Contents

Chapter 1

Preliminary results

1.1 Complex surfaces

The setting in which our work takes place is that of complex analytic surfaces. The exact definition of "surface" will vary from chapter to chapter. For this first chapter, we will adopt the weakest notion:

Definition We will call *surface* a Hausdorff, reduced and irreducible complex space of dimension 2, and a *smooth surface* a connected complex manifold, of complex dimension 2.

We will say that a smooth surface is *kählerian* if it is so as complex manifold, that is, if it is given a hermitian metric such that its associated real $(1,1)$-form is d-closed. A surface will be called *algebraic* or *projective* if it admits a closed immersion in a complex projective space.

We recall some well known facts on complex varieties. Every projective variety is compact and kählerian, since it inherits the Fubini-Study metrics on $\mathbb{P}^n$. Chow's theorem encloses in this point of view the algebraic subvarieties of $\mathbb{P}^n$, justifying terminology in the definition:

Theorem 1.1.1 (Chow [**?**]). *An analytic subvariety of $\mathbb{P}^n$ is the zero locus of a finite number of polynomials. In particular, if an analytic variety admits a closed immersion in some $\mathbb{P}^n$, then it is algebraic.*

Given the definition, one is lead to ask "how many" are the non-isomorphic surfaces, and "how many" of them are kählerian (resp. algebraic). To answer these questions, and frame the central part of our work, we enunciate here the classification of Enriques and Kodaira. To do so, we will introduce some concepts, especially regarding numerical invariants, that will be fundamental for the next chapters, too. We will not prove the facts stated in this chapter, since it would lead us too far from our objective.

1.2 Bimeromorphic equivalence

First, we want to clarify which is the purpose of the Enriques-Kodaira classification. As it often happens, it is too much to ask for a classification up to isomorphism; in our circumstance, it is much more profitable to ask for a classification up to *bimeromorphic equivalences*, notion that, as an additional result, will allow us to extend without any effort the classification to the case of singular surfaces. Let us begin with a preliminary definition.

Definition Let X, Y be surfaces. A proper holomorphic and surjective map $\pi\colon X \to Y$ is said to be a *bimeromorphic map* if there exist proper analytic subsets $T \subset X$, $S \subset Y$ such that the restriction $\pi\colon X \setminus T \to Y \setminus S$ is a biholomorphism.

Clearly, this is not a good notion to define an equivalence: it requires π to be definite on all of X, so that if it had an inverse of the same nature it would lead to an analytic isomorphism. On the other hand, we have two important classes of maps that are bimeromorphic maps, so we are lead to take this notion as the first step to define equivalence. The first of these classes is given by normalizations. A normalization is the given of a normal variety X_{norm} and a finite and surjective morphism $\nu\colon X_{\mathrm{norm}} \to X$, such that ν is an isomorphism out of the points over $N = \{x \in X \mid \mathcal{O}_{X,x} \text{ is not normal}\}$. Hence, for our definition, a normalization is in particular a bimeromorphic map; furthermore, the normalization is unique up to isomorphism (this is a trivial consequence of the universal property of the normalization). Thus, in what follows, when we are interested in classification questions we may always assume that the varieties we are working with are normal. The other important class of maps that turn out to be bimeromorphic is that of *blowups*. In particular, we will see how important are to our purposes the blowups in a point, called *monoidal transformations* or σ*-processes*.

We now come to the precise definition of bimeromorphic equivalence. We state it for normal surfaces, since up to bimeromorphic maps, we can always replace any variety with its normalization.

Definition Let X and Y be normal surfaces. We will say that X and Y are *bimeromorphically equivalent* if there is a triple (Z, π_1, π_2) where Z is another normal surface, and $\pi_1 \colon Z \to X$, $\pi_2 \colon Z \to Y$ are bimeromorphic maps.

It is really an equivalence relation: if there exist two surfaces Z_1 and Z_2, maps that make X_1 and X_2 equivalent via Z_1, and maps that make X_2 and X_3 equivalent via Z_2, then also X_1 and X_3 are equivalent via an adequate subvariety Z of the normalization of the fiber product $Z_1 \times_{X_2} Z_2$ (which by itself would be not normal, in general).

The main reasons for which we assume all spaces to be normal lie in the following theorems:

Theorem 1.2.1 ([**?**]). *Let $\pi \colon X \to Y$ be a proper holomorphic and surjective maps between surfaces, with X normal. Then there exists a discrete subset $S \subset Y$ such that the restriction to the complement $\pi|\pi^{-1}(Y \setminus S)$ is biholomorphic.*

Theorem 1.2.2 (Riemann, Levi [**?**]). *Let Y be a normal surface, and $S \subset Y$ be an analytic subset of codimension at least 2 (that is, a discrete subset). Then every meromorphic (resp. holomorphic) function defined on $Y \setminus S$ extends uniquely to a meromorphic (resp. holomorphic) function on Y.*

Thanks to these theorems we can define a map $\pi_* \colon \mathcal{F}_X \to \mathcal{F}_Y$ (where $\mathcal{F}_X$ may be both the ring of holomorphic functions or the field of meromorphic functions on X), composing a function $f \in \mathcal{F}_X$ with π^{-1} (where this makes sense). In this way we have a function on $Y \setminus S$, hence a unique function in $\mathcal{F}_Y$. For the same reason, the composition $\pi_* \circ \pi^*$ is the identity; the map π_* is clearly injective, hence *the assumption of normality makes the maps π_* and π^* induce isomorphisms between the rings of holomorphic functions and the fields of meromorphic functions.*

Obviously, meromorphic maps are particular examples of meromorphic equivalences, taking as the other map the identity.

Now that we have a good notion of equivalence between surfaces, let us see what reductions we can make thanks to it, other than the already discussed normality. The first one is about non-singularity: differently from the case of curves, taking normalization does not lead to a smooth surface (already in the algebraic case, Serre's theorem tells that normality is equivalent to being Cohen-Macaulay and regular in codimension 1, but in general there will be punctual singularities). However, we have:

Theorem 1.2.3. *Let X be a normal surface. Then there exists a bimeromorphic map $\pi\colon Y \to X$, with Y a smooth surface.*

(For the proof of this theorem, and of the facts that follows, we refer to [**?**], chap. III, par. 1–7). The disingularization so obtained is not unique, though. For example, if X is already smooth, the blowup of any point of X leads to another smooth surface, that maps bimeromorphically on X. But this is essentially the only possible exception:

Theorem 1.2.4. *Let $\pi\colon Y \to X$ be the σ-process in p. The pre-image $\pi^{-1}(p)$ is a rational curve E such that $E^2 = -1$ (this kind of curves is said to be exceptional curves of the first kind, or (-1)-curve).*

On the other hand, if E is a rational curve with $E^2 = -1$, there exists a unique monoidal transformation such that E is its exceptional curve.

In the theorem, E^2 denotes the self-intersection product; for the definition of intersection product and its main properties, we refer to [**?**], chap. II, par. 9. We will often refer to the second part of this theorem (also known as the Castelnuovo contraibility criterion) saying that "contracting (-1)-curves does not create singularities". Thanks to this theorem, one can obtain the following:

Theorem 1.2.5. *Let X be a normal surface. Then there exists a unique smooth surface Y with a bimeromorphic map $\pi\colon Y \to X$ with the property that π does not contract any (-1)-curve. Such resolution of singularities is said to be* minimal.

Given a complex surface X, then, we have first constructed its unique normalization, and then a canonical desingularization, obtaining a smooth surface. The last reduction we do is to take a *minimal* representative:

Definition A surface X is said to be *minimal* if it does not contain any (-1)-curve.

Given a complex *compact* surface, we can always find a minimal model by contracting its (-1)-curves: we just have to choose a (-1)-curve, contract it, and then iterate on the surface so obtained. The process must terminate, since one can show that the second Betti number drops by 1 at each step. Such process, in general, will *not* be unique, since it may depend on the order we choose to contract the curves (when there are at least two intersecting);

in the greatest part of the cases we see that the surface we obtain is uniquely determined, but it is not always so.

To understand the purpose of this definition, we enunciate the following factorization lemma:

Lemma 1.2.6. *Let $\pi\colon Y \to X$ be a bimeromorphic map between smooth surfaces. Then there are only two possibilities:*

1. *π is an isomorphism;*

2. *there exists a factorization $\pi = \pi' \circ \sigma\colon Y \to Y' \to X$, with $\sigma\colon Y \to Y'$ a σ-process.*

This fundamental lemma tells us that, after contracting the (-1)-curves, bimeromorphic maps become isomorphisms. Maybe, it is even more explicative in the following form: up to an isomorphism, every bimeromorphic map is composition of a locally finite number of σ-process. In particular, a surface is minimal if and only if every bimeromorphic map from it to another surface is an isomorphism. We have the following corollary:

Theorem 1.2.7. *Two smooth surfaces are bimeromorphically equivalent if and only if there exists a smooth surface Z and maps $\pi_1\colon Z \to X$, $\pi_2\colon Z \to Y$ that are compositions of σ-processes.*

This result allows us to verify readily when a number we define is an invariant of bimeromorphic equivalence.

1.3 Numerical invariants

The first step to speak of classification is clearly to define invariants for bimeromorphic equivalence. We define here the classical ones, and precise up to which transformations they are invariants. In what follows, X will always be a surface. Let us start with some notation:

- $\mathcal{O}_X$ is the structural sheaf of X;

- $\mathcal{O}_X^*$ is the sheaf of germs of never-vanishing holomorphic functions on X;

- $\mathcal{T}_X$ is the holomorphic tangent bundle of X;

- Ω^i_X is the sheaf of germs of holomorphic i-forms on X ($i = 1, 2$), that is the sheaf of germs of sections in the bundle $\bigwedge^i \mathcal{T}_X^\vee$
- $\mathcal{K}_X$ is the canonical line bundle on X, i.e. the holomorphic 1-vector bundle $\bigwedge^2 \mathcal{T}_X^\vee$.

With the convention $\Omega^0_X = \mathcal{O}_X$, we define:

$$h^{p,q}(X) = h^q(\Omega^p_X) = \dim_{\mathbb{C}} H^q(\Omega^p_X)$$

In particular, we will use the classical names of *geometric genus* for

$$p_g(X) = h^{0,2}(X)$$

and *irregularity* for

$$q(X) = h^{0,1}(X).$$

We obtain the *holomorphic Euler characteristic* $\chi(\mathcal{O}_X)$ and the *arithmetic genus* $p_a(X)$:

$$\chi(\mathcal{O}_X) = 1 + p_a(X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - q(X) + p_g(X)$$

For the following, we refer to [**?**], chap. I, par. 9, and the references there cited:

Theorem 1.3.1. *Let X be a complex manifold of dimension ≥ 2, $p\colon \tilde{X} \to X$ a σ-process and $E = p^{-1}(p_0)$ the exceptional curve. Then $\mathcal{N}_{E/\tilde{X}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and:*

1. *p induces an isomorphism between the fields of meromorphic functions on X and $\tilde{X}$;*
2. *$p_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_X$, and $R^i p_*(\mathcal{O}_{\tilde{X}}) = 0$ for $i \geq 1$;*
3. *$p^*\colon H^i(X, \mathcal{O}_X) \to H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is an isomorphism for all $i \geq 0$;*
4. *$p^*\colon H^i(X, \mathbb{Z}) \to H^i(\tilde{X}, \mathbb{Z})$ is bijective for $i = 1$ and injective for $i = 2$. Furthermore,*

$$H^2(\tilde{X}, \mathbb{Z}) \cong p^*(H^2(X, \mathbb{Z})) \oplus \mathbb{Z}c_1(\mathcal{O}_{\tilde{X}}(E));$$

5. *For every $a \in H^2(X, \mathbb{Z})$ we have $p_! p^*(a) = a$;*

6. *$p^*\colon H^1(X,\mathcal{O}_X^*) \to H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is injective, and thus $Pic(\tilde{X})$ is isomorphic to the product of $Pic(X)$ and the infinite cyclic subgroup generated by $\mathcal{O}_{\tilde{X}}(E)$;*

7. *$\mathcal{K}_{\tilde{X}} = p^*(\mathcal{K}_X) \otimes \mathcal{O}_{\tilde{X}}((\dim X - 1)E)$;*

8. *p induces an isomorphism $p^*\colon \Gamma(X,\mathcal{K}_X^{\otimes m}) \to \Gamma(\tilde{X},\mathcal{K}_{\tilde{X}}^{\otimes m})$ for all $m \geq 1$, so if X is compact, $P_m(\tilde{X}) = P_m(X)$, for $m \geq 1$ and $\mathrm{kod}(\tilde{X}) = \mathrm{kod}(X)$.*

This theorem implies that the first two, hence also the remaining ones, from the classical invariants are preserved by monoidal transformations, hence by every bimeromorphic equivalence. In the same way, we obtain as invariants the *plurigenera*, defined as a generalization of the geometric genus:

$$P_n = h^0(\mathcal{K}_X^{\otimes n}).$$

The analogy with the geometric genus follows from Serre's duality applied to the structural sheaf, $h^0(\mathcal{K}_X) = h^2(\mathcal{O}_X)$, so that $P_1 = p_g$.

Knowing all of the plurigenera is often too much to ask for, but they are used to define a very useful invariant that is related to their asymptotic growth. To do so, we introduce the canonical ring:

Definition Let X be a smooth surface. The direct sum

$$R(X) = \mathbb{C} \oplus \bigoplus_{m\geq 1} \Gamma(X,\mathcal{K}_X^{\otimes m})$$

is a graded ring with the natural product, called *canonical ring* of X.

We have the following theorem (for its proof we refer to [**?**]):

Theorem 1.3.2. *The canonical ring of a smooth surface is finitely generated. In particular, the transcendence degree of its field of fractions is finite.*

Definition We define the *Kodaira dimension* of X as:

$$\mathrm{kod}(X) = \begin{cases} -\infty & \text{se } R(X) \cong \mathbb{C} \\ \mathrm{trdeg}(R(X)) - 1 & \text{altrimenti} \end{cases}$$

Clearly, this coincides with the dimension of the variety defined by $\mathbf{Proj}(R(X))$. One can also define the Kodaira dimension for a curve C, setting $\mathrm{kod}(C) = -\infty$ if C is rational, $\mathrm{kod}(C) = 0$ if C is elliptic and $\mathrm{kod}(C) = 1$ otherwise.

Before enunciating its connection with plurigenera, let us also define the *algebraic dimension*:

Definition The algebraic dimension $\mathrm{a}(X)$ is the transcendence degree of the field of meromorphic function on X.

We collect in the following theorem the links between these invariants, and other properties of the Kodaira dimension:

Theorem 1.3.3. *Let X be a smooth surface. Then:*

1. $\mathrm{kod}(X) \leq \mathrm{a}(X) \leq \dim X$
2. *The Kodaira dimension is uniquely determined by the asymptotic growth of the plurigenera. In particular:*

$$\begin{aligned} \mathrm{kod}(X) = -\infty &\iff P_m(X) = 0 \quad \text{for all } m \geq 1 \\ \mathrm{kod}(X) = 0 &\iff P_m(X) \in \{0,1\} \text{ for all } m \geq 1, \text{ but it is not constantly } 0 \\ \mathrm{kod}(X) = 1 &\iff P_m(X) \text{ has linear growth for } m \to \infty \\ \mathrm{kod}(X) = 2 &\iff P_m(X) \text{ has quadratic growth for } m \to \infty \end{aligned}$$

3. *For all $r > 0$ the linear system $|rK_X|$ (where K_X is the canonical divisor of X) determines a meromorphic function to a projective space $\phi_{rK_X} \colon X \dashrightarrow \mathbb{P}^N$; then the Kodaira dimension of X equals the maximum dimension of the images of ϕ_{rK_X}, $r > 0$.*
4. *If C_1, C_2 are curves, then* $\mathrm{kod}(C_1 \times C_2) = \mathrm{kod}(C_1) + \mathrm{kod}(C_2)$.
5. *If $f \colon Y \to X$ is a holomorphic and generically finite map, then $P_n(Y) \geq P_n(X)$ for every $n \geq 1$, hence the same inequality holds for Kodaira dimension; if furthermore f is an unbranched covering, then* $\mathrm{kod}(X) = \mathrm{kod}(Y)$.
6. Iitaka's conjecture $C_{2,1}$*: if $f \colon X \to S$ is a proper, holomorphic and surjective map, with S a smooth curve, then*

$$\mathrm{kod}(X_s) + \mathrm{kod}(S) \leq \mathrm{kod}(X)$$

where X_s is the general fiber.

The proof of the first 5 parts can be found in [**?**]; for the last one, we refer to [**?**], chap. III, par. 18. For the definition and basic properties of linear systems and rational maps obtained by them, cfr [**?**], pages. 176-180.

The theorem implies that the Kodaira dimension is a bimeromorphic invariant; the same holds for algebraic dimension thanks to theorem (1.3.1). Algebraic dimension also gives us a very useful way to check if a surface is algebraic (cfr. [**?**], chap. IV, par. 6):

Theorem 1.3.4. *A surface X is algebraic if and only if* $\mathrm{a}(X) = 2$.

Together with this one, other useful criteria for projectivity are proved. We collect them here in the following:

Theorem 1.3.5. *Let X, Y be complex compact surfaces. Then:*

1. *X is projective if and only if there is on X a divisor L with $L^2 > 0$;*
2. *Let $\tilde{X}$ be obtained from X by blowing up a point. Then X is projective if and only if $\tilde{X}$ is projective;*
3. *Let $f\colon X \to Y$ be a finite map. Then X is projective if and only if Y is projective.*

We end this overview on invariants citing two more, but of topological nature: the cohomology sequences relate dimension of groups of cohomology with coefficients in the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^*$ with Betti numbers, so it is natural to consider these as further invariants. Clearly, by Poincaré duality and by the hypothesis of connectedness, we obtain readily

$$b_0(X) = b_4(X) = 1, \quad b_1(X) = b_3(X),$$

hence we can consider only the first and second Betti numbers. These are topological invariants, but a priori can change through bimeromorphic maps. However, for surfaces, again by theorem (1.3.1), $b_1(X)$ *is a bimeromorphic invariant, too.* On the other hand, in general $b_2(X)$ *is not.* The following (not easy) theorem holds:

Theorem 1.3.6. *Let X be a surface. Then if X is kählerian, $q = h^{0,1} = h^{1,0}$, and $b_1(X) = 2q$; if X is not kählerian, $h^{1,0} = q - 1$, and $b_1(X) = 2q - 1$.*

In particular, X is kählerian if and only if $b_1(X)$ is even.

The proof of these facts may be found in [**?**], chap. IV, par. 2 and 3.

1.4 Fibrations of compact surfaces

In this short section we introduce the definitions and some basic concepts regarding fibrations. They play an important role in the classification of Enriques and Kodaira, and will be the central object of our study. For this section, we limit ourselves to the case in which X is compact. The general case will be treated more accurately in the next chapters.

Definition Let X be a compact surface. A fibration is a surjective holomorphic map with connected fibers $f\colon X \to S$, where S is a smooth and connected curve.

Call $A \subset X$ the analytic subspace defined by $\{\mathrm{d}f = 0\}$. Clearly, A is a proper subset if the map is not constant. Hence, it is union of a finite number of irreducible components $A_1, \dots, A_k$ (that are in particular connected subsets), of dimension 0 or 1. Every irreducible component must be mapped by f to a unique point, since the restriction $f|A_i\colon A_i \to S$ is a map between complex spaces with identically zero differential, hence it must be constant. So f has only a finite number of critical values, that correspond to the singular fibers; in particular, almost every fiber is smooth.

So, we have a finite number of "special" curves over the critical values of f, and we will call the remaining fibers "general fibers". In the singular fibers, both the analytic and the topological structure may change; on the other hand, all the general fibers are diffeomorphic, and in particular the genus of the general fiber is well defined. This follows from:

Theorem 1.4.1 (Ehresmann)**.** *Let M, N be manifolds, $f\colon M \to N$ a proper and surjective differentiable submersion. Then f is a locally trivial fibration, i.e. M is locally a product of the base N times the fiber, which is unique up to diffeomorphisms.*

In our case, we can take $N = S \setminus A$ and $M = f^{-1}(N)$. The theorem applies since these subsets are obviously connected, and on them the map is differentiable with surjective differential; f is obviously proper also on this restriction. Hence, the general fibers are diffeomorphic, as stated.

Definition We will say that a surface is *ruled* if it is the total space of a fiber bundle with fiber $\mathbb{P}^1$ and structural group $PGL(2, \mathbb{C})$ over a smooth connected curve S.

We will say that a surface is *elliptic* if it admits a fibration with smooth elliptic general fiber.

1.5 Enriques-Kodaira classification

We are finally in position to state the theorem of classification by Enriques and Kodaira. This is simply:

Theorem 1.5.1. *Every minimal compact surface belongs to one of the ten classes listed below.*

Note that the coarsest invariant we use to distinguish among classes is the Kodaira dimension, and not algebraic dimension, as it may seem more natural. There are several reasons to do so: first of all, there are classes of surfaces which are defined in a very natural way, like complex tori, which have all the same Kodaira dimension but whose algebraic dimension may vary (in the example, every torus X has $\mathrm{kod}(X) = 0$ but may assume every value of $\mathrm{a}(X)$ among the possibilities 0, 1, 2). Furthermore, one could show that the minimal representative we have introduced is unique if $\mathrm{kod}(X) \geq 0$, so that taking the Kodaira dimension as first invariant cases of non-uniqueness are allowed only in the first groups.

For the proof of the facts listed in this section, we refer to [**?**]. In particular, the classification theorem is proved in chapter VI.

1. *Minimal rational surface*: $\mathrm{kod}(X) = -\infty$, $b_1(X) = 0$, $a(X) = 2$.

   These are the surfaces bimeromorphic to $\mathbb{P}^2$. Clearly, $\mathbb{P}^2$ is a minimal model, but this is precisely one of the cases in which there are more than one non-isomorphic minimal models. One can show that the minimal models distinct from $\mathbb{P}^2$ are always ruled surfaces on $\mathbb{P}^1$, which in turn are known to be a numerable family, $\{\Sigma_n\}_{n\in\mathbb{N}}$, called Hirzebruch surfaces. The surface Σ_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, that is clearly a minimal model, while Σ_1 is $\mathbb{P}^2$ blown up a point, hence it is not minimal. On the other hand, one can see that every Σ_n with $n \geq 2$ is minimal, as well, completing the list of minimal representatives.

   Clearly, one can compute the invariants on $\mathbb{P}^2$, to obtain that the Kodaira dimension is $-\infty$ (the dual of the canonical bundle in $\mathbb{P}^2$ is ample, hence no positive multiple of $\mathcal{K}_{\mathbb{P}^2}$ can have sections). The first Betti number is zero (since $\mathbb{P}^1$ is homeomorphic to S^2), and the algebraic dimension is clearly 2.

2. *Class VII minimal surfaces*: $\mathrm{kod}(X) = -\infty$, $b_1(X) = 1$, $a(X) \in \{0, 1\}$.

These are by definition the surfaces with Kodaira dimension $-\infty$ and first Betti number equal to 1. Hence, they are never kählerian, nor projective. One could show that they always admit a unique minimal model.

The algebraic dimension is thus at most 1; one can see that the ones with algebraic dimension 1 are Hopf surfaces, that is, their universal cover is analytically isomorphic to $\mathbb{C}^2 \setminus \{0\}$ (the typical model, that historically was the first example of these surfaces, is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the relation equivalence generated by $(z_1, z_2) \sim (2z_1, 2z_2)$). However, it is known that there exist surfaces of class VII without meromorphic global sections but the constants, but a satisfying classification is not yet available.

3. *Ruled surfaces over curves with genus* $g > 0$: $\mathrm{kod}(X) = -\infty$, $b_1(X) = 2g$, $a(X) = 2$.

   Even in this case the minimal model in not unique. In particular, ruled surfaces on a curve S are bimeromorphic to $S \times \mathbb{P}^1$, and these are not pairwise bimeromorphic, hence every product $S \times \mathbb{P}^1$ (which have $\mathrm{kod}(X) = -\infty$ by additivity) are in this class, if S is not rational. Furthermore, it is easy to prove that every ruled surface is algebraic, and this concludes the computation of the invariants.

4. *Enriques surfaces*: $\mathrm{kod}(X) = 0$, $b_1(X) = 0$, $a(X) = 2$.

   They are, by definition, the surfaces with $q(X) = 0$ (that clearly, since $b_1(X) \in \{2q(X) - 1, 2q(X)\}$, is equivalent to $b_1(X) = 0$) and such that the canonical bundle has order exactly 2 in the Picard group. It can be shown that they are always projective, and the Kodaira dimension is clearly 0, since we have $P_m = 1$ for even m.

5. *Bi-elliptic (or hyperelliptic) surfaces*: $\mathrm{kod}(X) = 0$, $b_1(X) = 2$, $a(X) = 2$.

   Every surface X that admit a locally trivial fibration, with base and fiber both of genus 1 and such that $b_1(X) = 2$ belongs to this class. These surfaces are completely classified: it turns out that the canonical bundle is an element of torsion in $\mathrm{Pic}(X)$, and the order is 2, 3, 4 or 6. Hence the Kodaira dimension is 0. Furthermore, they are always projective (more precisely, they are always constructable as quotients of the product of two elliptic curves).

6. *Kodaira surfaces*: $\operatorname{kod}(X) = 0$, $b_1(X) \in \{1, 3\}$, $a(X) = 1$.

   The *primary* Kodaira surfaces admit a locally trivial fibration, with base and fiber of genus 1, but we ask the first Betti number to be 3. In particular, they are never kählerian.

   The *secondary* Kodaira surfaces have a primary Kodaira surface as unbranched covering. One can see that the first Betti number must always be 1 (hence they are not kählerian as well), and they admit a fibration with elliptic fiber and rational base. In particular, they give an example in which the IItaka's $C_{2,1}$ inequality is strict.

7. *K*3 *surfaces*: $\operatorname{kod}(X) = 0$, $b_1(X) = 0$, $a(X) \in \{0, 1, 2\}$.

   They are, by definition, the surfaces with $q(X) = 0$ (equivalently, $b_1(X) = 0$) and whose canonical bundle is trivial in $\operatorname{Pic}(X)$. They are always kählerian, but there are $K3$ surfaces of any given algebraic dimension.

8. *Complex tori*: $\operatorname{kod}(X) = 0$, $b_1(X) = 4$, $a(X) \in \{0, 1, 2\}$.

   They are quotients of $\mathbb{C}^2$ by a lattice of rank 4. Clearly, they all inherit the kählerian structure of $\mathbb{C}^2$, but they can have any possible algebraic dimension. The topological structure is uniquely determined by this description, hence $b_1(X) = 4$.

   This is also the last class in which fibrations with elliptic base and fiber appear: it turns out that the total space of such fibration is necessarily a bi-elliptic surface, a primary Kodaira surface or a complex torus.

9. *Minimal properly elliptic surfaces*: $\operatorname{kod}(X) = 1$, $\mathrm{a}(X) \in \{1, 2\}$.

   They are by definition the elliptic surfaces of Kodaira dimension 1. Clearly, the first Betti number is not uniquely determined: at least there is every product of curves, one elliptic and one of genus $g \geq 2$, and in this way we obtain every even value for b_1 greater or equal to 6. As a matter of fact, there exist also non-algebraic, or even non-kählerian properly elliptic surfaces, even though always $\mathrm{a}(X) \geq 1$, since $\mathrm{a}(X) \geq \operatorname{kod}(X)$.

   We will see in next chapters that every surface with Kodaira dimension equal to 1 is elliptic, hence this is the only class of surfaces with $\operatorname{kod}(X) = 1$.

10. *Surfaces of general type*: $\mathrm{kod}(X) = 2$, $b_1(X)$ pari, $a(X) = 2$.

    By definition, these are the surfaces with Kodaira dimension equal to 2. Necessarily, they are all algebraic (hence kählerian). As the name suggests, it is a big class, and there is no complete classification available. There exist inequalities for the pair of integers K_X^2 (the self-intersection of the canonical divisor) and $\chi(\mathcal{O}_X)$, known as Noether's inequalities, that leave a "sector" of the lattice $\mathbb{Z}\times\mathbb{Z}$ as possible values. It has been proved that almost every pair of integers which satisfy these inequalities are effectively reached, but it is not known if this holds for every pair.

Chapter 2

Generalities on fibrations

In this chapter we will discuss briefly the general theory of fibration of surfaces. Our first objective is to prove the so called Zariski's lemma, that will be a fundamental tool in our approach to the theory of elliptic surfaces. In doing so, we will use a modified version of the Perron-Frobenius theorem (lemma 2.1.1); it is natural to use it to obtain a corollary on graphs, that will be used in the next chapter, but we include it here for sake of coherence in the style of the proofs. Still have to complete the introduction

In this chapter, by *surface* we will always mean a connected complex manifold, not necessarily compact.

2.1 Zariski's Lemma

We will start by proving a lemma similar to the Perron-Frobenius theorem (but much easier to prove); the idea is that we will apply this lemma to the matrix representing the edges of a connected graph of n vertices, with the (non-negative) entries out of the diagonal counting the number of edges connecting two given vertices and property (3) below expressing the connectedness of the graph.

Lemma 2.1.1. *Let Q be a symmetric bilinear form on the $\mathbb{Q}$-vector space $V = \mathbb{Q}^n$, given by the matrix $Q = (q_{ij})$, and call N its annihilator. Suppose that:*

1. $q_{ij} \geq 0$ *for all* $i \neq j$*;*

2. *Q is negative semi-definite (and we will write for short $Q \leq 0$),* or *the annihilator N contains a vector $x = (x_1, \ldots, x_n)$ all of whose entries are positive, $x_j > 0$ $\forall j = 1, \ldots, n$*

3. *If there is a partition $I \sqcup J = \{1, \ldots, n\}$ with the property that $\forall i \in I$, $\forall j \in J$ $q_{ij} = 0$, then the partition is trivial, i.e. $I = \varnothing$ or $J = \varnothing$.*

Then, either Q is negative definite (and we write $Q < 0$), which means that N is trivial, or N is a 1-dimensional space, spanned by a vector $z = (z_1, \ldots, z_n)$ such that $z_j > 0$ $\forall j = 1, \ldots, n$ (of course, under the second hypothesis in (2) *this just says that N consists only of multiples of x).*

Proof. Recall that the annihilator of a semi-definite quadratic form coincides with the kernel of the associated matrix.

Firstly, let us assume the first hypothesis in (2). Then, either $N = 0$, and we are done, or N contains a non-zero vector, $y = (y_1, \ldots, y_n)$. We note that this also implies that $x = (|y_1|, \ldots, |y_n|) \in N$, thus reducing us to the second case: indeed, since of course $y_i^2 = |y_i|^2$, and $q_{ij} \geq 0$ for $i \neq j$, we have:

$$0 \geq \sum_{i,j} q_{ij} |y_i||y_j| \geq \sum_{i,j} q_{ij} y_i y_j.$$

So, if y belongs to the annihilator N, so does x, and this fullfills the requirements of the second hypothesis.

Now, let us assume the second part of (2). Let $y = (y_1, \ldots, y_n) \in N$ be another element of the annihilator, and $z = (z_1 = |y_1|, \ldots, z_n = |y_n|) \in N$, so that for all $j = 1, \ldots, n$ we have $\sum_i q_{ji} y_i = \sum_i q_{ji} |y_i| = 0$. Set $I = \{i \in I \ : \ y_i \neq 0\}$. If $j \notin I$, every term appearing in the last summation is non negative, so they all have to be zero, which implies that, if $y_i \neq 0$, then $q_{ji} = 0$ necessarily; but this exactly means that:

$$\forall i \in I \ \forall j \notin I \ q_{ji} = 0$$

which, by property (2), implies $I = \varnothing$ (that is, $y = 0$) or $I = \{1, \ldots, n\}$, so that y has no non-zero term; this in turn implies that $\dim N = 1$, since otherwise it would have a non-trivial intersection with the hyperplane defined by $x_i = 0$. $\square$

We insert here a consequence of this lemma, regarding graph theory, that will be used as a fundamental step in the theorem of classification by Kodaira at the beginning of the next chapter. First, we begin with a little terminology.

Definition By a (connected, non-oriented) graph we mean the data of a set of vertices $\{v_1, \dots, v_n\}$, and one of edges, each one connecting two vertices, called its ends, such that for every two vertices v, w there is a path of edges connecting them, i.e. a set $\{e_1, \dots, e_h\}$ with e_1 having v as an end, e_h having w as an end and and e_i having an end in common with e_j. A *loop at* v_i will mean an edge that starts and ends in the same vertex v_i; a *cycle* will mean a non-empty path of edges $\{e_1, \dots, e_h\}$ connecting a vertex v to itself; we will say that a graph has *multiple edges* if there are two or more edges with the same ends. The number of edges having v as an end, is called the *weight* of v.

We say that a graph G *fully* contains a sub-graph H if H is obtained choosing a subset of the vertices and all of the edges in G between any two of them.

Firstly, we will associate to a graph a symmetric $n \times n$ matrix $Q = (q_{ij})$: we put

$$q_{ii} = -2 + 2 \cdot \#\{\text{loops at } v_i\}, \quad q_{ij} = \#\{\text{edges connecting } v_i \text{ and } v_j\}.$$

The first link between this construction and the previous lemma is given by the following two lists of example, that will prove to be crucial to our study.

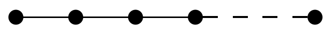
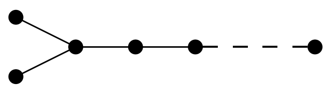
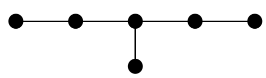
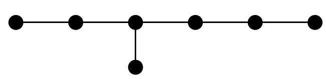
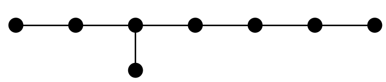
Definition A *Dynkin diagram* (of A-D-E type) is a graph of the type listed in table 2.1; the table also shows the associated matrix to each diagram. The subscript refers to the number of vertices, e.g. A_n has n vertices.

An *extended Dynkin diagram* is a graph of the type listed in table 2.2; the number attached to each vertex is called the *multiplicity* of that vertex. The subscript refers to the number of vertices minus one, e.g. $\tilde{A}_n$ has $n+1$ vertices.

A well known calculation (cfr. [**?**]) shows that all the matrices arising from a Dynkin diagram are negative definite. On the other hand, the matrices arising from extended Dynkin diagrams are negative semi-definite; this implies that we can apply the preceding lemma to them and obtain that the annihilator of the associated quadratic form is 1-dimensional. By another immediate verification, we obtain a generator of the annihilator:

Remark 2.1.2. The kernel of the matrix associated to an extended Dynkin diagram is a 1-dimensional space, generated by a (positive) integer linear combination of the vertices, with coefficients given by the multiplicities shown in table 2.2.

Table 2.1: Dynkin diagrams

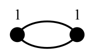
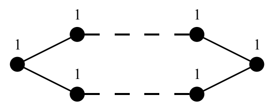
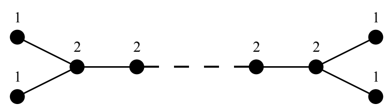
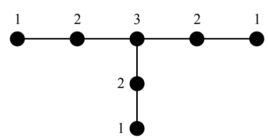
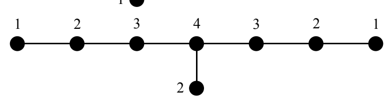
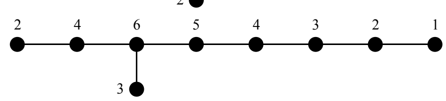
Name	Diagram	Matrix
A_n $(n > 0)$		$\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$
D_n $(n > 3)$		$\begin{pmatrix} \ddots & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & 1 \\ & & 1 & -2 & \\ & & 1 & -2 & \end{pmatrix}$
E_6		$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & 1 & \\ & & 1 & -2 & & \\ & & 1 & & -2 & 1 \\ & & & & 1 & \ddots \end{pmatrix}$
E_7		
E_8		

We are now in position to prove the following:

Proposition 2.1.3. *Let G be a connected graph whose associated matrix defines a negative semi-definite quadratic form. Then:*

1. *if the form is negative definite, then G is a Dynkin diagram;*

Table 2.2: Extended Dynkin diagrams

Name	Diagram
$\tilde{A}_0$	
$\tilde{A}_1$	
$\tilde{A}_n$ $(n \geq 2)$	
$\tilde{D}_n$ $(n \geq 4)$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	

2. *otherwise, G is an extended Dynkin diagram.*

Proof. Let Q be the quadratic form associated to G. We begin with a remark: if G fully contains a sub-graph H which is an extended Dynkin diagram, then in fact $H = G$ and so G is. This follows immediately from lemma 2.1.1, applied to Q, since this lemma gives a linear combination of some of the vertices of G that is in the annihilator, and so all the vertices must appear in this combination with a positive coefficient.

Let us first exclude some cases: if G contains a loop at a vertex v, then it contains a copy of $\tilde{A}_0$. This containment must be full, since otherwise the restriction of Q to the linear subspace generated by v would be positive definite (since the element of the diagonal would be a strictly positive even integer, by definition). So, G is of type $\tilde{A}_0$. In the following, G will contain no loop.

If G contains a multiple edge, say between vertices v and w, then G contains a copy of $\tilde{A}_1$; again, this must be a full containment, since otherwise we would have $n \geq 3$ edges between v and w and the restriction to the span of v and w would be a quadratic form of determinant $4 - n^2 < 0$, which cannot be defined. So, G is of type $\tilde{A}_1$. In the following, G will contain no multiple edge.

If G contains a cycle, let H be a minimal one, i.e. such that every vertex in H is connected via an edge of G to exactly other 2 vertex in H. Then H gives a fully contained sub-graph of G, and H is a graph of type $\tilde{A}_n$, $n \geq 2$, so G is. So, in the following, G will not contain any cycle, either, i.e. will be a tree. This in particular implies that any (connected) sub-graph of G is fully contained in it.

If G has a vertex of weight 4 or more, G (fully) contains a diagram of type $\tilde{D}_4$, and so must coincide with it. Then, we may assume that every vertex has weight no more than 3. If there are at least two vertices of weight 3, G contains a graph of type $\tilde{D}_n$, $n \geq 5$. If there is none, G is necessarily of type A_n, for some n.

We are left with the cases in which G has exactly one vertex of weight 3, i.e. G is a tripod. We can call p, q, r the number of edges in each leg of the tripod, and order them so that $1 \leq p \leq q \leq r$. The case $p = q = 1$ exactly corresponds to diagrams of type D_n; if $p \geq 2$ then G contains (and so is) a graph of type $\tilde{E}_6$. So we may assume $p = 1$, $q \geq 2$. If $q = 2$, then, depending on r, G may be of type E_6, E_7, E_8 or $\tilde{E}_8$. Finally, if $q \geq 3$ (and so also $r \geq 3$), then G contains, and therefore is, a graph of type $\tilde{E}_7$, and this conclude all of the possibilities. □

Remark 2.1.4. We observe that, dropping the hypothesis of Q being negative semi-definite, we obtain with the same proof that every connected graph G either contains or is contained in an extended Dynkin diagram.

We conclude this section by finally stating and proving Zariski's lemma, which brings us back again in the setting of fibrations:

Lemma 2.1.5 (Zariski's lemma)**.** *Let $f\colon X \to S$ be a fibration of a smooth surface, with connected fibers. Let X_s be one of the fibers, and write it as a combination of distinct irreducible components: $X_s = \sum n_i C_i$. Then:*

1. *$C_i.X_s = 0$, for all i;*
2. *if $D = \sum m_i C_i$, $m_i \in \mathbb{Z}$, then $D^2 \leq 0$;*
3. *in the same hypotheses, $D^2 = 0$ if and only if there exist $p, q \in \mathbb{Z}$ such that $pD = qX_s$ (we will write shortly $D = rX_s$, $r \in \mathbb{Q}$).*

Proof. The first result is a trivial consequence of the well-known and constantly used fact that in every curve, a divisor can be replaced via linear equivalence with another one having disjoint support. So, the fiber X_s may be replaced by a linear combination of different fibers, none of which intersects C_i.

Consider now the vector space, say W, with basis C_i, regarded as symbols. The intersection product induces a symmetric bilinear form Q on W. We may apply lemma 2.1.1 to Q: the first hypothesis is obviously satisfied from the definition, and the annihilator of Q contains the vector $X_s = \sum m_i C_i$ by the previous point; finally, the third hypothesis is granted by the connectedness of the fiber. We obtain that Q is negative semi-definite, and that its annihilator is generated (over $\mathbb{Q}$) by X_s, thus completing the proof. □

2.2 Generalities on fibrations

To be able to work locally, we first want to generalize the concept of fibration introduced in the first chapter to non-compact curves and surfaces. We thus define:

Definition A fibration of a (not necessarily compact) surface X is a *proper* map $f : X \to S$, were S is a smooth and connected complex space of dimension 1.

The condition that f be proper allows us to recover much of the properties we demand a fibration to have. Firstly, we want to recover the fact that the points over which a singular lies fiber form a discrete set in S, which does not follow from the same reasoning as in chapter 1. Again we have that the subset $A = \{x \in X \mid df(x) = 0\}$ is a proper analytic subset of X, which may have an infinite number of irreducible components; still, this union is locally finite, and each component is mapped to a single point, as in chapter 1. For every point $s \in S$, since the fiber is compact due to the properness hypothesis, we thus find a neighborhood of the fiber in which A has only a finite number of irreducible components; so the set of points in S over which a singular fiber lies form a closed and nowhere dense set.

Remark 2.2.1. Fibrations of surfaces are flat.

Proof. A map between smooth varieties $X \to Y$ is flat if and only if the fibers have constant dimension (cfr. [**?**]). In our case, it is easy to see that this holds: if we take a point $s \in S$, X_s cannot surely have dimension 2 because the fibration is not constant. But it cannot even be a finite set: if it were so, locally around a point in this set we would have a subspace of the 2-dimensional complex ball $B_2(\mathbb{C})$ defined by only one equation $f(x) = s$, which cannot be compact by Hartogs theorem. $\square$

Before stating the next remark, we recall the following (cfr. [**?**]):

Theorem 2.2.2 (Stein factorization theorem)**.** *Let X, Y be complex spaces, $f\colon X \to Y$ a proper and analytic map. Then f admits a unique factorization $f = h \circ g$*

$$f\colon X \to Z \to Y$$

such that:

1. *$g\colon X \to Z$ is an analytic, proper and surjective map, with $g_*\mathcal{O}_X = \mathcal{O}_Z$ and with connected fibers;*
2. *$h\colon Z \to Y$ is a holomorphic finite map.*

Furthermore, if X is normal, then Z is too.

Remark 2.2.3. If f is a fibration, then always $f_*\mathcal{O}_X = \mathcal{O}_S$. In particular, every holomorphic function on X is the pullback of some function on S.

Proof. We can apply the Stein factorization theorem, to obtain another curve T and a factorization $f = g \circ h\colon X \to T \to S$, with $h_*\mathcal{O}_X = \mathcal{O}_T$. Then the map $g\colon T \to S$ must be surjective and have connected fibers, since f is surjective with connected fibers; so g must also be injective, because all fibers have the same dimension 0. Then g is a bijective holomorphic map between complex manifolds of dimension 1, and must thus be an isomorphism (every injective holomorphic map has invertible differential, and thus the inverse is holomorphic by the inverse function theorem).

The second assertion follows from taking global sections of both sheaves. $\square$

We note that, as in chapter 1, Ehresmann's theorem obviously applies, so we still have a well defined genus of the general fiber. To make use of this fact also on the singular fibers, we note the following:

Remark 2.2.4. The arithmetic genus is the same for every fiber, including the singular ones.

Proof. This follow easily from the adjunction formula. It states that, if C is an irreducible curve on a surface X and K_X is the canonical divisor of X, then:

$$p_a(C) = 1 + \frac{1}{2}((K_X + C).C), \text{ or equivalently } \chi(C) = (K + C).C. \tag{2.1}$$

Since given a finite set on S and a divisor D we can always find another divisor D' linearly equivalent to D but missing the given set, we can substitute any fiber with a linear combination of non-singular fibers in the same linear equivalence class; then, combining adjunction formula with the additivity of χ we have the thesis. $\square$

Remark 2.2.5. We remark a couple of facts on compact complex curves, which in our situation will apply to the fibers of the map. Let F be such a curve, possibily reducible or non reduced. Firstly, $H^2(F, \mathbb{Z}) \cong \mathbb{Z}^N$, where N is the number of distinct irreducible components, and thus has no torsion. Let F_{red} is the reduced curve obtained discarding multiplicities; then, since F is compact, both the space $H^0(F_{\text{red}}, \mathcal{O}_{F_{\text{red}}})$ and the space $H^0(F_{\text{red}}, \mathcal{O}^*_{F_{\text{red}}})$ are made of locally constant functions; if F_{red} is the reduced curve obtained discarding multiplicities, the exponential map for F_{red} is surjective on global sections, and we obtain the injection in the exponential long exact sequence:

$$H^1(F, \mathbb{Z}) = H^1(F_{\text{red}}, \mathbb{Z}) \hookrightarrow H^1(F_{\text{red}}, \mathcal{O}_{F_{\text{red}}}).$$

But this coincides with the composition of natural maps $H^1(F, \mathbb{Z}) \to H^1(F, \mathcal{O}_F) \to H^1(F_{\text{red}}, \mathcal{O}_{F_{\text{red}}})$, hence we have also the injection:

$$H^1(F, \mathbb{Z}) \hookrightarrow H^1(F, \mathcal{O}_F).$$

We are now interested in the study of multiple fibers. If we write a fiber as $X_s = \sum n_i C_i$, we say that it is a *multiple fiber* of multiplicity n if $\gcd(n_i) = n$. We can thus write $X_s = nF$, where F an effective divisor on X. We will show that the hypothesis that $n > 1$ has strong implications on the topology of the fiber.

Lemma 2.2.6. *Let $S = \Delta \subset \mathbb{C}$ the unit disk, and $X_0 = nF$ the only singular fiber of f, of multiplicity $n > 1$. Then $\mathcal{O}_X(F)$ and $\mathcal{O}_F(F)$ are both torsion bundles of order exactly n.*

Proof. Both bundles are torsion of order a divisor of n: since $\mathcal{O}_F(F) = \mathcal{O}_X(F)|F$, it suffices to show that $\mathcal{O}_X(F)^{\otimes n} \cong \mathcal{O}_X$, but this bundle is equal to $\mathcal{O}_X(nF) = \mathcal{O}_X(X_s)$ which is isomorphic to $\mathcal{O}_X$ since the Picard group of Δ is trivial, and thus every fiber is linearly equivalent to 0.

$\mathcal{O}_X(F)$ cannot be of order strictly less than n: if it were so, there would be a function $g\colon X \to \mathbb{C}$, vanishing along X_0 with an order less than $z \circ f$, z being the coordinate on $\mathbb{C}$. But this is impossible by remark (2.2.3).

To show that $\mathcal{O}_F(F)$ has order exactly n, too, we firstly shrink Δ: in fact, as is shown in [**?**], fixed an analytic subspace F of a complex space X, we can always find a triangulation of X so that F is a subcomplex; this implies that we can find a neighborhood of F which can be deformation retracted on F (see, for example, [**?**], Prop A.5), and since in our situation F is compact, we can actually find it of the form $f^{-1}(\varepsilon\Delta)$, for some $\varepsilon > 0$. Thus, shrinking, we can suppose that the maps $H^i(X, \mathbb{Z}) \to H^i(F, \mathbb{Z})$ are isomorphisms, and we have a diagram:

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d] & H^1(X,\mathcal{O}_X^*) \arrow[r] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[d, equal] \\
H^1(F,\mathbb{Z}) \arrow[r, hook] & H^1(F,\mathcal{O}_F) \arrow[r] & H^1(F,\mathcal{O}_F^*) \arrow[r] & H^2(F,\mathbb{Z}) = \mathbb{Z}^N
\end{tikzcd}$$

where the horizontal map in the lowest-left corner is injective by the previous remark. Now we do some diagram chasing. Start from $[\mathcal{O}_X(F)] \in H^1(X, \mathcal{O}_X^*)$; by what we have already shown, it is a torsion bundle, so it must map to $0 \in H^2(X, \mathbb{Z})$, since by previous remark $H^2(X, \mathbb{Z}) = H^2(F, \mathbb{Z})$ has no torsion. Then the class $[\mathcal{O}_X(F)]$ has a preimage $\xi \in H^1(X, \mathcal{O}_X)$. Suppose that there is an $m|n$ such that $\mathcal{O}_F(mF) = \mathcal{O}_F$. We want to show that in fact $m = n$, and we will achieve this by proving that $\mathcal{O}_X(mF) \cong \mathcal{O}_X$. This is equivalent to saying that $m\xi$ maps to $0 \in H^1(X, \mathcal{O}_X^*)$, that is, it is the image of some element $x \in H^1(X, \mathbb{Z})$. Let us now draw two diagrams, one showing this situation, and the other after multiplying by m (positions are as in the diagram above; in particular, on the left only second and third columns are represented, and on the right only the first three):

$$\begin{tikzcd}
\xi \arrow[r, mapsto] \arrow[d, mapsto] & {[\mathcal{O}_X(F)]} \arrow[d, mapsto] \\
\xi|F \arrow[r, mapsto] & {[\mathcal{O}_F(F)]}
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, mapsto, dotted] \arrow[d, equal] & m\xi \arrow[r, mapsto] \arrow[d, mapsto] & m[\mathcal{O}_X(F)] \arrow[d, mapsto] \\
c \arrow[r, mapsto] & m(\xi|F) \arrow[r, mapsto] & m[\mathcal{O}_F(F)] = 0
\end{tikzcd}$$

The existence of c in the diagram mapping to $m(\xi|F) \in H^1(F, \mathcal{O}_F)$ is guaranteed by our hypothesis that $\mathcal{O}_F(F)$ is of order m. Our claim is that this c is in fact mapped to $m\xi \in H^1(X, \mathcal{O}_X)$. Surely, $n\xi$ has a pre-image $z \in H^1(X, \mathbb{Z})$, since it maps to $0 \in H^1(X, \mathcal{O}_X^*)$. Then, the first square reads:

$$\begin{tikzcd}
z \arrow[r, mapsto] & n\xi \arrow[d, mapsto] \\
\frac{n}{m}c \arrow[r, mapsto] & n(\xi|F)
\end{tikzcd}$$

But the composition $H^1(X, \mathbb{Z}) \hookrightarrow H^1(F, \mathcal{O}_F)$ is injective, so it must be $z = \frac{n}{m}c$. Call y the image of c in $H^1(X, \mathcal{O}_X)$; then $\frac{n}{m}(y - m\xi) = n\xi - n\xi = 0$. But $H^1(X, \mathcal{O}_X)$ is a $\mathbb{C}$-vector space, so actually $y = m\xi$, as required.

□

Corollary 2.2.7. *Let X_s be a simply connected fiber. Then it cannot be a multiple fiber.*

Proof. Let $X_s = nF$. By the exponential long exact sequence, since $H^1(F, \mathbb{Z}) = 0$, we realize $\mathrm{Pic}(F)$ as a subgroup of the torsion-free group $H^2(F, \mathbb{Z})$; so it has no torsion, and we conclude by the previous lemma applied to $\mathcal{O}_F(F)$. □

A technical reduction that will prove to be very useful is that of considering *relatively minimal* fibrations, that is, fibrations that do not contain any (-1)-curve as a component of any fiber. It is quite obvious that, given a fibration, we can always construct a relatively minimal one: simply go on contracting all the (-1)-curves contained in any fibers, until there is none left; this process must stop, since any fiber has a finite number of irreducible components. However, we do not want that the resulting surface depends on our choice of which (-1)-curve blow down, when an ambiguity arises. We show that this cannot happen, provided that the genus of the general fiber is strictly positive:

Proposition 2.2.8. *Let $f\colon X \to S$ be a fibration such that the genus of the general fiber is strictly positive. Then, there is a unique nonsingular surface Y such that we can write $f = g \circ h\colon X \to Y \to S$, with g a relatively minimal fibration.*

Proof. As noted, we only have to prove uniqueness.

Indeterminacy in the blow downs could arise only if we had two (-1)-curves C_1, C_2 with non-trivial intersection, both contained in a fiber X_s. Assume this is the case. Then:

$$(C_1 + C_2)^2 = -2 + 2C_1.C_2 \geq 0$$

and the equality holds if and only if C_1 and C_2 meet transversally at a single point; but this must be the case, since by Zariski's lemma the intersection product on the fiber is semi-negative defined. Then, by the last statement of Zariski's lemma, this forces the fiber to be of the form $n(C_1 + C_2)$. But $C_1 \cup C_2$ is topologically a bouquet of S^2, which is simply connected, so by corollary (2.2.7), $n = 1$. This means that X_s is given by two rational curves meeting transversally in one point, so $p_a(X_s) = 0$, against our hypothesis (where we have used remark (2.2.4)).

□

We mention here a result due to Grauert and Fischer. For a proof, cfr. [**?**].

Theorem 2.2.9 (Local Triviality theorem of Grauert-Fischer)**.** *Let $X \to S$ be a fibration with only smooth fibers. Then it is locally trivial (i.e. each point of s has a neighborhood over which the fibration is biholomorphically equivalent to a fiber bundle) if and only if all the smooth are analytically isomorphic.*

Let us now turn our attention to base change. If we have a fibration over the unit disk $f\colon X \to \Delta$, and we call t the coordinate on Δ, we want to consider the fibration obtained by the change of coordinates $s^n = t$. To be more precise, we refer to the *n-th root fibration of f* as the map obtained by the procedure:

$$\begin{tikzcd} X^{(n)} \arrow[r, "\tau''"] \arrow[d, "f^{(n)}"] & X'' \arrow[r, "\tau'"] \arrow[d, "f''"] & X' \arrow[r, "\tau"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \Delta \arrow[r, equal] & \Delta \arrow[r, equal] & \Delta \arrow[r, "\delta_n"] & \Delta \end{tikzcd}$$

where $X' = \Delta \times_\Delta X$ is the fiber product with respect to the map $\delta_n \colon \Delta \to \Delta$, $\delta_n(s) = s^n$, X'' is its normalization, and $X^{(n)}$ the minimal desingularization of X''. The induced map $f^{(n)}$ induces a fibration structure on $X^{(n)}$, obviously.

We end this section with a result that can be found in [**?**], chap. III, par. 3. We state it in general, but we insert it in this section since the typical situation in which we will use it is when $f \colon X \to S$ is a fibration and $\mathcal{S} = \mathcal{O}_X$ (that is flat: see [**?**]).

Theorem 2.2.10. *Let X, Y be reduced complex spaces, $f \colon X \to Y$ a proper holomorphic map. If $\mathcal{S}$ is a coherent sheaf on X, flat over Y (i.e. for every $x \in X$, $\mathcal{S}_x$ is a flat $\mathcal{O}_{f(x)}$-module), we have:*

1. *the Euler characteristic $\chi(\mathcal{S}_y)$ is locally constant;*
2. *the dimension $h^q(X_y, \mathcal{S}|X_y)$ is an upper semi-continues function;*
3. *for each $q \geq 0$, if $h^q(X_y, \mathcal{S}|X_y)$ is locally constant, then $R^q f_* \mathcal{S}$ is locally free and has the base change property, i.e. we have an isomorphism:*

$$(R^q f_* \mathcal{S})_y / \mathcal{I}_y \cdot (R^q f_* \mathcal{S})_y \cong H^q(X_y, \mathcal{S}|X_y)$$

2.3 Surfaces with Kodaira dimension 1

The purpose of this section is to prove the following theorem (which was anticipated as part of the theorem of classification of Enriques and Kodaira in chapter 1):

Theorem 2.3.1. *Let X be a minimal compact surface with Kodaira dimension 1. Then X is elliptic, i.e. it admits a surjective morphism to a smooth curve $p \colon X \to S$ whose general fiber is a smooth curve of genus 1.*

We recall that, by theorem (1.3.3), the only possibilities for $\mathrm{a}(X)$ are to be 1 or 2. In the latter case, then, X is a projective surface by theorem (1.3.4). We will treat the two cases separately.

2.3.1 Algebraic surfaces of Kodaira dimension 1

We start recalling the Riemann-Roch theorem for surfaces and the Hodge index theorem:

Theorem 2.3.2 (Riemann-Roch, cfr. [?])**.** *Let L be a divisor on X, K the canonical divisor. Then*

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - L.K)$$

Theorem 2.3.3 (Hodge index theorem, cfr. [?], chap. V, par. 1)**.** *Let X be a* projective *variety, H an ample divisor. If D is a divisor such that $D.H = 0$, then either $D.E = 0$ for all divisors E, or $D^2 < 0$.*

We insert here a general remark on surfaces of non-negative Kodaira dimension, that will be used in the proof:

Remark 2.3.4. Let X be a minimal (non necessarily projective) surface with $\mathrm{kod}(X) \geq 0$. Then, for every effective divisor D on X, $K.D \geq 0$.

Proof. It suffices to show the thesis when D is an irreducible curve. By contradiction, suppose that there exists an irreducible curve C such that $K.C < 0$. By the adjunction formula (2.1), this implies $C^2 = 2p_a(C) - 2 - K.C \geq -1$; but equality is ruled out by minimality hypothesis, since it would also imply $p_a(C) = 0$, and so $C^2 \geq 0$. Then, for every effective divisor E, we have obviously $C.E \geq 0$, writing $E = nC + F$ with $n \geq 0$ and F effective with $\mathrm{supp}(F) \not\supseteq \mathrm{supp}(C)$.

We thus have shown that no effective divisor can be linearly equivalent to K, i.e. $|K| = \varnothing$; but also $|nK| = \varnothing$, since $nK.C < 0$, too; this means precisely $\mathrm{kod}(X) = -\infty$. □

Lemma 2.3.5. *Let X be a minimal projective surface of Kodaira dimension ≥ 0, and K its canonical divisor. Then:*

1. *$K^2 \geq 0$.*
2. *If $K^2 > 0$, then $\mathrm{kod}(X) = 2$.*
3. *If $K^2 = 0$ and $P_r \geq 2$, after writing $rK \equiv Z + M$, where Z is the fixed part of the sistem $|rK|$ and M is the mobile part, we have:*

$$K.Z = K.M = Z^2 = Z.M = M^2 = 0$$

Proof. 1. Since $\mathrm{kod}(X) \geq 0$, there exists $r \geq 1$ such that $|rK| \neq \varnothing$. Then the conclusion follows from remark (2.3.4), since $rK.K \geq 0$.

2. Let X be embedded in some projective space, and let H be an hyperplane section. Let us apply Riemann-Roch theorem to $L = rK$:

$$h^0(rK) + h^0((1-r)K) \geq \frac{1}{2}K^2r^2 + O(r) \to \infty$$

as r tends to infinity. Then, by remark (2.3.4), $H.K \geq 0$; furthermore, by Hodge index theorem, we cannot have equality, since H is ample and $K^2 > 0$. Then, for sufficiently large r, $(H + (1-r)K).H < 0$, so that $H + (1-r)K$ cannot have global sections. But this in turn implies that $h^0(rK)$ has a quadratic growth, i.e. the surface has Kodaira dimension 2.

3. By their definition, both Z and M are effective; then, by the previous remark, $K.Z \geq 0$ and $K.M \geq 0$. But also $0 = rK^2 = K.Z + K.M$, hence $K.Z = K.M = 0$. Since M has no fixed part of codimension 1, we can move it inside his linear equivalence class to show that $M^2 \geq 0$ and also $Z.M \geq 0$. But we also have $0 = rK.M = Z.M + M^2$, and so $Z.M = M^2 = 0$. Finally, $Z^2 = (rK - M)^2 = 0$. □

Proposition 2.3.6. *Let X be a projective surface of Kodaira dimension 1. Then:*

1. *We have $K^2 = 0$;*

2. *X is an elliptic surface.*

Proof. 1. By the first statement in lemma (2.3.5), we must have $K^2 \leq 0$. But we easily see that already $\mathrm{kod}(X) \geq 0$ implies $K^2 \geq 0$: we only have to choose an $r > 0$ such that $P_r \geq 1$, so that there is an effective divisor $D \in |rK|$, and apply remark (2.3.4) to obtain $rK^2 = D.K \geq 0$.

2. Let r be an integer such that $P_r \geq 2$ so that the image of ϕ_{rK} is a curve. As in the lemma, write $rK \equiv Z + M$, where Z is the fixed part, M the mobile part. By the second statement in the lemma, $M^2 = 0$, so that M has no base points. Hence it defines a morphism (which is by definition simply ϕ_{nK}) $f\colon X \to C \subset \mathbb{P}^N$. Taking the Stein factorization, we have:

$$f\colon X \xrightarrow{p} B \xrightarrow{g} C \subset \mathbb{P}^N$$

where p has connected fibers and g is finite. Let F be a fiber of p. The divisor M is the pull-back of the hyperplane section of C, so it is a sum of fibers of p with positive coefficient, too. Again by lemma (2.3.5), $K.M = 0$, and so also $K.F = 0$. Applying the adjunction formula, then, we must have $p_a(F) = 1$, as required. □

2.3.2 Surfaces with algebraic dimension 1

These can be characterized as the non projective (compact) surfaces which admit a non trivial meromorphic function $f\colon X \dashrightarrow \mathbb{P}^1$. We recall that to give a meromorphic function $f\colon X \dashrightarrow \mathbb{P}^1$ is the same as giving a nonsingular surface Y, together with maps $\phi\colon Y \to X$ and $\rho\colon Y \to \mathbb{P}^1$.

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X & & \mathbb{P}^N \end{tikzcd}$$

By lemma (1.2.6), π is the composition of an isomorphism and a finite number of blow-up's.

Proposition 2.3.7. *Let X be a compact minimal surface of algebraic dimension 1. Then X is elliptic.*

Proof. Let $f\colon X \dashrightarrow \mathbb{P}^1$ be given by a diagram as above. Then the holomorphic map ρ must be surjective (the image must be open, since ρ is holomorphic, but also closed, since Y is compact). We apply the Stein factorization to ρ, and obtain a map to a smooth curve $h\colon Y \to S$, with connected fibers. We will prove that its general fiber F is elliptic.

If $g(F) \geq 2$, by adjunction formula we would have $K_Y.F > 0$, and so $(K_Y + nF)^2 > 0$ for n sufficiently large. But by theorem (1.3.5), this would imply that Y is projective and so also that X is, against our hypotheses.

The possibility $g(F) = 0$ (and $K_Y.F < 0$) is automatically ruled out if we assume $\mathrm{kod}(X) \geq 0$, so that the remark (2.3.4) applies. However, our statement holds true even without any hypothesis on $\mathrm{kod}(X)$: let us take a fiber $Y_s = \sum_1^m n_i C_i$. Then we claim that in fact it must be reduced and irreducible, i.e. $m = n_1 = 1$. Suppose the contrary holds; then we have

$$-2 = (K_Y + Y_s).Y_s = \sum_{i=1}^{m} n_i (K_Y.C_i)$$

which implies $m \geq 2$, otherwise either $n_1 = 1$ and we have in fact a smooth rational fiber, or $n_1 = 2$ and $K_Y.C_1 = -1$, which would lead to a non-integer arithmetic genus of C_1. Then there must be an i such that $K_Y.C_i < 0$, but also $C_i^2 < 0$ by Zariski's lemma (2.1.5), but applying adjunction formula to C_i this forces it to be a (-1)-curve, against our hypothesis of minimality. Hence every fiber of the fibration is reduced and irreducible, and of arithmetic genus 0, so they are all rational. Then, by theorem (2.2.9), the fibration is locally trivial, i.e. a fiber bundle with rational fiber. Otherwise said, Y is ruled; in particular, it is projective, hence also X is, against our hypothesis.

We want to deduce that X is an elliptic surface: if we prove that every exceptional tree is mapped to a point by h we are done, since then the map h descends to a map $h'\colon X \to S$, which has the same fiber away from points blown up by π. But this is true not only for the exceptional curves on Y, but for all curves C on Y: otherwise, we would have $C.F > 0$, and so $(C + nF)^2 > 0$, again implying that Y is projective. □

2.4 Direct image sheaves

In this section we deduce a useful lemma which together with theorem (2.2.10) will be used to deduce that the sheaves $R^i f_* \mathcal{O}_X$ (for $i = 0, 1$), and other sheaves related to them, are locally free and have the base change property. We start with a preliminary definition:

Definition Let C be a compact effective divisor on a surface X. Then C is said to be m-connected if for each non-trivial decomposition $C = C_1 + C_2$, with C_i effective, $C_1.C_2 \geq m$.

Remark 2.4.1. If there exists $m \geq 1$ such that C is m-connected, then it is also connected, but the converse needs not hold if there are multiple components with negative self-intersection.

Lemma 2.4.2. *Let C be a compact effective divisor on a surface X, $\mathcal{L}$ a line bundle on C, whose restriction to any irreducible component of C has degree 0. Let $s \in H^0(\mathcal{L})$, and write $C = C_1 + C_2$, with $C_1 \leq C$ a maximal divisor satisfying $s|C_1 = 0$. Then*

$$C_1.C_2 \leq 0.$$

Proof. By the definition of C_1, $s \in H^0(\mathcal{I}_{C_1} \cdot \mathcal{L}) = H^0(\mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L})$. Taken an irreducible component E of C_2, we can restrict everything to it, obtaining an injective map $s\colon \mathcal{O}_E \to \mathcal{O}_E(-C_1) \otimes \mathcal{L}$; so we have an exact sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_E(-C_1) \otimes \mathcal{L}|E \to \mathcal{Q}_E \to 0, \tag{2.2}$$

where $\mathcal{Q}_E$ has support in a finite set. We have:

$$-E.C_1 = \deg(\mathcal{O}_E(-C_1)) = \deg(\mathcal{O}_E(-C_1) \otimes \mathcal{L}|E) = \\ \chi(\mathcal{O}_E(-C_1) \otimes \mathcal{L}|E) - \chi(\mathcal{O}_E) = h^0(\mathcal{Q}_E) \geq 0,$$

where we have used that $\mathcal{L}|E$ has degree 0, Riemann-Roch on E, and the fact that $\mathcal{Q}$ is supported on points. Summing on the components of C_2 we obtain the thesis. $\square$

Lemma 2.4.3 (Ramanujam's Lemma)**.** *Let $\mathcal{L}$ be as above. If C is 1-connected, then $h^0(\mathcal{L}) \leq 1$, and equality holds if and only if $\mathcal{L} \cong \mathcal{O}_C$.*

Proof. Let us start from the case $\mathcal{L} = \mathcal{O}_C$. One inequality being trivial, we only have to prove that $h^0(\mathcal{O}_C) \leq 1$. This fact is trivial for C reduced and connected, because then every function on C is constant on each component, and since C is connected we have the thesis. If C is not reduced, though, this would be false without the stronger hypothesis of 1-connectedness, since there could be functions vanishing on a component with some order less than the multiplicity of the component. However, if $f \in H^0(\mathcal{O}_C)$ is not identically zero, and we write $C = C_1 + C_2$, with the same notations as above, then $C_2 \neq 0$, obviously, and so the previous lemma and the hypothesis of 1-connectedness imply $C_1 = 0$. This proves that no non-zero function can vanish of any order on the support of a curve contained in C, hence f must be constant for the same reasons as above.

If now $\mathcal{L} \ncong \mathcal{O}_C$, suppose by contradiction that there is $s \in h^0(\mathcal{L}) \setminus \{0\}$. If we write $C = C_1 + C_2$ with the same notation as above, we have $C_1 = 0$, $C = C_2$. In such circumstance, the analog of the sequence (2.2) is:

$$0 \to \mathcal{O}_C \to \mathcal{L} \to \mathcal{Q} \to 0.$$

But since $\mathcal{L}$ has degree 0 restricted to any irreducible component of C, by Riemann-Roch, $\chi(\mathcal{L}) = \chi(\mathcal{O}_C) \implies h^0(\mathcal{Q}) = \chi(\mathcal{Q}) = 0$. But this means $\mathcal{Q} = 0$, i.e. $\mathcal{L} \cong \mathcal{O}_C$. $\square$

Lemma 2.4.4. *Let $f\colon X \to S$ be a fibration, not necessarily with connected fibers. Then $h^i(\mathcal{O}_{X_s})$ ($i = 0, 1$) is independent of s.*

Proof. Firstly, let f be connected, so that $h^0(\mathcal{O}_{X_s}) \geq 1$, and we have equality for every regular value s. Assume that there is s such that $h^0(\mathcal{O}_{X_s}) > 1$. Then by Ramanujam's lemma (2.4.3) X_s cannot be 1-connected; but by

Zariski's lemma (2.1.5) every non-multiple fiber is 1-connected (because if $X_s = C_1 + C_2$, then $0 = C_1^2 + C_2^2 + 2C_1.C_2$, and if C_i is not a rational multiple of X_s then $C_i^2 < 0$), so $X_s = nF$ must be a multiple fiber, with F 1-connected.

Let us consider, for ν such that $1 \leq \nu \leq n-1$, the exact sequences (coming from the decomposition sequence for $(\nu+1)F = \nu F + F$):

$$0 \to \mathcal{O}_F(-\nu F) \to \mathcal{O}_{(\nu+1)F} \to \mathcal{O}_{\nu F} \to 0. \tag{2.3}$$

By lemma (2.2.6), $\mathcal{O}_F(F)$ is torsion of order n, hence all the bundles $\mathcal{O}_F(-\nu F)$ are non-trivial, and of order dividing n. Again by Ramanujam's lemma (2.4.3), we then must have $h^0(\mathcal{O}_F(-\nu F)) = 0$; then, passing to cohomology in (2.3), we obtain

$$h^0(\mathcal{O}_{X_s}) = h^0(\mathcal{O}_{nF}) \leq h^0(\mathcal{O}_{(n-1)F}) \leq \ldots \leq h^0(\mathcal{O}_F)$$

The last term equals 1 by 1-connectedness, hence $h^0(\mathcal{O}_{X_s}) = 1$, as required.

Let us now turn to non-connected fibrations. Take the Stein factorization,

$$f\colon X \xrightarrow{\rho} T \xrightarrow{\gamma} S,$$

so that γ is a ramified covering, and ρ is connected; by the previous point, for all $t \in T$, $h^0(\rho^{-1}(t)) = 1$. If $s \in S$ is not a ramification point, then $h^0(\mathcal{O}_{X_s}) = \deg \gamma = d$; if s is a ramification point, write $\gamma^{-1}(s) = \{t_1, \ldots, t_k\}$, and call ν_i the ramification order of t_i. Then $\nu_1 + \ldots + \nu_k = d$, and $X_s = \nu_1 \rho^{-1}(t_1) + \ldots + \nu_k \rho^{-1}(t_k)$, hence it is enough to show that $h^0(\mathcal{O}_{\nu\rho^{-1}(t)}) = \nu$, for every $t \in T$, $\nu \in \mathbb{N}$. But this follows from the same exact sequences as above

$$0 \to \mathcal{O}_{\rho^{-1}(t)}((1-\nu)\rho^{-1}(t)) \to \mathcal{O}_{\nu\rho^{-1}(t)} \to \mathcal{O}_{(\nu-1)\rho^{-1}(t)} \to 0,$$

this time using the fact (lemma (2.2.6)) that, for all ν, $\mathcal{O}_{\rho^{-1}(t)}((1-\nu)\rho^{-1}(t)) \cong \mathcal{O}_{\rho^{-1}(t)}$. Passing to cohomology we have an exact sequence:

$$0 \to H^0(\mathcal{O}_{\rho^{-1}(t)}) \cong \mathbb{C} \to H^0(\mathcal{O}_{\nu\rho^{-1}(t)}) \to H^0(\mathcal{O}_{(\nu-1)\rho^{-1}(t)}) \to 0.$$

By induction, then, $h^0(\mathcal{O}_{\nu\rho^{-1}(t)}) = \nu$.

This completes the proof for $i = 0$; but first part of theorem (2.2.10) ensures that the characteristic of $\mathcal{O}_{X_s}$ is locally constant, hence the result for $i = 1$ follows. $\square$

Corollary 2.4.5. *In the same hypotheses, the sheaves $f_*\mathcal{O}_X$ and $R^1f_*\mathcal{O}_X$ are locally free and have the base-change property.*

Proof. It is an immediate consequence of theorem (2.2.10), since by definition $\mathcal{O}_{X_s} = \mathcal{O}_X|X_s$. □

2.5 Relative duality

Let $f\colon X \to S$ be a fibration. The *dualizing sheaf* of f is the sheaf

$$\omega_{X/S} = K_X \otimes f^*(K_S^\vee).$$

Proposition 2.5.1. *The sheaves $f_*\omega_{X/S}$, $R^1f_*\omega_{X/S}$, f_*K_X and $R^1f_*K_X$ are locally free and have the base change property.*

Proof. We will prove this by means of theorem (2.2.10). We have to prove that $h^0(K_X|X_s)$ and $h^1(K_X|X_s)$ are independent of s; this will imply the thesis also for $\omega_{X/S}$, since $f^*(K_S^\vee)$ is a linear combination of fibers, hence vanishes when restricting to a fiber. We recall the definition of the canonical bundle for a singular curve C on a surface X:

$$\omega_C = K_X \otimes \mathcal{O}_C(C).$$

For $C = X_s$, however, we have already observed that $\mathcal{O}_{X_s}(X_s) \cong \mathcal{O}_{X_s}$, so that we have $\omega_{X_s} \cong K_X|X_s$. We can now use Serre's duality and have

$$h^0(\omega_{X_S}) = h^1(\mathcal{O}_{X_s}), \quad h^1(\omega_{X_S}) = h^0(\mathcal{O}_{X_s}),$$

which are independent of s by lemma (2.4.4). □

Corollary 2.5.2. *The canonical bundle of an elliptic fibration is* vertical*, i.e. a linear combination of components of fibers.*

Proof. The sheaf f_*K_X is locally free by the proposition. If we take a stalk $(f_*K_X)_s$, by base change it has dimension

$$\mathrm{rank}(f_*K_X) = h^0(K_X|X_s) = h^0(\omega_{X_s}) = 1.$$

Hence f_*K_X is an invertible sheaf, thus its sections are zero only on a discrete set and K_X can have no transversal components. □

We now cite a theorem, proven by Grothendieck in greater generality, whose proof can be found in [**?**], chap. III, par. 12:

Theorem 2.5.3 (Relative duality theorem)**.** *Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. There is an isomorphism*

$$f_*(\mathcal{F}^\vee \otimes \omega_{X/S}) \cong (R^1 f_* \mathcal{F})^\vee$$

Since $f_*\omega_{X/S}$ is a locally free sheaf, we may calculate its degree. This is actually quite a difficult thing to do, and we refer to [**?**], chap III, par. 18 for a proof.

Theorem 2.5.4. *Let $f\colon X \to S$ be a fibration with X and S compact and such that all fibers have strictly positive genus. Then*

$$\deg(f_*\omega_{X/S}) \geq 0$$

and we have equality if and only if one of the following holds:

- *the fibration f is locally trivial;*
- *the genus of the general fiber is 1 and the only singular fibers are multiples of nonsingular curves.*

2.6 Picard-Lefschetz monodromy

We present here briefly the so-called Picard-Lefschetz monodromy, regarding monodromy around a semi-stable fiber. We will not prove the main theorem, called Picard-Lefschetz formula, for which we refer to [**?**, **?**].

Let $f\colon X \to \Delta$ be a fibration, with only a singular fiber X_0, and suppose that it is reduced and has no singularities but ordinary double points. We have already cited while dealing with multiple fibers that the pair (X, X_0) is triangulable, so that X_0 is a deformation retract of its small neighborhoods. Hence we have canonical isomorphisms:

$$\iota^*\colon H^i(X, \mathbb{Z}) \to H^i(X_0, \mathbb{Z}), \quad \iota_*\colon H_i(X_0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

We are interested in the study of local monodromy around 0. To explain this concept, take a base of 1-cycles $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for a smooth fiber X_s. Then, if we take a loop γ winding once in a counterclockwise direction around

0, we can extend continuously the basis along the arc, and, after completing the loop, we obtain a new basis $\{a'_1, \dots, a'_g, b'_1, \dots, b'_g\}$ of $H_1(X_s, \mathbb{Z})$. This construction is well-defined since by Ehresmann's theorem the fibration is locally trivial out of 0, so the choice of two different arcs in a small neighborhood of any point in γ does not affect the homology class of the cycles chosen. For the same reason, the obtained basis depends in homology only on the basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ and on the homotopy class of γ.

To enunciate the Picard-Lefschetz formula, we need to introduce the concept of the *vanishing cycles*. The idea is that under the right condition the map induced by inclusion $H_1(X_s, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \cong H_1(X_0, \mathbb{Z})$ is surjective, and we would like to call vanishing cycles those lying in the kernel of this map. To make all this precise, we start by working locally.

Let $x_i \in X_0$ be a point. By hypothesis, it is a node, so we can find a neighborhood U and local coordinates (u, v) such that f is given by $f(u, v) = u^2 + v^2$. Let $\varepsilon > 0$ be small enough. Out of the point $x_i = (0, 0)$ the fiber $X_0 \cap U$ is smooth, so we can choose a retraction $r\colon U \to X_0 \cap U$ such that outside of the ball $B_i = \{|u|^2 + |v|^2 < 2\varepsilon\}$ it induces diffeomorphisms between $X_s \cap U$ and $X_0 \cap U$, at least for $|s| < \varepsilon$. In figure 2.1 we have draw a local picture of X_s for various values of $|s|$, and it should be quite clear that far enough from the singular point this is a retraction.

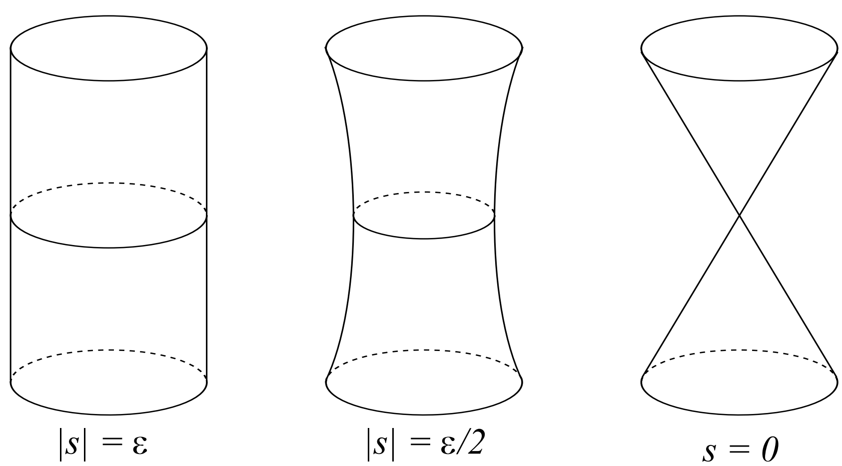


Figure 2.1: Local picture of X_s for various $|s|$

Let s be a positive real number such that $0 < s < \varepsilon$; then the circle

$$S_i = \{(u, v) \in B_i \mid u^2 + v^2 = s, \Im u = \Im v = 0\}$$

is a deformation retract of $B_i \cap X_s$, which (as in figure 2.1) is homeomorphic to $S^1 \times (0,1)$. Letting s tend to 0, this choice of S_i is continuous and tend to the single point x_i; since the construction of the homotopy r we made above does not pose any constraint on what happens inside B_i, we can also choose r so that it induces a diffeomorphism $(X_s \cap U) \setminus S_i \to X_0 \setminus \{x_i\}$. After doing so around each singular point, glueing together we are given a diffeomorphism $X_s \setminus \bigcup_i S_i \to X_0 \setminus \bigcup_i \{x_i\}$. Let $e_i = [S_i] \in H_1(X_s, \mathbb{Z})$; we call this a vanishing cycle. Note that, since $[S_i]$ has not any orientation given, it is determined only up to sign.

By the above construction, we have a topological construction of X_s as a connected sum of the components of the normalization $\tilde{X}_0$ of X_0: $\tilde{X}_0$ "looks like" X_0, except that near the double point it has two separated components. The singular fiber X_0 is reconstructed from $\tilde{X}_0$ deleting two small circular disks around the preimages of x_i and glueing at their place a two-sided cone; if instead we glue a cylinder, we obtain a 2-manifold homeomorphic to X_s (we have shown what happens locally in figure 2.2).

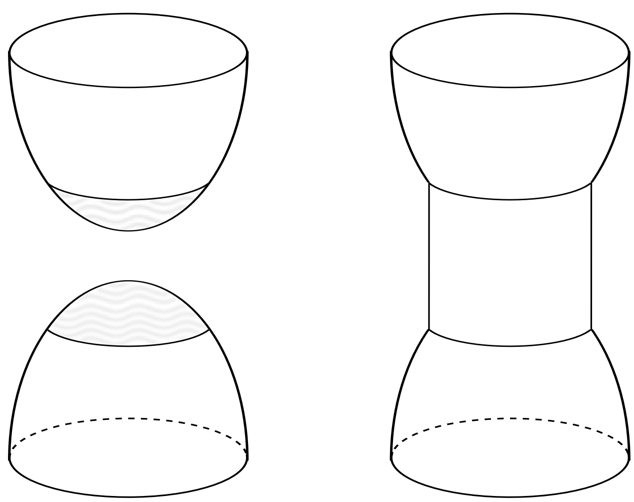


Figure 2.2: Construction of X_s from $\tilde{X}_0$

The construction of X_s and X_0 from $\tilde{X}_0$, gives us an equivalence of their homology, taking X_s "modulo S_i" and X_0 "modulo x_i". Formally, this means that the retraction gives a natural isomorphism between $H_1(X_s, \bigcup S_i; \mathbb{Z})$ and

$H_1(X_0, \bigcup x_i; \mathbb{Z})$. We then have a commutative diagram with exact rows:

$$\begin{tikzcd}
\bigoplus_i H_1(S_i,\mathbb{Z}) \arrow[r, "\iota_*"] \arrow[d] & H_1(X_s,\mathbb{Z}) \arrow[r, "p_1"] \arrow[d, "r_*"] & H_1(X_s, \bigcup S_i;\mathbb{Z}) \arrow[r, "\delta_1"] \arrow[d, equal] & \bigoplus H_0(S_i,\mathbb{Z}) \arrow[d, equal] \\
0 \arrow[r] & H_1(X_0,\mathbb{Z}) \arrow[r, "p_2"] & H_1(X_0, \bigcup\{x_i\};\mathbb{Z}) \arrow[r, "\delta_2"] & \bigoplus H_0(x_i,\mathbb{Z})
\end{tikzcd}$$

Let us call V the image of ι_* in $H_1(X_s, \mathbb{Z})$; it is the subspace generated by vanishing cycles. Then, by the exactness of the first row and injectivity of p_2, $V = \ker(p_1) = \ker(r_*)$. Furthermore, the images of p_1 and p_2 are the same, since by the previous construction of X_s from X_0 it is obvious that $\ker(\delta_1) = \ker(\delta_2)$, so that r_* is surjective. We thus have an exact sequence:

$$0 \to V \to H_1(X_s, \mathbb{Z}) \xrightarrow{r_*} H_1(X_0, \mathbb{Z}) \to 0. \tag{2.4}$$

We finally state the Picard-Lefschetz formula:

Theorem 2.6.1 (Picard-Lefschetz Formula). *For each $a \in H_1(X_s, \mathbb{Z})$ the monodromy action $T\colon H_1(X_s, \mathbb{Z}) \to H_1(X_s, \mathbb{Z})$ is given by*

$$T(a) = a - \sum_i (a.e_i)e_i$$

For the proof we refer to [**?**, **?**]. Note that, even though the vanishing cycles are determined only up to sign, the expression is not dependent on the choices.

We end this section by citing a result that will be very useful, determining the fixed points of the action of monodromy. We state it in cohomology, setting in which there is the induced monodromy $T(\alpha) = \alpha \circ T$, for $\alpha \in H^1(X_s, \mathbb{Z}) \cong \mathrm{Hom}(H_1(X_s, \mathbb{Z}); \mathbb{Z})$.

Theorem 2.6.2. *Let $f\colon X \to \Delta$ be a fibration with only singular fiber X_0, possibly not reduced or with non-nodal intersections. Then the image of the restriction*

$$\mathrm{restr}\colon H^1(X, \mathbb{Z}) \to H^1(X_s, \mathbb{Z})$$

is the subgroup of the invariants under the monodromy action.

This theorem is an easy consequence of the Picard-Lefschetz formula, when we assume that the fibers are reduced and with only nodal intersections; we will use it when this assumption does not hold, though. For a proof of this fact we refer to [**?**], theorem $A.1$, and [**?**].

Chapter 3

Elliptic surfaces

Questo verrà spostato come quarto capitolo

In this chapter we start the study of elliptic surfaces. In first section we prove the classification, due to Kodaira, of the possible types of singular fibers arising. The second section anticipates several results from next chapter about Weierstrass fibrations (relatively minimal elliptic fibrations with a section) that will be very useful to construct examples. The third section deals with monodromy around a singular fiber, and how this is related to the map associating to every non-singular fiber its J-value. Sections 4 and 5 are devoted to the construction of explicit examples of all the types of singular fibers allowed by the classification of Kodaira, and to the calculation of monodromy around a non-multiple fiber. In section 6 we prove a theorem of classification of elliptic fibrations without multiple fibers, and in section 7 we compute the Kodaira dimension of elliptic surfaces (with respect to other numerical invariants), placing them in the frame of Enriques-Kodaira classification.

In this chapter, by a *surface* we will mean a connected complex manifold of dimension 2; when we will be interested in study of singular objects, we will refer to them explicitely as complex spaces of dimension 2.

3.1 Kodaira's classification of singular fibers

The purpose of this section is to prove the following:

Theorem 3.1.1. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration of a smooth surface. Then the non-multiple fibers are classified according to table*

3.1; furthermore, the components of the reducible ones appearing in the table are (-2)*-curves. With the notation adopted in the table, if the multiplicity is greater than* 1*, then the fiber is a multiple of divisors of type* I_n*, for some* $n \geq 0$*.*

We start with a general lemma on reduced irreducible curves of arithmetic genus 1, that is basic for our work since will describe all the reduced irreducible fiber.

Lemma 3.1.2. *Let* C *be an irreducible reduced curve with* $p_a(C) = 1$*. Then* C *can only be one of the following:*

- *a smooth elliptic curve;*
- *a rational curve with a node;*
- *a rational curve with a cusp.*

Proof. Let $\nu : \tilde{C} \to C$ be the normalization of C. Then we have the following exact sequence of sheaves on C:

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \mathcal{S} \to 0$$

where $\mathcal{S} = \nu_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C$ is concentrated on the singular points of C. Then, by additivity of exact sequences, and since $\mathcal{S}$ is concentrated on points and hence cannot have cohomology higher than the 0 level, $\chi(\mathcal{O}_C) = \chi(\nu_*(\mathcal{O}_{\tilde{C}})) - h^0(\mathcal{S})$, which leads to

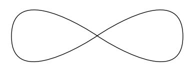
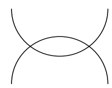
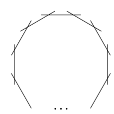
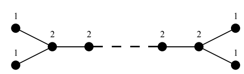
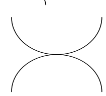
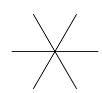
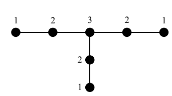
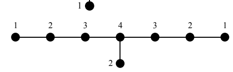
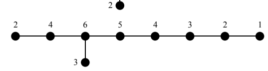
$$p_a(C) = p_a(\tilde{C}) + \delta(C), \quad \delta(C) = \sum_{x \in C} \dim_{\mathbb{C}}(\nu_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C)_x$$

Now, since C is irreducible $\tilde{C}$ is connected: one way to see this is that by one of the definitions of irreducibility, $C \setminus \{\text{singular points}\}$ is still connected, hence also $\tilde{C}$ is, since it is its closure in the blown-up space where C is embedded. This implies $p_a(\tilde{C}) \geq 0$.

By the definition of $\delta(C)$, it is always non-negative, and vanishes if and only if C is already non-singular. Then there remain only two possibilities: $\delta(C) = 0$, and C is smooth; $\delta(C) = 1$, and $p_a(\tilde{C}) = 0$. Furthermore, one can compute $\delta(C)$ as

$$\delta(C) = \sum_x \frac{1}{2} \mathrm{m}_x(\mathrm{m}_x - 1)$$

Table 3.1: Kodaira's table of singular fibers

Name	Description	Picture
I_0	smooth elliptic	
I_1	nodal rational	
I_2	two rational curves meeting transversally in two points	
I_n, $n \geq 2$	n rational curves meeting transversally in a cycle (dual graph: $\tilde{A}_n$)	
I_n^*, $n \geq 0$	$n+5$ rational curves; dual graph $\tilde{D}_{n+4}$	
II	cuspidal rational curve	
III	two rational curves meeting at a point of order 2	
IV	three rational curves meeting at one point, each one transversally to each other	
IV^*	7 rational curves; dual graph $\tilde{E}_6$	
III^*	8 rational curves; dual graph $\tilde{E}_7$	
II^*	9 rational curves; dual graph $\tilde{E}_8$	

where the sum is take over all points of C, *including those infinitely near* (cfr. [**?**], Theorem 3.9). Since a point has multiplicity 1 if and only if it is smooth, the only possibility is that there is only one point, which in fact is a double point and is resolved after a single blow-up. One can exploit this information (see, for example, [?], Chap. II, par. 8), and see that the singularity is analytically isomorphic to the one in $x^2 + y^n = 0$ $(n \geq 2)$, that is effectively a node or a cusp, as stated. □

Proof of theorem (3.1.1). The question is local, so we may assume that $S = \Delta \subset \mathbb{C}$ is the unit disk, and the only possibly singular fiber X_0 lies above 0.

Firstly, we classify *irreducible and reduced* fibers: they are irreducible curves of arithmetic genus 1, so they must be either smooth elliptic, or rational with a node, or rational with a cusp (in Kodaira's notation, as reported in table 3.1, type I_0, I_1 and II respectively) by lemma (3.1.2).

Let us now assume that $X = \sum n_i C_i$ is reducible, possibly non-reduced, but *not a multiple fiber*. We first prove the second assertion of the theorem, that is, every C_i is rational with self-intersection -2. By the adjunction formula, we have

$$K_X.C_i = 2p_a(C_i) - 2 - C_i^2, \tag{3.1}$$

which leads to (using again $p_a(X_0) = 1$ and adjunction formula as above)

$$0 = K_X.X_0 = \sum n_i K_X.C_i = \sum n_i(2p_a(C_i) - 2 - C_i^2). \tag{3.2}$$

The fiber is not irreducible nor multiple, so X_0 cannot be a multiple of any of the C_i; then, by Zariski's lemma (2.1.5), we must have $C_i^2 \leq -1$. Furthermore, by the assumption of relative minimality $C_i^2 = -1$ forces $p_a(C_i) > 0$, so that each term appearing in the sum is non-negative. But then every term must be 0, so that each C_i is a (-2)-curve.

Again by Zariski's lemma, we have, for $i \neq j$, $(C_i + C_j)^2 \leq 0$, so that $C_i.C_j \leq 2$. If equality holds, we also have $X_0 = C_i + C_j$ (using again the assumption of non-multiple fibers), and this means that we have two rational curves intersecting either transversally in 2 points (fiber of type I_2) or of multiplicity 2 in one point (fiber of type III). Otherwise, $C_i.C_j$ can be either 0 or 1, so we are left with rational curves, every distinct two of them meeting transversally. We can draw the dual graph, with a vertex for each rational curve and an edge connecting any two intersecting curves; we can attach to each vertex a positive integer, its multiplicity. But then, again by Zariski's lemma, the quadratic form associated to this graph (that is the one induced by intersection product) is semi-negative definite, and we can apply proposition (2.1.3). The quadratic form has obviously a non-trivial annihilator (generated by the whole fiber), so it must be attached to an extended Dynkin diagram. Then, the dual graph can be one of $\tilde{A}_n$ ($n \geq 2$), $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$; $\tilde{A}_2$ can be the dual graph of either fibers of type I_3 or of type IV, while there is no ambiguity for $\tilde{A}_n$ ($n \geq 3$), $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$: they represent, respectively, I_{n+1}, I^*_{n-4}, IV^*, III^* and II^*.

We are left only with the case of a multiple fiber, $X_0 = mF$. Then, $p_a(F) = 0$ by the adjunction formula ($K_X.F = 0$, since it must be 0 when multiplied by m). Then all the arguments made for a non-multiple fiber X_0 apply to F as well, so that F must figure in the table 3.1. But by corollary (2.2.7), F must be a non simply-connected real 2-variety. This excludes all the cases but I_n, for $n \geq 0$, since the other are, topologically, either S^2, or a bouquet of 2 copies of S^2, or a tree made of intersecting S^2 (that, homotopically, is still a bouquet of some copies of S^2). $\square$

3.2 Weierstrass fibrations

We introduce here the Weierstrass fibrations. They are a class of elliptic fibrations with additional properties which make their study much easier, and they will be the object of our interest in the next chapter. We introduce them here because a great number of interesting examples are of this form, and also because they can be used to classify the elliptic fibrations without multiple fibers, as we will see in section 3.6.

Definition Let X be a reduced normal complex space of dimension 2, and S a smooth connected curve (not necessarily compact). A *Weierstrass fibration* is a flat proper and surjective map $X \to S$ such that every fiber is reduced and irreducible of arithmetic genus 1, i.e. is either smooth of genus 1, or rational with a node, or rational with a cusp, with a smooth general fiber, admitting a section $\sigma\colon S \to X$ not passing through the node or cusp of any singular fiber.

The definition is apparently quite different from what we have treated until now, so some remarks are in order. Let us begin by noting that the hypothesis for an elliptic surface to have a section is not trivial:

Proposition 3.2.1. *Every elliptic surface with a section is algebraic and has no multiple fiber.*

Proof. Take a fiber X_s, and call S_0 the image of the section. Then $X_s.F > 0$, so that for n big enough

$$(X_s + nF)^2 > 0.$$

By theorem (1.3.5) this implies that X is projective.

If $\sigma\colon S \to X$ is a section, then σ meets every regular fiber with multiplicity 1; but the multiplicity of intersection with each fiber must be the same, so there cannot be multiple fibers. □

The definition of Weierstrass fibration allows X to have singularities; however, the hypotheses of X to be reduced and normal allow us to have a minimal desingularization to obtain a smooth surface. There is also a canonical way to associate to every minimal elliptic surface without multiple fibers a Weierstrass fibration:

Remark 3.2.2. If one starts with a smooth and minimal elliptic surface $f\colon X \to S$ admitting a section $s\colon S \to X$, after contracting all the singular irreducible components of fibers that do not meet the image of s we obtain a Weierstrass fibration with only rational double points.

Proof. Since the section meets the general fiber in exactly one point and transversally, σ must meet any singular fiber in a component of multiplicity 1. By simple inspection of table 3.1, this leads to an easy description of the singular fibers not meeting σ: if the fiber is of type I_n, removing a component leads to $n-1$ rational curve meeting with dual graph A_{n-1}; if it is of type I_n^* we are left with $n+4$ rational curves meeting with dual graph D_{n+4}, etc. In any circumstance, we are always left with a Dynkin diagram, corresponding to table 3.2, which always contract to an ADE-singularity, that in turn is a rational double point.

The curve obtained after the contraction is rational with a node if we started from type I_n and rational with a cusp if we started from type I_n^*, IV^*, III^*, II^*, III or IV. This is simply because a component of a fiber of type I_n meet the remaining components in two distinct points, in each with multiplicity one, while in all the other cases a glimpse at their drawings (or dual diagrams) shows that they always meet the divisor made by the other components in a point with multiplicity 2: it is tangent to a smooth rational curve in type III, transversal to two distinct $\mathbb{P}^1$'s in type IV, and transversal to a $\mathbb{P}^1$ with multiplicity 2 in the remainin cases. By (3.2.1) the obtained surface is algebraic, and all other properties are obvious. □

We can see this construction as a map F from the set of smooth minimal elliptic fibrations over S with a section to the set of Weierstrass fibrations over S. It clearly has a one-sided inverse: if we take the minimal resolution of singularities of the surface so-obtained, we get again the starting surface, by uniqueness of the minimal resolution. We indicate this process of minimal resolution of singularities of a Weierstrass fibration as a map G from the set of Weierstrass fibrations to that of smooth minimal elliptic fibrations with a section. Hence, F is injective and G is surjective. However, this is not a two-sided inverse, since $F \circ G$ is not the identity in general, i.e. F is not surjective. We will discuss this problem in full details in next chapter, and for now we limit ourselves to note that if X is a Weierstrass fibration with singularities that are not rational double points, it cannot be hit by F, by the previous remark. We say that a Weierstrass fibration is in *minimal form* if it is in the image of F; hence, studying minimal smooth elliptic fibrations with section over S is the same as studying Weierstrass fibrations in minimal form. Given any Weierstrass fibration $X \to S$, the process of applying $F \circ G$ will be called "putting in minimal form".

With these remarks in mind, we see that the only real assumption to be allowed to work in terms of Weierstrass fibrations is the existence of a section defined on all of S. As we have seen, in some situation there can be none, e.g. if the surface is not algebraic or if it has a multiple fiber. However, if there is no multiple fiber, locally a section always exists: if X_s is a non multiple singular fiber, by inspection of table 3.1 it has at least a component of multiplicity 1, so we can find a point on X_s with a neighborhood where f has non zero differential. Taking local coordinates given by the implicit function theorem, then, we obtain a section defined in a neighborhood of

Table 3.2: Reducible fibers minus a component of multiplicity 1

Fiber type	Dual graph
I_n, $n \geq 2$	A_{n-1}
I_n^*	D_{n+4}
III	A_1
IV	A_2
IV^*	E_6
III^*	E_7
II^*	E_8

s. Hence, when we work locally around a non-multiple fiber, we can always assume that we have a Weierstrass fibration.

We now present some results about Weierstrass fibrations that will be proved in the next chapter. We include them here since they make it much easier to give examples of any type of non-multiple singular fibers and more generally to develop results in local study.

Let $f\colon X \to C$ be a Weierstrass fibration with section σ, and let $S_0 = \operatorname{range}(\sigma)$. Then always $R^1 f_* \mathcal{O}_X \cong f_* N_{S_0/X}$. We call its dual the *fundamental line bundle*

$$\mathcal{L} = (f_* N_{S_0/X})^\vee \cong (R^1 f_* \mathcal{O}_X)^\vee. \tag{3.3}$$

By the second characterization, it does not depend on the section chosen. It can also be shown that if we are given a Weierstrass fibration, and $\mathcal{L}$ is its fundamental line bundle, than $\mathcal{L}^4$ and $\mathcal{L}^6$ admit sections A, B respectively, that locally (on every trivializing open set of $\mathcal{L}$) are constructed as the coefficients in the elliptic general curve $y^2 = x^3 + Ax + B$. In the same way, they are unique up to a common constant multiple λ, acting as $(A, B) \sim (\lambda^4 A, \lambda^6 B)$; furthermore, the *discriminant* $\Delta = 4A^3 + 27B^2$ is not identically zero, and vanishes exactly correspondingly to the singular fibers of X. We call *Weierstrass data* over S any triple $(\mathcal{L}, A, B)$ such that (A, B) is a section of $\mathcal{L}^4 \oplus \mathcal{L}^6$; two set of Weierstrass data are said to be isomorphic if there is an isomorphism between the line bundles, making the A, B sections of the first correspond to the respective sections of the second. Then the construction of the Weierstrass data starting from a Weierstrass fibration is effectively bijective, modulo isomorphism:

$$\left\{ \begin{array}{c} \text{Weierstrass} \\ \text{data} \\ \text{over } \mathbb{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Weierstrass} \\ \text{fibrations} \\ \text{over } \mathbb{C} \end{array} \right\}$$

The first important result of the Weierstrass fibrations we will use is the so-called a-b-δ table. This is simply table 3.3 which we present below: given a Weierstrass fibration in minimal form $X \to \Delta$, with no singular fiber except possibly X_0, we list the possibility of the singular fibers as in table 3.1, attaching to each fiber type the order of vanishing in 0 of the section A of $\mathcal{L}^4$, B of $\mathcal{L}^6$, Δ of $\mathcal{L}^{12}$, which we call a, b, δ, respectively. Furthermore, if we write, as is usual for elliptic curves,

$$J(X_0) = J(A, B) = \frac{4A^3}{4A^3 + 27B^2} \tag{3.4}$$

then we can consider J as a meromorphic function $J : \Delta \to \mathbb{P}^1$. In the table we also report the values that J can take at each type of fiber, and the multiplicity $m(J)$ with which they take that value.

Table 3.3: a-b-δ table for Weierstrass fibrations in minimal form

Name	a	b	δ	J	$m(J)$
	0	0	0	$\neq 0, 1, \infty$	?
I_0	$a \geq 1$	0	0	0	$3a$
	0	$b \geq 1$	0	1	$2b$
I_1	0	0	1	∞	1
I_n, $n \geq 1$	0	0	n	∞	n
	2	3	6	$\neq 0, 1, \infty$	?
I_0^*	$a \geq 3$	3	6	0	$3a - 6$
	2	$b \geq 4$	6	1	$2b - 6$
I_n^*, $n \geq 0$	2	3	$n + 6$	∞	n
II	$a \geq 1$	1	2	0	$3a - 2$
III	1	$b \geq 2$	3	1	$2b - 3$
IV	$a \geq 2$	2	4	0	$3a - 4$
IV^*	$a \geq 3$	4	8	0	$3a - 8$
III^*	3	$b \geq 5$	9	1	$2b - 9$
II^*	$a \geq 4$	5	10	0	$3a - 10$

The main result that the table shows is that the values of a, b and δ are enough to characterize the type of the singular fiber, hence its name. By inspection of the table, we see that always $a \leq 3$ or $b \leq 5$. This is not a coincidence, since we will prove:

Proposition 3.2.3. *Let $X \to S$ be a Weierstrass fibration. Then the following are equivalent:*

1. *$X \to S$ is in minimal form;*
2. *X has only rational double points as singularity;*
3. *there is no point $s \in S$ such that $\mathrm{m}_s(A) \geq 4$ and $\mathrm{m}_s(B) \geq 6$ (where m_s stands for the multiplicity of the function at the point s).*

Furthermore, one can see that putting a Weierstrass fibration in minimal form locally is the same as performing the transformations $a \mapsto a - 4$, $b \mapsto b - 6$ and, hence, $\delta \mapsto \delta - 12$; more precisely, evidently two Weierstrass fibrations have the same minimal model if and only if they are birationally equivalent, and we will show that two set of Weierstrass data $(\mathcal{L}_1, A_1, B_1)$ and $(\mathcal{L}_2, A_2, B_2)$ determine birationally equivalent surfaces if and only if there are two line bundles $\mathcal{M}_1$, $\mathcal{M}_2$ and sections $f_i \in H^0(\mathcal{M}_i)$ such that

$$(\mathcal{L} \otimes \mathcal{M}_1, A_1 f_1^4, B_1 f_1^6) \cong (\mathcal{L} \otimes \mathcal{M}_2, A_2 f_2^4, B_2 f_2^6).$$

We will need one last technique to make the local study of elliptic surfaces easier. We expect that the local behavior of the fibration depends not only on the fiber type, but also on the disposition of fibers in a neighborhood. Formally, we define the *germ* of a fiber X_s, as usual, as the equivalence class of fibrations restricted to the preimage of a neighborhood of s. We already know by theorem (2.2.9) that if all the fibers in the neighborhood are isomorphic then the fibration is locally trivial, so the germ is identified. This generalizes: if we call $\mathrm{m}_s(f)$ the multiplicity with which the function f takes the value $f(s)$, with the convention that if f is locally constant then $m_s(f) = \infty$, then we have:

Proposition 3.2.4. *Let X be an elliptic surface with J-map J, $s \in S$ a point. The germ of the fiber X_s of an elliptic surface with section is determined by $J(s)$, $\mathrm{m}_s(J)$ and the singular fiber type. In particular, the whole of the entries of table 3.3 determines the germ of the fiber.*

This proposition, together with table 3.3, allows us to make a table of the so-called normal forms of all the Weierstrass fibrations in minimal form. We list the result in table 3.4.

The verification of the table is straightforward. Let us take X_0 of type I_0, so that $\Delta(0) \neq 0$. Then, by table 3.3 we have three cases: if both $A(0)$

Table 3.4: Normal forms for local Weierstrass fibrations in minimal form

Fiber	$J(s)$	$A(s)$	$B(s)$
I_0	0	0	1
	1	1	0
	$j \neq 0, 1$	$-3j(j-1)$	$2j(j-1)^2$
	s^{3n}	s^n	1
	$1+s^{2n}$	1	s^n
	$j+s^n$, $j \neq 0, 1$	$-3(j+s^n)(j+s^n-1)$	$2(j+s^n)(j+s^n-1)^2$
I_n, $n \geq 1$	s^{-n}	$-3(1-s^n)$	$2(1-s^n)^2$
I_0^*	0	0	s^3
	1	s^2	0
	$j \neq 0, 1$	$-3j(j-1)s^2$	$2j(j-1)^2 s^3$
	s^{3n}	s^{n+2}	s^3
	$1+s^{2n}$	s^2	s^{n+3}
	$j+s^n$, $j \neq 0, 1$	$-3s^2(j+s^n)(j+s^n-1)$	$2s^3(j+s^n)(j+s^n-1)^2$
I_n^*, $n \geq 1$	s^{-n}	$-3s^2(1-s^n)$	$2s^3(1-s^n)^2$

Fiber	$J(s)$	$A(s)$	$B(s)$	Fiber	$J(s)$	$A(s)$	$B(s)$
II	0	0	s	IV^*	0	0	s^4
	s^{3n+1}	s^{n+1}	s		s^{3n+1}	s^{n+3}	s^4
III	1	s	0	III^*	1	s^3	0
	$1+s^{2n+1}$	s	s^{n+2}		$1+s^{2n+1}$	s^3	s^{n+5}
IV	0	0	s^2	II^*	0	0	s^5
	s^{3n+2}	s^{n+2}	s^2		s^{3n+2}	s^{n+4}	s^5

and $B(0)$ are non zero, then $J(0) = j \neq 0, 1$, and we only have to distinguish between the multiplicities; so either $J \equiv j$, or $J = j + s^n$, for some $n > 0$. If A vanishes of some order n, then $B(0) \neq 0$ and $J(0) = 0$, so that either $J \equiv 0$ or $J = s^{3n}$. Lastly, if B vanishes of some order n, then A does not, and $J(0) = 1$. Then either $J \equiv 1$ or $J = 1 + s^{2n}$. We then can choose arbitrarily the function A and B so that they respect the given multiplicities and determine the given value of J.

If X_0 is of type I_n, then we must have $A(0) \neq 0$, $B(0) \neq 0$ but $J(0) = \infty$, with a pole of order n. The natural choice is $J(s) = s^{-n}$, and a possible consequent choice of A and B is shown.

If X_0 is of type I_0^* we can obtain all the possibility simply by multiplying A by s^2 and B by s^3. This do not affect the value of J, but modify the fiber type accordingly (we will say that we have performed a *quadratic twist*; we refer to next chapter for a discussion on this tool). In the same way, we get the row I_n^* from that of I_n.

Types II, III, IV are treated similarly to case I_0: in cases II and IV we distinguish depending on the multiplicity of A, in case III on the multiplicity of B; the bounds are those given by table 3.3, and the multiplicity of J is deduced. Since all the possible values for this appear in the table, we have the classification. As above, by multiplying A by s^2 and B by s^3 we obtain all the corresponding $*$ fiber types.

3.3 Homological invariant, local monodromy and J map

3.3.1 Local Monodromy

Let $\Delta \subset \mathbb{C}$ be the unit one-dimensional disk, and $f\colon X \to \Delta$ an elliptic fibration, so that if we fix a point $s \neq 0$, the homology group $H_1(X_s, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. As we did in section 2.6, let us choose a loop $\gamma\colon [0, 1] \to S$, winding once around the origin in a counterclockwise direction, starting and ending at s; if a_0, b_0 are generators for $H_1(X_s, \mathbb{Z})$, we can choose continuously generators a_t, b_t of $X_{\gamma(t)}$, and considering the new base $\{a_1, b_1\}$ of $H_1(X_s, \mathbb{Z})$ we obtain an automorphism $T\colon H_1(X_s, \mathbb{Z}) \to H_1(X_s, \mathbb{Z})$, called the *local monodromy* around 0. Clearly, this element depends only on the homotopy class of the arc γ and on the basis chosen, and any arc between two points s, s' gives isomorphisms between the homology groups; thus, at least the conjugacy class of $T \in \mathrm{GL}(2,\mathbb{Z})$ is well defined. We note that the monodromy action can be equally well defined by the action in cohomology, given by $T(\alpha) = \alpha \circ T$, for each $\alpha \in H^1(X_s, \mathbb{Z})$.

Fix a holomorphic 1-form ω on X_s. Then, integration of ω along elements of $H_1(X_s, \mathbb{Z})$ gives an isomorphism between $H_1(X_s, \mathbb{Z})$ and the lattice Λ_s of the periods of the elliptic curve. Composing with Poincaré duality isomorphism, we can see the isomorphisms given by continuation along γ as lattices moving inside the fixed space $\mathbb{C} = H^1(X_{\gamma(t)}, \mathcal{O}_{X_{\gamma(t)}})$; in this way we see that the complex structure is preserved, and so the orientation of a basis $\{\tau_1, \tau_2\}$ of Λ_s is preserved by the monodromy automorphism. This implies that $\det T = 1$, hence that the monodromy is a representation

$$\rho\colon \pi_1(\Delta \setminus \{0\}) \to \mathrm{SL}(2,\mathbb{Z}),$$

well defined up to the choice of a base of $H_1(X_s, \mathbb{Z})$. So, at this point, we have an element of $\mathrm{SL}(2,\mathbb{Z})$ unique up to conjugacy in $\mathrm{GL}(2,\mathbb{Z})$; if we fix an orientation of X_s and require the basis to be oriented, it is unique up to conjugacy in $\mathrm{SL}(2,\mathbb{Z})$.

We postpone the calculation of the local monodromy to sections 3.4 and 3.5. The result will show that local monodromy is a very useful invariant: it depends only on the type of the singular fiber, and not on its neighborhood, and uniquely identifies the type of singular fiber.

3.3.2 The function $\tau(s)$

Let $f\colon X \to S$ be an elliptic surface; take a finite non empty set $\{s_1, \ldots, s_k\}$ containing all the singular values of f, and let $S^* = S \setminus \{s_1, \ldots, s_k\}$. As before, we know that for every $s \in S$, $H^1(X_s, \mathcal{O}_{X_s}) \cong \mathbb{C}$ and if furthermore we take $s \in S^*$, we can see $H^1(X_s, \mathbb{Z})$ as a lattice in it. Then, if we write its periods as $\{1, \tau(s)\}$, where $\tau(s)$ is a point in the upper half plane $\mathfrak{h}$ and is uniquely determined up to the usual action of $\Gamma = \mathbb{P}\mathrm{SL}(2,\mathbb{Z})$, we have defined a multivalued map

$$\tau\colon S^* \to \mathfrak{h}.$$

We call this the τ-map of the elliptic fibration. We want to show that it is in fact holomorphic. It is well known (see, for example, [**?**]) that, fixed a smooth fiber X_s, for any oriented bases of 1-cycles $\{a_s, b_s\}$ of $H_1(X_s, \mathbb{Z})$, and any choice of a holomorphic 1-form ω_s on X_s, τ is given by:

$$\tau(s) = \frac{\int_{a_s} \omega_s}{\int_{b_s} \omega_s}. \tag{3.5}$$

Lemma 3.3.1. *If $X \to S$ is a fibration with only irreducible and reduced fibers such that S is non-compact, then the canonical bundle is trivial.*

Proof. We have already seen (cfr. corollary (2.5.2)) that K_X is a linear combination of components of fibers. If S is not compact, then any divisor on it is linearly equivalent to 0, hence if every fiber is irreducible the canonical bundle is trivial, too. □

Remark 3.3.2. By means of canonical bundle formula (theorem (3.7.1)) we can strengthen this result asking only that the fibration has no multiple fiber, but allowing reducible or non-reduced ones.

Proposition 3.3.3. *The map $\tau\colon S^* \to \mathfrak{h}$ is a holomorphic multivalued map.*

Proof. Since we required $\{s_1, \ldots, s_k\}$ to be non-empty, S^* is not compact, so $K_X \cong \mathcal{O}_X$ by lemma (3.3.1) has a never vanishing holomorphic global section, i.e. there is a holomorphic never vanishing 2-form ψ. Chosen a point $s \in S^*$, we take a coordinate neighborhood U with the coordinate z, and we cover $f^{-1}(U)$ with coordinate neighborhoods V_λ, each with coordinates (u_λ, z). Writing

$$\psi = \psi_\lambda(u_\lambda, z)\mathrm{d}u_\lambda \wedge \mathrm{d}z,$$

for each $t \in U$ with coordinate $z(t)$, we can consider $\psi_\lambda(u_\lambda, z(t))\mathrm{d}u_\lambda$ as a 1-form on $X_t \cap V_\lambda$; these patch together since they are determined uniquely by ψ modulo $\mathrm{d}z$, giving a well defined holomorphic 1-form depending holomorphically on t:

$$\omega_t = \psi(u_\lambda, z(t))\mathrm{d}u_\lambda.$$

Furthermore, by Ehresmann's theorem, the fibration in this neighborhood is topologically trivial, i.e. we have a fiber-preserving homeomorphism $f^{-1}(U) \to U \times X_s$; hence we can choose for each $t \in U$ a basis of 1-cycles $\{a_t, b_t\}$ "moving" a chosen base on $H_1(X_s, \mathbb{Z})$, obtaining a family that depends only continuously on each variable u_λ, but holomorphically on z, since the homeomorphism is the identity in this variable. This shows that, around the point s, the map $\tau(u)$ given by formula (3.5) is continue and also holomorphic (since derivatives in the variable $z(t)$ depend only on $\omega(t)$), hence globally we obtain a multivalued holomorphic function. □

3.3.3 J-map and global monodromy

It is a very well known fact about elliptic surfaces (see, for example, [**?**]) that any elliptic curve E is isomorphic to a plane elliptic curve with equation $y^2 = x^3 + Ax + B$, with A, B complex numbers uniquely determined up to the equivalence $(A, B) \sim (\lambda^4 A, \lambda^6 B)$, with $\lambda \in \mathbb{C}^*$. One then defines:

$$J(E) = J(A, B) = \frac{4A^3}{4A^3 + 27B^2}.$$

Obviously, equivalent (A, B) give the same J, that is thus a well-defined function of E. Conversely, two elliptic curves with the same J unequal to 0 or 1 are isomorphic. We can now define a function $j\colon \mathfrak{h} \to \mathbb{C}$ (where $\mathfrak{h}$ is the upper half plane)

$$j(\tau) = J(\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)).$$

This is a holomorphic covering, ramified only over 0 and 1. Thus, if we restrict, we obtain a non ramified covering

$$j\colon \mathfrak{h} \setminus j^{-1}(\{0, 1\}) \to \mathbb{C} \setminus \{0, 1\}$$

with deck transformation group isomorphic to $\Gamma = \mathbb{P}\mathrm{SL}(2, \mathbb{Z})$, which acts with the usual action on $\mathfrak{h}$. This, in particular, gives a map $\alpha\colon \pi_1(\mathbb{C} \setminus \{0, 1\}) \to \mathbb{P}\mathrm{SL}(2, \mathbb{Z})$, since every covering has deck transformation group naturally isomorphic to a quotient group of the π_1 of the base.

Definition Let $f\colon X \to S$ be an elliptic fibration without multiple fibers, and $\{s_1, \ldots, s_k\} \subset S$ a subset containing all the singular values of f. Let $S^* = S \setminus \{s_1, \ldots, s_k\}$; we call the J-map of f the map $J\colon S^* \to \mathbb{C}$ obtained by

$$J(s) = J(X_s).$$

Remark 3.3.4. If f is put in Weierstrass normal form, then this J coincides with the function defined in equation (3.4)

Using the above definition of j, we can equally define $J(s) = j(\tau(s))$, where τ is the τ-map of f; in this way, it follows readily from proposition (3.3.3) that $J\colon S \to \mathbb{C}$ is holomorphic and single-valued.

Suppose now given a compact curve S and a nonconstant map $J\colon S \to \mathbb{P}^1$. Let $\{s_1, \ldots, s_k\}$ be a finite set containing $J^{-1}(\{0, 1, \infty\})$, and $S^* = S \setminus \{s_1, \ldots, s_k\}$. The restriction gives a map $S^* \to \mathbb{C} \setminus \{0, 1\}$, and passing to fundamental groups also $J_*\colon \pi_1(S^*) \to \pi_1(\mathbb{C} \setminus \{0,1\})$. We can compose with the α map above, thus obtaining a map (which we still call J_*)

$$J_*\colon \pi_1(S^*) \to \mathbb{P}\mathrm{SL}(2,\mathbb{Z}).$$

Let now $S = \Delta$ be the one-dimensional disk, $\Delta^* = \Delta \setminus \{0\}$, and J be the J-map of an elliptic surface $X \to \Delta$, which we assume to be non-constant; then, after shrinking Δ if necessary, we can also assume that X_0 is the only possibly singular fiber and that $j(s) \neq 0, 1$ for all $s \in \Delta^*$. If we call $\rho : \pi_1(\Delta^*) \to \mathrm{SL}(2,\mathbb{Z})$ the local monodromy representation, we have a commutative diagram (up to conjugacy):

$$\begin{tikzcd} & \mathrm{SL}(2,\mathbb{Z}) \arrow[d] \\ \pi_1(\Delta^*) \arrow[ur, "\rho"] \arrow[r, "J_*"] & \mathbb{P}\mathrm{SL}(2,\mathbb{Z}) \end{tikzcd}$$

i.e. the local monodromy is one of the 2 possible lifts of J_*. Using the description of local monodromy as moving lattices inside $\mathbb{C}$, it is easy to see that the diagram actually commutes: the vertical arrow $\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ can be seen as the representation of $\mathrm{SL}(2,\mathbb{Z})$ as the action on $\mathfrak{h} \setminus j^{-1}(\{0,1\})$; if we fix a normalized basis $\{1, \tau(s)\}$ of $H_1(X_s, \mathbb{Z})$ (that is, we take one branch of the multivalued τ-map), the monodromy gives another base $\{\xi, \eta\}$ with the same orientation. Normalizing, we obtain another element $\tau'(s)$, that is, another branch of the τ-map. Obviously, the action of the monodromy on a basis of the lattice is faithful, but after normalizing the basis obtained we may have identified $(-1, -\tau'(s))$ with $(1, \tau'(s))$; thus, we are regarding the local monodromy as an element of $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$. It is easy to see that the map J_* does exactly the same: given a point $s \in \Delta^*$ and a loop winding once in a counterclockwise direction, applying J_* we firstly obtain a loop in $\mathbb{C} \setminus \{0,1\}$, based in $J(s)$, and we lift it to an arc in $\mathfrak{h} \setminus j^{-1}(\{0,1\})$ which we can take to start in $\tau(s)$; then, the ending point will be exactly $\tau'(s)$, giving the desired commutativity.

We observe that all the constructions we have done until now easily extend to the case when $f\colon X \to S$ is a fibration without multiple fibers over a compact curve, and we take $\{s_1, \ldots, s_k\} \supseteq J^{-1}(\{0,1\})$ and $S^* = S \setminus \{s_1, \ldots, s_k\}$. Considering the isomorphism given by the class of a loop in S^* we obtain as in the local case an element of $\mathrm{SL}(2,\mathbb{Z})$; we call the resulting representation

$$G\colon \pi_1(S^*) \to \mathrm{SL}(2,\mathbb{Z})$$

the *global monodromy* of the fibration.

Definition Let S be a compact curve, J a meromorphic non constant function $J\colon S \to \mathbb{P}^1$, fix k points $s_1, \dots, s_k$ so that $J^{-1}(\{0,1,\infty\}) \subseteq \{s_1, \dots, s_k\}$ and let $S^* = S \backslash \{s_1, \dots, s_k\}$, as usual. Let G be a representation $G\colon \pi_1(S^*) \to \mathrm{SL}(2,\mathbb{Z})$. We say that G *belongs to* J if we obtain a commutative diagram

$$\begin{tikzcd} & \mathrm{SL}(2,\mathbb{Z}) \arrow[d] \\ \pi_1(S^*) \arrow[ur, "G"] \arrow[r, "J_*"] & \mathbb{P}\mathrm{SL}(2,\mathbb{Z}) \end{tikzcd}$$

With this language, we have shown:

Proposition 3.3.5. *Let $\pi : X \to S$ be an elliptic surface without multiple fibers, call J its J-map and let $\{s_1, \dots, s_k\}$ be such that $J^{-1}(\{0,1,\infty\}) \subseteq \{s_1, \dots, s_k\}$. Then the homological invariant belongs to J.*

3.4 Semi-stable fibrations: fibers of type I_n

We start here the study of the singular fibers allowed by table 3.1 and of the local monodromy around them. Note that table 3.4 deals with the existence of such fibers, at least in the non-multiple case. We start our study from the semi-stable fibers:

Definition Let $f\colon X \to S$ be a fibration with non-rational general fiber. A fiber X_s is said to be *semi-stable* if it satisfies all of the following:

1. X_s is reduced;
2. the only singularities of X_s are nodes;
3. X_s contains no (-1)-curves

If it also satisfies the condition:

4. X_s contains no (-2)-curves

then the fiber is said to be *stable*. A fibration is said to be stable (resp. semi-stable) if all of its fibers are.

A simple inspection of the table shows that all of the * fiber types are not reduced, while types II, III and IV have crossings that are not nodes. So the only semi-stable fibers are those of type I_n, $n \geq 0$; furthermore, only I_0 and I_1 are stable.

Let us begin our study from the easiest possible case, that of type I_0. Of course, we have already mentioned that the general fiber is always smooth, so every elliptic fibration have plenty of these fibers. Locally, we can construct an elliptic fibration over Δ explicitly by considering the quotient of $\mathbb{C} \times \Delta$ by the action of $\mathbb{Z} \oplus \mathbb{Z}$:

$$(m, n) \cdot (c, s) = (c + m + nz(s), s),$$

where we take $z \colon \Delta \to \mathfrak{h}$ holomorphic, with image in the upper half plane. Then the action is free, and we obtain a smooth surface considering the second projection: $X \to \Delta$. The fibers are smooth elliptic curves, with periods $\langle 1, z(s) \rangle$. More interesting, if we take any elliptic fibration, locally around any smooth fiber it always has this form: the only non-trivial fact is that we can always choose z to be holomorphic, and this is assured by proposition (3.3.3).

For fibrations with only smooth fibers, the local monodromy is obviously trivial, by the construction of monodromy (essentially because by Ehresmann's theorem the fibration is locally homoeomorphically isomorphic to the trivial one).

Let us now move to the next case, that of fibers of type I_1. By the results in section 3.2 we can construct them in Weierstrass form, taking the surface

$$\left\{[x_0 : x_1 : x_2], s \in \mathbb{P}^2 \times \Delta \mid x_0 x_2^2 = x_1^3 + A(s) x_0^2 x_1 + B(s) x_0^3\right\}$$

where $A(s)$, $B(s)$ are holomorphic functions, none of which vanishes at 0 and such that $4A^3(0) + 27B^2(0) = 0$. For example, we can take $A(s) = -3(1-t)$ and $B(S) = 2(1-t)^2$; we will show in next chapter that, up to biholomorphic isomorphism of fibration, this is the only possible case (cfr. table 3.4). We claim that in this case the matrix A_{I_1} representing monodromy in $\mathrm{SL}(2, \mathbb{Z})$ is given by

$$A_{I_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will use Picard-Lefschetz formula, stated in theorem (2.6.1). We recall that we have an exact sequence (2.4):

$$0 \to V \to H_1(X_s, \mathbb{Z}) \xrightarrow{r_*} H_1(X_0, \mathbb{Z}) \to 0,$$

with r_* induced by the inclusion in X, composed with the isomorphism given by the retraction $r\colon X \to X_0$. In our situation, the genus of the smooth fiber X_s is 1, so that $H_1(X_s, \mathbb{Z})$ is a free abelian group of rank 2; furthermore, since X_0 is of type I_1, its fundamental group is free of rank 1. So the sequence must split, and we may take a basis:

$$H_1(X_s, \mathbb{Z}) = \langle e, v \rangle$$

where e is the only vanishing cycle with some orientation, and v is a generator for the fundamental group of X_0 (identified with a fixed cycle in X_s that maps on it). By the Picard-Lefschetz formula,

$$T(e) = e, \quad T(v) = v - (e.v)e.$$

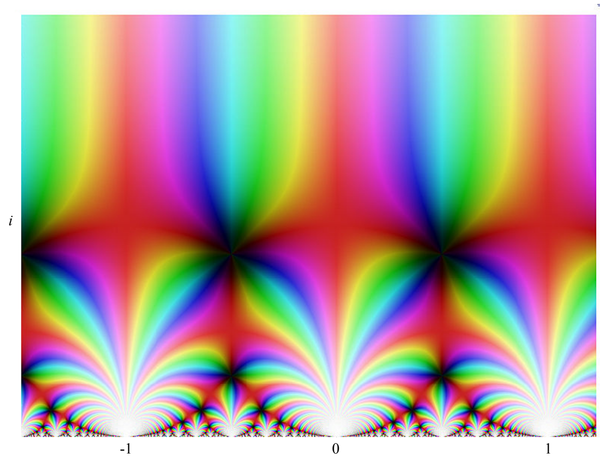
Since $\{e, v\}$ is a basis, though, we have necessarily $(e.v) = \pm 1$ (writing e and v with respect to the canonical basis of cycles of X_s, we see that $(e.v)$ equals the determinant of the base change matrix); hence we have:

$$A_{I_1} = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

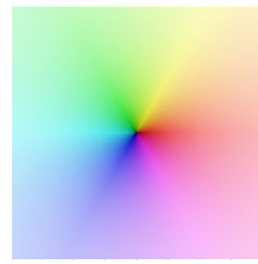
Note that the sign of $(e.v)$ is not well determined, since e and v have been given arbitrary orientation. One could try to study the induced orientation given by the complex structure on X, but we can avoid doing so using the results of subsection 3.3.3. There, we proved that under the assumption that the J-map is non-constant the monodromy can be interpreted, losing the distinction between $\pm A_{I_1}$, as the map $J_*\colon \pi_1(\Delta^*) \to \mathbb{P}\mathrm{SL}(2, \mathbb{Z})$. By inspection of table 3.4, we see that for X_0 of type I_1 we can always take $J(s) = s^{-1}$. If we take a loop in a counterclockwise direction around 0 and compose with J, we obtain in $\mathbb{C} \setminus \{0, 1\}$ a loop in positive direction around ∞, i.e. a loop around both 0 and 1 in a *clockwise* direction. We now have to take a lift to $\mathfrak{h} \setminus j^{-1}(\{0, 1\})$; since in any case we have a translation, for convenience we choose a starting point $\tau \in B$, where B is the fundamental region

$$B = \left\{ z \in \mathfrak{h} \mid |z| > 1,\ -\frac{1}{2} < \Re z < \frac{1}{2} \right\}.$$

Then we must have $A_{I_1} z = z \pm 1$ by the preceding, and an explicit calculation shows that it is $A_{I_1} = z + 1$: we simply have to note that if we take an arc $\gamma\colon [0, 1] \to \mathfrak{h}$ with $\gamma(t) = z + t$, where $\Im z$ is big enough (actually, $\Im z > 1$



(a) The j function



(b) Legend of colors (logarithmic scale)

Figure 3.1: The j function

will suffice), then the arc $j \circ \gamma$ is a loop winding once in clockwise direction. This can be done explicitly with the definition of the j function, and we have represented the values of j in the upper half plane in figure 3.1. We remark that, thanks to the identification with the action on $\mathbb{P}\mathrm{SL}(2, \mathbb{Z})$ we used Picard-Lefschetz formula only to determine that there is at least one cycle fixed by the monodromy action.

A similar discussion could be carried through without problems also for the I_b singular fiber type: they can be constructed via a Weierstrass equation, and we can calculate their monodromy with Picard-Lefschetz formula. However, it is more convenient to observe that fibers of type I_b are constructed starting from a fibration with a singular fiber of type I_1 and then taking b-th root fibration. The normalization of the fiber product $X \times_\Delta \Delta$ has a singularity of type A_{b-1} over the double point of X_0, which is resolved with a string of (-2)-curves meeting with dual graph A_{b-1} (cfr. [**?**], theorem III.7.1); if we also take into account the proper transform of the singular fiber X_0, we obtain b curves meeting with dual graph $\tilde{A}_{b-1}$, which is thus a fiber of type I_b. Alternatively, using Weierstrass fibrations, taking the b-th root fibration amounts to making the coordinate change $s = t^b$. From table 3.4, then, this leads to $J(t) = t^{-b}$, that is, a singular fiber of type I_b.

Either using Picard-Lefschetz formula or via the construction above, it follows that

$$A_{I_b} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

since one has either to consider all the vanishing cycles (and there are b of them) in the first case, or to base change of b-th order, that essentially means to wind up b times instead of one.

Again by inspection of table 3.4, taking the special construction of I_b as a base change of I_1 is not a restriction, since locally we can always assume this is the case.

3.5 Unstable fibers

In the first sub-section we deal with unstable non-multiple fibers. Again, the existence (together with a full classification) is guaranteed by table 3.4, so we concentrate on the study of monodromy. In the second sub-section we give constructions showing that multiple fibers of every possible type actually exist.

3.5.1 Germs without multiple fibers

In compiling table 3.4, we have listed all the possible non-multiple fibers occurring, and also the germs in which they can occur, giving explicit examples of each type; we now want to calculate the monodromy around those fibers. The result is shown in table 3.5: note in particular that it depends only on the fiber type, and not on the germ of the fiber. Furthermore, since there are no two distinct fiber types with the same (conjugacy class of) local monodromy, we instantly deduce the following by proposition (3.2.4):

Corollary 3.5.1. *Let $f\colon X \to S$ be an elliptic fibration. The germ of a fiber X_s is uniquely determined by $J(s)$, $\mathrm{m}_s(J)$ and the local monodromy around s.*

We have already seen that the monodromy around the semi-stable fibers is as indicated. To deduce the other ones, we use the n-th root fibration. Recalling its definition, when the base curve is simply Δ (as will be in our situation), this means simply to do the coordinate change $s = t^n$, to obtain another (in general singular) surface over the disk with coordinate t; then one

Table 3.5: Representatives for local monodromy

Fiber	Local Monodromy	Fiber	Local monodromy
I_n	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	I_n^*	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

normalizes the result and takes the minimal desingularization. If we start with a fibration in Weierstrass form with some coefficients $A(s)$, $B(s)$, then obviously the new coefficients will be $A(t^n)$, $B(t^n)$; this creates an algebraic surface that is locally a complete intersection and that has at most one singularity at the point $(x, y, t) = (0, 0, 0)$, so there is no need to normalize, and as we have noted to deduce the minimal form we only have to subtract $4k$ from a and $6k$ from b, with k maximum so that we still have non-negative values.

Furthermore, we have classified all possible normal forms above, so the complete classification follows with immediate calculation, and we report the result in table 3.6. Again, we note the fact that the fiber after we make the base change depends only on the singular fibers we have before, and not on the germ.

To check that the base change really operates as shown, using notations of table 3.4, we distinguish among several cases. The normal forms of I_0 clearly lead to another of the same type (possibly changing the multiplicity of J); those of I_b have functions J, A, B containing terms of the form $s^{\pm b}$, so that I_b is transformed to I_{nb}. Those of I_b^* ($b \geq 0$) differ for the preceding by a factor s^2 on the A and s^3 on the B; these can be canceled together if and only if we apply a base change with an even order n.

All the other cases are treated in the same way. If for example we take the germ of a singular fiber of type II, then we may write the A and B

Table 3.6: Fibers after taking the n-root

Before	n	After	Before	n	After
I_b	$n \geq 1$	I_{nb}	I_b^*	even	I_{nb}
				odd	I_{nb}^*
II	0 (mod 6)	I_0	II^*	0 (mod 6)	I_0
	1 (mod 6)	II		1 (mod 6)	II^*
	2 (mod 6)	IV		2 (mod 6)	IV^*
	3 (mod 6)	I_0^*		3 (mod 6)	I_0^*
	4 (mod 6)	IV^*		4 (mod 6)	IV
	5 (mod 6)	II^*		5 (mod 6)	II
III	0 (mod 4)	I_0	III^*	0 (mod 4)	I_0
	1 (mod 4)	III		1 (mod 4)	III^*
	2 (mod 4)	I_0^*		2 (mod 4)	I_0^*
	3 (mod 4)	III^*		3 (mod 4)	III
IV	0 (mod 3)	I_0	IV^*	0 (mod 3)	I_0
	1 (mod 3)	IV		1 (mod 3)	IV^*
	2 (mod 3)	IV^*		2 (mod 3)	IV

function of the associated Weierstrass fibration as $A = s^{n+1}$, $B = s$; then, after performing a base change of order 2 we obtain a fiber with $a = 2n+2$, $b = 2$, which by table 3.3 must be of type IV. If the order is 3, we obtain $a = 3n+3$, $b = 3$; again we have only one possibility, namely I_0^*. We go on in the same way, until we finally reach $b = 6$, which can be reduced to $b = 0$, that is a fiber of type I_b. But since $6a+6 > 4$, the only possibility is type I_0. The same method works equally well for all fiber types: in table 3.3 all types of fibers are in fact determined only by a and b, except the cases $a = b = 0$ that is common to all of the I_b and $a = 2$, $b = 3$, that is common to all of the I_b^*. But it is immediate to see that none of the fibers of type II to IV^* can be transformed to fibrations having such values of a and b, so we are done.

Now, let $T\colon H_1(X_s, \mathbb{Z}) \to H_1(X_s, \mathbb{Z})$ be the local monodromy around the singular fiber X_0. Clearly, passing to the n-th root fibration we obtain a monodromy that is equal to T^n, since winding once around the origin in the new fibration corresponds to winding n times in the starting one. If we call $A_{I_b}, A_{I_b^*}, A_{II}, \dots, A_{IV^*}$ the elements of $\mathrm{SL}(2,\mathbb{Z})$ corresponding to the monodromy action around fibers of type $I_b, I_b^*, II, \dots, IV^*$ respectively (for some choice of the normal form, that will result to be without consequences by our calculations), by table 3.6 we have:

$$\mathrm{ord}A_{II} = \mathrm{ord}A_{II^*} \,|\, 6; \quad \mathrm{ord}A_{III} = \mathrm{ord}A_{III^*} \,|\, 4; \quad \mathrm{ord}A_{IV} = \mathrm{ord}A_{IV^*} \,|\, 3;$$
$$\mathrm{ord}A_{I_0^*} \,|\, 2; \quad \mathrm{ord}A_{I_b^*} = \infty \text{ (if } b > 0.)$$

Furthermore, by theorem (2.6.2), since all these fibers are simply connected, there can be no invariant cycles. Hence none of the matrices above can be the identity, and we have immediately equalities for the order of $A_{I_0^*}$ and of A_{IV} and A_{IV^*}; this implies also that the order of A_{II} and A_{II^*} is 6, and that of A_{III} and A_{III^*} must be 4 since a quadratic base change leads to I_0^* which is not the identity.

It is well known (see for example [**?**]) that the modular group $\Gamma = \mathbb{P}\mathrm{SL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$ is generated by the classes of the two matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The elements S and ST obviously generate as well, but we have more: Γ is isomorphic to the free product of $\langle S \rangle$ and $\langle ST \rangle$. The order of ST is 3; one can check directly that this implies that the only torsion elements in Γ are conjugate to S, ST and $(ST)^{-1}$: if, for short, we denote by x the generator of C_2 and by y the generator of C_3, then if in $C_2 * C_3$ we have a relation $(y^{a_1} x y^{a_2} \dots y^{a_{k-1}} x y^{a_k})^n = 1$, it is clearly necessary that $a_1 \equiv 2a_k \pmod 3$. Then, by induction, one also obtains $a_2 \equiv a_{k-1}$, etc., so that if the element is not trivial there must be an odd number of terms, and the expression within brackets must be conjugate to the middle element, which is either x or y or y^2.

We may reinterpret this in $\mathrm{SL}(2,\mathbb{Z})$: if we have a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ that has a finite order n, then its class in Γ has order n if n is odd, and $n/2$ if n is even, since the only element of order 2 in $\mathrm{SL}(2,\mathbb{Z})$ is $-\mathbb{I}_2$, that is the trivial class in Γ. Then the only possibilities of orders of torsion in $\mathrm{SL}(2,\mathbb{Z})$ are 2, 3, 4 and 6. Furthermore, thanks to the explicit description in Γ, we can list them in $\mathrm{SL}(2,\mathbb{Z})$, too, and we do so in table 3.7.

It is easy to see that there are no other: the map $\mathrm{SL}(2,\mathbb{Z}) \to \Gamma$ is 2-to-1, so the 3 elements of torsion in Γ are lifted to 6 elements in $\mathrm{SL}(2,\mathbb{Z})$, plus we have the lift of the trivial class in Γ. Hence we have determined

Table 3.7: Elements of torsion in $\mathrm{SL}(2,\mathbb{Z})$

Order	Matrices	
2	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	
3	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$
4	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
6	$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$

the local monodromy around a fiber of type I_0^*, and to determine those of fibers of type $II, \dots, IV^*$ we only need to decide which matrix in each line correspond to the * fiber and which to the other. As in section 3.4, this is topologically a matter of orientation choice, since on each line the matrices are conjugated by an the element of $\mathrm{GL}(2,\mathbb{Z})$ that interchanges the elements of the basis. The same reasoning we did in that section with the J_* map allows us to deduce something, but not to conclude: first of all, the results developed in section 3.3 require that the J-map is not constant, fact that is not always assured for fibers of type $II \dots, IV^*$. Furthermore, also when it is non-constant the J-map distinguishes between II and II^* and between IV and IV^*, but not between III and III^*. When that method works, however, it is a straightforward calculation to show that the results are as cited in table 3.5, for example taking the fundamental region

$$\left\{\tau \in \mathfrak{h} \mid |\tau| > 1,\ -\frac{1}{2} < \Re(\tau) < \frac{1}{2}\right\}.$$

Then $J = 0$ corresponds to the two complex numbers on its border $\pm\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and, for example for a fiber of type II with multiplicity of J equal to $3n+1$ (cfr. table 3.4), winding once (in a counterclockwise direction) around the origin in the s-disk corresponds to winding $3n+1$ times around 0 in $\mathbb{C}\setminus\{0,1\}$, that in turn corresponds to winding n times around $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ plus a third of a circle, always in a counterclockwise direction. Computing explicitly the action of the 2 matrices A_{II} and A_{II^*} to a point $-\frac{1}{2}+\varepsilon+\left(\frac{\sqrt{3}}{2}+\varepsilon\right)i$, shows that the result has real part $< -\frac{1}{2}$ only for the matrix A_{II}, as required.

Da finire: restano i casi $II, \ldots, IV^*$ con J costante, che almeno in teoria si riescono a fare esplicitamente trovando una base orientata di X_1, scrivendo esplicitamente l'omeomorfismo di $f^{-1}(S^1 \setminus \{1\})$ con $(S^1 \setminus \{1\}) \times X_1$ e componendo con il trasporto lungo di questo per trovare come cambia la base; e i casi III e III^* che invece non so proprio come fare. Di certo la monodromia di III e quella di III^* sono diverse (una inversa dell'altra), ma bisognerebbe almeno mostrare che la monodromia non dipende da n, nella scrittura in forma normale di III

3.5.2 Monodromy of fibers I_b^*

By table 3.6, these are changed to fibers of type I_{2b} by base change of order 2. By what we have already shown, there, if $A_{I_b^*}$ is the matrix representing monodromy in $\mathrm{SL}(2,\mathbb{Z})$, modulo conjugacy we have:

$$A_{I_b^*}^2 = \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} \implies A_{I_b^*} = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

But we can apply theorem (2.6.2), since I_b^* is simply connected, so that there cannot be any fixed cycle. This forces:

$$A_{I_b^*} = -\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

3.5.3 Examples of multiple fibers

Lastly, we want to show the existence of all the multiple fibers that, according to theorem (3.1.1), are admissible. We start with type mI_0, i.e. fibers of type $X_0 = mF$ with F smooth elliptic. Take $Y = \mathbb{C} \times \Delta / L$, where $L = \mathbb{Z} + \mathbb{Z} \cdot z(s)$ is a collection of lattices given by a holomorphic function $z(s) = z_0 + c \cdot s^{mh}$, for some h positive integer, $c \in \mathbb{C}$ and $z_0 \in \mathfrak{h}$ with $\Im(z_0) > |c|$. In $\mathbb{C} \times \Delta$ we have the automorphism:

$$(c, s) \mapsto \left(c + \frac{1}{m}, \exp\left(\frac{2\pi i}{m}\right) s\right)$$

which generates a group of automorphisms isomorphic to $\mathbb{Z}/m\mathbb{Z}$, which descends to an action on Y (note that for this to be well defined we had to take the exponent of the function $z(s)$ to be a multiple of m), that is evidently free. Hence, the quotient is a smooth surface X. This is an elliptic surface with projection $[(c,s)] \mapsto s^m$, and the fiber over 0 is of type mI_0, because of the identification that, over 0, transforms the elliptic fiber with periods $\langle 1, z_0 \rangle$ in m copies of the elliptic curve with periods $\left\langle \frac{1}{m}, z_0 \right\rangle$.

Let us now turn our attention to fibers of type mI_b, $b > 0$. The idea of their construction is similar to that of mI_0, and we are going to explain it, but there are some technical details which would take us too far. In particular, we refer to [**?**], Chap. V, Sections 9 and 10 for the proof of the following fact: given a non-singular fibration Y over Δ, with singular fiber over 0 of type I_{mb}, call $Y_0^{\#}$ the set of its regular points, which is isomorphic as a group to $\mathbb{C}^* \times \mathbb{Z}/mb\mathbb{Z}$. Then there is, locally around Y_0, an action acting as $C_i \mapsto C_{i+b}$ on $Y_0^{\#}$ and as a translation on the nearby regular fibers. If one composes this action with the same automorphism as above:

$$(c,s) \mapsto \left(c, \exp\left(\frac{2\pi i}{m}\right) s \right),$$

one gets a group of automorphisms isomorphic to $\mathbb{Z}/m\mathbb{Z}$ acting without fixed points on Y. The quotient is then a smooth surface, and the fiber in 0 has clearly type mI_b, just as above.

3.6 Classification of elliptic surfaces without multiple fibers

Given a curve S, a meromorphic function J on it (taking values different from $0, 1, \infty$ on $S^* \subset S$) and a representation $G\colon \pi_1(S^*) \to \mathrm{SL}(2,\mathbb{Z})$ *belonging* to J (the *homological invariant*), we define $\mathcal{F}(J,G)$ to be the family of all the elliptic surfaces without multiple fibers with J-invariant J and homological invariant G, modulo isomorphism. The purpose of this section is to prove the existence of a unique element of $\mathcal{F}(J,G)$ (called the "basic member" by Kodaira) admitting a section. Furthermore, this element will allow us to classify all elliptic fibrations without multiple fibers on S with given invariants J, G.

In the following, whenever we have a meromorphic function J on S, we will indicate with $\{s_1, \dots, s_k\} = S \setminus S^* \neq \varnothing$ a chosen set of points of S such that for all $s \in S^*$, $J(s) \neq 0, 1, \infty$.

Lemma 3.6.1. *Fix a non-constant meromorphic function J on S. Then the number of homological invariants $G\colon \pi(S^*) \to \mathrm{SL}(2,\mathbb{Z})$ belonging to J is 2^{2g+k-1}.*

Proof. We simply have to observe that the fundamental group of a surface of genus $2g$ with k punctures is generated by $2g+k$ elements $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_k$ with the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdot \ldots \cdot a_g b_g a_g^{-1} b_g^{-1} c_1 \cdot \ldots \cdot c_k = 1.$$

Then, since $k > 0$, we can choose arbitrarily the images of all of the $a_i's$ and $b_i's$, and also of all of the $c_i's$ for $i = 1, \dots, k-1$; then the image of c_k can always be coherently chosen, and is uniquely determined. Fixed J, we have chosen all the images of the generator, up to the sign; hence the number of liftings is exactly 2^{2g+k-1}. □

Before stating the main theorem of this section, we note that the data of a conjugacy class of representation G is equivalent to that of an isomorphism class of locally constant sheaves over S^* (which thus we will also call G), with fibers isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$: the equivalence is given by considering the sheaf of abelian groups that over $s \in S^*$ has fiber $H^1(X_s, \mathbb{Z})$, with the monodromy action given by G.

Theorem 3.6.2. *Given a non-constant meromorphic function J on S and a representation $G\colon \pi_1(S^*) \to \mathrm{SL}(2,\mathbb{Z})$ belonging to J, there exists a unique minimal elliptic fibration with section $X = X(J,G)$ with J-map J and homological invariant G (the* basic member*). Furthermore this surface is algebraic.*

Before entering into the details of the proof, we prove a lemma of local uniqueness.

Lemma 3.6.3. *Let $f\colon X \to \Delta$ be an elliptic fibration such that all fibers are smooth, and with a given section σ. If ϕ is an automorphism of fibrations (i.e. a fiber preserving biholomorphism $X \to X$) fixing σ and acting trivially on all of the $H^1(X_s, \mathbb{Z})$, then $\phi = \mathrm{id}_X$.*

Proof. Consider on Δ the two sheaves $\mathcal{N} = R^1 f_* \mathbb{Z}_X$ and $\mathcal{E} = R^1 f_* \mathcal{O}_X$. Since Δ is simply connected, we have that the exponential sequence is exact on global sections, and passing to the long exact sequence we obtain the injection $\mathcal{N} \subseteq \mathcal{E}$. Furthermore, by corollary (2.4.5), $\mathcal{E}$ is locally free of rank 1, $\mathcal{E}_s \cong \mathbb{C}$, while $\mathcal{N}_s = H^1(X_s, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since every fiber is smooth, it is an abelian curve (with origin fixed by the section), hence

$$X_s \cong \mathrm{Pic}^0(X_s) = \mathcal{E}_s/\mathcal{N}_s.$$

So if ϕ fixes the origin and also fixes the lattice $\mathcal{N}_s$, it is the identity on each fiber, and thus $\phi = \mathrm{id}_X$. $\square$

Proof of theorem (3.6.2). The local existence is easy: first of all, if we are given a meromorphic map J, we can consider it to be given by two section $[s : t]$ of $\mathcal{L} = J^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then considering the Weierstrass data $(\mathcal{L}, -3t(t-s)s^2, 2t(t-s)^2s^3)$ we have:

$$J\big(-3t(t-s)s^2, 2t(t-s)^2s^3\big) = \frac{-108t^3(t-s)^3s^6}{-108t^3(t-s)^3s^6 + 108t^2(t-s)^4s^6} = \frac{t}{t-(t-s)} = \frac{t}{s} = J.$$

Fixed J, locally there are only 2 possible values for the conjugacy class of the monodromy representation, and the computations we have done in previous sections imply that for each type of J both are realized.

Local uniqueness is a consequence of the calculation of monodromy action: it is corollary (3.5.1). Hence we now want to focus on global aspects. Take a covering of open disks $\{U_i\}$ of S, chosen so that each of the points $\{s_1, \dots, s_k\}$ lies in exactly one of the U_i's, and that the intersection of any two or three disks is either empty or connected and simply connected. By the local statements, we can choose uniquely $f_i : X_i \to U_i$ with J-map equal to $J|U_i$, local monodromy given by the sheaf G and section σ_i. This fixes, out of singular points, an identification between the sheaf $R^1 f_{i*}\mathbb{Z}_{X_i}$ and the sheaf $G|U_i$; call it α_i.

Over $U_i \cap U_j$ we can apply local uniqueness result, since f_i and f_j have the same J-map and trivial monodromy; we obtain an isomorphism of fibrations $\phi_{ij}\colon X_j \to X_i$ mapping the section σ_j to σ_i, which also respects the identifications of sheaves above: $\alpha_i \circ \phi_{ij*} = \alpha_j$ on $U_j \cap U_i$. Moreover, the isomorphism ϕ_{ij} with these requirements is unique by lemma (3.6.3).

If we take three indices i, j, k, then $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki}$ is an automorphism of $X_i|(U_i \cap U_j \cap U_k)$, which again preserves the fibration, the section and acts as identity on the sheaf of $H^1(X_s, \mathbb{Z})$; hence it is the identity, again by lemma (3.6.3), and we can use these isomorphisms to glue together the

X_i's and obtain an elliptic section $f\colon X \to S$ as required. Furthermore this X is unique, since it must be unique after restriction at each U_i and the isomorphisms used to glue them together are unique as well.

The surface X thus constructed is algebraic by proposition (3.2.1). □

We conclude the section with the classification of elliptic surfaces without multiple fibers on a curve S having given invariants J and G:

Proposition 3.6.4. *Let J and G be as above, with G belonging to J. Then there is a bijection*

$$\mathcal{F}(J,G) \leftrightarrow H^1(S, \Sigma(X(J,G)))$$

where $\Sigma(X(J,G))$ is the sheaf of abelian groups of local sections of $X(J,G)$.

Proof. The sheaf in the statement is a sheaf of abelian groups since $X(J,G)$ is a smooth surface, so each section must meet fibers at their regular points.

Let $X, B \in \mathcal{F}(J,G)$ be elliptic surfaces over a curve S with given invariants, B being the basic member of the family. We choose a covering $\{U_i\}$ of S as in the proof of the theorem, i.e. so that each $s_i \notin S^*$ lies in exactly one U_i, and for each i,j,k, $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$ are connected and simply connected. The fibrations $X|U_i$ and $B|U_i$ have the same J-map and monodromy, hence there is an isomorphism

$$c_i\colon B|U_i \to X|U_i.$$

This isomorphism is not unique, since it depends on the choice of a local section (equivalently of B or X). Clearly, if we put $\phi_{ij} = c_j^{-1}c_i$, we obtain a 1-cocycle of the sheaf of local sections of B.

The described correspondence is bijective: if we start with a 1-cocycle ϕ_{ij}, we can use it to construct $X \in \mathcal{F}(J,G)$ as in the theorem above, starting from the unique $B|U_i$ and glueing together the local sections by means of ϕ_{ij}; cocycle relations ensure that we obtain a well-defined notion. Modifying the cocycle by a coboundary results in modifying X by an isomorphism. □

3.7 Elliptic fibrations in Enriques-Kodaira classification

We start this section by proving the canonical bundle formula. Then we use it to obtain information on the invariants of an elliptic surface, to place it inside the Enriques-Kodaira classification, in particular understanding the possible Kodaira dimension of the surface. In this section, $f\colon X \to S$ will always be a relatively minimal elliptic fibration, and we will call its multiple fibers $X_{s_1} = m_1F_1, \dots, X_{s_k} = m_kF_k$.

Theorem 3.7.1 (Canonical bundle formula)**.** *With the above notations, the canonical bundle is given by the formula:*

$$K_X = f^*\left(K_S \otimes (R^1 f_*\mathcal{O}_X)^\vee\right) \otimes \mathcal{O}_X\left(\sum (m_i - 1)F_i\right)$$

Proof. We apply relative duality (theorem (2.5.3)) to the sheaf $\mathcal{O}_X$, and tensor with K_S:

$$K_S \otimes (R^1 f_*\mathcal{O}_X)^\vee = K_S \otimes f_*\omega_{X/S} = f_*K_X,$$

where the last equality comes from the projection formula. Then we can take f^* on both sides, we have:

$$K_X = f^*(K_S \otimes (R^1 f_*\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(D),$$

where D is the zero divisor of the natural morphism

$$\lambda\colon f^*(f_*K_X) \to K_X.$$

Since K_X is the sum of irreducible components of fibers whose image in S forms a discrete set, in a neighborhood of X_s we can take D_s as a canonical divisor. Then for each irreducible component C of X_s, $D_s.C = 0$: if X_s is irreducible this comes directly from adjunction formula, while if it is not then we must also recall that C is a (-2)-curve by theorem (3.1.1). In particular, $D_s^2 = 0$, and by Zariski's lemma (2.1.5) D is a (rational) multiple of X_s. Hence, if X_s is not a multiple fiber, we must have $D_s = mX_s$; but this implies that $K_X|X_s$ is a multiple of X_s, hence $\lambda|X_s$ is an isomorphism, and so $D_s = 0$. We have thus proved that D has support contained in that of the multiple fibers.

For similar reasons, λ cannot vanish on X_s to an order greater or equal to the multiplicity of the fiber, i.e. $D = \sum n_iF_i$, with $n_i < m_i$. Otherwise, every 2-form on the neighborhood of X_s of the form $f^{-1}(U)$, being of type $f^*\omega$, with $\omega \in \Gamma(U, f_*K_X)$, would vanish on X_s of order at least the multiplicity of X_s. But this is absurd, since we can take one of minimal vanishing order, and divide it by an equation for the fiber X_s.

By the adjunction formula, then:

$$\omega_{F_i} = K_X \otimes \mathcal{O}_{F_i}(F_i) = \mathcal{O}_{F_i}(F_i)^{\otimes(n_i+1)}.$$

Since F_i is a curve of type I_{b_i}, $b_i \geq 0$, its canonical bundle is trivial (trovare una referenza nel caso $b_i > 0$). By lemma (2.2.6), $\mathcal{O}_{F_i}(F_i)$ is a torsion bundle of order m_i; so $m_i \mid (n_i + 1)$, which implies equality, $n_i = m_i - 1$. □

Proposition 3.7.2. *We have the equality:*

$$\deg\left((R^1 f_* \mathcal{O}_X)^\vee\right) = \chi(\mathcal{O}_X)$$

Proof. By [?] there is a first quadrant spectral sequence

$$E_2^{p,q} \cong H^p(S, R^q f_* \mathcal{O}_X) \Longrightarrow H^{p+q}(X, \mathcal{O}_X)$$

This spectral sequence has non-zero terms only in places $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, hence degenerates at E_2 level. We inspect it diagonal by diagonal: for $p + q = 0$ it simply leads to $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_S)$, which we already knew. The diagonal $p + q = 2$ gives $H^2(\mathcal{O}_X) \cong H^1(R^1 f_* \mathcal{O}_X)$, and finally for $p + q = 1$ we obtain the exact sequence:

$$0 \to H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_X) \to H^0(R^1 f_* \mathcal{O}_X) \to 0. \tag{3.6}$$

Using the additivity of the χ, this implies

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) - \chi(R^1 f_* \mathcal{O}_X),$$

and the result follows applying the Riemann-Roch theorem on the two invertible sheaves $\mathcal{O}_S$ and $R^1 f_* \mathcal{O}_X$. □

Proposition 3.7.3. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal elliptic fibration over $\mathbb{P}^1$. Then:*

1. *we have* $\mathrm{kod}(X) = 0 \iff K_X^{\otimes 12} \cong \mathcal{O}_X$*;*
2. *we have* $\mathrm{kod}(X) = -\infty \iff P_{12}(X) = 0$.

Proof. 1. Clearly, if $K_X^{\otimes 12} = \mathcal{O}_X$, then $\mathrm{kod}(X) = 0$, so we only have to prove the converse. Let us call $\mathcal{N}$ the sheaf appearing in theorem (3.7.1), $\mathcal{N} = f^*(K_s \otimes (R^1 f_* \mathcal{O}_X)^\vee)$. Then, by proposition (3.7.2), we have $\deg \mathcal{N} = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) - 2$. Furthermore, all points on $\mathbb{P}^1$ are linearly equivalent; if we take n to be multiple of all of the multiplicity of fibers m_i, so that nK_X is made only of multiple of fibers, we can write

$$nK_X \sim n\left(\chi(\mathcal{O}_X) - 2 + k - \sum_{i=1}^{k} \frac{1}{m_i}\right) X_s. \tag{3.7}$$

If the number within brackets were strictly positive, by Riemann-Roch on S, $P_n(X)$ would tend to infinity, i.e. $\mathrm{kod}(X) \geq 1$. But it being strictly negative would imply $\mathrm{kod}(X) = -\infty$, since for all n multiple of $m = \mathrm{lcm}(m_i)$ we would have $|nK_X| = \varnothing$; hence:

$$\chi(\mathcal{O}_X) - 2 + k - \sum_{i=1}^{k} \frac{1}{m_i} = 0 \tag{3.8}$$

By proposition (3.7.2) and theorem (2.5.4), we have $\chi(\mathcal{O}_X) \geq 0$, and the equality holds if and only if either the fibration is locally trivial, or it has no singular fibers apart multiple ones. Since $k - \sum_i 1/m_i > 0$ if there is at least a singular fiber, there is only a finite number of cases possibly occurring: first of all, either $\chi(\mathcal{O}_X) = 0$ or $\chi(\mathcal{O}_X) = 1$. Using $\sum 1/m_i \leq k/2$,

$$\chi(\mathcal{O}_X) = 1 \implies k = 2,\ m_1 = m_2 = 2.$$

If $\chi(\mathcal{O}_X) = 0$, from $\sum(1 - 2/k - 1/m_i) = 0$ it follows that $3 \leq k \leq 4$; furthermore,

$$\chi(\mathcal{O}_X) = 0,\ k = 4 \implies m_1 = m_2 = m_3 = m_4 = 2.$$

If $k = 3$ we have a few more possibilities, namely the following are admissible:

$$\chi(\mathcal{O}_X) = 0,\ k = 3 \implies \begin{cases} m_1 = m_2 = m_3 = 3; \\ m_1 = 2,\ m_2 = m_3 = 4; \\ m_1 = 2,\ m_2 = 3,\ m_3 = 6 \end{cases}.$$

Since in each case $\mathrm{lcm}(m_i) \,|\, 12$, we have always $K_X^{\otimes 12} = \mathcal{O}_X$.

2. Again, one direction is obvious: if $\mathrm{kod}(X) = -\infty$, then $P_{12}(X) = 0$. Suppose now that $P_{12} = 0$. Proceding as above, we may write

$$12K_X \sim (12\chi(\mathcal{O}_X) - 24)\, X_s + 12\sum_{i=1}^{k}(m_i - 1)F_i.$$

We want to split the last sum in a combination of fibers and a "residual" part $\sum r_i F_i$, with $r_i < m_i$. To this aim, we take l_i to be the least positive integer such that $l_i m_i \geq 12$ (this implies $l_i \in \{1, 2, 3, 4, 6\}$), and set

$$r_i = l_i m_i - 12 < m_i$$

In this way we obtain $12(m_i - 1) = m_i(12 - l_i) + r_i$, so that

$$12K_X \sim (12\chi(\mathcal{O}_X) - 24 + \sum_{i=1}^{k}(12 - l_i))X_s + \sum_{i=1}^{k} r_i F_i.$$

Since $P_{12}(X) = 0$ and $r_i \geq 0$, the term within brackets must be negative, i.e.

$$\sum(12 - l_i) < 24 - 12\chi(\mathcal{O}_X). \tag{3.9}$$

Recalling $l_i \leq 6$ and $\chi(\mathcal{O}_X) \geq 0$ (by proposition (3.7.2) and theorem (2.5.4)), this implies $k \leq 3$. If $k = 2$, the inequality implies $\chi(\mathcal{O}_X) = 0$, too, hence we obtain in (3.7) that the term within bracket is strictly negative; then $\mathrm{kod}(X) = -\infty$. If $k = 3$, (3.9) implies $l_1 + l_2 + l_3 > 12 + 12\chi(\mathcal{O}_X)$, hence $\chi(\mathcal{O}_X) = 0$ and, assuming $l_1 \geq l_2 \geq l_3$, we have $l_1 \geq 6$ and $l_2 \geq 4$. We are left with three possibilities:

$$\begin{aligned}
&l_1 = l_2 = 6 \implies m_1 = m_2 = 2;\\
&l_1 = 6, l_2 = l_3 = 4 \implies m_1 = 2, m_2 = m_3 = 3;\\
&l_1 = 6, l_2 = 4, l_3 = 3 \implies m_1 = 2, m_2 = 3, m_3 \in \{4, 5\}.
\end{aligned}$$

In each of the three cases, $\sum 1/m_i > 1$, hence the term within brackets in (3.7) is negative again. □

We now want to relate the Kodaira dimension of an elliptic surface over a possibly non-rational curve S with some other numerical invariant. The discussion in the case of a rational base suggests us to take as invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(S) - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right)\right)$$

This is indeed a good choice:

Proposition 3.7.4. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration, with X compact. Then* $\mathrm{kod}(X) \leq 1$. *Furthermore:*

- $\operatorname{kod}(X) = -\infty \iff \delta(f) < 0$;
- $\operatorname{kod}(X) = 0 \iff \delta(f) = 0$;
- $\operatorname{kod}(X) = 1 \iff \delta(f) > 0$.

The following are sufficient conditions for X to have Kodaira dimension 1:

1. *the genus $g(S) \geq 2$;*
2. *the genus $g(S) = 1$, and f is not locally trivial.*

Proof. Let $m = \operatorname{lcm}(m_i)$, in the usual notation. Then, as above, $K_X^{\otimes \mu m}$ is a linear combination of fibers for each $\mu \in \mathbb{N}$. We may write $K_X^{\otimes \mu m} = f^*(D^{\otimes \mu})$, where

$$D = K_S^{\otimes m} \otimes (R^1 f_* \mathcal{O}_X)^{\otimes(-m)} \otimes \mathcal{O}_S \left(\sum_{i=1}^{k} (m_i - 1) \frac{m}{m_i} s_i \right).$$

In particular, $\deg D = m\delta(f)$. Since $h^0(f^* D^{\otimes \mu}) = h^0(D^{\otimes \mu})$, we have proved that in any case $\operatorname{kod}(X) \leq 1$. Furthermore, applying Riemann-Roch on S, we obtain readily that if $\delta(f) > 0$ then $\operatorname{kod}(X) = 1$.

If $\delta(f) < 0$, then for each $\mu \in \mathbb{N}$ we have $\deg(D^{\otimes \mu}) < 0$, so that $h^0(D^{\otimes \mu}) = 0$; hence $\operatorname{kod}(X) = -\infty$.

If $\delta(f) = 0$, for each $\mu \in \mathbb{N}$, either $D^{\otimes \mu}$ is trivial or $h^0(D^{\otimes \mu}) = 0$. Hence $\operatorname{kod}(X) \leq 0$. To complete the proof of the first statement, it only remains to show that if $\operatorname{kod}(X) = -\infty$, then $\delta(f) < 0$. One way to see this is by means of Iitaka conjecture $C_{2,1}$ (see theorem (1.3.3)): if $\operatorname{kod}(X) = -\infty$, since $\operatorname{kod}(X_s) = 0$ it must be $\operatorname{kod}(S) = -\infty$, i.e. S is rational; then we reduce to the conditions of proposition (3.7.3), in the proof of which we already noted that $\operatorname{kod}(X) = 0 \iff \delta(f) = 0$.

Let us prove 1. If $g(S) \geq 2$, then $\deg(K_S) > 0$; since we also know $\deg((R^1 f_* \mathcal{O}_X)^\vee) \geq 0$, we obtain $\deg D > 0$, and so also that $\operatorname{kod}(X) = 1$. Note that this fact follows immediately from IItaka conjecture, too (theorem (1.3.3)); actually, theorem (2.5.4), which we are using repeatedly, is a main step in the proof of Iitaka conjecture.

Let us now suppose $g(S) = 1$, so that K_S is trivial. If $\deg(D) \leq 0$, then actually it is an equality and we must also have $\deg(R^1 f_* \mathcal{O}_X)^\vee = 0$ and $\sum (m_i - 1)(m/m_i) = 0$. Hence we cannot have any singular fiber, and by theorem (2.5.4) the fibration must be locally trivial. By the first part of this proposition, then, under the hypotheses of 2, $\operatorname{kod}(X) = 1$. $\square$

Remark 3.7.5. The last statement of the proposition is the beginning of a classification of the type of an elliptic surface (in the sense of the classification of Enriques and Kodaira), using as basic invariant the genus of the base curve S. We have already seen that the surface is general elliptic if $g(S) \geq 2$ or $g(S) = 1$ and $f\colon X \to S$ is not locally trivial; if f is locally trivial, then several possibilities may happen: X could be an abelian surface, a primary Kodaira surface or a bi-elliptic surface. We refer to [**?**], chap. V, sec. 5, for a proof. The case of $g(S) = 0$ was treated in proposition (3.7.3); in the case of X having a section, we will see in next chapter a complete list of the possible surfaces arising as Weierstrass fibrations: this includes trivial fibrations (i.e. product $X_s \times \mathbb{P}^1$), rational surfaces, $K3$ surfaces and properly elliptic surfaces, depending on the degree of the fundamental line bundle $\mathcal{L}$. This treatment, of course, deals only with the algebraic surfaces.

Chapter 4

Weierstrass fibrations

Questo verrà spostato come terzo capitolo. Manca l'introduzione

In this chapter by "surface" we will mean a normal reduced irreducible complex space of dimension 2. A "smooth surface" will be a connected complex manifold of dimension 2.

4.1 The Weierstrass equation

We recall from section 3.2 that a Weierstrass fibration is a reduced normal complex space of dimension 2 with a flat proper and surjective map to a smooth connected curve S, such that all fibers are irreducible of arithmetic genus 1, with a given section not meeting any singular point of any fiber. There, we proved that such surfaces are all algebraic and have no multiple fibers (cfr. proposition (3.2.1)), and constructed two maps F and G between the sets of smooth minimal elliptic surfaces over S with a section and that of the Weierstrass fibrations over S (both sets taken up to isomorphism): F contracts every component of the singular fibers which do not meet the section, and G resolves the singularities. By uniqueness of minimal resolution, $G \circ F$ is the identity, while we have called $F \circ G$ the process of "putting in minimal form". We also mentioned Weierstrass data to achieve the classification contained in the "$a-b-\delta$-table", and we began our study of Weierstrass fibrations by making explicit their construction.

4.1.1 Weierstrass equation for an elliptic curve

Any elliptic curve can be put into Weierstrass form. We recall its construction here because its direct generalization will supply to our necessities. Let E be such a curve, with a base point P. Call, for any $n \in \mathbb{N}$, $V_n = H^0(E, \mathcal{O}_E(nP))$; by Riemann Roch, we have

$$V_0 \cong V_1 \cong \mathbb{C}, \quad \dim V_n = n \ \forall n \geq 1.$$

We can interpret $V_0 = V_1$ as the constant functions on E, and V_n as the vector space of meromorphic functions having at most a pole of order n in p, and holomorphic elsewhere, so that naturally $V_i \subseteq V_{i+1}$. In this way we have a multiplication $V_i \otimes V_j \to V_{i+j}$ (that induces one from $\mathrm{Sym}^k V_i$ to V_{ki}).

Lemma 4.1.1. *1. There is $y \in V_3 \setminus V_2$ such that y^2 is in the image of $\mathrm{Sym}^3(V_2)$;*

2. there is $x \in V_2 \setminus V_1$ and some $a \in V_4$, $b \in V_6$ such that

$$y^2 = x^3 + ax + b;$$

3. If (x_1, y_1) and (x_2, y_2) are two pairs of elements satisfying (1) *and* (2)*, then there is a nonzero element $\lambda \in \mathbb{C}$ such that $x_2 = \lambda^2 x_1$ and $y_2 = \lambda^3 y_1$.*

Proof. With the above identifications, we can take as a base of V_0 and V_1 the constant function 1. Take any $f \in V_2 \setminus V_1$ and $g \in V_3 \setminus V_2$. Then $\{1, f\}$ is a base of V_2, $\{1, f, g\}$ of V_3, and $\{1, f, g, f^2, fg, f^3\}$ is a basis of V_6, since $f^2 \in V_4 \setminus V_3$, $fg \in V_5 \setminus V_4$ and $f^3 \in V_6 \setminus V_5$. The required equation is obtained by considering the expression of $g^2 \in V_6$ in terms of this base:

$$g^2 = a_6 f^3 + a_5 fg + a_4 f^2 + a_3 g + a_2 f + a_0. \tag{4.1}$$

Note also that $a_6 \neq 0$, since $g^2 \notin V_5$. It is easy to deduce from this the desired equation, first replacing f by $a_6 f$ and g by $a_6^2 g$ to get $a_6 = 1$, and then calling $y = g - \frac{a_5 f + a_3}{2}$ and $x = f + \frac{b_2}{3}$, thus proving (1) and (2). Furthermore, we obtain new bases $\{1, x\}$ of V_2, $\{1, x, y\}$ of V_3, and $\{1, x, x^2, x^3\}$ of $\mathrm{Sym}^3(V_2)$.

Let us now discuss uniqueness; if (x, y) are as above, clearly $(\lambda^2 x, \lambda^3 y)$ satisfy (1) and (2), as well. We have to prove that there are none else, so suppose (x_1, y_1) is another pair satisfying (1) and (2) (with some other A_1,

B_1 in an analogous of the equation in (2)). Writing $y_1 = \alpha y + \beta x + \gamma$, we obtain the relation

$$y_1^2 = \alpha^2(x^3 + Ax + B) + 2(\beta x + \gamma)y + (\beta x + \gamma)^2,$$

that belongs to $\mathrm{Sym}^3 V_2$ if and only if the coefficient of y vanishes, i.e. $\beta = \gamma = 0$. Hence y_1 is a multiple of y. Writing $x = \delta x + \epsilon$ and using the equation of (x_1, y_1), this leads to

$$\alpha^2(x^3 + Ax + b) = \alpha^2 y^2 = (\delta x + \epsilon)^3 + A_1(\delta x + \epsilon) + B_1.$$

The x^2 term on the right has to disappear, and since $\delta \neq 0$ we must have $\epsilon = 0$. Therefore also x_1 is a multiple of x, and comparing the coefficients of x^3 also $\delta^3 = \alpha^2$, i.e. $(x_1, y_1) = (\lambda^2 x, \lambda^3 y)$. $\square$

Corollary 4.1.2. *There are elements $A, B \in \mathbb{C}$ such that E is defined by the equation $y^2 = x^3 + Ax + B$. This pair is unique up to the action $\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B)$.*

Proof. Since the divisor P is ample on E, we have the isomorphism of affine curves

$$E \setminus \{P\} \cong \mathrm{Spec} R, \qquad R = \bigcup_{n=0}^{\infty} V_n.$$

We have bases $\{1, x, x^2, \ldots, x^m, y, xy, x^2y, \ldots, x^{m-2}y\}$ of V_{2m} and $\{1, x, x^2, \ldots, x^m, y, xy, x^2y, \ldots, x^{m-1}y\}$ of V_{2m+1}. Hence x and y generate R, and

$$R = \mathbb{C}[x, y] \big/ (y^2 - x^3 - Ax - B),$$

since it gives the correct Hilbert function for R (the y^2 terms can be substituted by a polynomial in x of the same degree).

The uniqueness is a consequence of the uniqueness statement for x and y: if we write another equation $y_1^2 = x_1^3 + A_1 x_1 + B_1$, by the lemma there exists $\lambda \in \mathbb{C}^*$ such that $(x_1, y_1) = (\lambda^2 x, \lambda^3 y)$. Substituting this values in the equation, we obtain $A_1 = \lambda^4 A$ and $B_1 = \lambda^6 B$. $\square$

Definition We call the pair (x, y) so defined a *Weierstrass basis* for (E, p). An equation of the form $y^2 = x^3 + Ax + B$ is called a *Weierstrass equation* for E. We define the *discriminant* as

$$\Delta = 4A^3 + 27B^2,$$

and call (A, B) the *Weierstrass coefficients* of E.

A direct computation shows that the since curve is nonsingular, then $\Delta = 0$, and also that any expression $y^2 = x^3 + Ax + B$ with $\Delta \neq 0$ gives a nonsingular elliptic curve.

4.1.2 The fundamental line bundle

Let $f\colon X \to S$ be a Weierstrass fibration. Thanks to the existence of a section, we can gather a lot of information about direct images of various sheaves on X. Let us start from the exact sequence for the normal bundle of the image of the section $S_0 \subset X$:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(S_0) \to N_{S_0/X} \to 0.$$

Applying f_*, we obtain the long exact sequence:

$$0 \to f_*\mathcal{O}_X \to f_*\mathcal{O}_X(S_0) \to f_*N_{S_0/X} \xrightarrow{\alpha} R^1f_*\mathcal{O}_X \to R^1f_*\mathcal{O}_X(S_0) \to 0, \quad (4.2)$$

with $R^1f_*N_{S_0/X} = 0$ since $N_{S_0/X}$ is supported on S_0, and the fibers of f restricted to S_0 have dimension 0. Furthermore, $h^1(\mathcal{O}_X(nS_0)|X_s) = 0$ for all $n \in \mathbb{N}$ and $s \in S$, and by Riemann Roch (which holds for the embedded curve X_s even when it is singular, or even not reduced: see [**?**], theorem II.3.1) also $h^0(\mathcal{O}_X(nS_0)|X_s) = n$ for each $n > 0$; hence theorem (2.2.10) applies, and for $n > 0$ the sheaf $f_*\mathcal{O}_X(nS_0)$ is locally free of rank n and $R^1f_*\mathcal{O}_X(nS_0) = 0$. Hence also the last term in (4.2) is zero, and α is surjective. Since $f_*N_{S_0/X}$ and $R^1f_*\mathcal{O}_X$ are line bundle on S, α must be an isomorphism:

$$\alpha\colon f_*N_{S_0/X} \cong R^1f_*\mathcal{O}_X,$$

thus proving (3.3). Recall that we call the *fundamental line bundle* of f the dual of this line bundle. Furthermore, since α is injective, we must have $f_*\mathcal{O}_X(S_0) \cong f_*\mathcal{O}_X \cong \mathcal{O}_S$.

For each n, we have $\mathcal{O}_X((n-1)S_0) \subseteq \mathcal{O}_X(nS_0)$, hence also $f_*\mathcal{O}_X((n-1)S_0) \subseteq f_*\mathcal{O}_X(nS_0)$. These are locally free sheaves, for $n \geq 1$, and are isomorphic to $\mathcal{O}_S$ if $n = 1$. For the other values of n, we have:

Lemma 4.1.3. *For every $n \geq 2$ there is a short exact sequence*

$$0 \to f_*\mathcal{O}_X((n-1)S_0) \to f_*\mathcal{O}_X(nS_0) \to \mathcal{L}^{-n} \to 0.$$

Proof. Start from the exact sequence

$$0 \to \mathcal{O}_X((n-1)S_0) \to \mathcal{O}_X(nS_0) \to \mathcal{O}_{S_0}(nS_0) \to 0,$$

and apply f_*. For $n \geq 2$, we have seen that $R^1 f_* \mathcal{O}_X((n-1)S_0) = 0$, so we have

$$0 \to f_*\mathcal{O}_X((n-1)S_0) \to f_*\mathcal{O}_X(nS_0) \to f_*\mathcal{O}_{S_0}(nS_0) \cong \mathcal{L}^{-n} \to 0$$

□

We now apply the results obtained in the previous subsection. Let us consider a single fiber, say over s; then $(f_*\mathcal{O}_X(3S_0))_s \cong H^0(\mathcal{O}_X(3S_0)|X_s) = V_3$, and here we can choose canonically 3 directions, namely those determined by 1, x and y. The latter two elements are not well defined, but since the indeterminacy lies only in a non-zero multiple, the directions they determine is. These directions give naturally a basis at every fiber, so determining a splitting of $f_*\mathcal{O}_X(3S_0)$. Furthemore, this splitting makes the exact sequence of lemma (4.1.3), in the case $n = 3$ split, too, since fiber by fiber the first map corresponds to the inclusion of the span of $1, x$, hence the cokernel is clearly generated by the class of y. More generally, we have the following:

Lemma 4.1.4. *For every $n \geq 2$ we have the splitting*

$$f_*\mathcal{O}_X(nS_0) \cong \mathcal{O}_X \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \oplus \ldots \oplus \mathcal{L}^{-n}.$$

Proof. On each fiber, for $n = 2$ the splitting is determined by 1 and x, and for $n = 3$ by 1, x and y, as we have already said. As noted, these splittings are exactly given by the maps in the exact sequence above, so that effectively

$$f_*\mathcal{O}_X(2S_0) \cong \mathcal{O}_X \oplus \mathcal{L}^{-2}, \quad f_*\mathcal{O}_X(3S_0) \cong \mathcal{O}_X \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}.$$

In general, if $n = 2m$ is even, the splitting is given by the spans of $1, x, x^2, \ldots, x_m, y, xy, \ldots, x^{m-2}y$, while if $n = 2m + 1$ is odd by $1, x, \ldots, x^m, y, xy, \ldots, x^{m-1}y$. Again, these basis determine the splitting of the exact sequence of lemma (4.1.3), so that the result is as stated. □

4.1.3 Weierstrass data

We are now in position to define the Weierstrass data $(\mathcal{L}, A, B)$ for a Weierstrass fibration $f\colon X \to S$.

Fix an open cover $\{U_i\}$ of S, so that $\mathcal{L}$ is trivialized on each U_i, and for each i pick a basis section e_i for $\mathcal{L}|U_i$. Then e_i^{-n} is a basis section for $\mathcal{L}^{-n}|U_i$. We call α_{ij} the transition functions for $\mathcal{L}$ with respect to the base $\{e_i\}$, i.e. $e_i = \alpha_{ij} e_j$ on $U_i \cap U_j$.

We start by choosing $f_i \in H^0(f_*\mathcal{O}_X(2S_0)|U_i)$ such that f_i projects to e_i^{-2}, and $g_i \in H^0(f_*\mathcal{O}_X(3S_0))$ that projects to e_i^{-3}, in terms of the decomposition of lemma (4.1.4). Then we find sections $a_0, a_2, \ldots a_6$ of $\mathcal{O}_{U_i}$ so that on U_i we have an equation formally analogous to (4.1); in this setting, it is an equality of sections of $f_*\mathcal{O}_X(6S_0)|U_i$. For our choice of f_i and g_i, projecting to $\mathcal{L}^{-6}|U_i$ one findes that $a_6 = 1$.

A local Weierstrass basis (x_i, y_i) is now obtained by completing the square in g_i and the cube in f_i as in the proof of lemma (4.1.1). Since this process affects only terms of order lower than 2 in g_i and 3 in f_i, the so-obtained basis still has the property that x_i projects to e_i^{-2} in $\mathcal{L}^{-2}|U_i$ and y_i to e_i^{-3} in $\mathcal{L}^{-3}|U_i$. Furthermore, as above, they are local generators for the direct summands of $f_*\mathcal{O}_X(3S_0)|U_i$ isomorphic to $\mathcal{L}^{-2}|U_i$ and $\mathcal{L}^{-3}|U_i$, respectively. Hence we have:

Lemma 4.1.5. *For each i there is a local Weierstrass basis (x_i, y_i) which transform by $x_i = \alpha_{ij}^{-2} x_j$ and $y_i = \alpha_{ij}^{-3} y_j$.*

The process of constructing the basis also gave us the local Weierstrass coefficients (A_i, B_i). On the intersection $U_i \cap U_j$ we can impose the transformation:

$$x_i^3 + A_i x_i + B_i = y_i^2 = \alpha_{ij}^{-6} y_j^2 = \alpha_{ij}^{-6}(x_j^3 + A_j x_j + B_j).$$

Hence A_i and B_i transform by $A_i = \alpha_{ij}^{-4} A_j$ and $B_i = \alpha_{ij}^{-6} B_j$. Therefore we obtain global sections A of $\mathcal{L}^{-4}$ and B of $\mathcal{L}^{-6}$ by patching together $\{A_i e_i^4\}$ and $\{B_i e_i^6\}$.

Definition The pair of sections (A, B) of $\mathcal{L}^4 \oplus \mathcal{L}^6$ are called the *Weierstrass coefficients* for the Weierstrass fibration $f\colon X \to S$. We call the *discriminant* of the fibration the section

$$\Delta = 4A^3 + 27B^2 \in H^0(\mathcal{L}^{12}).$$

Lemma 4.1.6. *Given a Weierstrass fibration $f\colon X \to S$, the discriminant is not identically zero; moreover, the Weierstrass coefficients are well defined up to the action of $H^0(S, \mathcal{O}_S^*)$ given by $\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B)$. Hence the discriminant is well defined up to a 12-th power in $H^0(S, \mathcal{O}_S^*)$.*

Proof. We have already seen that the discriminant vanishes if and only if the fiber is singular, and this cannot happen for any fiber (we have proved that on smooth surfaces the general fiber is smooth, and here it suffices to restrict the fibration to the points of S over which no singular point of X lies). The uniqueness can be checked locally, where it is the same as corollary (4.1.2). $\square$

Definition Let S be a smooth curve. A triple $(\mathcal{L}, A, B)$ where $\mathcal{L}$ is a line bundle on S and (A, B) are global sections of $\mathcal{L}^4 \oplus \mathcal{L}^6$ such that the section $\Delta = 4A^3 + 27B^2$ is not identically zero is called *Weierstrass data.*

A morphism between two Weierstrass data $(\mathcal{L}_1, A_1, B_1)$, $(\mathcal{L}_2, A_2, B_2)$ is a morphism of line bundles $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\phi^4(A_1) = A_2$ and $\phi^6(B_1) = B_2$.

We have proved that if $f\colon X \to S$ is a Weierstrass fibration over S, then the triple $(\mathcal{L}, A, B)$ where $\mathcal{L} = (R^1 f_* \mathcal{O}_X)^\vee$ and (A, B) are Weierstrass coefficients, is a Weierstrass data, and the zeros of the section Δ determine the singularity of the fibers. Conversely, given Weierstrass data $(\mathcal{L}, A, B)$ over S, we take a sufficiently fine open cover $\{U_i\}$ of S that trivialize $\mathcal{L}$, and construct local surfaces defined by the Weierstrass equations $y_i^2 = x_i^3 + A_i x_i + B_i$ (where x_i, y_i are local coordinates and A_i, B_i are the restrictions of A, B). Then we can patch them together, obtaining a Weierstrass fibration with the given Weierstrass data.

Given a Weierstrass fibration $f\colon X \to S$, the Weierstrass data it determines are (by lemma (4.1.6)) well determined up to isomorphism (of Weierstrass data), since in the definition of isomorphism if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ then an isomorphism $\mathcal{L} \to \mathcal{L}$ is multiplication by a never vanishing section of $\mathcal{O}_S$. Hence we have a bijection between the following sets, up to isomorphism on both sides:

$$\left\{ \begin{array}{c} \text{Weierstrass} \\ \text{data over } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Weierstrass} \\ \text{fibrations over } S \end{array} \right\}.$$

Lemma 4.1.7. *Let $(\mathcal{L}, A, B)$ be Weierstrass data over a compact curve S. Then $\deg \mathcal{L} \geq 0$. If $\deg \mathcal{L} = 0$, then either $\mathcal{L}^4$ or $\mathcal{L}^6$ is trivial, hence $\mathcal{L}$ is of torsion in $\mathrm{Pic}(S)$ of order 1, 2, 3, 4 or 6.*

Proof. The bundle $\mathcal{L}^{12}$ has the non-zero section Δ, so its degree must be non-negative. If it is zero, then since one between A and B must be non-zero (otherwise Δ would be as well), we have that either $\mathcal{L}^4$ or $\mathcal{L}^6$ is a locally free sheaf with degree 0 and a non-zero section, hence it is trivial. □

Lemma 4.1.8. *The number of singular fibers of a Weierstrass fibration over a compact curve S is 12 times the degree of the fundamental line bundle, counting properly.*

Proof. This is simply the degree of the discriminant Δ, i.e. the number of its zeros, multiplicities taken into account (this is the meaning of "counting properly"). □

4.2 Other ways of representing a Weierstrass fibration

In the first part of this section we discuss two more representation of a Weierstrass fibration (as a divisor in a $\mathbb{P}^2$ bundle over S and as a double cover of a ruled surface); then we discuss Weierstrass data in minimal form, and what it means to put a generic surface in minimal (or normal) form.

4.2.1 Representation as a divisor in a $\mathbb{P}^2$-bundle

Let $f\colon X \to S$ be a Weierstrass fibration, and call S_0 the divisor given by the section, $(\mathcal{L}, A, B)$ the associated Weierstrass data. We have seen that $f_*\mathcal{O}_X(3S_0) \cong \mathcal{O}_S \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$. Let $\phi\colon f^*f_*\mathcal{O}_X(3S_0) \to \mathcal{O}_X(3S_0)$ be the natural map; it is surjective as map of sheaves, since $\mathcal{O}_X(3S_0)$ is locally generated by sections 1, x, y, where x, y is a local Weierstrass basis.

Now, it is well known (cfr. [**?**], Proposition II.7.11) that if $f\colon X \to Y$ is a morphism and $\mathcal{E}$ is a locally free coherent sheaf on Y, then to give an invertible sheaf $\mathcal{N}$ on X and a surjective map of sheaves $f^*\mathcal{E} \to \mathcal{N} \to 0$ is the same as giving a morphism of X to $\mathbb{P}(\mathcal{E})$ as schemes over Y. In our case, calling $\mathcal{E} = f_*\mathcal{O}_X(3S_0)$ and $\mathcal{N} = \mathcal{O}_X(3S_0)$, we obtain a map $g\colon X \to \mathbb{P}(f_*\mathcal{O}_X(3S_0))$, which is a $\mathbb{P}^2$-bundle on S. Moreover, if p is the structure map $p\colon \mathbb{P} \to S$, then $p \circ g = f$.

In fact, in our circumstance g is an embedding: since it respects fibers, it suffices to check it on each fiber of f; we have maps $(g_s, g_s^{\#})\colon (X_s, \mathcal{O}_{X_s}) \to$

$(\mathbb{P}_s, \mathcal{O}_{\mathbb{P}_s})$, and since $\mathcal{O}_{X_s}$ is generated by the global sections 1, x, y the map on structure sheaves is surjective. Then, via g, we can see X as a divisor inside the $\mathbb{P}^2$-bundle $\mathbb{P}$ over S.

Let (A, B) be the Weierstrass coefficients for X over S. We obtain a global equation for X in $\mathbb{P}$ by $x_0x_1^2 = x_1^3 + Ax_0^2x_1 + Bx_0^3$, i.e. if we take an open cover on S that trivializes $\mathcal{L}$ and we restrict A and B on it, the resulting equation describes locally X inside $\mathbb{P}$. The section S_0 is defined by $x_0 = x_2 = 0$, since it corresponds to the unit in X_s as an abelian variety, that in Weierstrass notation is the point at infinity along the y axis. We can interpret (x_0, x_1, x_2) as global sections of $(\mathcal{O}_S, \mathcal{L}^{-2}, \mathcal{L}^{-3})$.

Proposition 4.2.1 (Canonical bundle formula)**.** *The canonical class of a Weierstrass fibration is*

$$\omega_X \cong f^*(\omega_S \otimes \mathcal{L}) \tag{4.3}$$

Proof. Since the surface X may be singular, one here has to use the adjunction formula to determine it, knowing the canonical bundle of $\mathbb{P}$. This is $p^*(\omega_S \otimes \mathcal{L}^{-5})(-3)$, then we obtain the stated formula. □

Corollary 4.2.2. *Let K_X be the canonical bundle of X. Then*

$$K_X^2 = 0$$

Proof. This is clear, since it is a linear combination of fibers. □

Lemma 4.2.3. *A Weierstrass fibration X is a product of S with a smooth elliptic curve if and only if $\mathcal{L} \cong \mathcal{O}_S$.*

Proof. If $\mathcal{L} \cong \mathcal{O}_S$, then $f_*\mathcal{O}_X(3S_0) \cong \mathcal{O}_S^3$, hence $\mathbb{P} \cong \mathbb{P}^2 \times S$; furthermore, A and B are constants, so X is the product of a smooth elliptic curve with S.

Conversely, if $X = E \times S$ is a product (with E an elliptic curve, say with base point e_0), we want to show that $\mathcal{L} \cong \mathcal{O}_S$. Note that the normal bundle N_{X/S_0} to the "horizontal" section $\{(e_0, s) \mid s \in S\}$ must be trivial since S_0 is a fiber of the projection on E. Now, $\mathcal{L}$ is the direct image of N_{X/S_0}, and it does not depend on the section chosen, so it is trivial, too. □

Remark 4.2.4. We do not need this for the proof, but actually the normal bundle to *any* section is trivial, since there is a fiber-preserving automorphism of X sending any given section to the horizontal one, given by the group structure.

Corollary 4.2.5. *The presence of non-zero global sections of $\mathcal{L}^{-1}$ distinguish whether X is a product or not. More precisely, we have:*

$$h^0(C, \mathcal{L}^\vee) = \begin{cases} 0 & \text{if } X \text{ is not a product} \\ 1 & \text{if } X \text{ is a product} \end{cases}$$

Proof. Since $\mathcal{L}$ has non-negative degree, if its dual has a non-zero section it must be the trivial bundle (and in that case, the dimension of the space of sections is obviously 1). By lemma (4.2.3), this determines whether X is a product or not. □

4.2.2 The representation as a double cover of a ruled surface

Let $R = \mathbb{P}(f_*\mathcal{O}_X(2S_0)) = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^{-2})$; then R is a ruled surface over S, and we call $q\colon R \to S$ the structure map. As above, the natural surjective map $f^*f_*\mathcal{O}_X(2S_0) \to \mathcal{O}_X(2S_0)$ gives a map $g\colon X \to R$ of varieties over S. Locally, X is, as usual, defined by $y^2 = x^3 + Ax + B$, where x is a section of $\mathcal{L}^{-2}$ and y of $\mathcal{L}^{-3}$; hence, the natural map $X \to R$ is given by sending $(x, y) \mapsto x$. Therefore, it is a double covering, branched over the trisection T defined in R (with homogeneous coordinate $[y_0 : y_1]$) by $y_1^3 + Ay_0^2y_1 + By_0^3 = 0$ and over the section given by $y_0 = 0$. Note that these two branch loci are disjoint.

We have the natural involution given by $(x, y) \mapsto (x, -y)$; on the fibers of f, this coincides with the inverse map in the group law of the elliptic curve. Therefore, we can write for short $R = X/\{\pm 1\}$.

The trisection T is a divisor in R with associated line bundle $(q^*\mathcal{L}^6)(3)$, and the section $y_0 = 0$ corresponds to the line bundle $\mathcal{O}_R(1)$, so the branch locus of g is a divisor with line bundle $(q^*\mathcal{L}^6)(4)$. The theory of cyclic coverings (cfr. [**?**], chap. I, par. 17) implies that if $g\colon X \to R$ is a double covering, and D is the branch divisor, then we can write $\mathcal{O}_Y(D) = \mathcal{N}^{\otimes 2}$, for some line bundle $\mathcal{N}$, and $g_*\mathcal{O}_X \cong \mathcal{O}_R \oplus \mathcal{N}^{-1}$. In our circumstance, this implies that $g_*\mathcal{O}_X \cong \mathcal{O}_R \oplus (q^*\mathcal{L}^{-3})(-2)$, the second factor being the sub-line bundle of $g_*\mathcal{O}_X$ locally generated by y.

When T intersects a fiber of R in 3 distinct points, the corresponding fiber of f is a smooth elliptic curve, and conversely: singular fibers are those where $\mathrm{d}f = 0$, and this happens if and only if the image of $\mathrm{d}g$ is contained in the $\mathbb{P}^1$-fiber of $Q\colon R \to S$, i.e. if the trisection T intersects it with multiplicity strictly greater than 1. We will see in next subsections that the local behavior of T and R determine the type of the singular fiber completely.

4.2.3 Weierstrass data in minimal form

We recall that a Weierstrass fibration is said to be in minimal form if and only if it is in the image of the map F from the set of smooth minimal elliptic fibrations with a section, that acts on every reducible fiber contracting the components not meeting the section. We have a map G on the other direction resolving singularities, that is only a one-sided inverse, i.e. $G \circ F = id$. In fact, F is not surjective, and in this subsection we precisely study what its range is. We start by proving proposition (3.2.3). We do so by means of the discussed representations, and the exact statement goes as follows:

Proposition 4.2.6. *Let $f\colon X \to S$ be a Weierstrass fibration over S, with Weierstrass data $(\mathcal{L}, A, B)$. The following are equivalent:*

1. *the fibration f is in minimal form;*
2. *the surface X has only rational double points;*
3. *the trisection T of the ruled surface $R = X/\{\pm 1\}$ has no triple tacnodes;*
4. *there is no point $s \in S$ such that $a = \mathrm{m}_s(A) \geq 4$ and $b = \mathrm{m}_s(B) \geq 6$.*

Proof. (1) $\implies$ (2): We already proved this (remark (3.2.2)).

(2) $\implies$ (1): Assume that $f\colon X \to S$ is not in minimal form. Then X cannot be smooth, obviously. Let X_s be a fiber in which a singularity of X lies, and from now on we work locally around this fiber. Call $\alpha\colon \tilde{X} \to X$ the minimal resolution of singularity. If the composition $f \circ \alpha\colon \tilde{X} \to S$ is relatively minimal, then α must simply be the contraction of some of the components of the singular fiber not meeting the section, and since by assumption X has only irreducible fibers, all of them must be contracted. Hence $f\colon X \to S$ would be in minimal form, that is excluded by hypothesis. So we may assume that $f \circ \alpha$ is not relatively minimal; call $\beta : \tilde{X} \to Y$ the minimalization (i.e. the map given by proposition (2.2.8)). This β must contract the proper transform of the original irreducible fiber X_s of f, since otherwise $\tilde{X}$ would not be the minimal desingularization of X.

Now we want to rebuild this construction from the other side. Starting from Y, we have a sequence of blow-ups that lead to $\tilde{X}$, and at some stage we must blow up the point of intersection of the section with the fiber of Y, otherwise β would not contract the proper transform of X_s. To go from $\tilde{X}$ to

X we have to contract all the components of the fiber $(\tilde{X})_s = \alpha^{-1}(f^{-1}(s))$, except that arising from the blow-up of the point that in Y is the intersection of the fiber and the section. Hence we are at least blowing down the entire fiber of Y, that has arithmetic genus one, so X has a singularity that is not a rational double point (they arise from contracting curves of type A-D-E, that have arithmetic genus 0).

(2) $\iff$ (3): It is well known (see [?], chap. III, par. 7) that if $g\colon X \to R$ is a double cover of surfaces, with R smooth, ramified over D, then X is smooth if and only if D is, and X has only rational double points (i.e. singularities of type A-D-E) if and only if D has only simple singularities, that are double or triple points whose strict transform under the subsequent blowups that resolve them have only double points at each stage. This can be rephrased by saying that the singularities of D have multiplicity smaller than 4, and there are no triple tacnodes. In our case, D has two disjoint components, T and $y_0 = 0$; this last component being smooth, we only need to apply the result to T, which is given by an equation of order 3, hence cannot have points of multiplicity 4 or more.

(3) $\iff$ (4): A direct computation shows that (3) implies (4): taken a local coordinate t near $s \in S$, such that t^4 divides A and t^6 divides B, then $x^3 + t^4C(t)x + t^6D(t) = 0$ has a triple tacnode at $(0,0)$. Indeed, we take (a chart of the) blowing up by putting $x = ut$; we then divide by t^3 to obtain the equation for the proper transform of T after blowing up the origin:

$$u^3 + ut^2C(t) + t^3D(t) = 0. \tag{4.4}$$

But this is again a triple point, against our assumption. Exactly the same computation also shows the converse: it is clear that in (4.4) if A had order less than 4 or B less than 6, we would had a double point (that, by the way, would correspond to one of the simple curve singularity E_7 or E_8), hence $(0,0)$ is not a tacnode. Of course, there can be no tacnodes in other points, either, since there is no x^2 term in the equation defining T. □

Due to this proposition, we will say that Weierstrass data $(\mathcal{L}, A, B)$ over S is in minimal form if for every $s \in S$, we have $\mathrm{m}_s(A) \leq 3$ or $\mathrm{m}_s(B) \leq 5$. As anticipated, we will refer to the process $F \circ G$ as putting the Weierstrass fibration into minimal (or normal) form. We now prove what this means in terms of Weierstrass data. We already mentioned (and used) this in chapter 3, where we said that locally, in terms of multiplicities a, b, one just had to do the substitution $a \mapsto a - 4$ and $b \mapsto b - 6$, as long as they both remain non-negative. More precisely, we have:

Lemma 4.2.7. *Let $f\colon X \to S$ be a Weierstrass fibration with data $(\mathcal{L}, A, B)$. For each $s \in S$, define $n_s = \max\{n \geq 0 \mid \mathrm{m}_s(A) \geq 4n \text{ and } \mathrm{m}_s(B) \geq 6n\}$. Let D be the divisor $D = \sum_s n_s s$, and f a section of $\mathcal{O}_S(D)$, with exactly D as locus of zeros. Then the Weierstrass data for the normal form of f is*

$$\left(\mathcal{L}(-D), \frac{A}{f^4}, \frac{B}{f^6}\right). \tag{4.5}$$

Proof. The definition is chosen so that $\frac{A}{f^4}$ and $\frac{B}{f^6}$ are holomorphic sections of, respectively, $\mathcal{L}(-D)^4$ and $\mathcal{L}(-D)^6$; furthermore, by the maximality of n_s, at each point the Weierstrass data in (4.5) are in minimal form. This data determines the unique Weierstrass fibration in minimal form birational to X: we may check uniqueness on regular fibers only, where it is precisely corollary (4.1.2). □

We may use this lemma to determine when the surfaces given by two set of Weierstrass data over a compact curve S determine birational surface. We recall that two projective varieties X, Y are said to be birational if there are an open subset $U \subseteq X$ and a morphism $\varphi\colon U \to Y$ which induces an isomorphism of function fields $K(Y) \cong K(X)$; we may as well take U to be the largest open on which it is defined. In the case of dimension 2, we have not used this definition when dealing with general surfaces, since it is equivalent to the definition of bimeromorphic surfaces we gave in chapter 1 only in the case of algebraic surfaces. To see this, it suffices to take $\Gamma_0 \subset U \times Y \subseteq X \times Y$, the graph of ϕ, and consider Γ, the normalization of the closure of Γ_0 in $X \times Y$. Since X and Y are embedded in a projective space, $X \times Y$ is, so that Γ is again a normal variety of dimension 2, thus implying that X and Y are bimeromorphically equivalent. Thanks to lemma (4.2.7), to check if two surfaces are birational it suffices to put both in normal form and check if the surfaces so obtained are isomorphic:

Corollary 4.2.8. *Two sets of Weierstrass data $(\mathcal{L}_1, A_1, B_1)$, $(\mathcal{L}_2, A_2, B_2)$ determine birational surfaces if and only if there are line bundles M_1, M_2 on S and sections $f_1 \in H^0(M_1)$, $f_2 \in H^0(M_2)$ such that*

$$(\mathcal{L}_1 \otimes M_1, A_1 f_1^4, B_1 f_1^6) \cong (\mathcal{L}_2 \otimes M_2, A_2 f_2^4, B_2 f_2^6).$$

Proof. If such bundles and sections exist, the surfaces are birational, again by corollary (4.1.2). Conversely, if the two sets of data determine isomorphic surfaces, they become the same when brought to normal form. Explicitly, there are divisors D_1, D_2 on S and sections f_1 of $\mathcal{O}_S(D_1)$, f_2 of $\mathcal{O}_S(D_2)$ such that

$$(\mathcal{L}_1(-D_1), A_1/f_1^4, B_1/f_1^6) \cong (\mathcal{L}_2(-D_2), A_2/f_2^4, B_2/f_2^6).$$

Tensoring both sides with $\mathcal{O}_S(D_1+D_2)$ and multiplying the A_i's by $f_1^4 f_2^4$ and the B_i's by $f_1^6 f_2^6$, we have the thesis, with $M_1 = \mathcal{O}_S(D_2)$, $M_2 = \mathcal{O}_S(D_1)$. □

4.3 Weierstrass fibration in Enriques-Kodaira classification

In this section we compute the standard invariants of a Weierstrass fibration over a compact curve S, in order to find their place inside the classification of Enriques and Kodaira. We have already determined Kodaira dimension without the assumption of the fibration having a section, but here we see that under this assumption it is easy to get a much finer classification.

We start from the determination of the $h^i(\mathcal{O}_X)$.

Proposition 4.3.1. *Let $f\colon X \to S$ be a Weierstrass fibration, and call g the genus of S. Then the irregularity and the geometrical genus of X are given by:*

$$q = h^1(X, \mathcal{O}_X) = \begin{cases} g & \text{if } X \text{ is not a product} \\ g+1 & \text{if } X \text{ is a product} \end{cases}$$

$$p_g = h^2(X, \mathcal{O}_X) = \begin{cases} g + \deg \mathcal{L} - 1 & \text{if } X \text{ is not a product} \\ g + \deg \mathcal{L} & \text{if } X \text{ is a product} \end{cases}$$

In particular, $\chi(\mathcal{O}_X) = \deg \mathcal{L}$.

Proof. We have already proved all that is necessary to deduce this conclusions in proposition (3.7.2): computing dimensions in short exact sequence (3.6), given by the Leray spectral sequence, we obtain $q = h^1(\mathcal{O}_S) + h^0(R^1 f_* \mathcal{O}_X)$. The first summand is g, and we have computed the second one corollary (4.2.5). Applying proposition (3.7.2),

$$p_g(X) = \chi(\mathcal{O}_X) + q(X) - 1 = q(X) + \deg \mathcal{L} - 1,$$

and substituting the value of $q(X)$ found above we have the result. □

Using Noether's formula together with this proposition and corollary (4.2.2), we have the formula for the Euler number (i.e. the second Chern number):

$$\mathrm{e}(X) = 12 \deg \mathcal{L}. \tag{4.6}$$

We now calculate the plurigenera. We already know the value of $P_1 = p_g$; for the remaining cases, then:

Proposition 4.3.2. *Let $f\colon X \to S$ be a Weierstrass fibration with base curve S of genus g. For $n \geq 2$, the plurigenera of X are:*

1. *If $g = 0$ then*

$$P_n(X) = \begin{cases} 0 & \text{if } \deg \mathcal{L} \leq 1 \\ 1 + n(k-2) & \text{if } \deg \mathcal{L} = k \geq 2 \end{cases}$$

2. *If $g = 1$ and $\deg \mathcal{L} = 0$, call $t \in \{1, 2, 3, 4, 6\}$ the order of torsion of $\mathcal{L}$ in $\mathrm{Pic}(S)$ (cfr. lemma (4.1.7)). Then*

$$P_n(X) = \begin{cases} 1 & \text{if } t \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

   If instead $g = 1$ and $\deg \mathcal{L} \geq 1$, then

$$P_n(X) = n \deg \mathcal{L}$$

3. *Finally, if $g \geq 2$,*

$$P_n(X) = n(2g - 2 + \deg \mathcal{L}) + 1 - g.$$

Proof. By applying the canonical bundle formula (4.2.1),

$$\begin{aligned} P_n(X) &= h^0(X, \omega_X^{\otimes n}) = h^0(X, f^*(\omega_S \otimes \mathcal{L})^{\otimes n}) = h^0(S, \omega_S^{\otimes n} \otimes \mathcal{L}^{\otimes n}) = \\ &= \chi(\omega_S^{\otimes n} \otimes \mathcal{L}^{\otimes n}) + h^1(\omega_S^{\otimes n} \otimes \mathcal{L}^{\otimes n}). \end{aligned}$$

We now apply Riemann Roch to the first summand, and duality to the second one, and obtain:

$$P_n(X) = n(2g - 2 + \deg \mathcal{L}) + 1 - g + h^0(\omega_S^{1-n} \otimes \mathcal{L}^{-n}).$$

This directly gives the thesis if $g \geq 2$ or $g = 1$ and $\deg \mathcal{L} \geq 1$, since in that case the line bundle $\omega_S^{1-n} \otimes \mathcal{L}^{-n}$ has negative degree, hence it can have no global holomorphic section. If $g = 1$ and $\deg \mathcal{L} = 0$, we have $P_n(X) = n \deg \mathcal{L} + h^0(\mathcal{L}^{-n})$, which is the desired equation (the only line bundle with $\deg = 0$ and a global holomorphic section is the trivial one). Finally, if $g = 0$ the line bundles are determined by their degree only; if $\deg \mathcal{L} \geq 2$ the bundle $\omega_S^{1-n} \otimes \mathcal{L}^{-n}$ has no sections, and this leads to the thesis; if $\deg \mathcal{L} \leq 1$, by Riemann Roch $h^0(\omega_S^{1-n} \otimes \mathcal{L}^{-n}) = n + \deg \mathcal{L} - 2$, hence $P_n(X) = 0$, as required. □

We can now determine the position of Weierstrass fibration inside Enriques-Kodaira classification:

Proposition 4.3.3. *According to the genus of the base curve and the degree of the canonical bundle, the surface X is of the following type:*

1. *Let $g = 0$. Then X is:*
   - *a product $E \times \mathbb{P}^1$ if* $\deg \mathcal{L} = 0$*;*
   - *a rational surface if* $\deg \mathcal{L} = 1$*;*
   - *a $K3$ surface if* $\deg \mathcal{L} = 2$*;*
   - *a properly elliptic surface if* $\deg \mathcal{L} \geq 3$.
2. *Let $g = 1$. Then X is:*
   - *a product (i.e. an abelian surface) if $\mathcal{L} \cong \mathcal{O}_S$;*
   - *a bi-elliptic surface if $\mathcal{L}$ is torsion of order* 2, 3, 4 *or* 6*. In this case, the order of K_X is the same as that of $\mathcal{L}$;*
   - *a properly elliptic surface if* $\deg \mathcal{L} \geq 1$.
3. *Let $g \geq 2$. Then X is a properly elliptic surface.*

Proof. We have already seen that (3) holds (cfr. proposition (3.7.4)). However, this follows clearly from proposition (4.3.2), and from the same proposition it also follows that the classification for properly elliptic surfaces (i.e. those with Kodaira dimension 1) is as stated.

If $g = 0$, $\deg \mathcal{L} = 2$ then the plurigenera are definitively equal to 1, hence the Kodaira dimension is 1; if $\deg \mathcal{L} \leq 1$, then $\mathrm{kod}(X) = 0$, and we can distinguish between rational surfaces and ruled surfaces over an elliptic curve either by the means of $c_2(X)$ or $b_1(X) = 2q(X)$. Finally, if $\deg \mathcal{L} = 2$, X is a surface with Kodaira dimension equal to 0 and $c_2 = 24$, i.e. a $K3$-surface.

If $g = 1$ and $\deg \mathcal{L} = 0$, then the Kodaira dimension is clearly 0; since $c_2(X) = 0$ it may be a bi-elliptic surface or a torus, and we conclude recalling that X is a product if and only if $\mathcal{L} \cong \mathcal{O}_S$ (lemma (4.2.3)). □

In the rational case, we have an even more explicit description of the surface. We can construct examples of rational elliptic surfaces in the following way: we take a pencil of cubics in $\mathbb{P}^2$, let's say $\{\lambda C + \mu D\}$, with smooth general element; this pencil has 9 base points (counted with multiplicities), and we blow them up. After doing so we are left with a space X obtained as $\mathbb{P}^2$ with 9 points blown up, and each point of X lies in exactly one cubic of the form $\lambda C + \mu D$; sending this point to $[\lambda, \mu] \in \mathbb{P}^1$ gives a surjective map to $\mathbb{P}^1$, whose fiber is an elliptic curve. We see now that every rational elliptic surface arises in this way:

Lemma 4.3.4. *Let $f\colon X \to \mathbb{P}^1$ be a rational elliptic surface with section. Then X coincides (as a fibration) with a 9-fold blowup of the plane $\mathbb{P}^2$ at the base points of a pencil of generically smooth cubic curves.*

Proof. Let $\varphi\colon X \to M$ the blow-down of X to a minimal model M (i.e. the contraction of all the (-1)-curves). As mentioned in chapter 1, while speaking of the Enriques-Kodaira classification, the minimal models for rational surfaces are completely classified, and consist of the Hirzebruch surfaces Σ_n, for n a non-negative integer different from 1, plus $\mathbb{P}^2$; hence M must be one of these.

The canonical bundle formula (proposition (4.2.1)) yelds $K_X \cong f^*(\omega_{\mathbb{P}^1} \otimes \mathcal{L}) \cong f^*((-2 + \deg \mathcal{L})s) = -X_s$, where X_s is any fiber, since all points in $\mathbb{P}^1$ are linearly equivalent. Hence for every rational curve E on X, adjunction formula yelds $E^2 \geq -2$; so M cannot be any of the Σ_n with $n \geq 3$ (since the section of Σ_n has auto-intersection $-n$). If M is $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then any blow-up of M can be blown-down to $\mathbb{P}^2$, as well; while if it is Σ_2 then the blow-ups needed to get to X cannot be in the section (since it has auto-intersection -2, and its proper transform would then have auto-intersection (-3)), and any blow-up of Σ_2 not at the section can again be blown down to $\mathbb{P}^2$. Therefore, since X is not itself ruled, we may assume that $M = \mathbb{P}^2$.

The pencil $|X_s|$ on X descends to a pencil on $\mathbb{P}^2$; this has generically smooth element, since the same holds for $|X_s|$, and is contained in $|-K_{\mathbb{P}^2}|$, hence it is made of cubics. It is clear that blowing up the base points and constructing the elliptic surface as above gives us X back again. $\square$

We end this section by computing the Hodge diamond of X.

Lemma 4.3.5. *In the same notations as above, if X is not a product than the Hodge diamond is:*

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g+\deg\mathcal{L}-1 & & 10\deg\mathcal{L}+2g & & g+\deg\mathcal{L}-1 \\ & g & & g & \\ & & 1 & & \end{array}$$

If X is a product, the Hodge diamond is:

$$\begin{array}{ccccc} & & 1 & & \\ & g+1 & & g+1 & \\ g+\deg\mathcal{L} & & 10\deg\mathcal{L}+2g+2 & & g+\deg\mathcal{L} \\ & g+1 & & g+1 & \\ & & 1 & & \end{array}$$

Proof. Since for Kähler surfaces $h^{0,1} = h^{1,0}$, in both cases the first two lines are computed, hence by duality also the last two. We also computed $p_g(X) = h^{0,2} = h^{2,0}$, so only $h^{1,1}(X)$ is still unknown. But this follows since the sum with alternating signs of the Hodge numbers is the Euler number $\mathrm{e}(X)$. $\square$

4.4 The a-b-δ table

The purpose of this section is to prove the a-b-δ table (3.3). To achieve this result, we will show that, when regarding a Weierstrass fibration as a double cover of a ruled surface, the singularity type of the fiber is determined by the singularities of the trisection and the ways it intersects the fiber of the ruled surface R. Then we will relate the nature of this intersection with the singularity type on the Weierstrass fibration Y, that in turn gives us information on the fiber type in the smooth minimal elliptic fibration X.

In this section, we will distinguish more carefully between a smooth minimal elliptic surface and the Weierstrass fibration it determines. So, we will call $X \to S$ a smooth minimal elliptic surface with section, and Y the surface obtained after collapsing the components of fibers not meeting the section. Again, let $R = Y/\{\pm 1\}$, and call T the trisection (a divisor in R), and Z the section in R. The union of these two divisors is the branch locus. Then, by proposition (4.2.6), T has no triple tacnodes, hence it has singularities of type *a-d-e* (see [**?**], chap. II, par. 8 for a proof). We list these singularities in table 4.1, together with local models and a geometric description; in the notation, T is the curve having the singularity, E is the exceptional divisor after blowing up the singularity, and $\overline{T}$ is the proper transform of T.

What we want to prove is that the Kodaira fiber type of the elliptic fibration f is determined only by the singularity of T, and the relative position of the fiber of R with respect to tangents to T at the singular points. The precise statement is as follows:

Lemma 4.4.1. *Let T be the trisection of R, F a fiber of R, and G the fiber of f over F. Then:*

1. *If $T|F = p + q + r$ consists of three distinct points of F, then G has type I_0 (i.e. it is a smooth elliptic curve).*

Table 4.1: *a-d-e* classification of curve singularities

Name	Local equation	Geometric description
a_0	$x = 0$	Smooth point
a_1	$x^2 = y^2$	Ordinary node
a_2	$x^2 = y^3$	Ordinary cusp
a_n	$x^2 = y^{n+1}$	Higher order cusp; n equals the *Milnor number* of the equation $f(x, y)$, that is $\dim_{\mathbb{C}} \mathcal{O}_{(0,0)} / \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$
d_4	$yx^2 = y^3$	Ordinary triple point (three distinct tangents)
d_n	$yx^2 = y^{n-1}$ $(n \geq 5)$	Triple point with two tangents. $\overline{T}$ meets E in two points, one smooth and one singular of type a_{n-5} (hence, if $n = 5$, it is smooth and tangent to E)
e_6	$x^3 = y^4$	Triple point with one tangent, such that $\overline{T}$ is smooth and meets E of order 3 at one point
e_7	$x^3 = xy^3$	Triple point with one tangent, such that $\overline{T}$ has an ordinary node (type a_1) with E as one of his tangents
e_8	$x^3 = y^5$	Triple point with one tangent, such that $\overline{T}$ has an ordinary cusp (type a_2) where it is tangent to E.

2. *Assume that* $T|F = p+2q$, *with* $p \neq q$. *Then* q *is at worst a double point of* T, *and if it is double then* F *is not one of its tangents. Moreover:*

   (a) *if* T *is smooth at* q, *then* G *has type* I_1 *(nodal rational curve);*

   (b) *if* T *has a double point at* q *of type* a_{n-1}, *then* G *has type* I_n.

3. *Assume* $T|F = 3p$. *Then* p *is at worst a triple point of* T. *Furthermore:*

   (a) *if* T *is smooth at* p, *than* G *has type* II *(a cuspidal rational curve);*

   (b) *if* T *is double at* p, *of type* a_n, *then* F *must be one of the tangents to* T *at* p *(otherwise* $(T.F)_p = 2$*), and* $n \leq 2$ *(otherwise* $(T.F)_p \geq 4$*).*

      i. *If* (T,p) *is of type* a_1, *then* G *is of type* III*;*
      ii. *if* (T,p) *is of type* a_2, *then* G *is of type* IV*;*

   (c) *if* T *is triple at* p, *then* F *is not one of its tangents.*

      i. *If* (T,p) *is of type* d_n, *then* G *is of type* I^*_{n-4}*;*
      ii. *if* (T,p) *is of type* e_6, *then* G *is of type* IV^**;*
      iii. *if* (T,p) *is of type* e_7, *then* G *is of type* III^**;*
      iv. *if* (T,p) *is of type* e_8, *then* G *is of type* II^*.

Proof. The geometric description of the tangents is obvious: if T meets $F \cong \mathbb{P}^1$ in three distinct points then T it is smooth and meets them transversely, if it meets F in a point of multiplicity 2 then either it is smooth and tangent or has a node, and F is not one of its tangents, and if it meets F in a point of multiplicity 3, than it is either a triple point with F none of its tangents, or a double point with F as one of its tangents, or a smooth flex (with tangent F). So we only have to prove that this information determines the type of the fiber G. We already explained (1) in subsection 4.2.2.

We recall that the type of $G \subset X$ determines uniquely the rational singularity arising in Y when contracting the components not meeting the fiber. We have listed the type of singularity arising in table 3.2; it is clear from that table that also a partial converse is true: if the singularity in Y is of type D_n or E_n for some n, or of type A_n with $n \geq 3$, then the singularity completely determine the type of G. More precisely, if we call T_Y the type of singularity in Y and T_G the type of the fiber G, we have:

$$\begin{array}{llll} T_Y = A_0 & \iff T_G = I_0, I_1, II & T_Y = D_n & \iff T_G = I^*_{n+4} \\ T_Y = A_1 & \iff T_G = I_2, III & T_Y = E_6 & \iff T_G = IV^* \\ T_Y = A_2 & \iff T_G = I_3, IV & T_Y = E_7 & \iff T_G = III^* \\ T_Y = A_n, n \geq 3 & \iff T_G = I_{n+1} & T_Y = E_8 & \iff T_G = II^* \end{array} \tag{4.7}$$

It is well known (see [?], chap. 3, par. 7) that a double cover of surfaces, with smooth base and branched over a curve with an *a*-*d*-*e* singularity gives rise to an *A*-*D*-*E* surface singularity of the corresponding type (a_n corresponds to A_n with the same n, d_n to D_n, etc.); hence in each entry we can call T_Y the singularity type of the trisection T too (note that these are usually indicated with small letters instead of capital ones). Hence the proof is almost complete: (1) holds, (2b) follows from what we have said for each $n \geq 4$, and also all of (3c); we only have to distinguish between types I_0, I_1 and II, between I_2 and III and between I_3 and IV. Since G being smooth is equivalent to T meeting F in three points, we can ignore the possibility I_0.

We want to distinguish according to the number of points the trisection meets the fiber in, i.e. we want to show that if G is of type I_n, then always $\#(T \cap F) = 2$, while if G is of type II, III or IV then $\#(T \cap F) = 1$. This is clear for the following reason: if G has type I_n, then after passing to Y (i.e. contracting all but one of its components) we obtain a rational curve with a node, while if G is of type II, III or IV we obtain a rational curve with a cusp; but this curve has to arise as the lift of F, hence if it is nodal F has to meet T in two distinct points (locally, the branches of the node), while if the fiber is cuspidal F must meet T in only one point. This concludes the proof. □

4.4.1 The proof of a-b-δ table

We are now ready to verify the a-b-δ table 3.3. The first step is to prove that the values of a, b and δ determine the type of the singular fiber (this method goes under the name of "Tate's algorithm"). We recall that in writing the table we assumed the fibration to be in minimal form, hence $a \leq 3$ or $b \leq 5$ must always hold. We keep the same notation as in lemma (4.4.1), that is, F is a fiber of the ruled surface R and G is the corresponding fiber of the smooth elliptic surface X; the Weierstrass fibration obtained contracting components will be called again Y.

Let us start from the first line: clearly, $\delta = 0$ is equivalent to G being smooth. Furthermore, $a = 0$ and $b > 0$ clearly implies $\delta = 0$; always $\delta \geq \min\{3a, 2b\}$, and in particular $\delta > 0$ if both $a > 0$ and $b > 0$. This proves the first line, i.e. that a, b and δ suffice to determine if the fiber is smooth (actually, δ suffices in this case) and if the fiber is smooth the values of a, b and δ must be as stated.

Our next claim is that G is of type I_n, for some $n > 0$, if and only if $a = b = 0$ and $\delta > 0$. First, notice that the vanishing of both A and B is equivalent to the singular point of the fiber being (in the usual affine coordinates (x, y)) in $(0, 0)$: $B = 0$ is needed for the point to be on the curve, and $A = 0$ for the vanishing of the differential with respect to coordinate x. This in turn is equivalent to the singularity being a cusp (otherwise, translating to have the singularity in $x = 0$ makes a non-vanishing term of second order in x appear, leading to a node), i.e. as in the proof of remark (3.2.2), to G being of type I_n. To complete the second and third lines (those of type I_n), there only remains to check that under these hypotheses, $n = \delta$. By lemma (4.4.1), this is equivalent to the trisection T having a singularity of type $a_{\delta-1}$. We compute this explicitly; let us start with the local coordinates (x, t) such that $F = \{t = 0\}$ and $T = \{x^3 + A(t)x + B(t)\}$. Since $\Delta(0) = 0$ while $A(0) \neq 0$ and $B(0) \neq 0$, and since (A, B) are determined up to a $\lambda \neq 0$ acting as $(\lambda^4 A, \lambda^6 B)$, we can choose arbitrarily $A(0) = -3$; then, $\lambda^4 = -3/A$ and we can take $\lambda^6 = \sqrt{-27/A^3} = 4/B^2$, i.e. $B(0) = 2$. Hence the trisection is given by

$$F(x, t) = x^3 + (-3 + f(t))x + (2 + g(t)) = 0, \quad f(0) = g(0) = 0.$$

Then the singular point is $(x, t) = (1, 0)$, and the other point where T meets F is $(-2, 0)$, and there T is smooth and transverse. Hence, locally in t, we can make a coordinate change of the type $x \mapsto x - \xi(t)$, where $x = \xi(t)$ is a local equation for T around $(-2, 0)$ given by the implicit function theorem, so that T is made of two branches, one around $(1, 0)$ (of the same analytic type of the original one) and the other of the form $\{x = -2\}$. Hence, near $(1, 0)$ we are left with a function that is a polynomial of degree 2 in the x coordinate, vanishing of order 2 in $(1, 0)$, i.e.

$$G(x, t) = x^2 + (-2 + \alpha(t))x + (1 + \beta(t)) = 0, \quad \alpha(0) = \beta(0) = 0.$$

Then the discriminant takes the form $\Delta = \alpha^2 - 4(\alpha + \beta)$, and putting $z = x + 1 - \frac{\alpha}{2}$ one gets the desired equation $z^2 = \frac{\Delta}{4}$ (otherwise, one could write the derivatives of $G(x,t)$ to compute its Milnor number, that is $\delta - 1$ since $\mathbb{C}[x,t]/(G_x, G_t) \cong \mathbb{C}[t]/(\Delta'))$.

We have thus analyzed all of the cases with $\#(T \cap F) = 2, 3$, hence from now on we will assume that T meets F in one point only (of multiplicity 3). In the usual coordinates (x,t) this point will always be $(0,0)$. The first thing to show is that if a, b and δ are as in the fourth and fifth rows of table 3.3, then we have a fiber of type I_n^* of the corresponding type. It is clear from the equation of T that it has a triple point in $(0,0)$ if and only if both $a \geq 2$ and $b \geq 3$; by lemma (4.4.1), this distinguishes between I_n^*, II^*, III^* and IV^* on one side, and II, III and IV on the other. Furthermore, by the classification of curve singularities (cfr. table 4.1) D_n singularities correspond to triple points with 2 or 3 tangents, and E_n singularities to triple points with only one tangent. In our equation, this amounts to saying that if $a = 2$ or $b = 3$, then the singularity in T is of type D_n, hence the fiber G is of type I_n^*, for some n. What we want to prove now is that if this holds, and given δ, then the singularity is of type $D_{\delta-2}$, so that the fiber has type $I_{\delta-6}$.

If $a \geq 3$ we have a homogeneous part of order 3 of the type $x^3 + t^3 = 0$; if $b \geq 4$ it is of type $x(x^2 + t^2)$. In both situations, there are 3 distinct tangents, and again from table 4.1, this means a singular point of type D_4, that corresponds to I_0^*. So we can ignore this case, and suppose $a = 2$, $b = 3$. If $A(t) = \lambda t^2 + \ldots$, $B(t) = \mu t^3 + \ldots$, then $\delta > 6$ is equivalent to $4\lambda^3 + 27\mu^2 = 0$, that is, the discriminant of $x^3 + \lambda x + \mu$ vanishes. This in turns means that there are only 2 distinct tangents, i.e. that we have a singularity D_n, $n \geq 5$; so we have completed the fourth line of the a-b-δ table, too.

Let us now suppose that $a = 2$, $b = 3$ and $\delta > 6$; we claim that then T has a singularity of type $D_{\delta-2}$, so that the fiber has type $I_{\delta-6}$. We know that there are exactly 2 distinct tangents; hence one must be of multiplicity 1. With an analytic change of coordinates, we can suppose that locally it is given by $\{x = 0\}$. Then the local equation for T becomes

$$F(x,t) = x(x^2 + \alpha(t)x + \beta(t)).$$

The discriminant now takes the form $(\alpha^2 - 4\beta)\beta^2$, and the expression between brackets is a simple double point, hence can be put again in the form $(t^2 + x^{n+1})$, where n is the Milnor number

$$n = \dim_{\mathbb{C}} \mathcal{O}_{(0,0)} \big/ (2x + \alpha(t), x\alpha'(t) + \beta'(t)) = \dim_{\mathbb{C}} \mathbb{C}[t]/(\alpha(t)\alpha'(t) - 2\beta(t)).$$

Modulo invertible elements, the expression here is the derivative of $\alpha^2 - 4\beta$, hence $\delta = n + 1 + 2\mathrm{m}_0(\beta)$. The result is proved once we observe that $\mathrm{m}_0(\beta) =$

2, since $x^2 + \alpha(t)x + \beta(t)$ must have one tangent with multiplicity 2 distinct from $x = 0$.

Let us analyze the remaining cases. From the discussion of the case I_n, we must always have $a \geq 1$ and $b \geq 1$. Then clearly T being smooth at the intersection point is equivalent to $x^3 + A(t)x + B(t)$ having a non-zero homogeneous part of degree 1, i.e. $b = 1$; by lemma (4.4.1) this is also equivalent to G being of type II, as desired. Hence from now on we can assume $b \geq 2$.

The remaining cases with a double point in the intersection (i.e. those in which $a \leq 1$ or $b \leq 2$) are easy to handle: there can be either a node or a cusp, and the first one arises if and only if $a = 1$ (otherwise there cannot be two distinct tangents); hence $a = 1$ if and only if G is of type III, and $a \geq 2$, $b = 2$ if and only if G is of type IV.

Finally, we are left with the case in which T has a triple point, with only one tangent, and G is of type II^*, III^* or IV^*; since in the expression for T there is no quadratic part in x, the tangent must be $x = 0$, i.e. $a \geq 3$ and $b \geq 4$. Taking into account that $a \leq 3$ or $b \leq 5$, there are only three classes of possibilities ($a = 3$, b arbitrary; $b = 4$, a arbitrary; $b = 5$, a arbitrary). If we write $A = t^a\alpha$, $B = t^b\beta$, then after performing a blow-up at the origin, we are left with $x^3 + t^{a-2}\alpha + t^{b-3}\beta$. In such a way, the three (classes of) possibilities remaining for a and b are put into correspondence with the curve singularities for the proper transform of the trisection after a blow-up (respectively, A_1, A_0 and A_2), and hence with the singularity of T and the types of the fiber G (respectively, III, II and IV). This concludes the proof that a, b and δ are as in table 3.3, and so that they determine the fiber type.

The values for J and $\mathrm{m}(J)$ are all clear, simply putting in the definition of $J = 4A^3/\Delta$ the values of A and B, and simplifying the right powers of t when possible. This concludes the proof of the validity of table 3.3.

4.5 The J-map

4.5.1 Quadratic twists and J-map

We have seen in corollary (4.2.8) a necessary and sufficient condition for two sets of Weierstrass data $(\mathcal{L}_i, A_i, B_i)$ to give birational surfaces. From this description it is clear that birational surfaces have the same J-map: $J(A, B) = J(Af^4, Bf^6)$ since f^{12} factors out of both the numerator and denominator. However, the converse need not hold, since for the same reason also $J(A,B) = J(Af^2, Bf^3)$, and this form is more accurate:

Proposition 4.5.1. *Let $(\mathcal{L}_1, A_1, B_1)$, $(\mathcal{L}_2, A_2, B_2)$ be Weierstrass data over S, and let J_1, J_2 be the corresponding J-maps. Assume that neither of them is identically 0 or 1; then the following are equivalent:*

1. *$J_1 = J_2$;*
2. *there exist line bundles M_1 and M_2 on S and non-zero sections $f_i \in H^0(M_i^2)$ such that $(\mathcal{L}_1 \otimes M_1, A_1 f_1^2, B_1 f_1^3) \cong (\mathcal{L}_2 \otimes M_2, A_2 f_2^2, B_2 f_2^3)$.*

Proof. We have already seen that (2) implies (1). For the converse, assume $J_1 = J_2$. None of A_i, B_i can vanish identically (otherwise J_i would be identically 0 or 1), hence $A_1^3 B_2^2 = A_2^3 B_1^2$. The conclusion follows considering $M_1 = L_1^2 \otimes L_2^3$, $M_2 = L_1^3 \otimes L_2^2$, $f_1 = A_1 B_2$ and $f_2 = A_2 B_1$. □

Definition A *quadratic twist* on the Weierstrass data $(\mathcal{L}, A, B)$ is the action of replacing it with the Weierstrass data $(\mathcal{L} \otimes M, Af^2, Bf^3)$, for M line bundle on S and f a non-zero section of M^2.

In these terms, unless J is identically 0 or 1, then Weierstrass data with the same J-map are equal "up to quadratic twists".

A special case of the quadratic twist operation arises considering M of order 2 in $\mathrm{Pic}(S)$. Then $M^2 \cong \mathcal{O}_S$, and we can take $f = 1$, to replace $(\mathcal{L}, A, B)$ with $(\mathcal{L} \otimes M, A, B)$. For example, if S is an elliptic curve and we start with an abelian surface (i.e. $\mathcal{L} \cong \mathcal{O}_S$), if we perform a quadratic twist of this type we obtain (M, A, B), that is an bi-elliptic surfaces by proposition (4.3.3).

Finally, note that if we perform a quadratic twist twice with the same pair (M, f) we obtain a surface that is birationally equivalent to that from which we have started; this suggests that, if we define properly a 2-group, this could lead to an action of this group to the set of birational classes of Weierstrass fibration. This is indeed the case, as we will see subsequently.

4.5.2 The double cover group

Definition Let S be a curve, and $C \subseteq S$ and arbitrary subset. A *double cover pair* on S relative to C is a pair (M, f) where M is a line bundle on

S and f is a non-zero section of $M^{\otimes 2}$, whose zero locus (regarded only as a set) is contained in C.

An isomorphism between two double cover pairs (M_1, f_1) and (M_2, f_2) is an isomorphism of bundles $\alpha\colon M_1 \to M_2$, such that the induced map $\alpha^2\colon M_1^2 \to M_2^2$ transports the section f_1 to f_2. We call $\mathcal{A}_C$ the set of isomorphism classes of double cover pairs over S relative to C, and indicate with $[M, f]$ the isomorphism class of (M, f).

We would like this to be a group, since it carries the whole information needed to define an action on the birational classes of Weierstrass data. We start by defining a product on $\mathcal{A}_C$: we set $[M_1, f_1] \cdot [M_2, f_2] = [M_1 \otimes M_2, f_1 f_2]$. This is clearly well defined, since an isomorphism, say, $\alpha M_2 \to M_2'$ gives another one $\beta = 1 \otimes \alpha\colon M_1 \otimes M_2 \to M_1 \otimes M_2'$, that clearly transform f_2 in the corresponding way. It is associative and commutative because taking tensor products is. Furthermore, there is the 2-sided unit $[\mathcal{O}_S, 1]$. Hence, we have a monoid, but this is not yet a 2-group as we require.

Let $\mathcal{B}_C$ be the submonoid of $\mathcal{A}_C$ given by the set of isomorphism classes $[M, f] \in \mathcal{A}_C$ where $f = g^2$, where g is some section of M. It is clear that this is indeed a submonoid. We want to take the quotient (in the category of monoids). We put on $\mathcal{A}_C$ the equivalence relation $a_1 \sim a_2$ if and only if there exist elements $b_1, b_2 \in \mathcal{B}_C$ such that $a_1 b_1 = a_2 b_2$. More explicitly, $[M_1, f_1]$ becomes equivalent to $[M_2, f_2]$ if and only if there exist bundles N_1, N_2, and non-zero sections g_1 and g_2 respectively, such that $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$. It is clear that this is an equivalence relation (this holds true more generally in the category of monoids).

Definition We call $\mathrm{Doub}_C(S)$ the set of equivalence classes, denoted by $\{M, f\}$, of elements in $\mathcal{A}_C$ modulo the equivalence relation defined above by means of $\mathcal{B}_C$. If there are no restriction on the zero locus of f, i.e. $C = S$, we simply write $\mathrm{Doub}(S)$ instead of $\mathrm{Doub}_S(S)$.

It is easy to see that this time we have a group, and precisely an abelian 2-group: the product clearly descends to equivalence classes, and it is well defined and obviously associative and commutative; every element is self-inverse, since $\{M, f\} \cdot \{M, f\} = \{M^{\otimes 2}, f^2\} \in \mathcal{B}_C$.

Lemma 4.5.2. *Every element of* $\mathrm{Doub}_C(S)$ *has a unique representative* $[M, f]$ *in* $\mathcal{A}_C$ *with the divisor of zeroes* $(f)_0$ *of* f *reduced.*

Proof. Let us start with the existence. Let $\{M, f\}$ be an element of $\mathrm{Doub}_C(S)$, and write $(f)_0 = D + 2E$, with D reduced, and both D and E effective divisors. It is clearly always possible to write a divisor uniquely in such a way, and $(f)_0$ is reduced if and only if $E = 0$. Let s be a section of $\mathcal{O}_S(E)$ whose zero locus is exactly E. Let $M' = M(-E)$, and $f' = f/s^2$, that is clearly a holomorphic section of $(M')^2$. Now, $\{M', f'\} = \{M, f\}$ (in $\mathrm{Doub}_C(S)$), since in $\mathcal{A}_C$ they differ by $[\mathcal{O}_S(E), s^2]$, which is in $\mathcal{B}_C$, and $(f')_0$ is reduced.

We can now turn to uniqueness. Let $[M_1, f_1]$ and $[M_2, f_2]$ be elements in $\mathcal{A}_C$, with $(f_1)_0$ and $(f_2)_0$ reduced and which are the same modulo $\mathcal{B}_C$ (i.e. $\{M_1, f_1\} = \{M_2, f_2\}$). By definition, there are line bundles N_1, N_2 and sections g_i of N_i such that $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$. Let $E_i = (g_i)_0$; then we have $D_1 + 2E_1 = D_2 + 2E_2$. But this are divisors of the same section (modulo isomorphism), so we must in fact have $D_1 = D_2$ and $E_1 = E_2$ for the stated uniqueness of this decomposition of divisor. This forces $N_1 \cong N_2$, and g_1/g_2, which is a holomorphic non-zero section of $N_1 \otimes N_2^{-1}$ must be constant. Tensoring both sides with N_2^{-1} we then obtain an isomorphism

$$\left[M_1 \otimes N_1 \otimes N_2^{-1}, f_1 \frac{g_1^2}{g_2^2}\right] \cong \left[M_2, f_2\right],$$

which leads in fact to $[M_1, f_1] \cong [M_2, f_2]$. □

We now concentrate to the case S compact and C finite. Then we can compute the order of $\mathrm{Doub}_C(S)$:

Lemma 4.5.3. *Suppose S is a compact curve, and C is finite. Then*

$$|\mathrm{Doub}_C(S)| = \begin{cases} 2^{2g} & \text{if } S = \varnothing \\ 2^{2g+|S|-1} & \text{otherwise} \end{cases}$$

Proof. By the previous lemma, we only need to count the double cover pairs (M, f) with $(f)_0$ reduced and contained inside S as a set. Let us first consider the divisor $(f)_0$: if S is empty, there is only one choice (namely, $(f)_0 = \varnothing$). Otherwise, since f is a section of M^2, the degree of $(f)_0$ must be even, and the possible choices for reduced divisors are in bijection with the subsets of even cardinality of C. Hence the number of possible choices is

$$\sum_{i \geq 0} \binom{|S|}{2i} = 2^{|S|-1}.$$

The divisor determines uniquely the line bundle M^2 and its section f, up to scalar multiplication. Given M^2, the possible choices for M are the number of elements in $\mathrm{Pic}(S)$ of 2-torsion (provided that there is at least one; but for divisors on a curve it is necessary and sufficient for the existence that the degree is even). This is the same as the elements of order dividing 2 in $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$, which are 2^{2g}, and the result follows. □

Now that we have a group, we want to relate it with Weierstrass data. We call $\mathbb{BW}$ the set of birational equivalence classes of Weierstrass data over S (obviously, two set of Weierstrass data are said to be birational if they determine birational surfaces). We remark that this set is in natural bijection with the set of Weierstrass fibration over S in minimal form, and then, in turn, to that of smooth elliptic fibrations over S with section; this is obvious, since each Weierstrass fibration has exactly one minimal model in its birational equivalence class, by lemma (4.2.7). We denote the birational class of $(\mathcal{L}, A, B)$ as $[\mathcal{L}, A, B]$. We define a "twisting" by the means of double cover pairs:

$$(M, f) \cdot (\mathcal{L}, A, B) = (\mathcal{L} \otimes M, Af^2, Bf^3). \tag{4.8}$$

We now see that this action passes to $\mathrm{Doub}_C(S)$, and the action so induced is free:

Proposition 4.5.4. *Let $C \subseteq S$ be arbitrary. Then the action defined by* (4.8) *induces a free action of* $\mathrm{Doub}_C(S)$ *on* $\mathbb{BW}$*.*

Proof. It is clear that letting a double cover pair (M, f) act on two set of Weierstrass data $(\mathcal{L}_1, A_1, B_1)$ and $(\mathcal{L}_2, A_2, B_2)$ which are birational, the results are still birational, by means of corollary (4.2.8). Moreover, if two double cover pairs are isomorphic, then they act identically on $\mathbb{BW}$ (the Weierstrass data obtained are isomorphic), hence we get an action of $\mathcal{A}_C$ on $\mathbb{BW}$. But again by corollary (4.2.8) we see that $\mathcal{B}_C$ maps a set of Weierstrass data to another which is birational to the starting one, hence the action descends to $\mathrm{Doub}_C(S)$.

Finally, we want to show that we actually obtained a free action. Suppose then that $\{M, f\}[\mathcal{L}, A, B] = [\mathcal{L}, A, B]$, so that there are line bundles N_i, each with section g_i, such that $(\mathcal{L} \otimes M \otimes N_1, Af^2 g_1^4, Bf^3 g_1^6) \cong (\mathcal{L} \otimes N_2, Ag_2^4, Bg_2^6)$. Hence there is an isomorphism $\alpha\colon \mathcal{L} \otimes N_2 \to \mathcal{L} \otimes M \otimes N_1$ making the sections correspond; we can rewrite this as an isomorphism $\beta\colon N_2 \otimes N_1^{-1} \to M$ such that β^4 carries g_2^4/g_1^4 to f^2 and β^6 carries g_2^6/g_1^6 to f^3. This in particular shows that g_2/g_1 is holomorphic, not only meromorphic; furthermore, β^2 carries g_2^2/g_1^2 to f, i.e. that $[M,f] = [N_2 \otimes N_1^{-1}, g_2^2/g_1^2]$ (as elements of $\mathcal{A}_C$), hence it belongs to $\mathcal{B}_C$. So $\{M,f\}$ is trivial in $\mathrm{Doub}_C(S)$, as wanted. $\square$

We have already noticed that the J map of a birational class of Weierstrass fibrations is well-defined. With the notation introduced, we can rephrase proposition (4.5.1) in the following way:

Proposition 4.5.5. *Let $[\mathcal{L}, A, B]$ be an element of $\mathbb{BW}$ whose J-map is not identically 0 or 1. Then the set of elements of $\mathbb{BW}$ with the same J-map is exactly the orbit of $[\mathcal{L}, A, B]$ under the action of $\mathrm{Doub}(S)$.*

Let us denote with $\mathbb{BW}'$ the set of birational classes of Weierstrass fibrations, with J-map not identically 0 or 1. Then we have:

Proposition 4.5.6. *Associating to each Weierstrass fibration its J map induces a bijection between*

$$\mathbb{BW}'/\mathrm{Doub}(S) \longleftrightarrow \{J \colon S \to \mathbb{P}^1 \mid J \not\equiv 0,\ J \not\equiv 1\}$$

Proof. We have already shown that two Weierstrass fibrations with J-map non identically 0 or 1 are birational if and only if they are the same modulo $\mathrm{Doub}(S)$, hence the map is well defined and injective. We have also proved that it is surjective, at the beginning of the proof of theorem (3.6.2): the (immediate) calculation we did there proved that if $J \colon S \to \mathbb{P}^1$ is not constantly 0 or 1 and given by two sections $[s:t]$ of $\mathcal{L} = J^*\mathcal{O}_{\mathbb{P}^1}(1)$, then J is realized as the J-map of the Weierstrass data $(\mathcal{L}, -3t(t-s)s^2, 2t(t-s)^2s^3)$. $\square$

Remark 4.5.7. The data used to construct the surface above is the pull-back of the data $(\mathcal{O}_{\mathbb{P}^1}(1), -3t(t-s)s^2, 2t(t-s)^2s^3)$, where this time we denote with $[s:t]$ the homogeneous coordinates on $\mathbb{P}^1$. This data defines a rational elliptic surface with section over $\mathbb{P}^1$, and by means of table 3.3 we can identify its singular fibers. Modulo invertible terms, the discriminant is $s^7t^2(t-s)^3$, hence it vanishes over 3 points, 0, 1 and ∞, of orders 2, 3 and 7 respectively. More precisely, in 0 we have $(a,b,\delta) = (1,1,2)$, hence there is a singular fiber of type II, in 1 $(a,b,\delta) = (1,2,3)$, hence there is a singular fiber of type III, and finally in ∞ we have $(a,b,\delta) = (2,3,7)$, that corresponds to a singular fiber of type I_1^*. The J-map is obviously the identity $\mathbb{P}^1 \to \mathbb{P}^1$.

The surface over $\mathbb{P}^1$ described in the remark is somehow a universal Weierstrass fibration in minimal form, meaning that every other Weierstrass fibration in minimal form with J-map not constantly 0 or 1 can be derived from this by the action of $\mathrm{Doub}(S)$. More precisely, and using lemma (4.5.2) we have the following:

Proposition 4.5.8. *Let $(\mathcal{L}, A, B)$ be a set of Weierstrass data in minimal form. Let J be the J map for the associated smooth elliptic fibration, and assume that J is not identically 0 or 1. Then there is a double cover pair (M, f) on S with $(f)_0$ reduced, unique up to isomorphism, such that $(\mathcal{L}, A, B)$ is the version in minimal form of the twist of $(J^*\mathcal{O}_{\mathbb{P}^1}(1), -3t(t-s)s^2, 2t(t-s)^2s^3)$ by (M, f).*

Otherwise said, we have chosen an element of each class $\mathbb{BW}'/\mathrm{Doub}(S)$ via the bijection given by (4.5.6), depending only on the sections t, s, defining J.

4.5.3 The transfer of $*$ process

We want to elaborate the concepts given above locally. Let us suppose $S = \Delta$ is a disk with center 0, so that $\mathrm{Pic}(\Delta)$ is trivial and double cover pairs relative to $C = \{0\}$ are all isomorphic to $(\mathcal{O}_\Delta, t^n)$ for some n (hence in this case $\mathcal{A}_{\{0\}} \cong \mathbb{Z}$). If we pass to the quotient, $\mathrm{Doub}_{\{0\}}(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$, since only the parity of n is meaningful. It is generated by the non-trivial element $\{\mathcal{O}_C, t\}$, and we want to study the effect of performing a quadratic twist by this non-trivial element.

Let $(\mathcal{O}_\Delta, A(t), B(t))$ be Weierstrass data over Δ. Performing the quadratic twist, we obtain $(\mathcal{O}_\Delta, t^2A(t), t^3B(t))$; by means of a-b-δ table 3.3, we can then study how this modify the fiber type over 0. We are replacing (a, b, δ) with $(a+2, b+3, \delta+6)$, and putting this in minimal form amounts to subtract $(4, 6, 12)$ once if after the quadratic twist we have obtained illegal values $a \geq 4$ and $b \geq 6$. Otherwise said, to obtain Weierstrass data in minimal form we are choosing the right representative of the generator of $\mathrm{Doub}_{\{0\}}$ in $\mathcal{A}_{\{0\}}$ between $(\mathcal{O}_\Delta, t)$ and $(\mathcal{O}_\Delta, t^{-1})$.

We thus obtain the following rule for how the fiber changes upon performing a quadratic twist:

$$\begin{array}{rcl} I_n & \longleftrightarrow & I_n^* \\ II & \longleftrightarrow & IV^* \\ III & \longleftrightarrow & III^* \\ IV & \longleftrightarrow & II^* \end{array}$$

This explains why this process is referred to as "transfer of $*$".

Let us now suppose that S is compact. Then if $(\mathcal{L}, A, B)$ is Weierstrass data over S, and (M, f) is a double cover pair with $(f)_0$ reduced, if we take a small disk around a point where f vanishes, we are locally in the same situation as above. Hence the twist by the double cover pair (M, f) operates the transfer of $*$ at each fiber over a point where f vanishes. Moreover, we know that the degree of $(f)_0$ must be even, thus the number of transfer of $*$ happening is even as well; hence the number of $*$-fibers of any Weierstrass fibration is invariant by quadratic twists. We say that a Weierstrass fibration is $*$-even or $*$-odd depending on whether the minimalization has an even or odd number of $*$-fibers. Since this property is invariant by quadratic twists, this is in fact a property of the J-map, and not of the fibration itself (provided J not constantly 0 or 1).

Given a divisor with even degree D, we can always find a fiber bundle M such that $M^{\otimes 2} \cong \mathcal{O}_S(D)$ (actually, we can find 2^{2g} of them, as already noted). Hence, given a Weierstrass fibration, call E the set of those points s of S such that X_s is a $*$-type fiber. If $\deg E$ is even, we can take the divisor D to be reduced with support equal to E, otherwise we take it with support equal to $E \setminus \{P\}$, where P is a point of E. Then, if we take f a section of $M^{\otimes 2}$ vanishing exactly on D, twisting by $\{M, f\} \in \mathrm{Doub}(S)$ we get a Weierstrass fibration with at most one $*$-fiber (and if there is one or none is a fact independent of the twisting chosen). We say a Weierstrass fibration with at most one $*$-fiber to be $*$-*minimal*.

We say that Weierstrass data $(\mathcal{L}, A, B)$ in normal form is J-*minimal* if $\deg(\mathcal{L})$ is minimal among all the sets of Weierstrass data in normal form having the same associated J-map.

Lemma 4.5.9. *Let $(\mathcal{L}, A, B)$ be Weierstrass data in normal form over a compact curve S, such that the associated J-map is not identically 0 or 1. Then $(\mathcal{L}, A, B)$ is $*$-minimal if and only if it is J-minimal.*

Proof. One direction is straightforward: if $(\mathcal{L}, A, B)$ is not $*$-minimal, then there are at least two $*$-fibers, say over s_1 and s_2 (distinct points). Then we take the divisor $D = s_1 + s_2$, and a line bundle M such that $M^2 \cong \mathcal{O}_S(s_1+s_2)$, with a section f of M^2 with zero divisor $s_1 + s_2$. Performing the twist by $\{M^{-1}, 1/f\}$ we obtain a Weierstrass data that is still in normal form (since the multiplicities of A and B decrease in the process) and with the same J-map of degree exactly one less, hence our starting data cannot be J-minimal.

For the other direction, let us start with an easy remark: if $(\mathcal{L}, A, B)$ is any set of Weierstrass data, and $(\mathcal{L}', A', B')$ is the data obtained by putting it in minimal form, then always $\deg(\mathcal{L}') \leq \deg(\mathcal{L})$, and equality holds if and only if $\mathcal{L}$ was already in minimal form. This is a trivial consequence of lemma (4.2.7), since $\mathcal{L}' = \mathcal{L}(-D)$, where D is an effective divisor.

Now, let $(\mathcal{L}_1, A_1, B_1)$ be a $*$-minimal set of Weierstrass data in normal form, such that its J-function is not constantly 0 or 1. Suppose there is $(\mathcal{L}_2, A_2, B_2)$ with the same J-map and J-minimal. In particular $\deg(\mathcal{L}_2) \leq \deg(\mathcal{L}_1)$, and $(\mathcal{L}_2, A_2, B_2)$ is $*$-minimal for the first part of the lemma; we want to show that in fact $\deg(\mathcal{L}_2) = \deg(\mathcal{L}_1)$. By proposition (4.5.1) there are line bundles M_1, M_2 and sections f_i of M_i such that

$$(\mathcal{L}_1 \otimes M_1, A_1 f_1^2, B_1 f_1^3) \cong (\mathcal{L}_2 \otimes M_2, A_2 f_2^2, B_2 f_2^3). \tag{4.9}$$

Clearly, the Weierstrass fibration we are dealing with have the same $*$-parity; furthermore, since they are $*$-minimal, either both have no $*$-fiber or both have only one. In the first case we readily conclude: the order of vanishing of $A_1 f_1^2$ and of $A_2 f_2^2$ coincide, and the same holds for the $B_i f_i^3$, but since for every point s either $\mathrm{m}_s(A_i) \leq 2$ or $\mathrm{m}_s(B_i) \leq 3$, we deduce that the divisor of zeroes coincide: $(f_1)_0 = (f_2)_0$. In particular, $M_1 \cong M_2$, hence, taking degrees in (4.9), we obtain $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$.

The conclusion follows without much effort also in the case $(\mathcal{L}_1, A_1, B_1)$ and $(\mathcal{L}_2, A_2, B_2)$ both have exactly one $*$-fiber. Call s_i the point over which the fibration $(\mathcal{L}_i, A_i, B_i)$ has a $*$-fiber; if $s_1 = s_2$ we can conclude as above. Otherwise, we have anyhow that $(f_1)_0 + s_1 = (f_2)_0 + s_2$, again by comparing the order of vanishing of the A_i's and those of the B_i's. Call $E = (f_1)_0 - s_2 = (f_2)_0 - s_1$, that is an effective divisor if $s_1 \neq s_2$. Then we can multiply both sides of (4.9) by $(\mathcal{O}(-E), g^{-2}, g^{-3})$, where g is a section of $\mathcal{O}(E)$ vanishing exactly on E, to obtain

$$(\mathcal{L}_1 \otimes N_1, A_1 h_1^2, B_1 h_1^3) \cong (\mathcal{L}_2 \otimes N_2, A_2 h_2^2, B_2 h_2^3),$$

where $h_i = f_i/g$. But now $(h_1)_0 = s_2$, $(h_2)_0 = s_1$, hence N_1 and N_2 have the same degree, and we conclude as above. $\square$

Remark 4.5.10. We note that the second part of the proof can be easily adapted to prove that if two Weierstrass fibrations with the same J-map (not identically 0 or 1) have the same number of $*$-fibers, then their fundamental bundles have the same degree.

4.5.4 Proof of table 3.4

We are finally in position to prove the last one of the results stated in section 3.2, that is, that table 3.4 lists all the possible configurations around a fiber (the "germ" of the fiber), up to analytic isomorphism. We already proved that the key point is proving proposition (3.2.4). For convenience, we state it here again, before proving it.

Proposition 4.5.11. *The germ of the fiber X_s of an elliptic curve with section $f\colon X \to S$ is determined by $J(s)$, $\mathrm{m}_s(J)$ and the singular fiber type.*

Proof. The question is local, so we assume that $S = \Delta$ is a small disk around the singular value of f, which we assume to be 0, as usual. Assume first that J is not identically 0 or 1. Then, up to analytic isomorphism, J is determined locally by its value $J(0)$ and multiplicity $\mathrm{m}_0(J)$. Since we are dealing with smooth surfaces, that means that we are interested in Weierstrass fibrations in minimal form only, the set of fibrations with a given J-map and without singular fibers out of $\{0\}$ is in bijection with $\mathrm{Doub}_{\{0\}}(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$, by propositions (4.5.4) and (4.5.5). We have seen in subsection 4.5.3 that the two elements of this group exactly distinguish between two types of fiber, one with the $*$, and the other without. Hence, knowing $J(0)$, $\mathrm{m}_0(J)$ and the fiber type (or even only if it is a $*$-type or not) characterizes the germ completely.

Let us now suppose that J is identically 0. This means that A is identically 0, and up to analytic isomorphism we may suppose B to be in the form t^b, so that the Weierstrass equation becomes $y^2 = x^3 + t^b$. Minimality forces $b \leq 5$; by a-b-δ table 3.3, each value corresponds a different singular fiber type, hence again knowing the singular fiber type is enough. The same reasoning applies to the case $J \equiv 1$: in this circumstance, up to analytic isomorphism, we have the equation $y^2 = x^3 + t^a x$, with $a \leq 3$ by minimality; again by inspection of the a-b-δ table these are in bijection with the singular fiber types. $\square$