] of a subspace arrangement such that its complement is not torsion-free. This arrangement is composed with six codimensional-5 coordinate subspaces in $\mathbb{R}^{10}$ (we make complete computation of the cohomology of the complement by using Goresky-MacPherson formula).

In section 3.2 we define generalized $d-$configuration spaces $\mathcal{M}(\mathcal{A})^{(d)}$, and the generalized Salvetti complex $\mathbf{S}^{(d)}$, whose cells correspond to all *chains* $(C \prec F_1 \prec \ldots \prec F_d)$, where C is a chamber and the F_i's are facets of the induced stratification $\Phi(\mathcal{A})$ of $\mathbb{R}^n$ (and $\prec$ is the standard face-ordering in $\Phi(\mathcal{A})$).

In chapter 4 we present the reduction of the complex $\mathbf{S} = \mathbf{S}^{(1)}$ using discrete Morse theory, following [SS07]. We define a system of polar coordinates in $\mathbb{R}^n$, and the induced polar ordering on the stratification $\Phi(\mathcal{A})$. Next, we define a gradient vector field Γ on the set of cells of $\mathbf{S}$; the critical cells of Γ are in one-to-one correspondence with the cells of a new CW-complex, which has the same homotopy type as $\mathbf{S}$. One can verify that the number of critical cells of dimension k equals the $k-$th Betti number, so the latter CW-complex is minimal.

The main original part of our thesis is contained in the last two chapters.

In chapter 5 we consider the two-dimensional case. For any affine line arrangement $\mathcal{A}$, we give explicit *minimal* attaching maps for the minimal two-complex corresponding to the polar gradient vector field. After considering the central case, the proof is by induction on the number of 0-dimensional facets of $\mathcal{A}$.

Of course, presentations of the fundamental group of the complement follow straightforward from these explicit boundary formulas.

In chapter 6 we apply discrete Morse theory to the complex $\mathbf{S}^{(d)}$. Even if the philosophy here is similar to that used for $d = 1$, the extension to the case $d > 1$ is not trivial. To construct a gradient field on $\mathbf{S}^{(d)}$, we have to consider on the ith-component of the chains $(C \prec F_1 \prec \ldots \prec F_d) \in \mathbf{S}^{(d)}$ either the polar ordering which is induced on the arrangement "centered" in the $(i+1)$th-component of the chain, or the opposite of such ordering, according to the parity of $d - i$. Then we use a double induction over d and the dimension of a sub-arrangement of $\mathcal{A}$.

Several examples are considered in order to better illustrate our results.

Chapter 2

Definitions and prerequisites

2.1 Hyperplane arrangements and Orlik-Solomon algebra

In this chapter we recall some basic definitions about hyperplane arrangements. A complete tractment may be founded in [OT92].

Definition 2.1.1 *Let $\mathbb{K}$ be a field, that we always suppose to be $\mathbb{C}$ or $\mathbb{R}$, and let V be a vector space of dimension l on $\mathbb{K}$. An hyperplane H in V is an affine subspace of dimension $l-1$. An* hyperplane arrangement $\mathcal{A} = \{H_i\}_{i \in I}$ *is a finite set of hyperplanes of V.*

When $\mathcal{A}$ is an hyperplane arrangement in $V = R^2$, the arrangement is called a line arrangement*.*

More generally, a *subspace arrangement* is a finite set of affine subspaces of V with no dimension restriction. Without any further specifications, by "arrangements" we just mean hyperplane arrangements.

Definition 2.1.2 *For all subspace arrangements $\mathcal{A}$ define the complement $\mathcal{M}(\mathcal{A})$ as the topological space:*

$$\mathcal{M}(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$$

An important class of arrangements is the following:

Definition 2.1.3 *If $T = \cap_{H \in \mathcal{A}} H \neq \emptyset$ then $\mathcal{A}$ is called central.*

Definition 2.1.4 *Chose for each hyperplane $H \in \mathcal{A}$ a polynomial α_H of degree 1 (defined up to a constant) such that $H = ker\alpha_H$.*

The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called the defining polynomial of $\mathcal{A}$.

We put here two examples of arrangement:

Example 1 *Let* $\mathbb{K} = \mathbb{R}$ *and* $V = \mathbb{R}^n = (x_1, \dots, x_n)$. *The* Boolean arrangement *in* V *is defined by:*

$$Q(\mathcal{A}) = x_1 x_2 \cdots x_n.$$

Example 2 *Let* $\mathbb{K} = \mathbb{R}$ *and* $V = \mathbb{R}^n$. *For* $1 \leq i < j \leq l$ *let* $H_{i,j} = ker(x_i - x_j)$. *The* braid arrangement *is defined by:*

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq l} (x_i - x_j)$$

Now we introduce a combinatorial poset associated to an arrangement, which is a lattice in the central case. For a background about lattices and posets see for example [K07].

Let us recall here the definition of *order complex* of a poset.

For a poset P, the set $S \subset P$ is called a *chain*, if S is totally ordered with respect to the partial order of P.

Definition 2.1.1 *Let* P *be a poset. Define* $K(P)$ *as the abstract simplicial complex, whose vertices are all elements of* P*, and whose simplices are all finite chains of* P*, including the empty chain. The complex* $K(P)$ *is called the* order complex *of* P.

Remark 2.1.5 *Observe that the order complex* $K(P)$ *of a poset* P *having a maximal (minimal) element is contractible, because it is a cone over the point in* $K(P)$ *corresponding to the maximal (minimal) element of* P.

We define $\mathcal{L} := \mathcal{L}(\mathcal{A})$ as a poset whose elements are all the subspaces of V of the form:

$$L = H_{j_1} \cap \cdots \cap H_{j_k}, \quad H_{j_l} \in \mathcal{A}.$$

The partial ordering in $\mathcal{L}$ is given by

$$L \prec L' \text{ iff } L' \subset L,$$

so there is a minimum element, corresponding to the empty intersection, which is the whole space V. Moreover, when $\mathcal{A}$ is a central arrangement, we have a maximum $L_0 := \bigcap_{H \in A} H$. The *rank* $rk(L)$ of a subspace $L \in \mathcal{L}$ is its codimension; the rank of $\mathcal{L}$ is the rank of L_0, and we also set $rk(\mathcal{A}) := rk(\mathcal{L}(\mathcal{A}))$.

In the central case $\mathcal{L}$ has the properties of a geometric lattice, that we call the *intersection lattice*. For X, Y in $\mathcal{L}$, the *meet* is given by $X \wedge Y = \cap\{Z \in \mathcal{L} | X \cup Y \subset Z\}$, while the *join* is defined by $X \vee Y = X \cap Y$. The *atoms* of this lattice are the hyperplanes $H \in \mathcal{L}$, for which we have $rk(H) = 1$.

The arrangement is called *essential* when $rk(\mathcal{A}) = n = dim(V)$, i.e. when L_0 reduces to a single point.

Example 3 *The lattice $\mathcal{L}(\mathcal{A})$ of the Boolean arrangement in $\mathbb{R}^n$ can be described as follows: let $H_i = ker(x_i)$ and let $I = \{i_1, \ldots, i_p\}$ where $1 \leq i_1 < \cdots < i_p \leq l$. Let $H_I = H_{i_1} \cap \cdots \cap H_{i_p}$. The lattice $\mathcal{L}(\mathcal{A})$ consists of the 2^l subspaces H_I for all subsets I.*

Example 4 *The lattice $\mathcal{L}(\mathcal{A})$ of the braid arrangement in $\mathbb{R}^n$ is the partition lattice Π_n, i.e. the set whose elements are all set partitions of the set $[n] = \{1, \ldots, n\}$. This can be partially ordered by refinement. This poset has a minimal element $\{1\}\{2\}\cdots\{n\}$ and a maximal element $[n]$. See [OT92] for the proof.*

We can also define a finer poset $\Phi := \Phi(\mathcal{A}) := (\{F\}, \prec)$ whose elements are the strata (also called *facets*) of the stratification induced on V by $\mathcal{A}$, where, as before:

$$F \prec F' \text{ iff } F' \subset cl(F).$$

The atoms (*chambers*) of $\Phi(\mathcal{A})$ are the connected components of $\mathcal{M}(\mathcal{A})$.

We have a map $\Phi \to \mathcal{L}$ which associates to a facet F its *support* $|F|$, which is by definition the subspace generated by F (in a different language, this is the standard map between an oriented matroid and its underlying matroid, see [OT92]). We define the rank function on Φ via this map:

$$rk(F) := rk(|F|) = codim(F).$$

Definition 2.1.6 *Let $(\mathcal{A}, V)$ be an arrangement. If $\mathcal{B} \subset \mathcal{A}$ is a subset, then $(\mathcal{B}, V)$ is called a subarrangement. For $L \in \mathcal{L}(\mathcal{A})$ define a subarrangement $\mathcal{A}_L$ of $\mathcal{A}$ by:*

$$\mathcal{A}_L = \{H \in \mathcal{A} | L \subset H\}.$$

Define an arrangement $(\mathcal{A}^L, L)$ in L by:

$$\mathcal{A}^L = \{L \cap H | H \in \mathcal{A} \setminus \mathcal{A}_L \text{ and } L \cap H \neq \emptyset\}.$$

Remark 2.1.7 *$\mathcal{A}_L$ is an arrangement in V of rank equal to $rk(L)$. $\mathcal{A}^L$ is an arrangement inside L itself (of rank $rk(\mathcal{A}) - rk(L)$). Let $\Phi_L := \Phi(\mathcal{A}_L)$, $\Phi^L := \Phi(\mathcal{A}^L)$ be the induced stratifications of V, L respectively. There is a map $pr_L : \Phi \to \Phi_L$, taking F' into the unique stratum containing it, and a map $j^L : \Phi^L \to \Phi$ just given by the inclusion.*

Fixing a facet F, set also $\Phi_F = \{ F' \in \Phi : F' \prec F\}$. It is easy to see that the restriction $\varphi_F := pr_{|F|}|\Phi_F : \Phi_F \to \Phi_{|F|}$ is a dimension-preserving bijection of posets.

2.1.1 Coxeter arrangements

We shall now recall some basic facts about Coxeter groups and reflection groups following [B81] (see also [H90]).

We shall denote by W a multiplicative group and by Σ a subset of generators of W such that $\Sigma = \Sigma^{-1}$ and $1 \notin \Sigma$.

Definition 2.1.8 *The pair* (W, Σ) *is said to be a* Coxeter system *if it satisfies the following condition: let* $m(s, s')$ *be the order of* ss'*, with* $s, s' \in \Sigma$*; let* I *be the set of the couples* (s, s') *such that* $m(s, s')$ *is finite. The set of generators* Σ *and the relations* $(ss')^{m(s,s')} = 1$ *for* $(s, s') \in I$ *give a presentation of* W*.*

When (W, Σ) is a Coxeter system, we shall also say that W is a *Coxeter group.*

Example 5 *Let* $\mathcal{S}_n$ *be the symmetric group with* $n \geq 2$*, let* s_i *be the transposition of* i *and* $i + 1$ *with* $1 \leq i < n$ *and let* $\Sigma = \{s_1, \ldots, s_{n-1}\}$*. Then* $(\mathcal{S}_n, \Sigma)$ *is a Coxeter system.*

Definition 2.1.9 *Let* J *be a set. A* Coxeter graph of type J *is a graph having* J *as set of vertices and whose edges are labelled by integers* ≥ 3 *or by the symbol* ∞*.*

To every Coxeter system (W, Σ) we can associate a Coxeter graph of type Σ by joining s and s' with an edge whenever $m(s, s') \geq 3$ or $m(s, s') = \infty$. The number $m(s, s')$ will be the label of the edge. It follows that two vertices $s \neq s'$ are not connected by an edge if and only if s and s' commute. Since the label 3 occurs frequently, it is usually omitted when drawing pictures.

Example 6 *The Coxeter graph associated with the Coxeter group* $(\mathcal{S}_n, \Sigma)$ *of example 5 is given by:*

$$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\cdots\circ\!-\!\!-\!\circ\!-\!\!-\!\circ \qquad (n-1 \text{ vertices})$$

It can be shown that any Coxeter graph is the graph associated with a Coxeter system.

A Coxeter system (W, Σ) is called *irreducible* if the graph underlying the associated Coxeter graph is connected and non-empty. That is to say that Σ in non-empty and there do not exist two different subsets Σ' and Σ'' in Σ such that each element of Σ' commutes with each element of Σ''.

Definition 2.1.10 *Let* $w \in W$*. The smallest integer* $q \geq 0$ *such that* w *is the product of a sequence of* q *elements of* Σ *is called the* length of w *with respect to* Σ*, and it is denoted by* $\ell_\Sigma(w)$*.*

For any part Γ of Σ, W_Γ will denote the subgroup of W generated by Γ. All groups obtainable in this way are called *parabolic subgroups*. The same name will be given also to all the subgroups conjugate under W with a parabolic subgroup.

Proposition 2.1.11 *Let* (W, Σ) *be a Coxeter system. Let* $\Gamma \subset \Sigma$*. The following statements hold.*

(i) (W_Γ, Γ) *is a Coxeter system.*

(ii) Viewing W_Γ *as a Coxeter group with length function* ℓ_Γ*,* $\ell_\Sigma = \ell_\Gamma$ *on* W_Γ*.*

(iii) Define $W^\Gamma \stackrel{\text{def}}{=} \{w \in W | \ell(ws) > \ell(w) \text{ for all } s \in \Gamma\}$*. Given* $w \in W$*, there is a unique* $u \in W^\Gamma$ *and a unique* $v \in W_\Gamma$ *such that* $w = uv$*. Their lengths satisfy* $\ell(w) = \ell(u) + \ell(v)$*. Moreover,* u *is the unique element of smallest length in the coset* wW_Γ*.*

¿From now on we consider only finite Coxeter groups.

Definition 2.1.12 *The* Poincaré polynomial of W *is the polynomial in the indeterminate* q *given by*

$$W(q) = \sum_{w \in W} q^{\ell(w)}.$$

More generally, for an arbitrary subset X of W we define $X(q) = \sum_{w \in X} q^{\ell(w)}$. For $\Gamma \subset \Sigma$, $W_\Gamma(q)$ coincides with the Poincaré polynomial of the group W_Γ. It is an immediate consequence of part *(iii)* of Proposition 2.1.11 that

$$W(q) = W^\Gamma(q) W_\Gamma(q).$$

Let us now see how Coxeter groups are related to reflection groups. We begin with recalling what is meant by a reflection in a real vector space V.

Definition 2.1.13 *A* reflection s *is an endomorphism of* V *such that* $1 - s$ *has rank equal to* 1 *and* $s^2 = 1$.

By definition, the kernel of $1-s$, i.e. the set of points x with $s(x) = x$, is a hyperplane H_s so that V is the direct sum of H_s and of a line L which is in the (-1)-eigenspace. Conversely, it is easy to see that if V is the direct sum of a hyperplane H and a line $L = \mathbb{R}\alpha$, with $\alpha \in V$, and s_α is an endomorphism of V such that $x = s_\alpha(x)$ for every $x \in H$ and $s_\alpha(y) = -y$ for every $y \in L$, then s_α is a reflection. It is sufficient to observe that if $v = x + y$, then it must be $s_\alpha(v) = x - y$. The line L is the kernel of $1 + s$.

Let B be a positive definite symmetric bilinear form on V. B is invariant under a reflection s if and only if $Ker(1-s)$ and $Ker(1+s)$ are orthogonal with respect to B.In this case $Ker(1-s)$ and $Ker(1+s)$ are non-degenerate. Conversely, if H is a hyperplane in V then there exists one and only one reflection s_H preserving B and inducing the identity on H. It is called the *orthogonal reflection associated with* H. If $a \neq 0$ is a vector orthogonal to H then $B(a,a) \neq 0$ and s_H is given by the formula

$$s_H(x) = x - 2\frac{B(x,a)}{B(a,a)} a$$

for every $x \in V$.

A finite group generated by reflections is usually called a *finite reflection group.*

Example 7 *$\mathcal{S}_n$ can be thought of as a finite reflection group acting on $\mathbb{R}^n$ (endowed with the standard scalar product) in the following way. Let $\{e_i\}_{1 \leq i \leq n}$ denote the standard basis of $\mathbb{R}^n$. The transposition of i and j acts as a reflection sending $e_i - e_j$ to its negative and fixing point-wise the orthogonal complement.*

We say that a finite reflection group W is *essential relative to* V if W acts on V with no nonzero fixed points.

For example $\mathcal{S}_n$ is not essential on $\mathbb{R}^n$ for it fixes the line generated by $e_1 + \cdots + e_n$. However $\mathcal{S}_n$ is essential relative to $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$.

If W is a finite reflection group acting on a Euclidean space V, as we have seen, each reflection s_α in W determines a reflecting hyperplane H_α and a line $L_\alpha = \mathbb{R}\alpha$ orthogonal to it. It turns out that W permutes the collection of all such lines. Indeed it is a result that if $w \in W$ then $s_{w\alpha} = w s_\alpha w^{-1}$ belongs to W whenever s_α does. It must be observed that only the lines L_α are determined by W, not the vectors α.

Definition 2.1.14 *A* root system Φ *is a finite set of nonzero vectors in* V*, called* roots *satisfying the conditions:*

(R1) Φ *generates the space* V

(R2) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ *for all* $\alpha \in \Phi$*;*

(R3) $s_\alpha \Phi = \Phi$ *for all* $\alpha \in \Phi$*.*

Given a root system Φ we can consider the group W generated by all reflections s_α, $\alpha \in \Phi$. Then W is finite, because it is contained in the permutation group of Φ. Conversely, any finite reflection group can be realized in this way, possibly for many different choices of Φ. Finally, every reflection in W is of the form s_α for some α in Φ.

Since a root system Φ may be extremely large compared with the dimension of V, we are lead to look for a linearly independent subset of Φ.

Definition 2.1.15 *A subset* Δ *of a root system* Φ *is said to be a* simple system *if* Δ *is a vector space basis for the* $\mathbb{R}$*-span of* Φ *in* V *and if moreover each* $\alpha \in \Phi$ *is a linear combination of elements of* Δ *with coefficients all of the same sign.*

It is a theorem that simple systems exist. Moreover, if Δ is a simple system in a root system Φ with associated reflection group W, then W is generated by those s_α for which $\alpha \in \Delta$.

So far we have described the way in which a finite reflection group W acts on V. Let us now describe the structure of W as an abstract group.

Theorem 1 *Let* Δ *be a simple system of* Φ*. Let* W *be the associated finite reflection group. Then* W *is generated by the set* $\Sigma = \{s_\alpha | \alpha \in \Delta\}$*, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \ (\alpha, \beta \in \Delta),$$

where $m(\alpha, \beta)$ *denotes the order of* $s_\alpha s_\beta$ *in* W*.*

In other words any finite reflection group is a finite Coxeter group.

Conversely, if we consider a finite Coxeter system (W, Σ), we can give W a geometrical representation as a group generated by reflections in a Euclidean space. The idea is to begin with a real vector space V having a basis $\{\alpha_s | s \in \Sigma\}$ in one-to-one correspondence with Σ

and then to impose a geometry on V in such a way that the "angle" between α_s and $\alpha_{s'}$ will be compatible with the given $m(s, s')$. To this aim we define a symmetric bilinear form B on V by setting

$$B(\alpha_s, \alpha_{s'}) = \begin{cases} -\cos \frac{\pi}{m(s,s')} & \text{if } m(s, s') \in \mathbb{Z}, \\ -1 & \text{if } m(s, s') = \infty. \end{cases}$$

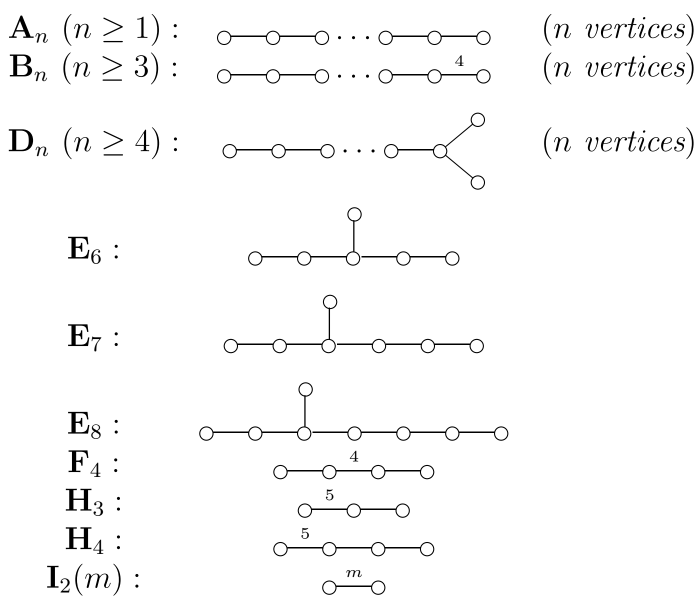
For each $s \in S$ we can now define a reflection $\sigma_s : V \to V$ by the rule

$$\sigma_s(x) = x - 2B(\alpha_s, x)\alpha_s.$$

Theorem 2 *Let W be a Coxeter group. The following conditions are equivalent:*
(a) W is finite.
(b) The bilinear form B is positive definite.
(c) W is a finite reflection group.

It follows that the classification of finite Coxeter groups coincides with that of all finite groups generated by reflections in a finite dimensional real vector space. This classification is completely known.

Theorem 3 *If (W, Σ) is a finite irreducible Coxeter system, its graph is one of the following ones:*



We recall that $\mathbf{A}_n$ is isomorphic to $\mathcal{S}_{n+1}$ and $\mathbf{I}_2(m)$ is isomorphic to the dihedral group $\mathcal{D}_m$.

Definition 2.1.16 *Given $V = \mathbb{R}^n$ and $\mathbf{W}$ a Coxeter group, the set $\mathcal{H}$ of its reflecting hyperplanes in V is an hyperplane arrangement. An hyperplane arrangement whose hyperplanes are given by the elements of a Coxeter group is called a* Coxeter arrangement *or a* reflection arrangement*.*

Let us conclude this section by emphasizing the role of the reflecting hyperplanes.

In the rest of this section W will denote a finite reflection group and $\mathcal{H}$ will be the set of its reflecting hyperplanes.

Definition 2.1.17 *Let R denote the equivalence relation defined by setting, for any two points x and y in V, xRy if for every $H \in \mathcal{H}$ either $x \in H$ and $y \in H$ or x and y belong to the same open half-space determined by H. The equivalence classes of R are called* facets of V with respect to *$\mathcal{H}$.*

Definition 2.1.18 *Every facet C of V with respect to $\mathcal{H}$ that is not contained in any hyperplane of $\mathcal{H}$ is called a* chamber *of V. The facets contained in the adherence of C are called* faces *of C. The hyperplanes supporting a face of C are said to be the* walls *of C.*

Remark 2.1.19 *Observe that the previous definition coincides with the definition of facets of stratification $\Phi(\mathcal{A})$ (see section 2.1).*

Proposition 2.1.20 *Let C be a chamber. The following statements hold.*

i) For every $x \in V$, there exists an element $w \in W$ such that $w(x) \in \overline{C}$.

ii) For every chamber C', there exists a unique element $w \in W$ such that $w(C) = C'$.

iii) A simple system of roots for W is given by those roots orthogonal to some wall of C and lying on the same side of the wall as C.

iv) Let $w \in W$ and let H be a wall of C. If $\ell(s_H w) > \ell(w)$ then the chambers C and $w(C)$ are on the same side of H.

2.1.2 Orlik-Solomon Algebra

In this section we assume that $\mathcal{A}$ is an arrangement of hyperplanes in a vector space V of dimension n over a field $\mathbb{K}$ and that $\mathcal{A}$ is central.

We associated to $\mathcal{A}$ a graded anticommutative algebra $A(\mathcal{A})$ over a commutative ring $\mathcal{K}$, the so called *Orlik-Solomon algebra*. The algebra $A(\mathcal{A})$ was first defined in [S87] (see also [OT92]).

Definition 2.1.21 *Let $\mathcal{A}$ be an arrangement over $\mathbb{K}$. We can associate to any hyperplane H an element e_H. Let $\mathcal{K}$ be a commutative ring. Let*

$$E_1 = \bigoplus_{H \in \mathcal{A}} \mathcal{K} e_H$$

and let

$$E = E(\mathcal{A}) = \Lambda(E_1)$$

be the exterior algebra of E_1.

Write $uv = u \wedge v$ and note that $e_H^2 = 0$, $e_H e_K = -e_K e_H$ for $H, K \in \mathcal{A}$. The algebra E is graded. If $|\mathcal{A}| = n$, then

$$E = \bigoplus_{p=0}^{n} E_p,$$

where $E_0 = \mathcal{K}$, E_1 agrees with its earlier definition and E_p is spanned over $\mathcal{K}$ by all $e_{H_1} \cdots e_{H_p}$ with $H_k \in \mathcal{A}$.

Definition 2.1.22 *Define a $\mathcal{K}$-linear map $\partial_E = \partial : E \longrightarrow E$ by $\partial(1) = 0$, $\partial(e_H) = 1$ and for $p \geq 2$*

$$\partial(e_{H_1} \cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots e_{\hat{H_k}} \cdots e_{H_p}$$

for all $H_1, \cdots, H_p \in \mathcal{A}$.

(E, ∂_E) is a chain complex (see [OT92]).

Definition 2.1.23 *Given a p-tuple of hyperplanes, $S = (H_1, \cdots, H_p)$, write $|S| = p$,*

$$e_S = e_{H_1} \cdots e_{H_p} \in E, \qquad \cap S = H_1 \cap \cdots \cap H_p.$$

Since $\mathcal{A}$ is central, $\cap S \in \mathcal{L}(\mathcal{A})$ for all S.

If $p = 0$, we agree that $S = ()$ is the empty tuple, $e_S = 1$, and $\cap S = V$.
Since the rank function on $\mathcal{L}$ is codimension, it is clear that $rk(\cap S) \leq |S|$.

Definition 2.1.24 *Call S* independent *if $rk(\cap S) = |S|$ and* dependent *if $rk(\cap S) < |S|$.*

Geometrically the tuple S is independent if the hyperplanes of S are in general position.

Let $\mathbf{S}_p$ denote the set of all p-tuples $(H_1, \cdots, H_p)$ and $\mathbf{S} = \cup_{p \geq 0} \mathbf{S}_p$. Then we define $I = I(\mathcal{A})$ the ideal of E generated by $\partial(e_S)$ for all dependent $S \in \mathbf{S}$.

Definition 2.1.25 *Set $A = A(\mathcal{A}) = E/I$, let us define $\partial_A : A \longrightarrow A$ by $\partial_A(\varphi(u)) = \varphi(\partial_E(u))$ where $\varphi : E \longrightarrow A$ is the natural homomorphism.*

Since $\partial_E I \subseteq I$ then ∂_A is well defined and (A, ∂_A) is a chain complex, in particular it's acyclic (see [OT92]).

The construction of $A(\mathcal{A})$ in the affine case is quite similar (see [OT92]).

In [S87] Orlik and Solomon prove a very interesting result:

Theorem 4 *If $\mathcal{A}$ is a complex arrangement then*

$$A(\mathcal{A}) \simeq H^*(\mathcal{M}(\mathcal{A}), \mathbb{Z})$$

as a graded algebra.

2.1.3 The Broken Circuit Basis

Next we show that the $\mathcal{K}$-algebra $A(\mathcal{A})$ is a free $\mathcal{K}$-module by constructing a standard $\mathcal{K}$-basis.

We introduce an arbitrary linear order $\dashv$ in $\mathcal{A}$. Call a p-tuple $S = (H_1, \cdots, H_p)$ *standard* if $H_1 \dashv \cdots \dashv H_p$. Notice that $E = E(\mathcal{A})$ has a $\mathcal{K}$-basis consisting of all e_S with standard S.

Definition 2.1.26 *A p-tuple $S = (H_1, \cdots, H_p)$ is a* circuit *if it is minimally dependent. Thus $(H_1, \cdots, H_p)$ is dependent, but for $1 \leq k \leq p$ the (p-1)-tuple $(H_1, \cdots, \hat{H}_k, \cdots, H_p)$ is independent.*

Given $S = (H_1, \cdots, H_p)$, let $\max(S)$ be the maximal element of S in the linear order $\dashv$ in $\mathcal{A}$.

Definition 2.1.27 *A standard p-tuple $S \in \mathbf{S}$ is a* broken circuit *if there exists $H \in \mathcal{A}$ such that* $\max$*(S)* $\dashv$ *H and (S, H) is a circuit.*

Definition 2.1.28 *A standard p-tuple S is called* χ-independent *if it does not contain any broken circuit. Define*

$$\mathcal{C}_p = \{S \in \mathbf{S}_p : S \text{ is standard and } \chi - \text{independent}\}.$$

Let $\mathcal{C} = \cup_{p \geq 0} \mathcal{C}_p$.

Definition 2.1.29 *The* broken circuit module *$C = C(\mathcal{A})$ is defined as follows. Let $C_0 = \mathcal{K}$, and for $p \geq 1$ let C_p be the free $\mathcal{K}$-module with basis $\{e_S \in E : S \in \mathcal{C}_p\}$. Let $C = C(\mathcal{A}) = \oplus_{p \geq 0} C_p$. Then $C(\mathcal{A})$ is a free $\mathcal{K}$-module.*

It is clear that every broken circuit is obtained by deleting the maximal element in a standard circuit. By definition $C(\mathcal{A})$ is a submodule of $E(\mathcal{A})$. If we define $\psi : C(\mathcal{A}) \longrightarrow A(\mathcal{A})$ the restriction of $\varphi : E(\mathcal{A}) \longrightarrow A(\mathcal{A})$ we have (see [S87])

Theorem 5 *The map $\psi : C(\mathcal{A}) \longrightarrow A(\mathcal{A})$ is an isomorphism of graded $\mathcal{K}$-module. The set*

$$\{e_S + I \in A(\mathcal{A}) | S \text{ is standard and } \chi - \text{independent}\}$$

is a basis for $A(\mathcal{A})$ as a graded $\mathcal{K}$-module.

In the affine case we can define analogously the Orlik-Solomon algebra and the broken circuit basis. See [OT92] for details.

2.2 Salvetti's complex

Now we present a cell complex that has the same homotopy type of the complement $\mathcal{M}(\mathcal{A})$ of the complexification of a real arrangement $\mathcal{A}_{\mathbb{R}}$.

The construction we are going to present can be found in [S87], where we refer for all proofs.

Let $\mathcal{A}_{\mathbb{R}}$ be a finite real arrangement in $\mathbb{R}^n$. The arrangement induces a stratification $\Phi(\mathcal{A})$ of the space $\mathbb{R}^n$ in facets, as we have seen in section 2.1. Let $\mathbf{Q}$ be the dual cell complex of Φ. We can realize $\mathbf{Q}$ inside $\mathbb{R}^n$ associating to each facet F^j of codimension j a point $v(F^j) \in F^j$ and considering the simplexes:

$$\sigma(F^{i_0}, \dots, F^{i_j}) = \{\sum_{k=0}^{j} \lambda_k v(F^{j_k}) \mid \sum \lambda_k = 1, \lambda_k \in [0,1]\}$$

where $F^{i_k} \prec F^{i_{k+1}}$ for $k = 0, \dots, j-1$. We define the j-cell $e^j(\widetilde{F}^j)$, dual to $\widetilde{F}^j$, as the union $\bigcup \sigma(F^0, \dots, F^{j-1}, \widetilde{F}^j)$, over all the chains $F^0 \prec \cdots \prec \widetilde{F}^j$. Hence we have $\mathbf{Q} = \bigcup e^j(F^j)$, where the union is taken over all facets in Φ.

We can think of the 1-skeleton $\mathbf{Q}_1$ as a graph and we define the combinatorial distance between two vertexes v, v' of $\mathbf{Q}$ as the minimum number of edges in an edge-path connecting v to v'. For each cell e^j we indicate by $V(e^j) = \mathbf{Q}_0 \cap e^j$ the 0-skeleton of e^j. We have:

Lemma 2.2.1 *For every vertex $v \in \mathbf{Q}_0$ and for every cell $e^i \in \mathbf{Q}$, there exists a unique vertex $\underline{w}(v, e^i) \in V(e^i)$ of minimal distance from v, that is:*

$$d(v, \underline{w}(v, e^i)) < d(v, v') \qquad \forall v' \in V(e^i) \setminus \{\underline{w}(v, e^i)\}$$

If $e^j \subset e^i$ then $\underline{w}(v, e^j) = \underline{w}(\underline{w}(v, e^i), e^j)$. □

Take a cell $e^j = e^j(F^j) = \bigcup \sigma(F^0, \dots, F^{j-1}, F^j)$ of $\mathbf{Q}$ and let $v \in V(e^j)$. We can map the simplex $\sigma(F^0, \dots, F^j)$ in $\mathbb{C}^n$ by the application

$$\phi_{v,e^j}(\sum \lambda_k v(F^k)) = \sum \lambda_k v(F^k) + i \sum \lambda_l(\underline{w}(v, e^k) - v(F^k)).$$

One can prove that ϕ_{v,e^j} gives an embedding of e^j in $\mathcal{M}(\mathcal{A})$. We call $E^j(e^j, v)$ the image of the map ϕ_{v,e^j}, and we define the *Salvetti complex* for the arrangement $\mathcal{A}$ as the union

$$\mathbf{S} = \bigcup E^j(e^j, v),$$

where the union is taken over all e^j and v.

Theorem 6 ([S87]) *The CW-complex $\mathbf{S}$ is homotopy equivalent to the complement $\mathcal{M}(\mathcal{A})$.* □

We remark that the fact that the maps ϕ_{v,e^j} glue together in the proper way is a consequence of Lemma 2.2.1.

2.2.1 Examples

In this section we present some examples that we will use for all this tractation:

Example 8 *In figure 2.1 is represented a central arrangement in $V = \mathbb{R}^2$ with six lines. Here is drawed also a polar system given by $(0, V)$ (see section 4.1). Plane hyperplane arrangement are also called line-arrangements.*

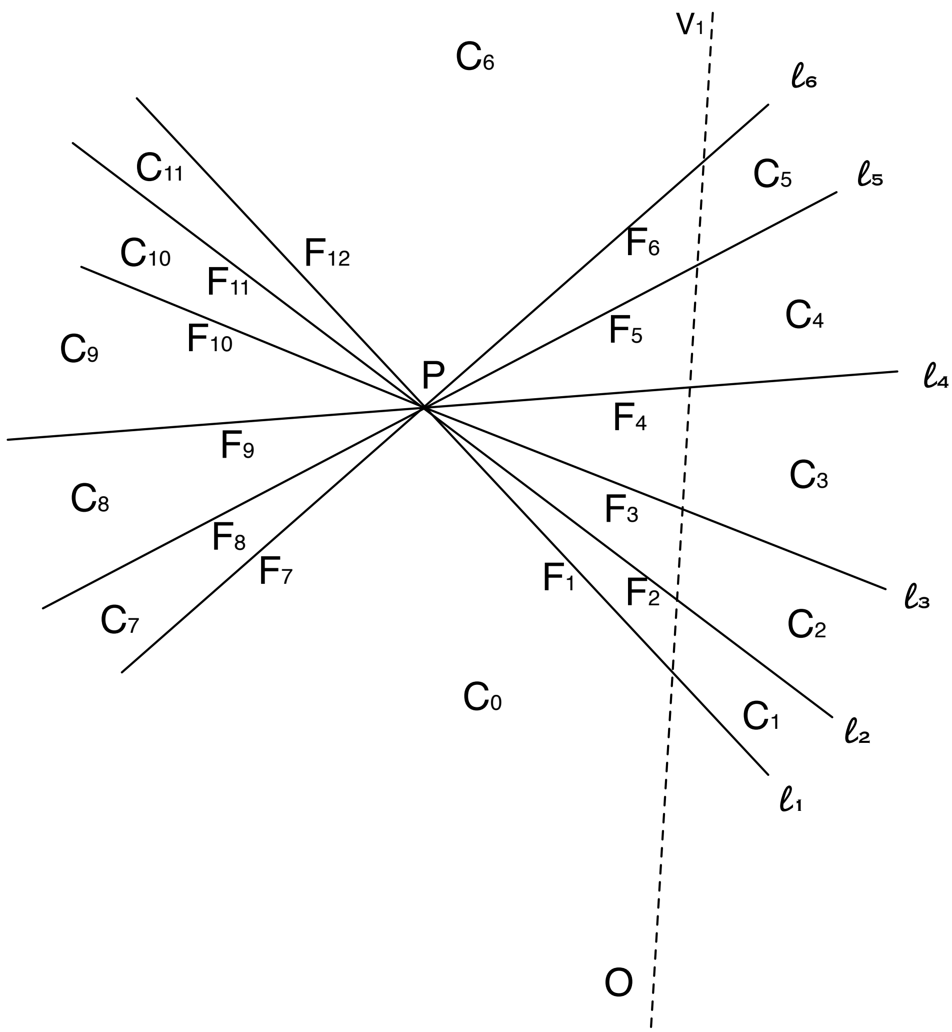


Figure 2.1: The central case with six line.

The complex $\mathbf{S}^{(1)}$ is composed by the following cells:

1. *12 cells of dimension 0:* $e(C_0, C_0)$, $e(C_1, C_1)$, $e(C_2, C_2)$, $e(C_3, C_3)$, $e(C_4, C_4)$, $e(C_5, C_5)$, $e(C_6, C_6)$, $e(C_7, C_7)$, $e(C_8, C_8)$, $e(C_9, C_9)$, $e(C_{10}, C_{10})$, $e(C_{11}, C_{11})$,

2. *24 cells of dimension 1:* $e(C_0,F_1)$, $e(C_0,F_7)$, $e(C_1,F_1)$, $e(C_1,F_2)$, $e(C_2,F_2)$, $e(C_2,F_3)$, $e(C_3,F_3)$, $e(C_3,F_4)$, $e(C_4,F_4)$, $e(C_4,F_5)$, $e(C_5,F_5)$, $e(C_5,F_6)$, $e(C_6,F_6)$, $e(C_6,F_{12})$, $e(C_7,F_7)$, $e(C_7,F_8)$, $e(C_8,F_8)$, $e(C_8,F_9)$, $e(C_9,F_9)$, $e(C_9,F_{10})$, $e(C_{10},F_{10})$, $e(C_{10},F_{11})$, $e(C_{11},F_{11})$, $e(C_{11},F_{12})$;

3. *12 cells of dimension 2:* $e(C_0,P)$, $e(C_1,P)$, $e(C_2,P)$, $e(C_3,P)$, $e(C_4,P)$, $e(C_5,P)$, $e(C_6,P)$, $e(C_7,P)$, $e(C_8,P)$, $e(C_9,P)$, $e(C_{10},P)$, $e(C_{11},P)$.

Example 9 *In Figure 2.2 is represented the arrangement obtained from the reflection arrangement of type A_3 (see section 2.1.1) by the standard deconing construction (see for instance [OT92]).*

The complex $\mathbf{S}^{(1)}$ is composed by the following cells:

1. *12 cells of dimension 0:* $e(C_0,C_0)$, $e(C_1,C_1)$, $e(C_2,C_2)$, $e(C_3,C_3)$, $e(C_4,C_4)$, $e(C_5,C_5)$, $e(C_6,C_6)$, $e(C_7,C_7)$, $e(C_8,C_8)$, $e(C_9,C_9)$, $e(C_{10},C_{10})$, $e(C_{11},C_{11})$;

2. *30 cells of dimension 1:* $e(C_0,F_1)$, $e(C_0,F_{11})$, $e(C_1,F_1)$, $e(C_1,F_2)$, $e(C_1,F_6)$, $e(C_2,F_2)$, $e(C_2,F_3)$, $e(C_3,F_3)$, $e(C_3,F_4)$, $e(C_4,F_4)$, $e(C_4,F_5)$, $e(C_4,F_8)$, $e(C_5,F_5)$, $e(C_5,F_{10})$, $e(C_6,F_6)$, $e(C_6,F_7)$, $e(C_6,F_{12})$, $e(C_7,F_7)$, $e(C_7,F_8)$, $e(C_7,F_9)$, $e(C_8,F_9)$, $e(C_8,F_{10})$, $e(C_8,F_{15})$, $e(C_9,F_{11})$, $e(C_9,F_{12})$, $e(C_9,F_{13})$, $e(C_{10},F_{13})$, $e(C_{10},F_{14})$, $e(C_{11},F_{14})$, $e(C_{11},F_{15})$;

3. *20 cells of dimension 2:* $e(C_0,P_3)$, $e(C_1,P_1)$, $e(C_1,P_3)$, $e(C_2,P_1)$, $e(C_3,P_1)$, $e(C_4,P_1)$, $e(C_4,P_2)$, $e(C_5,P_2)$, $e(C_6,P_1)$, $e(C_6,P_3)$, $e(C_6,P_4)$, $e(C_7,P_1)$, $e(C_7,P_2)$, $e(C_7,P_4)$, $e(C_8,P_2)$, $e(C_8,P_4)$, $e(C_9,P_3)$, $e(C_9,P_4)$, $e(C_{10},P_4)$, $e(C_{11},P_4)$.

Example 10 *In section 2.1.1 we have defined the reflection arrangements of type A_n. The reflection arrangement of type A_3 is an arrangement $\mathcal{A}\in\mathbb{R}^3=(x_1,x_2,x_3)$ given by the section of the arrangement $\mathcal{B}=\{\{x_1=x_2\},\{x_1=x_3\},\{x_2=x_3\},\{x_1=x_4\},\{x_2=x_4\},\{x_3=x_4\}\}\in\mathbb{R}^4$ with the $3-$plane $\{(x_1,x_2,x_3,x_4)|x_1+x_2+x_3+x_4=1\}$ orthogonal to the line $l:x_1=x_2=x_3=x_4$. Figure 2.3 represents the section of $\mathcal{A}$ with the $2-$plane $z=100$, while figure 2.4 represents the section with the $2-$plane $z=-100$. In figure 2.4 we indicate with $\overline{F}$ the opposite of a bounded facet F of figure 2.3. The center of this arrangement is a facet of dimension 0, not indicated in our figures and corresponding to the origin of $\mathbb{R}^4$, that we call P.*

The complex $\mathbf{S}^{(1)}$ is composed by the following cells:

1. *24 cells of dimension 0:* $e(C_0,C_0)$, $e(C_1,C_1)$, $e(C_2,C_2)$, $e(C_3,C_3)$, $e(C_4,C_4)$, $e(C_5,C_5)$, $e(C_6,C_6)$, $e(C_7,C_7)$, $e(C_8,C_8)$, $e(C_9,C_9)$, $e(C_{10},C_{10})$, $e(C_{11},C_{11})$, $e(C_{12},C_{12})$, $e(C_{13},C_{13})$, $e(C_{14},C_{14})$, $e(C_{15},C_{15})$, $e(C_{16},C_{16})$, $e(C_{17},C_{17})$, $e(\overline{C}_7,\overline{C}_7)$, $e(\overline{C}_8,\overline{C}_8)$, $e(\overline{C}_9,\overline{C}_9)$, $e(\overline{C}_{10},\overline{C}_{10})$, $e(\overline{C}_{12},\overline{C}_{12})$, $e(\overline{C}_{13},\overline{C}_{13})$;

2. *72 cells of dimension 1:* $e(C_0,F_1)$, $e(C_0,F_7)$, $e(C_0,F_{20})$, $e(C_1,F_1)$, $e(C_1,F_2)$, $e(C_1,\overline{F}_{13})$, $e(C_2,F_2)$, $e(C_2,F_3)$, $e(C_2,\overline{F}_{18})$, $e(C_3,F_3)$, $e(C_3,F_4)$, $e(C_3,F_9)$, $e(C_4,F_4)$, $e(C_4,F_5)$, $e(C_4,\overline{F}_{22})$, $e(C_5,F_5)$, $e(C_5,F_6)$, $e(C_5,F_{12})$, $e(C_6,F_6)$, $e(C_6,F_{14})$, $e(C_6,\overline{F}_7)$, $e(C_7,F_7)$,

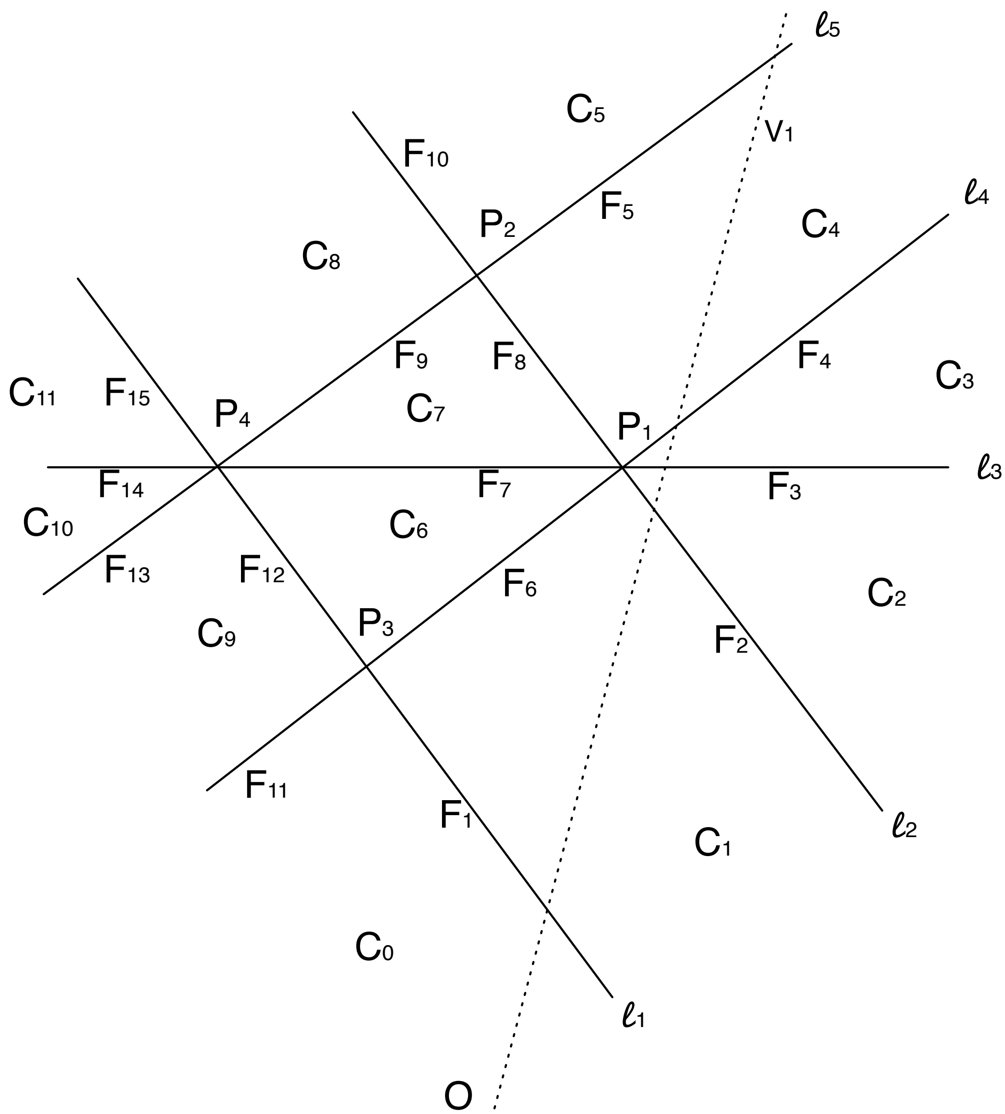


Figure 2.2: Deconing A_3

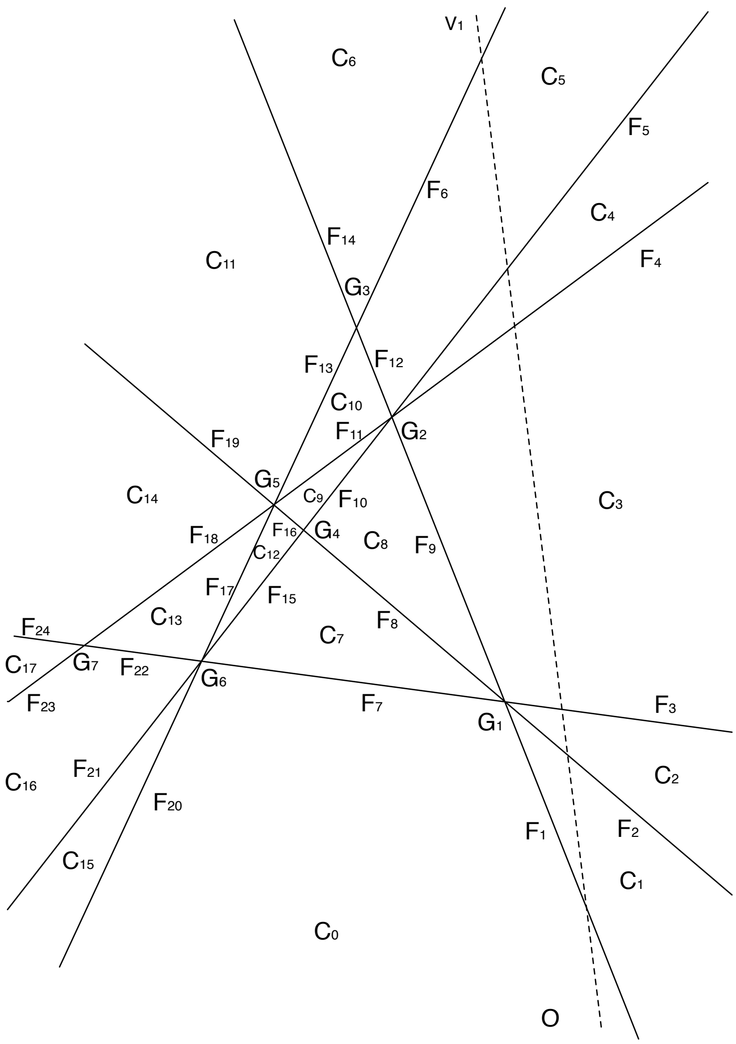


Figure 2.3: An upper section of the reflection arrangement A_3

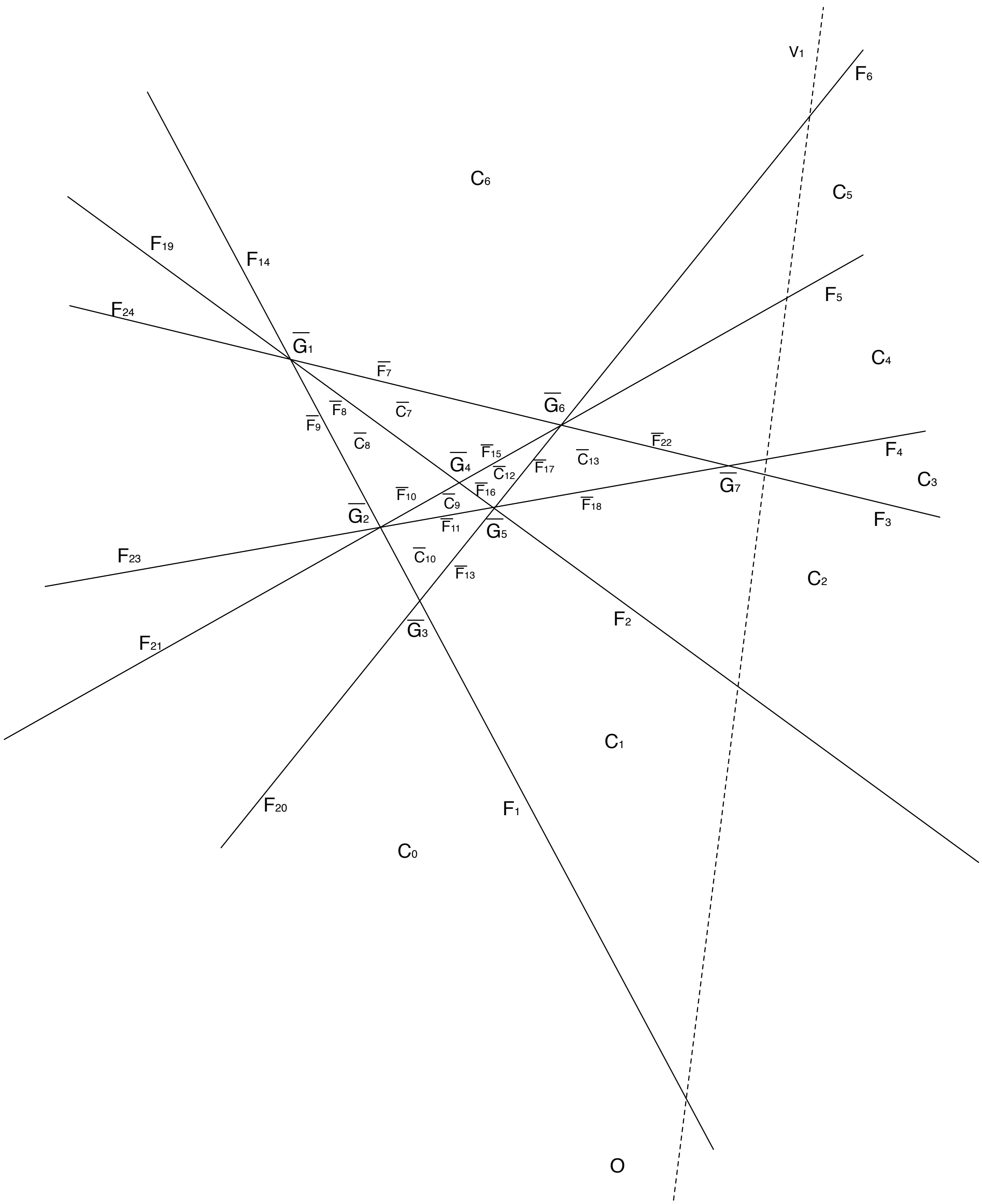


Figure 2.4: A lower section of the reflection arrangement A_3

$e(C_7, F_8)$, $e(C_7, F_{15})$, $e(C_8, F_8)$, $e(C_8, F_9)$, $e(C_8, F_{10})$, $e(C_9, F_{10})$, $e(C_9, F_{11})$, $e(C_9, F_{16})$, $e(C_{10}, F_{11})$, $e(C_{10}, F_{12})$, $e(C_{10}, F_{13})$, $e(C_{11}, F_{13})$, $e(C_{11}, F_{14})$, $e(C_{11}, F_{19})$, $e(C_{12}, F_{15})$, $e(C_{12}, F_{16})$, $e(C_{12}, F_{17})$, $e(C_{13}, F_{17})$, $e(C_{13}, F_{18})$, $e(C_{13}, F_{22})$, $e(C_{14}, F_{18})$, $e(C_{14}, F_{19})$, $e(C_{14}, F_{24})$, $e(C_{15}, F_{20})$, $e(C_{15}, F_{21})$, $e(C_{15}, \overline{F}_{12})$, $e(C_{16}, F_{21})$, $e(C_{16}, F_{22})$, $e(C_{16}, F_{23})$, $e(C_{17}, F_{23})$, $e(C_{17}, F_{24})$, $e(C_{17}, \overline{F}_9)$, $e(\overline{C}_7, \overline{F}_{15})$, $e(\overline{C}_7, \overline{F}_7)$, $e(\overline{C}_7, \overline{F}_8)$, $e(\overline{C}_8, \overline{F}_{10})$, $e(\overline{C}_8, \overline{F}_9)$, $e(\overline{C}_8, \overline{F}_8)$, $e(\overline{C}_9, \overline{F}_{16})$, $e(\overline{C}_9, \overline{F}_{10})$, $e(\overline{C}_9, \overline{F}_{11})$, $e(\overline{C}_{10}, \overline{F}_{13})$, $e(\overline{C}_{10}, \overline{F}_{11})$, $e(\overline{C}_{10}, \overline{F}_{12})$, $e(\overline{C}_{12}, \overline{F}_{17})$, $e(\overline{C}_{12}, \overline{F}_{15})$, $e(\overline{C}_{12}, \overline{F}_{16})$, $e(\overline{C}_{13}, \overline{F}_{18})$, $e(\overline{C}_{13}, \overline{F}_{22})$, $e(\overline{C}_{13}, \overline{F}_{17})$;

3. *72 cells of dimension 2:* $e(C_0, G_1)$, $e(C_0, G_6)$, $e(C_0, \overline{G}_3)$, $e(C_1, G_1)$, $e(C_1, \overline{G}_3)$, $e(C_1, \overline{G}_5)$, $e(C_2, G_1)$, $e(C_2, \overline{G}_5)$, $e(C_2, \overline{G}_7)$, $e(C_3, G_1)$, $e(C_3, G_2)$, $e(C_3, \overline{G}_7)$, $e(C_4, G_2)$, $e(C_4, \overline{G}_7)$, $e(C_4, \overline{G}_6)$, $e(C_5, G_2)$, $e(C_5, G_3)$, $e(C_5, \overline{G}_6)$, $e(C_6, G_3)$, $e(C_6, \overline{G}_6)$, $e(C_6, \overline{G}_1)$, $e(C_7, G_1)$, $e(C_7, G_4)$, $e(C_7, G_6)$, $e(C_8, G_1)$, $e(C_8, G_2)$, $e(C_8, G_4)$, $e(C_9, G_2)$, $e(C_9, G_4)$, $e(C_9, G_5)$, $e(C_{10}, G_2)$, $e(C_{10}, G_3)$, $e(C_{10}, G_5)$, $e(C_{11}, G_3)$, $e(C_{11}, G_5)$, $e(C_{11}, \overline{G}_1)$, $e(C_{12}, G_4)$, $e(C_{12}, G_5)$, $e(C_{12}, G_6)$, $e(C_{13}, G_5)$, $e(C_{13}, G_6)$, $e(C_{13}, G_7)$, $e(C_{14}, G_5)$, $e(C_{14}, G_7)$, $e(C_{14}, \overline{G}_1)$, $e(C_{15}, G_6)$, $e(C_{15}, \overline{G}_2)$, $e(C_{15}, \overline{G}_3)$, $e(C_{16}, G_6)$, $e(C_{16}, G_7)$, $e(C_{16}, \overline{G}_2)$, $e(C_{17}, G_7)$, $e(C_{17}, \overline{G}_1)$, $e(C_{17}, \overline{G}_2)$, $e(\overline{C}_7, \overline{G}_6)$, $e(\overline{C}_7, \overline{G}_1)$, $e(\overline{C}_7, \overline{G}_4)$, $e(\overline{C}_8, \overline{G}_4)$, $e(\overline{C}_8, \overline{G}_1)$, $e(\overline{C}_8, \overline{G}_2)$, $e(\overline{C}_9, \overline{G}_5)$, $e(\overline{C}_9, \overline{G}_4)$, $e(\overline{C}_9, \overline{G}_2)$, $e(\overline{C}_{10}, \overline{G}_5)$, $e(\overline{C}_{10}, \overline{G}_2)$, $e(\overline{C}_{10}, \overline{G}_3)$, $e(\overline{C}_{12}, \overline{G}_6)$, $e(\overline{C}_{12}, \overline{G}_5)$, $e(\overline{C}_{12}, \overline{G}_4)$, $e(\overline{C}_{13}, \overline{G}_7)$, $e(\overline{C}_{13}, \overline{G}_6)$, $e(\overline{C}_{13}, \overline{G}_5)$;

4. *24 cells of dimension 3:* $e(C_0, P)$, $e(C_1, P)$, $e(C_2, P)$, $e(C_3, P)$, $e(C_4, P)$, $e(C_5, P)$, $e(C_6, P)$, $e(C_7, P)$, $e(C_8, P)$, $e(C_9, P)$, $e(C_{10}, P)$, $e(C_{11}, P)$, $e(C_{12}, P)$, $e(C_{13}, P)$, $e(C_{14}, P)$, $e(C_{15}, P)$, $e(C_{16}, P)$, $e(C_{17}, P)$, $e(\overline{C}_7, P)$, $e(\overline{C}_8, P)$, $e(\overline{C}_9, P)$, $e(\overline{C}_{10}, P)$, $e(\overline{C}_{12}, P)$, $e(\overline{C}_{13}, P)$.

2.3 Discrete Morse theory on CW-complexes

We recall here some of the main definitions and results from [F98], [F02], where Morse theory from a combinatorial viewpoint was first developed.

Let $\mathcal{C}$ be a finite *regular* $CW-$complex. Let K be the collection of cells of $\mathcal{C}$, partially ordered by

$$\sigma \ < \ \tau \quad \Leftrightarrow \quad \sigma \subset \tau.$$

As usual, denote by K_p the $p-$skeleton of K.

Definition 2.3.1 *A discrete Morse function on $\mathcal{C}$ is a function*

$$f : K \longrightarrow \mathbb{R}$$

satisfying for all $\sigma^{(p)} \in K_p$ the following two conditions

$$\begin{array}{lrcl} (i) & \sharp\{\tau^{(p+1)} > \sigma^{(p)} \mid f(\tau^{(p+1)}) \le f(\sigma^{(p)})\} & \le & 1 \\ (ii) & \sharp\{v^{(p-1)} < \sigma^{(p)} \mid f(\sigma^{(p)}) \le f(v^{(p-1)})\} & \le & 1 \end{array}$$

Actually, one shows that, if f satisfies (i) and (ii) above, then for any given cell of K at least one between (i), (ii) is a strict inequality.
The analog of a critical point of index p in standard Morse theory is here a *critical cell* of dimension p: a p-cell $\sigma^{(p)}$ is critical iff both the cardinalities in i), ii) are zero.

Let $m_p(f)$ denote the number of critical $p-$cells of f. Then one has

Proposition 2.3.2 *$\mathcal{C}$ is homotopy equivalent to a CW-complex which has exactly $m_p(f)$ cells of dimension p.*

The *discrete gradient vector field* Γ_f of a Morse function f over K is the set of all pairs of cells for which the exception in definition 2.3.1 happens:

$$\Gamma_f \ = \ \{(\sigma^{(p)}, \tau^{(p+1)}) | \ \sigma^{(p)} < \tau^{(p+1)}, \ f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\}.$$

Since, for any given cell, at most one between (i), (ii) in 2.3.1 is an equality, it follows that each cell belongs to at most one pair of Γ_f.

A general definition of discrete vector field is the following.

Definition 2.3.3 *A discrete vector field Γ on $\mathcal{C}$ is a collection of pairs of cells $(\sigma^{(p)}, \tau^{(p+1)}) \in \mathcal{C} \times \mathcal{C}$ such that $\sigma^{(p)} < \tau^{(p+1)}$ and such that each cell of $\mathcal{C}$ belongs to at most one pair of Γ.*

Remark 2.3.4 *A discrete vector field is sometimes calle also a* matching *(see for example [K07]).*

For Γ as above, define a Γ-path as a sequence of cells

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \cdots, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)} \tag{2.1}$$

such that for each $i = 0, \cdots, r$ one has $(\sigma_r^{(p)}, \tau_r^{(p+1)}) \in \Gamma$ and $\sigma_i^{(p)} \neq \sigma_{i+1}^{(p)} < \tau_i^{(p+1)}$.
The $\Gamma-$path is closed (and non-trivial) iff $\sigma_0^{(p)} = \sigma_{r+1}^{(p)}, \ r \geq 0$. One has:

Theorem 7 *A discrete vector field Γ is the gradient vector field of a discrete Morse function on $\mathcal{C}$ iff there are no non-trivial closed $\Gamma-$paths.*

We find convenient to generalize the definition of critical cell, in the case of a general discrete field.

Definition 2.3.5 *Given a discrete field Γ on $\mathcal{C}$, define a* critical cell *in $\mathcal{C}$ as a cell $\sigma \notin \Gamma$.*

Chapter 3

Subspace arrangements

Now we want to present a result about general subspace arrangements.

3.1 Goresky-McPherson's formula

Let $\mathcal{A}$ be a subspace arrangement in a real vectorial space V of dimension n, and let $\mathcal{M}(\mathcal{A})$ be the complement. Let $(\mathcal{L}(\mathcal{A}), \prec)$ be the intersection poset.

Notation 3.1.1 *We denote as $<$ the stricted inequality $\precneqq$ on the poset $\mathcal{L}(\mathcal{A})$.*

Goresky-McPherson's formula (see [GMP88], pag.238) allows us to compute the homology with integer coefficients of the complement. In formulas:

$$H_i(\mathcal{M}(\mathcal{A}); \mathbb{Z}) = \bigoplus_{v \in \mathcal{L}(\mathcal{A})} H^{n-d(v)-i-1}(K(\mathcal{L}_<), K(\mathcal{L}_{(v,V)}), \mathbb{Z}). \tag{3.1}$$

Here $d(v)$ denotes the dimension of an element $v \in \mathcal{L}(\mathcal{A})$, $K(P)$ is the order complex of the poset P (see section 2.1), $\mathcal{L}_<$ denotes the set $\mathcal{L}_< = \{w \in \mathcal{L} | w < v\}$, and $\mathcal{L}_{(v,V)}$ is the set $\mathcal{L}_{(v,V)} = \{w \in \mathcal{L} | V < w < v\}$.

Now we need some general remarks for the calculation that we are going to do in section 3.1.1:

Remark 3.1.2 *The complex $K(\mathcal{L}_<)$ is contractible, because it is the order complex of a poset with a minimum element (see remark 2.1.5)*

By the exact sequence of the pair $(K(\mathcal{L}_<), K(\mathcal{L}_{(v,V)}))$ we have that the sequence:

$$\cdots \to H^{i-1}(K(\mathcal{L}_<); \mathbb{Z}) \to H^{i-1}(K(\mathcal{L}_{(v,V)}); \mathbb{Z}) \to$$

$$\to H^i(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)}); \mathbb{Z}) \to H^i(K(\mathcal{L}_{<v}); \mathbb{Z}) \to \cdots . \tag{3.2}$$

is exact.

Remark 3.1.3 *When $i > 1$ we have $H^{i-1}(\mathcal{L}_{<v};\mathbb{Z}) = 0$, so the exact sequence* (3.2) *becomes:*

$$0 \to H^{i-1}(K(\mathcal{L}_{(v,V)});\mathbb{Z}) \to H^i(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) \to 0.$$

It follows that $H^{i-1}(K(\mathcal{L}_{(v,V)});\mathbb{Z}) \cong H^i(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z})$ for $i > 1$.

Remark 3.1.4 *We have $H^0(K(\mathcal{L}_{<v});\mathbb{Z}) = \mathbb{Z}$ and $H^0(K(\mathcal{L}_{(v,V)});\mathbb{Z}) = \mathbb{Z}^k$, where k is the number of component of $K(\mathcal{L}_{(v,V)})$.*

If $i = 1$ then (3.2) *gives:*

$$\begin{array}{ccccccccc} & & & & 0 & \to & H^0(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) & \xrightarrow{f} & \\ \to & H^0(K(\mathcal{L}_{<v});\mathbb{Z}) & \to & H^0(K(\mathcal{L}_{(v,V)});\mathbb{Z}) & \to & H^1(K(\mathcal{L}_{<v};\mathbb{Z}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) & \to & 0. \\ & 1 & \mapsto & (1,1,\ldots 1) & & & & \end{array}$$

It follows $H^1(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) = \mathbb{Z}^{k-1}$.

Let us note that the maps f and g are injective, the first one because the sequence is exact, and the second one because of its form.

It follows that $ker(g) = Im(f) = 0$ and so $H^0(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) = 0$.

Notation 3.1.5 *We define $H^{-1}(\emptyset, \emptyset;\mathbb{Z}) = 0$.*

3.1.1 An example

We have seen in section 2.1.3 that, given an hyperplane arrangement $\mathcal{A}$, its homology is a free $\mathbb{Z}-$module. This is not true for general subspace arrangements. In this section we use Goresky-McPherson's formula for the calculation of homology of a particular subspace arrangement.

In [J94] we found an example of an hyperplane arrangement with torsion homology: there the example is calculated using some spectral sequence. Here we want to write explicitly the calculation of the homology groups using Goresky-McPherson's formula.

In our example $V = \mathbb{R}^{10}$. Let's consider the coordinates $(x_1, \ldots, x_{10})$ in V. Our arrangement is given by $\mathcal{A} = \{A_i\}_{i=1\cdots6}$, where $A_1, \ldots, A_6$ are the subspaces:

$A_1 = \{x_1 = x_2 = x_3 = x_4 = x_5 = 0\}$
$A_2 = \{x_1 = x_2 = x_6 = x_7 = x_8 = 0\}$
$A_3 = \{x_1 = x_3 = x_6 = x_9 = x_{10} = 0\}$
$A_4 = \{x_2 = x_4 = x_7 = x_9 = x_{10} = 0\}$
$A_5 = \{x_3 = x_5 = x_7 = x_8 = x_9 = 0\}$
$A_6 = \{x_4 = x_5 = x_6 = x_8 = x_{10} = 0\}$.

We have the following intersection poset $\mathcal{L}$:

							0								dim=0
		C_{125}	C_{126}	C_{134}	C_{136}	C_{145}	C_{234}	C_{235}	C_{246}	C_{356}	C_{456}				dim=1
B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{23}	B_{24}	B_{25}	B_{26}	B_{34}	B_{35}	B_{36}	B_{45}	B_{46}	B_{56}	dim=2
				A_1	A_2	A_3	A_4	A_5	A_6						dim=5
							V								dim=10

where $B_{ij} = A_i \cap A_j$ and $C_{ijk} = A_i \cap A_j \cap A_k$. Observe that the 6 elements A_i have dimension 5, the fifteen elements B_{ij} have dimension 2 and the ten elements C_{ijk}'s have dimension 1.

So let us consider all elements $v \in \mathcal{L}$. By $G_1(B_{ij})$, $G_2(C_{ijk})$, $G_3(0)$ we denote the contractible simplicial complexes of dimension respectively 1, 2 and 3 which are the order complexes respectively of the posets $\mathcal{L}_{<B_{ij}}$, $\mathcal{L}_{<C_{ijk}}$ and $\mathcal{L}_{<0}$. By $G'_1(C_{ijk})$ and $G'_2(0)$ we denote the simplicial complexes of dimension respectively 1 and 2 which are the order complexes respectively of the poset $\mathcal{L}_{(C_{ijk},V)}$ and of the poset $\mathcal{L}_{(0,V)}$. Denote by S^i the sphere of dimension i and by G_0 the topological space coresponding to a point. We need the following lemma:

Lemma 3.1.6 *The simplicial complex $G'_1(C_{ijk})$ is an $1-$sphere S^1.*
The simplicial complex $G'_2(0)$ is a projective plane.

Proof. The first part of the lemma follows from the fact that $G'_1(C_{ijk})$ is the order complex of the subposet $\{A_i, A_j, A_k, B_{ij}, B_{ik}, B_{jk}\} \subset \mathcal{L}$ with the induced order. So the order complex $G'_1(C_{ijk})$ is given by the six $0-$simplexes corresponding to these six elements, with the six $1-$simplexes given by the pairs $\{(A_i, B_{ij}), (A_j, B_{ij}), (A_i, B_{ik}), (A_k, B_{ik}), (A_j, B_{jk}), (A_k, B_{jk})\}$. Since each $0-$simplex belongs to the boundary of exactly two $1-$simplexes, then $G'_1(C_{ijk})$ is homotopical equivalent to S^1.

For the second part of the lemma, note before that the subposet $\mathcal{L}_{(0,V)}$ equals $\mathcal{L} \setminus \{0, V\}$ with the induced order.

In the order complex we have a point for any element of $\mathcal{L}_{(0,V)}$. Let us consider before the $0-$simplex corresponding to C_{ijk}. This point of $G'_2(0)$ is in the boundary of exactly six $2-$simplexes, i.e. the simplexes given by the triples:

$$\{(C_{ijk}, B_{ij}, A_i), (C_{ijk}, B_{ij}, A_j), (C_{ijk}, B_{ik}, A_i), (C_{ijk}, B_{ik}, A_k), (C_{ijk}, B_{jk}, A_j), (C_{ijk}, B_{jk}, A_k)\}.$$

We can easily see that we can glue together these $2-$simplexes along the edges containing the $0-$simplex C_{ijk}, obtaining an exagon with vertices $\{A_i, B_{ij}, A_j, B_{jk}, A_k, B_{ik}, A_k\}$. The edges of this exagon are $\{(B_{ij}, A_i), (B_{ij}, A_j), (B_{ik}, A_i), (B_{ik}, A_k), (B_{jk}, A_j), (B_{jk}, A_k)\}$. So we have 10 exagons, one for each C_{ijk}, moreover any edge in the order complex is in the boundary of exactly two of these exagons. We must glue them together according to the identification prescribed by the common edges. We obtain a simplicial complex homeomorphic to the projective plane. This can be viewed comparing figure 3.1.

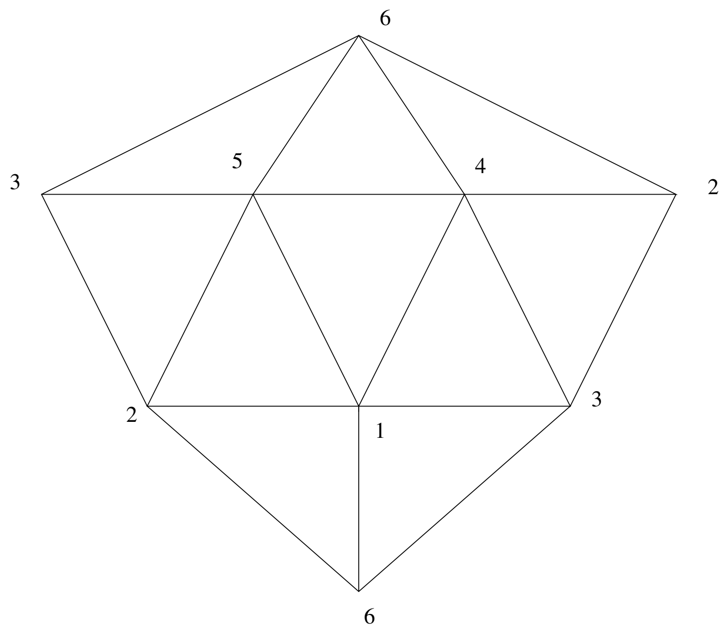


Figure 3.1: The projective plane $\mathbb{R}P^2$

$\square$

Now we define as usual $\mathcal{M}(\mathcal{A}) = V \setminus \cup\mathcal{A}$ and, by formula (3.1) with $i = 0$ we have:

$$H_0(\mathcal{M}(\mathcal{A}), \mathbb{Z}) = \bigoplus_{v \in \mathcal{L}} H^{9-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)}); \mathbb{Z}).$$

For any $v \in \mathcal{L}(\mathcal{A})$ we compute the corresponding $H^{9-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)}), \mathbb{Z})$:

1. V: $H^{-1}(\emptyset, \emptyset; \mathbb{Z}) = \mathbb{Z}$;

2. A_i: $H^4(G_0, \emptyset; \mathbb{Z}) = H^4(G_0; \mathbb{Z}) = 0$;

3. B_{ij}: $H^7(G_1, S^0; \mathbb{Z}) = H^6(S^0, \mathbb{Z}) = 0$;

4. C_{ijk}: $H^8(G_2, G_1'; \mathbb{Z}) = H^7(G_1'; \mathbb{Z}) = 0$ for dimensional reasons;

5. 0: $H^9(G_3, G_2'; \mathbb{Z}) = H^8(G_2'; \mathbb{Z}) = 0$ as before.

In cases 3. 4. 5. we used remark 3.1.3. So we obtained $H^0(\mathcal{M};\mathbb{Z}) = \mathbb{Z}$, as we could immaginate.

Now let us consider higher homology spaces.

Observe that for $v = V$ we have $H^{n-d(v)-i-1}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}) =$
$= H^{-i-1}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z})$ and for $i \geq 1$ we have $-i-1 \leq -2$. It follows that for $i \geq 1$ the element $V \in \mathcal{L}$ doesn't give contribute in the direct sum of the formula (3.1).

For $i = 1$ we have:

$$H_1(\mathcal{M},\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{8-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

With the same notations as before, and using remark 3.1.3 we have:

1. A_i: $H^3(G_0,\emptyset;\mathbb{Z}) = 0$;

2. B_{ij}: $H^6(G_1,S^0;\mathbb{Z}) = H^5(S^0,\mathbb{Z}) = 0$;

3. C_{ijk}: $H^7(G_2,G_1';\mathbb{Z}) = H^6(G_1';\mathbb{Z}) = 0$;

4. 0: $H^8(G_3,G_2';\mathbb{Z}) = H^7(G_2';\mathbb{Z}) = 0$.

It follows that $H_1(\mathcal{M};\mathbb{Z}) = 0$.

Let's continue by calculating:

$$H_2(\mathcal{M},\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{7-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. A_i: $H^2(G_0,\emptyset;\mathbb{Z}) = 0$;

2. B_{ij}: $H^5(G_1,S^0;\mathbb{Z}) = H^4(S^0,\mathbb{Z}) = 0$;

3. C_{ijk}: $H^6(G_2,G_1';\mathbb{Z}) = H^5(G_1';\mathbb{Z}) = 0$;

4. 0: $H^7(G_3,G_2';\mathbb{Z}) = H^6(G_2';\mathbb{Z}) = 0$.

It follows that $H_1(\mathcal{M};\mathbb{Z}) = 0$.

$$H_3(\mathcal{M},\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{6-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. A_i: $H^1(G_0,\emptyset;\mathbb{Z}) = 0$;

2. B_{ij}: $H^4(G_1,S^0;\mathbb{Z}) = H^3(S^0,\mathbb{Z}) = 0$;

3. C_{ijk}: $H^5(G_2,G_1';\mathbb{Z}) = H^4(G_1';\mathbb{Z}) = 0$;

4. 0: $H^6(G_3,G_2';\mathbb{Z}) = H^5(G_2';\mathbb{Z}) = 0$.

It follows that $H_2(\mathcal{M};\mathbb{Z}) = 0$.

$$H_4(\mathcal{M},\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{5-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. A_i: $H^0(G_0,\emptyset;\mathbb{Z}) = H^0(G_0,\mathbb{Z}) = \mathbb{Z}$;
2. B_{ij}: $H^3(G_1,S^0;\mathbb{Z}) = H^2(S^0,\mathbb{Z}) = 0$;
3. C_{ijk}: $H^4(G_2,G_1';\mathbb{Z}) = H^3(G_1';\mathbb{Z}) = 0$;
4. 0: $H^5(G_3,G_2';\mathbb{Z}) = H^4(G_2';\mathbb{Z}) = 0$.

It follows that $H_4(\mathcal{M};\mathbb{Z}) = \mathbb{Z}^6$.

With an argument similar to the one that we gave for the element V it's possible to see that for $i \geq 5$ any element of the form $A_j \in \mathcal{L}$ doesn't give contributes in the direct sum of the formula (3.1). So we arrived to:

$$H_5(\mathcal{M};\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{5-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. B_{ij}: $H^2(G_1,S^0;\mathbb{Z}) = H^1(S^0,\mathbb{Z}) = 0$;
2. C_{ijk}: $H^3(G_2,G_1';\mathbb{Z}) = H^2(G_1';\mathbb{Z}) = 0$;
3. 0: $H^4(G_3,G_2';\mathbb{Z}) = H^3(G_2';\mathbb{Z}) = 0$.

It follows that $H_5(\mathcal{M};\mathbb{Z}) = 0$.

$$H_6(\mathcal{M};\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{4-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. B_{ij}: $H^1(G_1,S^0;\mathbb{Z}) = \mathbb{Z}$;
2. C_{ijk}: $H^2(G_2,G_1';\mathbb{Z}) = H^1(S^1;\mathbb{Z}) = \mathbb{Z}$;
3. 0: $H^3(G_3,G_2';\mathbb{Z}) = H^2(G_2';\mathbb{Z}) = 0$.

2. and 3. follow from lemma 3.1.6, while 1. follows from remark 3.1.4.

It follows that $H_6(\mathcal{M};\mathbb{Z}) = \mathbb{Z}^{25}$.

Now we can delete also B_{ij}'s from our calculation, because they doesn't give contribute to the direct sum for $i \geq 7$.

$$H_7(\mathcal{M};\mathbb{Z}) = \bigoplus_{v\in\mathcal{L}} H^{3-d(v)}(K(\mathcal{L}_{<v}), K(\mathcal{L}_{(v,V)});\mathbb{Z}).$$

1. C_{ijk}: $H^1(G_2,G_1';\mathbb{Z}) = 0$;

2. 0: $H^2(G_3, G_2'; \mathbb{Z}) = H^1(G_2'; \mathbb{Z}) = \mathbb{Z}_2$.

2. follows from lemma 3.1.6, while 1. follows from remark 3.1.4 and lemma 3.1.6.
It follows that $H_7(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}_2$.
All others H_i's are equal to 0 by remark and 3.1.4.
So we have seen that $H_7(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}_2$ and $\mathcal{M}$ is a topological space with torsion homology.

3.2 Generalized configuration spaces

Definition 3.2.1 *Let $\mathcal{A}_{\mathbb{R}} = \{H_i\}_{i\in I}$ a real arrangement in $V = \mathbb{R}^n$ with defining polynomial $Q(\mathcal{A}_{\mathbb{R}})$. The complexified arrangement is in $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{C}} \mathbb{C}$ and consist of the hyperplanes $\{H \otimes_{\mathbb{R}} \mathbb{C} | H \in \mathcal{A}_{\mathbb{R}}\}$, and so $Q(\mathcal{A}_{\mathbb{R}}) = Q(\mathcal{A}_{\mathbb{C}})$.*

More generally, we can define the d-complexified of an arrangement $\mathcal{A}$.

Let $\mathcal{A} = \{H_j\}_{j\in J}$ be a finite arrangement of linear hyperplanes in $V := \mathbb{R}^n$. We introduce a coordinate $x \in V$ and coordinates $(x_1, ..., x_d)$, $x_i \in V$, in V^d, $d > 0$. Each hyperplane is given by a linear equation $H_j = \{x \in V |\ a_j \cdot x = 0\}$, $a_j \in V \setminus \{0\}$. For each $d > 0$, one has the $d-$*complexification* $\mathcal{A}^{(d)} \subset V^d$ of $\mathcal{A}$, given by the collection of linear codimension-d *subspaces*

$$(H_j)^{(d)} \ := \ \{(x_1, \dots, x_d) \mid a_j \cdot x_k = 0, \ k = 1, \dots, d\}$$

(when $d = 2$ one has the standard complexification $\mathcal{A}_{\mathbb{C}} \subset \mathbb{C}^n$).

The $d-$*generalized configuration space* associated to $\mathcal{A}$ is the complement to the subspace arrangement

$$\mathcal{M}^{(d)} \ = \ \mathcal{M}(\mathcal{A})^{(d)} := \ V^d \setminus \bigcup_{H\in\mathcal{A}} H^{(d)} \ .$$

Remark 3.2.2 *This construction can be generalized to an arrangement of affine real hyperplanes. If $H_j = \{x \in V \mid a_j x = b_j\}$, $a_j \in V \setminus 0$, $b_j \in \mathbb{R}$ then define:*

$$(H_j)^{(d)} := \{(x_1, \dots, x_d) \mid a_j x_1 = b_j, \ a_j x_k = 0, \ k = 2, \dots, d\}.$$

All the constructions which follow hold in this more general case, with few modifications.

In the present case we can also define the poset $\mathcal{L}(\mathcal{A}^{(d)})$ exactly in the same way that for arrangements in section 2.1, where the codimensional minimal elements are the elements of set $\{(H_j)^{(d)}\}_{j\in J}$. One can easily see that $\mathcal{L}(\mathcal{A}^{(d)})$ and $\mathcal{L}(\mathcal{A})$ are composed by the same elements, and they are different only for the ranks of the elements.

Generalized configuration spaces are particular types of subspace arrangements. Their homology is torsion-free, because of Goresky-McPherson's formula and by remark 3.2.2.

3.3 Cell decomposition of configuration spaces

In this part we recall some of the main constructions in [DCS00], without further citations. In this paper the authors generalize the construction of section 2.2.

Let $\mathcal{A} = \{H_j\}_{j\in J}$ be a finite arrangement of linear hyperplanes in $V := \mathbb{R}^n$. Let $\Phi(\mathcal{A})$ be the stratification induced by the arrangement and $\mathcal{L}(\mathcal{A})$ the intersection poset (see section 2.1). Let $\mathcal{M}^{(d)} = \mathcal{M}^{(d)}(\mathcal{A})$ be the $d-$generalized configuration space associated to $\mathcal{A}$ (see section 3.2).

Let Φ^d be the product of d copies of Φ, $d \geq 0$, and let

$$\Phi^{(d)} \;=\; \{(F_1, \ldots, F_d) \in \Phi^d : \; F_1 \prec \ldots \prec F_d\}$$

be the set of $d-$chains in Φ (repetitions in the chain are allowed). Then $\Phi^{(d)}$ corresponds to a stratification of the space V^d as follows: to each $\mathcal{F} = (F_1, \ldots, F_d)$ in $\Phi^{(d)}$ it corresponds the stratum $\hat{\mathcal{F}}$ in V^d given by:

$$\hat{\mathcal{F}} \;:=\; \{(x_1, \ldots, x_d) \in V^d : \; x_1 \in F_d \;,\; x_k \in \varphi_{F_{d-k+2}}(F_{d-k+1}) \;,\; k = 2, \ldots, d\}.$$

Where the map $\phi|_F$ is defined in section 2.1.

One has:

Proposition 3.3.1 *(i) Each $\hat{\mathcal{F}}$ is homeomorphic to an open cell*
(ii) $\bigcup_{\mathcal{F}\in\Phi^{(d)}} \;\hat{\mathcal{F}} \;=\; V^d$
(iii) $\hat{\mathcal{F}} \cap \hat{\mathcal{G}} \;=\; \emptyset$ *if* $\mathcal{F} \neq \mathcal{G}$
(iv) $cl(\hat{\mathcal{F}}) \cap \hat{\mathcal{G}} \;\neq\; \emptyset$ *iff* $cl(\hat{\mathcal{F}}) \supset \hat{\mathcal{G}}$.
(v) $\mathcal{M}^{(d)} \;=\; \bigcup_{\{\mathcal{F}\in\Phi^{(d)} \;:\; F_1 \;\; is\ a\ chamber\ of \;\; \Phi\}} \;\hat{\mathcal{F}}$

For $\mathcal{F} = (F_1, \ldots, F_d)$, one has

$$codim(\mathcal{F}) \;:=\; codim(\hat{\mathcal{F}}) \;=\; \sum_{i=1}^{d} codim(F_i).$$

The partial ordering on $\Phi^{(d)}$ is given similarly to the one on $\Phi^{(1)}$, i.e. by:

$$\mathcal{F} \prec \mathcal{G} \quad \text{iff} \quad \hat{\mathcal{G}} \subset cl(\hat{\mathcal{F}}).$$

This has the following characterization:

Lemma 3.3.2 *For* $\mathcal{F} = (F_1, \ldots, F_d)$, $\mathcal{G} = (G_1, \ldots, G_d) \in \Phi^{(d)}$ *one has*

$$\mathcal{F} \prec \mathcal{G} \quad \textit{iff}\ F_d \prec G_d \quad \textit{and} \;\; pr_{|F_{i+1}|}(F_i) \prec pr_{|F_{i+1}|}(G_i)$$

in the stratification $\Phi_{|F_{i+1}|}$, $i = d-1, \ldots, 1$.

Remark 3.3.3 *In the case $d = 1$ with the boundary condition given in lemma 3.3.2 specializes to: $e(D, G)$ is in the boundary of $e(C, F)$ iff*

i) $G \prec F$

ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_G$, that is $D = \varphi_{G^{-1}}(pr_{|G|}(C))$.

Part (v) of proposition 3.3.1 gives us the poset corresponding to the induced stratification of the generalized configuration space $\mathcal{M}^{(d)}$ which is

$$\Phi_0^{(d)} := \{\mathcal{F} = (F_1, \dots, F_d) \in \Phi^{(d)} : \ rk(F_1) = 0\}$$

while the union $\bigcup_{H \in A} H^{(d)}$ of the $d-$complexified subspaces correspond to the poset

$$\Phi_+^{(d)} := \{\mathcal{F} = (F_1, \dots, F_d) \in \Phi^{(d)} : \ rk(F_1) > 0\}.$$

Taking one point $v(\mathcal{F})$ inside each stratum $\hat{\mathcal{F}}$, one obtains

Proposition 3.3.4 *For each chain $\mathcal{C} = (\mathcal{F}_0 \precneqq \cdots \precneqq \mathcal{F}_k)$ of $\Phi^{(d)}$, the join in V^d*

$$s(\mathcal{C}) := \bigvee_{\mathcal{G} \in \mathcal{C}} v(\mathcal{G})$$

is a $k-$dimensional affine simplex. For $\mathcal{F} \in \Phi^{(d)}$ the set

$$e(\mathcal{F}) := \bigcup_{\mathcal{C} \prec \mathcal{F}} s(\mathcal{C})$$

where the union is over all chains $\mathcal{C}$ of $\Phi^{(d)}$ which have $\mathcal{F}$ as an upper bound, is a triangulated cell in V^d of dimension equal to $codim(\mathcal{F})$. The cell $e(\mathcal{F})$ is dual to the stratum $\hat{\mathcal{F}}$, intersecting it in exactly one point. Its boundary is given by

$$\partial(e(\mathcal{F})) = \bigcup_{\mathcal{C} \prec \mathcal{F},\ \mathcal{F} \notin \mathcal{C}} s(\mathcal{C}) = \bigcup_{\mathcal{F}' \precneqq \mathcal{F}} e(\mathcal{F}')$$

The set

$$\mathbf{Q}^{(d)} := \bigcup_{\mathcal{F} \in \Phi^{(d)}} e(\mathcal{F})$$

is a cellular $n'd$-ball in V^d (a regular cell complex) dual to the stratification, where $n' := rk(L_0)$.

Remark 3.3.5 *It follows from lemma 3.3.2 that if the first element F_1 of $\mathcal{F}$ is a chamber, then also the first element of any $\mathcal{G} \prec \mathcal{F}$ is a chamber.*

Definition 3.3.6 *We denote by* $\mathbf{S}^{(d)}$ *the subcomplex of* $\mathbf{Q}^{(d+1)}$ *whose cells correspond to* $\Phi_0^{(d+1)}$:

$$\mathbf{S}^{(d)} \ := \ \cup_{\mathcal{F}\in\Phi_0^{(d+1)}} \ e(\mathcal{F})$$

(Observe that in case $d = 1$ we have $S^{(1)} = \mathbf{S}$, the complex of section 2.2; see also [BZ92], [OT92]).

Set $\Phi_0 \subset \Phi$ as the set of chambers of Φ. There is an inclusion $\Phi_0^{(d+1)} \subset \Phi_0 \times \Phi^{(d+1)}$: indeed

$$\Phi_0^{(d+1)} \ = \ \{\mathcal{F} = (C, \mathcal{F}') \ \in \Phi_0 \times \Phi^{(d)} : \ \mathcal{F}' = (F_1, \dots, F_d), C \prec F_1\}.$$

The map $(C, \mathcal{F}') \ \rightarrow \ (\varphi_{F_1}^{-1}(pr_{|F_1|}(C)), \mathcal{F}')$ gives a "projection" $\Phi_0 \times \Phi^{(d)} \rightarrow \Phi_0^{(d+1)}$. The image of $(C, \mathcal{F}')$ through such projection will be denoted also by $[C, \mathcal{F}']$. In general, given a chamber C and a facet F in $\Phi^{(d+1)}$ we will use the notation by:

$$C.F \ := \ \varphi_{F^{-1}}(pr_{|F|}(C))$$

which is a uniquely defined chamber containing F in its boundary. So, one has also $[C, \mathcal{F}'] = (C.F_1, \mathcal{F}')$.

By using the previous projection, we can write a cell $e(\mathcal{F})$ of $\mathbf{S}^{(d)}$, $\mathcal{F} = [C, \mathcal{F}']$, as $e[C, \mathcal{F}']$. The "real projection":

$$pr_{\Re} : \ V^{d+1} \rightarrow V^d : \ (x_1, \dots, x_{d+1}) \rightarrow \ (x_1, \dots, x_d)$$

induces a map

$$pr_{\Re} : \ \Phi^{(d+1)} \ \rightarrow \ \Phi^{(d)} : \ \mathcal{F} = (F_1, \mathcal{F}') \rightarrow \mathcal{F}'$$

and a map

$$pr_{\Re} : \ \mathbf{Q}^{(d+1)} \rightarrow \mathbf{Q}^{(d)} : \ e(\mathcal{F}) \rightarrow e(\mathcal{F}').$$

which restricts to a surjective map (which we continue to call $pr_{\Re}$)

$$pr_{\Re} : \ \mathbf{S}^{(d)} \rightarrow \mathbf{Q}^{(d)} : \ e[C, \mathcal{F}'] \rightarrow e(\mathcal{F}') \ .$$

One has:

Theorem 8 *(i)* $\mathbf{S}^{(d)}$ *is a deformation retract of* $\mathcal{M}^{(d+1)}$.
(ii) The map $pr_{\Re}|_{e[C,\mathcal{F}']}$ *is a homeomorphism between* $e[C, \mathcal{F}'] \subset \mathbf{S}^{(d)}$ *and* $e(\mathcal{F}') \subset \mathbf{Q}^{(d)}$. *There are as many cells of* $\mathbf{S}^{(d)}$ *over* $e(\mathcal{F}')$ *as the chambers* $C \prec F_1$ *(here* $\mathcal{F}' = (F_1, \dots, F_d)$*).*
(iii)

$$\partial(e[C, \mathcal{F}']) \ = \bigcup_{\mathcal{F}'' \ \prec \ \mathcal{F}', \ codim(\mathcal{F}'') \ = \ codim(\mathcal{F}')-1} e[C, \mathcal{F}''].$$

(iv) Fixing $C \in \Phi_0$ *the map* $\psi_C \ : \ \mathbf{Q}^{(d)} \ \rightarrow \ \mathbf{S}^{(d)}$ *defined by:*

$$\psi_C|_{e(\mathcal{F}')} \ = \ (pr_{\Re}|_{e[C,\mathcal{F}']})^{-1}$$

is a cellular embedding of $\mathbf{Q}^{(d)}$ *inside* $\mathbf{S}^{(d)}$.
(v) dim $e[C, \mathcal{F}'] \ = \ dim \ e(\mathcal{F}') \ = \ codim_{V^d}(\mathcal{F}')$. *In particular* $dim(\mathbf{S}^{(d)}) = dim(\mathbf{Q}^{(d)}) = n'd$.

Remark 3.3.7 *All results here generalize (with easy modifications) to the case where $\mathcal{A}$ is a locally finite arrangement of affine hyperplanes in $\mathbb{R}^n$.*

We consider now the natural inclusion $j_d : V^{d+1} \rightarrow V^{d+2}$, which induces an injection of posets $j_d : \Phi^{(d+1)} \rightarrow \Phi^{(d+2)}$:

$$\mathcal{F} = (F_1, \dots, F_{d+1}) \rightarrow \mathcal{F}^1 = (F_1, F_1, \dots, F_{d+1})$$

so $\hat{\mathcal{F}}^1$ is the unique stratum of $\Phi^{(d+2)}$ which contains $\hat{\mathcal{F}}$. This map preserves the codimension iff $F_1 \in \Phi_0$, i.e. iff $\mathcal{F} \in \Phi_0^{(d+1)}$. The induced map on the dual (call it again j_d)

$$j_d : \mathbf{Q}^{(d+1)} \rightarrow \mathbf{Q}^{(d+2)} : \; e(F_1, \dots F_{d+1}) \rightarrow e(F_1, F_1, \dots, F_{d+1})$$

is a dimension preserving cellular map when restricted to $\mathbf{S}^{(d)}$, so it identifies $\mathbf{S}^{(d)}$ to a subcomplex of $\mathbf{S}^{(d+1)}$.

Theorem 9 $\mathbf{S}^{(d+1)}$ *is obtained from* $\mathbf{S}^{(d)}$ *by attaching cells of dimension grater or equal than* $d+1$*.*

Notice that there are cells of dimension $d+1$ in $\mathbf{S}^{(d+1)} \setminus \mathbf{S}^{(d)}$, precisely those of the form

$$e(C, F, \dots, F), \; codim(F) = 1, \; C \in \Phi_0.$$

Since $\mathcal{M}^{(d+1)}$ is obtained from V^{d+1} by removing subspaces of codimension $d+1$ one gets

Corollary 3.3.8 *i) The space* $\mathbf{S}^{(d)}$ *is* $d-1$ *connected.*
ii) The limit space

$$\mathbf{S}^{(\infty)} := \lim_{d \to \infty} \mathbf{S}^{(d)}$$

is contractible.

3.3.1 Examples

In section 2.2.1 we have presented the Salvetti's complex $\mathbf{S}^{(1)}$ in the case of the complexification of three particular real hyperplane arrangements. Here we consider again such real arrangements and write explicitly the complexes $\mathbf{S}^{(2)}$ and $\mathbf{S}^{(3)}$.

Example 11 *Consider the arrangement* $\mathcal{A} \in \mathbb{R}^2$ *of example 8 (see figure 2.1). The complex* $\mathbf{S}^{(2)}$ *is composed by the following cells:*

- *12 cells of dimension 0: The cells of form* $e(C_i, C_i, C_i)$ *where* $C_i \in \Phi(\mathcal{A})$ *is a chamber, i.e. a facet of codimension 0;*
- *24 cells of dimension 1: The cells of form* $e(C_i, C_i, F_j)$ *where* C_i *is a chamber, and* $F_j \in \Phi(\mathcal{A})$ *is a facet of codimension 1 in the boundary of* C_i*, i.e.* $C_i \prec F_j$*;*

- *24+12 cells of dimension 2: 12 cells of form* $e(C_i, C_i, P)$ *where* C_i *is a chamber, and* $P \in \Phi(\mathcal{A})$ *is the center of* $\mathcal{A}$*, i.e. the only facet of codimension 2 in* $\Phi(\mathcal{A})$*; 24 cells of form* $e(C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *24 cells of dimension 3: The cells of form* $e(C_i, F_j, P)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*.*

- *12 cells of dimension 4: The cells of form* $e(C_i, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*;*

The complex $\mathbf{S}^{(3)}$ *is composed by the following cells:*

- *12 cells of dimension 0: The cells of form* $e(C_i, C_i, C_i, C_i)$ *where* C_i *is a chamber;*

- *24 cells of dimension 1: The cells of form* $e(C_i, C_i, C_i, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *24+12 cells of dimension 2: 12 cells of form* $e(C_i, C_i, C_i, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*; 24 cells of form* $e(C_i, C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *24+24 cells of dimension 3: 24 cells of form* $e(C_i, F_j, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*; 24 cells of form* $e(C_i, C_i, F_j, P)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*;*

- *24+12 cells of dimension 4: 24 cells of form* $e(C_i, F_j, F_j, P)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 12 cells of form* $e(C_i, C_i, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*;*

- *24 cells of dimension 5: The cells of form* $e(C_i, F_j, P, P)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*;*

- *12 cells of dimension 6: 12 cells of form* $e(C_i, P, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*.*

Example 12 *Consider the arrangement* $\mathcal{A} \in \mathbb{R}^2$ *of example 9 and the induced stratification* $\Phi(\mathcal{A})$ *of* $\mathbb{R}^3$ *(see figure 2.2). The complex* $\mathbf{S}^{(2)}$ *is composed by the following cells:*

- *12 cells of dimension 0: The cells of form* $e(C_i, C_i, C_i)$ *where* $C_i \in \Phi(\mathcal{A})$ *is a chamber, i.e. a facet of codimension 0;*

- *30 cells of dimension 1: The cells of form* $e(C_i, C_i, F_j)$ *where* C_i *is a chamber, and* $F_j \in \Phi(\mathcal{A})$ *is a facet of codimension 1 in the boundary of* C_i*, i.e.* $C_i \prec F_j$*;*

- *30+20 cells of dimension 2: 20 cells of form* $e(C_i, C_i, P_k)$ *where* C_i *is a chamber, and* $P_k \in \Phi(\mathcal{A})$ *is a facet of codimension 2 (a point) in the boundary of* C_i*, i.e.* $C_i \prec P_k$*; 30 cells of form* $e(C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *40 cells of dimension 3: The cells of form* $e(C_i, F_j, P_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* P_k *is a facet of codimension 2 in the boundary of* F_j*, i.e.* $F_j \prec P_k$*;*

- *20 cells of dimension 4: The cells of form* $e(C_i, P_k, P_k)$*, where* C_i *is a chamber and* P_k *is a facet of codimension 2 in the boundary of* C_i*;*

The complex $\mathbf{S}^{(3)}$ *is composed by the following cells:*

- *12 cells of dimension 0: The cells of form* $e(C_i, C_i, C_i, C_i)$ *where* C_i *is a chamber;*

- *30 cells of dimension 1: The cells of form* $e(C_i, C_i, C_i, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *30+20 cells of dimension 2: 20 cells of form* $e(C_i, C_i, C_i, P_k)$ *where* C_i *is a chamber, and* P_k *is a facet of codimension 2 in the boundary of* C_i*; 30 cells of form* $e(C_i, C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*;*

- *30+40 cells of dimension 3: 30 cells of form* $e(C_i, F_j, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*; 40 cells of form* $e(C_i, C_i, F_j, P_k)]$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* P_k *is a facet of codimension 2 in the boundary of* F_j*;*

- *40+20 cells of dimension 4: 40 cells of form* $e(C_i, F_j, F_j, P_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* P_k *is a facet of codimension 2 in the boundary of* F_j*; 20 cells of form* $e(C_i, C_i, P_k, P_k)$ *where* C_i *is a chamber, and* P_k *is a facet of codimension 2 in the boundary of* C_i*;*

- *40 cells of dimension 5: The cells of form* $e(C_i, F_j, P_k, P_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* P_k *is a facet of codimension 2 in the boundary of* F_j*;*

- *20 cells of dimension 6: The cells of form* $e(C_i, P_k, P_k, P_k)$ *where* C_i *is a chamber, and* P_k *is a facet of codimension 2 in the boundary of* C_i*;*

Example 13 *Take the arrangement* $\mathcal{A} \in \mathbb{R}^3$ *of example 10 and the induced stratification* $\Phi(\mathcal{A})$ *of* $\mathbb{R}^3$ *(see figures 2.3 and 2.4). The complex* $\mathbf{S}^{(2)}$ *is composed by the following cells:*

- *24 cells of dimension 0: The cells of form* $e(C_i, C_i, C_i)$ *where* $C_i \in \Phi(\mathcal{A})$ *is a chamber, i.e. a facet of codimension 0.*

- *72 cells of dimension 1: The cells of form* $\mathrm{e}(C_i, C_i, F_j)$ *where* C_i *is a chamber, and* $F_j \in \Phi(\mathcal{A})$ *is a facet of codimension 1 in the boundary of* C_i*, i.e.* $C_i \prec F_j$*.*

- *72+72 cells of dimension 2: 72 cells of form* $\mathrm{e}(C_i, C_i, G_k)$ *where* C_i *is a chamber, and* $G_k \in \Phi(\mathcal{A})$ *is a facet of codimension 2 in the boundary of* C_i*, i.e.* $C_i \prec G_k$*; 72 cells of form* $\mathrm{e}(C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*.*

- *24+144 cells of dimension 3: 24 cells of form* $\mathrm{e}(C_i, C_i, P)$ *where* C_i *is a chamber, and* $P \in \Phi(\mathcal{A})$ *is the center of* $\mathcal{A}$*, i.e. the only facet of codimension 3 in* $\Phi(\mathcal{A})$*; 144 cells of form* $\mathrm{e}(C_i, F_j, G_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* G_k *is a facet of codimension 2 in the boundary of* F_j*, i.e.* $F_j \prec G_k$*.*

- *72+72 cells of dimension 4: 72 cells of form* $\mathrm{e}(C_i, F_j, P)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 72 cells of form* $\mathrm{e}(C_i, G_k, G_k)$*, where* C_i *is a chamber and* G_k *is a facet of codimension 2 in the boundary of* C_i*.*

- *72 cells of dimension 5: The cells of form* $\mathrm{e}(C_i, G_k, P)$*, where* C_i *is a chamber,* G_k *is a facet of codimension 2 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*.*

- *24 cells of dimension 6: The cells of the form* $\mathrm{e}(C_i, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*.*

The complex $\mathbf{S}^{(3)}$ *is composed by the following cells:*

- *24 cells of dimension 0: The cells of form* $\mathrm{e}(C_i, C_i, C_i, C_i)$ *where* C_i *is a chamber.*

- *72 cells of dimension 1: The cells of form* $\mathrm{e}(C_i, C_i, C_i, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*.*

- *72+72 cells of dimension 2: 72 cells of form* $\mathrm{e}(C_i, C_i, C_i, G_k)$ *where* C_i *is a chamber, and* G_k *is a facet of codimension 2 in the boundary of* C_i*; 72 cells of form* $\mathrm{e}(C_i, C_i, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*.*

- *24+144+72 cells of dimension 3: 24 cells of form* $\mathrm{e}(C_i, C_i, C_i, P)$ *where* C_i *is a chamber and* P *is the center of* $\mathcal{A}$*; 144 cells of form* $\mathrm{e}(C_i, C_i, F_j, G_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* G_k *is a facet of codimension 2 in the boundary of* F_j*; 72 cells of form* $\mathrm{e}(C_i, F_j, F_j, F_j)$ *where* C_i *is a chamber, and* F_j *is a facet of codimension 1 in the boundary of* C_i*.*

- *72+72+144 cells of dimension 4: 72 cells of form* $\mathrm{e}(C_i, C_i, F_j, P)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 72 cells of form* $\mathrm{e}(C_i, C_i, G_k, G_k)$*, where* C_i *is a chamber and* G_k *is a facet of codimension 2 in the boundary of* C_i*; 144 cells of form* $\mathrm{e}(C_i, F_j, F_j, G_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* G_k *is a facet of codimension 2 in the boundary of* F_j*.*

- *72+72+144 cells of dimension 5: 72 cells of form* $e(C_i, C_i, G_k, P)$*, where* C_i *is a chamber,* G_k *is a facet of codimension 2 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 72 cells of form* $e(C_i, F_j, F_j, P)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 144 cells of form* $e(C_i, F_j, G_k, G_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* G_k *is a facet of codimension 2 in the boundary of* F_j*.*

- *24+144+72 cells of dimension 6: 24 cells of form* $e(C_i, C_i, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*; 144 cells of form* $e(C_i, F_j, G_k, P)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*, and* G_k *is a facet of codimension 2 in the boundary of* F_j *and* P *is the center of* $\mathcal{A}$*; 72 cells of form* $e(C_i, G_k, G_k, G_k)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i*.*

- *72+72 cells of dimension 7: 72 cells of form* $e(C_i, G_k, G_k, P)$*, where* C_i *is a chamber,* G_k *is a facet of codimension 2 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*; 72 cells of form* $e(C_i, F_j, P, P)$*, where* C_i *is a chamber,* F_j *is a facet of codimension 1 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*.*

- *72 cells of dimension 8: The cells of form* $e(C_i, G_k, P, P)$*, where* C_i *is a chamber,* G_k *is a facet of codimension 2 in the boundary of* C_i *and* P *is the center of* $\mathcal{A}$*.*

- *24 cells of dimension 9: The cells of form* $e(C_i, P, P, P)$ *where* C_i *is a chamber, and* P *is the center of* $\mathcal{A}$*.*

Chapter 4

Discrete Morse theory on hyperplane arrangements

We now apply the theory of section 2.3 to Hyperplane Arrangements (all results in this chapter come from [SS07], which we recall without further citations). We set in this chapter $\mathbf{S} := \mathbf{S}^{(1)}$, the case of standard complexification presented in section 2.2.

4.1 Polar coordinates

Start with an orthonormal frame

$$O, \mathbf{e}_1, ..., \mathbf{e}_n,\ O \in V$$

of the Euclidean $n-$dimensional space V and consider the two flags of subspaces

$$V_i \ = < \mathbf{e}_1, ..., \mathbf{e}_i >, \quad i = 0, ..., n \ (V_0 = 0)$$

and

$$W_i \ = < \mathbf{e}_i, ..., \mathbf{e}_n >, \quad i = 1, ..., n$$

Let $P_i,\ i = 1, ..., n$, be the orthogonal projection of the point $P \in V$ into W_i. Define the "polar coordinates" $(\theta_0, \theta_1, \ldots, \theta_{n-1})$ of P in the following way. Let

$$\theta_{n-1} \in (-\pi, \pi]$$

be the angle that OP_{n-1} forms with $\mathbf{e}_{n-1}$ (in the 2-plane W_{n-1}). Let then

$$\theta_i \in [0, \pi], \quad i = 1, ..., n-2$$

be the angle that OP_i makes with e_i. Let also θ_0 be the *modulus* $\|OP\|$ (notice: the coordinates are defined only for $i \leq max\{j : \ P_j \neq 0\}$).

Let $B(\Phi)$ be the union of bounded facets in Φ. The coordinate subspace V_i , $i = 1, ..., n$ is divided by V_{i-1} into two components:

$$V_i \setminus V_{i-1} \ = \ V_i(0) \cup V_i(\pi)$$

where

$$V_i(0) \ = \ \{P : \ \theta_i(P) = 0\}$$

and

$$V_i(\pi) \ = \ \{P : \ \theta_i(P) = \pi\}$$

(this makes sense for $i = n$ too, setting θ_n as the angle between P_n and $\mathbf{e}_n$). More generally, we indicate by $(i \leq n-1)$

$$V_i(\bar{\theta}_i, ..., \bar{\theta}_{n-1}) \ := \ \{P : \ \theta_i(P) = \bar{\theta}_i, ..., \theta_{n-1}(P) = \bar{\theta}_{n-1}\} \tag{4.1}$$

where by convention $\bar{\theta}_j = 0$ or $\pi \ \Rightarrow \ \bar{\theta}_k = 0$ for all $k > j$; so in particular, $V_i(0) = V_i(0, ..., 0)$ and $V_i(\pi) = V_i(\pi, 0, ..., 0)$ ($n - i$ components). The space $V_i(\bar{\theta})$, $\bar{\theta} := (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$, is an $i-$dimensional open half-subspace in the euclidean space V, and we denote by $|V_i(\bar{\theta})|$ the subspace which is spanned by it.

For all $\delta \in (0, \pi/2)$ the space

$$\tilde{B} := \ \tilde{B}(\delta) := \ \{P : \ \theta_i(P) \in (0, \delta), \ i = 1, ..., n-1, \ \theta_0(P) > 0\}$$

is an open cone contained in $\mathbb{R}^n_+$.

Definition 4.1.1 *We say that a system of polar coordinates in $V = \mathbb{R}^n$, defined by an origin O and a base $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$, is* generic *with respect to the arrangement $\mathcal{A}$ if it satisfies the following conditions:*

i) the origin O is contained in a chamber C_0 of $\mathcal{A}$, where C_0 is relatively open at infinity (i.e. its intersection with the hyperplane at infinity is a chamber);

ii) there exist $\delta \in (0, \pi/2)$ such that

$$B(\Phi) \ \subset \ \tilde{B} = \tilde{B}(\delta);$$

(therefore, for each facet $F \in \Phi$ one has $F \cap \tilde{B} \neq \emptyset$);

iii) (transversality) subspaces $V_i(\bar{\theta}) = V_i(\bar{\theta}_i, ..., \bar{\theta}_{n-1})$ which intersect $clos(\tilde{B})$ (so $\bar{\theta}_j \in [0, \delta]$ for $j = i, ..., n-1$) are generic with respect to $\mathcal{A}$, in the sense that, for each codim$-k$ subspace $L \in \mathcal{L}(\mathcal{A})$,

$$i \geq k \ \Rightarrow \ V_i(\bar{\theta}) \cap L \cap clos(\tilde{B}) \neq \emptyset \ and \ dim(|V_i(\bar{\theta})| \cap L) \ = \ i - k.$$

One has:

Theorem 10 *For each unbounded chamber C which is relatively open at infinity, the set of points $O \in C$ such that there exists a polar coordinate system centered in O and generic with respect to $\mathcal{A}$ forms a non-empty open subset of C.*

4.2 Polar orderings

Fix a system of generic polar coordinates, associated to a center O and frame $\mathbf{e}_1, ..., \mathbf{e}_n$. Let $\delta > 0$ be the number coming from definition 4.2.1. We denote for brevity $\bar{B} := clos(\tilde{B}(\delta))$. Each point P has polar coordinates $P \equiv (\theta_0, \theta_1, ..., \theta_{n-1})$, where we use the convention $\theta_0 := \rho$.

Notice that (4.1) makes sense also for $i = 0$, being

$$V_0(\bar{\theta}_0, \bar{\theta}_1, ..., \bar{\theta}_{n-1})$$

given by a single point P with

$$\theta_0(P) = \bar{\theta}_0,\ \theta_1(P) = \bar{\theta}_1,\ ...\ ,\ \theta_{n-1}(P) = \bar{\theta}_{n-1}.$$

Given a codimension$-k$ facet $F \in \Phi$, let us denote by

$$F(\theta) := F(\theta_i, ..., \theta_{n-1}) := F \cap V_i(\theta_i, ..., \theta_{n-1}), \quad \theta_j \in [0, \delta],\ j = i, ..., n-1$$

(notice: $F = F(\theta) = F \cap V_n$ with $\theta = \emptyset$.)

By genericity conditions, if $i \geq k$ then $F(\theta)$ is either empty or it is a codimension $k + n - i$ facet contained in $V_i(\theta_i, ..., \theta_{n-1})$.

Let us set, for every facet $F(\theta)$,

$$i_{F(\theta)} \ :=\ min\{j \geq 0 :\ V_j \cap clos(F(\theta)) \neq \emptyset\}.$$

Still by genericity, setting $L := |F(\theta)|$, one has

$$L \cap V_j \ \neq\ \emptyset \ \Leftrightarrow\ j \geq codim(F(\theta))$$

so also

$$i_{F(\theta)} \ \geq\ codim(F(\theta)). \tag{4.2}$$

When the facet $F(\theta) := F(\theta_i, ..., \theta_{n-1}),\ i > 0$, is not empty and $i_{F(\theta)} \geq i$ (i.e., $clos(F(\theta)) \cap V_{i-1} = \emptyset$), then among its vertices (0$-$dimensional facets in its boundary) there exists, still by genericity, a unique one

$$P := P_{F(\theta)} \in clos(F(\theta)) \tag{4.3}$$

such that

$$\theta_{i-1}(P) \ =\ min\{\theta_{i-1}(Q) :\ Q \in clos(F(\theta))\} \tag{4.4}$$

(of course, $P_{F(\theta)} = F(\theta)$ if $dim(F(\theta)) = 0$, i.e. $i = k$).

When $i_{F(\theta)} < i$ then the point P of (4.3) is either the origin 0 ($\Leftrightarrow$ $i_{F(\theta)} = 0 \Leftrightarrow F$ is the base chamber C_0) or it is the unique one such that

$$\theta_{i_{F(\theta)}-1}(P) \ =\ min\{\theta_{i_{F(\theta)}-1}(Q) :\ Q \in clos(F(\theta)) \cap V_{i_{F(\theta)}}\} \tag{4.5}$$

Definition 4.2.1 *Given any facet $F(\theta) = F(\theta_i, ..., \theta_{n-1})$ let us denote by*

$$P_{F(\theta)} \in clos(F(\theta))$$

the minimum vertex of $clos(F(\theta)) \cap\ V_{i_{F(\theta)}}$ (as in (4.3))
(for $F \in \Phi$ we briefly write P_F).

We associate to the facet $F(\theta)$ the $n-$vector of polar coordinates of $P_{F(\theta)}$

$$\Theta(F(\theta)) \ := \ (\theta_0(F(\theta)), ..., \theta_{i_{F(\theta)}-1}(F(\theta)), 0, ..., 0)$$

($n - i_{F(\theta)}$ zeroes) where we set

$$\theta_j(F(\theta)) := \theta_j(P_{F(\theta)}), \quad j = 0, ..., i_{F(\theta)} - 1.$$

We want to define another ordering over the poset $(\Phi, \prec)$. We give a recursive definition, actually ordering all facets in $V_i(\theta)$ for any given $\theta = (\theta_i, ..., \theta_{n-1})$.

Definition 4.2.2 *We define the* polar ordering *as follows: given $F,\ G\ \in \Phi$, and given $\bar\theta = (\bar\theta_i, ..., \bar\theta_{n-1})$, $0 \le i \le n$, $\bar\theta_j \in [0, \delta]$ for $j \in i, .., n-1$, ($\bar\theta = \emptyset$ for $i = n$) such that $F(\bar\theta), G(\bar\theta) \neq \emptyset$, we set*

$$F(\bar\theta) \ \lhd \ G(\bar\theta)$$

iff one of the following cases occurs:

i) $P_{F(\bar\theta)} \neq P_{G(\bar\theta)}$. Then $\Theta(F(\bar\theta)) < \Theta(G(\bar\theta))$, where we are considering the anti-lexicographic ordering of the coordinates (i.e., the lexicographic ordering starting from the last coordinate).

ii) $P_{F(\bar\theta)} = P_{G(\bar\theta)}$. Then either

iia) $dim(F(\bar\theta)) = 0$ (so $P_{F(\bar\theta)} = F(\bar\theta)$) and $F(\bar\theta) \neq G(\bar\theta)$ (so $dim(G(\bar\theta)) > 0$)

or

iib) $dim(F(\bar\theta)) > 0$, $dim(G(\bar\theta)) > 0$. In this case let $i_0 := i_{F(\bar\theta)} = i_{G(\bar\theta)}$.
When $i_0 \ge i$ one can write

$$\Theta(F(\bar\theta)) = \Theta(G(\bar\theta)) = (\tilde\theta_0, ..., \tilde\theta_{i-1}, \bar\theta_i, ..., \bar\theta_{i_0-1}, 0, ..., 0).$$

Then $\forall \epsilon,\ 0 < \epsilon << \delta$, it must happen

$$F(\tilde\theta_{i-1} + \epsilon, \bar\theta_i, ..., \bar\theta_{i_0-1}, 0, ..., 0) \ \lhd \ G(\tilde\theta_{i-1} + \epsilon, \bar\theta_i, ..., \bar\theta_{i_0-1}, 0, ..., 0).$$

If $i_0 < i$ then one can write

$$\Theta(F(\bar\theta)) = \Theta(G(\bar\theta)) = (\tilde\theta_0, ..., \tilde\theta_{i_0-1}, 0, ..., 0).$$

Then $\forall \epsilon,\ 0 < \epsilon << \delta$, it must happen

$$F(\tilde\theta_{i_0-1} + \epsilon, 0, ..., 0) \ \lhd \ G(\tilde\theta_{i_0-1} + \epsilon, 0, ..., 0).$$

($n - i_0$ zeroes) □

Condition (iib) says that one has to move a little bit the suitable $V_j(\theta')$ which intersects $clos(F(\theta))$ and $clos(G(\theta))$ in the point $P(F(\theta)) = P(G(\theta))$ (according to (4.4) or (4.5)), and consider the facets which are obtained by intersection with this moved subspace.

It is quite clear from the definition that irreflexivity and transitivity hold for $\lhd$ so we have

Theorem 11 *Polar ordering $\lhd$ is a total ordering on the facets of $V_i(\bar{\theta})$, for any given $\bar{\theta} = (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$. In particular (taking $\bar{\theta} = \emptyset$) it gives a total ordering on Φ.* $\square$

One has

Theorem 12 *Each codimension-k facet $F^k \in \Phi$ ($k < n$) such that $F^k \cap V_k = \emptyset$ has the following property: among all codimension-$(k+1)$ facets G^{k+1} with $F^k \prec G^{k+1}$, there exists a unique one F^{k+1} such that*

$$F^{k+1} \lhd F^k.$$

If $F^k \cap V_k \neq \emptyset$ (so $F^k \cap V_k = P(F^k)$) then

$$F^k \lhd G^{k+1}, \quad \forall G^{k+1} \text{ with } F^k \prec G^{k+1}.$$

4.2.1 Polar orderings for plane arrangements

For a line arrangement, the polar ordering assume a very simpler form, that we report here for the convenience of the reader.

A system of polar coordinates in $\mathbb{R}^2$ is defined by an origin O and a line V_1 containing O; we call by $V_1(\theta_1)$ the line which is obtained by a rotation (say counterclockwise) of V_1 with rotation angle θ_1 (therefore $V_1(0) = V_1$). Then a point P in $\mathbb{R}^2$ has polar coordinates (θ_0, θ_1) $(0 \leq \theta_0,\ 0 \leq \theta_1 < 2\pi)$ if it lies on $V_1(\theta_1)$ and its distance from O is θ_0.

Definition 4.2.1 *We say that a system of polar coordinates in $\mathbb{R}^2$, defined by an origin O and a line V_1 containing O, is* generic *with respect to the arrangement $\mathcal{A}$ if:*

i) the origin O is contained in a chamber C_0 of $\mathcal{A}$;

ii) there exists $\delta \in (0, \pi/2)$ such that the union of the bounded facets of $\Phi(\mathcal{A})$ lies in the open positive cone $C(0, \delta)$ delimited by the lines $V_1(0)$ and $V_1(\delta)$;

iii) the lines $V_1(\theta)$ with $\theta \in [0, \delta]$ are generic with respect to $\mathcal{A}$; this means that, for each line $l \in \mathcal{A}$, $V_1(\theta) \cap l$ is a point which belongs to $\overline{C(0, \delta)}$;

iv) the line $V_1(\theta)$ (with $\theta \in [0, \delta]$) contains at most one 0-dimensional facet of $\Phi(\mathcal{A})$.

As before, the origin O of coordinates must belong to an unbounded chamber.

Definition 4.2.2 *Given a facet $T \in \Phi(\mathcal{A})$, let $\overline{\theta_1}(T) \in [0, 2\pi)$ be defined by*

$$\overline{\theta_1}(T) = inf\{\theta_1 \mid V_1(\overline{\theta_1}) \cap T \neq \emptyset\}.$$

Definition 4.2.3 *Given two facets $F, G \in \Phi(\mathcal{A})$, we say that $F \lhd G$ if one of the following conditions holds:*

i) $\overline{\theta_1}(F) < \overline{\theta_1}(G)$*;*

ii) $\overline{\theta_1}(F) = \overline{\theta_1}(G)$ *and F is a point while G is not a point;*

iii) $\overline{\theta_1}(F) = \overline{\theta_1}(G)$ *and for all sufficiently small $\epsilon > 0$ one has that $V_1(\overline{\theta_1}(F) + \epsilon) \cap F$, $V_1(\overline{\theta_1}(G) + \epsilon) \cap G$ are not empty and the first set contains a point which is closer to the origin O than any one of the points of the latter set.*

In figures 2.1 and 2.2 we indicated the base point O and the line V_1. The lines $\{l_i\}_{i=1\ldots n}$ are ordered according to polar ordering, i.e., $i < j$ if and only if $l_i \cap V_1$ is closer to the origin than $l_j \cap V_1$. We have $\Phi(\mathcal{A}) = \{C_i, F_j, P_k\}$, where C_i are the 0-codimensional facets, F_j are the 1-codimensional facets and P_k are the 2-codimensional facets. Among the facets of the same codimension, the indexes are induced by polar ordering, i.e. $i < i'$ if and only if $C_i \lhd C_{i'}$ and so for the others facets.

4.3 A minimal CW-complex

We consider here the regular CW-complex $\mathbf{S} = \mathbf{S}^{(1)}$ and we define a combinatorial gradient vector field Γ over $\mathbf{S}$. One can describe Γ as a collection of pairs of cells

$$\Gamma \subset \{(e, f) \in \mathbf{S} \times \mathbf{S} \mid dim(f) = dim(e) + 1,\ e \in \partial(f)\}$$

so that Γ decomposes into its dimension-p components

$$\Gamma = \bigsqcup_{p=1}^{n} \Gamma^p, \qquad \Gamma^p \subset \mathbf{S}_{p-1} \times \mathbf{S}_p$$

($\mathbf{S}_p$ being the $p-$skeleton of $\mathbf{S}$).

We give the following recursive definition:

Definition 4.3.1 (Polar Gradient) *We define a combinatorial gradient field Γ over $\mathbf{S}$ in the following way:*

the $(j+1)-th$ component Γ^{j+1} of Γ, $j = 0, ..., n-1$, is given by all pairs

$$(e(C, F^j), e(C, F^{j+1})), \quad F^j \prec F^{j+1}$$

(same chamber C) such that

1. $F^{j+1} \lhd F^j$

2. $\forall F^{j-1} \prec F^j$ *such that* $C \prec F^{j-1}$ *the pair*

$$(e(C \prec F^{j-1}), e(C \prec F^j)) \notin \ \Gamma^j$$

Condition 2 of 4.3.1 is empty for the 1-dimensional part Γ^1 of Γ, so

$$\Gamma^1 = \{(e(C \prec C), e(C \prec F^1)) \ : \quad F^1 \lhd C\}.$$

According to the definition of generic polar coordinates, only the base-chamber C_0 intersects the origin $O = V_0$, so by Theorem 12 all $0-$cells $e(C \prec C),\ C \neq C_0$, belong to exactly one pair of Γ^1.

Theorem 13 *One has:*

1. Γ *is a combinatorial vector field on* $\mathbf{S}$ *which is the gradient of a discrete Morse function.*

2. *The pair*

$$(e(C \prec F^j), e(C \prec F^{j+1})), \quad F^j \prec F^{j+1}$$

belongs to Γ *iff the following conditions hold:*

(a) $F^{j+1} \lhd F^j$

(b) $\forall\ F^{j-1}$ *such that* $C \prec F^{j-1} \prec F^j$, *one has* $F^{j-1} \lhd F^j$.

3. *Given* $F^j \in \Phi$, *there exists a chamber* C *such that the cell* $[C \prec F^j]$ *is the second factor of a pair in* (Γ) *iff there exists* $F^{j-1} \prec F^j$ *with* $F^j \lhd F^{j-1}$. *More precisely, for each chamber* C *such that there exists* F^{j-1} *with*

$$C \prec F^{j-1} \prec F^j, \quad F^j \lhd F^{j-1} \tag{$*$}$$

the pair $(e(C \prec \bar{F}^{j-1}), e(C \prec F^j)) \in \Gamma$, *where* $\bar{F}^{j-1}$ *is the maximum* $(j-1)-$*facet (with respect to polar ordering) satisfying conditions (*).*

4. *The set of* $k-$*dimensional critical cells is given by*

$$\begin{aligned} Sing_k(\mathbf{S}) \ &= \\ &= \ \{e(C \prec F^k) \ : \ F^k \cap V_k \neq \emptyset,\ F^j \lhd F^k,\ \forall\ C \prec F^j \precneqq F^k\}. \end{aligned} \tag{4.6}$$

Equivalently, $F^k \cap V_k$ *is the maximum (in polar ordering) among all facets of* $C \cap V_k$.

We can check that:

Remark 4.3.2 *We can immediately check that there is only one singular 0-dimensional cell, namely $e(C_0, C_0)$, since all the other 2-cells C have in their boundary a 1-codimensional facet F such that $F \lhd C$.*

In the case of a line arrangement (see section 2.1), we have two component of Γ: $\Gamma = \Gamma_1 \sqcup \Gamma_2$. The next remark will be useful in chapter 5.

Remark 4.3.3 *The component $\Gamma_1 \subset \Gamma$ is a maximal oriented tree, with root $e(C_0, C_0)$, in the 1-skeleton of $\mathbf{S}$ and we can associate to each 0-cell $e(C, C)$ of $\mathbf{S}$ an unique path γ_C of Γ_1 which connects $e(C, C)$ to $e(C_0, C_0)$ and is oriented from $e(C, C)$ to $e(C_0, C_0)$.*

For later use in chapter 6, we state here two results directly following from the proof of part 1. of Theorem 13.

Take a $\Gamma-$path in $\mathbf{S}$:

$$e(C_1, F_1^k), e(C_1, F_1^{k+1}), \ldots, e(C_m, F_m^k), e(C_m, F_m^{k+1}), e(C_{m+1}, F_{m+1}^k) \tag{4.7}$$

Here the pair $(e(C_i, F_i^k), e(C_i, F_i^{k+1}))$ is an element of Γ, and $e(C_i, F_i^k)$ is in the boundary of $e(C_{i-1}, F_{i-1}^{k+1})$.

According to Theorem 7 we have to show that, if the path (4.7) is closed, (i.e. if $e(C_{m+1}, F_{m+1}^k)$ equals to $e(C_1, F_1^k)$), then it is trivial, i.e. $F_i^k = F_{i+1}^k$, $F_i^{k+1} = F_{i+1}^{k+1}$, and $C_i = C_{i+1}$ $(i = 1, \ldots, m-1)$.

The proof directly follows from the following two claims.

Claim 1 *Given a triple of consecutive cells in (4.7) of the form:*

$$e(C_i, F_i^{k+1}), e(C_{i+1}, F_{i+1}^k), e(C_{i+1}, F_{i+1}^{k+1}). \tag{4.8}$$

we have that $F_{i+1}^{k+1} \unlhd F_i^{k+1}$.

Claim 2 *Given a quadruple of consecutive cells in (4.7) of the form:*

$$e(C_i, F_i^k), e(C_i, F^{k+1}), e(C_{i+1}, F_{i+1}^k), e(C_{i+1}, F^{k+1}). \tag{4.9}$$

we have $F_i^k \unlhd F_{i+1}^k$.

Corollary 4.3.4 *Once a polar ordering is assigned, the set of singular cells is described only in terms of it by*

$$\begin{aligned} Sing_k(\mathbf{S}) \ := \ \{ e(C \prec F^k) \ : \\ a) \quad F^k \lhd F^{k+1}, \quad \forall \quad F^{k+1} \quad s.t. \quad F^k \prec F^{k+1} \\ b) \quad F' \lhd F^k, \quad \forall \quad F' \quad s.t. \quad C \prec F' \prec F^k \ \} \end{aligned}$$

□

Remark 4.3.5 *The construction of Theorem 13 gives an explicit additive basis for the homology and for the cohomology in terms of the singular cells in* $\mathbf{S}$. *We can call it a* polar basis *(relative to a given system of generic polar coordinates).*

Remark 4.3.6 *The minimality of the associated Morse complex is obtained by the one-to-one correspondence between singular cells and the set of all the chambers of* Φ, *and the well-known formula* $\sum b_i = |\{chambers\}|$ *(see [Z75]).*

4.4 Examples

Here we will present some examples illustrating the previous tractation.

Example 14 *We consider the line arrangement of figure 2.1. Here we want to write explicitly the polar ordering on the stratification* $\Phi(\mathcal{A})$ *and the reduction of the complex* $\mathbf{S}^{(1)}$ *(see example 8).*

The system of polar coordinates here is given by O *and* e_1, *where* $V_1 = < e_1 >$.

The total ordering in $\Phi(\mathcal{A})$ *is:*

$$C_0 \lhd F_1 \lhd C_1 \lhd F_2 \lhd C_2 \lhd F_3 \lhd C_3 \lhd F_4 \lhd C_4 \lhd F_5 \lhd C_5 \lhd F_6 \lhd C_6 \lhd P \lhd$$
$$\lhd F_7 \lhd C_7 \lhd F_8 \lhd C_8 \lhd F_9 \lhd C_9 \lhd F_{10} \lhd C_{10} \lhd F_{11} \lhd C_{11} \lhd F_{12}.$$

Recall that the complex $\mathbf{S}^{(1)}$ *has 48 cells. The polar gradient has two components:* $\Gamma = \Gamma_1 \sqcup \Gamma_2$. *Following definition 4.3.1 we can see that:*

- Γ_1 *is composed by 11 pairs of type* $(e(C_i, C_i), e(C_i, F_j))$, *with* $F_j \lhd C_i$, *i.e. by the pairs:*
  $\{(e(C_1, C_1), e(C_1, F_1)),\ (e(C_2, C_2), e(C_2, F_2)),\ (e(C_3, C_3), e(C_3, F_3)),$
  $(e(C_4, C_4), e(C_4, F_4)),\ (e(C_5, C_5), e(C_5, F_5)),\ (e(C_6, C_6), e(C_6, F_6)),$
  $(e(C_7, C_7), e(C_7, F_7)),\ (e(C_8, C_8), e(C_8, F_8)),\ (e(C_9, C_9), e(C_9, F_9)),$
  $(e(C_{10}, C_{10}), e(C_{10}, F_{10})),\ (e(C_{11}, C_{11}), e(C_{11}, F_{11}))\}$;

- Γ_2 *is composed by 7 pairs of type* $(e(C_i, F_j), e(C_i, P))$, *with* $P \lhd F_j$ *and* $e(C_i, F_j)$ *does not belong to any pair of* Γ_1, *i.e. by the pairs:*
  $\{(e(C_0, F_7), e(C_0, P)), (e(C_6, F_{12}), e(C_6, P)), (e(C_7, F_8), e(C_7, P)), (e(C_8, F_9), e(C_8, P)),$
  $(e(C_9, F_{10}), e(C_9, P)), (e(C_{10}, F_{11}), e(C_{10}, P)), (e(C_{11}, F_{12}), e(C_{11}, P))\}$.

So the set of critical cells $Sing(\mathbf{S})$ *is composed by the cells which do not appear in any pair of* Γ, *i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0)$;

- *6 critical 1-cells* $e(C_0, F_1), e(C_1, F_2), e(C_2, F_3), e(C_3, F_4), e(C_4, F_5), e(C_5, F_6)$, *which correspond to those 1-cells* $e(C_i, F_j)$ *with* $F_j \cap V_1 \neq \emptyset$ *and* $C_i \lhd F_j$;

- *5 critical 2-cells* $e(C_1,P), e(C_2,P), e(C_3,P), e(C_4,P), e(C_5,P),$ *, which correspond to those 2-cells* $e(C_i,P)$ *with* P *is the maximum (in polar ordering) among all facets of* C*.*

Remark 4.4.1 *Observe that the number of all cells of* $\mathbf{S}^{(1)}$ *is equal to the number of critical cells plus twice the number of the pairs of* Γ*, indeed* $1+6+5+2*(11+7)=48$*.*

Remark 4.4.2 *The previous example can be easily generalized to the case of a central line arrangement of* n *lines. In this case the critical cells are given by:*

- *1 critical 0-cell* $e(C_0,C_0)$*;*
- n *critical 1-cells* $e(C_0,F_1), e(C_1,F_2)\dots e(C_{n-1},F_n)$*;*
- $n-1$ *critical 2-cells* $e(C_1,P), e(C_2,P)\dots e(C_{n-1},P)$*.*

Example 15 *We take here the line arrangement of figure 2.2. The polar ordering on the stratification* $\Phi(\mathcal{A})$ *and the reduction of the complex* $\mathbf{S}^{(1)}$ *(see example 9) are as follows.*

The system of polar coordinates here is given by O *and* V_1 *.*

Here the total ordering in $\Phi(\mathcal{A})$ *is:*

$$C_0 \lhd F_1 \lhd C_1 \lhd F_2 \lhd C_2 \lhd F_3 \lhd C_3 \lhd F_4 \lhd C_4 \lhd F_5 \lhd C_5 \lhd P_1 \lhd F_6 \lhd C_6 \lhd F_7 \lhd C_7 \lhd F_8 \lhd$$

$$\lhd P_2 \lhd F_9 \lhd C_8 \lhd F_{10} \lhd P_3 \lhd F_{11} \lhd C_9 \lhd F_{12} \lhd P_4 \lhd F_{13} \lhd C_{10} \lhd F_{14} \lhd C_{11} \lhd F_{15}.$$

Recall that the complex $\mathbf{S}^{(1)}$ *has 62 cells. The polar gradient has two components:* $\Gamma = \Gamma_1 \sqcup \Gamma_2$*. Following definition 4.3.1 we can see that:*

- Γ_1 *is composed by 11 pairs of type* $(e(C_i,C_i), e(C_i,F_j))$*, with* $F_j \lhd C_i$*, i.e. by the pairs:*
  $\{(e(C_1,C_1),e(C_1,F_1)),\ (e(C_2,C_2),e(C_2,F_2)),\ (e(C_3,C_3),e(C_3,F_3)),$
  $(e(C_4,C_4),e(C_4,F_4)),\ (e(C_5,C_5),e(C_5,F_5)),\ (e(C_6,C_6),e(C_6,F_6)),$
  $(e(C_7,C_7),e(C_7,F_7)),\ (e(C_8,C_8),e(C_8,F_9)),\ (e(C_9,C_9),e(C_9,F_{11})),$
  $(e(C_{10},C_{10}),e(C_{10},F_{13})),\ (e(C_{11},C_{11}),e(C_{11},F_{14}))\}$*;*
- Γ_2 *is composed by 14 pairs of type* $(e(C_i,F_j), e(C_i,P_k))$*, with* $P_k \lhd F_j$ *and* $e(C_i,F_j)$ *does not belong to any pair of* Γ_1*, i.e. by the pairs:*
  $\{(e(C_0,F_{11}),e(C_0,P_3)),\ (e(C_1,F_6),e(C_1,P_1)),\ (e(C_4,F_8),e(C_4,P_1)),$
  $(e(C_5,F_{10}),e(C_5,P_2)),\ (e(C_6,F_7),e(C_6,P_1)),\ (e(C_6,F_{12}),e(C_6,P_3)),$
  $(e(C_7,F_8),e(C_7,P_1)),\ (e(C_7,F_9),e(C_7,P_2)),\ (e(C_8,F_{10}),e(C_8,P_2)),$
  $(e(C_8,F_{15}),e(C_8,P_4)),\ (e(C_9,F_{12}),e(C_9,P_3)),\ (e(C_9,F_{13}),e(C_9,P_4)),$
  $(e(C_{10},F_{14}),e(C_{10},P_4)),\ (e(C_{11},F_{15}),e(C_{11},P_4))\}.$

So the set of critical cells $Sing(\mathbf{S})$ *is given by the cells not appearing in any pair of* Γ*, i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0)$*;*
- *5 critical 1-cells* $e(C_0, F_1), e(C_1, F_2), e(C_2, F_3), e(C_3, F_4), e(C_4, F_5)$*, which correspond to those 1-cells* $e(C_i, F_j)$ *with* $F_j \cap V_1 \neq \emptyset$ *and* $C_i \lhd F_j$*;*
- *6 critical 2-cells* $e(C_1, P_3), e(C_2, P_1), e(C_3, P_1), e(C_4, P_2), e(C_6, P_4), e(C_7, P_4)$*, which correspond to those 2-cells* $e(C_i, P_k)$ *with* P_k *is the maximum (in polar ordering) among all facets of* C*.*

Remark 4.4.3 *As in previous example we have that the number of all cells of* $\mathbf{S}^{(1)}$ *is equal to the number of critical cells plus twice the number of the pairs of* Γ*, indeed* $1 + 5 + 6 + 2 * (11 + 14) = 62$*.*

Example 16 *Consider the arrangement* $\mathcal{A} \in \mathbb{R}^3$ *of example 10 (see figures 2.3 and 2.4). The system of polar coordinates here is given by* O*,* V_1 *and* V_2*. Here* V_2 *is the* $2-$*plane in* $\mathbb{R}^3$ *which corresponds to the plane* $z = 100$*, i.e. the plane of figure 2.3, while* V_1 *is the line indicated in figure 2.3.*

To give the total ordering in $\Phi(\mathcal{A})$ *we proceed as follows: the cells intersecting the plane* V_2 *comes before al the others cells, and they are ordered as in the case of a line arrangement with* O*,* V_1 *as polar coordinates system. The center* P *comes after these cells and before of the other ones. For ordering the other ones we have to move the plane* V_2 *fixing the line* V_1*, until meeting the center* P*, and then we have to pass to the other side and ordering all the cells comparing in figure 2.4 as in the case of a line arrangement with* O*,* V_1 *as polar coordinates system. So the polar ordering in* A_3 *is given by:*

$$C_0 \lhd F_1 \lhd C_1 \lhd F_2 \lhd C_2 \lhd F_3 \lhd C_3 \lhd F_4 \lhd C_4 \lhd F_5 \lhd C_5 \lhd F_6 \lhd C_6 \lhd G_1 \lhd F_7 \lhd C_7 \lhd F_8 \lhd C_8 \lhd F_9 \lhd G_2 \lhd$$

$$\lhd F_{10} \lhd C_9 \lhd F_{11} \lhd C_{10} \lhd F_{12} \lhd G_3 \lhd F_{13} \lhd C_{11} \lhd F_{14} \lhd G_4 \lhd F_{15} \lhd C_{12} \lhd F_{16} \lhd G_5 \lhd F_{17} \lhd C_{13} \lhd F_{18} \lhd C_{14} \lhd$$

$$\lhd F_{19} \lhd G_6 \lhd F_{20} \lhd C_{15} \lhd F_{21} \lhd C_{16} \lhd F_{22} \lhd G_7 \lhd F_{23} \lhd C_{17} \lhd F_{24} \lhd P \lhd \overline{G}_7 \lhd \overline{F}_{18} \lhd \overline{C}_{13} \lhd \overline{F}_{22} \lhd \overline{G}_6 \lhd \overline{F}_{17} \lhd$$

$$\lhd \overline{C}_{12} \lhd \overline{F}_{15} \lhd \overline{C}_7 \lhd \overline{F}_7 \lhd \overline{G}_5 \lhd \overline{F}_{13} \lhd \overline{C}_{10} \lhd \overline{F}_{11} \lhd \overline{G}_4 \lhd \overline{F}_{10} \lhd \overline{C}_8 \lhd \overline{F}_8 \lhd \overline{G}_3 \lhd \overline{F}_{12} \lhd \overline{G}_2 \lhd \overline{F}_9 \lhd \overline{G}_1$$

Recall that the complex $\mathbf{S}^{(1)}$ *has 192 cells. The polar gradient has components:* $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3$*. Following definition 4.3.1 we can see that:*

- Γ_1 *is composed by 23 pairs of type* $(e(C_i, C_i), e(C_i, F_j))$*, with* $F_j \lhd C_i$*;*
- Γ_2 *is composed by 43 pairs of type* $(e(C_i, F_j), e(C_i, G_k))$*, with* $G_k \lhd F_j$ *and* $e(C_i, F_j)$ *does not belong to any pair of* Γ_1*.*
- Γ_3 *is composed by 18 pairs of type* $(e(C_i, G_k), e(C_i, P))$*, with* $P \lhd G_k$ *and* $e(C_i, G_k)$ *does not belong to any pair of* Γ_2*.*

So the set of critical cells $Sing(\mathbf{S})$ *is composed by the cells not appearing in any pair of* Γ*, i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0)$*;*

- *6 critical 1-cells* $e(C_0, F_1), e(C_1, F_2), e(C_2, F_3), e(C_3, F_4), e(C_4, F_5), e(C_5, F_6)$*, which correspond to those 1-cells* $e(C_i, F_j)$ *with* $F_j \cap V_1 \neq \emptyset$ *and* $F_j \cap V_1$ *is the maximum (in polar ordering) among all facets of* $C \cap V_1$*;*

- *11 critical 2-cells* $e(C_1, G_1)$*,* $e(C_2, G_1)$*,* $e(C_3, G_2)$*,* $e(C_4, G_2)$*,* $e(C_5, G_3)$*,* $e(C_7, G_6)$*,* $e(C_8, G_4)$*,* $e(C_9, G_5)$*,* $e(C_{10}, G_5)$*,* $e(C_{12}, G_6)$*,* $e(C_{13}, G_7)$*, which correspond to those 2-cells* $e(C_i, G_k)$ *with* $G_k \cap V_2 \neq \emptyset$ *and* $G_k \cap V_2$ *is the maximum (in polar ordering) among all facets of* $C \cap V_2$*.*

- *6 critical 3-cells* $e(C_7, P), e(C_8, P), e(C_9, P), e(C_{10}, P), e(C_{12}, P), e(C_{13}, P)$*, which correspond to those 2-cells* $e(C_i, P)$ *with* P *is the maximum (in polar ordering) among all facets of* C*.*

Remark 4.4.4 *As in previous examples we can calculate the number of cells as* $1+6+11+6+2*(23+43+18) = 192$*.*

Chapter 5

Explicit description of the attaching maps for plane arrangements

This chapter presents the results of [GMS09].

Given an affine line arrangement (see chapter 2.1) $\mathcal{A} = \{l_i\}_{i=1\dots n}$ in $\mathbb{R}^2$, let $\mathcal{M}(\mathcal{A})$ be its complement. In the previous chapter we described the cells of the minimal complex. In this chapter we want to present the explicit description of the attaching maps of the cells of this complex.

5.1 Notations

Notation 5.1.1 *An 1-cell $e(C, F)$ of $\mathbf{S}$ is called $\overline{\gamma}_{i,j}$ if the following conditions are both satisfied :*

- *$F \subset l_i$ and l_i does not separate C from C_0;*
- *there are exacly j 2-codimensional facets on l_i which are lower than F in the polar ordering.*

To each $\overline{\gamma}_{i,j} = e(C, F)$ we associate a loop $\gamma_{i,j}$ in the 1-skeleton of $\mathbf{S}$:

$$\gamma_{i,j} = \Gamma_C^{-1}\overline{\gamma}_{i,j}\Gamma_{C'}$$

where C' is the other chamber having F in its boundary and Γ_C, $\Gamma_{C'}$ are as in remark 4.3.3.

The critical 1-cells (see section 4.3) correspond to $\overline{\gamma}_{1,0} \dots \overline{\gamma}_{n,0}$. For simplicity we denote $\gamma_{i,0}$ by γ_i. We also use the same symbols $\gamma_{i,j}$ and γ_i for the induced homotopy classes in $\pi_1(\mathcal{M}(\mathcal{A}))$, when the meaning is clear from the context.

So the 1-skeleton of the minimal complex is a bouquet of n loops represented by $\gamma_1 \dots \gamma_n$ (a true bouquet is obtained collapsing the maximal tree Γ_1, see remark 4.3.3).

Our goal is to explicitly describe the attaching maps of the 2-cells in the minimal complex, as they derive from the discrete vector field Γ.

Definition 5.1.1 *Let P be a 2-codimensional facet of $\Phi(\mathcal{A})$.*

Define $I(P) = \{k \in 1, \ldots, n | P \in l_k,\ or\ P \notin l_k\ and\ l_k\ does\ not\ separate\ O\ from\ P\}$.

We associate to P the sequence of words in the γ_i's:

$$A_j(P) = \prod_{\substack{k=1 \\ k \in I(P)}}^{j-1} \gamma_k$$

for $j = 1 \ldots n$.

We define also $A_j(\infty) = 1$ for all $j > 0$.

Definition 5.1.2 *Given any words $\alpha_1 \ldots \alpha_p$ in the γ_i's, we define for each $i = 1, \ldots, p-1$:*

$$W_{i+1}(\alpha_1, \ldots, \alpha_p) = \alpha_{i+1} \cdots \alpha_p \alpha_1 \ldots \alpha_i (\alpha_1 \cdots \alpha_p)^{-1}.$$

5.2 Central case

First, we consider a central line arrangement with lines $\{l_i\}_{i=1\ldots n}$ ordered according to the polar ordering. So we have $\Phi(\mathcal{A}) = \{C_i, F_j, P\}_{0 \le i \le 2n-1, 1 \le j \le 2n}$ (see example 8 and remark 4.4.2).

Notation 5.2.1 *We put $\overline{\delta}_1 = e(C_1, F_1), \ldots \overline{\delta}_{2n-1} = e(C_{2n-1}, F_{2n-1})$. We put also $\overline{\delta}_{2n} = e(C_n, F_{2n})$ (see figure 5.1).*

As before, we associate to each $\overline{\delta}_i$ the loop δ_i defined by $\Gamma_{C_i}^{-1} \overline{\delta}_i \Gamma_{C_{i-1}}$ if $1 \le i \le 2n-1$ and by $\Gamma_{C_n}^{-1} \overline{\delta}_{2n} \Gamma_{C_{2n-1}}$ if $i = 2n$.

Remark 5.2.2 *We observe that the $\overline{\delta_i}$'s, for $1 \le i \le 2n-1$, are associated to elements of Γ_1, so they induce trivial loops δ_i in the fundamental group.*

As a preliminary result we have:

Lemma 5.2.3 *Let $\mathcal{A}$ be a central arrangement.*

In the fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ we have $\gamma_{i,1} = (\gamma_1 \ldots \gamma_{i-1})\gamma_i(\gamma_1 \ldots \gamma_{i-1})^{-1}$ for $1 \le i \le n$; furthermore $\delta_{2n} = 1$.

Proof. The 1-cell $\overline{\gamma}_{i,1} \in \mathbf{S}$ belongs to the boundary of the 2-cell $e(C_{2n-i}, P)$ for $1 \le i < n$, while $\overline{\gamma}_{n,1}$ belongs to $\partial(e(C_0, P))$. The boundary of $e(C_{2n-i}, P)$ can be written as:

$$\overline{\delta}_{2n-i} \overline{\delta}_{2n-i-1} \ldots \overline{\delta}_{n+1} \overline{\gamma}_1 \overline{\gamma}_2 \ldots \overline{\gamma}_i \overline{\delta}_{i+1}^{-1} \ldots \overline{\delta}_n^{-1} \overline{\gamma}_{1,1}^{-1} \ldots \overline{\gamma}_{i-1,1}^{-1} \overline{\gamma}_{i,1}^{-1}.$$

By remark 5.2.2 the above expression induces the following equality in the fundamental group:

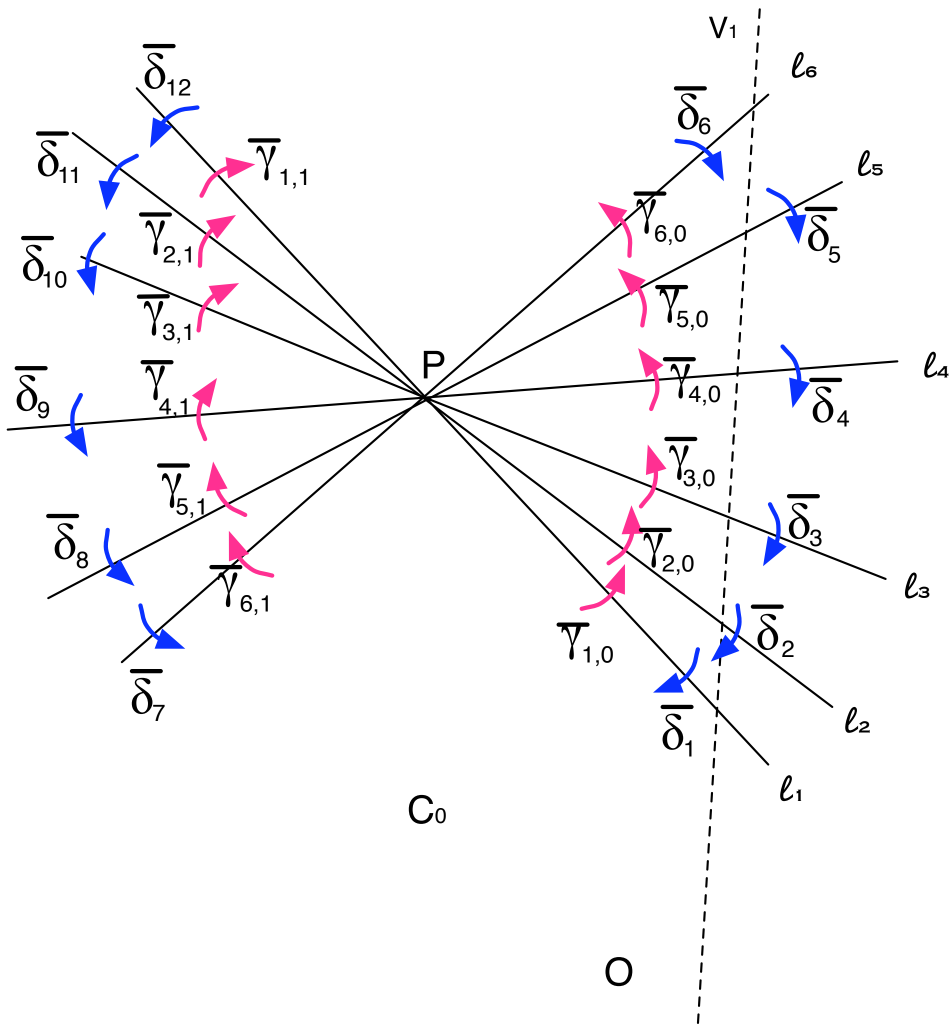


Figure 5.1: A figure illustrating notation 5.1.1 in central case.

$$\gamma_{i,1} = \gamma_1\gamma_2 \ldots \gamma_i\gamma_{1,1}^{-1} \ldots \gamma_{i-1,1}^{-1}. \tag{5.1}$$

In the case $n = 1$ the formula 5.1 gives $\gamma_{1,1} = \gamma_1$ which is our thesis.

The thesis follows by induction. The cases of δ_{2n} and $\gamma_{n,1}$ are analogous.

qed

The above lemma can be immediately generalized to the affine case. First we introduce the following notation:

Notation 5.2.4 *Let $\mathcal{A}$ be any line arrangement. Let $\overline{\gamma}_{i,j} = e(C, F)$ with $j \geq 0$ a 1-cell of* **S** *as in notation 5.1.1. We call $P_{\gamma_{i,j}}$ the 2-codimensional facet which follows F in the polar ordering restricted to l_i (if it exists, and in this case we will call it "the farthest end" of F).*

Moreover, given a 2-codimensional facet P, and given a line l_i passing through P we denote by $j(i, P)$ the index such that $P_{\gamma_{i,j(i,P)}} = P$. We also put $P_{\gamma_{i,k}} = \infty$ if $k < 0$.

Lemma 5.2.5 *Consider $\overline{\gamma}_{i_I,J}$ with $J > 0$ and its "nearest end" $P_{\gamma_{i_I,J-1}} = P$. Let $l_{i_1} \ldots l_{i_p}$ be the lines passing through P ordered according to the polar ordering. Then in the fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ we have*

$$\gamma_{i_I,J} = (\gamma_{i_1,j(i_1,P)} \cdots \gamma_{i_{I-1},j(i_{I-1},P)})\gamma_{i_I,J-1}(\gamma_{i_1,j(i_1,P)} \cdots \gamma_{i_{I-1},j(i_{I-1},P)})^{-1}.$$

Proof. The proof is totally analogous to the one of lemma 5.2.3. □

Now we can state the formula of attaching maps in central case:

Theorem 14 *Given a critical 2-cell $e(C_i, P)$, its attaching map in the minimal complex is given by:*

$$W_{i+1}(\gamma_1, \ldots, \gamma_n) = \gamma_{i+1} \cdots \gamma_n\gamma_1 \cdots \gamma_i(\gamma_1 \cdots \gamma_n)^{-1}.$$

Proof. The boundary of $e(C_i, P)$ in **S** is written as:

$$\overline{\gamma}_{i+1} \cdots \overline{\gamma}_n\overline{\delta}_{2n}\overline{\delta}_{2n-1} \cdots \overline{\delta}_{2n-i+1}\overline{\gamma}_{i+1,1}^{-1} \cdots \overline{\gamma}_{n,1}^{-1}\overline{\delta}_1^{-1} \cdots \overline{\delta}_i^{-1}.$$

By remark 5.2.2 and second part of lemma 5.2.3, the above expression induces in the fundamental group:

$$\gamma_{i+1} \cdots \gamma_n\gamma_{i+1,1}^{-1} \cdots \gamma_{n,1}^{-1}.$$

Then one can substitute the $\gamma_{j,1}$'s using the formulas of lemma 5.2.3 and the thesis follows. □

5.3 Affine case

Now we consider the affine case. The following lemma is a direct generalization of theorem 5.2 :

Lemma 5.3.1 *Consider a 2-codimensional facet P in $\Phi(\mathcal{A})$ and let us denote by $l_{i_1} \dots l_{i_p}$ the lines passing through P ordered according to the polar ordering. Given a critical 2-cell $e(C,P)$, delimited by l_{i_k}, $l_{i_{k+1}}$, its attaching map in $\mathbf{S}$ is homotopic to:*

$$W_{k+1}(\gamma_{i_1,j(i_1,P)} \dots \gamma_{i_p,j(i_p,P)}) =$$
$$= \gamma_{i_{k+1},j(i_{k+1},P)} \dots \gamma_{i_p,j(i_p,P)} \gamma_{i_1,j(i_1,P)} \dots \gamma_{i_k,j(i_k,P)} (\gamma_{i_1,j(i_1,P)} \dots \gamma_{i_p,j(i_p,P)})^{-1}.$$

Proof. The attaching map of $e(C,P)$ in $\mathbf{S}$ is homotopic to:

$$\left(\gamma_{i_{k+1},j(i_{k+1},P)} \gamma_{i_{k+2},j(i_{k+2},P)} \cdots \gamma_{i_p,j(i_p,P)}\right) \left(\gamma_{i_{k+1},j(i_{k+1},P)+1} \gamma_{i_{k+2},j(i_{k+2},P)+1} \cdots \gamma_{i_p,j(i_p,P)+1}\right)^{-1}.$$

The remaining part of the proof is completely analogous to the one of theorem 5.2, using lemma 5.2.5.

□

Let us consider a critical 2-cell $e(C,P)$. We want to know the attaching map of the corresponding 2-cell of the minimal complex.

Define $\gamma'_{P,j} = A_j(P)\gamma_j A_j(P)^{-1}$.

Lemma 5.3.2 *Let $\overline{\gamma}_{i,j} = e(C,F)$ be a 1-cell as in notation 5.1.1. We have:*

$$\gamma_{i,j} = A_i(P_{\gamma_{i,j-1}})\gamma_i A_i(P_{\gamma_{i,j-1}})^{-1} = \gamma'_{P_{\gamma_{i,j-1}},i}.$$

Proof. By induction on (i,j), where the couples are considered with the lexicographic ordering: $(h,k) < (s,t)$ if and only if $h < s$, or $h = s$ and $k < t$.

For all i, we have $\gamma_{i,0} = \gamma_i$ and the thesis is proved when $j = 0$.

Let's consider now the case $i = 1$. We proceed by induction on j. As we know the thesis is true for $j = 0$. Suppose the theorem true for $j-1$, i.e. $\gamma_{1,j-1} = \gamma_1$. Let $l_{i_1} \dots l_{i_p}$ be the lines passing through $P_{\gamma_{1,j-1}}$ (by construction we have $i_1 = 1$). By lemma 5.2.5 $\gamma_{1,j} = \gamma_{1,j-1} = \gamma_1$ and we have done.

We can now study the general case: consider $\gamma_{i_I,J}$ and assume our claim true for $\gamma_{i',j}$ with all j and $i' < i_I$, or for $\gamma_{i_I,j'}$ for $j' < J$. Let us call by $l_{i_1} \dots l_{i_p}$ the lines passing through $P; = P_{\gamma_{i_I,J-1}}$ ordered according to the polar ordering. By lemma 5.2.5, we obtain:

$$\gamma_{i_I,J} = (\gamma_{i_1,j(i_1,P)} \dots \gamma_{i_{I-1},j(i_{I-1},P)})\gamma_{i_I,J-1}(\gamma_{i_1,j(i_1,P)} \dots \gamma_{i_{I-1},j(i_{I-1},P)})^{-1}. \tag{5.2}$$

In this formula all pairs of indexes of the γ's are lower than (i_I, J) in the lexicographic ordering, so we can use the inductive hypothesis in formula (5.2) to obtain:

$$\gamma_{i_I,J} = A_{i_1}(P_{\gamma_{i_1,j(i_1,P)-1}})\gamma_{i_1}(A_{i_1}(P_{\gamma_{i_1,j(i_1,P)-1}}))^{-1} \cdots$$

$$\cdots A_{i_{I-1}}(P_{\gamma_{i_{I-1},j(i_{I-1},P)-1}})\gamma_{i_{I-1}}(A_{i_{I-1}}(P_{\gamma_{i_{I-1},j(i_{I-1},P)-1}}))^{-1}\cdot A_{i_I}(P_{\gamma_{i_I,J-2}})\gamma_{i_I}A_{i_I}(P_{\gamma_{i_I,J-2}})^{-1}\cdot$$
$$\cdot\Big(A_{i_1}(P_{\gamma_{i_1,j(i_1,P)-1}})\gamma_{i_1}(A_{i_1}(P_{\gamma_{i_1,j(i_1,P)-1}}))^{-1}\cdots A_{i_{I-1}}(P_{\gamma_{i_{I-1},j(i_{I-1},P)-1}})\gamma_{i_{I-1}}(A_{i_{I-1}}(P_{\gamma_{i_{I-1},(i_{I-1},P)-1}})^{-1})\Big)^{-1}.$$

Now let us denote for simplicity $P_k = P_{\gamma_{i_k,j(i_k,P)-1}}$ and take a piece of the expression above of the type:

$$A_{i_k}(P_k)\gamma_{i_k}(A_{i_k}(P_k))^{-1}A_{i_{k+1}}(P_{k+1})\gamma_{i_{k+1}}(A_{i_{k+1}}(P_{k+1}))^{-1}.$$

We observe that $A_{i_{k+1}}(P_{k+1})$ is a product that includes all the factors of $A_{i_k}(P_k)$. To prove this, suppose $i < i_k$ and $i \in I(P_k)$, i.e. $P_k \in l_i$, or $P_k \notin l_i$ and l_i does not separate O from P_k. If $i \notin I(P_{k+1})$ then $P_{k+1} \notin l_i$ and l_i separates P_{k+1} from O. But in this case, being $i < i_k$ (and $\theta_1(l_i \cap l_{i_k}) \leq \theta_1(P_k)$), l_i meets $l_{i_{k+1}}$ in a point belonging to the segment $[P, P_{k+1}]$ and this is a contradiction because P_{k+1} and P are ends of a same codimensional-1 facet. So it must be $i \in I(P_{k+1})$.

Then $B_{i_{k+1}}(P_{k+1}) := (A_{i_k}(P_{k+1}))^{-1}(A_{i_{k+1}}(P_{k+1}))$ is a product:

$$B_{i_{k+1}}(P_{k+1}) = \prod_{\substack{a=i_k+1\\ a\in I(P_{k+1})}}^{i_{k+1}-1} \gamma_a.$$

This gives:

$$\gamma_{i_I,J} = A_{i_1}(P_1)\gamma_{i_1}B_{i_2}(P_2)\gamma_{i_2}\ldots B_{i_{I-1}}(P_{I-1})\gamma_{i_{I-1}}B_{i_I}(P_I)\gamma_{i_I}\cdot$$
$$\cdot\big(A_{i_1}(P_1)\gamma_{i_1}B_{i_2}(P_2)\gamma_{i_2}\ldots B_{i_{I-1}}(P_{I-1})\gamma_{i_{I-1}}B_{i_I}(P_I)\big)^{-1}.$$

To complete our proof we need to show that

$$A_{i_I}(P) = A_{i_1}(P_1)\gamma_{i_1}B_{i_2}(P_2)\gamma_{i_2}\ldots B_{i_{I-1}}(P_{I-1})\gamma_{i_{I-1}}B_{i_I}(P_I).$$

First of all observe that γ_{i_k} appears in the two members of the equality for all $1 \leq k < I$. Now let us consider γ_i with $i \neq i_k$ for all $1 \leq k < I$ and prove that if a γ_i appears in the left member of the equality, then it appears also in the right one and viceversa.

Consider the left part of the above equality, and take γ_i with $i \in I(P)$ and $i_k < i < i_{k+1}$. It means $P \notin l_i$ and l_i does not separate O from P. Then γ_i is a factor of $B_{i_{k+1}}(P_{k+1})$. Indeed suppose there is a line l_i verifying $i_k < i < i_{k+1}$, $i \in I(P)$ but γ_i is not a factor of $B_{i_{k+1}}(P_{k+1})$. So $P_{k+1} \notin l_i$ and l_i separates P_{k+1} from 0. Then, as before, l_i must meet l_{k+1} in a point belonging to the segment $[P, P_{k+1}]$, but it is impossible because P and P_{k+1} are ends of the same codimensional-1 facet of $\Phi(\mathcal{A})$.

Conversely, consider the right member of the above equality. Take a factor γ_i of $B_{i_{k+1}}(P_{k+1})$, with $i_k < i < i_{k+1}$. If we have $P_{k+1} \in l_i$, then $i \in I(P)$; else we have $P_{k+1} \notin l_i$ and l_i does not separate P_{k+1} from 0. Also in this case we can conclude that $i \in I(P)$.

This completes the proof.

□

Theorem 15 *Let us consider a critical cell $e(C, P)$. Let $l_{i_1} \dots l_{i_p}$ be the lines through P ordered according to the polar ordering, and let l_{i_k}, $l_{i_{k+1}}$ be the lines delimiting the chamber C. The attaching map of the 2-cell $e(C, P)$ in the minimal complex is given by:*

$$W_{k+1}(\gamma'_{P_{\gamma_{i_1,j(i_1,P)-1}},i_1}, \dots, \gamma'_{P_{\gamma_{i_p,j(i_p,P)-1}},i_p}).$$

Proof. By lemma 5.3.1 we obtain that the attaching map is represented by:

$$W_{k+1}(\gamma_{i_1,j(i_1,P)}, \dots, \gamma_{i_p,j(i_p,P)}).$$

Then the thesis follows immediately using lemma 5.3.2.

□

5.4 Example: deconing A_3

Let us consider the arrangement $\mathcal{A}$ as in Figure 2.2. The line l_3 has two triple points and there are two pairs of parallel lines l_1, l_2 and l_4, l_5. Recall the polar ordering on $\Phi(\mathcal{A})$, and the critical cells as presented in example 9.

With our notations, we have $\overline{\gamma}_{1,1} = e(C_9, F_{12})$, $\overline{\gamma}_{1,2} = e(C_{11}, F_{15})$, $\overline{\gamma}_{2,1} = e(C_7, F_8)$, $\overline{\gamma}_{2,2} = e(C_8, F_{10})$, $\overline{\gamma}_{3,1} = e(C_6, F_7)$, $\overline{\gamma}_{3,2} = e(C_{10}, F_{14})$, $\overline{\gamma}_{4,1} = e(C_1, F_6)$, $\overline{\gamma}_{4,2} = e(C_0, F_{11})$, $\overline{\gamma}_{5,1} = e(C_7, F_9)$, $\overline{\gamma}_{5,2} = e(C_9, F_{13})$. (see figure 5.2)

First we want to give an example of application of lemma 5.3.2. Consider the 1-cell $\gamma_{5,2}$. It appairs in the boundary of the 2-cell $e(C_9, P_4)$. Let's write this boundary:

$$\partial(e(C_9, P_4)) = \overline{\gamma}_{1,1}\overline{\gamma}_{3,1}\overline{\gamma}_{5,1}\overline{\gamma}_{1,2}^{-1}\overline{\gamma}_{3,2}^{-1}\overline{\gamma}_{5,2}^{-1}.$$

This gives the equality in π_1:

$$\gamma_{5,2} = \gamma_{1,1}\gamma_{3,1}\gamma_{5,1}\gamma_{1,2}^{-1}\gamma_{3,2}^{-1}.$$

In the same way, computing the boundary of the 2-cell $e(C_{11}, P_4)$ we have:

$$\gamma_{1,2} = \gamma_{1,1}$$

and computing the boundary of the 2-cell $e(C_{10}, P_4)$ we have:

$$\gamma_{3,2} = \gamma_{1,1}\gamma_{3,1}\gamma_{1,1}^{-1}.$$

Substituting we obtain:

$$\gamma_{5,2} = \gamma_{1,1}\gamma_{3,1}\gamma_{5,1}\gamma_{3,1}^{-1}\gamma_{1,1}^{-1} = \gamma_{1,1}\gamma_{3,1}\gamma_{5,1}(\gamma_{1,1}\gamma_{3,1})^{-1}.$$

Now we can continue similarly substituting the $\gamma_{i,j}$ with $j = 1$: $\gamma_{1,1} = \gamma_1$, $\gamma_{3,1} = \gamma_2\gamma_3\gamma_2^{-1}$, $\gamma_{5,1} = \gamma_2\gamma_5\gamma_2^{-1}$. We obtain:

$$\gamma_{5,2} = \gamma_1\gamma_2\gamma_3\gamma_2^{-1}\gamma_2\gamma_5\gamma_2^{-1}(\gamma_1\gamma_2\gamma_3\gamma_2^{-1})^{-1} = \gamma_1\gamma_2\gamma_3\gamma_5(\gamma_1\gamma_2\gamma_3)^{-1}$$

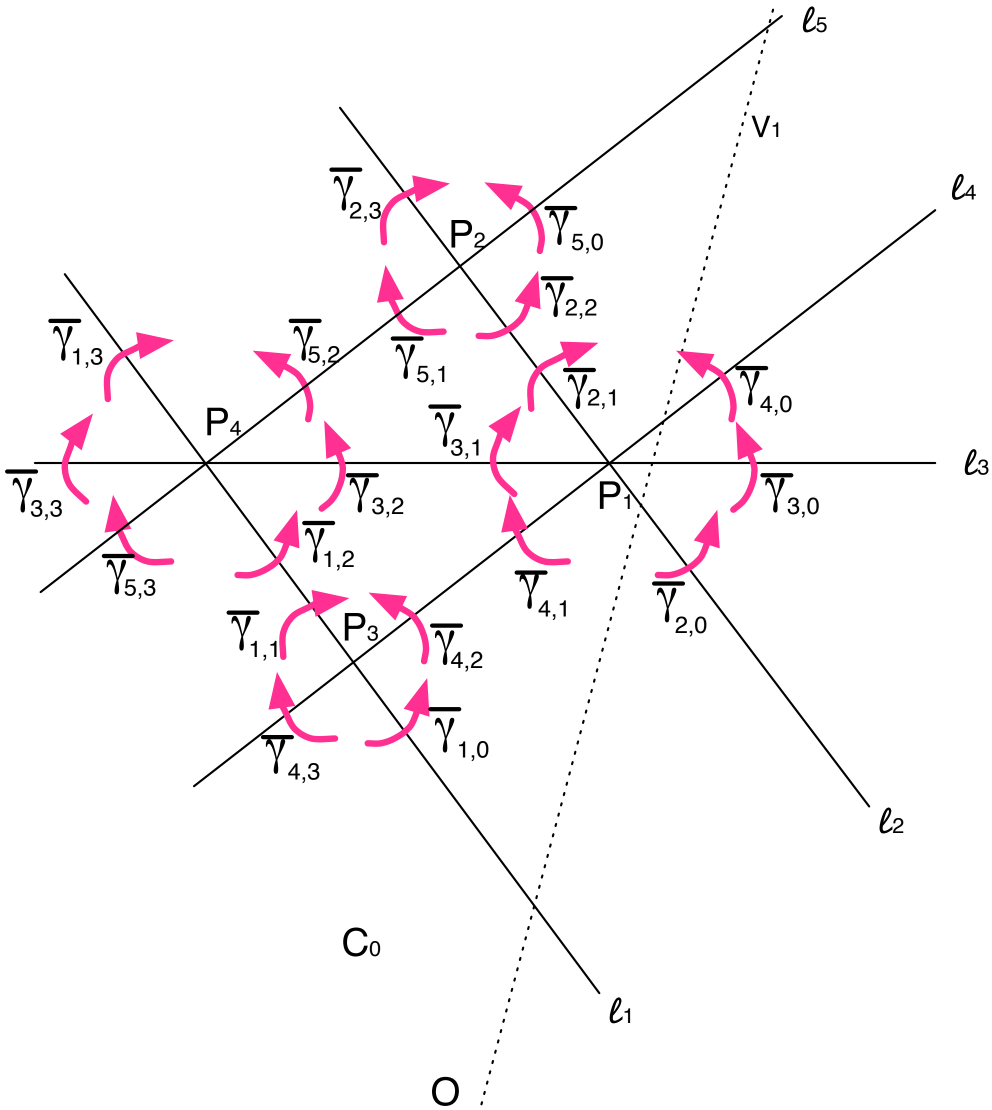


Figure 5.2: A figure illustrating notation 5.1.1 in our example.

as wanted.

Let us write now an example of application of theorem 15.

The attaching map of the critical cell $e(C_1, P_3)$ is:

$$\partial(e(C_1,P_3)) = \overline{\gamma}_{4,1}e(C_6,F_{12})\overline{\gamma}_{4,2}{}^{-1}e(C_1,F_1)^{-1}.$$

This induces in π_1 an attaching map homotopic to:

$$\gamma_{4,1}\gamma_{4,2}^{-1}.$$

But $\gamma_{4,1} = \gamma_2\gamma_3\gamma_4(\gamma_2\gamma_3)^{-1}$ and $\gamma_{4,2} = \gamma_1\gamma_{4,1}\gamma_1^{-1} = (\gamma_1\gamma_2\gamma_3)\gamma_4(\gamma_1\gamma_2\gamma_3)^{-1}$ by lemma 5.3.2. Substituting we obtain:

$$\partial(e(C_1,P_3)) = \gamma_2\gamma_3\gamma_4(\gamma_2\gamma_3)^{-1}(\gamma_1\gamma_2\gamma_3)\gamma_4^{-1}(\gamma_1\gamma_2\gamma_3)^{-1} = \gamma'_{P_1,4}\gamma_1(\gamma_1\gamma'_{P_1,4})^{-1} = W_2(\gamma'_{\infty,1},\gamma'_{P_1,4}).$$

Another example is given by:

$$\partial(e(C_6,P_4)) = \overline{\gamma}_{3,1}\overline{\gamma}_{5,1}e(C_8,F_{15})\overline{\gamma}_{3,2}{}^{-1}\overline{\gamma}_{5,2}{}^{-1}e(C_6,F_{12})^{-1},$$

that induces in π_1 an attaching map homotopic to:

$$\gamma_{3,1}\gamma_{5,1}\gamma_{3,2}^{-1}\gamma_{5,2}^{-1}.$$

But $\gamma_{3,1} = \gamma_2\gamma_3\gamma_2^{-1}$, $\gamma_{5,1} = \gamma_2\gamma_5\gamma_2^{-1}$, $\gamma_{5,2} = \gamma_1\gamma_2\gamma_3\gamma_5(\gamma_1\gamma_2\gamma_3)^{-1}$ and $\gamma_{3,2} = \gamma_1\gamma_2\gamma_3(\gamma_1\gamma_2)^{-1}$ by lemma 5.3.2. Substituting we obtain:

$$\partial(e(C_6,P_4)) = \gamma_2\gamma_3\gamma_2^{-1}\gamma_2\gamma_5\gamma_2^{-1}\gamma_1\gamma_2\gamma_3^{-1}\gamma_2^{-1}\gamma_2\gamma_5^{-1}\gamma_2^{-1}\gamma_2\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1} =$$
$$= (\gamma'_{P_1,3}\gamma'_{P_2,5}\gamma'_{P_3,1})(\gamma'_{P_3,1}\gamma'_{P_1,3}\gamma'_{P_2,5})^{-1} = W_2(\gamma'_{P_3,1},\gamma'_{P_1,3},\gamma'_{P_2,5}).$$

Chapter 6

Discrete Morse theory of configuration spaces

Consider the $d-$generalized configuration space (see 3.2) associated to an hyperplane arrangement $\mathcal{A} = \{H_i\}_{i\in I}$ in $\mathbb{R}^n$. In this chapter we reduce the complex $\mathbf{S}^{(d)}$ to a minimal complex, using Morse theory, as we have seen in chapter 4, where it is presented the case $d = 1$. In particular, we obtain an explicit basis, called "polar basis", for the homology of the complement.

We refer to section 2.1 for the main definitions and notations and to section 3.3 for the definition of the CW-complex $\mathbf{S}^{(d)}$. The results of this chapter appear in [MS09]. At the end of the chapter we present some explicit examples.

6.1 A minimal CW-complex

Lemma 6.1.1 *Let $L \in \mathcal{L}(\mathcal{A})$ be a codimension k subspace. Then the system $V_0, \ldots, V_k$ gives a generic system of polar coordinates for the arrangement $\mathcal{A}_L \cap V_k := \{H \cap V_k |\ H \in \mathcal{A}\}$ in V_k, so it induces a polar ordering $\lhd_L$ on Φ_L. The system $V_k \cap L, \ldots, V_n \cap L$ gives a generic system of polar coordinates for the arrangement $\mathcal{A}^L$ on L, inducing a polar ordering $\lhd^L$ on Φ^L.*

One has that $\lhd^L$ coincides with the restriction $\lhd_{|\Phi^L}$ of the polar ordering $\lhd$ to $\Phi^L \subset \Phi$.

Proof. In the case of Φ_L, there are no bounded facets, except for the point $L \cap V_k$, and the genericity condition of definition 4.2.1 reduces to transversality, i.e. point iii) of such definition; which is included in the genericity condition for the given system $V_0, \ldots, V_n$, taking into account that $\mathcal{L}(\mathcal{A}_L) \subset \mathcal{L}(\mathcal{A})$.

For Φ^L, just remark that by the genericity of the given system, $V_i \cap L \neq \emptyset$ only for $i \geq k$, so by the genericity of the original system one gets the genericity of the restricted one $V_k \cap L, \ldots, V_n \cap L$. Now notice that if $(\theta'_0, \ldots, \theta'_{n-k})$ are the polar coordinates of a point P in L with respect to the system $V_k \cap L, \ldots, V_n \cap L$ and $(\theta_0, \ldots, \theta_n)$ are the polar coordinates of P with respect to $V_0, \ldots, V_n$, then there exist smooth functions:

$$\alpha_j : ([0, \epsilon))^{n-j+1} \to \mathbb{R}_{\geq 0} :\ (\theta_j, \ldots, \theta_n) \mapsto \theta'_j,\ j = k, \ldots, n,$$

with $\epsilon > \delta$ (δ as in 4.2.1) such that:

$$\theta'_i \;=\; \alpha_{i+k}(\theta_{i+k}, \ldots, \theta_n), \quad i = 0, \ldots, n-k$$

and the function:

$$\theta \;\longmapsto\; \alpha_{i+k}(\theta, \theta_{i+k+1}, ..., \theta_n)$$

is strictly increasing in $[0, \epsilon)$ ($i = 0, \ldots, n-k$). This shows the last assertion of the lemma, by definition of polar ordering (see section 4.2). □

Definition 6.1.2 *Given $G \in \Phi$, we set also $\lhd_G := \lhd_{|G|}$.*

Definition 6.1.3 *Given $G \in \Phi$, and being Φ_G as defined in section 2.1, we define an involution:*

$$\begin{aligned} op_G : \Phi_G &\;\rightarrow\; \Phi_G \\ F &\;\mapsto\; op_G(F) \end{aligned}$$

where $op_G(F)$ is the unique facet which is symmetric to F with respect to $supp(G)$. In other terms (using the maps φ_G, $pr_{|G|}$ defined in section 2.1):

$$op_G(F) \;:=\; \varphi_G^{-1}(-(pr_{|G|}(F))).$$

Here we notice that, for a central arrangement, every facet F has a unique opposite $-F$ with respect to the center.

Definition 6.1.4 *Define the* opposite *polar ordering $\lhd_G^{op}$ in Φ_G as:*

$$F \lhd_G^{op} F' \Leftrightarrow op_G(F) \lhd_G op_G(F')$$

$F, F' \in \Phi_G$.

Definition 6.1.5 *For all arrangements $\mathcal{A}$, all polar orderings $\lhd$ on $\mathcal{A}$, and all $d \geq 1$, we define the degree-d discrete field*

$$\Gamma_{(d)} := \Gamma_{(d)}(\mathcal{A}, \lhd)$$

on the complex $\mathbf{S}^{(d)}(\mathcal{A})$. Assume by recurrence that $\Gamma_{(d')}(\mathcal{A}^L)$ has been defined for $d' < d$, for any $\mathcal{A}^L \subset \mathcal{A}$, $L \in \mathcal{L}(\mathcal{A})$, for the induced polar ordering $\lhd^L$ (see lemma 6.1.1). Then the $k-$dimensional part $\Gamma_{(d)}^k(\mathcal{A})$ is given by the set of pairs of cells in $\mathbf{S}^{(d)}(\mathcal{A})$

$$(e(\mathcal{F}), e(\mathcal{F}'))$$

where $dim(e(\mathcal{F})) = k-1$, $dim(e(\mathcal{F}')) = k$, $\mathcal{F}' \prec \mathcal{F}$ (so $e(\mathcal{F}) \subset \partial(e(\mathcal{F}'))$), and the two flags differ only in a single position:

$$\mathcal{F} = (C, F_1, \ldots, F_{i-1}, F_i^j, F_{i+1}, \ldots, F_d),$$

$$\mathcal{F}' = (C, F_1, \ldots, F_{i-1}, F_i^{j+1}, F_{i+1}, \ldots, F_d)$$

with $F_i^j \prec F_i^{j+1}$ *(j, $j+1$ denote codimensions). Moreover, such pairs satisfy the following conditions (6.1),(6.2):*

$$\begin{array}{l} \textit{for } i < d,\ e(F_i^j, F_{i+1}, \ldots, F_d) \quad \textit{is a critical cell in the complex } \mathbf{S}^{(d-i)}(\mathcal{A}^L), \\ \quad \textit{endowed with the discrete } (d-i)- \textit{ vector field } \Gamma_{(d-i)}(\mathcal{A}^L), \\ \quad \textit{with } L := |F_i^j| \quad (\textit{i.e., } e(F_i^j, F_{i+1}, \ldots, F_d) \notin \Gamma_{(d-i)}(\mathcal{A}^L), \textit{ see} \\ \quad \textit{def. (2.3.5)}) \end{array} \tag{6.1}$$

Set $l = d - i$; then:

$$\begin{array}{l} \textit{for even } l \\ \quad F_i^{j+1} \lhd_{F_{i+1}} F_i^j \quad \textit{and} \quad F_i^j = max_{\lhd_{F_{i+1}}}\{F \mid F_{i-1} \prec F \prec F_i^j\}; \\ \\ \textit{for odd } l \\ \quad F_i^{j+1} \lhd^{op}_{F_{i+1}} F_i^j \quad \textit{and} \quad F_i^j = max_{\lhd^{op}_{F_{i+1}}}\{F \mid F_{i-1} \prec F \prec F_i^j\}. \end{array} \tag{6.2}$$

For $i = d$ there is no F_{i+1} in the first condition of (6.2), which is to be considered in this case as defined by using the given polar ordering $\lhd$.

We have $\Gamma_{(d)} = \oplus_{k=1}^{n'd} \Gamma_{(d)}^k$, $n' = rk(\mathcal{A})$.

Definition 6.1.6 *Let $L \in \mathcal{L}(\mathcal{A})$ be a codimension k subspace.*

Set $\Gamma_{(d)}^L$ as the degree-d discrete field of the arrangement $\mathcal{A}^L$ with respect to the polar ordering $\lhd^L = \lhd_{|L}$.

Set $\Gamma_{L,(d)}$ as the degree-d discrete field of the arrangement $\mathcal{A}_L \cap V^k$ with respect to the polar ordering $\lhd_L$ (see lemma (6.1.1)).

Theorem 16 *One has:*

1. *$\Gamma_{(d)}$ is a discrete vector field;*
2. *$\Gamma_{(d)}$ is a gradient field of a discrete Morse function;*
3. *the critical cells of $\Gamma_{(d)}$ are the following ones, depending on the parity of d:*

$$e(C, F^k, \ldots, F^k) \tag{6.3}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ critical cell for $(\Gamma_{(1)}, \lhd)$, if d is odd;

$$e(op_{F^k}(C), F^k, \ldots, F^k) \tag{6.4}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ critical cell for $(\Gamma_{(1)}, \lhd)$, if d is even.

The following theorem is an immediate consequence of theorem 16:

Theorem 17 *1. The configuration space $\mathcal{M}^{(d)}(\mathcal{A})$ is a minimal space ($d \geq 1$).*

2. The cohomology of $\mathbf{S}^{(d)}$ (or of $\mathcal{M}^{(d+1)}$), $d \geq 1$, is concentrated in dimension

$$id \ , \ i = 0 \dots n \ .$$

The Betti numbers are given by

$$B_{id}(\mathbf{S}^{(d)}) = B_i(\mathbf{S}^{(1)}).$$

Proof of theorem 17. Case $d = 1$ is considered in section 4.3. For $d > 1$ minimality follows immediately from the gap between the dimensions of the critical cells.

□

Notation 6.1.7 *Given $L \in \mathcal{L}(\mathcal{A})$, with $codim(L) = k$, notice that the cells of $\mathbf{S}^{(d-1)}(\mathcal{A}^L)$ have the form $e(\mathcal{F})$, where $\mathcal{F}$ is a flag whose first element is some facet F^k with $|F^k| = L$ (in fact, F^k is a chamber in the arrangement $\mathcal{A}^L$).*

Let

$$\lambda : \ \mathbf{S}^{(d-1)}(\mathcal{A}^L) \ \dashrightarrow \mathbf{S}^{(d)}(\mathcal{A})$$

be the correspondence defined by

$$\lambda(e(F^k, F_2, \dots, F_{d-1})) := \ \{e(C, F^k, F_2, \dots, F_{d-1}) | \ C \prec F^k\} \subset \mathbf{S}^{(d)}(\mathcal{A}).$$

When Γ' is a discrete field over $\mathbf{S}^{(d-1)}(\mathcal{A}^L)$, we have an induced field $\lambda_*(\Gamma')$ over $\mathbf{S}^{(d)}(\mathcal{A})$: for each pair

$$(e(F^k, F_2, \dots, F_{d-1}), e(F^k, F_2', \dots, F_{d-1}')) \in \ \Gamma',$$

take all pairs of the shape

$$(e(C, F^k, F_2, \dots, F_{d-1}), e(C, F^k, F_2', \dots, F_{d-1}')), \ \textit{for all chambers } C \prec F^k.$$

In particular, when L is the whole space V, λ gives a map which to a discrete field on $\mathbf{S}^{(d-1)}(\mathcal{A})$ associates a discrete field on $\mathbf{S}^{(d)}(\mathcal{A})$ (in this case $k = 0$ so one can add just one chamber $C = F^0$). Here $\lambda = j_{d-1} : \mathbf{S}^{(d-1)} \to \mathbf{S}^{(d)}$ defined at the end of section 3.3.

Proof of theorem 16. We proceed by double induction on the degree d and the rank of the arrangement $\mathcal{A}$.

The case $d = 1$, any rank, is exactly given by theorem 13, section 4.3.

The case of rank 0, any d, is trivial since here $\mathcal{A} = \emptyset$ and Φ has only one facet $C = V$; so, $\mathbf{S}^{(d)}$ is given by the unique point $e(C, \dots, C)$, and $\Gamma^{(d)} = \emptyset$ verifies the thesis.

We assume the theorem holds when the arrangement has rank lower than $rk(\mathcal{A})$ and arbitrary degree, or when the arrangement has the same rank as $\mathcal{A}$ but the degree d' is lower than d.

To prove 1), we partition $\Gamma_{(d)}$ into subsets, each of these being a discrete field; this will suffice to prove that $\Gamma_{(d)}$ is a field.

Consider first all cells in $\mathbf{S}^{(d)}$ with $codim(F_1) = 0$, that is of the kind

$$e(C, C, F_2, \dots, F_d). \tag{6.5}$$

This set bijectively corresponds to $\mathbf{S}^{(d-1)}$ by j_{d-1}.

By induction, $\Gamma_{(d-1)}$ is a discrete field, so $\lambda_*(\Gamma_{(d-1)}) \subset \Gamma_{(d)}$ is also a discrete field (see 6.1.7). The cells of $j_{d-1}(\mathbf{S}^{(d-1)}) \subset \mathbf{S}^{(d)}$ (of the shape (6.5)) which are not contained in $\lambda_*(\Gamma_{(d-1)})$ are exactly given by

$$j_{d-1}(Sing(\mathbf{S}^{(d-1)})) = \{j_{d-1}(e(\mathcal{F})) : \quad e(\mathcal{F}) \text{ critical cell of } \mathbf{S}^{(d-1)}\}. \tag{6.6}$$

By induction, the cells (6.6) will be of the following form:

$$\begin{gathered} e(C, C, F^h, \dots, F^h) \\ \text{with } e(C, F^h) \text{ critical for } (\Gamma_{(1)}, \lhd) \text{ if } d \text{ is even;} \end{gathered} \tag{6.7}$$

$$\begin{gathered} e(op_{F^h}(C), op_{F^h}(C), F^h, \dots, F^h) \\ \text{with } e(C, F^h) \text{ critical for } (\Gamma_{(1)}, \lhd) \text{ if } d \text{ is odd} \end{gathered} \tag{6.8}$$

(of course, d even implies $d-1$ odd and conversely).

Notice that the unique 0-cell in (6.6) is $e(C_0, \dots, C_0)$, where C_0 is the unique chamber containing the origin V_0 of the polar system. All the other $0-cells$ in $\mathbf{S}^{(d)}$ (all having analog shape $e(C, \dots, C)$, $C \neq C_0$) belong to the image of j_{d-1}.

It will be useful to give an equivalent condition for a cell $e(C, F^h) \in \mathbf{S}$ to be critical, which come easily from point 4) of theorem 13:

$$F^h \cap V^h \neq \emptyset \text{ and } pr_{|F^h|}(C) \cap V^{h-1} \neq \emptyset \text{ and bounded.} \tag{6.9}$$

Now let $L \in \mathcal{L}(\mathcal{A})$ be a codim-k subspace. Consider all cells in $\mathbf{S}^{(d)}$ of the form

$$\{e(C, F^k, F_2, \dots, F_d) \mid F^k \subset L\}. \tag{6.10}$$

By induction $\Gamma^L_{(d-1)}$ (6.1.6) is a discrete vector field. By the definition of $\Gamma_{(d)}$, one has that $\lambda_*(\Gamma^L_{(d-1)})$ is contained into $\Gamma_{(d)}$ and by 6.1.7 it is a discrete field.

The remaining cells of $\mathbf{S}^{(d)}$ of shape (6.10) which are not in $\lambda_*(\Gamma^L_{(d-1)})$ (i.e. the cells in $\lambda(Sing(\mathbf{S}^{(d-1)}(\mathcal{A}^L))))$ have the form

$$\begin{gathered} e(C, F^k, F^h, \ldots, F^h) \\ \text{with } e(F^k, F^h) \text{ critical cell for } (\Gamma^L_{(1)}, \lhd^L) \text{ for even } d; \end{gathered} \tag{6.11}$$

$$\begin{gathered} e(C, op_{F^h}(F^k), F^h, \ldots, F^h) \\ \text{with } e(F^k, F^h) \text{ critical cell for } (\Gamma^L_{(1)}, \lhd^L) \text{ for odd } d. \end{gathered} \tag{6.12}$$

In particular for $h = k$ they remain all cells $e(C, F^k_0, \ldots, F^k_0)$ for all $C \prec F^k_0$, where F^k_0 is the unique facet in L such that $F^k_0 \cap V^k \neq \emptyset$.

When $k < h$ condition (6.9) for critical cells translate as:

$$\begin{gathered} V^h \cap L \cap F^h = V^h \cap F^h \neq \emptyset \\ \text{and} \\ pr_{|F^h|}(F^k) \cap V^{h-1} \cap L = pr_{|F^h|}(F^k) \cap V^{h-1} \neq \emptyset \quad \text{and bounded.} \end{gathered} \tag{6.13}$$

Let $L',\ L'' \in \mathcal{L}(\mathcal{A})$. Remark that $\Gamma^{L'}_{(d-1)}$ and $\Gamma^{L''}_{(d-1)}$ have no common cell if $L' \neq L''$ (by definition, the first facet defining a cell has support respectively L', L''). When L is the whole space V, clearly $\Gamma_{(d-1)} = \Gamma^L_{(d-1)}$. If $L = L_0 = \cap\{H_i\}$ is the center of the arrangement, $\Gamma^L_{(d-1)} = \emptyset$ and the unique critical cell is $e_0 := e(L, \ldots, L)$. In this case $\lambda(e_0) = \{e(C, L, \ldots, L)|\ C \text{ any chamber}\}$.

Summarizing the previous discussion, we have by induction that

$$\Gamma'_{(d)} = \bigcup_{L \in \mathcal{L}(\mathcal{A})} (\lambda_*(\Gamma^L_{(d-1)}) \subset \Gamma_{(d)}$$

is a discrete field; the set $\mathcal{E}$ of cells of $\mathbf{S}^{(d)}$ which do not belong to $\Gamma'_{(d)}$ are given by

$$\begin{gathered} e(C, F^k, F^h, \ldots, F^h) \\ \text{with } F^h \cap V^h \neq \emptyset,\ pr_{|F^h|}(F^k) \cap V^{h-1} \neq \emptyset \text{ and bounded, even } d, \end{gathered} \tag{6.14}$$

$$\begin{gathered} e(C, op_{F^h}(F^k), F^h, \ldots, F^h) \\ \text{with } F^h \cap V^h \neq \emptyset,\ pr_{|F^h|}(F^k) \cap V^{h-1} \neq \emptyset \text{ and bounded, odd } d, \end{gathered} \tag{6.15}$$

when $k < h$; by

$$\begin{gathered} e(C, F^h, \ldots, F^h) \\ \text{with } F^h \cap V^h \neq \emptyset \end{gathered} \tag{6.16}$$

when $h = k$.

The proof of parts 1) and 3) of theorem 16 will follow from the following lemma.

Lemma 6.1.8 *Let $k < h$. For each cell $e(C, F^k, F^h, \ldots, F^h) \in \mathcal{E}$, there exists either*

$$F^{k+1} \text{ such that } F^k \prec F^{k+1} \prec F^h \text{ (if } k+1 = h \text{ then } F^{k+1} = F^h\text{)} \tag{6.17}$$

or

$$F^{k-1} \text{ such that } C \prec F^{k-1} \prec F^k \text{ (if } k = 1 \text{ then } F^{k-1} = C\text{)} \tag{6.18}$$

such that $e(C, F^{k+1}, F^h, \ldots, F^h) \in \mathcal{E}$ (resp. $e(C, F^{k-1}, F^h, \ldots, F^h) \in \mathcal{E}$) and the pair $(e(C, F^k), e(C, F^{k+1}))$ (resp. $(e(C, F^{k-1}), e(C, F^k))$) belongs to the field $\Gamma_{L,(1)}$ which is defined by using the ordering

$$\lhd_L^{op} \quad , \text{ for even } d;$$

or

$$\lhd_L \quad , \text{ for odd } d.$$

($L := |F^h|$).

Case $k = h$. For each cell $e(C, F^k, \ldots, F^k) \in \mathcal{E}$, there exists F^{k-1} verifying (6.18) and such that $e(C, F^{k-1}, F^k, \ldots, F^k)$ belongs to $\mathcal{E}$ and the pair $(e(C, F^{k-1}), e(C, F^k))$ belongs to the field $(\Gamma_{L,(1)}, \lhd_L^{op})$ for d even, respectively to $(\Gamma_{L,(1)}, \lhd_L)$ for d odd, ($L = |F^k|$) iff $e(C, F^k)$ is non critical for $(\Gamma_{L,(1)}, \lhd_L^{op})$ for d even, respectively for $(\Gamma_{L,(1)}, \lhd_L)$ for d odd.

Proof of lemma 6.1.8. We show first that $e(C, F^k)$ is not critical for $\Gamma_{L,(1)}$ (with the suitable ordering defined above).

Assume d even. Condition (6.14), giving $pr_{|F^h|}(F^k) \cap V^{h-1} \neq \emptyset$ and bounded, is equivalent to

$$pr_{|F^h|}(op_{F^h}(F^k)) \cap V^{h-1} = \emptyset. \tag{6.19}$$

If $e(C, F^k)$ were critical for $(\Gamma_{L,(1)}, \lhd_L^{op})$ then $pr_{|F^h|}(op_{F^h}(F^k)) \cap V^k \neq \emptyset$. Since $V^k \subset V^{h-1}$ this is impossible.

If d is odd, condition (6.15), giving $pr_{|F^h|}(op_{F^h}(F^k)) \cap V^{h-1} \neq \emptyset$ and bounded, is equivalent to $pr_{|F^h|}(F^k) \cap V^{h-1} = \emptyset$. If $e(C, F^k)$ were critical for $(\Gamma_{L,(1)}, \lhd_L)$ then $pr_{|F^h|}(F^k) \cap V^k \neq \emptyset$, and this also is impossible.

Assume that there exists F^{k+1} as in (6.17) such that the pair

$$(e(C, F^k), e(C, F^{k+1})) \in \Gamma_{L,(1)}.$$

We have to show that $e(C, F^{k+1}, F^h, \ldots, F^h) \in \mathcal{E}$.

In case $k + 1 < h$, if condition (6.14) holds for F^k (d even) then it clearly holds for all facets F^{k+1} in the boundary of F^k and such that $F^{k+1} \prec F^h$, and analog for condition (6.15).

When $k + 1 = h$, then by (6.16) all cells $e(C, F^h, \ldots, F^h)$ with $F^h \cap V^h \neq \emptyset$ belong to $\mathcal{E}$.

Now let us suppose there exists F^{k-1} as in (6.18) such that

$$(e(C, F^{k-1}), e(C, F^k)) \in \Gamma_{L,(1)}$$

and we have to show that $e(C, F^{k-1}, F^h, \dots, F^h) \in \mathcal{E}$.

Let d be even. By the definition of $(\Gamma_{L,(1)}, \lhd_L^{op})$ (section 4.3) one must have $F^k \lhd_L^{op} F^{k-1}$, that is $op_{F^h}(F^{k-1}) \;\; \lhd_L \;\; op_{F^h}(F^k)$. Since for F^k condition (6.19) holds, we must have (by definition of polar ordering) also

$$pr_{|F^h|}(op_{F^h}(F^{k-1})) \cap V^{h-1} = \emptyset$$

which is equivalent (as said above) to (6.14), so it gives the thesis in this case.

The case d odd is proved in the same way, so as the last assertion of the theorem for case $k = h$. $\square$

By lemma 6.1.8 it follows both parts 1) and 3) of theorem 16. Indeed, the pairs of cells of $\Gamma_{(d)}$ coming from lemma 6.1.8 clearly do not have common cells and, together with the previous field $\Gamma'_{(d)}$, exhaust all $\Gamma_{(d)}$. This proves part 1). Part 3) follows from the last part of lemma 6.1.8.

We now come to part 2) of the theorem. We use the characterizing property for gradient fields stated in theorem 7 of section 2.3.

Take a $\Gamma_{(d)}-$path in $\mathbf{S}^{(d)}$:

$$e(C_1, \mathcal{F}_1^k), e(C_1, \mathcal{F}_1^{k+1}), \dots, e(C_m, \mathcal{F}_m^k), e(C_m, \mathcal{F}_m^{k+1}), e(C_{m+1}, \mathcal{F}_{m+1}^k) \tag{6.20}$$

where by $\mathcal{F}_i^k$ we indicate the flag $(F_{i,1}^{j_{i,1}}, \dots, F_{i,d}^{j_{i,d}})$, whose codimension is $k = \sum_{l=1}^d j_{i,l}$, and by $\mathcal{F}_i^{k+1}$ we indicate the flag $(F_{i,1}^{j'_{i,1}}, \dots, F_{i,d}^{j'_{i,d}})$, whose codimension is $k+1 = \sum_{l=1}^d j'_{i,l}$ $(i = 1, \dots, m)$. Here the pair $(e(C_i, \mathcal{F}_i^k), e(C_i, \mathcal{F}_i^{k+1}))$ is an element of $\Gamma_{(d)}$, and $e(C_i, \mathcal{F}_i^k)$ is in the boundary of $e(C_{i-1}, \mathcal{F}_{i-1}^{k+1})$.

We have to prove that if the path (6.20) is closed (i.e. if $e(C_{m+1}, \mathcal{F}_{m+1}^k)$ equals to $e(C_1, \mathcal{F}_1^k)$), then such path is trivial, i.e. $\mathcal{F}_i^k = \mathcal{F}_{i+1}^k$, $\mathcal{F}_i^{k+1} = \mathcal{F}_{i+1}^{k+1}$, and $C_i = C_{i+1}$ $(i = 1, \dots, m-1)$.

Given a polar ordering $\blacktriangleleft$ over an arrangement $\mathcal{A}'$, introduce the ordering $\blacktriangleleft_{lex}$ among the cells of $\mathbf{S}^{(d)}(\mathcal{A}')$:

$$e(F_0, F_1, \dots, F_d) \blacktriangleleft_{lex} e(F'_0, F'_1, \dots, F'_d)$$

$(codim(F_0) = codim(F'_0) = 0)$ iff, being $\underline{k}$ the last position where $F_k \neq F'_k$, one has

$$F_k \blacktriangleleft_{\underline{k}} F'_k,$$

where $\blacktriangleleft_{\underline{k}}$ equals either $\blacktriangleleft_{|F_{k+1}|}$ or $\blacktriangleleft_{|F_{k+1}|}^{op}$ according to the parity of $d - \underline{k}$.

We will prove the following claim:

Claim 3 *Given a triple of consecutive cells in (6.20) of the form:*

$$e(C_i, \mathcal{F}_i^{k+1}), e(C_{i+1}, \mathcal{F}_{i+1}^k), e(C_{i+1}, \mathcal{F}_{i+1}^{k+1}). \tag{6.21}$$

we have that $\mathcal{F}_{i+1}^{k+1} \unlhd_{lex} \mathcal{F}_i^{k+1}$.

We prove this claim inductively on d. The base of the induction is the case $d = 1$, i.e. claim 1 of section 4.3.

We now suppose the statement true for $d-1$ and prove it for d.

By splitting a flag $\mathcal{F} = (\mathcal{G}, F_d)$ into a $(d-1)-$flag $\mathcal{G}$ and the last facet F_d, the triple (6.21) will become:

$$e(C_i, \mathcal{G}_i^{h_i'}, F_{i,d}^{l_i'}), e(C_{i+1}, \mathcal{G}_{i+1}^{h_{i+1}}, F_{i+1,d}^{l_{i+1}}), e(C_{i+1}, \mathcal{G}_{i+1}^{h_{i+1}'}, F_{i+1,d}^{l_{i+1}'}).$$

Now we have to distinguish several cases.

1. If $l_i' = l_{i+1} = l_{i+1}' = l$, then $F_{i,d}^l \prec F_{i+1,d}^l$ and so $F_{i,d}^l = F_{i+1,d}^l$ for an argument of dimension. We have also that $h_i' - 1 = h_{i+1} = h_{i+1}' - 1 = h$, so we can rewrite the triple as:

$$e(C_i, \mathcal{G}_i^{h+1}, F^l), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^l), e(C_{i+1}, \mathcal{G}_{i+1}^{h+1}, F^l).$$

   By inductive hypothesis we have that $\mathcal{G}_{i+1}^{h+1} (\lhd_{F^l}^{op})_{lex} \mathcal{G}_i^{h+1}$, and so also $\mathcal{F}_{i+1}^{k+1} \lhd_{lex} \mathcal{F}_i^{k+1}$.

2. If $l_i' - 1 = l_{i+1} = l_{i+1}' - 1 = l$, so $h_i' = h_{i+1} = h_{i+1}' = h$ and we can write the triple as:

$$e(C_i, \mathcal{G}_i^h, F_{i,d}^{l+1}), e(C_{i+1}, \mathcal{G}_{i+1}^h, F_{i+1,d}^l), e(C_{i+1}, \mathcal{G}_{i+1}^h, F_{i+1,d}^{l+1}).$$

   This case follows from claim 1 applied to

$$e(F_{i,d-1}, F_{i,d}^{l+1}), e(F_{i+1,d-1}, F_{i+1,d}^l), e(F_{i+1,d-1}, F_{i+1,d}^{l+1}).$$

3. If $l_i' - 1 = l_{i+1} = l_{i+1}' = l$, so $h_i' = h_{i+1} = h_{i+1}' - 1 = h$ and we can write the triple as:

$$e(C_i, \mathcal{G}_i^h, F_{i,d}^{l+1}), e(C_{i+1}, \mathcal{G}_{i+1}^h, F_{i+1,d}^l), e(C_{i+1}, \mathcal{G}_{i+1}^{h+1}, F_{i+1,d}^l).$$

   Here $(e(C_{i+1}, \mathcal{G}_{i+1}^h, F_{i+1,d}^l), e(C_{i+1}, \mathcal{G}_{i+1}^{h+1}, F_{i+1,d}^l)) \in \Gamma_{(d)}$, so, by condition (6.1), there exists an index $j \geq 1$ so that $e(F_{i+1,d-j}, F_{i+1,d-j+1}, \ldots, F_{i+1,d}^l)$ is critical for $\Gamma_{(j)}^L$, where $L = \mid F_{i+1,d-j} \mid$; we have in particular $F_{i+1,d}^l \cap V^l \neq \emptyset$. The definition of polar ordering gives $F_{i+1,d}^l \lhd F_{i,d}^{l+1}$, so $\mathcal{F}_{i+1}^{k+1} \lhd_{lex} \mathcal{F}_i^{k+1}$ as required.

4. If $l_i' = l_{i+1} = l_{i+1}' - 1 = l$, so $F_{i,d}^l = F_{i+1,d}^l$ as above, and $h_i' - 1 = h_{i+1} = h_{i+1}' = h$, so we can write the triple as:

$$e(C_i, \mathcal{G}_i^{h+1}, F^l), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^l), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^{l+1}).$$

   Since $(e(C_{i+1}, \mathcal{G}_{i+1}^h, F^l), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^{l+1}) \in \Gamma_{(d)})$ then condition (6.2) gives $F^{l+1} \lhd F^l$ and $\mathcal{F}_{i+1}^{k+1} \lhd_{lex} \mathcal{F}_i^{k+1}$ as required.

This proves the claim. If the path (6.20) is closed, it follows that all the flags of codimension $k+1$ equal a fixed flag $\mathcal{F}^{k+1}$.

We can rewrite (6.20) as follows:

$$e(C_1, \mathcal{F}_1^k), e(C_1, \mathcal{F}^{k+1}), \ldots, e(C_m, \mathcal{F}_m^k), e(C_m, \mathcal{F}^{k+1}), e(C_1, \mathcal{F}_1^k). \tag{6.22}$$

So $\mathcal{F}_i^k \prec \mathcal{F}^{k+1}, \ i = 1, \ldots, m$.

We will now prove another claim:

Claim 4 *Given a quadruple of consecutive cells in (6.22) of the form:*

$$e(C_i, \mathcal{F}_i^k), e(C_i, \mathcal{F}^{k+1}), e(C_{i+1}, \mathcal{F}_{i+1}^k), e(C_{i+1}, \mathcal{F}^{k+1}). \tag{6.23}$$

we have $\mathcal{F}_i^k \trianglelefteq_{lex} \mathcal{F}_{i+1}^k$.

We prove this claim inductively on d. The base of the induction is the case $d = 1$, i.e. claim 2 of section 4.3.

We now suppose the statement true for $d-1$ and prove it for d.

As before, we can rewrite the quadruple as:

$$e(C_i, \mathcal{G}_i^{h_i}, F_{i,d}^{l_i}), e(C_i, \mathcal{G}^{h+1}, F^{l+1}), e(C_{i+1}, \mathcal{G}_{i+1}^{h_{i+1}}, F_{i+1,d}^{l_{i+1}}), e(C_{i+1}, \mathcal{G}^{h+1}, F^{l+1}).$$

Let us distinguish several cases:

1. if $l_i = l_{i+1} = l+1$ then, by definition of $\Gamma_{(d)}$, it follows $F_{i,d}^{l+1} = F^{l+1} = F_{i+1,d}^{l+1}$ and $h_i = h_{i+1} = h$, so the quadruple becomes:

   $$e(C_i, \mathcal{G}_i^h, F^{l+1}), e(C_i, \mathcal{G}^{h+1}, F^{l+1}), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^{l+1}), e(C_{i+1}, \mathcal{G}^{h+1}, F^{l+1}).$$

   By inductive hypothesis we have $\mathcal{G}_i^h (\triangleleft_{F^{l+1}}^{op})_{lex} \mathcal{G}_{i+1}^h$, and consequently also $\mathcal{F}_i^k \triangleleft_{lex} \mathcal{F}_{i+1}^k$.

2. If $l_i = l_{i+1} = l$ then $h_i = h_{i+1} = h+1$ and the quadruple become:

   $$e(C_i, \mathcal{G}^{h+1}, F_{i,d}^l), e(C_i, \mathcal{G}^{h+1}, F^{l+1}), e(C_{i+1}, \mathcal{G}^{h+1}, F_{i+1,d}^l),$$
   $$, e(C_{i+1}, \mathcal{G}^{h+1}, F^{l+1})$$

   (the fact that all the $\mathcal{G}$'s coincide here follows directly from the definition of $\Gamma_{(d)}$ applied to both pairs of the field in (6.23)).

   By claim 2, applied to the quadruple

   $$e(G, F_{i,d}^l), e(G, F^{l+1}), e(G, F_{i+1,d}^l), e(G, F^{l+1}),$$

   where G is the last facet of $\mathcal{G}$, it follows $F_{i,d}^l \trianglelefteq F_{i+1,d}^l$, which concludes this case.

3. if $l_i = l_{i+1} + 1 = l + 1$ then $F_{i,d}^{l+1} = F^{l+1}$, moreover $h_i + 1 = h_{i+1} = h + 1$. So the quadruple become:

$$e(C_i, \mathcal{G}_i^h, F^{l+1}), e(C_i, \mathcal{G}^{h+1}, F^{l+1}), e(C_{i+1}, \mathcal{G}^{h+1}, F_{i+1}^l), e(C_{i+1}, \mathcal{G}^{h+1}, F^{l+1}).$$

   By condition (6.2) we have $F^{l+1} \lhd F_{i+1}^l$, so $\mathcal{F}_i^k \lhd_{lex} \mathcal{F}_{i+1}^k$ as required.

4. if $l_i + 1 = l_{i+1} = l + 1$ then $F_{i+1,d}^{l+1} = F^{l+1}$, moreover $h_i = h_{i+1} + 1 = h + 1$. So the quadruple become:

$$e(C_i, \mathcal{G}^{h+1}, F_{i,d}^l), e(C_i, \mathcal{G}^{h+1}, F^{l+1}), e(C_{i+1}, \mathcal{G}_{i+1}^h, F^{l+1}), e(C_{i+1}, \mathcal{G}^{h+1}, F^{l+1}).$$

   It follows from condition (6.1) that there exists an index j such that $e(F_{i+1,d-j}^p, F_{i+1,d-j+1} = F^{l+1}, \ldots, F^{l+1})$ is critical for $\Gamma_{(j)}^L$, with $L = |F_{i+1,d-j}^p|$. So $F^{l+1} = max_{\lhd}\{F|\ F_{i+1,d-j} \prec F \prec F^{l+1}\}$.

   We have

$$\mathcal{G}_{i+1}^h = (F_{i+1,1}, \ldots, F_{i+1,j-1}, F_{i+1,j}^p, F_{i+1,j+1}, \ldots, F_{i+1,d-1}),$$

$$\mathcal{G}^{h+1} = (F_{i+1,1}, \ldots, F_{i+1,j-1}, F_{i+1,j}^{p+1}, F_{i+1,j+1}, \ldots, F_{i+1,d-1}).$$

   Then we have also $F_{i+1,j}^p \prec F_{i+1,j}^{p+1} \prec F_{i,d}^l \prec F^{l+1}$, and it must be $F_{i,d}^l \lhd F^{l+1}$.

   This is not possible since condition (6.2) gives $F^{l+1} \lhd F_{i,d}^l$.

Then all $\mathcal{F}_i^k$ coincide.

It remains to see that all the chambers C_i's coincide. We will show that $C_i = C_{i+1}$ for $i = 1 \ldots m-1$.

In (6.22) we can take a triple of consecutive cells of the form:

$$e(C_i, \mathcal{F}^k), e(C_i, \mathcal{F}^{k+1}), e(C_{i+1}, \mathcal{F}^k)$$

where the pair $(e(C_i, \mathcal{F}^k), e(C_i, \mathcal{F}^{k+1}))$ is in $\Gamma_{(d)}$ and $e(C_{i+1}, \mathcal{F}^k)$ is in the boundary of $e(C_i, \mathcal{F}^{k+1})$. If we write $\mathcal{F}^k = (F_1, \mathcal{F}')$ then this triple become:

$$e(C_i, F_1, \mathcal{F}'), e(C_i, \mathcal{F}^{k+1}), e(C_{i+1}, F_1, \mathcal{F}').$$

By condition of boundary we have $pr_{|F_1|}(C_{i+1}) \prec pr_{|F_1|}(C_i)$ that, by an argument of dimension, become $pr_{|F_1|}(C_{i+1}) = pr_{|F_1|}(C_i)$. Moreover $C_i \prec F_1$ and $C_{i+1} \prec F_1$. So they must be equals.

This completes the proof.

$\square$

Remark 6.1.9 *The* $d-$polar basis *which we find here is new, being different from* $\mathbb{Z}$*-bases which are known for example in the case of braid arrangement. As far as we know, a precise description of a* $\mathbb{Z}$*-basis is not written anywhere for other arrangements.*

Remark 6.1.10 *Regarding the case of* Coxeter arrangements, *related to a Coxeter group* W*, (see section 2.1.1)a knowledge of a minimal basis for the cohomology of* $\mathcal{M}^{(d)}$ *should also be useful in the study of the cohomology of the orbit space* $\mathcal{M}^{(d)}/W$*.*

6.2 Examples

Here we will present some examples illustrating the previous theory.

In section 4.4 we described, for some particular examples, the reduction of the complex $\mathbf{S}^{(1)}$ into a minimal complex by using the gradient vector field $\Gamma = \Gamma_{(1)}$. Here we consider again these examples and show how one can reduce the complex $\mathbf{S}^{(d)}$, for $d = 2, 3$ (see example 11), into a minimal complex, by using the gradient vector field $\Gamma_{(d)}$.

Example 17 *Consider the arrangement* $\mathcal{A}$ *of example 14 (see figure 6.1).*

Case $d = 2$*.*

Recall that the complex $\mathbf{S}^{(2)}$ *has 108 cells. The polar gradient* $\Gamma_{(2)}$ *has components:* $\Gamma_{(2)} = \Gamma^1_{(2)} \sqcup \Gamma^2_{(2)} \sqcup \Gamma^3_{(2)} \sqcup \Gamma^4_{(2)}$*. Following definition 6.1.5 we can see that:*

- $\Gamma^1_{(2)}$ *is composed by 11 pairs of type* $(e(C_i, C_i, C_i), e(C_i, C_i, F_j))$*, with* $F_j \lhd C_i$*, i.e.* $(e(C_i, C_i), e(C_i, F_j)) \in \Gamma^1_{(1)}$*;*
- $\Gamma^2_{(2)}$ *is composed by two types of pairs: 7 pairs of type* $(e(C_i, C_i, F_j), e(C_i, C_i, P))$*, with* $P \lhd F_j$ *and* $e(C_i, C_i, F_j)$ *does not belong to any pair of* $\Gamma^1_{(2)}$*, i.e.* $(e(C_i, F_j), e(C_i, P)) \in \Gamma^2_{(1)}$*; 6 pairs of type* $(e(C_i, C_i, F_j), e(C_i, F_j, F_j))$ *with* $e(C_i, F_j)$ *critical for* $\Gamma_{(1)}$*;*
- $\Gamma^3_{(2)}$ *is composed by two types of pairs: 12 pairs of type* $(e(C_i, F_j, F_j), e(C_i, F_j, P))$*, with* $P \lhd F_j$*; 5 pairs of type* $(e(C_i, C_i, P), e(C_i, F_j, P))$ *with* $F_j \lhd^{op}_P C_i$*, i.e.* $C_i \lhd F_j$*, and* $e(C_i, P)$ *is critical for* $\Gamma_{(1)}$*, i.e.* $C_i \cap V_1 \neq \emptyset$ *and bounded;*
- $\Gamma^4_{(2)}$ *is composed by 7 pairs of type* $(e(C_i, F_j, P), e(C_i, P, P))$ *with* $P \lhd^{op}_P F_j$*, and* $e(C_i, F_j, P)$ *is not in a pair of* $\Gamma^3_{(2)}$*.*

It follows that the set of critical cells $Sing(\mathbf{S}^{(2)})$ *is composed by the cells not appearing in any pair of* $\Gamma_{(2)}$*, i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0, C_0)$*;*
- *6 critical 2-cells* $e(C_1, F_1, F_1)$*,* $e(C_2, F_2, F_2)$*,* $e(C_3, F_3, F_3)$*,* $e(C_4, F_4, F_4)$*,* $e(C_5, F_5, F_5)$*,* $e(C_6, F_6, F_6)$*;*
- *5 critical 4-cells* $e(C_7, P, P)$*,* $e(C_8, P, P)$*,* $e(C_9, P, P)$*,* $e(C_{10}, P, P)$*,* $e(C_{11}, P, P)$*.*

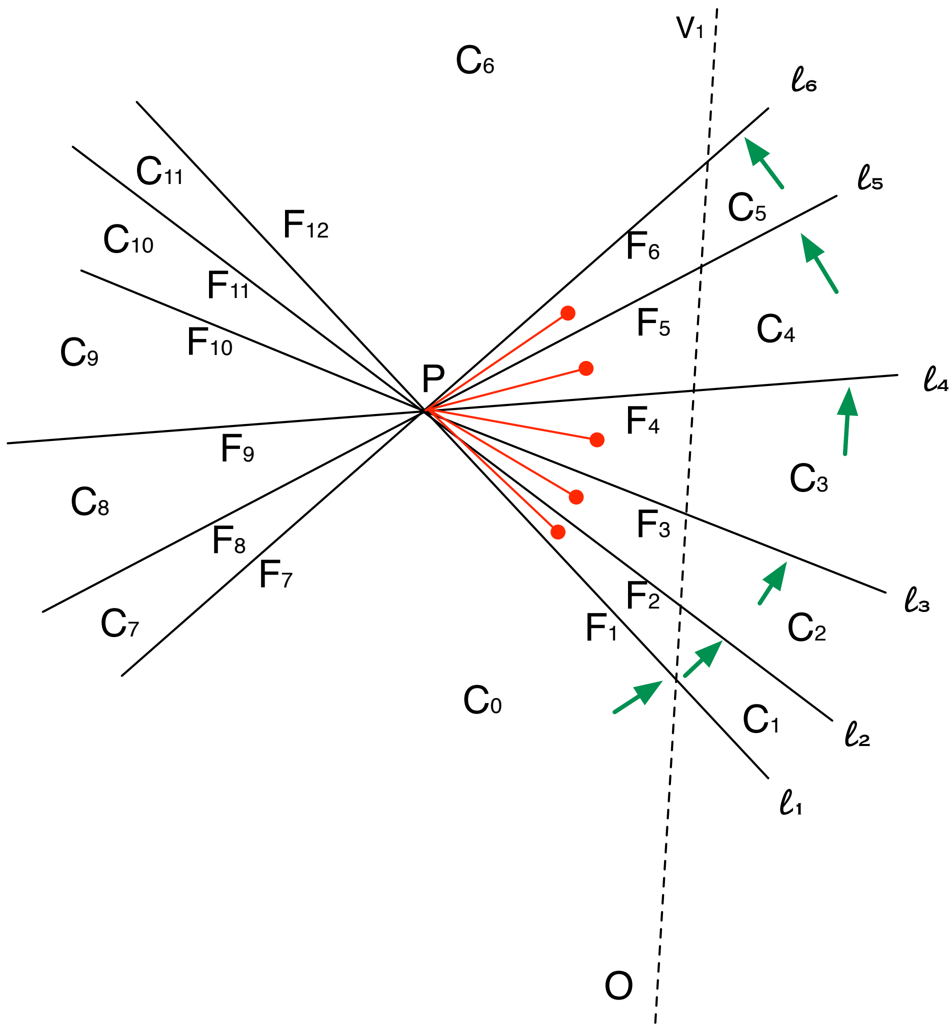


Figure 6.1: The central case with six line. Here we indicated the critical cells for the case d odd

Remark 6.2.1 *Here $d = 2$ is even, so the set of critical cell is composed by:*

$$\{e(op_{F^k}(C), F^k, F^k)\}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ critical cell for $(\Gamma_{(1)}, \lhd)$, according to theorem 16.

Remark 6.2.2 *Observe that the number of all cells of $\mathbf{S}^{(2)}$ is equal to the number of critical cells plus twice the number of the pairs of Γ, indeed $1+6+5+2*(11+7+6+12+5+7) = 108$.*

Remark 6.2.3 *The previous example can be easily generalized to the case of a central line arrangement of n lines. In this case the critical cells are given by:*

- *1 critical 0-cell $e(C_0, C_0, C_0)$;*
- *n critical 1-cells $e(C_1, F_1, F_1), e(C_2, F_2, F_2) \ldots e(C_n, F_n, F_n)$;*
- *$n - 1$ critical 2-cells $e(C_{n+1}, P, P), e(C_{n+2}, P, P) \ldots e(C_{2n-1}, P, P)$.*

Case $d = 3$.

Recall that the complex $\mathbf{S}^{(3)}$ has 192 cells. The polar gradient $\Gamma_{(3)}$ has components: $\Gamma_{(3)} = \Gamma^1_{(3)} \sqcup \Gamma^2_{(3)} \sqcup \Gamma^3_{(3)} \sqcup \Gamma^4_{(3)} \sqcup \Gamma^5_{(3)} \sqcup \Gamma^6_{(3)}$. Following definition 6.1.5 we can see that:

- *$\Gamma^1_{(3)}$ is composed by 11 pairs of type $(e(C_i, C_i, C_i, C_i), e(C_i, C_i, C_i, F_j))$, with $F_j \lhd C_i$, i.e. $(e(C_i, C_i, C_i), e(C_i, C_i, F_j)) \in \Gamma^1_{(2)}$;*
- *$\Gamma^2_{(3)}$ is composed by two types of pairs: 7 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, C_i, P))$, with $P \lhd F_j$ and $e(C_i, C_i, F_j)$ does not belong to any pair of $\Gamma_{(1)}$, i.e. $(e(C_i, C_i, F_j), e(C_i, C_i, P)) \in \Gamma^2_{(2)}$; 6 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, F_j, F_j))$ with $e(C_i, F_j)$ critical for $(\Gamma_{(1)})$, i.e. $(e(C_i, C_i, F_j), e(C_i, F_j, F_j)) \in \Gamma^2_{(2)}$;*
- *$\Gamma^3_{(3)}$ is composed by three types of pairs: 6 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, F_j, F_j, F_j))$, with $e(C_i, F_j, F_j)$ critical for $\Gamma_{(2)}$; 12 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, C_i, F_j, P))$ with $P \lhd F_j$, i.e. $(e(C_i, F_j, F_j), e(C_i, F_j, P)) \in \Gamma^3_{(2)}$; 5 pairs of type $(e(C_i, C_i, C_i, P), e(C_i, C_i, F_j, P))$ with $F_j \lhd^{op}_P C_i$, i.e. $C_i \lhd F_j$, and $e(C_i, P)$ is critical for $(\Gamma^1_{(1)})$, i.e.$(e(C_i, C_i, P), e(C_i, F_j, P)) \in \Gamma^3_{(2)}$;*
- *$\Gamma^4_{(3)}$ is composed by two types of pairs: 7 pairs of type $(e(C_i, C_i, F_j, P), e(C_i, C_i, P, P))$, with $(e(C_i, F_j, P), e(C_i, P, P)) \in \Gamma^4_{(2)}$; 12 pairs of type $(e(C_i, F_j, F_j, F_j), e(C_i, F_j, F_j, P))$ with $P \lhd F_j$. Notice that in $\Gamma_{(3)}$ there are no pair of type $(e(C_i, C_i, F_j, P), e(C_i, F_j, F_j, P))$ because no cell of the form $e(C_i, F_j, P)$ is critical for $\Gamma_{(2)}$;*
- *$\Gamma^5_{(3)}$ is composed by two types of pairs: 5 pairs of type $(e(C_i, C_i, P, P), e(C_i, F_j, P, P))$ with $e(C_i, P, P)$ critical for $\Gamma_{(2)}$, i.e. $C_i \cap V_1 = \emptyset$, and $F_j \lhd C_i$; 12 pairs of type $(e(C_i, F_j, F_j, P), e(C_i, F_j, P, P))$ with $P \lhd^{op}_P F_j$, i.e. $F_j \lhd P$;*

- $\Gamma^6_{(3)}$ *is composed by 7 pairs of type* $(e(C_i, F_j, P, P), e(C_i, P, P, P))$ *with* $P \lhd F_j$*, and* $e(C_i, F_j, P, P)$ *is not in a pair of* $\Gamma^5_{(3)}$.

So the set of critical cells $Sing(\mathbf{S}^{(3)})$ *is composed by the cells not compairing in any pair of* $\Gamma_{(3)}$*, i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0, C_0, C_0)$*;*

- *6 critical 3-cells* $e(C_0, F_1, F_1, F_1), e(C_1, F_2, F_2, F_2), e(C_2, F_3, F_3, F_3), e(C_3, F_4, F_4, F_4),$ $e(C_4, F_5, F_5, F_5), e(C_5, F_6, F_6, F_6)$*;*

- *5 critical 6-cells* $e(C_1, P, P, P), e(C_2, P, P, P), e(C_3, P, P, P), e(C_4, P, P, P),$ $e(C_5, P, P, P)$.

Remark 6.2.4 *Observe that the number of all cells of* $\mathbf{S}^{(3)}$ *is equal to the number of critical cells plus twice the number of the pairs of* Γ*, indeed* $1 + 6 + 5 + 2 * (11 + 7 + 6 + 6 + 12 + 5 + 7 + 12 + 5 + 12 + 7) = 192$.

Remark 6.2.5 *Here* $d = 3$ *is odd, and so the set of critical cell is composed by:*

$$\{e(C, F^k, F^k, F^k)\}$$

with $e(C, F^k) \in \mathbf{S}^{(1)}$ *critical cell for* $(\Gamma_{(1)}, \lhd)$*, according to theorem 16.*

Remark 6.2.6 *The previous example can be easily generalized to the case of a central line arrangement of* n *lines.*

Example 18 *Here we consider the arrangement* $\mathcal{A}$ *of example 15 (see figure 6.2).*

Case $d = 2$.

Recall that the complex $\mathbf{S}^{(2)}$ *has 152 cells.*

The polar gradient $\Gamma_{(2)}$ *has components:* $\Gamma_{(2)} = \Gamma^1_{(2)} \sqcup \Gamma^2_{(2)} \sqcup \Gamma^3_{(2)} \sqcup \Gamma^4_{(2)}$*. Following definition 6.1.5 we can see that:*

- $\Gamma^1_{(2)}$ *is composed by 11 pairs of type* $(e(C_i, C_i, C_i), e(C_i, C_i, F_j))$*, with* $F_j \lhd C_i$*, i.e.* $(e(C_i, C_i), e(C_i, F_j)) \in \Gamma^1_{(1)}$*;*

- $\Gamma^2_{(2)}$ *is composed by two types of pairs: 14 pairs of type* $(e(C_i, C_i, F_j), e(C_i, C_i, P_k))$*, with* $P_k \lhd F_j$ *and* $e(C_i, C_i, F_j)$ *does not belong to any pair of* $\Gamma_{(1)}$*, i.e.* $(e(C_i, F_j), e(C_i, P_k)) \in \Gamma^2_{(1)}$*; 5 pairs of type* $(e(C_i, C_i, F_j), e(C_i, F_j, F_j))$ *with* $e(C_i, F_j)$ *critical for* $\Gamma_{(1)}$*;*

- $\Gamma^3_{(2)}$ *is composed by two types of pairs: 20 pairs of type* $(e(C_i, F_j, F_j), e(C_i, F_j, P_k))$*, with* $P_k \lhd F_j$*; 6 pairs of type* $(e(C_i, C_i, P_k), e(C_i, F_j, P_k))$ *with* $F_j \lhd^{op}_{P_k} C_i$*, i.e.* $C_i \lhd F_j$*, and* $e(C_i, P_k)$ *is critical for* $\Gamma^1_{(1)}$*;*

- $\Gamma^4_{(2)}$ *is composed by 14 pairs of type* $(e(C_i, F_j, P_k), e(C_i, P_k, P_k))$ *with* $F_j \lhd P_k$*, and* $e(C_i, F_j, P_k)$ *is not in a pair of* $\Gamma^3_{(2)}$.

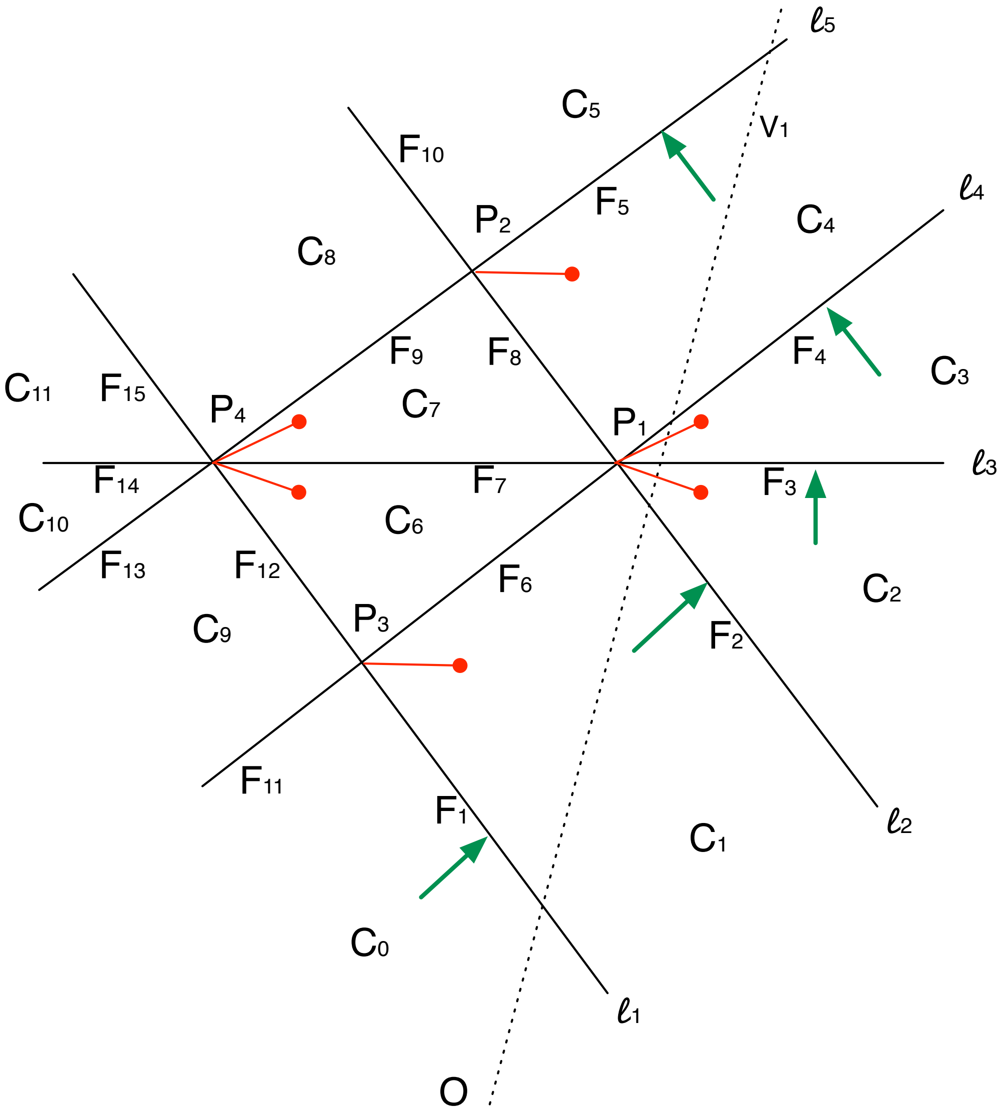


Figure 6.2: Deconing A_3 and critical cells for the case d odd.

So the set of critical cells $Sing(\mathbf{S})$ is given by all cells which do not appear in any pair of Γ, i.e. by the following cells:

- *1 critical 0-cell* $e(C_0, C_0, C_0)$;
- *5 critical 1-cells* $e(C_1, F_1, F_1), e(C_2, F_2, F_2), e(C_3, F_3, F_3), e(C_4, F_4, F_4), e(C_5, F_5, F_5)$;
- *6 critical 2-cells* $e(C_6, P_1, P_1), e(C_7, P_1, P_1), e(C_8, P_2, P_2), e(C_9, P_3, P_3), e(C_{10}, P_4, P_4), e(C_{11}, P_4, P_4)$.

Remark 6.2.7 *As in previous example we have that the number of all cells of $\mathbf{S}^{(2)}$ is equal to the number of critical cells plus twice the number of the pairs of Γ, indeed* $1+5+6+2*(11+14+5+20+6+14) = 152$.

Case $d = 3$.

Recall that the complex $\mathbf{S}^{(3)}$ has 282 cells.

The polar gradient $\Gamma_{(3)}$ has components: $\Gamma_{(3)} = \Gamma^1_{(3)} \sqcup \Gamma^2_{(3)} \sqcup \Gamma^3_{(3)} \sqcup \Gamma^4_{(3)} \sqcup \Gamma^5_{(3)} \sqcup \Gamma^6_{(3)}$. *Following definition 6.1.5 we can see that:*

- $\Gamma^1_{(3)}$ *is composed by 11 pairs of type* $(e(C_i, C_i, C_i, C_i), e(C_i, C_i, C_i, F_j))$, *with* $F_j \lhd C_i$, *i.e.* $(e(C_i, C_i, C_i), e(C_i, C_i, F_j)) \in \Gamma^1_{(2)}$;
- $\Gamma^2_{(3)}$ *is composed by two types of pairs: 14 pairs of type* $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, C_i, P_k))$, *with $P_k \lhd F_j$ and $e(C_i, C_i, F_j)$ does not belong to any pair of $\Gamma_{(1)}$, i.e.* $(e(C_i, C_i, F_j), e(C_i, C_i, P_k)) \in \Gamma^2_{(2)}$; *5 pairs of type* $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, F_j, F_j))$ *with $e(C_i, F_j)$ critical for $\Gamma_{(1)}$ i.e.* $(e(C_i, C_i, F_j), e(C_i, F_j, F_j)) \in \Gamma^2_{(2)}$;
- $\Gamma^3_{(3)}$ *is composed by three types of pairs: 5 pairs of type* $(e(C_i, C_i, F_j, F_j), e(C_i, F_j, F_j, F_j))$, *with $e(C_i, F_j, F_j)$ critical for $\Gamma_{(2)}$; 20 pairs of type* $(e(C_i, C_i, F_j, F_j), e(C_i, C_i, F_j, P_k))$ *with $P_k \lhd F_j$, i.e.* $(e(C_i, F_j, F_j), e(C_i, F_j, P_k)) \in \Gamma^3_{(2)}$; *6 pairs of type* $(e(C_i, C_i, C_i, P_k), e(C_i, C_i, F_j, P_k))$ *with $C_i \lhd F_j$, and $e(C_i, P_k)$ is critical for $\Gamma_{(1)}$, i.e.*$(e(C_i, C_i, P_k), e(C_i, F_j, P_k)) \in \Gamma^3_{(2)}$;
- $\Gamma^4_{(3)}$ *is composed by two types of pairs: 14 pairs of type* $(e(C_i, C_i, F_j, P_k), e(C_i, C_i, P_k, P_k))$, *with* $(e(C_i, F_j, P_k), e(C_i, P_k, P_k)) \in \Gamma^4_{(2)}$; *20 pairs of type* $(e(C_i, F_j, F_j, F_j), e(C_i, F_j, F_j, P_k))$ *with* $P_k \lhd F_j$;
- $\Gamma^5_{(3)}$ *is composed by two types of pairs: 6 pairs of type* $(e(C_i, C_i, P_k, P_k), e(C_i, F_j, P_k, P_k))$ *with $e(C_i, P_k, P_k)$ critical for $\Gamma_{(2)}$, i.e. $C_i \cap V_1 = \emptyset$ and unbounded, and $F_j \lhd C_i$; 20 pairs of type* $(e(C_i, F_j, F_j, P_k), e(C_i, F_j, P_k, P_k))$ *with* $F_j \lhd P_k$;
- $\Gamma^6_{(3)}$ *is composed by 14 pairs of type* $(e(C_i, F_j, P_k, P_k), e(C_i, P_k, P_k, P_k))$ *with* $P_k \lhd F_j$, *and $e(C_i, F_j, P_k, P_k)$ is not in a pair of* $\Gamma^5_{(3)}$.

So the set of critical cells $Sing(\mathbf{S})$ is composed by the cells not appearing in any pair of $\Gamma_{(3)}$, i.e. by the following cells:

- *1 critical 0-cell $e(C_0, C_0, C_0, C_0)$;*
- *5 critical 3-cells $e(C_0, F_1, F_1, F_1), e(C_1, F_2, F_2, F_2), e(C_2, F_3, F_3, F_3), e(C_3, F_4, F_4, F_4), e(C_4, F_5, F_5, F_5)$;*
- *6 critical 6-cells $e(C_1, P_3, P_3, P_3), e(C_2, P_1, P_1, P_1), e(C_3, P_1, P_1, P_1), e(C_4, P_2, P_2, P_2), e(C_6, P_4, P_4, P_4), e(C_7, P_4, P_4, P_4)$.*

Remark 6.2.8 *As in previous example we have that the number of all cells of $\mathbf{S}^{(3)}$ is equal to the number of critical cells plus twice the number of the pairs of Γ, indeed $1 + 5 + 6 + 2 * (11 + 14 + 5 + 5 + 20 + 6 + 14 + 20 + 6 + 20 + 14) = 282$.*

Example 19 *We take now the arrangement $\mathcal{A} \in \mathbb{R}^3$ of example 10 (see figures 6.3 and 6.4).*
Case $d = 2$.
Recall that the complex $\mathbf{S}^{(2)}$ has 648 cells.
The polar gradient $\Gamma_{(2)}$ has components: $\Gamma_{(2)} = \bigsqcup_{i=1}^{6} \Gamma^i_{(2)}$. By definition 6.1.5 we have:

- *$\Gamma^1_{(2)}$ is composed by 23 pairs of type $(e(C_i, C_i, C_i), e(C_i, C_i, F_j))$, with $F_j \lhd C_i$, i.e. $(e(C_i, C_i), e(C_i, F_j)) \in \Gamma^1_{(1)}$;*
- *$\Gamma^2_{(2)}$ is composed by two types of pairs: 43 pairs of type $(e(C_i, C_i, F_j), e(C_i, C_i, G_k))$, with $G_k \lhd F_j$ and $e(C_i, C_i, F_j)$ does not belong to any pair of $\Gamma^1_{(2)}$, i.e. $(e(C_i, F_j), e(C_i, G_k)) \in \Gamma^2_{(1)}$; 6 pairs of type $(e(C_i, C_i, F_j), e(C_i, F_j, F_j))$ with $e(C_i, F_j)$ critical for $\Gamma_{(1)}$;*
- *$\Gamma^3_{(2)}$ is composed by three types of pairs: 60 pairs of type $(e(C_i, F_j, F_j), e(C_i, F_j, G_k))$, with $G_k \lhd F_j$; 11 pairs of type $(e(C_i, C_i, G_k), e(C_i, F_j, G_k))$ with $F_j \lhd^{op}_{G_k} C_i$, and $e(C_i, G_k)$ is critical for $\Gamma_{(1)}$; 18 pairs of type $(e(C_i, C_i, G_k), e(C_i, C_i, P))$, with $P \lhd G_k$ and $e(C_i, C_i, G_k)$ does not belong to any pair of $\Gamma^2_{(2)}$, i.e. $(e(C_i, G_k), e(C_i, P)) \in \Gamma^3_{(1)}$;*
- *$\Gamma^4_{(2)}$ is composed by three types of pairs: 48 pairs of type $(e(C_i, F_j, G_k), e(C_i, F_j, P))$ with $P \lhd G_k$ and $G_k = max_{\lhd}\{F | C_i \prec F \prec G_k\}$; 25 pairs of type $(e(C_i, F_j, G_k), e(C_i, G_k, G_k))$ with $G_k \lhd^{op}_{G_k} F_j$ and $e(F_j, G_k)$ critical for $\Gamma^{|F_j|}_{(1)}$, i.e. $G_k \lhd P$; 6 pairs of type $(e(C_i, C_i, P), e(C_i, F_j, P))$ with $F_j \lhd^{op}_P C_i$ and $e(C_i, P)$ critical for $\Gamma_{(1)}$, i.e. $C_i \cap V_2 \neq \emptyset$ and bounded;*
- *$\Gamma^5_{(2)}$ is composed by two types of pairs: 18 pairs of type $(e(C_i, F_j, P), e(C_i, G_k, P))$ with $e(F_j, P)$ critical for $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$, i.e. $F_j \cap V_2 \neq \emptyset$ and bounded, and $G_k \lhd^{op}_P F_j$; 36 pairs of type $(e(C_i, G_k, G_k), e(C_i, G_k, P))$ with $P \lhd G_k$;*
- *$\Gamma^6_{(2)}$ is composed by 18 pairs of type $(e(C_i, G_k, P), e(C_i, P, P))$, with $P \lhd^{op}_P G_k$, such that $e(C_i, G_k, P)$ is not in a pair of $\Gamma^3_{(2)}$;*

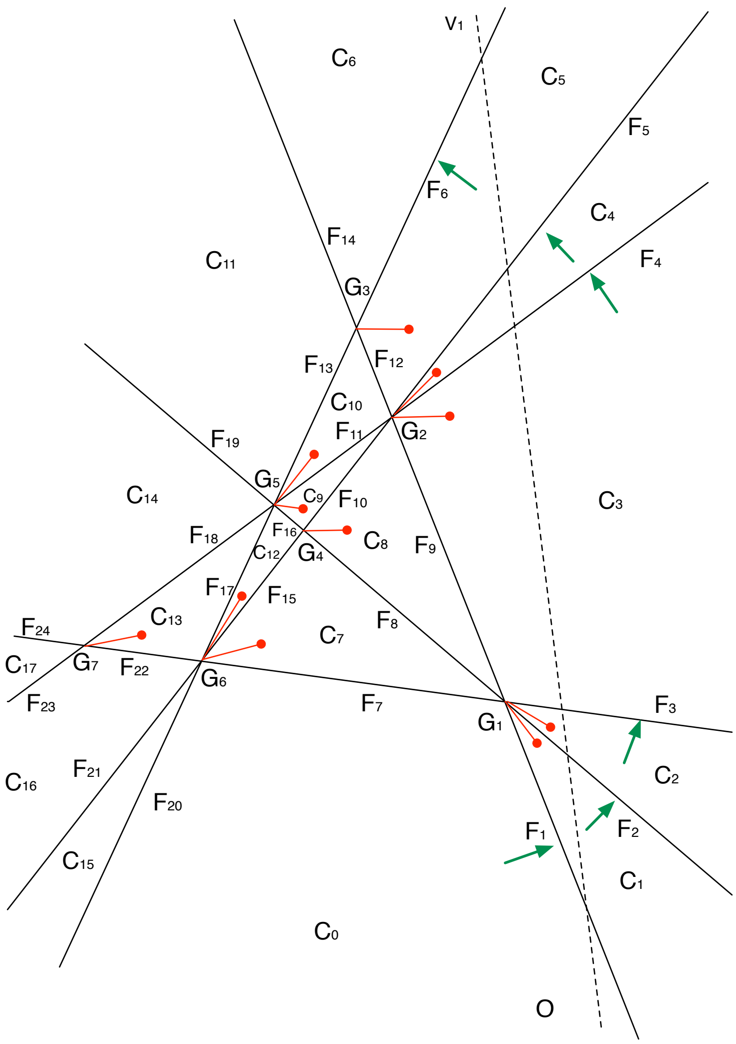


Figure 6.3: An upper section of A_3 and critical cells in case d odd.

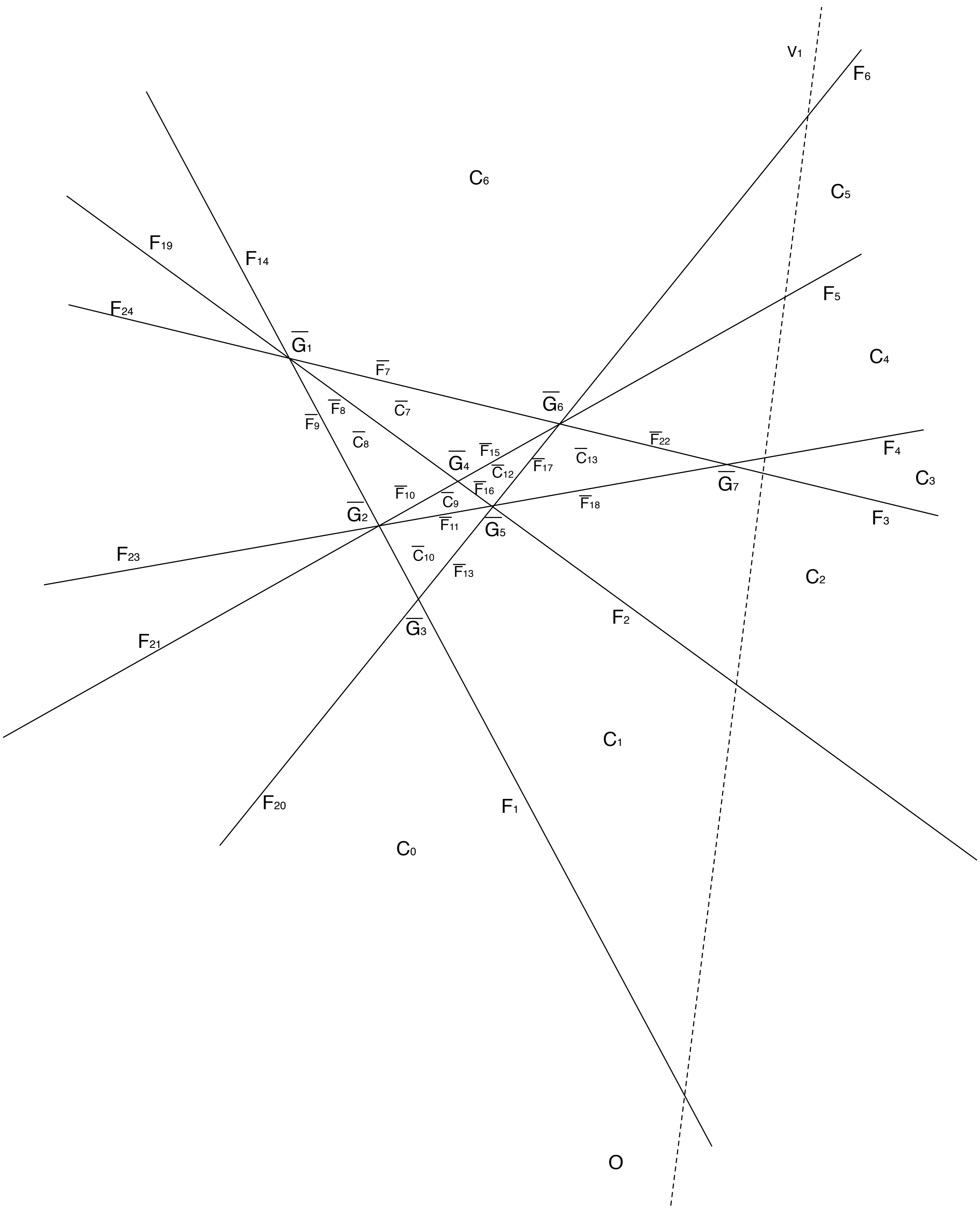


Figure 6.4: A lower section of A_3.

So the set of critical cells $Sing(\mathbf{S})$ is given by the cells not appearing in any pair of $\Gamma_{(3)}$, i.e. by the following cells:

- *1 critical 0-cell $e(C_0, C_0, C_0)$;*
- *6 critical 2-cells $e(C_1, F_1, F_1), e(C_2, F_2, F_2), e(C_3, F_3, F_3), e(C_4, F_4, F_4), e(C_5, F_5, F_5), e(C_6, F_6, F_6)$;*
- *11 critical 4-cells $e(C_7, G_1, G_1)$, $e(C_8, G_1, G_1)$, $e(C_9, G_2, G_2)$, $e(C_{10}, G_2, G_2)$, $e(C_{11}, G_3, G_3)$, $e(C_{12}, G_4, G_4)$, $e(C_{13}, G_5, G_5)$, $e(C_{14}, G_5, G_5)$, $e(C_{15}, G_6, G_6)$, $e(C_{16}, G_6, G_6)$, $e(C_{17}, G_7, G_7)$;*
- *6 critical 6-cells $e(\overline{C}_7, P, P), e(\overline{C}_8, P, P), e(\overline{C}_9, P, P), e(\overline{C}_{10}, P, P), e(\overline{C}_{12}, P, P), e(\overline{C}_{13}, P, P)$.*

Remark 6.2.9 *As in previous examples we can calculate the number of cells as $1+6+11+6+2*(23+43+6+60+11+18+48+25+6+18+36+18) = 648$.*

Case $d = 3$.
Recall that the complex $\mathbf{S}^{(3)}$ has 1536 cells.
The polar gradient $\Gamma_{(3)}$ has components: $\Gamma_{(3)} = \bigsqcup_{i=1}^{9} \Gamma_{(3)}^{i}$. Following definition 6.1.5 we can see that:

- *$\Gamma_{(3)}^{1}$ is composed by 23 pairs of type $(e(C_i, C_i, C_i, C_i), e(C_i, C_i, C_i, F_j))$, with $F_j \lhd C_i$, i.e. $(e(C_i, C_i, C_i), e(C_i, C_i, F_j)) \in \Gamma_{(2)}^{1}$;*
- *$\Gamma_{(3)}^{2}$ is composed by two types of pairs: 43 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, C_i, G_k))$, with $G_k \lhd F_j$ and $e(C_i, C_i, C_i, F_j)$ does not belong to any pair of $\Gamma_{(3)}^{1}$, i.e. $(e(C_i, C_i, F_j), e(C_i, C_i, G_k)) \in \Gamma_{(2)}^{2}$; 6 pairs of type $(e(C_i, C_i, C_i, F_j), e(C_i, C_i, F_j, F_j))$ with $e(C_i, F_j)$ critical for $\Gamma_{(1)}$, i.e. $(e(C_i, C_i, F_j), e(C_i, F_j, F_j)) \in \Gamma_{(2)}^{2}$;*
- *$\Gamma_{(3)}^{3}$ is composed by four types of pairs: 60 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, C_i, F_j, G_k))$, with $G_k \lhd F_j$, i.e. $(e(C_i, F_j, F_j), e(C_i, F_j, G_k)) \in \Gamma_{(2)}^{3}$; 11 pairs of type $(e(C_i, C_i, C_i, G_k), e(C_i, C_i, F_j, G_k))$ with $F_j \lhd_{G_k}^{op} C_i$, and $e(C_i, G_k)$ is critical for $\Gamma_{(1)}^{1}$, i.e. $(e(C_i, C_i, G_k), e(C_i, F_j, G_k)) \in \Gamma_{(2)}^{3}$; 18 pairs of type $(e(C_i, C_i, C_i, G_k), e(C_i, C_i, C_i, P))$, with $P \lhd G_k$ and $e(C_i, C_i, C_i, G_k)$ does not belong to any pair of $\Gamma_{(3)}^{2}$, i.e. $(e(C_i, C_i, G_k), e(C_i, C_i, P)) \in \Gamma_{(2)}^{3}$; 6 pairs of type $(e(C_i, C_i, F_j, F_j), e(C_i, F_j, F_j, F_j))$ with $e(C_i, F_j, F_j)$ critical for $\Gamma_{(2)}$;*
- *$\Gamma_{(3)}^{4}$ is composed by four types of pairs: 48 pairs of type $(e(C_i, C_i, F_j, G_k), e(C_i, C_i, F_j, P))$ with $P \lhd G_k$ and $G_k = max_{\lhd}\{F | C_i \prec F \prec G_k\}$, i.e. $(e(C_i, F_j, G_k), e(C_i, F_j, P)) \in \Gamma_{(2)}^{4}$; 25 pairs of type $(e(C_i, C_i, F_j, G_k), e(C_i, C_i, G_k, G_k))$ with $G_k \lhd_{G_k}^{op} F_j$ and $e(F_j, G_k)$ critical for $\Gamma_{(1)}^{|F_j|}$, i.e. $(e(C_i, F_j, G_k), e(C_i, G_k, G_k)) \in \Gamma_{(2)}^{4}$;*

6 pairs of type $(e(C_i, C_i, C_i, P), e(C_i, C_i, F_j, P))$ *with* $F_j \lhd_P^{op} C_i$ *and* $e(C_i, P)$ *critical for* $\Gamma_{(1)}$*, i.e.* $(e(C_i, C_i, P), e(C_i, F_j, P)) \in \Gamma^4_{(2)}$*; 60 pairs of type* $(e(C_i, F_j, F_j, F_j), e(C_i, F_j, F_j, G_k))$*, with* $G_k \lhd F_j$*. Notice that in* $\Gamma_{(3)}$ *there are no pair of type* $(e(C_i, C_i, F_j, G_k), e(C_i, F_j, F_j, G_k))$ *because no cell of the form* $e(C_i, F_j, G_k)$ *is critical for* $\Gamma_{(2)}$*;*

- $\Gamma^5_{(3)}$ *is composed by five types of pairs: 18 pairs of type* $(e(C_i, C_i, F_j, P), e(C_i, C_i, G_k, P))$ *with* $e(F_j, P)$ *critical for* $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$*, and* $G_k \lhd_P^{op} F_j$*, i.e.* $(e(C_i, F_j, P), e(C_i, G_k, P)) \in \Gamma^5_{(2)}$*; 36 pairs of type* $(e(C_i, C_i, G_k, G_k), e(C_i, C_i, G_k, P))$ *with* $P \lhd G_k$*, i.e.* $(e(C_i, G_k, G_k), e(C_i, G_k, P)) \in \Gamma^5_{(2)}$*; 11 pairs of type* $(e(C_i, C_i, G_k, G_k), e(C_i, F_j, G_k, G_k))$ *with* $F_j \lhd C_i$*, and* $e(C_i, G_k, G_k)$ *is critical for* $\Gamma_{(2)}$*; 48 pairs of type* $(e(C_i, F_j, F_j, G_k), e(C_i, F_j, F_j, P))$ *with* $P \lhd G_k$ *and* $G_k = max_{\lhd}\{F|F_j \prec F \prec G_k\}$*; 36 pairs of type* $(e(C_i, F_j, F_j, G_k), e(C_i, F_j, G_k, G_k))$ *with* $e(F_j, G_k)$ *critical for* $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$*,* $G_k \lhd^{op}_{G_k} F_j$*. Notice that in* $\Gamma_{(3)}$ *there are no pair of type* $(e(C_i, C_i, F_j, P), e(C_i, F_j, F_j, P))$ *because no cell of the form* $e(C_i, F_j, P)$ *is critical for* $\Gamma_{(2)}$*;*

- $\Gamma^6_{(3)}$ *is composed by four types of pairs: 18 pairs of type* $(e(C_i, C_i, G_k, P), e(C_i, C_i, P, P))$*, with* $P \lhd_P^{op} G_k$*, such that* $e(C_i, C_i, G_k, P)$ *is not in* $\Gamma^5_{(3)}$*, i.e.* $(e(C_i, G_k, P), e(C_i, P, P)) \in \Gamma^6_{(2)}$*; 72 pairs of type* $(e(C_i, F_j, G_k, G_k), e(C_i, F_j, G_k, P))$ *with* $P \lhd G_k$*; 25 pairs of type* $(e(C_i, F_j, G_k, G_k), e(C_i, G_k, G_k, G_k))$ *with* $G_k \lhd F_j$ *and* $e(F_j, G_k, G_k)$ *critical for* $\Gamma^{|F_j|}_{(2)}$*, i.e.* $G_k \lhd P$*; 24 pairs of type* $(e(C_i, F_j, F_j, P), e(C_i, F_j, G_k, P))$ *with* $e(F_j, P)$ *critical for* $\Gamma_{(1)}(\mathcal{A}^{|F_j|})$*, i.e.* $F_j \cap V_2 \neq \emptyset$ *and bounded, and* $G_k \lhd_P^{op} F_j$*. Notice that in* $\Gamma_{(3)}$ *there are no pair of type* $(e(C_i, C_i, G_k, P), e(C_i, F_j, G_k, P))$ *because no cell of the form* $e(C_i, G_k, P)$ *is critical for* $\Gamma_{(2)}$*;*

- $\Gamma^7_{(3)}$ *is composed by three types of pairs: 6 pairs of type* $(e(C_i, C_i, P, P), e(C_i, F_j, P, P))$ *with* $F_j \lhd C_i$*, and* $e(C_i, P, P)$ *is critical for* $\Gamma_{(2)}$*; 36 pairs of type* $(e(C_i, G_k, G_k, G_k), e(C_i, G_k, G_k, P))$ *with* $P \lhd G_k$*; 48 pairs of type* $(e(C_i, F_j, G_k, P), e(C_i, F_j, P, P))$*, with* $P \lhd_P^{op} G_k$*, and* $G_k = max_{\lhd_P^{op}}\{F|F_j \prec F \prec G_k\}$*. Notice that in* $\Gamma_{(3)}$ *there are no pair of type* $(e(C_i, F_j, G_k, P), e(C_i, G_k, G_k, P))$ *because no cell of the form* $e(F_j, G_k, P)$ *is critical for* $\Gamma^{|F_j|}_{(2)}$*;*

- $\Gamma^8_{(3)}$ *is composed by two types of pairs: 18 pairs of type* $(e(C_i, F_j, P, P), e(C_i, G_k, P, P))$ *with* $e(F_j, P, P)$ *critical for* $\Gamma_{(2)}(\mathcal{A}^{|F_j|})$*, i.e.* $F_j \cap V_2 = \emptyset$*, and* $G_k \lhd F_j$*; 36 pairs of type* $(e(C_i, G_k, G_k, P), e(C_i, G_k, P, P))$ *with* $P \lhd_P^{op} G_k$*;*

- $\Gamma^9_{(3)}$ *is composed by 18 pairs of type* $(e(C_i, G_k, P, P), e(C_i, P, P, P))$*, with* $P \lhd G_k$*, such that* $e(C_i, G_k, P, P)$ *is not in a pair of* $\Gamma^8_{(3)}$*;*

So the set of critical cells $Sing(\mathbf{S})$ *is composed by the cells not appearing in any pair of* Γ*, i.e. by the following cells:*

- *1 critical 0-cell* $e(C_0, C_0, C_0, C_0)$*;*

- *6 critical 3-cells* $e(C_0, F_1, F_1, F_1), e(C_1, F_2, F_2, F_2), e(C_2, F_3, F_3, F_3), e(C_3, F_4, F_4, F_4), e(C_4, F_5, F_5, F_5), e(C_5, F_6, F_6, F_6)$*;*

- *11 critical 6-cells* $e(C_1, G_1, G_1, G_1)$*,* $e(C_2, G_1, G_1, G_1)$*,* $e(C_3, G_2, G_2, G_2)$*,* $e(C_4, G_2, G_2, G_2)$*,* $e(C_5, G_3, G_3, G_3)$*,* $e(C_8, G_4, G_4, G_4)$*,* $e(C_9, G_5, G_5, G_5)$*,* $e(C_{10}, G_5, G_5, G_5)$*,* $e(C_7, G_6, G_6, G_6)$*,* $e(C_{12}, G_6, G_6, G_6)$*,* $e(C_{13}, G_7, G_7, G_7)$*;*

- *6 critical 9-cells* $e(C_7, P, P, P), e(C_8, P, P, P), e(C_9, P, P, P), e(C_{10}, P, P, P)$*,* $e(C_{12}, P, P, P)$*,* $e(C_{13}, P, P, P)$*.*

Remark 6.2.10 *As in previous examples we can calculate the number of cells as* $1 + 6 + 11 + 6 + 2 * (23 + 43 + 6 + 60 + 11 + 18 + 6 + 48 + 25 + 6 + 60 + 18 + 36 + 11 + 48 + 36 + 18 + 72 + 25 + 24 + 6 + 36 + 48 + 18 + 36 + 18) = 1536$*.*

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