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Ph.D. Program in Mathematics for Economic Decisions

Leonardo Fibonacci School

Ph.D. Thesis

"Branch and Bound" and "Branch and Reduce" approaches for a class of D.C. Programs

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Contents

List of Tables

Chapter 1

Introduction

The term Mathematical Programming refers to the study of problems having the aim of minimizing or maximizing a real function of real or integer variables, subject to constraints on the variables. In this light, the problems can be studied from both a theoretical point of view (that is their mathematical properties), an algorithmic point of view (development and implementation of algorithms to solve them) and an applicative one (application of properties and algorithms to real world problems).

Among the huge literature of Mathematical Programming, one of the topics more often approached and studied is the so called "d.c. programming", which deals with problems described by means of differences of convex functions.

The origin of d.c. programming can be represented by the pioneering paper by P. Hartman "On Functions Representable as a Difference of Convex Functions", published in 1959 on the Pacific Journal of Mathematics [31].

After a fast development in the years 70s-80s [23, 26, 27, 32, 54, 55, 77, 85, 86, 88, 94, 95], d.c. programming became a consolidated field in the years 90s [29, 57, 58, 60, 62].

The relevance of d.c. programming is not only theoretical but also lies on its usefulness in applicative problems (see for all [4, 5, 10, 29, 57, 62, 67, 68, 87, 100] and references therein).

In this Ph.D. thesis the following d.c. program is studied from both a theoretical and an algorithmic point of view:

$$P : \begin{cases} min\ f(x) = c(x) - \sum_{i=1}^{k} g_i(d_i^T x) \\ x \in X \subseteq \mathbb{R}^n \end{cases} \tag{1.1}$$

The set X is a polyhedron given by inequality constraints $Ax \le b$ and/or equality constraints $A_{eq}x = b_{eq}$ and/or box constraints $l \le x \le u$, where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $l, u \in \mathbb{R}^n$, $A_{eq} \in \mathbb{R}^{h\times n}$, $b_{eq} \in \mathbb{R}^h$, $d_i \in \mathbb{R}^n$ for all $i = 1, \dots, k$. The functions $c : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, k$, are convex and continuous. It is also assumed that there exists $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^k$ such that $\tilde{\alpha}_i \le d_i^T x \le \tilde{\beta}_i$ $\forall x \in X$ $\forall i = 1, \dots, k$.

In [82] the particular case of $c(x) = \frac{1}{2}x^T Qx + q^T x$, with $q \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ symmetric and positive semi-definite, is studied in the following form (where

$d_i^T x = y_i$ for all $i = 1, \dots, k$):

$$P_Q : \begin{cases} min\ f(x) = \frac{1}{2}x^T Qx + q^T x - \sum_{i=1}^{k} g_i(y_i) \\ (x, y) \in X \times Y \subseteq \mathbb{R}^n \times \mathbb{R}^k \\ Y = \{y \in \mathbb{R}^k : y_i = d_i^T x, x \in X\} \end{cases} \tag{1.2}$$

In [53] problem P_Q is studied in the particular case $y_i = x_i\ \forall i = 1, \cdots, n$, while in [81] the concave case $Q = 0$ is analyzed.

In this thesis this class of problems is computationally studied with a "branch and bound" and a "branch and reduce" approaches, pointing out the effectiveness of partitioning rules and of stack policies for managing the branches. In this light, the results appeared in the literature are extended and deepened on. This study can be virtually divided into three main steps.

First of all, problem (1.1) is solved by means of a rectangular branch and bound approach based on two fundamental rules: the stack policy needed to manage the stored subproblems and the partitioning criterion needed to create the subproblems themselves. In this light, we aim to analyze from a computational point of view different kinds of stacks and diverse partitioning criteria in order to point out their performance efficiency. Deeply speaking, the results provided in [53, 82] will be deepened on, devoting a particular attention to the so called "ω-subdivision process" and "largest distance bisections". In particular, it is pointed out that the "ω-subdivisions" provide the better performance just for small values of k, while when k increases they become the worst criterion among the compared ones. It is shown also that, for the three considered classes of functions, the "largest distance bisections" and the "ω-subdivisions" are not actually computationally equivalent as suggested in [53]. The obtained results are collected in [7] and submitted to *"Operations Research Letters"*.

In a second part of this study the class of programs P are approached by means of a branch and reduce method based on Lagrangean cuts. The main aim is to deepen on the study proposed in [7] by means of a branch and reduce approach based on some acceleration devices. In particular, it will be analyzed the opportunity of using Lagrangean cuts based on duality properties and of recalculating some of the bounds needed to determine the relaxed convex subproblems within the branch process. Seven different partitioning rules are also evaluated. It will be pointed out that the use of the proposed acceleration devices improves the results given in [7, 82] and deepens on the study in [53, 81]. It will be also shown that the "ω-subdivision" partitioning rule, which is commonly used in the literature, is not the better choice. The obtained results are collected in [8] and submitted to *"Central European Journal of Operations Research"*.

The third goal of this work is to deepen on the study of the branch and reduce method proposed in [8] by analyzing techniques aimed to emphasize the effectiveness of the Lagrangean cuts. In particular, there are some open problems related to the resize operations which deserve to be deepened on:

- how many resize operations have to be done in order to improve the performance of the method;

- how often the resize operations should be applied in order to decrease as much as possible the average CPU time.

In other words, it is wondered which is the most profitable set of indices to be used for applying the resize operations and whether they have to be applied to all of the evaluated subproblems or just to some of them. Within this study, the role of the seven different partitioning rules is also evaluated, pointing out that the "ω-subdivision" partitioning rule, commonly used in the literature, is not the better choice. Finally, for the sake of completeness, the proposed approach is compared with the so called "DCA" method by Le Thi Hoai An et al, which frequently appeared in the recent literature of global optimization. The obtained results are published in the *Journal of Computational and Applied Mathematics* (see [9]).

The thesis is divided into the following chapters. Chapter 2 contains a survey of the existent d.c. programming literature. Chapter 3 provides an extended presentation of the considered class of d.c. programs, the theory and the code of the algorithms, first the branch and bound version and then the branch and reduce modification. Chapter 4 contains the results of deep computational tests and the in-depht study of them. After the great impact of the priority stack is established, the so called "ω-subdivisions" and "largest distance bisections" are compared with five further different bisection criteria. The obtained computational results are presented and deeply analyzed. Chapter 5 contains a brief presentation of the DCA approach and its underlying theory. The results of the comparison between DCA and the branch and reduce method are proposed. Finally, for the sake of completeness two appendixes are provided. Appendix A contains some applications to real world problems, formulated as d.c. programs, and solved with the proposed branch and reduce algorithm. Appendix B contains the tables collecting all of the results of the deep preliminary tests that lead to obtain the ones exposed in Chapter 4.

The original results of this Ph.D. thesis have been collected in the following research papers:

1. R. Cambini, F. Salvi, (2009): *Solving a class of low rank d.c. programs via a branch and bound approach: a computational experience*, Technical Report n. 320, Department of Statistics and Applied Mathematics, University of Pisa, submitted to *Operations Research Letters*

2. R. Cambini, F. Salvi, (2009): *Solving a class of low rank d.c. programs via a branch and reduce approach: a computational study*, Technical Report n. 321, Department of Statistics and Applied Mathematics, University of Pisa, submitted to *Central European Journal of Operations Research*

3. R. Cambini, F. Salvi, (2009): *A branch and reduce approach for solving a class of low rank d.c. programs*, Journal of Computational and Applied Mathematics, vol.233, pp.492-501, ISSN 0377-0427