Transverse Gravitational Theories

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1 Introduccion

1.1 Equivalence principles, metric theories and universal coupling

The Principle of Equivalence, from the beginning, has played an important role in the development of gravitational theories: Newton himself dedicated a detailed discussion of it in the opening paragraphs of his "*Philosophiae naturalis principia matematica*". To Newton, the Principle of Equivalence demanded that the "mass" of **any** body, namely the property of any body (inertia) that regulates its response to an applied force, be equal to its "weight", that property that regulates its response to gravitation.
Bondi in 1957 coined the terms "inertial mass" (m_i) and "passive gravitational mass" (m_p) to refer to these quantities, so that Newton's second law and the law of gravitation take the form

$$\mathbf{F} = m_i \mathbf{a} \qquad \mathbf{F} = m_p \mathbf{g}.$$

The Principle of Equivalence can then be succinctly stated saying that

For any body $m_i = m_p$.

with a more precise statement it can be expressed by saying that

If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition.

By "uncharged test body" we mean an electrically neutral body with negligible self-gravitational energy.
Today Newton's Equivalence Principle is generally referred to as the **Weak Equivalence Principle (WEP)**.

According to the WEP, if all bodies fall with the same acceleration in an external gravitational field, then, to an observer in a freely falling elevator in the same gravitational field, the bodies should be unaccelerated (assuming that small effects due to inhomogeneities in the gravitational field can be made as small as desired by working in a sufficiently small elevator). Thus, insofar as their mechanical motion are concerned, the bodies will behave as if gravity were absent.
It was Einstein who added the key element to the WEP that revealed the path to General Relativity. Going one step further, he proposed that not only should mechanical laws behave in such an elevator as if gravity were absent, but so should **all** the laws of physics, including for example the

laws of electrodynamics: that is, “we [...] assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system” (Einstein 1907). Thus being at rest on the surface of the Earth is equivalent to being inside a spaceship (far from any sources of gravity) that is being accelerated by its engines. From this principle, Einstein deduced that free-fall is actually inertial motion. By contrast, in Newtonian mechanics, gravity is assumed to be a force, so that a person at rest on the surface of a (non-rotating) massive object is in an inertial frame of reference. Now this is called the **Einstein Equivalence Principle (EEP)**, and can be expressed with the statement that

The Weak Equivalence Principle is valid and the outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

By “local non-gravitational experiment” we mean any experiment performed in a shielded freely falling laboratory with negligible self-gravitational effects. The EEP is essentially composed by three different parts: WEP, Local Position Invariance (LPI) i.e. invariance under location change of the laboratory, and Local Lorentz Invariance (LLI) i.e. invariance under velocity change of the laboratory.

Today a third Equivalence Principle exists, which is much more restrictive than Einstein’s formulation: the **Strong Equivalence Principle (SEP)** states that

The gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution. The outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

The first part is a version of the WEP that applies to objects that exert a gravitational force on themselves, while the second part is the EEP restated to allow gravitational experiments and self-gravitating bodies.
Some powerful consequences [1] of the SEP are that the gravitational constant G must be the same everywhere in the universe, and that a fifth force beyond the known ones is not allowed. Anyway, some physicists have criticized the differences made between the EEP and the SEP because there is no universally accepted way to distinguish gravitational from non-gravitational experiments.

Today, in most gravitational theories, gravitation is a curved-spacetime phenomenon, i.e. must satisfy the postulates of "metric theories" which state that

Spacetime is endowed with a metric $g_{\mu\nu}$*. The world lines of test bodies are geodesics of that metric. In local freely falling frames, called local Lorentz frames, the non-gravitational laws of physics are those of special relativity.*

It is possible to argue that if a theory satisfies the EEP, it is a metric theory [1].
Metric theories are equivalent to those characterized by the so called **universal coupling**, that is the property that all non-gravitational field should couple in the same manner to a single gravitational field. It's only a matter of choosing to consider the metric $g_{\mu\nu}$ as a property of spacetime itself rather than as a field over a flat spacetime. Thus metric theories can differ from each other only in the number and type of additional gravitational fields they introduce and in the field equations that determine their structure and evolution. There may be other gravitational fields besides the metric which contribute to the curvature of spacetime; nevertheless, once determined the evolution of the metric, the only field that couples directly to matter is the metric itself.

1.2 Schiff's conjecture [1]

The three parts of EEP are so different in their empirical consequences that it is tempting to regard them as independent theoretical principles. Anyway, in 1960 Leonard Schiff conjectured that *any complete, self-consistent theory that embodies WEP necessarily embodies EEP*. By a complete self-consistent theory we mean a theory capable of predicting the results of any experiment of interest, giving the same result through whichever method is used.
A rigorous proof of the conjecture could give much stricter bounds to the violation of EEP. Anyway, so far only "plausibility" arguments have been found. One of the most elegant of these, for instance, assumes the conservation of energy:

Let's consider an idealized composite body made up of structureless test particles bounded by some non-gravitational force, which moves sufficiently slowly in a weak, static gravitational field to describe the motion in a quasi-Newtonian form (so that second order terms $\sim v^4, U^2$ can be neglected). If $U(\mathbf{x})$ is the gravitational potential and the composite body is small enough to be regarded as point-like, we can assume that the conserved energy function has the general form

$$E = M_R c^2 - M_R U(\mathbf{x}) + \frac{1}{2} M_R v^2 + o(v^2, U).$$

If we assume EEP violations, the speed of light could depend on the presence of gravity, so we don't set $c = 1$. The rest energy can be written as

$$M_R c^2 = M_0 c^2 - E_B(\mathbf{x}, \mathbf{v})$$

where M_0 is the sum of the rest masses and E_B is the binding energy that, expanded in powers of U and v^2, can be written as

$$E_B(\mathbf{x}, \mathbf{v}) = E_B^0 + \delta m_p^{ij} U_{ij}(\mathbf{x}) - \frac{1}{2} \delta m_I^{ij} v_i v_j \qquad \left(U_{ij}(\mathbf{x}) \equiv \int d^4x \frac{\rho(x - x')_i (x - x')_j}{|\mathbf{x} - \mathbf{x}'|^3} \right).$$

It can be shown that δm_p^{ij} and δm_I^{ij} (called anomalous passive and inertial mass tensors, which depend upon the internal structure of the body) are possible terms that give rise to EEP violation, since a freely falling observer, detecting the binding energy of the system, could detect the effects of his location and velocity. Let's prove that they give rise also to WEP violation, through a "gedanken experiment":
We start from n free particles of mass m_0 at rest at $\mathbf{x} = \mathbf{h}$; their conserved energy is

$$n m_0 (c^2 - U(\mathbf{h})).$$

We form a bound state and keep the energy released $E_B(\mathbf{h}, \mathbf{0})$ in a reservoir of free particles of mass m_0. Now the conserved energies of the bound state and the reservoir are respectively

$$[n m_0 c^2 - E_B(\mathbf{h}, \mathbf{0})][1 - U(\mathbf{h})/c^2] \qquad \text{and} \qquad E_B(\mathbf{h}, \mathbf{0})[1 - U(\mathbf{h})/c^2].$$

We let the stored particles and the bound system freely fall, with accelerations $\mathbf{g} = \nabla U$ and $\mathbf{a} = \mathbf{g} + \delta \mathbf{a}$ respectively, until $\mathbf{x} = 0$. Here we bring the systems at rest and put into the reservoir the kinetic energies collected

$$-[n m_0 - E_B(\mathbf{0}, \mathbf{v})/c^2] \mathbf{a} \cdot \mathbf{h} - \delta m_I^{ij} g_i h_j \qquad \text{and} \qquad - E_B(\mathbf{h}, \mathbf{0}) \mathbf{g} \cdot \mathbf{h}/c^2$$

(some kinematic identities have been used to substitute $\mathbf{v}$).
From the reservoir, that now has energy

$$E_B(\mathbf{h}, \mathbf{0})[1 - U(\mathbf{0})/c^2] - E_B^0 \mathbf{g} \cdot \mathbf{h}/c^2 - (n m_0 - E_B^0/c^2) \mathbf{a} \cdot \mathbf{h} - \delta m_I^{ij} g_i h_j$$

we extract enough energy ($E_B(\mathbf{0},\mathbf{0})[1 - U(\mathbf{0})/c^2]$) to deassemble the bound system, and enough energy ($-nm_0\mathbf{g}\cdot\mathbf{h}$) to carry the n particles back at $\mathbf{x} = \mathbf{h}$. The cycle is closed, and if energy is conserved the reservoir should be empty. This means that we must have

$$E_B(\mathbf{h},\mathbf{0}) - E_B(\mathbf{0},\mathbf{0}) - (nm_0 - E_B^0/c^2)\delta\mathbf{a}\cdot\mathbf{h} - \delta m_I^{ij} g_i h_j = 0.$$

Since

$$E_B(\mathbf{h},\mathbf{0}) - E_B(\mathbf{0},\mathbf{0}) = \delta m_p^{ij} \nabla U_{ij} \cdot \mathbf{h}$$

we get

$$a_i = g_i + \frac{\delta m_p^{jk}}{M_R}\partial_i U_{jk} - \frac{\delta m_I^{ij}}{M_R} g_j.$$

So, since the WEP would give $a_i = g_i$, WEP is violated unless $\delta m_p^{ij} = \delta m_I^{ij} = 0$. Schiff's conjecture, under the assumptions made, is proved. Anyway, the whole argument is valid only in the non-relativistic limit.

2 Transverse mass-less theories

2.1 What are Transverse Theories?

Einstein's General Relativity has the property to be invariant under a general diffeomorphism in the coordinates (**Diff invariance**). This property is manifest if we get to Einstein's equations starting from the variational principle of the Hilbert action

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2k^2}R + L_m\Big] \tag{2.1.1}$$

since $d^4x\sqrt{-g}$ is a scalar for generic diffeomorphic transformations.
Here we have defined

$$k^2 \equiv M_P^{-2} \equiv 8\pi G. \tag{2.1.2}$$

Anyway, it is maybe not well known that, four years after writing down the equations of General Relativity, Einstein also proposed a different set of equations, which are the traceless part of the ones of General Relativity. This different set of equations comes out from those theories which are now commonly called "unimodular theories" [3], in which the determinant of the metric g is fixed. Although Einstein never talks about an action principle (since he was actually interested only in the equations of motion), if we work in the variational formalism, unimodular theories constrain the allowed unimodular variations $\delta^u g^{\alpha\beta}$ to be such that

$$\delta^u g = 0 \tag{2.1.3}$$

or equivalently

$$g_{\mu\nu}\delta^u g^{\mu\nu} = 0. \tag{2.1.4}$$

This means that the unimodular variation can be expressed in terms of an unconstrained variation as

$$\delta^u g^{\alpha\beta} = \delta g^{\alpha\beta} - \frac{1}{4}g^{\alpha\beta}g_{\mu\nu}\,\delta g^{\mu\nu} \tag{2.1.5}$$

so that any variation of an action can be expressed as

$$\delta S = \frac{\delta S}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta} = \frac{\delta S}{\delta g^{\alpha\beta}}\Big(\delta^u g^{\alpha\beta} + \frac{1}{4}g^{\alpha\beta}g_{\mu\nu}\delta g^{\mu\nu}\Big). \tag{2.1.6}$$

Eventually, the restricted variation is just the trace-free part of the unconstrained variation:

$$\frac{\delta S}{\delta^u g^{\alpha\beta}} = \frac{\delta S}{\delta g^{\alpha\beta}} - \frac{1}{4}g_{\alpha\beta}g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}}\,. \tag{2.1.7}$$

Hence, calculating the restricted variation of the Hilbert action we get the equations

$$R_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R = k^2 \Big(T_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} T \Big) \tag{2.1.8}$$

which are exactly the second traceless set of equations proposed by Einstein.
It seems that this alternative set of equations carry less information than the well known Einstein equations, because the trace has been left out. But Einstein himself realized in 1919 that this unimodular theory is equivalent to General Relativity, with the Cosmological Constant appearing as an integration constant:

If we assume the energy-momentum to be covariantly conserved and using the contracted Bianchi identity $\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla_\mu R$ (where ∇^ν are covariant derivatives), deriving by ∇^β the whole equation we get

$$\frac{1}{4}\nabla_\alpha R = -\frac{k^2}{4}\nabla_\alpha T \tag{2.1.9}$$

which integrated gives

$$R + k^2 T = \text{constant} \equiv -4\Lambda. \tag{2.1.10}$$

Finally, if we substitute T in equation (2.1.8) we get exactly Einstein's General Relativity equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - g_{\alpha\beta}\Lambda = k^2 T_{\alpha\beta}. \tag{2.1.11}$$

Transverse Theories [2] are a bit different from unimodular theories: in Transverse Theories the determinant of the metric g is not fixed (it's dynamical), so that the variation $\delta g^{\alpha\beta}$ is not restricted by $\delta g = 0$. But **the action is invariant under transverse diffeomorphisms (TDiff)** in the sense that the gauge symmetry group of the Lagrangian is not the whole group of diffeomorphisms, but only the TDiff group.
The TDiff group is the group of the diffeomorphisms that leave the determinant of the metric g unchanged. We have

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \tag{2.1.12}$$

where, at the first order, for a linear transformation

$$x^\mu \longrightarrow x^\mu - \xi^\mu \tag{2.1.13}$$

we have

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\mu}\partial_\nu \xi^\rho + g_{\rho\nu}\partial_\mu \xi^\rho = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{2.1.14}$$

with ∇_μ denoting covariant derivatives. Hence, since invariance under TDiff means that $\delta g = 0$, we must have

$$\delta g = g g^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 2g\nabla_\mu \xi^\mu = 0 \tag{2.1.15}$$

i.e.

$$\nabla_\mu \xi^\mu = 0. \tag{2.1.16}$$

Vector fields inducing TDiff transformations can generically be represented as [4]

$$\xi^\mu = \epsilon^{\mu\nu\rho\sigma}\nabla_\nu \Omega_{\rho\sigma} \tag{2.1.17}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita controvariant tensor and $\Omega_{\rho\sigma}$ is an antisymmetric tensor. Since $\epsilon^{\mu\nu\rho\sigma}$ is completely antisymmetric, the contraction with the symmetric term $\nabla_\mu\nabla_\nu$ implies that

$$\nabla_\mu \xi^\mu = \epsilon^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\nu \Omega_{\rho\sigma} = 0. \tag{2.1.18}$$

Given a metric $g_{\mu\nu}$, we can split it into

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + k h_{\mu\nu} \tag{2.1.19}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric and $k^2 \equiv 8\pi G$ so that the deviation $h_{\mu\nu}$ from the flat metric can be regarded as a tensor field of dimension one.
The former is an exact definition of $h_{\mu\nu}$, so that the inverse metric can only be written as a formal power of series

$$g^{\mu\nu} = \eta^{\mu\nu} - k h^{\mu\nu} + k^2 h^\mu_\rho h^{\nu\rho} - k^3 h^{\mu\rho} h_{\rho\sigma} h^{\nu\sigma} + O(k^4). \tag{2.1.20}$$

When using the field $h_{\mu\nu}$ the indexes are always raised and lowered by the flat Minkowski metric.

The variation of $h_{\mu\nu}$, at the lowest orders,is given by

$$\delta h_{\mu\nu} = k^{-1}(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + h_{\mu\rho}\partial_\nu \xi^\rho + h_{\nu\rho}\partial_\mu \xi^\rho + \xi^\rho \partial_\rho h_{\mu\nu}. \tag{2.1.21}$$

If we consider weak fields, we can consider only the term proportional to k^{-1}: at the first order we can say that TDiff transformations are those gauge transformations

$$k\delta h_{\mu\nu} = (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \tag{2.1.22}$$

which satisfy

$$\partial_\mu \xi^\mu = 0. \tag{2.1.23}$$

2.2 Why Transverse Theories?

General Relativity is perfectly consistent as a classical theory, and insofar almost every experiment surprisingly agrees with the theoretical predictions. Anyway, two are the main problems Einstein's theory has to deal with: (*i*) the difficulties to extend the classical theory to consistent renormalizable quantum field theories, which have been so successful in describing all the other interactions; (*ii*) the Cosmological Constant problem: if the cosmological constant is the vacuum energy, why has it such a tiny value $\Lambda = M_D^4 \sim 10^{-48}\text{GeV}^4$ and isn't it of the order of the cutoff scale $M_P^4 \sim 10^{76}\text{GeV}^4$?
Mainly for these reasons General Relativity is not considered the definitive answer for Gravity, and many modifications have been considered (like for instance String Theories). So, one of the possibilities is to modify General Relativity postulating less spacetime symmetry, like in Transverse Theories, which leaves us with more freedom in writing the possible actions. More precisely, two are the main arguments that justify the interest in TDiff theories:

- Consistent propagation of a massless spin-2 graviton requires only TDiff invariance [3] [2] [5]:

In the standard "transverse-traceless gauge", the five polarizations $\epsilon_{\mu\nu}$ of a spin-2 symmetric tensor field must satisfy

$$\partial_\mu \epsilon^\mu_\nu = 0 \tag{2.2.1}$$
$$\epsilon^\mu_\mu = 0 \tag{2.2.2}$$

where $\epsilon^\mu_\nu \equiv \eta^{\mu\nu}\epsilon_{\mu\nu}$. Thus, for a mass-less particle with four-momentum $k^\mu = (k, 0, 0, k)$, in momentum space, the five polarizations can be written as

$$\begin{aligned} \epsilon^{\times}_{\mu\nu} &= e_1 \otimes e_2 + e_2 \otimes e_1 \\ \epsilon^{+}_{\mu\nu} &= e_1 \otimes e_1 - e_2 \otimes e_2 \\ \epsilon^{(1)}_{\mu\nu} &= k \otimes k \\ \epsilon^{(2)}_{\mu\nu} &= k \otimes e_1 + e_1 \otimes k \\ \epsilon^{(3)}_{\mu\nu} &= k \otimes e_2 + e_2 \otimes k. \end{aligned} \tag{2.2.3}$$

We notice that $\epsilon^{(1,2,3)}_{\mu\nu}$ are all of the form $k_\mu \xi_\nu + k_\nu \xi_\mu$ with $k_\mu \xi^\mu = 0$.

First, in the mass-less case, we would like to be left with the only two helicity-eigenstate polarizations (which are known to be $\epsilon^{+}_{\mu\nu} \pm \epsilon^{\times}_{\mu\nu}$). Next, we have to deal with the infinite degeneration arising from the "little group" problem; the "little group", which in this case is the group of the transformations that leave the four-momentum $(k, 0, 0, k)$ of a mass-less particle unchanged, has three generators: I_z (rotation around the z-axis) and I_{0x} , I_{0y} (boost along the x, y-axis plus a rotation around the y, x-axis to "neutralize the aberration" coming from the boost).
I_z unitarily acts in the right way on the standard helicity polarizations $\epsilon^{+}_{\mu\nu} \pm \epsilon^{\times}_{\mu\nu}$, giving only a phase $e^{\pm 2i\theta}$ under rotations of angle θ. But the infinite dimensional unitary representation of the non-compact transformations I_{0x} , I_{0y} lead to the appearance of infinite polarizations for any given momentum. We would like to solve the problem in a similar way as for the abelian case of Electrodynamics, i.e. through a gauge-invariant principle.
The standard helicity polarizations transform under I_{0x} , I_{0y} into the other three $\epsilon^{(1,2,3)}_{\mu\nu}$. Moreover, I_{0x} , I_{0y} leave the trace ϵ^{μ}_{μ} unchanged, since the trace is Lorentz-invariant. It is then straightforward to declare equivalent those polarizations which are related to one another by a standard gauge transformation

$$\epsilon_{\mu\nu} \longrightarrow \epsilon_{\mu\nu} - (k_\mu \xi_\nu + \kappa_\nu \xi_\mu) \tag{2.2.4}$$

which leaves the trace invariant, that is such that

$$k_\mu \xi^\mu = 0.$$

This way I_{0x} and I_{0y} become only gauge-invariant transformations, and the three polarizations $\epsilon^{(1,2,3)}_{\mu\nu}$ become "pure gauge".
To solve the problem only TDiff invariance is needed.

- In Transverse Theories it is possible to make the coupling of the metric $g_{\mu\nu}$ to the vacuum energy Λ as small as desired:

Since TDiff transformations leave the determinant of the metric g invariant, it is possible to substitute the term $\sqrt{-g}$ that appears in the Hilbert action with an arbitrary function $f(g)$. It means that we could write an action with a term

$$S_v = \int d^4x f(g)\Lambda \tag{2.2.5}$$

which allows us, playing with the function $f(g)$, to make the coupling between the metric and the vacuum energy arbitrarily small: Λ could even be of the order of M_P^4.

2.3 TDiff quadratic Lagrangian

We are going now to analyze the linearized theory.
The most general quadratic Lorentz-invariant local Lagrangian for a free massless symmetric tensor field $h_{\mu\nu}$ can be written as [2]

$$L = L_0 + c_1 L_1 + c_2 L_2 + c_3 L_3 \tag{2.3.1}$$

where

$$L_0 \equiv \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} \tag{2.3.2a}$$

$$L_1 \equiv -\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\rho h_\nu^\rho \tag{2.3.2b}$$

$$L_2 \equiv \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} \tag{2.3.2c}$$

$$L_3 \equiv -\frac{1}{4}\partial_\mu h \partial^\mu h. \tag{2.3.2d}$$

L_0 is strictly needed for the propagation of a spin-2 particle.

The variation of $h^{\mu\nu}$, up to total derivatives, gives

$$\delta L_0 = -\frac{1}{2}\Box h_{\mu\nu}\delta h^{\mu\nu} \tag{2.3.3a}$$

$$\delta L_1 = (\partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu)\delta h^{\mu\nu} \tag{2.3.3b}$$

$$\delta L_2 = -\frac{1}{2}(\partial_\mu\partial_\nu h^{\mu\nu}\delta h + \partial_\mu\partial_\nu h\delta h^{\mu\nu}) \tag{2.3.3c}$$

$$\delta L_3 = \frac{1}{2}\Box h\delta h. \tag{2.3.3d}$$

For TDiff gauge transformations, (2.1.22) with the constraint (2.1.23), we notice that

$$\delta L_2 = \partial_\mu\partial_\nu h^{\mu\nu}\partial_\rho\xi^\rho + \partial_\mu(\partial_\nu h\partial^\nu\xi^\mu) - \partial_\nu h\partial^\nu\partial_\mu\xi^\mu = 0 \tag{2.3.4a}$$

$$\delta L_3 = -\Box h\partial_\mu\xi^\mu = 0. \tag{2.3.4b}$$

So, **to have TDiff invariance it's only necessary that** $c_1 = 1$:

$$\delta L_0 + \delta L_1 = -(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial_\rho\partial^\rho h_{\mu\nu})(\partial^\mu\xi^\nu + \partial^\nu\xi^\mu) = 0 \tag{2.3.5}$$

because

$$\begin{cases} (\partial_\rho\partial_\mu h^\rho_\nu - \partial_\rho\partial^\rho h_{\mu\nu})\partial^\mu\xi^\nu = \partial_\rho\left[(\partial_\mu h^\rho_\nu - \partial^\rho h_{\mu\nu})\partial^\mu\xi^\nu\right] - (\partial_\mu h^\rho_\nu - \partial^\rho h_{\mu\nu})\partial_\rho\partial^\mu\xi^\nu = 0 \\ \partial_\rho\partial_\nu h^\rho_\mu\partial^\mu\xi^\nu = \partial_\nu(\partial_\rho h^\rho_\mu\partial^\mu\xi^\nu) - \partial_\rho h^\rho_\mu\partial^\mu\partial_\nu\xi^\nu = 0 \end{cases}$$

The most general quadratic TDiff-invariant Lagrangian is then

$$L = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu + \frac{c_2}{2}\partial_\mu h\partial_\nu h^{\mu\nu} - \frac{c_3}{4}\partial_\mu h\partial^\mu h. \tag{2.3.6}$$

2.4 Dynamical analysis

We will still work in the approximation of a linear theory.
As shown in section (2.2), the quantum theory of Gravitation is not unitary unless the Lagrangian is invariant under TDiff. Actually, we will show that the absence of TDiff symmetry leads to pathologies such as classical instabilities or the appearance of ghosts.
Let's use the "cosmological decomposition" for the field $h_{\mu\nu}$ in terms of scalars, vectors and tensors under spatial rotations [6]:

$$h_{00} = A \tag{2.4.1a}$$
$$h_{0i} = \partial_i B + V_i \tag{2.4.1b}$$
$$h_{ij} = \psi\delta_{ij} + \partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + t_{ij} \tag{2.4.1c}$$

with

$$\partial_i F^i = \partial_i V^i = \partial_i t^{ij} = t^i_i = 0. \tag{2.4.2}$$

With this decomposition, in the generic quadratic Lagrangian (2.3.1), scalars, vectors and tensors decouple from each other; working in momentum space [2]:

- The tensor t_{ij} only contributes to L_0:

$$L_t = \frac{1}{4}(\partial_\mu t^{ij})^2. \tag{2.4.3}$$

- The vectors contribute only to L_0 and L_1:

$$L_v = \frac{1}{2}\mathbf{k^2}(V^i - \dot{F}^i)^2 + \frac{1}{2}(c_1 - 1)(\mathbf{k^2}F^i + \dot{V}^i)^2. \tag{2.4.4}$$

The momenta conjugated to V^i and F^i are

$$\Pi^i_V = (c_1 - 1)(\mathbf{k^2}F^i + \dot{V}^i) \tag{2.4.5a}$$
$$\Pi^i_F = \mathbf{k^2}(\dot{F}^i - V^i) \tag{2.4.5b}$$

so that (2.4.4), for $c_1 \neq 1$, can be rewritten as

$$L_v = \frac{1}{2\mathbf{k^2}}\Pi^2_V + \frac{1}{2(c_1 - 1)}\Pi^2_F. \tag{2.4.6}$$

The Hamiltonian is given by

$$H_v = \frac{1}{2\mathbf{k^2}}(\Pi_F^i + \mathbf{k^2}V^i)^2 - \frac{1}{2(1-c_1)}\left[\Pi_V^i + (1-c_1)\mathbf{k^2}F^i\right]^2 + \frac{1-c_1}{2}k^4F^2 - \frac{1}{2}\mathbf{k^2}V^2. \tag{2.4.7}$$

Because of the alternating signs, the Hamiltonian is not bounded below, which leads to classical instability: from Hamilton's equations we have

$$\dot{\Pi}_F^i = \mathbf{k^2}\Pi_V^i \tag{2.4.8a}$$

$$\dot{\Pi}_V^i = -\Pi_F^i \tag{2.4.8b}$$

which give the general oscillatory solution

$$|\mathbf{k}|\Pi_V^i + i\Pi_F^i = C\exp[i(|\mathbf{k}|t + \phi_0)], \tag{2.4.9}$$

while taking the derivative of (2.4.5) with respect to t and using (2.4.8), we have

$$\ddot{V}^i + \mathbf{k}^2V^i = -\frac{c_1}{c_1-1}\Pi_F^i \tag{2.4.10a}$$

$$\ddot{F}^i + \mathbf{k}^2F^i = \frac{c_1}{c_1-1}\Pi_V^i \tag{2.4.10b}$$

which, for $c_1 \neq 0$ are the equations of forced oscillators with asymptotic solution

$$V^i + i|\mathbf{k}|F^i \sim \frac{Cc_1t}{(c_1-1)|\mathbf{k}|}\exp[i(|\mathbf{k}|t + \phi_0)]. \tag{2.4.11}$$

The solution, which grows linearly with time, is the evidence of classical instability.
Classical instability could be avoided setting $c_1 = 0$. But in this case the vectors V^i and F^i would decouple from each other and V^i would become a ghost, since

$$L_v(c_1 = 0) = \frac{1}{2}\mathbf{k}^2(\partial_\mu F^i)^2 - \frac{1}{2}(\partial_\mu V^i)^2. \tag{2.4.12}$$

Hence, the only possibility to avoid classical instabilities and ghosts is to set $c_1 = 1$, that is, to endow the Lagrangian with TDiff-invariance. In this case

$$L_v(c_1 = 1) = \frac{1}{2}\mathbf{k}^2(V^i - \dot{F}^i)^2. \tag{2.4.13}$$

The variation with respect to V^i gives the constraint

$$V^i - \dot{F}^i = 0 \tag{2.4.14}$$

which, substituted in (2.4.13), shows that there is no vector dynamics.

- The scalar Lagrangian, with $c_1 = 1$, is given by

$$\begin{aligned} L_s = &\frac{1}{4}\Big[(\partial_\mu A)^2 - 2\mathbf{k}^2(\partial_\mu B)^2 + 3(\partial_\mu \psi)^2 - 2\mathbf{k}^2\partial_\mu\psi\partial^\mu E + \mathbf{k}^4(\partial_\mu E)^2\Big] \\ &- \frac{1}{2}\Big[(\dot{A} + \mathbf{k}^2 B)^2 - \mathbf{k}^2\dot{B}^2 - \mathbf{k}^2\psi^2 + 2\mathbf{k}^4 E\psi - \mathbf{k}^6 E^2 + 2\mathbf{k}^2\dot{B}(\psi - \mathbf{k}^2 E)\Big] \\ &+ \frac{c_2}{2}\Big[(\dot{A} - 3\dot{\psi} + \mathbf{k}^2\dot{E})(\dot{A} + \mathbf{k}^2 B) - \mathbf{k}^2(A - 3\psi + \mathbf{k}^2 E)(\dot{B} - \psi + \mathbf{k}^2 E)\Big] \\ &- \frac{c_3}{4}\Big[\partial_\mu(A - 3\psi + \mathbf{k}^2 E)\Big]^2. \end{aligned} \tag{2.4.15}$$

The variation of B gives the constraint

$$2\psi = (c_2 - 1)(A - 3\psi + \mathbf{k}^2 E) = (c_2 - 1)h \tag{2.4.16}$$

that, substituted back in (2.4.15), gives the simple expression

$$L_s = -\frac{\Delta c_3}{4}(\partial_\mu h)^2 \tag{2.4.17}$$

where

$$\Delta c_3 \equiv c_3 - \frac{3c_2^2 - 2c_2 + 1}{2}. \tag{2.4.18}$$

Hence, the scalar sector contains a single degree of freedom, proportional to the trace.
Moreover, **to avoid ghosts, we must have the condition**

$$\Delta c_3 \leq 0. \tag{2.4.19}$$

In the special case where

$$\Delta c_3 = 0 \tag{2.4.20}$$

the scalar sector disappears, and we are left only with the tensor sector.

Let's see in the next chapter what does the condition $\Delta c_3 = 0$ mean.

2.5 Non-linear generalizations

The simplest way to generalize TDiff theories is to mix the Hilbert action with general functions of the determinant of the metric: since by definition TDiff transformations leave the determinant unchanged, these functions are also TDiff-invariant.
Hence, a general gravitational TDiff-invariant action could be of the form

$$S = \int d^4x \left(-\frac{1}{2k^2}\right) \Big(f_1(|g|)R + f_2(|g|)g^{\mu\nu}\partial_\mu g\partial_\nu g\Big). \tag{2.5.1}$$

To extend to non-linear theories the study of TDiff-invariance we could even make some particular choices: as seen in the previous section, in general TDiff quadratic theories there is a supplementary scalar degree of freedom, proportional to the trace h. Thus, the idea [7] is to split the metric degrees of freedom into the determinant g and a new metric

$$\hat{g}_{\mu\nu} \equiv |g|^{-1/4} g_{\mu\nu} \tag{2.5.2}$$

with fixed determinant $|\hat{g}| = 1$. Under arbitrary diffeomorphisms (2.1.13) the new metric transforms as

$$\delta\hat{g}_{\mu\nu} = \hat{\nabla}_\mu\xi_\nu + \hat{\nabla}_\nu\xi_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\rho\xi^\rho \tag{2.5.3}$$

where $\hat{\nabla}$ denotes covariant derivative with respect to $\hat{g}_{\mu\nu}$, and the indexes are raised and lowered by the new metric. Transverse diffeomorphisms are defined as those which satisfy

$$\hat{\nabla}_\mu\xi^\mu = 0. \tag{2.5.4}$$

But since $|\hat{g}| \equiv 1$, we have that $\hat{\Gamma}^\mu_{\mu\nu} = \partial_\nu\sqrt{|\hat{g}|} = 0$ so that condition (2.5.4) reduces to

$$\hat{\nabla}_\mu\xi^\mu = \partial_\mu\xi^\mu + \hat{\Gamma}^\mu_{\mu\nu}\xi^\nu = \partial_\mu\xi^\mu = 0. \tag{2.5.5}$$

Hence, under TDiff with the constraint (2.5.4) the metric $\hat{g}_{\mu\nu}$ transforms as a tensor:

$$\delta\hat{g}_{\mu\nu} = \hat{\nabla}_\mu\xi_\nu + \hat{\nabla}_\nu\xi_\mu \tag{2.5.6}$$

and also the determinant g, as expected, transforms as a scalar:

$$\delta g = \xi^\mu\partial_\mu g. \tag{2.5.7}$$

It can be shown [7] that the only terms which can be constructed from $\hat{g}_{\mu\nu}$ that behave as tensors are the geometric tensors $\hat{R}_{\mu\nu\rho\sigma}$ and its contractions, so that the most general TDiff-invariant action which contains at most two derivatives of the metric takes the form:

$$S = \int d^4x \left(-\frac{1}{2}\chi^2(g, \{\phi\})\hat{R} + L(g, \{\phi\}, \hat{g}_{\mu\nu}) \right) \tag{2.5.8}$$

where χ^2 is a scalar made out of the matter fields $\{\phi\}$ and the determinant g.
We thus notice that TDiff-invariant theories can be seen as "unimodular" (i.e. with fixed determinant) scalar-tensor theories, where g plays the role of an additional scalar.
The equations of motion must be calculated through a restricted variation of the metric, since the action is composed of a metric with fixed determinant $\hat{g} = 1$.

If we define

$$\bar{g}_{\mu\nu} \equiv \chi^2 \hat{g}_{\mu\nu} \tag{2.5.9}$$

so that

$$\sqrt{-\bar{g}} = \sqrt{-\chi^8 \hat{g}} = \chi^4 \tag{2.5.10}$$

we can go to the "Einstein frame": the new action reads

$$S = \int d^4x \sqrt{-\bar{g}} \left[-\frac{1}{2}R(\bar{g}_{\mu\nu}) + \frac{6}{\chi^2}\,\bar{g}_{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{\chi^4}\bar{L}(\chi, \{\phi\}, \bar{g}_{\mu\nu}) \right]. \tag{2.5.11}$$

Anyway, we have to implement the constraint (2.5.10), that can be done through a Lagrange multiplier $\Lambda(x)$. Hence we have

$$S = \int d^4x \sqrt{-\bar{g}} \left[-\frac{1}{2}R(\bar{g}_{\mu\nu}) + \frac{6}{\chi^2}\,\bar{g}_{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{\chi^4}\bar{L}(\chi, \{\phi\}, \bar{g}_{\mu\nu}) \right]$$

$$- \int d^4x \sqrt{-\bar{g}} \frac{1}{\chi^4}\Lambda + \int d^4x \Lambda. \tag{2.5.12}$$

We note that the invariance under full diffeomorphisms which treat $\bar{g}_{\mu\nu}$ as a metric and χ and Λ as scalars is only broken by the last term. We can show that this term is actually an integration constant, and not a parameter of the Lagrangian.

If we define the matter Lagrangian as

$$L_m \equiv \sqrt{-\bar{g}} \left[\frac{6}{\chi^2} \bar{g}_{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{1}{\chi^4} \bar{L}(\chi, \{\phi\}, \bar{g}_{\mu\nu}) - \frac{1}{\chi^4} \Lambda \right] + \Lambda \tag{2.5.13}$$

the Bianchi identities applied to the pure gravitational part $\sqrt{-\bar{g}}\frac{1}{2}R(\bar{g}_{\mu\nu})$ give, as in General Relativity, the conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.5.14}$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}} \frac{\delta L_m}{\delta \bar{g}^{\mu\nu}}. \tag{2.5.15}$$

On the other hand, since only the last term of (2.5.13) breaks Diff-invariance, the variation of the matter part of the action under a general coordinate transformation is given by

$$\delta L_m = \frac{\delta L_m}{\delta \chi}\delta\chi + \frac{\delta L_m}{\delta \psi}\delta\psi + \frac{\delta L_m}{\delta \Lambda}\delta\Lambda + \frac{\sqrt{-\bar{g}}}{2} T_{\mu\nu}\delta g^{\mu\nu} = \delta\Lambda = \xi^\mu \partial_\mu \Lambda. \tag{2.5.16}$$

If the equations of motion for χ, Λ and ψ are satisfied, i.e.

$$\frac{\delta L_m}{\delta \chi} = \frac{\delta L_m}{\delta \psi} = \frac{\delta L_m}{\delta \Lambda} = 0, \tag{2.5.17}$$

since $\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi \mu$, after partial integration we get

$$\xi^\mu (\partial_\mu \Lambda + \sqrt{-\bar{g}} \nabla^\nu T_{\mu\nu}) = 0 \tag{2.5.18}$$

that is, using (2.5.14),

$$\partial_\mu \Lambda = 0. \tag{2.5.19}$$

2.6 Lorentz covariance and compensators

Usually, *tensor densities of weight w* are defined in such a way that they get an extra factor of the Jacobian to the power w in the tensorial transformation law. For instance a scalar of weight w transforms as

$$\phi'(y) = D(y,x)^w \phi(x) \tag{2.6.1}$$

where

$$D(y,x) \equiv \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right). \tag{2.6.2}$$

A particular scalar density is the determinant of the metric g, which behaves as a scalar density of weight $w = -2$, that is

$$g(y) = \left(\frac{1}{D(y,x)}\right)^2 g(x). \tag{2.6.3}$$

TDiff transformations are actually those transformations with unitary Jacobian ($D(y,x) = 1$).
This means that as long as we assume that TDiff is the basic symmetry of nature, we do not distinguish tensor densities from real tensors.

Now, going back to the study of transverse theories, we can for instance take a general action of the form

$$S = \int d^4x \left(-\frac{1}{2k^2} f_1(g) R + f_m(g) L_m(g_{\mu\nu}, \{\phi\})\right). \tag{2.6.4}$$

It should be remarked that this action is not fully covariant, unless $f_1(g) = f_m(g) = \sqrt{-g}$, i.e. the theory is Diff-invariant.
If the action is assumed to take the form (2.6.4) in a particular reference system with some privileged coordinates denoted by $\bar{x}^\mu$, in general coordinates the action reads [4]

$$S = \int d^4x \frac{1}{C(x)}\Big(-\frac{1}{2k^2} f_1\left(g(x)C(x)^2\right) R(x) + f_m\left(g(x)C(x)^2\right) L_m\left(g_{\mu\nu}(x), \{\psi(x)\}\right)\Big) \tag{2.6.5}$$

where $C(x)$ is a scalar density of weight $w = 1$. This scalar density is sometimes [8] called a **compensator field**, and is introduced exactly to make the action Diff-invariant. A notorious example is the Stueckelberg field which renders gauge-invariant massive electrodynamics.
The original theory can always be recovered setting $C(x) = 1$, which looks like a particular gauge choice.

Let's see now what implications follow from the equations of motion of the compensator $C(x)$: the variation with respect to $C(x)$ gives

$$-\frac{1}{C^2}\left[f_m L_m - \frac{1}{2k^2} f_1 R\right] + \frac{1}{C}\left[\frac{\partial f_m}{\partial C} L_m - \frac{1}{2k^2}\frac{\partial f_1}{\partial C} R\right] = 0 \tag{2.6.6}$$

that can be rewritten as

$$\left[-\frac{f_m}{C} + \frac{\partial f_m}{\partial C}\right] L_m - \frac{1}{2k^2}\left[-\frac{f_1}{C} + \frac{\partial f_1}{\partial C}\right] R = 0. \tag{2.6.7}$$

As we shall see also in section (6.2), some problems or strong constraints rise when we want only one sector (i.e. the gravitational or the matter part) to have the restricted TDiff symmetry. When one sector is Diff-invariant (i.e. $f(x) = \sqrt{|x|}$), its compensator equations of motion are identically satisfied:

$$\left[-\frac{f}{C} + \frac{\partial f}{\partial C}\right] L = \left[-\frac{\sqrt{|g|C^2}}{C} + \frac{\partial \sqrt{|g|C^2}}{\partial C}\right] L = 0. \tag{2.6.8}$$

Hence, if we choose for instance only the gravitational sector to be Diff-invariant, equation (2.6.7) becomes

$$\left[-\frac{f_m}{C} + \frac{\partial f_m}{\partial C}\right] L_m = 0. \tag{2.6.9}$$

The solutions are given by

$$f_m(gC^2) \propto C \tag{2.6.10}$$

which implies $f_m(x) = \sqrt{|x|}$, i.e. also the matter sector has to be Diff-invariant. Or else

$$L_m \equiv 0. \tag{2.6.11}$$

As we shall see in section (6.4), L_m can be identified with the matter pressure, so that only theories in which the matter is pressure-less would be allowed.

In a similar way, if we choose only the matter sector to be Diff-invariant, we would find that whether the gravitational part has to be Diff-invariant as well, or the constraint $R = 0$.

3 From TDiff to enhanced symmetries [2]

3.1 Diff and Weyl symmetry

We still work in the linear approximation of mass-less fields.
For particular values of the parameters c_2, c_3 the Lagrangian can acquire enhanced symmetries: for instance, the case $c_2 = c_3 = 1$ corresponds to the Fierz-Pauli Lagrangian (L_{FP}), which is Diff-invariant.
Starting from the Fierz-Pauli Lagrangian, through a simple non-derivative field redefinition

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \lambda h \eta_{\mu\nu} \qquad (\lambda \neq -1/4) \tag{3.1.1}$$

where the condition $\lambda \neq -1/4$ is necessary for the transformation to be invertible, the parameters in the Lagrangian (2.3.6) change as

$$\begin{cases} c_2 \longrightarrow c_2 + 2\lambda(2c_2 - 1) \\ c_3 \longrightarrow c_3 + 2\lambda(4c_3 - c_2 - 1) + 2\lambda^2(8c_3 - 4c_2 - 1) \end{cases} \tag{3.1.2}$$

so that, starting from $c_2 = c_3 = 1$, the new parameters are related by

$$c_3 = \frac{3c_2^2 - 2c_2 + 1}{2} \qquad \text{with } c_2 \neq \frac{1}{2}. \tag{3.1.3}$$

This means that the condition (2.4.20) is satisfied, so that the scalar sector of the Lagrangian is absent. Lagrangians of the form (2.3.6) with the relation (3.1.3) between c_2 and c_3 are equivalent to the Fierz-Pauli Lagrangian.

Another possibility is to replace $h_{\mu\nu}$ in the Lagrangian (2.3.6) with the traceless part:

$$h_{\mu\nu} \longrightarrow \tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4} h \eta_{\mu\nu} \tag{3.1.4}$$

which is formally analogous to (3.1.1) with $\lambda = -1/4$, but can't be interpreted as a field redefinition since the trace h can't be recovered from the new field (3.1.4).
The Lagrangian is still TDiff-invariant, since the replacement (3.1.4) doesn't change the coefficients in front of L_0, L_1. Anyway, it becomes invariant under a new Weyl transformation (**WTDiff symmetry**):

$$\delta h^{\mu\nu} \equiv \frac{1}{2} \phi \eta^{\mu\nu}. \tag{3.1.5}$$

The WTDiff symmety is manifest, since the new field (3.1.4) is invariant under (3.1.5).

Using (3.1.2) with $\lambda = -1/4$ we immediately find that a WTDiff-invariant Lagrangian must be of the form (2.3.6) with

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{3}{8}. \tag{3.1.6}$$

Also in this case the condition (2.4.20) is satisfied, and there are no scalar dynamics.

3.2 Unicity of the enhancements

Let's show that Diff and WTDiff exhaust all possible enhancements of TDiff symmetry for a Lagrangian of the form (2.3.1):
Since the variation of L_0 (2.3.3a) involves a term $\Box h_{\mu\nu}$ where $h_{\mu\nu}$ are arbitrary, this term can cancel with other ones only if the transformation is of the form

$$\delta h^{\mu\nu} = (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) + \frac{1}{2}\phi\eta^{\mu\nu} \tag{3.2.1}$$

for some vector ξ^μ and some scalar ϕ. The vector can generically be decomposed as

$$\xi^\mu = \zeta^\mu + \partial^\mu \psi \qquad \text{with} \quad \partial_\mu \zeta^\mu = 0. \tag{3.2.2}$$

Then, using (2.3.3), after some calculations, we eventually find that

$$\begin{aligned} \delta L = {} & \zeta_\nu (c_1 - 1)\Box(\partial_\mu h^{\mu\nu}) \\ & + \Box\psi \frac{1}{2}\left[(c_3 - c_2)\Box h + (2c_1 - c_2 - 1)\partial_\mu\partial_\nu h^{\mu\nu}\right] \\ & + \phi\frac{1}{4}\left[(4c_3 - c_2 - 1)\Box h + 2(c_1 - 2c_2)\partial_\mu\partial_\nu h^{\mu\nu}\right]. \end{aligned} \tag{3.2.3}$$

TDiff corresponds to taking $c_1 = 1$ and setting $\phi = \psi = 0$. To enhance the symmetry, i.e. to have invariance under transformations involving non-vanishing ϕ and ψ, we have to cancel the terms involving $\partial_\mu\partial_\nu h^{\mu\nu}$ and $\Box h$:

$$\begin{cases} \frac{1}{2}(1 - 2c_2)\phi - \frac{1}{2}(c_2 - 1)\Box\psi = 0 \\ \frac{1}{4}(4c_3 - c_2 - 1)\phi + \frac{1}{2}(c_3 - c_2)\Box\psi = 0 \end{cases}$$

that is

$$
\begin{cases}
\Box\psi = \dfrac{1-2c_2}{c_2-1}\phi \\
\\
c_3 = \dfrac{3c_2^2-2c_2+1}{2}
\end{cases}
\qquad (3.2.4)
$$

The second equation in (3.2.4) is exactly the same as (3.1.3), so that any Lagrangian with enhanced symmetry is equivalent to the Fierz-Pauli Lagrangian, unless $c_2 = \frac{1}{2}$, $c_3 = \frac{3}{8}$, which corresponds to a WTDiff-invariant Lagrangian.

3.3 WTDiff versus Diff symmetry

We are now going to analyze whether, at the lowest order, a WTDiff-invariant theory is classically equivalent to General Relativity (Fierz-Pauli Lagrangian).
We have, from the definition (3.1.4), that

$$L_{WTD}(h_{\mu\nu}) \equiv L_{TD}(\tilde{h}_{\mu\nu}). \qquad (3.3.1)$$

Since the Fierz-Pauli Lagrangian is a particular TDiff-invariant Lagrangian, we can write

$$\frac{\delta S_{WTD}(h_{\mu\nu})}{\delta h^{\mu\nu}} = \frac{\delta S_{FP}(\tilde{h}_{\mu\nu})}{\delta \tilde{h}^{\rho\sigma}}\frac{\delta \tilde{h}^{\rho\sigma}}{\delta h^{\mu\nu}} = \frac{\delta S_{FP}(\tilde{h}_{\mu\nu})}{\delta \tilde{h}^{\rho\sigma}}\Big(\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{4}\eta^{\rho\sigma}\eta_{\mu\nu}\Big). \qquad (3.3.2)$$

Both through a Weyl and a Diff transformation we can, in WTDiff or Diff theories respectively, go to a gauge where $h = 0$, that is $\tilde{h}_{\mu\nu} = h_{\mu\nu}$. Thus the WTDiff equations of motion are simply the traceless part of the Fierz-Pauli equations of motion.
This means that a WTDiff theory is classically equivalent to Einstein's unimodular theory analyzed in section (2.1). Diff and WTDiff theories differ classically only by an integration constant.

Let us now consider the relation between the two symmetry groups: they act infinitesimally on $h_{\mu\nu}$ giving

$$\delta_D h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + 2\partial_\mu \partial_\nu \psi \tag{3.3.3}$$

$$\delta_{WTD} h_{\mu\nu} = \partial_\mu \tilde{\zeta}_\nu + \partial_\nu \tilde{\zeta}_\mu + \frac{1}{2}\phi \eta_{\mu\nu} \tag{3.3.4}$$

where the decomposition (3.2.2) has been used, and also $\partial_\mu \tilde{\zeta}^\mu = 0$.
The intersection of the two groups is given by

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + 2\partial_\mu \partial_\nu \psi = \partial_\mu \tilde{\zeta}_\nu + \partial_\nu \tilde{\zeta}_\mu + \frac{1}{2}\phi \eta_{\mu\nu}. \tag{3.3.5}$$

Taking the trace and the divergence of (3.3.5) we get

$$\Box \psi = \phi \tag{3.3.6}$$

$$\Box(\tilde{\zeta}_\mu - \zeta_\mu) = 2\Box \partial_\mu \psi - \tfrac{1}{2}\partial_\mu \phi \tag{3.3.7}$$

that yield

$$\Box(\tilde{\zeta}_\mu - \zeta_\mu) = \frac{3}{4}\Box \partial_\mu \psi. \tag{3.3.8}$$

Taking the derivative with respect to ν, symmetrizing with respect to μ and using (3.3.5) and (3.3.6), we finally get

$$\partial_\mu \partial_\nu \phi = 0 \Longrightarrow \quad \phi = a_\mu x^\mu + c. \tag{3.3.9}$$

This means that not ever Weyl transformation is a Diff transformation, but only those which satisfy (3.3.9).
Conversely, the subset of the Diff transformation that can be expressed as a Weyl transformation are those given by [9]:

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2}\partial_\rho \xi^\rho \eta_{\mu\nu} \tag{3.3.10}$$

4 Massive fields

4.1 Dynamical analysis

The most general mass term that can be added to the quadratic Lagrangian (2.3.6) takes the form [2]:

$$L_m = -\frac{1}{4}m_1^2 h_{\mu\nu}h^{\mu\nu} + \frac{1}{4}m_2^2 h^2. \tag{4.1.1}$$

If we set $m_1 \equiv 0$, only the scalar degree of freedom h is given a mass; if $-m_2^2 > 0$ is larger than the energy scales we are interested in, the extra scalar effectively decouples, and only the standard helicity polarizations of the graviton are allowed to propagate.
The matter Lagrangian L_m is still TDiff invariant, since under TDiff the variation is

$$\delta L_m = \frac{1}{2}m_2^2 h\delta h = m_2^2 h\partial_\mu \xi^\mu = 0. \tag{4.1.2}$$

If we allow $m_1 \neq 0$, in general the whole Lagrangian is not TDiff-invariant anymore. Let's make a dynamical analysis as in section (2.4). Using the "cosmological decomposition" (2.4.1),

- the tensor sector becomes

$$L_t = -\frac{1}{4}t^{ij}(\Box + m_1^2)t_{ij} \tag{4.1.3}$$

with the constraint $m_1^2 > 0$ to avoid tachyonic instabilities.

- The vector Lagrangian is given by

$$L_v = \frac{1}{2}\mathbf{k^2}(V^i - \dot{F}^i)^2 + \frac{1}{2}(c_1 - 1)(\mathbf{k^2}F^i + \dot{V}^i)^2 - \frac{1}{2}m_1^2\Big[\mathbf{k}^2(F^i)^2 - (V^i)^2\Big]. \tag{4.1.4}$$

The Hamiltonian, for $c_1 \neq 1$, is

$$H_v = \frac{1}{2\mathbf{k^2}}(\Pi_F^i + \mathbf{k^2}V^i)^2 - \frac{1}{2(1-c_1)}\left[\Pi_V^i + (1-c_1)\mathbf{k^2}F^i\right]^2 + \frac{1-c_1}{2}k^4F^2$$
$$-\frac{1}{2}\mathbf{k^2}V^2 + \frac{1}{2}m_1^2\Big[\mathbf{k}^2F^2 - V^2\Big] \tag{4.1.5}$$

which gives, as in section (2.4), tachyonic instabilities or ghosts: this can easily be seen noticing that the contribution proportional to $(V^i)^2$ is negative. Hence we must have $c_1 = 1$. The vector Lagrangian is thus given by

$$L_v = \frac{1}{2}\mathbf{k^2}(V^i - \dot{F}^i)^2 - \frac{1}{2}m_1^2\Big[\mathbf{k}^2(F^i)^2 - (V^i)^2\Big] \tag{4.1.6}$$

The variation of V^i gives the constraint

$$(\mathbf{k}^2 + m_1^2)V^i = \mathbf{k}^2\dot{F}^i \tag{4.1.7}$$

so that the vector sector can be rewritten as

$$L_v = -\frac{1}{2}\left(\frac{\mathbf{k}^2 m_1^2}{\mathbf{k}^2 + m_1^2}\right) F^i(\Box + m_1^2)F^i. \tag{4.1.8}$$

- The scalar Lagrangian, with $c_1 = 1$, is given by

$$L_s = L_s^0 - \frac{m_1^2}{4}(A^2 - 2\mathbf{k}^2B^2 + 3\psi^2 - 2\mathbf{k}^2\psi E + \mathbf{k}^4E^2) + \frac{m_2^2}{4}(A - 3\psi + \mathbf{k}^2E)^2 \tag{4.1.9}$$

where L_s^0 is the mass-less scalar Lagrangian (2.4.15).
The variation with respect to B leads to the constraint

$$m_1^2 B = (1 - c_2)(\dot{A} + \mathbf{k}^2\dot{E}) + (3c_2 - 1)\dot{\psi}. \tag{4.1.10}$$

Further, substituting E through the trace h and defining two new variables U and V:

$$\mathbf{k}^2E = h + 3\psi - A \tag{4.1.11a}$$

$$2A \equiv (3c_2 - 1)h + (4\mathbf{k}^2 - 3m_1^2)U \tag{4.1.11b}$$

$$2\psi \equiv (c_2 - 1)h - m_1^2(U - V) \tag{4.1.11c}$$

we can rewrite the scalar Lagrangian as

$$L_s = -\frac{\Delta c_3}{4}\dot{h}^2 + \frac{(3m_1^2 - 4\mathbf{k}^2)m_1^2}{8}(\dot{V}^2 - \dot{U}^2) + \frac{1}{8}W(h, U, V) \tag{4.1.12}$$

where Δc_3 is defined by (2.4.18) and

$$\begin{aligned}W \equiv 2\Big[&\mathbf{k}^2\Delta c_3 + m_2^2 - (3c_2^2 - 3c_2 + 1)m_1^2\Big]h^2 \\ &+ m_1^4(\mathbf{k}^2 - 3m_1^2)V^2 \\ &- m_1^2(8\mathbf{k}^4 - 11m_1^2\mathbf{k}^2 + 6m_1^4)U^2 \\ &+ 4m_1^2\mathbf{k}^2(3m_1^2 - 2\mathbf{k}^2)UV \\ &+ 2m_1^2(2c_2 - 1)\left[(3m_1^2 - 2\mathbf{k}^2)U - 2\mathbf{k}^2 V\right]h.\end{aligned} \tag{4.1.13}$$

From (4.1.12) we see that either U or V, wheter $4\mathbf{k}^2 < 3m_1^2$ or $4\mathbf{k}^2 > 3m_1^2$, are ghosts, unless

$$\Delta c_3 = 0$$

i.e. **the only possibility to avoid ghosts in a theory with $m_1 \neq 0$ is to enhance the symmetry of the kinetic term of the Lagrangian to Diff or WTDiff**: in this case h is non-dynamical and the variation of W with respect to h gives the constraints

$$m_2^2 = (3c_2^2 - 3c_2 + 1)m_1^2, \tag{4.1.14}$$

$$2\mathbf{k}^2 V = (3m_1^2 - 2\mathbf{k}^2)U. \tag{4.1.15}$$

With these constraints the ghosts disappear and we're left with only one scalar degree of freedom.

Counting the degrees of freedom, we find, as expected, that they correspond to the five polarizations of a spin-2 particle: 2 from the symmetric, transverse and traceless tensor t_{ij}, 2 from the transverse vector F^i and one from the scalar. Anyway, the tensor, vector and scalar Lagrangians we have written are not in a manifestly Lorentz-invariant form.

4.2 Diff-invariant kinetic term

As seen in the previous section, to have massive fields without ghosts or classical instabilities, the kinetic term must be invariant under Diff or WTDiff. Let's analyze the Diff-invariant case.
Without loss of generality, as seen in section (3.1), we can take $c_2 = c_3 = 1$. From (4.1.14) we have the usual Fierz-Pauli relation

$$m_1^2 = m_2^2 \tag{4.2.1}$$

Using (4.1.15) together with the definition (4.1.11c) we get

$$2\mathbf{k}^2\psi = m_1^2(3m_1^2 - 4\mathbf{k}^2)U \tag{4.2.2}$$

Hence, writing the whole scalar Lagrangian as function of ψ, we find

$$L_s = -\frac{3}{4}\psi(\Box + m_1^2)\psi \tag{4.2.3}$$

4.3 WTDiff-invariant kinetic term

In the special case where the kinetic term is WTDiff-invariant, we have, from (3.1.6), that $c_2 = \frac{1}{2}$. Hence the last term in (4.1.13) cancels, so that the trace of the metric doesn't mix with U and V. The consequence is that we the variation of h doesn't give a constraint between U and V, and thus the ghost in (4.1.12) is always present for $m_1 \neq 0$.

This means that the WTDiff theory cannot be deformed with the addition of a mass term for the graviton without provoking the appearance of a ghost.

5 Coupling to the matter

5.1 Gauge fixing

In Diff theories one usually chooses the harmonic gauge:

$$\omega_\mu \equiv \partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h = 0. \tag{5.1.1}$$

This gauge choice carries a free index μ, which leads to four independent conditions, and is possible thanks to the four degrees of freedom of a generic Diff transformation (2.1.14).
Conversely, in Transverse Theories, the TDiff restriction (2.1.16) leaves us with only three gauge degrees of freedom. Hence, the harmonic gauge can't be chosen, as well as any other gauge-fixing which is linear in the momentum k^μ [2]: indeed, the most general linear gauge-fixing condition can be written as

$$M^{\alpha\beta\gamma}h_{\beta\gamma} = 0 \tag{5.1.2}$$

with

$$M^{\alpha\beta\gamma} \equiv a_1(\eta^{\alpha\beta}\partial^\gamma + \eta^{\alpha\gamma}\partial^\beta) + a_2\eta^{\beta\gamma}\partial^\alpha. \tag{5.1.3}$$

In order to bring a generic metric $h_{\mu\nu}$ to the guage (5.1.2) through a transformation (2.1.22) we must have the condition on ξ^μ

$$M^{\alpha\beta\gamma}h_{\beta\gamma} = k^{-1}M^{\alpha\beta\gamma}(\partial_\beta\xi_\gamma + \partial_\gamma\xi_\beta). \tag{5.1.4}$$

But if only TDiff transformations are allowed, deriving with respect to α, the constraint $\partial_\mu\xi^\mu = 0$ gives

$$\partial_\alpha M^{\alpha\beta\gamma}h_{\beta\gamma} = k^{-1}2(2a_1 + a_2)\Box\partial_\mu\xi^\mu = 0. \tag{5.1.5}$$

This means that in general the gauge (5.1.2) can't be reached.

The simplest way to fix the gauge with only three independent conditions is to impose the transversality

$$\partial_\mu\omega^{\mu\nu} = 0 \tag{5.1.6}$$

to the antisymmetric tensor

$$\omega_{\mu\nu} \equiv \partial^\rho(\partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho}). \tag{5.1.7}$$

This is actually equivalent to projecting the harmonic gauge (5.1.1) on its transverse part, i.e. (in momentum space)

$$k^2\theta_{\mu\nu}\omega^{\nu} \equiv (k^2\eta_{\mu\nu} - k_\mu k_\nu)\omega^{\nu} = 0. \tag{5.1.8}$$

The most general quadratic TDiff Lagrangian, with the gauge-fixing and the ghost terms, can then be written as [10]:

$$L = L_0 + L_1 + c_2 L_2 + c_3 L_3 + L_m + L_{gf} + L_{gh} \tag{5.1.9}$$

where the gauge-fixing Lagrangian is

$$L_{fg} = B_\mu \partial_\nu \omega^{\mu\nu} - \frac{1}{2}\alpha B_\mu^2 \tag{5.1.10}$$

and the ghost Lagrangian, which decouples from the other fields, is

$$L_{gh} = -\bar{c}_\mu \Box^2 c^\mu. \tag{5.1.11}$$

We notice that the auxiliary field B_μ is dimensionless, so that the gauge-fixing parameter α must be dimensionful: we thus redefine

$$\alpha \equiv M^4. \tag{5.1.12}$$

The variation of the auxiliary field B_μ allows us to rewrite the gauge-fixing Lagrangian as

$$L_{gf} = \frac{1}{2M^4}(\partial_\mu \omega^{\mu\nu})^2 = \frac{1}{2M^4}(\partial_\mu \partial_\nu \partial_\rho h^{\nu\rho} - \Box \partial_\nu h^{\nu}_{\mu})^2. \tag{5.1.13}$$

To conclude we just give the BRST transformations for the different fields [10]:

$$\begin{aligned} \delta h_{\alpha\beta} &= \partial_\alpha \partial^\mu c_{\mu\beta} + \partial_\beta \partial^\mu c_{\mu\alpha} \\ \delta B_\mu &= 0 \\ \delta \bar{c}_\mu &= -B_\mu \\ \delta c_{\mu\nu} &= 0. \end{aligned} \tag{5.1.14}$$

The ghost and antighost are defined from the antisymmetric two-index ones:

$$c^\mu \equiv \partial_\nu c^{\nu\mu} \qquad \bar{c}^\mu \equiv \partial_\nu \bar{c}^{\nu\mu} \tag{5.1.15}$$

so that

$$\partial_\mu c^\mu = \partial_\mu \bar{c}^\mu = 0. \tag{5.1.16}$$

5.2 Propagators

We work now in momentum space.
Defining first the usual transverse and longitudinal projectors

$$\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{5.2.1}$$

$$\lambda_{\mu\nu} \equiv \frac{k_\mu k_\nu}{k^2}, \tag{5.2.2}$$

we define the following Barnes-Rivers projectors, symmetric in $(\mu\nu)$, $(\rho\sigma)$ and in the exchange $(\mu\nu \leftrightarrow \rho\sigma)$, [11]:

$$P_2 \equiv \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \tag{5.2.3a}$$

$$P_1 \equiv \frac{1}{2}(\theta_{\mu\rho}\lambda_{\nu\sigma} + \theta_{\mu\sigma}\lambda_{\nu\rho} + \theta_{\nu\rho}\lambda_{\mu\sigma} + \theta_{\nu\sigma}\lambda_{\mu\rho}) \tag{5.2.3b}$$

$$P_0^s \equiv \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \tag{5.2.3c}$$

$$P_0^w \equiv \lambda_{\mu\nu}\lambda_{\rho\sigma} \tag{5.2.3d}$$

$$P_0^{sw} \equiv \frac{1}{\sqrt{3}}\,\theta_{\mu\nu}\lambda_{\rho\sigma} \tag{5.2.3e}$$

$$P_0^{ws} \equiv \frac{1}{\sqrt{3}}\,\lambda_{\mu\nu}\theta_{\rho\sigma} \tag{5.2.3f}$$

$$P_0^\times \equiv P_0^{(ws)} + P_0^{(sw)}. \tag{5.2.3g}$$

Any symmetric operator can be written as

$$K = a_2 P_2 + a_1 P_1 + a_w P_0^w + a_s P_0^s + a_\times P_0^\times \tag{5.2.4}$$

whose inverse operator is given by

$$K^{-1} = \frac{1}{a_2}P_2 + \frac{1}{a_1}P_1 + \frac{1}{a_s a_w - a_\times^2}\left(a_s P_0^w + a_w P_0^s - a_\times P_0^\times\right) \tag{5.2.5}$$

provided

$$g(k) \equiv a_s a_w - a_\times^2 \neq 0. \tag{5.2.6}$$

The Lagrangian (5.1.9), without the ghost piece, can be rewritten as [2]

$$L = \frac{1}{4} h_{\mu\nu} K^{\mu\nu\rho\sigma} h_{\rho\sigma} \tag{5.2.7}$$

where

$$K^{\mu\nu\rho\sigma} = (k^2 - m_1^2)P_2 + (\frac{1}{4M^4}k^6 - m_1^2)P_1 + a_s P_0^s + a_w P_0^w + a_\times P_0^\times \tag{5.2.8}$$

with

$$a_s = (1 - 3c_3)k^2 - m_1^2 + 3m_2^2 \tag{5.2.9a}$$
$$a_w = (2c_2 - c_3 - 1)k^2 - m_1^2 + m_2^2 \tag{5.2.9b}$$
$$a_\times = \sqrt{3}(c_2 k^2 - c_3 k^2 + m_2^2). \tag{5.2.9c}$$

Thus the propagator is given by

$$\Delta = K^{-1} = \frac{P_2}{k^2 - m_1^2} + \frac{4M^4 P_1}{k^6 - 4M^4 m_1^2} + \frac{1}{g(k)}\left(a_s P_0^w + a_w P_0^s - a_\times P_0^\times\right) \tag{5.2.10}$$

where

$$g(k) = (2c_3 - 3c_2^2 + 2c_2 - 1)k^4 - 2m_2^2 k^2 + 2(2c_3 - c_2)m_1^2 k^2 + m_1^4 - 4m_1^2 m_2^2. \tag{5.2.11}$$

5.3 Coupling to the matter

If we consider a generic coupling to the matter of the form

$$\frac{1}{2}(\lambda_1 T^{\mu\nu} + \lambda_2 T \eta^{\mu\nu})h_{\mu\nu} \equiv \frac{1}{2} T_{tot}^{\mu\nu} h_{\mu\nu} \tag{5.3.1}$$

the interaction between different sources is completely characterized by [12]:

$$S_{int} = \int d^4k T_{tot}(k)^*_{\mu\nu} \Delta^{\mu\nu\rho\sigma} T_{tot}(k)_{\rho\sigma}. \tag{5.3.2}$$

If we consider conserved sources, i.e.

$$\partial_\mu T^{\mu\nu} = k_\mu T^{\mu\nu} = 0 \tag{5.3.3}$$

so that in the contractions of the projectors with $T_{\mu\nu}$ we have

$$\theta_{\mu\nu} T^{\mu\rho} = \eta_{\mu\nu} T^{\mu\rho} \tag{5.3.4a}$$

$$\lambda_{\mu\nu} T^{\mu\rho} = 0 \tag{5.3.4b}$$

and using the traces of the projectors

$$\mathrm{tr}\,\theta_{\mu\nu} = 3 \tag{5.3.5a}$$

$$\mathrm{tr}\,\lambda_{\mu\nu} = 1 \tag{5.3.5b}$$

$$\mathrm{tr}\,P_2 \equiv \eta^{\mu\nu}(P_2)_{\mu\nu\rho\sigma} = 0 \tag{5.3.5c}$$

$$\mathrm{tr}\,P_1 \equiv \eta^{\mu\nu}(P_1)_{\mu\nu\rho\sigma} = 0 \tag{5.3.5d}$$

$$\mathrm{tr}\,P_0^s \equiv \eta^{\mu\nu}(P_0^s)_{\mu\nu\rho\sigma} = \theta_{\rho\sigma} \tag{5.3.5e}$$

$$\mathrm{tr}\,P_0^w \equiv \eta^{\mu\nu}(P_0^w)_{\mu\nu\rho\sigma} = \lambda_{\rho\sigma} \tag{5.3.5f}$$

$$\mathrm{tr}\,P_0^\times \equiv \eta^{\mu\nu}(P_0^\times)_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}(\theta_{\rho\sigma} + 3\lambda_{\rho\sigma}) \tag{5.3.5g}$$

we find that

$$T^*_{tot} P_2 T_{tot} = \lambda_1^2 \left(T^*_{\mu\nu} T^{\mu\nu} - \frac{1}{3}|T|^2 \right) \tag{5.3.6}$$

$$T^*_{tot} P_1 T_{tot} = 0 \tag{5.3.7}$$

$$T^*_{tot} P_0^s T_{tot} = \left(\frac{\lambda_1^2}{3} + 2\lambda_1\lambda_2 + 3\lambda_2^2 \right) |T|^2 \tag{5.3.8}$$

$$T^*_{tot} P_0^w T_{tot} = \lambda_2^2 |T|^2 \tag{5.3.9}$$

$$T^*_{tot} P_0^\times T_{tot} = \frac{2}{\sqrt{3}}(\lambda_1\lambda_2 + 3\lambda_2^2)|T|^2. \tag{5.3.10}$$

Hence the interaction Lagrangian is

$$L_{int} = T^*_{tot} \Delta T_{tot} = \frac{\lambda_1^2}{k^2 - m_1^2} T^*_{\mu\nu} T^{\mu\nu} + \left(\frac{\tilde{P}_0}{g(k)} - \frac{\lambda_1^2}{3(k^2 - m_1^2)} \right) |T|^2 \tag{5.3.11}$$

where

$$\tilde{P}_0 = \frac{1}{3}\lambda_1^2 a_w + 2\lambda_1\lambda_2(a_w - \frac{a_\times}{\sqrt{3}}) + \lambda_2^2(3a_w + a_s - 2\sqrt{3}a_\times). \tag{5.3.12}$$

5.4 Massive Fierz-Pauli Lagrangian

In this case the parameters of the Lagrangian are given by

$$c_2 = c_3 = 1$$

and

$$m_1^2 = m_2^2.$$

Hence, from (5.2.11), we have that

$$g(k) = -3m_1^4 \tag{5.4.1}$$

which does not depend on the momentum k. This means that the contribution of $\tilde{P}_0$ to the interaction Lagrangian (5.3.11) corresponds only to a contact term, which doesn't contribute to interactions between different sources.

We are thus only left with the term involving P_2, which is

$$L_{int} = \frac{\lambda_1^2}{k^2 - m_1^2} \left(T^*_{\mu\nu} T^{\mu\nu} - \frac{1}{3}|T|^2 \right). \tag{5.4.2}$$

The factor $\frac{1}{3}$ in front of $|T|^2$, different from the familiar $\frac{1}{2}$ which is encountered in linearized General Relativity, produces the well known vDVZ discontinuity in the mass-less limit [13–15].

5.5 Full TDiff invariant Lagrangian

In this case we only set $m_1 = 0$. From (5.2.11) we have

$$g(k) = 2(\Delta c_3 k^2 - m_2^2)k^2 \tag{5.5.1}$$

where Δc_3 is given by (2.4.20). We notice that $g(k)$ is quartic in the momenta, while only the terms proportional to $\lambda_1\lambda_2$ and λ_2^2 in (5.3.12) are quadratic in the momenta; indeed, using (5.2.9):

$$\lambda_2^2(3a_w + a_s - 2\sqrt{3}a_\times) = -2\lambda_2^2 k^2,$$

$$2\lambda_1\lambda_2(a_w - \frac{a_\times}{\sqrt{3}}) = 2\lambda_1\lambda_2(c_2 - 1)k^2,$$

$$\frac{1}{3}\lambda_1^2 a_w = \frac{1}{3}\lambda_1^2 \left[(2c_2 - c_3 - 1)k^2 + m_2^2\right].$$

Hence, decomposing $g(k)^{-1}$ as

$$\frac{1}{g(k)} = \frac{1}{2m_2^2}\left(\frac{1}{k^2 - \frac{m_2^2}{\Delta c_3}} - \frac{1}{k^2}\right) \tag{5.5.2}$$

we find that

$$\frac{\tilde{P}_0}{g(k)} = -\frac{\lambda_1^2}{6k^2} - \left[\left(\lambda_2 + \frac{1-c_2}{2}\lambda_1\right)^2 + \frac{\lambda_1^2 \Delta c_3}{6}\right]\frac{1}{\Delta c_3 k^2 - m_2^2}. \tag{5.5.3}$$

Substituting in (5.3.11) and adding the contribution given by P_2, which is (5.4.2) with $m_1 = 0$, we find the interaction Lagrangian:

$$L_{int} = \frac{\lambda_1^2}{k^2}\left(T^*_{\mu\nu}T^{\mu\nu} - \frac{1}{2}|T|^2\right) - \left[\left(\lambda_2 + \frac{1-c_2}{2}\lambda_1\right)^2 + \frac{\lambda_1^2 \Delta c_3}{6}\right]\frac{|T|^2}{\Delta c_3 k^2 - m_2^2}. \tag{5.5.4}$$

We can see that in this case the mass-less interaction between conserved sources is the same as in standard linearized General Relativity, since we find the familiar factor $\frac{1}{2}$ in front of $|T|^2$.
In addition there is a massive scalar interaction, with an effective squared mass

$$m_{eff}^2 = \frac{m_2^2}{\Delta c_3} > 0 \tag{5.5.5}$$

(since both m_2^2 and Δc_3 must be negative according to our previous analysis) and an effective coupling

$$\lambda_{eff}^2 = -\frac{1}{\Delta c_3}\left[\left(\lambda_2 + \frac{1-c_2}{2}\lambda_1\right)^2 + \frac{\lambda_1^2 \Delta c_3}{6}\right]. \tag{5.5.6}$$

5.6 Mass-less Diff and WTDiff Lagrangian

We already now that the mass-less quadratic Diff-invariant Lagrangian is the lowest order of the General Relativity Lagrangian. We thus expect to have an interaction Lagrangian

$$L_{int} = \frac{\lambda_1^2}{k^2}\left(T^*_{\mu\nu}T^{\mu\nu} - \frac{1}{2}|T|^2\right). \tag{5.6.1}$$

From general arguments, also in mass-less WTDiff theories, we expect to have the same interaction Lagrangian, since Diff and WTDiff theories differ only by an integration constant (see section 3.3), but have the same degrees of freedom (see section 2.4).

Indeed, setting $\Delta c_3 = 0$, in both Diff and WTDiff theories, an additional term $m_2^2 h^2$ in the Lagrangian could be thought of as the additional gauge fixing which removes the redundancy under the supplementary Weyl or full Diff symmetry.
With $\Delta c_3 = 0$ in (5.5.4), the second term becomes a contact term, and the interaction, as expected, is given by (5.6.1).

6 Matter Lagrangian and the "active" energy-momentum tensor

6.1 Linear approximation

In the following, by (active) energy-momentum tensor (**EMT**) we mean the source of gravity, i.e. the term in the equations of motion that determines how the gravitational field is generated. To be more precise, given a generic Lagrangian

$$L = L_g(g_{\mu\nu}) + L_m(\{\phi\}, g_{\mu\nu}) \tag{6.1.1}$$

where L_g is the pure gravitational Lagrangian while L_m is the generic matter Lagrangian ($\{\phi\}$ denotes the set of all non-gravitational fields), the EMT is defined as

$$T_{\mu\nu} \equiv \frac{\delta L_m}{\delta g^{\mu\nu}} \,. \tag{6.1.2}$$

At a linear level, using the perturbation $h_{\mu\nu}$ upon the flat metric ($g^{\mu\nu} \approx \eta^{\mu\nu} - kh^{\mu\nu}$), the matter Lagrangian is $L_m(\{\phi\}, h_{\mu\nu})$ and the EMT is

$$kT_{\mu\nu} \equiv -\frac{\delta L_m}{\delta h^{\mu\nu}} \,. \tag{6.1.3}$$

To make an example, let's consider a scalar field ϕ which in a freely falling locally inertial reference system has the Lagrangian

$$L_m^0 = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{6.1.4}$$

Assuming symmetry under

$$\phi \longrightarrow -\phi$$

to avoid classical instabilities, the allowed matter Lagrangian, up to linear terms in h, can generically be written as [16]

$$\begin{aligned} L_m &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \\ &\quad + k\left(-\frac{\mu_1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\mu_2}{4}h\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\mu_3}{2}hV(\phi)\right). \end{aligned} \tag{6.1.5}$$

Since the variation of the scalar field ϕ, by a linear transformation (2.1.13), is given by

$$\delta\phi = -\xi^\mu \partial_\mu \phi \tag{6.1.6}$$

to have TDiff invariance it is necessary that $\mu_1 = 1$.
The variation with respect to ϕ gives the equations of motion for the matter field:

$$-\Box\phi - V'(\phi) + k\left[\partial_\mu h^{\mu\beta}\partial_\beta\phi + h^{\mu\beta}\partial_\mu\partial_\beta\phi \frac{\mu_2}{2}(\partial_\mu h \partial^\mu\phi + h\Box\phi) - \frac{\mu_3}{2} h V'(\phi)\right] = 0 \tag{6.1.7}$$

while the variation with respect to $-h^{\mu\nu}$ gives the EMT:

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{\mu_2}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{\mu_3}{2}\eta_{\mu\nu}V(\phi) \tag{6.1.8}$$

We can show that **in TDiff theories the active energy-momentum tensor is generally not conserved**:

$$\partial_\mu T^{\mu\nu} = \frac{1}{2}\partial^\nu\phi\Box\phi - \frac{\mu_2 - 1}{2}\partial^\nu\partial_\mu\phi\partial^\mu\phi + \frac{\mu_3}{2}\partial^\nu\phi V'(\phi). \tag{6.1.9}$$

Using equation (6.1.7) at the k^0-th order to substitute $V'(\phi)$ we get

$$\partial_\mu T^{\mu\nu} = -\frac{\mu_3 - 1}{2}\partial^\nu\phi\Box\phi - \frac{\mu_2 - 1}{2}\partial^\nu\partial_\mu\phi\partial^\mu\phi \tag{6.1.10}$$

which is in general different from 0 unless $\mu_2 = \mu_3 = 1$. This actually corresponds to a Diff-invariant matter Lagrangian.

Depending on which symmetry characterizes the gravitational Lagrangian L_g, consistency imposes some restrictions to the matter Lagrangian.
For instance, a WTDiff-invariant gravitational Lagrangian, which leads to trace-less equations of motion for the gravitational part, forces also the EMT to be trace-less:

$$\eta^{\mu\nu}T_{\mu\nu} = -\frac{1}{2}(2\mu_2 - 1)\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2\mu_3 V(\phi) = 0 \tag{6.1.11}$$

i.e.

$$\begin{cases} \mu_2 = \dfrac{1}{2} \\ \mu_3 = 0. \end{cases} \tag{6.1.12}$$

On the other hand, as we shall better see in the next section, a Diff-invariant gravitational Lagrangian forces the matter Lagrangian to be Diff-invariant as well.

We note that in any case **the TDiff EMT does not reduce in flat space to the canonical energy-momentum tensor (or equivalently to the Belinfante one)**, which is well known to be conserved and has the form

$$T^{can}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}L_m. \tag{6.1.13}$$

This means that in TDiff theories the EMT tensor does not convey the Noether current corresponding to translation invariance.

6.2 Non-linear theory

A general TDiff-invariant matter action can be written in the form (see section 2.5):

$$S_m = \int d^4x f_m(-g)L_m. \tag{6.2.1}$$

The energy-momentum tensor is then given by

$$T_{\mu\nu} = f_m(-g)\frac{\delta L_m}{\delta g^{\mu\nu}} - |g|f'_m(-g)L_m g_{\mu\nu}. \tag{6.2.2}$$

Let's study the conservation law of the EMT, knowing that the action is invariant under TDiff.

Since a generic TDiff transformation of the metric is given by (2.1.14) with ξ^μ given by (2.1.17), TDiff invariance requires that [4]

$$\begin{aligned} 0 = T^{\mu\nu}\delta_t g_{\mu\nu} = T^{\mu\nu}\,[&\epsilon^{\rho\alpha_2\alpha_3\alpha_4}\partial_{\alpha_2}\Omega_{\alpha_3\alpha_4}\partial_\rho g_{\mu\nu} \\ &+ g_{\mu\rho}\partial_\nu(\epsilon^{\rho\alpha_2\alpha_3\alpha_4}\partial_{\alpha_2}\Omega_{\alpha_3\alpha_4}) + g_{\nu\rho}\partial_\mu(\epsilon^{\rho\alpha_2\alpha_3\alpha_4}\partial_{\alpha_2}\Omega_{\alpha_3\alpha_4})]\,. \end{aligned} \tag{6.2.3}$$

Defining the antisymmetric tensor

$$\omega^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha_3\alpha_4}\Omega_{\alpha_3\alpha_4} \tag{6.2.4}$$

up to total derivatives equation (6.2.3) can be rewritten as

$$(2\partial_\nu\partial_\rho T^\rho_\mu - \partial_\mu g_{\rho\sigma}\partial_\nu T^{\rho\sigma})\omega^{\mu\nu} = 0 \tag{6.2.5}$$

that is

$$2(\partial_\nu\partial_\rho T^\rho_\mu - \partial_\mu\partial_\rho T^\rho_\nu) = \partial_\mu g_{\rho\sigma}\partial_\nu T^{\rho\sigma} - \partial_\nu g_{\rho\sigma}\partial_\mu T^{\rho\sigma}. \tag{6.2.6}$$

Equation (6.2.6) can be rewritten as

$$\partial_\nu\partial_\rho T^\rho_\mu - \frac{1}{2}\partial_\nu T^{\rho\sigma}\partial_\mu g_{\rho\sigma} = \partial_\mu\partial_\rho T^\rho_\nu - \frac{1}{2}\partial_\mu T^{\rho\sigma}\partial_\nu g_{\rho\sigma} \tag{6.2.7}$$

which shows the $\mu \leftrightarrow \nu$ symmetry of each member. This implies that

$$\partial_\rho T^\rho_\mu - \frac{1}{2}T^{\rho\sigma}\partial_\mu g_{\rho\sigma} = \partial_\mu \Phi \tag{6.2.8}$$

for some function Φ.
Using the well known formula valid for any symmetric tensor S^ν_μ [17]:

$$\nabla_\nu S^\nu_\mu = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}S^\nu_\mu\right) - \frac{1}{2}\partial_\mu g_{\rho\sigma}S^{\rho\sigma} \tag{6.2.9}$$

equation (6.2.8) can be rewritten as

$$\nabla_\nu\left(\frac{T^\nu_\mu}{\sqrt{|g|}}\right) = \frac{1}{\sqrt{|g|}}\partial_\mu\Phi. \tag{6.2.10}$$

On the other hand, equation (6.2.6) can be rewritten as

$$\partial_\nu\partial_\rho T^\rho_\mu + \frac{1}{2}\partial_\mu T^{\rho\sigma}\partial_\nu g_{\rho\sigma} = \partial_\mu\partial_\rho T^\rho_\nu + \frac{1}{2}\partial_\nu T^{\rho\sigma}\partial_\mu g_{\rho\sigma} \tag{6.2.11}$$

whose $\mu \leftrightarrow \nu$ symmetry implies that

$$\partial_\rho T^\rho_\mu + \frac{1}{2}\partial_\mu T^{\rho\sigma}g_{\rho\sigma} = \partial_\mu\Phi'. \tag{6.2.12}$$

Finally, from (6.2.12)−(6.2.8) we find that

$$\Phi' - \Phi = \frac{T}{2} \tag{6.2.13}$$

which alone can't ensure the conservation of the EMT.

On the other hand, if we take a Diff-invariant matter Lagrangian, the EMT is conserved: the simple requirement of TDiff-invariance gives

$$0 = T^{\alpha\beta}(\xi^\rho \partial_\rho g_{\alpha\beta} + g_{\alpha\rho}\partial_\beta \xi^\rho + g_{\beta\rho}\partial_\alpha \xi^\rho) \tag{6.2.14}$$

which, up to total derivatives, conveys the fact that

$$0 = \partial_\rho g_{\alpha\beta} T^{\alpha\beta} - \partial_\beta T^\beta_\rho - \partial_\alpha T^\alpha_\rho = -2\sqrt{|g|}\nabla_\alpha \left(\frac{T^\alpha_\rho}{\sqrt{|g|}}\right). \tag{6.2.15}$$

For this reason the EMT is usually defined as

$$T^{\text{GR}}_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} T^{\text{Diff}}_{\mu\nu}. \tag{6.2.16}$$

Also if we only take a Diff-invariant gravitational Lagrangian, that is the Hilbert Lagrangian, the EMT must be conserved: the equations of motion read

$$G_{\mu\nu} = \frac{2k^2}{\sqrt{|g|}} T_{\mu\nu} \tag{6.2.17}$$

where $G_{\mu\nu}$ is the Einstein tensor. But Bianchi identities ensure that

$$\nabla_\mu G^\mu_\nu = 0 \tag{6.2.18}$$

which also implies that

$$\nabla_\mu \left(\frac{T^\mu_\nu}{\sqrt{|g|}}\right) = 0. \tag{6.2.19}$$

Since in TDiff theories the EMT is generally not conserved, the assumptions made in chapter (5) on the conservation of the matter sources are only approximations.

6.3 The weight of energy and the Cosmological Constant problem

We are going now to show that in Transverse Theories it is possible to build theoretically consistent models in which the potential energy can have a tiny weight (that is, its coupling to the gravitational field), or even models in which the potential energy does not weigh at all [4].
Since the potential energy contains the vacuum energy, such models could solve the direct cosmological constant problem, as anticipated in section 2.2. Anyway, the models we are going to present are not expected to be realistic, but want only to be some examples that, *theoretically*, could solve the problem.

One of the possibilities could be the action (6.2.1). If we set

$$f_m(-g) \equiv 1 \tag{6.3.1}$$

the action reads

$$S_m = \int d^4x \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{6.3.2}$$

We can see that only the kinetic energy is coupled to the gravitational field, while the potential energy doesn't weigh at all. The energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \tag{6.3.3}$$

Only the kinetic part enters in the equations of motion.

Even a more general action than (6.2.1) could be written, giving different couplings to the determinant of the metric g for the different terms of the matter Lagrangian. For instance we could write

$$L_m = f_k(-g) \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - f_v(-g) V(\phi) \tag{6.3.4}$$

whose EMT is

$$T_{\mu\nu} = \frac{1}{2} f_k(-g) \partial_\mu \phi \partial_\nu \phi - \frac{f_k'(-g)}{2} |g| g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + f_v'(-g) |g| g_{\mu\nu} V(\phi). \tag{6.3.5}$$

The linearized form of the Lagrangian (6.3.4) is

$$L_m = \frac{f_k(1)}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - f_v(1)V(\phi)$$

$$+ k\left[-\frac{f_k(1)}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{f_k'(1)}{2}h\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - f_v'(1)hV(\phi)\right]. \quad (6.3.6)$$

If we require (6.1.4) we have the condition

$$f_k(1) = f_v(1) = 1 \quad (6.3.7)$$

and equation (6.3.6) becomes exactly the same as (6.1.5) with

$$\begin{cases} \mu_2 = 2f_k'(1) \\ \mu_3 = 2f_v'(1). \end{cases} \quad (6.3.8)$$

Also in this case, playing with the only function f_v, we are able to give the potential energy the desired weight.

6.4 Connections with ρ and p

If we assume the matter to be a perfect fluid, we can define the energy density, the pressure and the velocity as [18]

$$\rho \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) \quad (6.4.1a)$$

$$p \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \quad (6.4.1b)$$

$$u^\mu \equiv \frac{g^{\mu\nu}\partial_\nu\phi}{\sqrt{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}} \quad (6.4.1c)$$

so that

$$\partial_\mu\phi\partial_\nu\phi = (\rho + p)u_\mu u_\nu \quad (6.4.2)$$

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \rho + p \quad (6.4.3)$$

$$2V(\phi) = \rho - p. \quad (6.4.4)$$

Thus, if we start from a general matter Lagrangian of the form (6.3.4), the energy-momentum tensor (6.3.5) can be rewritten as

$$\begin{aligned} T_{\mu\nu} = & \frac{\rho}{2}\left[f_k(-g)u_\mu u_\nu - \left(f_k'(-g) - f_v'(-g)\right)|g|g_{\mu\nu}\right] \\ & + \frac{p}{2}\left[f_k(-g)u_\mu u_\nu - \left(f_k'(-g) + f_v'(-g)\right)|g|g_{\mu\nu}\right]. \end{aligned} \tag{6.4.5}$$

In the case of General Relativity, where $f_k(-g) = f_v(-g) = \sqrt{-g}$, the EMT (6.2.16) reduces to

$$T_{\mu\nu}^{\mathrm{GR}} = \frac{2}{\sqrt{|g|}}T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{6.4.6}$$

which, in flat space and in the matter rest frame, gives the well known expression

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p). \tag{6.4.7}$$

7 Particles and matter in Transverse Theories

7.1 Particle behavior

Let's examine the consequences of the hypothesis that the matter action is invariant only under TDiff.
Let' start from a Lagrangian of the form (6.3.4) with

$$V(\phi) = -\hat{m}^2\phi. \tag{7.1.1}$$

The equations of motion are given by

$$\partial_\mu(f_k(-g)g^{\mu\nu}\partial_\nu\phi) + f_v(-g)\hat{m}^2\phi = 0. \tag{7.1.2}$$

If we take a WKB expansion of the field in terms of the eikonal [19]

$$\phi = \mathrm{Re}\left[e^{i\left(\frac{\psi_0}{h}+\psi_1+\ldots\right)}\right] \tag{7.1.3}$$

and define

$$m^2 \equiv h^2\hat{m}^2 \tag{7.1.4}$$
$$k_\mu \equiv \partial_\mu\psi_0 \tag{7.1.5}$$
$$p_\mu \equiv \partial_\mu\psi_1 \tag{7.1.6}$$

then the dominant order (i.e. the geometrical optics approximation) in formal power of h is $O(h^{-2})$ and reads

$$f_k(-g)k^2 = f_v(-g)m^2 \tag{7.1.7}$$

while the second order approximation (physical optics), of order h^{-1}, yields

$$k\cdot p = \frac{\partial_\mu(f_k k^\mu)}{2f_v}\,. \tag{7.1.8}$$

The trajectories of the particles are geodesics only if $f_k = f_v$: from equation (7.1.7) we get

$$k^2 = m^2 \tag{7.1.9}$$

and since m is a constant parameter, deriving the former equation we have

$$\dot{k}_\mu \equiv k^\alpha \nabla_\alpha k_\mu = 0. \qquad (7.1.10)$$

Hence only if $f_k = f_v$ we can say that the passive gravitational mass is the same as the inertial mass, so that the Weak Equivalence Principle is satisfied.

7.2 Perfect fluid

For a matter Lagrangian

$$L_m = f_m(-g)\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \qquad (7.2.1)$$

the equations of motion are

$$\partial_\mu(f_m(-g)g^{\mu\nu}\partial_\nu\phi) + f_m(-g)V'(\phi) = 0. \qquad (7.2.2)$$

Since for a generic vector A^μ [17]

$$\partial_\mu A^\mu = \nabla_\mu A^\mu - A^\mu \frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}} \qquad (7.2.3)$$

identifying $A^\mu = f_m(-g)g^{\mu\nu}\partial_\nu\phi$, we can rewrite (7.2.2) as

$$\nabla_\mu\nabla^\mu\phi + V'(\phi) + g^{\mu\nu}\partial_\mu\phi\partial_\nu\chi = 0 \qquad (7.2.4)$$

where

$$\chi \equiv \log\frac{f_m(-g)}{\sqrt{-g}}\,. \qquad (7.2.5)$$

Now, indicating through a dot over a quantity its derivative in the direction of u^μ

$$\dot{f} \equiv u^\mu\nabla_\mu f, \qquad (7.2.6)$$

defining the optical expansion of the timelike congruence [20]

$$\theta \equiv \nabla_\mu u^\mu, \qquad (7.2.7)$$

multiplying by $\partial_\alpha \phi$ equation (7.2.4), and using the definitions (6.4.1), we find [19]

$$\frac{1}{2} u_\alpha (\dot{\rho} + \dot{p}) + u_\alpha \theta (\rho + p) + \frac{1}{2} \nabla_\alpha (\rho - p) + u_\alpha \dot{\chi} (\rho + p) = 0. \tag{7.2.8}$$

Taking into account that

$$\nabla_\alpha p = \nabla^\mu \phi \nabla_\mu \nabla_\alpha \phi - V'(\phi) \nabla_\alpha \phi = \frac{1}{2} u_\alpha (\dot{\rho} + \dot{p}) + \dot{u}_\alpha (\rho + p) - \frac{1}{2} \nabla_\alpha (\rho - p), \tag{7.2.9}$$

we arrive to

$$u_\alpha (\dot{\rho} + \dot{p}) + u_\alpha \theta (\rho + p) + \dot{u}_\alpha (\rho + p) - \nabla_\alpha p + u_\alpha \dot{\chi} (\rho + p) = 0. \tag{7.2.10}$$

Since $u_\alpha \dot{u}^\alpha = 0$, projecting along u^α equation (7.2.10), we get

$$\dot{\rho} + (\rho + p)(\theta + \dot{\chi}) = 0 \tag{7.2.11}$$

which, for a fluid verifying the equation of state $p = \omega \rho$ can be rewritten as

$$\dot{\rho} + (1 + \omega) \rho (\theta + \dot{\chi}) = 0. \tag{7.2.12}$$

We notice that this continuity equation differs by the last term from the one we find in General Relativity:

$$\dot{\rho} + (1 + \omega) \rho \theta = 0. \tag{7.2.13}$$

Since [20] $\theta = 3\frac{\dot{a}}{a}$, in General Relativity the continuity equation (7.2.13), which we can rewrite as

$$\frac{\dot{\rho}}{\rho} = -3(1 + \omega) \frac{\dot{a}}{a} \tag{7.2.14}$$

gives the well known behavior of pressureless matter ($\omega = 0$) and radiation ($\omega = \frac{1}{3}$)

$$\rho_m \sim a^{-3} \tag{7.2.15}$$

$$\rho_r \sim a^{-4}. \tag{7.2.16}$$

Instead in transverse theories the corresponding relationship is

$$\rho = \left(\frac{1}{a^3} \cdot \frac{\sqrt{-g}}{f_m(-g)} \right)^{1+\omega} \tag{7.2.17}$$

so that the redefined quantity

$$\rho' \equiv \rho e^{(1+\omega)\chi} \tag{7.2.18}$$

verifies the same continuity equation (7.2.13) as in General Relativity.

On the other hand, the transverse equation obtained by projecting through the transverse projector $h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ equation (7.2.10) reads

$$\dot{u}^\beta(\rho + p) = h^{\alpha\beta} \nabla_\alpha p \tag{7.2.19}$$

which is exactly the same equation we find in General Relativity.

8 Transverse Theories and experiments

8.1 Matter-graviton coupling for massive TDiff Lagrangian

Let us examine the special case where the gravitational Lagrangian is TDiff-invariant with a massive term. As seen in section (4.1), only a mass term

$$\frac{1}{4}m_2^2 h^2$$

is allowed, while it must be $m_1 = 0$.
For a conserved energy-momentum tensor coupled to gravity in the form (5.3.1):

$$L_I = \frac{1}{2}(\lambda_1 T^{\mu\nu} + \lambda_2 T\eta^{\mu\nu})h_{\mu\nu} \tag{8.1.1}$$

as seen in section (5.5), in momentum space the interaction between two different sources is given by (5.5.4):

$$L_{int} = \frac{\lambda_1^2}{k^2}\left(T^*_{\mu\nu}T^{\mu\nu} - \frac{1}{2}|T|^2\right) - \left[\left(\lambda_2 + \frac{1-c_2}{2}\lambda_1\right)^2 + \frac{\lambda_1^2\Delta c_3}{6}\right]\frac{|T|^2}{\Delta c_3 k^2 - m_2^2}. \tag{8.1.2}$$

Let's take as an example the linear matter Lagrangian (6.1.5):

$$L_m = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

$$+ k\left(-\frac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\mu_2}{4}h\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\mu_3}{2}hV(\phi)\right)$$

whose EMT is (6.1.8):

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{\mu_2}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{\mu_3}{2}\eta_{\mu\nu}V(\phi). \tag{8.1.3}$$

The coupling would be of the form (8.1.1) with $\lambda_1 = -k$ $\lambda_2 = 0$.

Unfortunately, if we take a general TDiff-invariant matter Lagrangian, as seen in sections (6.1, 6.2) the EMT is not conserved, so that the interaction Lagrangian should not be of the form (8.1.2) anymore.

Anyway, for the case we are analyzing, we notice that a conserved tensor can be defined as [16]

$$\Theta_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\left(\frac{1-\mu_2}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + (\mu_3 - 1)V(\phi)\right) =$$

$$= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\left(\frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - V(\phi)\right). \qquad (8.1.4)$$

In the particular case that $\mu_3 = 2\mu_2 - 1$ (which includes the Diff-invariant matter Lagrangian), the EMT (8.1.3) can be written in terms of the new one (8.1.4) and its trace:

$$T_{\mu\nu} = \Theta_{\mu\nu} + \frac{\mu_2 - 1}{2}\eta_{\mu\nu}\Theta \qquad (8.1.5)$$

Hence, in terms of the new EMT, the coupling to gravity is of the form (8.1.1) with

$$\lambda_1 = -k, \qquad \lambda_2 = -\frac{k}{2}(\mu_2 - 1). \qquad (8.1.6)$$

Now, the exchange of additional massive scalar degrees of freedom produces a Yukawa-like potential which is usually parameterized as [21]:

$$V(r) \sim \frac{1}{r}\left(1 + \alpha e^{-r/\lambda}\right) \qquad (8.1.7)$$

where the parameter α is the ratio between the scalar and the spin 2 couplings, while λ gives the range of the interaction, or equivalently the mass of the scalar exchanged. In our particular case

$$\alpha = -\frac{(\lambda_2 + \frac{1-c_2}{2}\lambda_1)^2}{\Delta c_3 \lambda_1^2} - \frac{1}{6} = -\frac{(\mu_2 - c_2)^2}{4\Delta c_3} - \frac{1}{6} \qquad (8.1.8)$$

$$\lambda^2 = \frac{\Delta c_3}{m_2^2}. \qquad (8.1.9)$$

We remember that, in order to avoid ghosts, one has to impose $\Delta c_3 < 0$, so that also $m_2^2 < 0$.

According to [21], there are important constraints on the strength of hypothetical Yukawa interactions for wide ranges of λ. Through (8.1.8) and (8.1.9) it is then possible to constrain the space of parameters of the linearized theory.
We will use figures 4, 5 and 9 of [21], which show allowed and excluded regions for α corresponding to the ranges $(10^{-2} \div 10^{14})m$, $(10^{-2} \div 10^{-6})m$ and $(10^{-6} \div 10^{-9})m$ respectively.

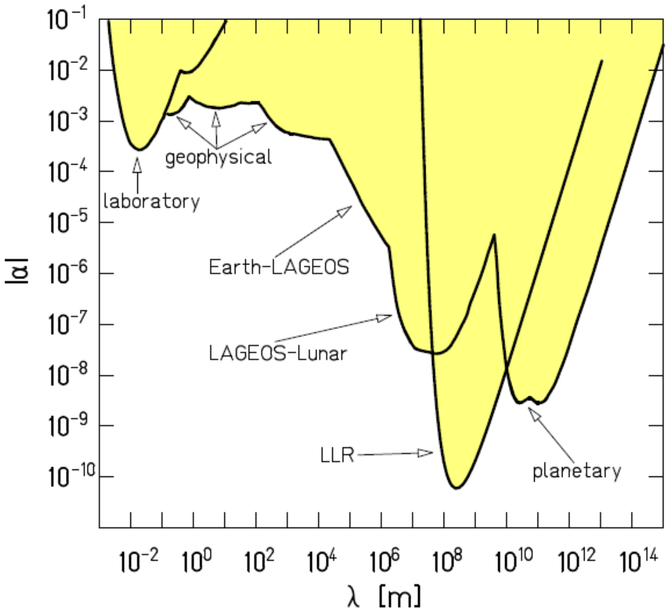

Figure 1: 95%-confidence-level constraints on inverse square law violating Yukawa interactions with $\lambda > 1$cm.

Since we are just interested in general behaviors and not in accurate results we will approximate the experimental curves by straight lines, so we have experimentally allowed regions of the form

$$|\alpha| < k\lambda^a \tag{8.1.10}$$

where k and a are extrapolated from the plots in [21].
There are however four parameters to play with, i.e. μ_2, m_2^2, c_2 and c_3. First, it is interesting to see the order of magnitude for the mass once we fix the values of c_2 and c_3. The result is plotted in Fig.4 [16].

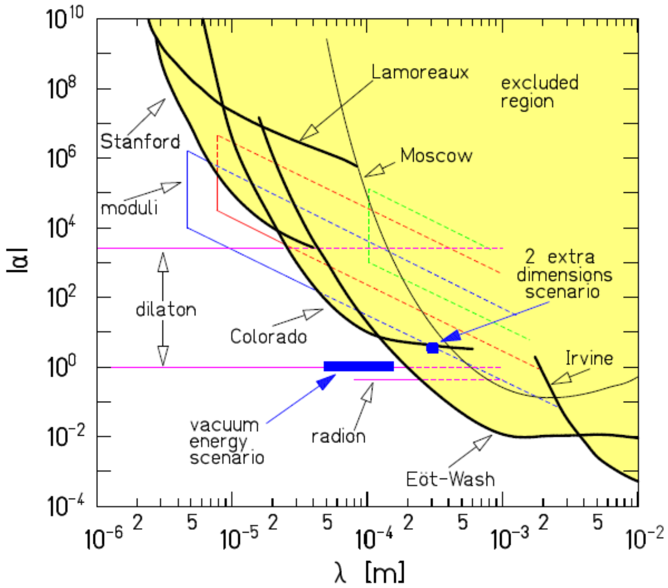

Figure 2: 95%-confidence-level constraints on inverse square law violating Yukawa interactions with $1\mu\text{m} < \lambda < 1\text{cm}$.

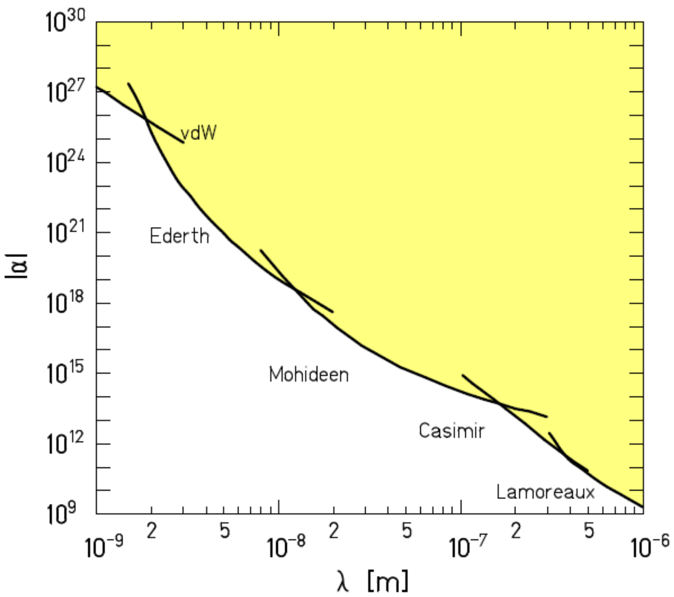

Figure 3: 95%-confidence-level constraints on inverse square law violating Yukawa interactions with $1\text{nm} < \lambda < 1\mu\text{m}$.

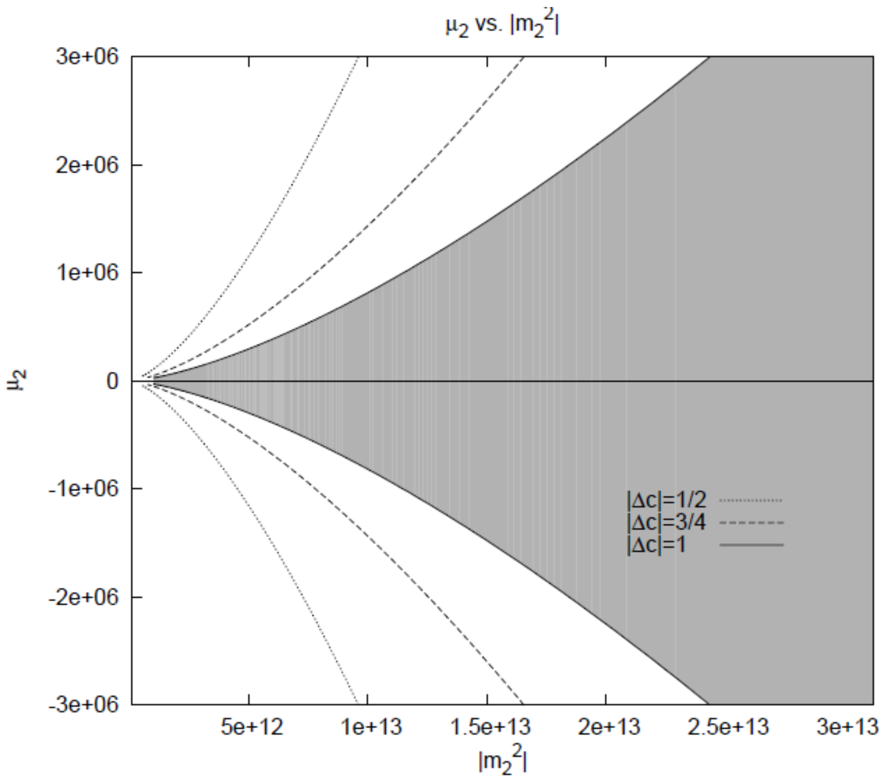

Figure 4: The shadowed region shows experimentally allowed regions for $|m_2^2|$ in m^{-2} and μ_2 for given values of c_2 and c_3 expressed in terms of Δc_3 and in the range $1\text{nm} < \lambda < 1\mu\text{m}$. For the dashed lines the shadowed region has to be considered extended until the dashed lines.

It can be seen that greater values for the mass are favored, being the lower bound around

$$|m_2^2| \sim 5 \cdot 10^{11}\text{m}^{-2} \sim 0,02\ \text{eV}^2. \tag{8.1.11}$$

Another possibility is to fix m_2^2 and μ_2 and see in the plane (c_2, c_3) how far from Diff-invariance we can move away, remembering that we have always to take into account the restriction $\Delta c_3 \leq 0$.
In the range 1nm$< \lambda < 1\mu$m there is no hope of seeing an experimental curve that appreciably deviates from the parabola $\Delta c_3 = 0$, because of (8.1.9) and the tiny value of λ. Let's then consider ranges for greater values of λ. Some examples of resulting plots are given in Figs 5,6 and 7, where has always been set $\mu_2 = 0$ since other values of μ_2 simply shift the experimental allowed curves along the parabola; but the qualitative results remain unchanged.

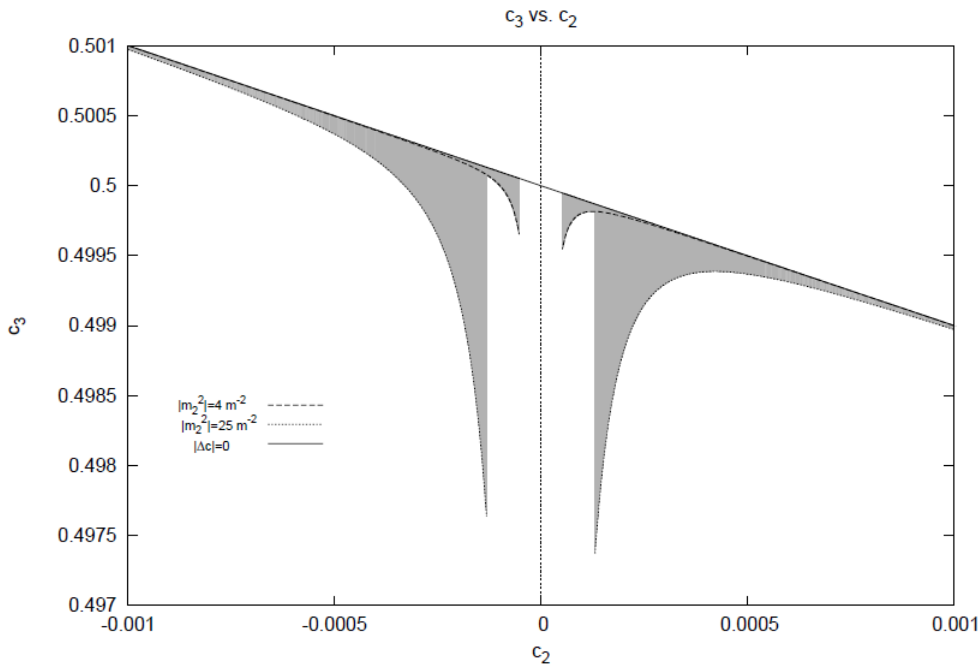

Figure 5: Experimentally allowed region in the plane (c_2, c_3), for a couple of values of the mass, in the range $10^{-6}\text{m} < \lambda < 10^{-2}\text{m}$. The plot is restricted to the zone where the curve appreciably deviates from the parabola $\Delta c_3 = 0$.

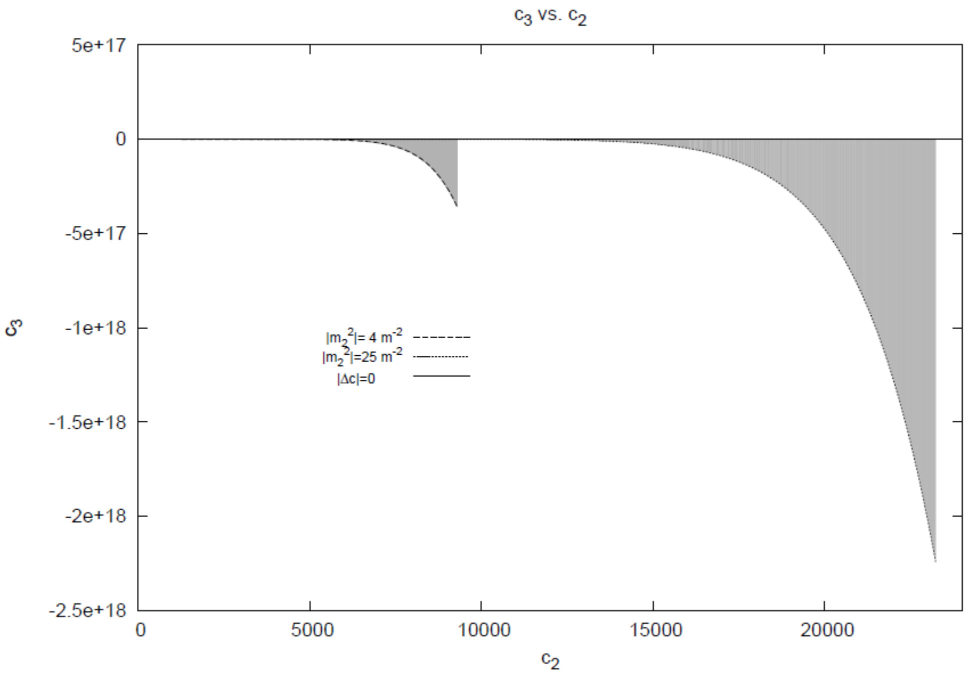

Figure 6: Experimentally allowed region in the plane (c_2, c_3), for a couple of values of the mass, in the range $10^{-2}\text{m} < \lambda < 10^{14}\text{m}$. We only show the positive c_2 branch. The parabola is indistinguishable from the c_2 axis.

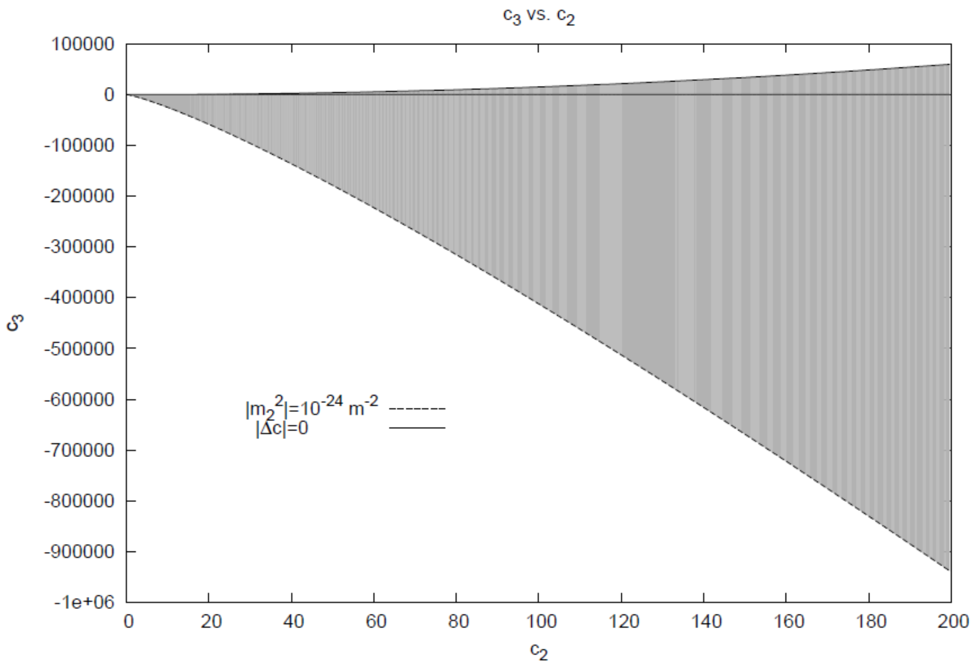

Figure 7: Experimentally allowed region in the plane (c_2, c_3), in the range $10^{-2}\mathrm{m} < \lambda < 10^{14}\mathrm{m}$ for a very tiny mass. That allows us to see the parabola, which was hidden in the previous figure.

8.2 Masses in Transverse theories

In physics we can distinguish three different concepts for what is commonly called "mass", with three deeply different meanings:

- The **inertial mass** m_i is an intrinsic property of a particle, independent of the environment, which enters in all its interactions, like other parameters (as may be the charge, the passive gravitational mass,...). Classically, the inertial mass is the property that regulates the response of a body to any applied force, and it's determined by applying a force to an object and measuring the acceleration that results from that force.
- The **passive gravitational mass** m_p is a kind of "charge" of the gravitational interaction, i.e. the property that regulates the response of a particle to an externally given gravitational field. It is determined by dividing an object's weight by its free-fall acceleration.
- The **active gravitational mass** m_a is the source of gravity, i.e. the property of a particle that regulates the "amount" of gravitational field generated by the particle. The gravitational field can be measured by allowing a small test object to freely fall and measuring its free-fall acceleration.

As already seen in section (1.1), the WEP postulates the equivalence between the inertial and the passive gravitational mass, implying universality of the acceleration of free fall. This is one of the best established experimental facts in physics, with a relative precision of at least 10^{-12}, as quoted in [22]. As seen in section (7.1), transverse theories with $f_k(|g|) = f_v(|g|)$ verify the Weak Equivalence Principle.

On the other hand, the equality of the active gravitational mass to the other two lies essentially on the third of Newton's laws, that is momentum conservation; in this case the experimental precision seems even better. It has indeed been recently claimed [23] that the bounds on relative violations of Newton's third law are $\sim 10^{-13}$.
To be specific, what is bound to be small is the difference of the quotient of the active and passive gravitational masses for distinct bodies (dubbed 1 and 2 in the following), that is

$$S(1,2) = \left| \left(\frac{m_a}{m_p} \right)_1 - \left(\frac{m_a}{m_p} \right)_2 \right| \leq 10^{-13}. \tag{8.2.1}$$

Inequality between active and passive gravitational masses is traduced in an unbalanced force that accelerates the center of mass of the interacting pair:

$$\vec{F}_{12} = S(1,2) G\, m_p^1\, m_p^2 \frac{\vec{r}_{12}}{r_{12}^3}. \tag{8.2.2}$$

Let's consider now what may be the active gravitational mass in Transverse Theories.
As seen in section (6.4), in Transverse theories the active EMT is given by (6.4.5):

$$\begin{aligned} T_{\mu\nu} = & \frac{\rho}{2} \left[f_k(|g|) u_\mu u_\nu - \left(f_k'(|g|) - f_v'(|g|) \right) |g| g_{\mu\nu} \right] \\ & + \frac{p}{2} \left[f_k(|g|) u_\mu u_\nu - \left(f_k'(|g|) + f_v'(|g|) \right) |g| g_{\mu\nu} \right]. \end{aligned}$$

If we define, as in General Relativity,

$$\tilde{T}_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} T_{\mu\nu} \tag{8.2.3}$$

we find

$$\begin{aligned} \tilde{T}_{\mu\nu} = & \rho \left[\frac{f_k(|g|)}{\sqrt{|g|}} u_\mu u_\nu - \left(f_k'(|g|) - f_v'(|g|) \right) \sqrt{|g|} g_{\mu\nu} \right] \\ & + p \left[\frac{f_k(|g|)}{\sqrt{|g|}} u_\mu u_\nu - \left(f_k'(|g|) + f_v'(|g|) \right) \sqrt{|g|} g_{\mu\nu} \right]. \end{aligned} \tag{8.2.4}$$

If we want to write a scalar source of gravitation, in order to identify it with the active gravitational mass, we can define [19]:

$$m_a \equiv \tilde{T}_{\mu\nu} u^\mu u^\nu \tag{8.2.5}$$

so that, in the case of General Relativity,

$$m_a^{\mathrm{GR}} = \left[(\rho + p) u_\mu u_\nu - p\, g_{\mu\nu} \right] u^\mu u^\nu = \rho. \tag{8.2.6}$$

In TDiff theories the EMT is given by (8.2.4), so that the active gravitational mass is given by

$$
\begin{aligned}
m_a = \rho &\left[\frac{f_k(|g|)}{\sqrt{|g|}} - \left(f_k'(|g|) - f_v'(|g|) \right) \sqrt{|g|} \right] \\
+ p &\left[\frac{f_k(|g|)}{\sqrt{|g|}} - \left(f_k'(|g|) + f_v'(|g|) \right) \sqrt{|g|} \right].
\end{aligned}
\tag{8.2.7}
$$

We can measure the relative difference between general relativistic and transverse active masses through the quantity [19]

$$
\delta \equiv \frac{m_a - m_a^{\mathrm{GR}}}{m_a^{\mathrm{GR}}}. \tag{8.2.8}
$$

From (8.2.7) we get

$$
\begin{aligned}
\delta = &\frac{f_k(|g|) - \sqrt{|g|}}{\sqrt{|g|}} - \left(f_k'(|g|) - f_v'(|g|) \right) \sqrt{|g|} \\
&+ \frac{p}{\rho} \left[\frac{f_k(|g|)}{\sqrt{|g|}} - \left(f_k'(|g|) + f_v'(|g|) \right) \sqrt{|g|} \right].
\end{aligned}
\tag{8.2.9}
$$

We notice that even in the non-relativistic cold limit where

$$
\frac{p}{\rho} \approx 0 \tag{8.2.10}
$$

we have

$$
\delta = \frac{f_k(|g|) - \sqrt{|g|}}{\sqrt{|g|}} - \left(f_k'(|g|) - f_v'(|g|) \right) \sqrt{|g|} \neq 0. \tag{8.2.11}
$$

In the special case that $f_k = f_v \equiv f_m$, we have

$$
m_a = \frac{f_m(|g|)}{\sqrt{|g|}} \rho + \left[\frac{f_m(|g|)}{\sqrt{|g|}} - 2 f_m'(|g|) \sqrt{|g|} \right] p, \tag{8.2.12}
$$

$$
\delta = \frac{f_m(|g|) - \sqrt{|g|}}{\sqrt{|g|}} + \frac{p}{\rho} \cdot \frac{f_m(|g|) - 2|g| f_m'}{\sqrt{|g|}}, \tag{8.2.13}
$$

$$
\delta = \frac{f_m(|g|) - \sqrt{|g|}}{\sqrt{|g|}} \qquad \left(\frac{p}{\rho} \to 0\right). \tag{8.2.14}
$$

We thus must conclude that in any case, if we admit that in General Relativity all three masses are equal, in the transverse models we are considering the active gravitational mass differs from the other two. This will eventually lead to a violation (represented by the quantity δ) of Newton's third law, that must be carefully tuned up in order for it to be compatible with experiments.
Anyway, the constraint (8.2.1) implies only

$$\delta_1 - \delta_2 \leq 10^{-13} \tag{8.2.15}$$

which does not constrain the observable δ itself.
It is nevertheless true that, because of the dependence of δ on the determinant of the metric g, the ratio between m_a and m_p will depend on the particular point in the spacetime. Since [19] we can identify the ratio between the active and passive gravitational mass with Newton's constant G, this will lead to a violation of the "constancy of constants": that is, a violation of the SEP (see section (1.1)).

9 Transverse Theories and experiments: the PPN formalism

9.1 The Newtonian approximation

In the solar system gravitation is weak enough for Newton's theory of gravity to adequately explain all but the most minute effects: to an accuracy of about one part in 10^5, light rays travel on straight lines at constant speed, and test bodies move according to

$$\mathbf{a} = \nabla U \tag{9.1.1}$$

where $\mathbf{a}$ is the body's acceleration and U is the Newtonian gravitational potential produced by the rest-mass density ρ_0: in "geometrized" units, in which the speed of light and the gravitational constant as measured far from the solar system are unity, U is given by

$$U(\mathbf{x}, t) = \int \frac{\rho_0(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{9.1.2}$$

so that

$$\nabla^2 U = -4\pi\rho_0. \tag{9.1.3}$$

The definition of "rest-mass density" is actually a bit misleading, since by ρ_0 we mean a measure of the number density of baryons n, and nothing more; it is defined as the product of n with some standard figure for the mass per baryon (μ_0) in some well defined standard state:

$$\rho_0 \equiv n\mu_0. \tag{9.1.4}$$

From the standpoint of a metric theory of gravity, where the metric and the equations of motion become the primary theoretical entities, Newtonian physics may be viewed as a first order approximation in the expansion of the metric: if we consider a test body momentarily at rest ($dx^i/dt = 0$) in a static external gravitational field, from the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{9.1.5}$$

we get

$$a^i = \frac{d^2x^i}{dt^2} = -\Gamma^i_{00} = \frac{1}{2}g^{ik}\partial_k g_{00}. \tag{9.1.6}$$

Far from the Newtonian system we know that, in an appropriately chosen coordinate system, the metric must reduce to the Minkowski metric:

$$g_{\mu\nu} \longrightarrow \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{9.1.7}$$

Hence, in the presence of a very weak gravitational field, equation (9.1.6) can yield Newtonian gravitation (i.e. eq. 9.1.1) only if

$$g_{00} \approx 1 - 2U, \tag{9.1.8a}$$

$$g^{ik} \approx -\delta^{jk}. \tag{9.1.8b}$$

Anyway, the Newtonian limit no longer suffices when we begin to demand accuracies greater then a part in 10^5. For example, it cannot account for Mercury's additional perihelion shift of $\sim 5 \cdot 10^{-7}$ radians per orbit. Thus we need a more accurate approximation to the metric $g_{\mu\nu}$, that goes beyond Newtonian theory: that is, the "Post-Newtonian" limit.

9.2 The Post-Newtonian limit

In the solar system, in geometrized units (where U is dimensionless), the Newtonian gravitational potential U is nowhere larger than 10^{-5} [1]:

$$U \lesssim 10^{-5}. \tag{9.2.1}$$

Planetary velocities are related to U by virial relations which yield

$$v^2 \lesssim U. \tag{9.2.2}$$

The matter making up the Sun and planets is under pressure p, but this pressure is generally smaller than the matter's gravitational energy density ρU (p/ρ is $\sim 10^{-5}$ in the Sun and $\sim 10^{-10}$ in the Earth):

$$\frac{p}{\rho_0} \lesssim U. \tag{9.2.3}$$

Other forms of energy in the solar system (compressional energy, radiation, thermal energy,...) are small. We can define the specific energy density Π, that is the ratio of other kinds of energy densities to rest-mass density:

$$\Pi \equiv \frac{\rho - \rho_0}{\rho_0}. \tag{9.2.4}$$

In the solar system, $\Pi \sim 10^{-5}$ in the Sun and $\Pi \sim 10^{-9}$ in the Earth. Hence we can say that

$$\Pi \lesssim U. \tag{9.2.5}$$

We can thus assume that the quantities above are all of the same "order of smallness", denoted by ϵ, so that

$$U \sim v^2 \sim \frac{p}{\rho_0} \sim \Pi \sim 10^{-5} \sim \epsilon^2. \tag{9.2.6}$$

Moreover, since the time evolution of the solar system is governed by the motion of its constituents, we have

$$\frac{\partial}{\partial t} \sim \mathbf{v} \cdot \nabla \tag{9.2.7}$$

which implies that

$$\frac{|\partial/\partial t|}{|\partial/\partial x|} \sim O(\epsilon). \tag{9.2.8}$$

The "Post-Newtonian limit" is an expansion of the metric in a formal power of series in ϵ, up to one order beyond the Newtonian expansion.
In this post-Newtonian expansion, terms odd in ϵ (i.e. terms whose total number of v's and $(\partial/\partial t)$'s is odd) like for instance

$$\int \frac{\rho_0(\mathbf{x}', t) v_j(\mathbf{x}', t)}{|\mathbf{x}' - \mathbf{x}|} d^3x' \sim \frac{M}{R} v \sim \epsilon^3$$

change sign under time reversal ($x^0 \to -x^0$), whereas terms even in ϵ do not. Time reversal also changes the sign of g_{0i}, but leaves g_{00} and g_{ik} unchanged. Therefore, g_{0i} must contain only terms odd in ϵ, whereas g_{00} and g_{ik} must contain only even terms. Actually, this ceases to be the case when radiation damping enters the picture, since time reversal converts outgoing waves into ingoing waves. However, radiation damping does not come into play until order ϵ^5 beyond Newtonian limit [24].

The Newtonian expansion, as seen in (9.1.8), is given by

$$g_{00} = 1 - 2U \tag{9.2.9a}$$
$$g_{0i} = 0 \tag{9.2.9b}$$
$$g_{ik} = -\delta_{jk} \tag{9.2.9c}$$

Hence, the post-Newtonian expansion is given by

$$g_{00} = 1 - 2U + O(\epsilon^4) \tag{9.2.10a}$$
$$g_{0i} = O(\epsilon^3) \tag{9.2.10b}$$
$$g_{ik} = -\delta_{jk} + O(\epsilon^2) \tag{9.2.10c}$$

How accurately does the post-Newtonian approximation agree with the metric theory it comes from? The fractional error will be $\lesssim \epsilon^2$ in quantities of post-Newtonian order and $\lesssim \epsilon^4$ in quantities od Newtonian order. For instance, since almost everywhere in the solar system $U \lesssim 10^{-6}$, it misrepresents the deflection of light by $\sim 10^{-6} \times$ (post-Newtonian deflection) $\sim 10^{-6}$ seconds of arc, and it ignores relativistic deformations of the Earth's orbit of magnitude $\sim 10^{-12} \times (1\ a.u.) \sim 10$cm.

9.3 Gravitational potentials

Each metric theory has its own post-Newtonian expansion of the metric. Despite the great differences between metric theories themselves, their post-Newtonian approximations are very similar; actually, so similar that one can construct a single post-Newtonian theory of gravity, that contains the post-Newtonian approximation of every conceivable metric theory as a special case: the most general post-Newtonian metric can be found by simply writing down metric terms composed of all possible post-Newtonian functionals of matter variables (Gravitational potentials), each multiplied by an arbitrary coefficient that may depend on the cosmological matching conditions and on other constants, and adding these terms to the Minkowski metric to obtain the physical metric. This all-inclusive post-Newtonian theory is called the **Parameterized Post-Newtonian Formalism (PPN formalism)**.
Unfortunately, there is an infinite number of such gravitational potentials, so that in order to obtain a formalism that is both useful and manageable, we must impose some restrictions to the possible terms to be considered, guided in part by a subjective notion of "reasonableness" and in part by evidence obtained from known gravitation theories. Some of these restrictions are obvious [1]:

- Only Newtonian and post-Newtonian terms are considered, with no higher terms.
- The potentials should tend to zero as the distance $|\mathbf{x} - \mathbf{x}'|$ between the field point $\mathbf{x}$ and a typical point $\mathbf{x}'$ inside the matter becomes large. This will guarantee that the metric becomes asymptotically Minkowskian.
- The coordinates are chosen so that the metric is dimensionless.
- In our coordinate system the spatial origin and initial moment of time are completely arbitrary, so the metric should contain no explicit reference to these quantities. This is guaranteed using functionals in which

the field point $\mathbf{x}$ always occurs in the combination $\mathbf{x} - \mathbf{x}'$ where $\mathbf{x}'$ is a point associated with the matter distribution, and making all time dependence in the metric terms implicit via the evolution of the matter variables and the possible cosmological matching parameters.

- The metric functionals should be generated by the quantities defined above
$$\rho_0,\ \Pi,\ p,\ v_i$$
and not by their gradients. This restriction is purely subjective, but no reason has yet arisen to remove it.

- A final, extremely subjective constraint is that the gravitational potentials should be "simple".

We have to consider that, writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}$, the metric corrections h_{00}, h_{0i} and h_{ij} should transform under spatial rotations as a scalar, vector and tensor respectively.
With these restrictions in mind, we can now write down the possible terms that may appear in the post-Newtonian metric:

- $\mathbf{h_{ij}}$ **to** $\mathbf{O}(\epsilon^{\mathbf{2}})$: it must behave as a three-dimensional tensor under spatial rotations. Thus the only terms that can appear are:

$$U\delta_{ij} \tag{9.3.1}$$
$$U_{ij} \equiv \int \frac{\rho_0'(x-x')_i(x-x')_j}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'. \tag{9.3.2}$$

- $\mathbf{h_{0i}}$ **to** $\mathbf{O}(\epsilon^{\mathbf{3}})$: it must behave as a three-vector under spatial rotations. Thus it can contain only the terms:

$$V_i \equiv \int \frac{\rho_0' v_i'}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \tag{9.3.3}$$

$$W_i \equiv \int \frac{\rho_0'[\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')](x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'. \tag{9.3.4}$$

- $\mathbf{h_{00}}$ **to** $\mathbf{O}(\epsilon^{\mathbf{4}})$: it must behave as a scalar under spatial rotations. The only terms we shall consider are:

$$U^2 \tag{9.3.5}$$

$$\Phi_W \equiv \int \frac{\rho_0'\rho_0''(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \cdot \left(\frac{\mathbf{x}'-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|} - \frac{\mathbf{x}-\mathbf{x}''}{|\mathbf{x}'-\mathbf{x}''|}\right) d^3x' d^3x'' \tag{9.3.6}$$

$$\Phi_1 \equiv \int \frac{\rho_0' v'^2}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{9.3.7}$$

$$\Phi_2 \equiv \int \frac{\rho_0' U'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{9.3.8}$$

$$\Phi_3 \equiv \int \frac{\rho_0' \Pi'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{9.3.9}$$

$$\Phi_4 \equiv \int \frac{p'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{9.3.10}$$

$$A \equiv \int \frac{\rho_0'[\mathbf{v}' \cdot (\mathbf{x}-\mathbf{x}')]^2}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \tag{9.3.11}$$

$$B \equiv \int \frac{\rho_0'}{|\mathbf{x}-\mathbf{x}'|} (\mathbf{x}-\mathbf{x}') \cdot \frac{d\mathbf{v}'}{dt} d^3x'. \tag{9.3.12}$$

Now, defining the superpotential χ as

$$\chi(\mathbf{x},t) \equiv -\int \rho_0' |\mathbf{x}-\mathbf{x}'| d^3x' \tag{9.3.13}$$

we can write down the following useful relationships valid to the post-Newtonian order:

$$\partial_i \partial_j \chi = -U\delta_{ij} + U_{ij} \tag{9.3.14a}$$

$$\partial_0 \partial_i \chi = V_i - W_i \tag{9.3.14b}$$

$$\partial_0^2 \chi = A + B - \Phi_1 \tag{9.3.14c}$$

$$\partial_i V_i = -\partial_0 U \tag{9.3.14d}$$

$$\nabla^2 \chi = -2U \tag{9.3.14e}$$

$$\nabla^2(\Phi_W + 2U^2 - 3\Phi_2) = 2(\partial_i \partial_j \chi)(\partial_i \partial_j U) \tag{9.3.14f}$$

$$\nabla^2 V_i = -4\pi\rho_0 v_i \tag{9.3.14g}$$

$$\nabla^2 \Phi_1 = -4\pi\rho_0 v^2 \tag{9.3.14h}$$

$$\nabla^2 \Phi_2 = -4\pi\rho_0 U \tag{9.3.14i}$$

$$\nabla^2 \Phi_3 = -4\pi\rho_0 \Pi \tag{9.3.14j}$$

$$\nabla^2 \Phi_4 = -4\pi p \tag{9.3.14k}$$

where also the conservation of baryon number has been used:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0. \tag{9.3.15}$$

9.4 The standard Post-Newtonian gauge

The general PPN metric, expanded through the gravitational potentials defined in the previous section, can be restricted by making use of the arbitrariness of the coordinates choice.
If we want to to retain the post-Newtonian character of the metric $g_{\mu\nu}$, that through an infinitesimal gauge transformation changes to

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \tag{9.4.1}$$

the functions ξ_μ have to satisfy the conditions [1]

- $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ have to be post-Newtonian functions, as the ones defined in the previous section.
- $\nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) \longrightarrow 0$ as $|x| \to +\infty$, so that the metric is still asymptotically Minkowskian.
- $|\xi^\mu|/|x^\mu| \longrightarrow 0$ as $|x^\mu| \to +\infty$.

The only "simple" function that has these properties is the gradient of the superpotential $\partial_\mu \chi(x)$. Thus, we choose

$$\xi_0 = \lambda_1 \partial_0 \chi \tag{9.4.2a}$$
$$\xi_i = \lambda_2 \partial_i \chi \tag{9.4.2b}$$

and obtain, to post-Newtonian order,

$$g_{ij}(x) \longrightarrow g_{ij}(x') - 2\lambda_2 \partial_i \partial_j \chi \tag{9.4.3a}$$

$$g_{0i}(x) \longrightarrow g_{0i}(x') - (\lambda_1 + \lambda_2)\partial_0 \partial_i \chi \tag{9.4.3b}$$

$$g_{00}(x) \longrightarrow g_{00}(x') - 2\lambda_1 \partial_0^2 \chi + 2\lambda_2 \Gamma^i_{00} \partial_i \chi. \tag{9.4.3c}$$

After some calculations and using (9.3.14), equations (9.4.3) yield [1]

$$g_{ij} \longrightarrow g_{ij} + 2\lambda_2 U \delta_{ij} - 2\lambda_2 U_{ij} \tag{9.4.4a}$$

$$g_{0i} \longrightarrow g_{0i} - (\lambda_1 + \lambda_2)(V_i - W_i) \tag{9.4.4b}$$

$$g_{00} \longrightarrow g_{00} - 2\lambda_1 (A + B - \Phi_1) - 2\lambda_2 (U^2 + \Phi_W - \Phi_2). \tag{9.4.4c}$$

Hence, by an appropriate choice of λ_1 and λ_2, we can eliminate certain terms from the post-Newtonian metric. The *Standard post-Newtonian gauge* that we will adopt is the gauge in which the spatial part is diagonal and isotropic (that is, g_{ij} contains no term U_{ij}) and in which g_{00} contains no term B. There is no physical significance in this gauge choice, but is only a matter of convenience.
In this gauge we're thus left with only 10 gravitational potentials in the post-Newtonian expansion of the metric, with 10 associated parameters.

9.5 PPN metric

Henceforth, we shall adopt the Will-Nordtvedt version of the PPN formalism, in which the PPN metric, expanded in ϵ^n $(kh^{(n)}_{\mu\nu})$ terms as in (9.2.10), reads

$$kh^{(2)}_{00} = -2U \quad (9.5.1)$$

$$\begin{aligned} kh^{(4)}_{00} = \; & 2\beta U^2 + \xi\Phi_W - (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\ & -2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 - 2(1 + \zeta_3)\Phi_3 \\ & -2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 + (\zeta_1 - 2\xi)A \end{aligned} \quad (9.5.2)$$

$$\begin{aligned} kh^{(3)}_{0i} = \; & \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i \\ & +\frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i \end{aligned} \quad (9.5.3)$$

$$kh^{(2)}_{ij} = -2\gamma U\delta_{ij}. \quad (9.5.4)$$

In the Will-Nordtvedt version the parameters have been chosen in such a way that the parameters have special physical significance [1]:

- γ measures how much space-curvature is produced by a unitary rest mass. In GR $\gamma = 1$.
- β measures how much "non-linearity" there is in the superposition law for gravity. In GR $\beta = 1$.
- $\xi \neq 0$ is consequence of LPI violations. In GR $\xi = 0$.
- α_1, α_2, α_3 measure LLI, that is, if there are preferred-frame effects. In GR $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
- α_3, ζ_1, ζ_2, ζ_3, ζ_4 measure violations in the conservation of the four-momentum. In GR $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

9.6 PPN Energy-momentum tensor

We shall consider a model in which the matter action is given by (6.2.1), with

$$L_m = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \tag{9.6.1}$$

so that the action is

$$S_m = \int d^4x f_m(|g|) \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{9.6.2}$$

The variation with respect to $g^{\mu\nu}$ yields the EMT; in this case, with the definitions (6.4.1), it is given by (6.4.5) with $f_k = f_v \equiv f_m$, that is

$$T_{\mu\nu} = \frac{f_m(|g|)}{2} (\rho + p) u_\mu u_\nu - |g| f'_m(|g|) p \, g_{\mu\nu} \tag{9.6.3}$$

where, from the definition (9.2.4),

$$\rho = \rho_0 (1 + \Pi). \tag{9.6.4}$$

In the matter rest-frame, considering that $h = O(\epsilon^2)$, the components of the EMT to the post-Newtonian order are given by

$$T^{RF}_{00} = \frac{f_m}{2} (1 + \Pi) \rho_0 + (\frac{f_m}{2} - f'_m) p + O(\epsilon^4) \tag{9.6.5a}$$

$$T^{RF}_{0j} = O(\epsilon^5) \tag{9.6.5b}$$

$$T^{RF}_{ij} = f'_m p \, \delta_{ij} + O(\epsilon^4). \tag{9.6.5c}$$

Since we want to work in the *standard post-Newtonian gauge*, we have to change reference system. The transformation rules from the matter rest-frame (coordinates denoted by $\omega^{\tilde{\beta}}$) to the PPN frame (coordinates denoted by dx^α) are given in [24]:

$$dx^\alpha = A^\alpha_{\tilde{\beta}} \omega^{\tilde{\beta}} \tag{9.6.6}$$

with

$$A^0_{\tilde{0}} = 1 + v^2/2 + U + O(\epsilon^4) \quad (9.6.7a)$$

$$A^0_{\tilde{j}} = v_j\left[1 + \frac{1}{2}v^2 + (2+\gamma)U\right] - \frac{7}{2}\Delta_1 V_j - \frac{1}{2}\Delta_2 W_j + O(\epsilon^5) \quad (9.6.7b)$$

$$A^j_{\tilde{0}} = v_j(1 + v^2/2 + U) + O(\epsilon^5) \quad (9.6.7c)$$

$$A^j_{\tilde{k}} = (1 - \gamma U)\delta_{jk} + \frac{1}{2}v_j v_k + O(\epsilon^4) \quad (9.6.7d)$$

where Δ_1 and Δ_2 denote some particular combinations of the PPN parameters.
Anyway, since we need the transformation rules for covariant components, we calculate the inverse tensor from the equation $A^{\tilde{\mu}}_{\alpha} A^{\alpha}_{\tilde{\nu}} = \delta^{\mu}_{\nu}$, finding

$$A^{\tilde{0}}_0 = 1 + v^2/2 - U + O(\epsilon^4) \quad (9.6.8a)$$

$$A^{\tilde{0}}_j = -v_j\left[1 + \frac{1}{2}v^2 + (1+2\gamma)U\right] + \frac{7}{2}\Delta_1 V_j + \frac{1}{2}\Delta_2 W_j + O(\epsilon^5) \quad (9.6.8b)$$

$$A^{\tilde{j}}_0 = -v_j(1 + v^2/2 + \gamma U) + O(\epsilon^5) \quad (9.6.8c)$$

$$A^{\tilde{k}}_j = (1 + \gamma U)\delta_{jk} + \frac{1}{2}v_j v_k + O(\epsilon^4). \quad (9.6.8d)$$

Hence, applying the transformation

$$T^{PPN}_{\alpha\beta} = A^{\tilde{\mu}}_{\alpha} A^{\tilde{\nu}}_{\beta} T^{RF}_{\tilde{\mu}\tilde{\nu}} \quad (9.6.9)$$

we obtain the components of the PPN energy-momentum tensor:

$$T_{00} = \frac{f_m}{2}\rho_0(1 + \Pi + v^2 - 2U) + (\frac{f_m}{2} - f'_m)p + O(\epsilon^4) \quad (9.6.10a)$$

$$T_{0j} = -\frac{f_m}{2}\rho_0 v_j + O(\epsilon^3) \quad (9.6.10b)$$

$$T_{ij} = \frac{f_m}{2}\rho_0 v_i v_j + f'_m p\,\delta_{ij} + O(\epsilon^4). \quad (9.6.10c)$$

Expanding $f_m(|g|)$ as

$$f_m(|g|) = f_m(1) + k f'_m(1) h + O(k^2) \quad (9.6.11)$$

we can rewrite the expressions (9.6.10) expanded in terms of order ϵ^n ($T^{(n)}_{\mu\nu}$):

$$T^{(0)}_{00} = \frac{f_m(1)}{2}\rho_0 \tag{9.6.12a}$$

$$T^{(2)}_{00} = \frac{f_m(1)}{2}\left[\rho_0(\Pi + v^2 - 2U) + p\right] + \frac{f'_m(1)}{2}\left[kh^{(2)}\rho_0 - 2p\right] \tag{9.6.12b}$$

$$T^{(1)}_{0j} = -\frac{f_m(1)}{2}\rho_0 v_j \tag{9.6.12c}$$

$$T^{(2)}_{ij} = \frac{f_m(1)}{2}\rho_0 v_i v_j + f'_m(1)p\,\delta_{ij}. \tag{9.6.12d}$$

9.7 PPN formalism and TDiff theories: linear approximation

For the gravitational sector, let's take the most general massless quadratic Lagrangian $\frac{1}{2}\cdot$(2.3.6). Then the total action of the model we are considering reads

$$S = \int d^4x \left[\frac{1}{8}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{1}{4}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu + \frac{c_2}{4}\partial_\mu h \partial_\nu h^{\mu\nu} - \frac{c_3}{8}\partial_\mu h\partial^\mu h \right.$$
$$\left. + f_(|g|)\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\right] \tag{9.7.1}$$

where $h_{\mu\nu}$ is defined by (2.1.19):

$$h_{\mu\nu} = k^{-1}(g_{\mu\nu} - \eta_{\mu\nu}) \tag{9.7.2}$$

and the indexes are always raised and lowered by the flat Minkowski metric.

Since from (2.1.20) the inverse metric $g^{\mu\nu}$ at the lowest order is

$$g^{\mu\nu} = \eta^{\mu\nu} - kh^{\mu\nu} + O(k^2) \tag{9.7.3}$$

for the gravitational part we have

$$\frac{\delta L_g}{\delta g^{\alpha\beta}} = -k^{-1}\frac{\delta L_g}{\delta h^{\alpha\beta}}\,. \tag{9.7.4}$$

The variation with respect to $g^{\alpha\beta}$ then gives the equations of motion:

$$-\frac{1}{4}\Box h_{\alpha\beta}+\frac{1}{4}(\partial_\mu\partial_\alpha h^\mu_\beta+\partial_\mu\partial_\beta h^\mu_\alpha)-\frac{c_2}{4}(\eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}+\partial_\alpha\partial_\beta h)+\frac{c_3}{4}\eta_{\alpha\beta}\Box h = kT_{\alpha\beta}. \quad (9.7.5)$$

If

$$c_2 \neq \frac{1}{2}, \quad (9.7.6)$$

using the trace of the equation, we can substitute the term $\partial_\mu\partial_\nu h^{\mu\nu}$ and rewrite (9.7.5) as

$$-\frac{1}{4}\Box h_{\alpha\beta} + \frac{1}{4}(\partial_\mu\partial_\alpha h^\mu_\beta + \partial_\mu\partial_\beta h^\mu_\alpha) - \frac{c_2}{4}\partial_\alpha\partial_\beta h + \frac{2c_3 - c_2^2 - c_2}{8(1-2c_2)}\eta_{\alpha\beta}\Box h =$$

$$= k\left(T_{\alpha\beta} - \frac{c_2}{2(2c_2-1)}\eta_{\alpha\beta}T\right). \quad (9.7.7)$$

We can now go to solve the equation at the lowest order.

9.7.1 00-component at the ϵ^2 order

Using also (9.2.8) the equation reads

$$\frac{1}{4}\nabla^2 h^{(2)}_{00} + \frac{c_2^2 + c_2 - 2c_3}{8(1-2c_2)}\nabla^2 h^{(2)} = k\left(T^{(0)}_{00} - \frac{c_2}{2(2c_2-1)}T^{(0)}\right). \quad (9.7.8)$$

Substituting the general PPN expansion of $kh_{\mu\nu}$ given by (9.5.1) and the expression for the EMT to the desired order, we get

$$-\left[1 - \frac{c_2^2 + c_2 - 2c_3}{2(2c_2-1)}(1-3\gamma)\right]\nabla^2 U = k^2 f_m(1)\left[1 - \frac{c_2}{2(2c_2-1)}\right]\rho_0 \quad (9.7.9)$$

where

$$k^2 = 8\pi \quad (9.7.10)$$

because we are using geometrized units in which $G \equiv 1$.

In passing we notice that when the whole theory is Diff-invariant

($c_2 = c_3 = f_m(1) = 1$), equation (9.7.9) gives the identity (9.1.3).
In the general TDiff-invariant case the possible solutions are

$$\gamma = \frac{1}{3} + \frac{2}{3} \cdot \frac{c_2(3f_m(1) - 2) - 2f_m(1) + 1}{c_2^2 + c_2 - 2c_3} \tag{9.7.11}$$

or else

$$\begin{cases} c_2^2 + c_2 - 2c_3 = 0 \\ f_m(1)\left(1 - \frac{c_2}{2(2c_2-1)}\right) = 1/2 \Rightarrow \quad f_m(1) = \frac{2c_2-1}{3c_2-2} \end{cases} \tag{9.7.12}$$

with $c_2 \neq \frac{2}{3}$ because otherwise the system (9.7.12) would have no solution.

9.7.2 ij-components at the ϵ^2 order

The equation reads

$$\frac{1}{4}\nabla^2 h_{ij}^{(2)} + \frac{1}{4}(\partial_l \partial_i h_j^{l(2)} + \partial_l \partial_j h_i^{l(2)}) - \frac{c_2}{4}\partial_i \partial_j h^{(2)} + \frac{c_2^2 + c_2 - 2c_3}{8(2c_2 - 1)}\delta_{ij}\nabla^2 h^{(2)} =$$

$$= k\left(T_{ij}^{(0)} + \frac{c_2}{2(2c_2 - 1)}\delta_{ij}T^{(0)}\right) \tag{9.7.13}$$

that, using (9.5.1) and (9.6.12), can be rewritten as

$$-\frac{1}{2}\gamma\delta_{ij}\nabla^2 U + \gamma\partial_i\partial_j U + \frac{c_2}{2}(1 - 3\gamma)\partial_i\partial_j U - \frac{c_2^2 + c_2 - 2c_3}{4(2c_2 - 1)}(1 - 3\gamma)\delta_{ij}\nabla^2 U =$$

$$= 8\pi\frac{c_2}{4(2c_2 - 1)}\delta_{ij} f_m(1)\rho_0. \tag{9.7.14}$$

The equation is equivalent to

$$\begin{cases} \gamma + \frac{c_2}{2}(1 - 3\gamma) = 0 \\ \frac{1}{2}\gamma + \frac{c_2^2+c_2-2c_3}{4(2c_2-1)}(1 - 3\gamma) = \frac{c_2}{2(2c_2-1)} f_m(1). \end{cases} \tag{9.7.15}$$

If we choose $c_2^2 + c_2 - 2c_3 \neq 0$, using also (9.7.11), we get the system

$$\begin{cases} \gamma = \frac{1}{3} + \frac{2}{3} \cdot \frac{c_2(3f_m(1)-2)-2f_m(1)+1}{c_2^2+c_2-2c_3} \\ \\ \gamma = \frac{c_2}{3c_2-2} \\ \\ (3c_2^2 - c_2 - 6c_3 + 2)\gamma = c_2^2 - c_2(2f_m(1) - 1) - 2c_3 \end{cases} \tag{9.7.16}$$

which has no possible solutions.
Hence, the only possible solution is to impose the constraint

$$c_2^2 + c_2 - 2c_3 = 0. \tag{9.7.17}$$

The system (9.7.15) thus becomes

$$\begin{cases} \gamma + \frac{c_2}{2}(1 - 3\gamma) = 0 \\ \\ \frac{1}{2}\gamma = \frac{c_2}{2(2c_2-1)} f_m(1) \end{cases} \tag{9.7.18}$$

which yields

$$\gamma = \frac{c_2}{3c_2 - 2} \tag{9.7.19}$$

$$f_m(1) = \frac{2c_2 - 1}{3c_2 - 2} \tag{9.7.20}$$

Equation (9.7.20), which is exactly the same expression given by the second equation of (9.7.12), is not a constraint on the parameters of the theory, but only a consequence of our choice to use the geometrized units in which $G = 1$ [1]. There is no physical constraint implied.
Equation (9.7.19) gives the value of the first PPN parameter.

On the other hand, if we want to choose $c_2 = \frac{1}{2}$ (which until now has been excluded by (9.7.6)), from equation (9.7.5), we get

$$\frac{1}{4}\nabla^2 h_{ij}^{(2)} - \frac{1}{4}\left(\partial_k\partial_i h_{kj}^{(2)} + \partial_k\partial_j h_{ki}^{(2)}\right) + \frac{1}{8}\delta_{ij}\partial_k\partial_l h_{kl}^{(2)} - \frac{1}{8}\partial_i\partial_j h^{(2)}$$

$$+ \frac{c_3}{4}\delta_{ij}\nabla^2 h^{(2)} = kT_{ij}^{(0)}. \tag{9.7.21}$$

Substituting the PPN expressions for h_{ij} and T_{ij} we get

$$-\left(\frac{3}{4}\gamma + \frac{c_3}{2}(1-3\gamma)\right)\delta_{ij}\nabla^2 U + \left(\gamma + \frac{1}{4}(1-3\gamma)\right)\partial_i\partial_j U = 0 \tag{9.7.22}$$

which is equivalent to

$$\begin{cases} \frac{3}{4}\gamma + \frac{c_3}{2}(1-3\gamma) = 0 \\ \gamma + \frac{1}{4}(1-3\gamma) = 0. \end{cases} \tag{9.7.23}$$

The system yields

$$c_3 = \frac{3}{8} \tag{9.7.24}$$

$$\gamma = -1. \tag{9.7.25}$$

The constraint on c_3 means that if we want to analyze the case $c_2 = \frac{1}{2}$, we are considering a WTDiff-invariant gravitational Lagrangian. Anyway, as seen in section 6.1, a WTDiff-invariant Lagrangian imposes the constraints (6.1.12) on the matter Lagrangian, which thanks to (6.3.8) mean that in the matter Lagrangian $f_k(|g|) \neq f_v(|g|)$. But this is not the case we are analyzing in our model.
Hence, hereafter we will consider only theories with

$$\begin{cases} 2c_3 = c_2^2 + c_2 \\ c_2 \neq \frac{1}{2}. \end{cases}$$

9.7.3 0i-components at the ϵ^3 order

The equation, using (9.7.17), is

$$\frac{1}{4}\nabla^2 h_{0i}^{(3)}+\frac{1}{4}(\partial_0\partial_k h_i^{k(2)}+\partial_0\partial_i h_0^{0(2)}+\partial_k\partial_i h_0^{k(3)})-\frac{c_2}{4}\partial_0\partial_i h^{(2)} = kT_{0i}^{(1)}. \quad (9.7.26)$$

Inserting the general PPN expansion of the metric, the l.h.s. reads

$$k^{-1}\left[\frac{1}{8}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi)\nabla^2 V_i+\frac{1}{8}(1+\alpha_2-\zeta_1+2\xi)\nabla^2 W_i\right.$$

$$+\frac{\gamma}{2}\partial_0\partial_i U-\frac{1}{2}\partial_0\partial_i U-\frac{1}{8}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi)\partial_k\partial_i V_k$$

$$\left.-\frac{1}{8}(1+\alpha_2-\zeta_1+2\xi)\partial_k\partial_i W_k+\frac{c_2}{2}(1-3\gamma)\partial_0\partial_i U\right]. \quad (9.7.27)$$

Using (9.3.14b), (9.3.14d) and (9.3.14e) so that

$$\partial_0\partial_i U = -\frac{1}{2}\nabla^2\partial_0\partial_i\chi = \frac{1}{2}\nabla^2(W_i - V_i) \quad (9.7.28)$$

$$\partial_i\partial_k V_k = -\partial_o\partial_i U = \frac{1}{2}\nabla^2(V_i - W_i) \quad (9.7.29)$$

$$\partial_i\partial_k W_k = \partial_i\partial_k V_k - \partial_0\partial_i\nabla^2\chi = \partial_0\partial_i U = \frac{1}{2}\nabla^2(W_i - V_i) \quad (9.7.30)$$

and substituting the expression for $T_{0i}^{(1)}$, the whole equation (9.7.26) becomes

$$\frac{1}{8}(4+\frac{\alpha_1}{2}-2c_2+6c_2\gamma)\nabla^2 V_i+\frac{1}{8}(4\gamma+\frac{\alpha_1}{2}+2c_2-6c_2\gamma)\nabla^2 W_i =$$

$$= -4\pi f_m(1)\rho_0 v_i. \quad (9.7.31)$$

The solution is given by

$$\begin{cases} 4\gamma+\frac{\alpha_1}{2}+2c_2-6c_2\gamma = 0 \\ \frac{1}{8}(4+\frac{\alpha_1}{2}-2c_2+6c_2\gamma) = f_m(1). \end{cases} \quad (9.7.32)$$

Inserting the value for γ given by (9.7.19), the system (9.7.32) yields

$$f_m(1) = \frac{2c_2 - 1}{3c_2 - 2} \tag{9.7.33}$$

$$\alpha_1 = 0. \tag{9.7.34}$$

The relation involving $f_m(1)$ is the same found in (9.7.20) and (9.7.12), while (9.7.34) gives the value of the second PPN parameter. This value of α_1 is the expected one, since a value of the PPN parameter $\alpha_1 \neq 0$ is consequence of theories with preferred frame-system.

Summarizing, in the linear approximation of the TDiff theory described by the action (9.7.1), we have found that the parameters of the theory have to obey the constraint

$$2c_3 = c_2^2 + c_2 \tag{9.7.35}$$

and the two PPN parameters

$$\gamma = \frac{c_2}{3c_2 - 2} \tag{9.7.36}$$

$$\alpha_1 = 0. \tag{9.7.37}$$

To find the other PPN parameters, we need to write a Lagrangian that contains cubic terms in $h_{\mu\nu}$. That's what we are going to do in the next section.

9.8 PPN formalism and TDiff theories: non-linear case

We shall now consider a model in which the gravitational action is given by (2.5.1). Hence the total action is

$$S = \int d^4x \left[\left(-\frac{1}{2k^2} \right) \left(f_1(|g|)R + f_2(|g|)g^{\mu\nu}\partial_\mu g \partial_\nu g \right) \right.$$

$$\left. + f_m(|g|) \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \right]. \tag{9.8.1}$$

In General Relativity we would have

$$f_1(|g|) = f_m(|g|) = \sqrt{|g|} \tag{9.8.2}$$
$$f_2(|g|) = 0. \tag{9.8.3}$$

9.8.1 Cubic Lagrangian

We are going now to expand the Lagrangian in (9.8.1) up to cubic terms in $h_{\mu\nu}$, using expressions (2.1.19) and (2.1.20) to expand the metric, and raising and lowering the indexes through the flat Minkowski metric.
We have also to expand the determinant of the metric as

$$g \approx -1 - kh + k^2 \tilde{h} \tag{9.8.4}$$

where

$$h = \eta^{\mu\nu} h_{\mu\nu} \tag{9.8.5}$$

$$\tilde{h} \equiv h_{12}^2 + h_{13}^2 + h_{23}^2 - h_{01}^2 - h_{02}^2 - h_{03}^2$$
$$+ h_{00}(h_{11} + h_{22} + h_{33}) - h_{11}h_{22} - h_{11}h_{33} - h_{22}h_{33}. \tag{9.8.6}$$

Thus a generic function $f(|g|)$ can be expanded as

$$f(|g|) = f(-g) \approx f(1 + kh - k^2\tilde{h})$$

$$\approx f(1) + kf'(1)h + k^2 \left(\frac{1}{2} f''(1)h^2 - f'(1)\tilde{h} \right). \tag{9.8.7}$$

Hence, the expansion of the gravitational Lagrangian

$$L_g = \left(-\frac{1}{2k^2}\right)\left(f_1(|g|)R + f_2(|g|)g^{\mu\nu}\partial_\mu g\partial_\nu g\right) \tag{9.8.8}$$

to the deisred order and without total derivatives, is

$$\begin{aligned}
L_g \approx & \frac{1}{8}f_1(1)\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \frac{1}{4}f_1(1)\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu + \frac{1}{2}f_1'(1)\partial_\mu h^{\mu\nu}\partial_\nu h \\
& - \frac{1}{8}\left(4f_1'(1) + 4f_2(1) - f_1(1)\right)\partial_\mu h\partial^\mu h + k\left[-\frac{1}{4}f_1(1)h^{\mu\nu}\partial_\rho h^\sigma_\nu\partial^\rho h_{\mu\sigma}\right. \\
& - \frac{1}{4}f_1(1)h^{\mu\nu}\partial_\rho h^\sigma_\nu\partial_\sigma h^\rho_\mu - \frac{1}{8}f_1(1)h^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + \frac{1}{2}f_1(1)h^{\mu\nu}\partial_\mu h_{\nu\rho}\partial_\sigma h^{\rho\sigma} \\
& + \frac{1}{2}f_1(1)h^{\mu\nu}\partial_\rho h^\rho_\mu\partial_\sigma h^\sigma_\nu - \frac{1}{2}f_1'(1)h^{\mu\nu}\partial_\mu h^\rho_\nu\partial_\rho h - \frac{1}{4}f_1'(1)h\partial_\mu h^{\nu\rho}\partial_\rho h^\mu_\nu \\
& + \frac{1}{8}f_1'(1)h\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{1}{4}\left(2f_1'(1) - f_1(1)\right)h^{\mu\nu}\partial_\rho h_{\mu\nu}\partial^\rho h \\
& - \frac{1}{2}f_1'(1)h^{\mu\nu}\partial_\rho h^\rho_\mu\partial_\nu h + \frac{1}{8}\left(4f_1'(1) + 4f_2(1) - f_1(1)\right)h^{\mu\nu}\partial_\mu h\partial_\nu h \\
& + \frac{1}{2}f_1''(1)h\partial_\mu h^{\mu\nu}\partial_\nu h - \frac{1}{8}\left(4f_1''(1) + 4f_2'(1) - f_1'(1)\right)h\partial_\mu h\partial^\mu h \\
& \left. - \frac{1}{2}f_1'(1)\partial_\mu h^{\mu\nu}\partial_\nu\tilde{h} + \frac{1}{2}\left(f_1'(1) + 2f_2(1)\right)\partial_\mu h\partial^\mu\tilde{h}\right].
\end{aligned} \tag{9.8.9}$$

Dividing by $f_1(1)$ in order to normalize the Lagrangian in such a way to have correspondence with (2.3.6), we can redefine

$$f_1(|g|) \equiv \frac{f_1(|g|)}{f_1(1)}, \qquad f_2(|g|) \equiv \frac{f_2(|g|)}{f_1(1)}, \qquad f_m(|g|) \equiv \frac{f_m(|g|)}{f_1(1)}, \tag{9.8.10}$$

identify

$$c_2 \equiv 2f_1'(1) \tag{9.8.11a}$$

$$c_3 \equiv 4f_1'(1) + 4f_2(1) - 1 \tag{9.8.11b}$$

and define

$$c_4 \equiv -4f_1''(1) \tag{9.8.11c}$$

$$c_5 \equiv 4f_2'(1) \tag{9.8.11d}$$

so that (9.8.9) becomes

$$\begin{aligned} L_g \approx & \frac{1}{8}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \frac{1}{4}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu + \frac{c_2}{4}\partial_\mu h^{\mu\nu}\partial_\nu h - \frac{c_3}{8}\partial_\mu h \partial^\mu h \\ & + k\left[-\frac{1}{4}h^{\mu\nu}\partial_\rho h^\sigma_\nu \partial^\rho h_{\mu\sigma} - \frac{1}{4}h^{\mu\nu}\partial_\rho h^\sigma_\nu \partial_\sigma h^\rho_\mu - \frac{1}{8}h^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma}\right. \\ & + \frac{1}{2}h^{\mu\nu}\partial_\mu h_{\nu\rho}\partial_\sigma h^{\rho\sigma} + \frac{1}{2}h^{\mu\nu}\partial_\rho h^\rho_\mu \partial_\sigma h^\sigma_\nu - \frac{c_2}{4}h^{\mu\nu}\partial_\mu h^\rho_\nu \partial_\rho h - \frac{c_2}{8}h\partial_\mu h^{\nu\rho}\partial_\rho h^\mu_\nu \\ & + \frac{c_2}{16}h\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{c_2 - 1}{4}h^{\mu\nu}\partial_\rho h_{\mu\nu}\partial^\rho h - \frac{c_2}{4}h^{\mu\nu}\partial_\rho h^\rho_\mu \partial_\nu h \\ & + \frac{c_3}{8}h^{\mu\nu}\partial_\mu h \partial_\nu h - \frac{c_4}{8}h\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{c_2 + 2c_4 - 2c_5}{16}h\partial_\mu h \partial^\mu h \\ & \left. -\frac{c_2}{4}\partial_\mu h^{\mu\nu}\partial_\nu \tilde{h} + \frac{c_3 - c_2 + 1}{4}\partial_\mu h \partial^\mu \tilde{h}\right]. \end{aligned} \tag{9.8.12}$$

In General Relativity

$$c_2 = c_3 = c_4 = 1 \tag{9.8.13a}$$

$$c_5 = 0. \tag{9.8.13b}$$

9.8.2 Equations of motion

The variation with respect to $h^{\alpha\beta}$ of the whole gravitational Lagrangian (9.8.12) yields

$$\frac{\delta L_g}{\delta h^{\alpha\beta}} = -\frac{1}{4}\Box h_{\alpha\beta} + \frac{1}{4}(\partial_\mu\partial_\alpha h^\mu_\beta + \partial_\mu\partial_\beta h^\mu_\alpha) - \frac{c_2}{4}(\eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} + \partial_\alpha\partial_\beta h) + \frac{c_3}{4}\eta_{\alpha\beta}\Box h$$

$$+k\left[\frac{1}{4}\partial_\mu h_{\alpha\nu}\partial^\mu h^\nu_\beta - \frac{1}{4}\partial_\mu h^\nu_\alpha\partial_\nu h^\mu_\beta + \frac{1}{4}(\partial_\mu h_{\nu\alpha}\partial_\beta h^{\mu\nu} + \partial_\mu h_{\nu\beta}\partial_\alpha h^{\mu\nu})\right.$$

$$-\frac{1}{8}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} + \frac{1}{4}\partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta} - \frac{1}{4}\left(\partial_\alpha h_{\mu\beta}\partial_\nu h^{\mu\nu} + \partial_\beta h_{\mu\alpha}\partial_\nu h^{\mu\nu}\right)$$

$$-\frac{1}{4}\left(\partial_\alpha h^{\mu\nu}\partial_\mu h_{\nu\beta} + \partial_\beta h^{\mu\nu}\partial_\mu h_{\nu\alpha}\right) + \frac{c_2}{8}(\partial_\alpha h^\mu_\beta\partial_\mu h + \partial_\beta h^\mu_\alpha\partial_\mu h) - \frac{c_2}{8}\partial_\mu h_{\alpha\beta}\partial^\mu h$$

$$+\frac{c_3+c_4}{8}\partial_\alpha h\partial_\beta h + \frac{c_2}{8}\eta_{\alpha\beta}\partial_\mu h^{\nu\rho}\partial_\nu h^\mu_\rho - \frac{3c_2-4}{16}\eta_{\alpha\beta}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu}$$

$$+\frac{c_2}{4}\eta_{\alpha\beta}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu - \frac{c_3}{4}\eta_{\alpha\beta}\partial_\mu h^{\mu\nu}\partial_\nu h - \frac{c_2+2c_4-2c_5}{16}\eta_{\alpha\beta}\partial_\mu h\partial^\mu h$$

$$+\frac{1}{4}(h_{\mu\alpha}\Box h^\mu_\beta + h_{\mu\beta}\Box h^\mu_\alpha) + \frac{1}{4}(h_{\mu\alpha}\partial_\nu\partial_\beta h^{\mu\nu} + h_{\mu\beta}\partial_\nu\partial_\alpha h^{\mu\nu}) + \frac{1}{4}h^{\mu\nu}\partial_\mu\partial_\nu h_{\alpha\beta}$$

$$-\frac{1}{4}\left(h^\mu_\alpha\partial_\mu\partial_\nu h^\nu_\beta + h^\mu_\beta\partial_\mu\partial_\nu h^\nu_\alpha\right) - \frac{1}{4}\left(h^{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\beta} + h^{\mu\nu}\partial_\mu\partial_\beta h_{\nu\alpha}\right)$$

$$-\frac{1}{2}\left(h^\mu_\alpha\partial_\beta\partial_\nu h^\nu_\mu + h^\mu_\beta\partial_\alpha\partial_\nu h^\nu_\mu\right) + \frac{c_2}{4}(h^\mu_\alpha\partial_\mu\partial_\beta h + h^\mu_\beta\partial_\mu\partial_\alpha h) - \frac{c_2}{8}h\Box h_{\alpha\beta}$$

$$+\frac{c_2}{8}(h\partial_\mu\partial_\alpha h^\mu_\beta + h\partial_\mu\partial_\beta h^\mu_\alpha) - \frac{c_2-1}{4}h_{\alpha\beta}\Box h + \frac{c_4}{8}h\partial_\alpha\partial_\beta h + \frac{c_2}{2}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\rho h^\rho_\nu$$

$$-\frac{c_2-1}{4}\eta_{\alpha\beta}h^{\mu\nu}\Box h_{\mu\nu} - \frac{c_3}{4}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\nu h + \frac{c_4}{8}\eta_{\alpha\beta}h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{c_2+2c_4-2c_5}{8}\eta_{\alpha\beta}h\Box h$$

$$\left.+\frac{c_2}{4}\partial_\alpha\partial_\beta\tilde{h} - \frac{c_3-c_2+1}{4}\eta_{\alpha\beta}\Box\tilde{h} + \frac{c_2}{4}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\partial_\mu\partial_\nu h^{\mu\nu} - \frac{c_3-c_2+1}{4}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\Box h\right]. \quad (9.8.14)$$

On the other hand, the variation with respect to $h^{\alpha\beta}$ of the matter Lagrangian yields

$$\frac{\delta L_m}{\delta h^{\alpha\beta}} = \frac{\delta L_m}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta h^{\alpha\beta}} = T_{\mu\nu}\Big(-k\delta^\mu_\alpha\delta^\nu_\beta + k^2(\delta^\mu_\alpha h^\nu_\beta + \delta^\mu_\beta h^\nu_\alpha)\Big) =$$

$$= -kT_{\alpha\beta} + k^2\left(T_{\alpha\mu}h^\mu_\beta + T_{\beta\mu}h^\mu_\alpha\right). \tag{9.8.15}$$

Hence the equations of motion are:

$$-\frac{1}{4}\Box h_{\alpha\beta} + \frac{1}{4}(\partial_\mu\partial_\alpha h^\mu_\beta + \partial_\mu\partial_\beta h^\mu_\alpha) - \frac{c_2}{4}(\eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} + \partial_\alpha\partial_\beta h) + \frac{c_3}{4}\eta_{\alpha\beta}\Box h$$

$$+ k\left[\frac{1}{4}\partial_\mu h_{\alpha\nu}\partial^\mu h^\nu_\beta - \frac{1}{4}\partial_\mu h^\nu_\alpha\partial_\nu h^\mu_\beta + \frac{1}{4}(\partial_\mu h_{\nu\alpha}\partial_\beta h^{\mu\nu} + \partial_\mu h_{\nu\beta}\partial_\alpha h^{\mu\nu}) - \frac{1}{8}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu}\right.$$

$$+ \frac{1}{4}\partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta} - \frac{1}{4}\left(\partial_\alpha h_{\mu\beta}\partial_\nu h^{\mu\nu} + \partial_\beta h_{\mu\alpha}\partial_\nu h^{\mu\nu}\right) - \frac{1}{4}\left(\partial_\alpha h^{\mu\nu}\partial_\mu h_{\nu\beta} + \partial_\beta h^{\mu\nu}\partial_\mu h_{\nu\alpha}\right)$$

$$+ \frac{c_2}{8}(\partial_\alpha h^\mu_\beta\partial_\mu h + \partial_\beta h^\mu_\alpha\partial_\mu h) - \frac{c_2}{8}\partial_\mu h_{\alpha\beta}\partial^\mu h + \frac{c_3+c_4}{8}\partial_\alpha h\partial_\beta h + \frac{c_2}{8}\eta_{\alpha\beta}\partial_\mu h^{\nu\rho}\partial_\nu h^\mu_\rho$$

$$- \frac{3c_2-4}{16}\eta_{\alpha\beta}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} + \frac{c_2}{4}\eta_{\alpha\beta}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu - \frac{c_3}{4}\eta_{\alpha\beta}\partial_\mu h^{\mu\nu}\partial_\nu h$$

$$- \frac{c_2+2c_4-2c_5}{16}\eta_{\alpha\beta}\partial_\mu h\partial^\mu h + \frac{1}{4}(h_{\mu\alpha}\Box h^\mu_\beta + h_{\mu\beta}\Box h^\mu_\alpha) + \frac{1}{4}(h_{\mu\alpha}\partial_\nu\partial_\beta h^{\mu\nu} + h_{\mu\beta}\partial_\nu\partial_\alpha h^{\mu\nu})$$

$$+ \frac{1}{4}h^{\mu\nu}\partial_\mu\partial_\nu h_{\alpha\beta} - \frac{1}{4}\left(h^\mu_\alpha\partial_\mu\partial_\nu h^\nu_\beta + h^\mu_\beta\partial_\mu\partial_\nu h^\nu_\alpha\right) - \frac{1}{4}\left(h^{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\beta} + h^{\mu\nu}\partial_\mu\partial_\beta h_{\nu\alpha}\right)$$

$$- \frac{1}{2}\left(h^\mu_\alpha\partial_\beta\partial_\nu h^\nu_\mu + h^\mu_\beta\partial_\alpha\partial_\nu h^\nu_\mu\right) + \frac{c_2}{4}(h^\mu_\alpha\partial_\mu\partial_\beta h + h^\mu_\beta\partial_\mu\partial_\alpha h) - \frac{c_2}{8}h\Box h_{\alpha\beta}$$

$$+ \frac{c_2}{8}(h\partial_\mu\partial_\alpha h^\mu_\beta + h\partial_\mu\partial_\beta h^\mu_\alpha) - \frac{c_2-1}{4}h_{\alpha\beta}\Box h + \frac{c_4}{8}h\partial_\alpha\partial_\beta h + \frac{c_2}{2}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\rho h^\rho_\nu$$

$$- \frac{c_2-1}{4}\eta_{\alpha\beta}h^{\mu\nu}\Box h_{\mu\nu} - \frac{c_3}{4}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\nu h + \frac{c_4}{8}\eta_{\alpha\beta}h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{c_2+2c_4-2c_5}{8}\eta_{\alpha\beta}h\Box h$$

$$\left.+\frac{c_2}{4}\partial_\alpha\partial_\beta\tilde{h} - \frac{c_3-c_2+1}{4}\eta_{\alpha\beta}\Box\tilde{h} + \frac{c_2}{4}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\partial_\mu\partial_\nu h^{\mu\nu} - \frac{c_3-c_2+1}{4}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\Box h\right]$$

$$= kT_{\alpha\beta} - k^2\left(T_{\alpha\mu}h^\mu_\beta + T_{\beta\mu}h^\mu_\alpha\right). \tag{9.8.16}$$

Using the trace of the equation and the constraint (9.7.17) to eliminate the terms $\eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}$ and $\eta_{\alpha\beta}\Box h$, we get the new equations of motion:

$$
\begin{aligned}
&-\frac{1}{4}\Box h_{\alpha\beta}+\frac{1}{4}(\partial_\mu\partial_\alpha h^\mu_\beta+\partial_\mu\partial_\beta h^\mu_\alpha)-\frac{c_2}{4}\partial_\alpha\partial_\beta h+k\left[\frac{1}{4}\partial_\mu h_{\alpha\nu}\partial^\mu h^\nu_\beta-\frac{1}{4}\partial_\mu h^\nu_\alpha\partial_\nu h^\mu_\beta\right.\\
&+\frac{1}{4}(\partial_\mu h_{\nu\alpha}\partial_\beta h^{\mu\nu}+\partial_\mu h_{\nu\beta}\partial_\alpha h^{\mu\nu})-\frac{1}{8}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu}-\frac{1}{4}\left(\partial_\alpha h_{\mu\beta}\partial_\nu h^{\mu\nu}+\partial_\beta h_{\mu\alpha}\partial_\nu h^{\mu\nu}\right)\\
&+\frac{1}{4}\partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta}-\frac{1}{4}\left(\partial_\alpha h^{\mu\nu}\partial_\mu h_{\nu\beta}+\partial_\beta h^{\mu\nu}\partial_\mu h_{\nu\alpha}\right)+\frac{c_2}{8}(\partial_\alpha h^\mu_\beta\partial_\mu h+\partial_\beta h^\mu_\alpha\partial_\mu h)\\
&-\frac{c_2}{8}\partial_\mu h_{\alpha\beta}\partial^\mu h+\frac{c_2^2+c_2+2c_4}{16}\partial_\alpha h\partial_\beta h+\frac{c_2-2}{8(2c_2-1)}\eta_{\alpha\beta}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu}\\
&+\frac{-c_2^3+c_2^2+2c_2+2(2-c_2)c_4-4c_5}{32(2c_2-1)}\eta_{\alpha\beta}\partial_\mu h\partial^\mu h+\frac{1}{4}(h_{\mu\alpha}\Box h^\mu_\beta+h_{\mu\beta}\Box h^\mu_\alpha)\\
&+\frac{1}{4}(h_{\mu\alpha}\partial_\nu\partial_\beta h^{\mu\nu}+h_{\mu\beta}\partial_\nu\partial_\alpha h^{\mu\nu})+\frac{1}{4}h^{\mu\nu}\partial_\mu\partial_\nu h_{\alpha\beta}-\frac{1}{4}\left(h^\mu_\alpha\partial_\mu\partial_\nu h^\nu_\beta+h^\mu_\beta\partial_\mu\partial_\nu h^\nu_\alpha\right)\\
&-\frac{1}{4}\left(h^{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\beta}+h^{\mu\nu}\partial_\mu\partial_\beta h_{\nu\alpha}\right)-\frac{1}{2}\left(h^\mu_\alpha\partial_\beta\partial_\nu h^\nu_\mu+h^\mu_\beta\partial_\alpha\partial_\nu h^\nu_\mu\right)\\
&+\frac{c_2}{4}(h^\mu_\alpha\partial_\mu\partial_\beta h+h^\mu_\beta\partial_\mu\partial_\alpha h)-\frac{c_2}{8}h\Box h_{\alpha\beta}+\frac{c_2}{8}(h\partial_\mu\partial_\alpha h^\mu_\beta+h\partial_\mu\partial_\beta h^\mu_\alpha)-\frac{c_2-1}{4}h_{\alpha\beta}\Box h\\
&+\frac{c_4}{8}h\partial_\alpha\partial_\beta h+\frac{c_2}{4(2c_2-1)}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\rho h^\rho_\nu-\frac{1}{4(2c_2-1)}\eta_{\alpha\beta}h^{\mu\nu}\Box h_{\mu\nu}\\
&-\frac{c_2^2}{8(2c_2-1)}\eta_{\alpha\beta}h^{\mu\nu}\partial_\mu\partial_\nu h-\frac{c_2^2+c_4}{8(2c_2-1)}\eta_{\alpha\beta}h\partial_\mu\partial_\nu h^{\mu\nu}+\frac{3c_2^2+(4-c_2)c_4-4c_5}{16(2c_2-1)}\eta_{\alpha\beta}h\Box h\\
&+\frac{c_2}{4}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\partial_\mu\partial_\nu h^{\mu\nu}-\frac{c_2^2-c_2+2}{8}\frac{\delta\tilde{h}}{\delta h^{\alpha\beta}}\Box h+\frac{c_2}{4}\partial_\alpha\partial_\beta\tilde{h}-\frac{c_2-2}{8(2c_2-1)}\eta_{\alpha\beta}\Box\tilde{h}\\
&\left.-\frac{c_2^2}{8(2c_2-1)}\frac{\delta\tilde{h}}{\delta h^{\rho\sigma}}\eta^{\rho\sigma}\eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}+\frac{c_2(c_2^2-c_2+2)}{16(2c_2-1)}\frac{\delta\tilde{h}}{\delta h^{\rho\sigma}}\eta^{\rho\sigma}\eta_{\alpha\beta}\Box h\right]\\
&=k\left(T_{\alpha\beta}-\frac{c_2}{2(2c_2-1)}\eta_{\alpha\beta}T\right)-k^2\left(T_{\alpha\mu}h^\mu_\beta+T_{\beta\mu}h^\mu_\alpha-\frac{c_2}{2c_2-1}\eta_{\alpha\beta}T_{\mu\nu}h^{\mu\nu}\right).
\end{aligned}
\tag{9.8.17}
$$

9.8.3 00-component at the ϵ^4 order

The 00-component of equation (9.8.17) at the desired order is

$$\frac{1-c_2}{4}\partial_0^2 h_{00}^{(2)} + \frac{1}{4}\nabla^2 h_{00}^{(4)} - \frac{1}{2}\partial_0\partial_i h_{0i}^{(3)} + \frac{c_2}{4}\partial_0^2 h_{ii}^{(2)}$$

$$+ k\left[+\frac{1}{4}\partial_i h_{ij}^{(2)}\partial_j h_{00}^{(2)} + \frac{c_2^3+7c_2-26c_2+2c_2c_4-4c_4+4c_5+16}{32(2c_2-1)}\left(\partial_i h_{00}^{(2)}\right)^2\right.$$

$$-\frac{c_2-2}{8(2c_2-1)}\left(\partial_i h_{jl}^{(2)}\right)^2 + \frac{c_2^3-c_2^2-2c_2-4c_4+4c_5+2c_2c_4}{32(2c_2-1)}\left(\partial_i h_{kk}^{(2)}\right)^2$$

$$-\frac{c_2^3+3c_2^2-4c_2+2c_2c_4-4c_4+4c_5}{16(2c_2-1)}\partial_i h_{jj}^{(2)}\partial_i h_{00}^{(2)}$$

$$+\frac{9c_2^2-30c_2-(4-c_2)c_4+4c_5+16}{16(2c_2-1)}h_{00}^{(2)}\nabla^2 h_{00}^{(2)} - \frac{c_2^2-4c_2+2}{8(2c_2-1)}h_{ij}^{(2)}\partial_i\partial_j h_{00}^{(2)}$$

$$-\frac{c_2^2-2c_2-(4-c_2)c_4+4c_5}{16(2c_2-1)}h_{ii}^{(2)}\nabla^2 h_{00}^{(2)} - \frac{c_2}{4(2c_2-1)}h_{ij}^{(2)}\partial_i\partial_k h_{jk}^{(2)}$$

$$-\frac{5c_2^2-12c_2-(4-c_2)c_4+4c_5+4}{16(2c_2-1)}h_{00}^{(2)}\nabla^2 h_{ii}^{(2)} + \frac{1}{4(2c_2-1)}h_{ij}^{(2)}\nabla^2 h_{ij}^{(2)}$$

$$+\frac{c_2^2}{8(2c_2-1)}h_{ij}^{(2)}\partial_i\partial_j h_{kk}^{(2)} - \frac{c_2^2+c_4}{8(2c_2-1)}h_{00}^{(2)}\partial_i\partial_j h_{ij}^{(2)} + \frac{c_2^2+c_4}{8(2c_2-1)}h_{kk}^{(2)}\partial_i\partial_j h_{ij}^{(2)}$$

$$-\frac{3c_2^2+(4-c_2)c_4-4c_5}{16(2c_2-1)}h_{ii}^{(2)}\nabla^2 h_{jj}^{(2)} + \frac{3c_2^3-5c_2^2+8c_2-4}{16(2c_2-1)}\left(\frac{\delta\tilde{h}}{\delta h_{00}}\right)^{(2)}\nabla^2\left(h_{00}^{(2)}-h_{ii}^{(2)}\right)$$

$$+\frac{c_2(3c_2-2)}{8(2c_2-1)}\left(\frac{\delta\tilde{h}}{\delta h_{00}}\right)^{(2)}\partial_i\partial_j h_{ij}^{(2)} + \frac{c_2^2}{8(2c_2-1)}\left(\frac{\delta\tilde{h}}{\delta h_{lm}}\right)^{(2)}\delta_{lm}\,\partial_i\partial_j h_{ij}^{(2)}$$

$$\left.+\frac{c_2(c_2^2-c_2+2)}{16(2c_2-1)}\left(\frac{\delta\tilde{h}}{\delta h_{ij}}\right)^{(2)}\delta_{ij}\,\nabla^2\left(h_{00}^{(2)}-h_{kk}^{(2)}\right) + \frac{c_2-2}{8(2c_2-1)}\nabla^2\tilde{h}^{(4)}\right] =$$

$$= k\left(T_{00}^{(2)} - \frac{c_2}{2(2c_2-1)}T^{(2)}\right) - k^2\left(\frac{3c_2-2}{2c_2-1}T_{00}^{(0)}h_{00}^{(2)}\right). \qquad (9.8.18)$$

From (9.8.6) we have that

$$\left(\frac{\delta\tilde{h}}{\delta h_{00}}\right)^{(2)} = h_{ii}^{(2)} \tag{9.8.19a}$$

$$\left(\frac{\delta\tilde{h}}{\delta h_{ij}}\right)^{(2)} \delta_{ij} = 3h_{00}^{(2)} - 2h_{ii}^{(2)} \tag{9.8.19b}$$

$$\begin{aligned}\nabla^2\tilde{h}^{(4)} = &\left(h_{ii}^{(2)}\nabla^2 h_{00}^{(2)} + h_{00}^{(2)}\nabla^2 h_{ii}^{(2)} + 2\partial_j h_{ii}\partial_j h_{00}^{(2)}\right)\\ &- \left(h_{11}^{(2)}\nabla^2 h_{22}^{(2)} + h_{22}^{(2)}\nabla^2 h_{11}^{(2)} + 2\partial_j h_{11}\partial_j h_{22}^{(2)}\right)\\ &- \left(h_{11}^{(2)}\nabla^2 h_{33}^{(2)} + h_{33}^{(2)}\nabla^2 h_{11}^{(2)} + 2\partial_j h_{11}\partial_j h_{33}^{(2)}\right)\\ &- \left(h_{33}^{(2)}\nabla^2 h_{22}^{(2)} + h_{22}^{(2)}\nabla^2 h_{33}^{(2)} + 2\partial_j h_{33}\partial_j h_{22}^{(2)}\right). \end{aligned} \tag{9.8.19c}$$

Moreover, from (9.3.14d), (9.3.14b) and (9.3.14e) we have that

$$\partial_0\partial_i V_i = -\partial_0^2 U \tag{9.8.20a}$$

$$\partial_0\partial_i W_i = \partial_0\partial_i V_i - \partial_0^2\nabla^2\chi = \partial_0^2 U. \tag{9.8.20b}$$

Hence, using (9.8.19), inserting the general PPN expansion for the metric (9.5.1) and for the EMT (9.6.12), and using the relations (9.8.20), the equations of motion (9.8.18) read

$$
\frac{c_2-1}{2}\partial_0^2 U+\frac{\xi}{4}\nabla^2\Phi_W-\frac{1}{4}(2\gamma+2+\alpha_3+\zeta_1-2\xi)\nabla^2\Phi_1-\frac{1}{2}(1+\zeta_3)\nabla^2\Phi_3
$$

$$
-\frac{1}{2}(3\gamma-2\beta+1+\zeta_2+\xi)\nabla^2\Phi_2-\frac{1}{2}(3\gamma+3\zeta_4-2\xi)\nabla^2\Phi_4+\frac{1}{4}(\zeta_1-2\xi)\nabla^2 A
$$

$$
+\beta U\nabla^2 U+\beta\left(\partial_i U\right)^2+\frac{1}{4}\left(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi\right)\partial_0^2 U-\frac{1}{4}\left(1+\alpha_2-\zeta_1+2\xi\right)\partial_0^2 U
$$

$$
-\frac{3c_2}{2}\gamma\,\partial_0^2 U+\frac{c_2^3+7c_2^2-26c_2+2c_2c_4-4c_4+4c_5+16}{8(2c_2-1)}\left(\partial_i U\right)^2
$$

$$
-\frac{3c_2^3+9c_2^2-32c_2+6c_2c_4-12c_4+12c_5+28}{4(2c_2-1)}\gamma\left(\partial_i U\right)^2
$$

$$
+\frac{9(c_2^3-c_2^2-6c_2-4c_4+4c_5+2c_2c_4+8)}{8(2c_2-1)}\gamma^2\left(\partial_i U\right)^2
$$

$$
+\frac{3c_2^3+6c_2^2-24c_2-(4-c_2)c_4+4c_5+16}{4(2c_2-1)}U\nabla^2 U
$$

$$
-\frac{3c_2^3+8c_2^2-28c_2+3c_2c_4-11c_4+12c_5+26}{2(2c_2-1)}\gamma U\nabla^2 U
$$

$$
-\frac{9c_2^3-18c_2^2+64c_2-9c_2c_4+30c_4-36c_5-72}{4(2c_2-1)}\gamma^2 U\nabla^2 U=
$$

$$
=8\pi\left[\frac{3c_2-2}{4(2c_2-1)}f_m(1)\rho_0\Pi+\frac{3c_2-2}{2(2c_2-1)}\Big(f_m(1)-(1-3\gamma)f_m'(1)\Big)\rho_0 U\right.
$$

$$
\left.+\frac{1}{2}f_m(1)\rho_0 v^2+\left(\frac{3c_2-2}{4(2c_2-1)}f_m(1)+\frac{1}{2c_2-1}f_m'(1)\right)p\right]. \qquad (9.8.21)
$$

Inserting in the equation the values found for γ (9.7.19), α_1 (9.7.34) and $f_m(1)$ (9.7.20), the solution is given by

$$\begin{cases} \xi\nabla^2\Phi_W = 0 \\ (\zeta_1 - 2\xi)\nabla^2 A = 0 \\ (\alpha_2 - \zeta_1 + 2\xi)\,\partial_0^2 U = 0 \\ \left(4\frac{2c_2-1}{3c_2-2} + \alpha_3 + \zeta_1 - 2\xi\right)\nabla^2\Phi_1 = -16\pi\,\frac{2c_2-1}{3c_2-2}\rho_0 v^2 \\ (1+\zeta_3)\nabla^2\Phi_3 = -4\pi\rho_0\Pi \\ \left(\frac{3c_2}{3c_2-2} + 3\zeta_4 - 2\xi\right)\nabla^2\Phi_4 = -16\pi\,\left(\frac{1}{4} + \frac{f'_m(1)}{2c_2-1}\right)p \\ \Big[2(2c_2-1)(3c_2-2)^2\beta - 35c_2^3 + 65c_2^2 - 46c_2 + 16 + 2c_4(c_2-2) + 4c_5\Big]\,(\partial_i U)^2 = 0 \\ 2\Big[35c_2^3 - 65c_2^2 + 46c_2 - 16 - 2c_4(c_2-2) - 4c_5 - (2c_2-1)(3c_2-2)^2\beta\Big]U\nabla^2 U \\ +(2c_2-1)(3c_2-2)^2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\nabla^2\Phi_2 = \\ = -8\pi\,(3c_2-2)^2\Big(2c_2 - 1 + 2f'_m(1)\Big)\rho_0 U. \end{cases} \tag{9.8.22}$$

The system (9.8.22) yields

$$\alpha_2 = \alpha_3 = \zeta_1 = \zeta_3 = \xi = 0$$

$$\zeta_2 = \frac{2}{3c_2-2}\left(2\frac{f'_m(1)}{f_m(1)} - 1\right) = 2\left(\frac{2}{2c_2-1}f'_m(1) - \frac{1}{3c_2-2}\right)$$

$$\zeta_4 = \frac{2}{3(3c_2-2)}\left(2\frac{f'_m(1)}{f_m(1)} - 1\right) = \frac{2}{3}\left(\frac{2}{2c_2-1}f'_m(1) - \frac{1}{3c_2-2}\right)$$

$$\beta = \frac{35c_2^3 - 65c_2^2 + 46c_2 - 16 - 2c_4(c_2-2) - 4c_5}{2(2c_2-1)(3c_2-2)^2}\,.$$

9.9 Summary and comparison with experiments

Using together the results found in sections (9.7) and (9.8), with the definitions (9.8.10) and (9.8.11), we can summarize the values of the PPN parameters that have been found for the general TDiff action (9.8.1):

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_3 = \xi = 0 \tag{9.9.1}$$

$$\zeta_2 = 2\left(\frac{2}{2c_2 - 1}f'_m(1) - \frac{1}{3c_2 - 2}\right) \tag{9.9.2}$$

$$\zeta_4 = \frac{2}{3}\left(\frac{2}{2c_2 - 1}f'_m(1) - \frac{1}{3c_2 - 2}\right) \tag{9.9.3}$$

$$\gamma = \frac{c_2}{3c_2 - 2} \tag{9.9.4}$$

$$\beta = \frac{35c_2^3 - 65c_2^2 + 46c_2 - 16 - 2c_4(c_2 - 2) - 4c_5}{2(2c_2 - 1)(3c_2 - 2)^2}\,. \tag{9.9.5}$$

As expected, since TDiff theories do not predict preferred-location effects, we find $\xi = 0$; and since also preferred-frame effects are not predicted, we find $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
On the other hand, the non-conservation of the EMT has already been studied in chapter (6), so that we expected to find values of the ζ parameters different from zero, which characterize non-conservative theories (with violations of the conservation of the total momentum).

We notice that in the special Diff-invariant case (General Relativity), i.e. when

$$c_2 = c_3 = c_4 = 1$$
$$c_5 = 0$$
$$f'_m(1) = \tfrac{1}{2}\,,$$

the PPN parameters reduce to the expected values:

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi = 0 \tag{9.9.6a}$$
$$\beta = \gamma = 1. \tag{9.9.6b}$$

There are strong experimental bounds on the values of the PPN parameters. Today's measured values of the PPN parameters are [25]:

$$|\gamma - 1| < 2,3 \cdot 10^{-5} \tag{9.9.7a}$$
$$|\beta - 1| < 2,3 \cdot 10^{-4} \tag{9.9.7b}$$
$$|\xi| < 10^{-3} \tag{9.9.7c}$$
$$|\alpha_1| < 10^{-4} \tag{9.9.7d}$$
$$|\alpha_2| < 4 \cdot 10^{-7} \tag{9.9.7e}$$
$$|\alpha_3| < 4 \cdot 10^{-20} \tag{9.9.7f}$$
$$|\zeta_1| < 2 \cdot 10^{-2} \tag{9.9.7g}$$
$$|\zeta_2| < 4 \cdot 10^{-5} \tag{9.9.7h}$$
$$|\zeta_3| < 10^{-7}. \tag{9.9.7i}$$

Let's first define the following quantities:

$$\Delta_2 \equiv c_2 - 1 \tag{9.9.8a}$$
$$\Delta_4 \equiv c_4 - 1 \tag{9.9.8b}$$
$$\Delta_f \equiv 2f'_m(1) - 1. \tag{9.9.8c}$$

Since γ is an injective function of c_2 (actually it is an hyperbole), the experimental value of γ does not leave much freedom to the parameter of the theory c_2: we can expand the expression for γ in terms of the small quantity Δ_2 as

$$\gamma \approx 1 - 2\Delta_2 \tag{9.9.9}$$

so that (9.9.7a) gives the limit

$$|\Delta_2| < 10^{-5}. \tag{9.9.10}$$

Hence, we can set $c_2 \approx 1$ in the other PPN parameters, so that

$$\zeta_2 \approx 2\Delta_f \tag{9.9.11a}$$

$$\zeta_4 \approx \frac{2}{3}\Delta_f \tag{9.9.11b}$$

$$\beta \approx 1 + \Delta_4 - 2c_5. \tag{9.9.11c}$$

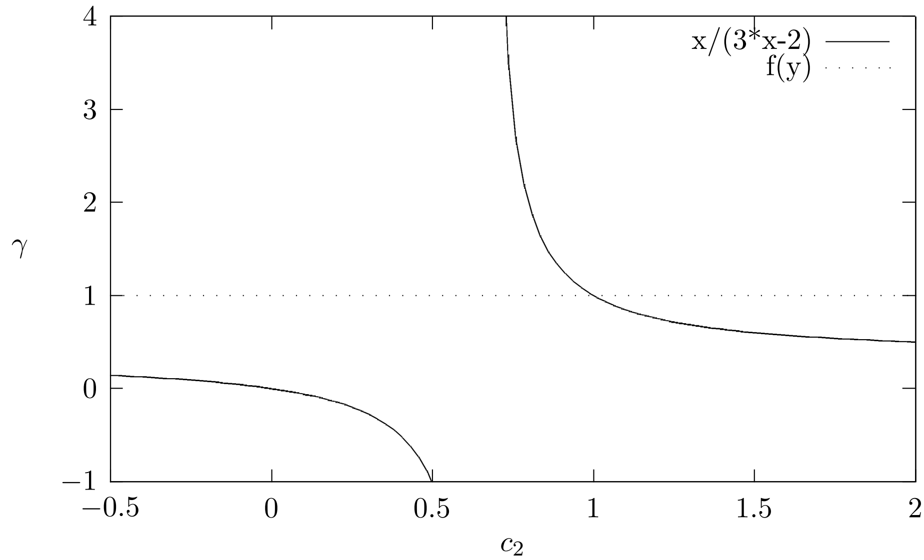

Figure 8: Value of γ in function of c_2 plotted with contiuous line.

The experimental values (9.9.7h) and (9.9.7b) give the bounds

$$|\Delta_m| < 2 \cdot 10^{-5} \tag{9.9.12a}$$

$$|\Delta_4 - 2c_5| < 2,3 \cdot 10^{-4}. \tag{9.9.12b}$$

Using the definitions (9.8.11) for the c_i parameters and (9.9.8) , the experimental bounds (9.9.10) and (9.9.12) can also be rewritten as

$$|2f_1'(1) - 1| < 10^{-5} \tag{9.9.13a}$$

$$|2f_m'(1) - 1| < 2 \cdot 10^{-5} \tag{9.9.13b}$$

$$\left| f_1''(1) + 2f_2'(1) + \frac{1}{4} \right| < 5,8 \cdot 10^{-5}. \tag{9.9.13c}$$

We have to impose the constraint (9.7.17):

$$2c_3 = c_2^2 + c_2 \tag{9.9.14}$$

which looks like an integrability condition. This constraint, together with (2.4.19), yields

$$c_2 \leq \frac{1}{2} \qquad \text{or} \qquad c_2 \geq 1. \tag{9.9.15}$$

This means that the observational bound (9.9.10) is actually

$$c_2 - 1 < 10^{-5}. \tag{9.9.16}$$

The measures of the PPN parameters impose strict bounds on the parameters of the theory c_2 and $f'_m(1)$, but not on the single parameters c_4 and c_5; only the combination $c_4 - 2c_5$ is constrained by experiments.

10 Conclusions

In this work we have investigated a natural scenario to study deviations from General Relativity, in order to give possible answers to the problems arising from Einstein's theory. Anyway, almost only classical effects have been investigated, since these are the most easily verifiable ones; it has not been verified if TDiff theories could be consistent renormalizable theories for quantum gravity.
Since consistent propagation of a massless spin-2 graviton requires less symmetry than that imposed in General Relativity, the straightforward path has been to consider a theory whose gauge-group is only TDiff, which leaves us much more freedom to write the possible actions. This is the smallest symmetry needed for the theory not to have classical instabilities, provided that the parameters of the linearized theory satisfy the condition

$$2c_3 \leq 3c_2^2 - 2c_2 + 1.$$

Consistency requires that both the gravitational and the matter sectors have to be whether Diff-invariant or TDiff-invariant, unless we impose the extremely strong integrability conditions $L_m = 0$ or $R = 0$. If we consider the TDiff-invariant case, the freedom we have to play with in the matter sector allows to give some theoretical solutions to direct Cosmological Constant problem.

Probably one of the greatest differences from Diff-invariant theories is the non-conservation of the source of gravity, that is, of the "active" energy-momentum tensor. This will eventually lead to violations of the conservation of the total momentum, of the equality between active and passive gravitational masses, and of the Strong Equivalence Principle; hence, experiments leave small room for deviations from General Relativity.
Other confrontations with experimental observations are given by the tests on the violation of the inverse-square-law, since the coupling of TDiff-invariant gravity to the matter predicts the propagation of an additional scalar degree of freedom, which modifies the standard spin-2 graviton interaction.

Finally, we have calculated the PPN expansion of the metric for some general TDiff theories (i.e. those which verify $f_k(|g|) = f_v(|g|)$ in the matter Lagrangian), since the PPN formalism is one of the most efficient tools to compare gravitational theories with experiments. In this part we have found an integrability condition on the parameters of the theory:

$$2c_3 = c_2^2 + c_2.$$

Moreover, the results and the confrontation with the measured PPN parameters pose some strict but not absolute bounds on the possible violations of Diff-invariance.

Concluding, we can say that no strong theoretical arguments have risen to exclude TDiff-invariance from being the fundamental symmetry of nature, although experiments show that the violations of Diff-invariance have to be small. Anyway, some freedom seems to be left, since for instance the terms $f_1''(1)$ and $f_2'(1)$ of the theory are not strictly bounded by observations (only the combination $f_1''(1) + 2f_2'(1)$ is bounded).
Moreover, not all possible TDiff-invariant Lagrangians have been taken into account in calculating the PPN parameters: for example, the case with $f_k(|g|) \neq f_v(|g|)$ in the matter Lagrangian, which includes WTDiff-invariant Lagrangians, have still to be considered. Such Lagrangians could better solve the direct Cosmological Constant problem (since the kinetic energy could have the same weight as in General Relativity) even if, in this case, test bodies seem not to follow geodesic trajectories.
Besides, the quantum behavior of Transverse Gravity should be better investigated.

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