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On the relations between discrete and continuous dynamics in $\mathbb{C}^2$

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Introduction

The theory of local holomorphic foliations in $\mathbb{C}^2$ deals with the study of complex dynamics and invariant curves (separatrices) of germs of holomorphic vector fields, regardless their parametrizations.

Let $\mathcal{F}$ be a germ of a holomorphic foliation at 0 defined by

$$X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}.$$

The origin is a singularity for $\mathcal{F}$ if $A(0,0) = B(0,0) = 0$. The Poincarè problem consists in the study of the existence of invariant curves through a point. If the origin is not singular the Cauchy-Kowaleskaya theorem provides the existence of a unique, non singular, invariant, holomorphic curve through the origin. Namely, the differential holomorphic equation

$$\begin{cases} \dot{x} = A(x,y) \\ \dot{y} = B(x,y) \end{cases} \qquad x(0) = y(0) = 0$$

has a unique analytic solution through $(0,0)$.

In case $(0,0)$ is a singularity the dynamics of $\mathcal{F}$ around $(0,0)$ was first studied at the end of the XIX century by Briot-Bouquet [14] and Dulac [30].

They studied the problem for a particular class of singularities, nowadays called reduced singularities. Let $(0,0)$ be a singularity for $\mathcal{F}$ and let $J^1_{(0,0)}X$ denote the linear part of the the vector field X which defines $\mathcal{F}$. If both eigenvalues are different from zero and their ratio is not a positive rational number, then $(0,0)$ is a singularity of type $(*1)$. If one of the eigenvalues of $J^1_{(0,0)}X$ is zero but the other is different from zero, then, we say that the singularity is reduced of type $(*2)$. Briot-Bouquet [14] and Dulac [30] proved that for a singularity of type $(*1)$ there exist exactly two separatrices through 0 which are not singular and intersect transversally at $(0,0)$. In case of a $(*2)$ singularity it can be proved that there exists a separatrix through 0, but there might exists another one.

In 1968 Seidenberg [51] shows the importance of this class of singularities. He proves that, after a finite number of blows-up, it is possible to reduce the foliation to one having only reduced or dicritical singularities, namely, singularities for which there exist infinitely many separatrices.

In 1982 Camacho and Sad [18] prove that through every singularity passes at least one, possibly singular, separatrix, thus completely solving the Poincarè problem.

The new ingredient in their proof is the possibility of relating the dynamics of a holomorphic foliation on a compact non singular separatrix to the topological properties of the separatrix itself. More precisely, if S is a complex compact non singular curve, on a complex surface M, which is invariant by a holomorphic foliation $\mathcal{F}$, then, for every point $p \in S$ one can define a complex number $\operatorname{Ind}(\mathcal{F}, S, p)$, called index, that *reads* the dynamics of $\mathcal{F}$ near p. The sum of all these indices is equal to the self intersection number of S, namely, to the way S sits into M. This "index theorem" was generalized to the case of a singular curve by Lins Neto [40] and Suwa [55] and in higher dimension by Lehmann [38] and Lehmann-Suwa [39]. By this index theorem, Camacho and Sad prove that the reduced foliation always admits a *good* reduced singularity. This means that there exists a reduced singularity through which passes a separatrix that projects to a separatrix for the original vector field.

Subsequently J. Cano in 1997 [20] found an easier strategy to find the *good* singularity in the resolved foliation. He introduces a special class of points among the points of the exceptional divisor S:

- a point $p \in S$ is of type (C_1) if S is not singular at p and

$$\mathrm{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}.$$

- a point $p \in S$ is of type (C_2) if S has exactly two irreducible non singular branches S_0, S_1 that intersect transversally at p and there exists a number $r > 0$ such that:

$$\mathrm{Ind}(\mathcal{F}, S_0, p) \in \mathbb{Q}_{\leq -\frac{1}{r}} := \{x \in \mathbb{Q} \mid x \leq -\frac{1}{r}\}$$
$$\mathrm{Ind}(\mathcal{F}, S_1, p) \notin \mathbb{Q}_{\geq -r} := \{x \in \mathbb{Q} \mid x \geq -r\}.$$

Cano proves that blowing-up a (C_1) or a (C_2) singularity one gets another (C_1) or a (C_2) singularity. Then, after a finite number of blows-up this gives a *good* singularity.

The study of local holomorphic foliations is very much related to the study of local holomorphic diffeomorphisms. In one direction, because the exponential flow of a holomorphic vector field is a holomorphic diffeomorphism and, on the other direction, because the holonomy along a separatrix of a holomorphic foliation, is a holomorphic diffeomorphism.

The local dynamics of diffeomorphisms in dimension one is mostly well understood (see e.g. [21]). The most interesting case is when $|f'(0)| = 1$, which corresponds to the local holonomy around a singularity for a local holomorphic foliation. In the other cases the map is, indeed, linearizable. When $f'(0) = 1$ (or more generally when $f'(0)$ is a root of 1) the map is said to be tangent to the identity and its dynamics is completely described by the

Leau and Fatou theorem. The dynamical picture appears as a flower where each petal is, alternatively, an attractive and repelling domain.

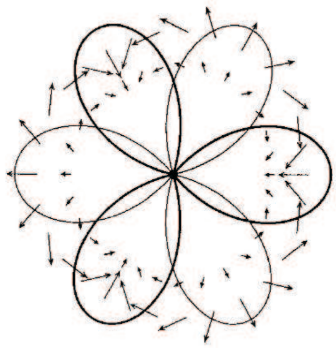


Figure 1: Leau-Fatou dynamics with three attractive petals and three repelling petals. The arrows go from z to $f(z)$.

When $f'(0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the germ is for almost every θ linearizable [52]. Cremer finds in [25] and [26] a family of maps that are not linearizable.

One of the goal of this work is to understand something more on dynamics of tangent to the identity maps in $\mathbb{C}^2$.

Ècalle [31] and Hakim [36] proved that, if f is a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ and $df_0 = Id$, then *generically* there exist $f-$invariant curves whose closure contains the origin and such that the dynamics is attractive (such curves are called parabolic curves for f in 0).

Successively Abate [2] proved the existence of such parabolic curves *for every* germ tangent to the identity in $\mathbb{C}^2$.

Abate technique retraces Seidenberg and Camacho-Sad strategy. In particular Abate introduces an index $\mathrm{Ind}(f, S, p)$ for a diffeomorphism f at a point p of a curve S of fixed points and gets an index theorem, similar to the

Camacho-Sad index one, that links the dynamics to the topological properties of the curve. Abate, as Camacho and Sad, blows-up the map and reduces to the case of a map with a smooth curve of fixed points for which he can use the index theorem.

This formal analogy with the continuous dynamics theory was studied in more detail in [3], [11], [12]. In [3] and [11] Abate, Bracci and Tovena relate discrete and continuous dynamics (see Chapter 2) associating to a map a family of local vector fields, whose flows approximate, at the first order, the map. In [2], [11] and [12] the crucial concept of tangentiality of a map along a curve of fixed points is introduced (and generalized in [3]), i.e. something analogous to the "continuous" concept of invariant curve for a vector field [28].

In [11] Bracci notes the possibility of simplifying Abate's technique, as Cano did in case of foliation. So the presence of a (C_1) or a (C_2) point guarantees the existence of a parabolic curve. A point that admits, in its resolution, a (C_1) or (C_2) point is called appropriate singularity.

At this point natural questions arise:

- by Camacho-Sad theorem we know that through every singularity of a holomorphic vector field a separatrix passes. So, when does another separatrix exist?

- given a diffeomorphism tangent to the identity with a singular curve of fixed points through which points does a parabolic curve pass?

We start with a holomorphic foliation (respectively a diffeomorphism tangent to the identity) with a separarix (respectively curve of fixed points) and we want to know if, and through which points, another separatrix (parabolic curve) passes.

A first step in this direction is made by Bracci, that in [11] examines the case of generalized cusps, i.e., curves of type $\{y^n = x^m\}$. The strategy consists in finding which conditions guarantee that a point is an appropriate singularity.

To make this we have to analyze the behavior of the index during the resolution of singularities of the foliation (map) but also of the separatrix (curve). This last analysis creates a lot of complications on the study of the evolution of the index, because we find points that can belong to one, two or even three irreducible branches of the total transform and in every of these cases the bheavior of the index changes.

An accurate study of the evolution of such index allows to state:

Theorem 0.1. *Let M be a complex two dimensional manifold, $\mathcal{F}$ a holomorphic foliation on some open subset of M, $S \subset M$ a possibly singular curve locally irreducible at a point $p \in M$, such that it is a separatrix for $\mathcal{F}$ at p. If $Ind(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists (at least) another separatrix for $\mathcal{F}$ at p.*

The answer we find to the second question is specular to the previous one:

Theorem 0.2. *Let M be a two dimensional complex manifold, $f : M \longrightarrow M$ a holomorphic map such that $Fix(f) = S$ with S a locally irreducible, possibly singular curve at a point $p \in M$. Assume that f is tangential on S and $Ind(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$. Then there exists (at least) a parabolic curve for f at p.*

These results generalize the results of Camacho-Sad and Abate. It is sufficient to perform a blow-up and remember that the self intersection of the exceptional divisor is -1 to find the conditions required by these theorems.

Both results are of local flavour, in fact they allow to find exactly through which points a separatrix (parabolic curve) passes. We note that this criterion of localization is the same in the two contexts.

All these facts underline, another time, the strict relation between the two settings.

If this connection is so deep, as it seems, how can we read definitions and objects of one setting in the other one?

Differently from [2], [3] and [11], in [28] we associated to a germ f of diffeomorphism tangent to the identity, having a curve S of fixed points, a formal vector field X such that $\exp(X) = f$.

Under this construction we have only to work with one vector field, instead of a family of vector fields as in [3] and [11], but we loose the convergence of the vector field. This construction is proposed even in a very recent preprint of Brochero Martinez, Cano and López Hernanz [15]. These authors consider the case of a map with an isolated singularity. They, as always, make a blow-up and reduce to a map with a particular curve of fixed points, i.e. the exceptional divisor. According to the *philosophical* idea that the natural setting is a map with a generic curve of fixed points, we start with this general assumption. So we find their results as particular cases.

Naturally, this construction is useful only if we can get an index theorem for the vector field X which is exacltly the *same* of that of the map's one. Such an index theorem might not exist for formal vector fields.

So, does the vector field X have some additional useful structure?

To answer to this question let observe what happens when the vector field is the blow-up of a formal one and the invariant curve is the exceptional divisor. If

$$X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

where $A, B \in \mathbb{C}[[x, y]]$ then, when we formally blow-up X, in the chart $x = u; y = uv$, the vector field becomes:

$$A(u, uv)\frac{\partial}{\partial u} + \left(\frac{B(u, uv) - vA(u, uv)}{u}\right)\frac{\partial}{\partial v}.$$

Now, it is easy to see that the coefficients of such vector field live in

$$\mathbb{C}[v][[u]] \oplus \mathbb{C}[v][[u]].$$

Namely, the coefficients are convergent (polynomial) in the coordinate *transversal* to the exceptional divisor $\{u = 0\}$. This structure is preserved even in the general case? The answer is positive and X is said transversely formal.

Theorem 0.3. *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ with $Fix(f) = S$, where S is a non singular complex curve and suppose f is tangential to S. Then, the formal vector field X such that* $\exp(X) = f$ *is transversely formal along the separatrix S, namely*

$$X \in \mathbb{C}\{x\}[[y]] \oplus \mathbb{C}\{x\}[[y]].$$

The transversely formality is the right condition to get an index theorem of Camacho Sad type [28].

Theorem 0.4. *Let X be a transversely formal vector field on a complex manifold of dimension two tangent to a compact, connected non singular curve $S \subset M$. Then, for every $p \in S$ there exists an index $Ind(X, S, p) \in \mathbb{C}$ such that:*

$$\sum_{p \in S} Ind(X, S, p) = S \cdot S.$$

The existence of such an index allows to get the same dynamical result found for holomorphic vector fields, even for transversely formal ones.

Theorem 0.5. *Let*

$$A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$$

be a transversely formal vector field with an invariant smooth curve S and let $p \in S$ be such that $Ind(X,S,p) \notin \mathbb{Q}^+ \cup \{0\}$. Then there exists another formal separatrix through p.

If we want to study by mean of this construction the dynamics of maps with a curve of fixed points we need that the vector field X does not have a discrete set of singularities on the separatrix S. To reduce to this case we have to algebraically normalize it. Such an operation will be made very often and we refer to the new vector field as the normalization of X.

In this way, the condition of tangentiality assumes an easy geometric interpretation: f is tangent to S if and obly if S is invariant for the normalization of X along S.

This construction shows that the study of the evolution of the index during the reduction is the same for maps and vector fields. So, even by this way, we recover Theorem 0.2.

All the techniques used to study the dynamics of maps until now, always, refer to foliation techniques.

So a natural question arise: can all the parabolic curves be found by a blow-up process? Abate and Tovena in [5] noticed that the parabolic curves found in this way have an additional structure, in fact they survive under blows-up. Such parabolic curves are called robust parabolic curves. Abate and Tovena find in [5] a family of self-maps of $\mathbb{C}^3$ tangent to the identity and with the origin as an isolated fixed point with parabolic curves but with no robust parabolic ones. So in dimension three the two concepts are not the same. Until now we do not know if such difference exists even in dimension two. However, when we try to find an upper bound for the number of parabolic curves, with our method, we have to restrict to robust parabolic curves.

In [29] we study the dynamics flows of vector fields tangent to the identity . Such a set, denoted by $\Phi_{\geq 2}(\mathbb{C}^2, 0)$, is a dense subset of the space of germs of maps tangent to the identity in $\mathbb{C}^2$. In this work we give a geometric interpretation of some concepts of discrete dynamics, such as tangentiality, and examine with more attention the relationship of some concepts, such as dicriticity, for vector fields and maps.

We find that the concept of dicriticity passes from the map to the associated vector field and viceversa.

Proposition 0.6. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a map tangent to the identity in $\mathbb{C}^{\ 2}$ and let X be the vector field such that $exp(X) = f$. Then f is dicritical in 0 if and only if X is dicritical in 0.*

So we have that a dicritical map has infinitely many robust parabolic curves if and only if the vector field has infinitely many separatrices.

For what concerns the non dicritical case the robust parabolic curves are exactly the parabolic curves living inside a separatrix of the associated vector field:

Proposition 0.7. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a holomorphic map and let X be vector field such that $\exp(X) = f$. Let φ be a robust parabolic curve. Then φ is contained in a formal separatrix of X. Conversely in every formal separatrix of X there exists at least one robust parabolic curve for f.*

If we can estimate the number of robust parabolic curves inside a separatrix and the number of separatrices, we get an upper bound for the number of robust parabolic curves. According to the work of Corral and Fernandez-Sanchez in [24] an upper bound for the number of separatrices exists and we find the following estimation for the number of robust parabolic curves.

Theorem 0.8. *Let* $f = (f_1, f_2) \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ *be a non-dicritical holomorphic map. Set* $\eta(f) := \max\{ord(f_1 - Id), ord(f_2 - Id)\}$ *and* $\mu(f)$ *the Milnor number of* f*. Then the number of robust parabolic curves is at most*

$$(\mu(f) + 1)(\eta^2(f) - \eta(f)).$$

Chapter 1

Continuous dynamics

In this chapter we recall the basic facts about holomorphic foliations. We start with the definition of a non singular foliation and then generalize it to the singular case, that is the most interesting case for this study. We end up by studying singular foliations in $\mathbb{C}^2$.

1.1 Holomorphic foliations

In this section we briefly recall the basic definitions of holomorphic foliations. For more details we refer to [16], [17] and [34].

Definition 1.1. *Let M be a complex manifold of complex dimension m. A holomorphic foliation $\mathcal{F}$ on M of complex codimension k is given by a holomorphic maximal atlas*

$$\{\varphi_j : U_j \to \varphi_j(U_j)\}_j$$

of M such that the transition maps are of the form:

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : &\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \\ &(x, y) \mapsto (g_{i,j}(x, y), h_{i,j}(y)) \end{aligned}$$

with $g_{i,j}$ and $h_{i,j}$ holomorphic maps. The charts φ_j are called trivialization charts.

Remark 1.2. *A holomorphic foliation $\mathcal{F}$ is a real codimension $2k$ foliation of M which is given by a holomorphic atlas of M whose transition maps preserve the complex structure.*

Let us consider a real foliation $\mathcal{F}$ of real codimension k on an $n+k$ manifold M, let $\varphi_j : U_j \to \varphi_j(U_j)$ be a chart as in the definition. The plaques of $\mathcal{F}$ in U are given by $\varphi^{-1}(\mathbb{R}^{2n} \times \{y\})$. Let P and $\tilde{P}$ be two plaques of the charts φ and $\tilde{\varphi}$. Then either $P \cap \tilde{P} = \emptyset$ or $P \cap (U \cap \tilde{U}) = \tilde{P} \cap (U \cap \tilde{U})$. On M we can define the following equivalence relation: two points $p, q \in M$ are equivalent if and only if there are some plaques $P_1, \cdots, P_r$ such that $p \in P_1$, $q \in P_r$ and $P_i \cap P_{i+1} \neq \emptyset \forall i$.

Definition 1.3. *A leaf of $\mathcal{F}$ is a class $[p] \subset M$ under the previous introduced equivalence relation.*

Definition 1.4. *The leaves of a holomorphic foliation $\mathcal{F}$ are the leaves of the underlying real foliation.*

Remark 1.5. *The leaves, endowed with the natural complex structure, become complex immersed submanifolds of M.*

We can now recall what a singular foliation is.

Definition 1.6. *A holomorphic foliation with singularities on a complex manifold M is a pair $\mathcal{F} = (\mathcal{F}', X)$ where $X \subsetneq M$ is a proper analytic subset of M with $codim(X) \geq 2$ and $\mathcal{F}'$ is a (non singular) holomorphic foliation on $M' = M \setminus X$.*

The leaves of $\mathcal{F}$ are the leaves of $\mathcal{F}'$ on M'. The set X is called the singular set of $\mathcal{F}$, $Sing(\mathcal{F}) = X$.

Definition 1.7. *A singular holomorphic foliation is called saturated if it is not possible to find an analytic subset $X' \subset X$ such that $\mathcal{F}'$ extends as a non singular holomorphic foliation to $M \setminus X'$.*

1.1.1 Holomorphic foliations on complex surfaces

In complex dimension two it is easy to relate holomorphic foliations to holomorphic vector fields [41], [47]. Let ζ be a holomorphic vector field in a neighborhood of the origin U and suppose that $Sing(\zeta) = \{0\}$. Let observe that if

$$\zeta = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$$

with $\{A(x,y) = 0\} \cap \{B(x,y) = 0\} = \{0\}$ then ζ defines a complex ordinary differential equation by:

$$\dot{\xi} = \zeta(\xi),$$

i.e.

$$\begin{cases} \dot{x} = A(x,y) \\ \dot{y} = B(x,y). \end{cases}$$

The set of solutions defines a singular foliation in U.

Viceversa, let $\mathcal{F}$ be a holomorphic foliation in a neighborhood U of the origin in $\mathbb{C}^2$ with $Sing(\mathcal{F}) = \{0\}$. Let be $p \in U \setminus \{0\}$ and define the function:

$$f : U \setminus \{0\} \to \mathbb{C}$$
$$p \mapsto f(p),$$

where $f(p)$ is the inclination of the complex line tangent to the leaf of $\mathcal{F}$ for p in p. Such a function is a meromorphic function $f : U \setminus \{0\} \to \mathbb{C}$. Now we recall two basic fact of complex analysis (see [35]):

Theorem 1.8 (Hartogs' Theorem)**.** *Let M be a complex manifold, $X \subset M$ an analytic subset of codimension bigger or equal then two. Let ω be* *a meromorphic q-form in $M \setminus X$. Then ω admits a unique meromorphic extension $\tilde{\omega}$ to M.*

Theorem 1.9 (Cartan's Theorem)**.** *In a neighborhood of a ball in $\mathbb{C}^n$ with $n \geq 2$ any meromorphic function f is the quotient of two holomorphic functions A and B such that $\{A = 0\} \cap \{B = 0\}$ has no components of positive dimension.*

From this two theorems we easily deduce that $f = \frac{A}{B}$ and

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)}.$$

so, the leaves of $\mathcal{F}$ are the integral curves of

$$A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}.$$

Proposition 1.10. *Any holomorphic foliation $\mathcal{F}$ in a neighborhood of the origin of $\mathbb{C}^2$ can be given by a holomorphic vector field ζ and the singularities of $\mathcal{F}$ are the singularities of the vector field.*

Let ζ be a holomorphic vector field in $U \subset \mathbb{C}^2$ such that $\{\zeta = 0\}$ contains some one dimensional component S and suppose S is described by a holomorphic function $h : U \to \mathbb{C}$, i.e., $S = \{h = 0\}$. We can always assume that h is reduced in the following sense: if a local function F vanishes on $\{\zeta = 0\}$ then $F = h^m F'$ where $m \in \mathbb{N}$ and $\{F' = 0\} \cap \{\zeta = 0\}$ has dimension zero. So we can write the vector field in the following way

$$h^m A_1 \frac{\partial}{\partial x} + h^n B_1 \frac{\partial}{\partial y}$$

and so

$$\zeta = h^{m'} \zeta_1,$$

where $m' = min\{m, n\}$ and ζ_1 is a holomorphic vector field in U such that $\{\zeta_1 = 0\}$ has dimension zero. Thus the foliation $\mathcal{F}_\zeta$ admits an extension $\mathcal{F}_{\zeta_1}$ which satisfies Codim(Sing(ζ_1)) = 2.

In conclusion we can always assume in case of foliations in $\mathbb{C}^2$ that:

(i) the foliation, locally, is given by a holomorphic vector field;

(ii) the foliation is saturated, namely, it has only isolated singularities.

Proposition 1.11. *Let $\mathcal{F}$ be a saturated holomorphic foliation on M^2. Then it exists an open covering $\{U_i\}$ of M such that $\mathcal{F}|_{U_i}$ is given by a holomorphic vector field ζ_i in U_i and the number of singularities of ζ_i is at most one. For any i, j such that $U_i \cap U_j \neq \emptyset$ there exists a holomorphic non vanishing function $h_{i,j} : U_i \cap U_j \to \mathbb{C}$ such that $\zeta_i = h_{i,j}\zeta_j$.*

Remark 1.12. *By the previous proposition we find a family of functions $h_{i,j}$ satisfying a cocycle condition. Thus defining a holomorphic line bundle L^* over M and a natural morphism $L \to T^{1,0}M$ (here L is the dual of L^*). This line bundle, L, is called the tangent bundle to the foliation. In fact, an equivalent way to define a holomorphic foliation on a two dimensional complex manifold M is by means of a holomorphic line bundle L over M and a morphism $\varphi : L \to T^{1,0}M$ [34].*

Let us observe that on the open set U_j the vector field v_j, defining the foliation, in the local coordinates (z, w), assumes the form:

$$v_j = A_j(z, w)\frac{\partial}{\partial z} + B_j(z, w)\frac{\partial}{\partial w}, \tag{1.1}$$

for some $A_j, B_j \in \mathcal{O}(U_j)$. Therefore, v_j, where $v_j \neq 0$, generates the kernel of the holomorphic $1-$form ω_j defined by:

$$\omega_j = B_j(z, w)dz - A_j(z, w)dw. \tag{1.2}$$

In conclusion a holomorphic foliation in $\mathbb{C}^2$ (or more generally on a complex two dimensional manifold) can be seen as an equivalence class of vector fields or as an equivalence class $[\{U_j\}_{j\in J}, \{\omega_j\}_{j\in J}]_\sim$ of holomorphic $1-$forms where $\{\{U_j\}_{j\in J}, \{\omega_j\}_{j\in J}\} \sim \{\{U_i'\}_{i\in I}, \{\omega_i'\}_{i\in I}\}$ if and only if for every i, j such that $U_j \cap U_i' \neq \emptyset$ it exists a function, $f \in \mathcal{O}^*(U_j \cap U_i')$, such that $\omega_j = f\omega_i'$.

1.2 Singularities and normal forms

A normal form for a holomorphic vector field is a particulary easy to handle expression of the vector field which we can obtain after some manipulations. If we are interested in the analytic (formal) structure of the vector field then the equivalence relation is the analytic (formal) conjugation.

Definition 1.13. *Two germs of holomorphic vector fields $(\mathbb{C}^2, 0)$, X_1, X_2 are holomorphically (formally)* ***conjugated*** *if there exists a germ of holomorphic (formal) diffeomorphism Φ of $(\mathbb{C}^2, 0)$ such that:*

$$d\Phi \circ X_1 \circ \Phi^{-1} = X_2.$$

If instead we were interested in the geometric structure determined by the vector field, i.e., in the foliation determined by the vector field, we can consider the following equivalence relation:

Definition 1.14. *Two germs of holomorphic vector fields in $(\mathbb{C}^2, 0)$, X_1, X_2 are holomorphical (formally)* ***equivalent*** *if there exists a germ of holomorphic (formal) diffeomorphism Φ of $(\mathbb{C}^2, 0)$ such that:*

$$d\Phi \circ X_1 = \Psi \cdot X_2 \circ \Phi,$$

where Ψ is a holomorphic (formal) function with $\Psi(0, 0) \neq 0$.

Remark 1.15. *Let λ_1, λ_2 be the eigenvalues of the linear part of the vector field X. Up to conjugation, we can assume that the linear part of the vector field is diagonal (or triangular). This linear part is equivalent (if the eigenvalues are not both zero) to:*

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda_2}{\lambda_1} \end{pmatrix}.$$

According to the type of eigenvalues of the vector field we can divide the singularities in the following way:

Definition 1.16. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ which is singular at the origin. Let λ_1, λ_2 be the eigenvalues of the linear part of X. The singularity is*

- *in the **Poincarè domain** if:*

  1. *$\lambda_1\lambda_2 \neq 0$; and*
  2. *$\frac{\lambda_1}{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}^-$.*

- *in the **Siegel domain** if:*

  1. *$\lambda_1\lambda_2 \neq 0$;*
  2. *$\frac{\lambda_1}{\lambda_2} \in \mathbb{R}^-$.*

- *a **saddle-node** if an eigenvalue is zero and the other is not zero;*

- *a **nilpotent** singularity if both eigenvalues are zero but the linear part is not zero i.e. the linear part is equivalent to $x_2 \frac{\partial}{\partial x_1}$.*

1.2.1 Poincarè domain

In this section we recall some results of Poincaré. The proofs can be found, e.g., in [19].

Theorem 1.17. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ singular at the origin. Suppose that the singularity lies in the Poincarè domain. If $\frac{\lambda_1}{\lambda_2} \notin \mathbb{N} \cup \frac{1}{\mathbb{N}}$ (where $\frac{1}{\mathbb{N}} = \{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$) the vector field is holomorphically conjugated to its linear part.*

The case in which $\frac{\lambda_1}{\lambda_2} \in \mathbb{N} \cup \frac{1}{\mathbb{N}}$ was studied by Poincarè and Dulac:

Theorem 1.18. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$. If $\frac{\lambda_1}{\lambda_2} = n \in \mathbb{N} \cup \frac{1}{\mathbb{N}}$, then X is holomorphically equivalent to:*

$$(nx_1 + ax_2^n)\frac{\partial}{\partial x} + x_2\frac{\partial}{\partial x_2}.$$

1.2.2 Siegel domain

In this section we assume the singularity is in the Siegel domain i.e. the quotient of the eigenvalues is negative real. For the proofs we refer to [19].

Theorem 1.19. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ which is singular at the origin. Suppose the singularity is in the Siegel domain. Then it is holomorphically conjugated to:*

$$(\lambda_1 x_1 + x_1 x_2 a(x_1, x_2))\frac{\partial}{\partial x_1} + (\lambda_2 x_2 + x_1 x_2 b(x_1, x_2))\frac{\partial}{\partial x_2},$$

where $a, b \in \mathbb{C}\{x_1, x_2\}$.

From a formal point of view we have:

Theorem 1.20. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ singular at the origin and suppose the singularity is in the Siegel domain, then:*

1. *if $\frac{\lambda_1}{\lambda_2} \in \mathbb{Q}^-$ the vector field is formally conjugated to:*

$$(\lambda_1 x_1 + \sum_{k\geq 1} \psi_{1,k} x_1^{nk+1} x_2^{km})\frac{\partial}{\partial x_1} + (\lambda_2 x_2 + \sum_{k\geq 1} \psi_{2,k} x_1^{kn} x_2^{km+1})\frac{\partial}{\partial x_2};$$

2. *if $\frac{\lambda_1}{\lambda_2} \in \mathbb{R}^- \setminus \mathbb{Q}$ then the vector field is formally linearizable.*

1.2.3 Saddle node

Saddle-node singularities were first studied by Briot, Bouquet and Dulac [19], [30].

Theorem 1.21. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ with a saddle node type singularity at the origin. Then is holomorphically conjugated to:*

$$(x_1 + x_2 g(x_1, x_2))\frac{\partial}{\partial x_1} + (x_2 h(x_1, x_2))\frac{\partial}{\partial x_2},$$

with $g, h \in \mathbb{C}\{x_1, x_2\}$.

from the formal point of view we can say something more on the structure of g and h.

Theorem 1.22. *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ with a saddle node singularity at the origin. Then is formally conjugated to:*

$$x_1^{p+1}\frac{\partial}{\partial x_1} + (x_2(1 + \omega x_1^p))\frac{\partial}{\partial x_2},$$

with $p \in \mathbb{N}$ and $\omega \in \mathbb{C}$.

1.2.4 Nilpotent singularity

In case of nilpotent singularities the problem of normal forms is more complicated and not completely solved. Generally, we have the Takens pre-normal forms:

Theorem 1.23 ([56])**.** *Let X be a holomorphic vector field in $(\mathbb{C}^2, 0)$ with a nilpotent singularity at the origin. Then is formally equivalent to:*

$$(2x_2 + x_1^p U(x_1))\frac{\partial}{\partial x_1} - n x_1^{n-1}\frac{\partial}{\partial x_2}, \tag{1.3}$$

where $2 \le n - 1$, $p \ge 2$ and $U(x_1) \in \mathbb{C}\{x_1\}$ with $U(0) \neq 0$.

We can suppose that, up to conjugation, every nilpotent singularity is of the form:

$$(x_2 + a(x_1))\frac{\partial}{\partial x_1} + b(x_1)\frac{\partial}{\partial x_2}$$

with $a(x_1) = a_r x_1^r + \cdots$ and $b(x_1) = b_{s-1}x_1^{s-1} + \cdots$.

Theorem 1.24 ([54]). *The pre-normal form of Takens is analytic (i.e. the series that reduces the vector field in the pre-normal form is convergent) if* $s < \infty$.

In order to specify the structure of the function U we have to make some distinctions.

The numbers r and s are not invariant for the vector field but their relations are invariant. Thus we can make a classification of nilpotent singularities according to these relations:

1. we have a **generalized cusp** if $b_{s-1} \neq 0$ and $s < 2r$;
2. we have a **generalized saddle-node** if $a_r \neq 0$ and $2r < s$;
3. we have a **generalized saddle** if $s = 2r, a_r \neq 0$ and $b_{s-1} \neq 0$.

Generalized cusp singularities have been classified by Stróżyna and Żołądek in [54]. To express such a classification we set:

$$\begin{aligned} X_H &= 2x_2\frac{\partial}{\partial x_1} + sx_1^{s-1}\frac{\partial}{\partial x_2}; \\ E_H &= 2x_1\frac{\partial}{\partial x_1} + sx_2\frac{\partial}{\partial x_2} \\ n_0 &= \frac{r}{s} - \frac{1}{2}. \end{aligned}$$

Theorem 1.25 ([54]). *Let X be a holomorphic vector field in* $(\mathbb{C}^2, 0)$ *with a generalized cusp singularity at the origin in the pre-normal form* (1.3). *Then* *is formally equivalent to one of the following normal forms* $J^s_{r,\phi}$*. These forms are classified by a formal series* $\phi(x) = \sum^* c_j x^j$ *(the star means that the summation is taken over a particular subset of integers), by the class modulo* s *of the exponent* $r \neq 0$ *or by* $r = \infty$.

- $J^s_{\infty,0} = X_H$;
- $J^s_{r,\phi} = X_H + x_1^{r-1}(1+\phi(x_1))E_H$, *where*

$$\sum{}^* = \sum_{j \neq 0, -r (mod\, s)},$$

  if $r < \infty$ *e* $n_0 \notin \mathbb{Z}$;
- *the vector field* (1.25) *where:*

$$\phi = c_{n_0} s x_1^{n_0 s},$$

  if $r < \infty$ *e* $n_0 \in \mathbb{Z}$;
- *the vector field* (1.25) *where:*

$$\sum{}^* = \sum_{j \in \{n_0 s, j_0\}} + \sum_{j > j_0 \quad j \neq 0, -r (mod\, s), \quad j \neq j_0 + n_0 s}.$$

If two vector field with normal forms $J^s_{r,\phi}$ *e* $J^{s'}_{r',\phi'}$ *are formally equivalent then* $r = r', s' = s'$ *and* $\phi(x_1) \equiv \phi(\alpha x_1)$ *for some constant* α *that satisfies* $\alpha^{2r-s} = 1$, *when* $r \neq \infty$.

The study of generalized saddle node is in [53]. Let consider:

$$n_0 = \frac{s}{r} - 2$$
$$E_H = x_1 \frac{\partial}{\partial x_1} + r y \frac{\partial}{\partial x_2}.$$

Theorem 1.26 ([53]). *Every holomorphic vector field with a generalized saddle node singularity at the origin is formally equivalent to one of the following normal forms* $J_r^{s,\phi}$. *Such forms are classified by the exponents* $s = 2r+1, 2r+2, \cdots, \infty$ *and by the formal series* $\phi(x_1) = \sum {}^{*} d_j x_1^j$:

- $J_r^{\infty,0} = (x_2 - x_1^r)\frac{\partial}{\partial x_1}$;
- $J_r^{s,\phi} = (x_2 - x_1^r)\frac{\partial}{\partial_1} + x_1^{s-r-1}(1+\phi(x_1))E_H$ *where*
$$\sum{}^{*} = \sum_{j \neq 0 (mod r)},$$
if $s < \infty$ *and* $n_0 \notin \mathbb{Z}$;
- $J_r^{s,\phi}$ *with* $\phi = d_{n_0} x_1^{n_0 r}$, *if* $s < \infty$ *and* $n_0 \in \mathbb{Z}$;
- $J_r^{s,\phi}$ *with:*
$$\sum{}^{*} = \sum_{j \in \{n_0 r, j_0\}} + \sum_{j > j_0,\ j \neq 0 (mod r),\ j \neq j_0 + n_0 r},$$
if $s < \infty$ *and* $n_0 \in \mathbb{Z}$ *and there exists a non zero coefficient* d_{j_0} *with* $j_0 \neq 0 mod r$.

If two vector field with a generalized saddle node singularity and with normal forms $J_r^{s,\phi}$ *and* $J_{r'}^{s',\phi'}$ *are formally equivalent then* $r = r', s = s'$ *and* $\phi'(x_1) \equiv \phi(\alpha x_1)$ *for some constant* α *such that* $\alpha^{s-2r} = 1$.

1.3 Desingularization theorem

1.3.1 Blow-up of a complex surface

The blow-up of a complex surface M at a point $p \in M$ consists of substituting the point p by a complex projective line, called **exceptional divisor**. In this way we find a new manifold of complex dimension two where, in place of p, we have a complex projective line that we can *interpret* as the set of tangent directions of M at p. Let M be a complex surface, p a point of M and let T_pM be the tangent space of M in p. Let:

$$[\cdot] : \mathrm{T}_pM \setminus \{0\} \longrightarrow \frac{\mathrm{T}_pM \setminus \{0\}}{\sim} \cong \mathbb{CP}(1)$$

where $\mathbb{CP}(1)$ is a complex projective line.

Let define:

$$M_p = (M - p) \mathring{\cup} \mathbb{CP}(1)$$

and introduce on it a structure of differentiable manifold as follows.

If U is a coordinate open set of M that does not contain p then the corresponding chart on M is even a chart on M_p. If, instead, U is a coordinate neighborhood of p in M and $\chi : U \subseteq M \longrightarrow \mathbb{C}^2$ is a chart on M such that $\chi(p) = (x(p), y(p)) = (0, 0)$ we define:

$$V_1 = (U - \chi^{-1}(\{0\} \times \mathbb{C})) \cup (\mathbb{CP}^1 - [K_1])$$

$$V_2 = (U - \chi^{-1}(\{0\} \times \mathbb{C})) \cup (\mathbb{CP}^1 - [K_2])$$

where $K_1 = \ker(dx)_p$ and $K_2 = \ker(dy)_p$, that is $K_1 =< [\frac{\partial}{\partial y}|_p] >$ and $K_2 =< [\frac{\partial}{\partial x}|_p] >$. Notice that $V_1 \cup V_2 = U$ and $V_1 \cap V_2 \neq \emptyset$. Let define:

$$\begin{aligned} \chi_1 : V_1 &\longrightarrow \mathbb{C}^2 \\ q &\longmapsto \left(x(q), \frac{y(q)}{x(q)}\right), \text{ if } q \in U - \chi^{-1}(\{0\} \times \mathbb{C}) \\ \left[\alpha_1 \frac{\partial}{\partial x}|_p + \alpha_2 \frac{\partial}{\partial y}|_p\right] &\longmapsto \left(0, \frac{\alpha_2}{\alpha_1}\right), \text{ if } \left[\alpha_1 \frac{\partial}{\partial x}|_p + \alpha_2 \frac{\partial}{\partial y}|_p\right] \in \mathbb{CP}^1 \setminus [K_1]. \end{aligned}$$

and:

$$\begin{aligned} \chi_2 : V_2 &\longrightarrow \mathbb{C}^2 \\ q &\longmapsto \left(\frac{x(q)}{y(q)}, y(q)\right) \text{ if } q \in U - \chi^{-1}(\{0\} \times \mathbb{C}) \\ \left[\alpha_1 \frac{\partial}{\partial x}|_p + \alpha_2 \frac{\partial}{\partial y}|_p\right] &\longmapsto \left(\frac{\alpha_1}{\alpha_2}, 0\right) \text{ if } \left[\alpha_1 \frac{\partial}{\partial x}|_p + \alpha_2 \frac{\partial}{\partial y}|_p\right] \in \mathbb{CP}^1 \setminus [K_2]. \end{aligned}$$

Then $\{(V_i, \chi_i)\}_{i=1,2}$ is an atlas for M_p.

The projection:

$$\begin{aligned} \pi_p : & M_p \longrightarrow M \\ & u \longmapsto \begin{cases} u & \text{if } u \in M \setminus \{p\} \\ p & \text{if } u \in \mathbb{CP}^1 \end{cases} \end{aligned}$$

is a holomorphic function and, in local coordinates, assumes the form:

$$\pi_p(u, v) = (u, uv) \text{ in } V_1 \ ,$$

$$\pi_p(u, v) = (uv, v) \text{ in } V_2.$$

1.3.2 Desingularization of a foliation

Let consider a germ of holomorphic foliation in $\mathbb{C}^2$ which is singular at the origin. We have seen that it can be expressed in the following form:

$$\begin{cases} \dot{x} = A(x, y) \\ \dot{y} = B(x, y) \end{cases}$$

where $A(0, 0) = B(0, 0) = 0$. If we denote by ν the algebraic multiplicity of this equation at $0 \in \mathbb{C}^2$, i.e. the least order of its expression at $0 \in \mathbb{C}^2$, then we can write the previous equation in the form:

$$\begin{cases} \dot{x} = A_\nu(x, y) + A_{\nu+1}(x, y) + \cdots \\ \dot{y} = B_\nu(x, y) + B_{\nu+1}(x, y) + \cdots \end{cases}$$

where A_i and B_i are the homogenous part, of degree i, of the power series expression of A and B.

In the chart of the blown-up manifold for which the projection assumes the form $\pi(u, v) = (u, uv)$ the foliation becomes:

$$\begin{cases} \dot{u} = A_\nu(u, uv) + A_{\nu+1}(u, uv) + \cdots \\ \dot{v} = \dfrac{\dot{y} - v\dot{u}}{u} = \dfrac{B_\nu(u, uv) + B_{\nu+1}(u, uv) + \ldots - v(A_\nu(u, uv) + \ldots)}{u}, \end{cases}$$

or, in a better form:

$$\begin{cases} \dot{u} = u^{\nu}(A_{\nu}(1,v) + uA_{\nu+1}(1,v) + \cdots) \\ \dot{v} = u^{\nu-1}(P_{\nu+1}(1,v) + u(\cdots)), \end{cases}$$

where $P_{\nu+1}(x,y) = xB_{\nu}(x,y) - yA_{\nu}(x,y)$. If we use the other chart we find another vector field that is equivalent to this one on the intersection. So, we have a global foliation on the blown-up manifold. We have two cases: the dicritical one and the non dicritical one.

In case $P_{\nu+1}(x,y) \equiv 0$ the singularity is called **dicritical** and the foliation becomes:

$$\begin{cases} \dot{u} = u^{\nu}(A_{\nu}(1,v) + uA_{\nu+1}(1,v) + \cdots \\ \dot{v} = u^{\nu}(\cdots). \end{cases}$$

Dividing by the factor u^{ν} we obtain a saturated foliation that is equivalent to the previous one outside $(u = 0)$. Let observe that in this case the exceptional divisor $(u = 0)$ is not invariant for the foliation.

In case $P_{\nu+1}(x,y) \not\equiv 0$ we have a **non dicritical** singularity and to get a saturated foliation we can only divide by $u^{\nu-1}$. In this way the curve $(u = 0)$ is an invariant curve for the foliation.

Let observe that in both cases we find a discrete set of singularities and so we can repeat the construction for every singularity. In this way we can reduce the vector field to have only a particular class of singularities: dicritical ones and reduced ones.

Definition 1.27. *Let M be a complex surface, $\mathcal{F}$ an holomophic foliation on M and $p \in M$, an isolated singularity of $\mathcal{F}$. The point p is a* ***reduced singularity*** *for $\mathcal{F}$ if one of the following conditions holds:*

(∗1) $\lambda_1 \neq 0, \lambda_2 \neq 0$ *and* $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+ \cup \{0\}$

(∗2) $\lambda_1 \neq 0, \lambda_2 = 0$ *or* $\lambda_1 = 0, \lambda_2 \neq 0$.

Theorem 1.28 (Seidenberg Theorem [16], [43], [51]). *After a finite number of blows-up we can reduced the foliation to one that has only dicritical singularities and reduced singularities.*

1.4 Separatrices of foliations

Let us consider the foliation:

$$\begin{cases} \dot{x} = 3y^2 \\ \dot{y} = 2x. \end{cases}$$

The foliation is singular at $(0,0)$, the curve $\{x^2 - y^3 = 0\} \setminus \{(0,0)\}$ is a leaf for the foliation and its closure $\{x^2 - y^3 = 0\}$ is a singular curve, containing $(0,0)$. In order to analyze such a situation we start with a definition:

Definition 1.29. *Let M be a complex surface, $\mathcal{F}$ a holomorphic foliation on M and p a point of M. A* ***local separatrix*** *of $\mathcal{F}$ at p is a germ of irreducible complex curve $C \subset M$, such that $p \in C$ and $C \setminus \{p\}$ is a leaf of $\mathcal{F} \mid_{M \setminus Sing(\mathcal{F})}$. Namely, if v_j represents the foliation $\mathcal{F}$ on the open set U_j that contains the point p, then:*

$$span_{\mathbb{C}}\{v_j(q)\} = T_q C.$$

Remark 1.30. *Let C be a separatrix for $\mathcal{F}$. If $j : C \longrightarrow M$ is the immersion of C in M and the foliation, $\mathcal{F}$, is defined by $\{\omega = 0\}$ with ω a holomorphic $1-$form on M, then $j^*(\omega \mid_{C \setminus \{p\}}) \equiv 0$.*

If the point p is not a singularity for the foliation $\mathcal{F}$ then the Cauchy-Kowaleskaya theorem assures the existence of exactly one non singular complex curve C such that $p \in C$ and C is a leaf of $\mathcal{F}$. This means that a separatrix C is such that $\mathrm{Sing}(C) \subset \mathrm{Sing}(\mathcal{F})$.

If the point p is a dicritical singularity then infinite separatrices pass trough it.

Lemma 1.31 ([16])**.** *Let M a complex surface and $\mathcal{F}$ an holomorphic foliation on it. Let p be a point of M. If p is a dicritical singularity for $\mathcal{F}$ then there exist infinitely many separatrices through p.*

In case of reduced singularity the problem of existence of separatrices can be solved using the normal forms we have described in the previous section.

if p is a reduced singularity of type $(*1)$ then there exist exactly two separatrices passing through p and they intersect each other transversally,

if p is a singularity of type $(*2)$ then there exist two formal separatrices passing through p, one is always convergent and it is called **strong separatrix**, and the other is generally only formal and it is called **weak separatrix**.

1.5 Camacho-Sad index theorem

In this section we recall the Camacho-Sad index theorem [18] in its general form [55], [39]. Let S be a singular curve on a complex surface M and let suppose that S is compact and irreducible. Let $\mathcal{F}$ be a holomorphic foliation on M with a discrete set of singularities on S. Let $S' := S \setminus (\mathrm{Sing}(S) \cup \mathrm{Sing}(\mathcal{F}))$.

Lemma 1.32 (Baum-Bott vanishing lemma for the Camacho-Sad theorem [39])**.** *There exists a connection ∇ for the normal bundle $N_{S'}$ of S' in M, such that its curvature $K \equiv 0$ on S'.*

By the previous lemma we can prove the following index theorem in case S is even singular [39] [55] (Camacho and Sad proved the result only in case S is not singular). The proof is based on the localization around the singularities of the curvature K of the previous connection.

Theorem 1.33 (Suwa, Lehmann [39], [55]). *Let S be a compact irreducible curve in M. Let $\mathcal{F}$ be a holomorphic foliation in M with $Sing(\mathcal{F})$ discrete and such that S is $\mathcal{F}-$invariant. Then for every point $p \in Sing(\mathcal{F}) \cap S$ there exists a complex number $Ind(\mathcal{F}, S, p) \in \mathbb{C}$ that depends only on the behavior of $\mathcal{F}$ near S in p such that:*

$$\sum_{p \in Sing(\mathcal{F}) \cap S} Ind(\mathcal{F}, S, p) = S \cdot S.$$

Moreover, if S is defined by $\{f = 0\}$ at p, and $\mathcal{F}$ is defined by $\omega = f d\tau + h df = 0$ at p, then:

$$Ind(\mathcal{F}, S, p) = \frac{1}{2\pi i} \int_L \frac{d\tau}{h}$$

where $L = S \cap S^3$ and S^3 is a sphere centered in p with a small *radius.*

As we can see in the following section it will be convenient to use this index together with blow-up techniques. So we describe the behavior of the index under blow-up.

Proposition 1.34. *Let M be a two dimensional complex manifold, $\mathcal{F}$ a holomorphic foliation, S an $\mathcal{F}$ -separatrix and $p \in Sing(S)$. We denote by $\pi : \tilde{M} \longrightarrow M$ the blow-up of M at p, by $\tilde{\mathcal{F}}$ the saturated foliation and by $D := \pi^{-1}(p)$ and $\hat{S} := \overline{\pi^{-1}(S \setminus \{p\})}$, respectively, the exceptional divisor and the strict transform of S. Then $\hat{S}$ is an $\tilde{\mathcal{F}}$ separatrix. Moreover if $\{\tilde{p}\} := D \cap \hat{S}$ then*

$$Ind(\tilde{\mathcal{F}}, \hat{S}, \tilde{p}) = Ind(\mathcal{F}, S, p) - m^2$$

where $m \geq 1$ is the multiplicity of S in p.

1.6 Camacho-Sad and Cano results

The general problem of the existence of a separatrix for holomorphic foliations on complex surfaces was solved by Cesar Camacho and Paulo Sad in 1982:

Theorem 1.35 (Camacho-Sad theorem [18]). *Let M be a complex surface and $\mathcal{F}$ a holomorphic foliation on M. Then through every singularity p of $\mathcal{F}$ passes at least a separatrix.*

We recall the steps of the proof of this results:

1. study of the existence of separatrices through reduced singularities,
2. Seidenberg reduction theorem,
3. Index Theorem,
4. combinatoric part.

By Seidenberg theorem we can assume, after a finite number of blows-up, that all singularities on the exceptional divisor are only of type $(*1)$ or $(*2)$. In fact, in case of presence of dicritical singularities Lemma 1.31 assures the existence of infinite separatrices and so the problem is solved. Now, the problem is that it could happen that all singularities of type $(*1)$ are intersections of irreducible components of the exceptional divisor and all singularities of type $(*2)$ are smooth points of the exceptional divisor that have not convergent weak separatrix. To prove that this situation cannot happen Camacho and Sad exploited an index theorem that link the singularities of $\mathcal{F}$ on every irreducible component of the exceptional divisor with the geometry of its immersion in the ambient space. By this tool and a combinatoric argument they proved that it must exist a *good* $(*1)$ singularity, i.e., a $(*1)$ singularity on an irreducible component of the exceptional divisor which is not a corner. Thus, through this singulariy, there passes a separatrix which is not contained in the exceptional divisor and this projects down into a separatrix for the original foliation.

We quote here a result of Cano that is extremely useful to generalize the Camacho-Sad result [20].

Definition 1.36. *Let M be a complex surface, $\mathcal{F}$ a holomorphic foliation on M and S a separatrix of $\mathcal{F}$ through $p \in M$.*

- *the point $p \in S$ is of type (C_1) if S is not singular at p and*

$$Ind(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}.$$

- *the point $p \in S$ is of type (C_2) if S has exactly two irreducible non singular branches S_0, S_1 that intersect transversally at p and there exists a number $r > 0$ such that:*

$$Ind(\mathcal{F}, S_0, p) \in \mathbb{Q}_{\leq -\frac{1}{r}} := \{x \in \mathbb{Q} \mid x \leq -\frac{1}{r}\}$$
$$Ind(\mathcal{F}, S_1, p) \notin \mathbb{Q}_{\geq -r} := \{x \in \mathbb{Q} \mid x \geq -r\}.$$

This class of points is stable under blow-up in the sense described by the following lemma.

Lemma 1.37 ([20])**.** *Let M be a complex surface, $\mathcal{F}$ an holomorphic foliation and S a separatrix for $\mathcal{F}$ passing through $p \in M$. Let suppose that $p \in S$ is a point of type (C_1) or (C_2). Let $\pi : \tilde{M} \longrightarrow M$ the blow-up of M at p, $\tilde{S} := \pi^{-1}(S)$ the total transform of S and $\tilde{\mathcal{F}}$ the saturated foliation associated to $\mathcal{F}$. Then there exists a point of type (C_1) or (C_2) on $\tilde{S}$.*

According to this lemma we get the following definition:

Definition 1.38. *Let M be a complex surface, $\mathcal{F}$ a holomorphic foliation on M and S a separatrix of $\mathcal{F}$ trough $p \in M$. The point p is an* ***appropriate singularity*** *for $\mathcal{F}$ if after a finite number of blow-ups there exists a point of type (C_1) or (C_2) on the total transform of S.*

In case we can find a (C_1) or a (C_2) point, Cano proves that if we blow-up only these points, after a finite number of blows-up, we eventually find a *good* $(*1)$ singularity and thus we get the existence of a separatrix.

Proposition 1.39 ([20]). *Let M be a complex surface, $\mathcal{F}$ a holomorphic foliation and S a separatrix of $\mathcal{F}$ through $p \in M$. Let suppose that $p \in S$ is an appropriate singularity for $\mathcal{F}$. Then there exists at least a separatrix through p.*

Chapter 2

Discrete dynamics

In this chapter we concentrate our attention on the dynamics of maps tangent to the identity in $\mathbb{C}^2$. Firstly, we recall how things work in dimension one.

2.1 Discrete dynamics in dimension one

We recall some basic facts about complex dynamics in dimension one. For more details see [21], [45]. A formal diffeomorphism is every formal series $\hat{\varphi} \in \mathbb{C}[[x]]$ such that $\hat{\varphi}(0) = 0$ and $\hat{\varphi}'(0) \neq 0$.

Proposition 2.1. *Let be $f = az + \ldots \in$ Diff$(\mathbb{C}, 0), a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then there exists a formal change of coordinates, $\hat{\varphi}$, such that:*

1. $\hat{\varphi}_* f(z) := \hat{\varphi}^{-1} \circ f \circ \hat{\varphi}(z) = az$, *if a is not periodic, i.e. $a^n \neq a$ for every $n \in \mathbb{N}$;*

2. $\hat{\varphi}_* f(z) = az$ *if a is of order $q \in \mathbb{N}^*$, $a^q = 1$, and $f^{\circ q} = Id$,*

3. $\hat{\varphi}_* f(z) = a \exp(X_{kq,\lambda})$ *with $k \in \mathbb{N}^*$ and $X_{p,\lambda} = \frac{z^{p+1}}{1+\frac{\lambda}{2\pi i} z^p} \frac{\partial}{\partial z}$ if a is of order $q \in \mathbb{N}^*$ but $f^{\circ q} \neq Id$.*

We are interested in the case where the conjugation is holomorphic.

Theorem 2.2 (Koenigs). *With the notations of the Proposition 2.1, if* $|a| \neq 1$ *then the conjugation to the linear part is holomorphic.*

From a dynamical point of view is interesting to know what happen to the iterates of the map. With this aim we introduce the following definition:

Definition 2.3. *Let be* $f : D \subset \mathbb{C} \to f(D) \subset D$ *an holomorphic map such that* $0 \in D$ *and* $f(0) = 0$. *The map is attractive on* D *if for every* $p \in D$

$$\lim_{n \to +\infty} f^{(n)}(p) = 0,$$

where $f^{(n)} := f \circ \cdots \circ f$ *If the map is a diffeomorphism and* f^{-1} *is attractive on* $f^{-1}(D)$ *then the map is called repelling on* D.

According to this definition, if $| f'(0) | \neq 1$, then the map is attractive (if $| f'(0) | < 1$) or repelling (if $| f'(0) | > 1$). So, from a dynamical point of view, the more interesting maps are those with $| f'(0) | = 1$.

In this class of holomorphic maps the most important, for the intent of this thesis, are the maps tangent to the identity i.e. such that $f'(0) = 1$. The dynamics of these maps is completely described by the Leau-Fatou flower Theorem [45].

Theorem 2.4 (Leau-Fatou flower theorem). *Let* $f(z) = z + a_k z^k + \ldots$, *with* $k \geq 2$ *and* $a_k \neq 0$, *be a holomorphic function fixing the origin. Then there are* $k - 1$ *disjoint domains* $D_1, \cdots, D_{k-1}$ *with the origin in their boundary, invariant under* g *(i.e.*$g(D_j) \subset D_j$*) and over which* g *is attractive.*

2.2 Dynamics in $\mathbb{C}^2$

We are interested in the study of the dynamics of germs of maps f of $(\mathbb{C}^2, 0)$ tangent to the identity, i.e., $df_0 = Id$. To find a dynamical behavior similar to the one dimensional case, first we generalize the concept of petals:

Definition 2.5. *Let M be a complex surface, $f : M \to M$ a holomorphic function and p a point in M. A* ***parabolic curve*** *for f at p is an injective holomorphic map $\phi : \Delta \to M$, with $\Delta := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$, satisfying:*

(1) *ϕ is continuous on $\bar{\Delta}$ and $p = \phi(1)$;*

(2) *$f(\phi(\Delta)) \subset \phi(\Delta)$;*

(3) *for all $q \in \phi(\Delta)$ $\lim_{n\to\infty} f^n(q) = p$.*

We want to analyze the existence and the number of this objects. To get this goal we convert the problem in terms of continuous dynamics.

In the following sections we recall how to construct this dictionary following [2], mainly [11] and [3].

2.2.1 Singularities of maps

Let M be a complex surface and p a point in M. Let $f : M \to M$ be a holomorphic function such that $f(p) = p$ and $df_p = Id$.
For every $z, w \in \mathcal{O}_p$ let us consider the $1-$form:

$$\omega^{w,z,p} := (w \circ f - w)dz - (z \circ f - z)dw,$$

and construct the family of germs of holomorphic foliations:

$$\Omega_{f,p} := \{\omega^{w,z,p} = 0 \mid w, z \in \mathcal{O}_p, dw_p \wedge dz_p \neq 0\}.$$

For every $\omega^{w,z,p} \in \Omega_{f,p}$ we can construct the form $\hat{\omega}^{w,z,p}$ dividing the given form $\omega^{w,z,p}$ by the greatest common divisor of the coefficients.

Remark 2.6. $\hat{\omega}^{w,z,p} = \omega^{w,z,p}$ *if and only if p is an isolated fixed point of f.*

In this way we have linked the study of discrete dynamics to the continuous one.

Definition 2.7. *Let M be a complex surface. Let p be a point in M and $f : M \to M$ a holomorphic function such that $f(p) = p$ and $df_p = Id$. The point p is a* ***singular point*** *for f if $\hat{\omega}^{w,z,p}[p] = 0$ for every $\omega^{w,z,p} \in \Omega_{f,p}$.*

Using this construction we can *translate to discrete dynamics* all the definitions given for foliations. In particular we still have points of type $(*1)$ and $(*2)$. Let us observe that the definition of $(*1)$ and $(*2)$ points is independent of the coordinates. The following lemma assures that the definition is well posed:

Lemma 2.8 ([11])**.** *Let $\tilde{w}, \tilde{z} \in \mathcal{O}_p$ be such that $d\tilde{w}_p \wedge d\tilde{z}_p \neq 0$ and let $z, w \in \mathcal{O}_p$ with $dz_p \wedge dw_p \neq 0$. Then there exists $u \in \mathcal{O}_p^*$ such that $\omega_p^{\tilde{z},\tilde{w},p} = u\omega_p^{z,w,p}$. In particular if both the eigenvalues of the linear part are non zero, the quotient $\frac{\lambda_1{}^{z,w,p}}{\lambda_2{}^{z,w,p}}$ is independent by z, w and if it exist z_0, w_0 such that one of the eigenvalues $\lambda_j^{z_0,w_0,p} = 0$ then it holds for every z, w.*

We can now give the definition of reduced singularities for maps.

Definition 2.9 ([2],[11])**.** *Let M be a complex surface, p a point of M and $f : M \to M$ a holomorphic map such that $f(p) = p$ and $df_p = Id$. The point p is a* ***reduced singularity*** *for f if it is a reduced singularity for some and hence for all $\omega^{z,w,p}$, i.e., one of the following conditions holds:*

$(*1)$ $\lambda_1^{z,w,p}\lambda_2^{z,w,p} \neq 0, \frac{\lambda_1^{z,w,p}}{\lambda_2^{z,w,p}} \notin \mathbb{Q}^+$,

$(*2)$ $\lambda_1^{z,w,p} = 0, \lambda_2^{z,w,p} \neq 0$ *or* $\lambda_2^{z,w,p} = 0, \lambda_1^{z,w,p} \neq 0$.

2.2.2 Curves of fixed points

Now we have to convert in the language of the foliation the case of a map with curve of fixed points. Let M be a complex surface and S a (possibly singular) irreducible curve in M. Let $f : M \to M$ be a holomorphic function such that $f|_S = Id_S$ and $f \neq Id_M$. Let U be an open set of M and $\phi : U \to \mathbb{C}^2$ a local chart on M. If $l \in \mathcal{O}_p$ is a defining function of the curve S near the point p then:

$$\phi \circ f \circ \phi^{-1} = Id + (l \circ \phi)^T G \tag{2.1}$$

for some germ $G = (G_1, G_2)$ of holomorphic function at p with $G|_{\phi(S)} \not\equiv (0,0)$ and $T \geq 1$.

The exponent T in the decomposition (2.1) is called **order** of f on S in p and we denote it with $T_p(f, S)$.

Remark 2.10 ([12])**.** *We observe that* $\forall q \in U \cap S$ *we have* $T_q(f,S) = T_p(f,S)$.

If $p \in U \cap (S \setminus \mathrm{Sing}(S))$ the defining function l of S in p is such that $dl_p \neq 0$. Let τ be a function on U such that $d\tau_p \wedge dl_p \neq 0$ (we say that τ is transverse to l in p). So we can construct:

$$\hat{\omega}^{l,\tau,p} := \frac{\tau \circ f - \tau}{al^T} dl - \frac{l \circ f - l}{al^T} d\tau,$$

with $a \in \mathcal{O}(U)$, $a|_S \not\equiv 0$. We see that S is a leaf of $\hat{\omega}^{l,\tau,p}$ if and only if $\frac{l \circ f - l}{l^T} \equiv 0$ modulo the ideal of functions identically vanishing on S at p.

Definition 2.11. *Let M be a complex surface, S an irreducible complex curve (possibly singular) in M. Let $f : M \to M$ be a holomorphic function such that $f|_S = Id_S$ and $f \neq Id_M$. If p is a point of S we say that f is* ***tangential*** *on S at p if for every function l that defines S near p:*

$$\frac{l \circ f - l}{l^T} \equiv 0 \quad mod\, \mathcal{I}(S)_p. \tag{2.2}$$

where $\mathcal{I}(S)_p$ is the ideal of functions identically vanishing on S at p.

Proposition 2.12 ([12])**.** *If the curve S is globally irreducible then f is non-tangential at $p \in S$ if and only if f is non-tangential at $q \in S$ for every $q \in S$.*

In terms of foliations we can read the tangentiality conditions in the following way:

Proposition 2.13 ([11]). *The map f is tangential on $S \cap U$ if and only if S is a leaf for the family of holomorphic foliations on U given by $\{\omega^{l,\tau}\}$ where:*

$$\omega^{l,\tau} := \frac{\tau \circ f - \tau}{l^T} dl - \frac{l \circ f - l}{l^T} d\tau,$$

for any defining function l of S and transverse to S τ.

Assumes f is tangential on S. If $p \in Sing(S)$ then $dl_p = 0$ for any defining function of S at p. Therefore in this case $\omega^{l,\tau}[p] = 0$ for any defining function l of S and any transverse τ. So p is a singularity for all the family of foliations $\omega^{l,\tau} = 0$. Also in this case we say that $p \in S$ is a singularity of f.

Now we can define even dicritical singularities for maps. To do this we have to define how to transfer the map f on the blown-up surface $\tilde{M}$. In [1] Abate proves that there exists a unique $\tilde{f} : \tilde{M} \to \tilde{M}$ such that $\pi \circ \tilde{f} = f \circ \pi$ and whose action on the exceptional divisor is induced by the action of df_p on $\mathbb{P}(T_pM)$.

Definition 2.14. *Let M be a complex surface, p a point of M and $f : M \to M$ a holomorphic function such that $f(p) = p$ and $df_p = Id$. Let $\pi : \tilde{M} \to M$ be the projection of the blow-up of M in p and let $D := \pi^{-1}(p)$ be the exceptional divisor. The point p is **dicritical** for f if $\tilde{f} : \tilde{M} \to \tilde{M}$ is non tangential on D.*

After these definitions we can expect that even a sort of reduction of singularities holds for maps. Indeed:

Theorem 2.15 ([11] [2]). *Let M be a complex surface, S a (possibly singular) curve in M. Let p be a singularity of M and $f : M \to M$ an holomorphic* *map such that* $f(p) = p$ *and* $df_p = Id$. *Then there exists a complex surface* $\tilde{M}$, *a holomorphic proper function* $\pi : \tilde{M} \to M$ *and a holomorphic function* $\tilde{f} : \tilde{M} \to \tilde{M}$ *such that:*

(1) $D := \pi^{-1}(p) = \cap_{i=1}^{n} D_i$ *where* D_i *are complex projective line that intersect each other transversally;*

(2) $\pi : \tilde{M} \setminus D \to M \setminus \{p\}$ *is a biolomorphism;*

(3) $\pi \circ \tilde{f} = f \circ \pi$;

(4) $\tilde{f} \mid_D = Id \mid_D$;

(5) $\tilde{f}$ *has only reduced singularities or dicritical singularities on* D.

2.2.3 Index Theorem

In the previous section we introduced a link between separatrices and curves of fixed points. We know that in the study of separatrices is extremely useful to have an index theorem. So one of the first step in the study of discrete dynamics is to find an index theorem [2], [3], [11].

Using the foliations $\{\omega^{l,\tau} = 0\}$ defined before and observing that all the $\{\omega^{l,\tau} = 0\}$ are equal to the *first tangential order in* S [2], [11], we find that:

Theorem 2.16 ([2] [11] [12]). *Let* M *be a complex surface,* S *a globally irreducible compact complex curve and* $f : M \to M$ *a holomorphic function such that* $f \mid_S = Id_S$ *and* f *is tangential on* S. *Then for every* $p \in S$ *there exists a number* $Ind(f, S, p) \in \mathbb{C}$, *that depends only on* f *near* S *in* p, *such that:*

$$\sum_{p \in S} Ind(f, S, p) = S \cdot S.$$

If l defines S in p and τ is transversal to l the index assumes the form:

$$Ind(f, S, p) = \frac{1}{2\pi i} \int_{\partial L} \left\{ \frac{l \circ f - l}{l(\tau \circ f - \tau)} \right\} d\tau,$$

where $L = S^3 \cap S$ with S^3 a sphere centered in p with enough small radius.

As for foliation we want to use the index theorem together with blow-up techniques. So it is useful to know the behavior of the index under blow-up.

Proposition 2.17 ([11], [12]). *Let M be a complex surface, $f : M \to M$ a holomorphic function and S a curve in M such that $f|_S = Id_S$ and f is tangential on S. Let $p \in S$ be a point such that S is irreducible at p and let $\pi : \tilde{M} \to M$ be the blow-up of M in p and $\tilde{f} : \tilde{M} \to \tilde{M}$ the induced map by f. If $D := \pi^{-1}(p)$ is the exceptional divisor and $\hat{S} := \overline{\pi^{-1}(S \setminus \{p\})}$ the strict transform of S then:*

(1) *the map $\tilde{f}$ is tangential on $\tilde{S}$,*

(2) *if $\{\tilde{p}\} = D \cap \tilde{S}$ then $Ind(\tilde{f}, \hat{S}, \tilde{p}) = Ind(f, S, p) - m^2$ where $m \geq 1$ is the multiplicity of S in p.*

2.2.4 Parabolic curves

The constructions made until now suggest to retrace the proof of Camacho-Sad [18] to get the existence of parabolic curves. To make this we need the last ingredient i.e., the solution of the problem for some particular cases.

If the point p is not a singularity the problem was solved by Abate [2]:

Proposition 2.18. *Let M a complex surface, S a curve in M and $f : M \to M$ a holomorphic map such that $Fix(f) = S$ near p. Let suppose that f is tangential on S. If p is not a singularity for f on S then p is not an attractive point for f and then there do not exist parabolic curves for p.*

If instead p is a singularity of f of type $(*1)$ a first result is achieved by Hakim [36] (see also Abate [2] where a slightly simpler proof is given):

Proposition 2.19. *Let M be a complex surface, $S \subset M$ a complex curve. Let $f : M \to M$ be a holomorphic map such that $f\mid_S = Id_S$. Let p be a non singular point of S and suppose f is tangential on S at p. If $Fix(f) = S$ near p and p is a reduced singularity of type $(*1)$ for f then there exists at least one parabolic curve for f at p.*

The last *special* case is the dicritical one:

Proposition 2.20 ([2], [11])**.** *Let M be a complex surface, $p \in M$. Let $f : M \to M$ be holomorphic and such that $f(p) = p$, $df_p = Id$. Then p is dicritical for f if and only if there exist infinitely many parabolic curves for f at p.*

Now we have all the ingredients for the general case. We can desingularized the map f around an isolated singularity. At the end, if we do not find dicritical singularities, we find a *good* $(*1)$ singularity and then there exists a parabolic curve that projects into a parabolic curve for f. With this strategy in mind we can follow Cano's construction as proposed by Bracci in [11].

Definition 2.21. *Let M a complex surface, $S \subset M$ a complex curve. Let $f : M \to M$ be a holomorphic map such that $Fix(f) = S$.*

- *the point $p \in S$ is of type (C_1) if S is non singular in p, the map f is tangential on S in p and*

$$Ind(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}.$$

- *the point $p \in S$ is of type (C_2) if S has two non singular branches S_0, S_1 that intersect transversally each other in p and it exists a number $r > 0$*

such that:

$$\begin{aligned} Ind(f, S_0, p) &\in \mathbb{Q}_{\leq -\frac{1}{r}} = \{x \in \mathbb{Q} \mid x \leq -\frac{1}{r}\} \\ Ind(f, S_2, p) &\notin \mathbb{Q}_{\geq -r} = \{x \in \mathbb{Q} \mid x \geq -r\}. \end{aligned}$$

Definition 2.22. *Let M be a complex surface, S a curve in M. Let $f : M \to M$ be a holomorphic function such that Fix$(f) = S$. Suppose f is tangential on S. The point p is an* ***appropriate singularity*** *for f if after a finite number of blows-up there exists a point of type (C_1) or (C_2) on the total transform of S.*

Proposition 2.23 ([11])**.** *Let M be a complex surface, $S \subset M$ a complex curve. Let $f : M \to M$ be a holomorphic map such that Fix$(f) = S$. Suppose f is tangential on S. Let $p \in S$ be an appropriate singularity for f then there exists at least one parabolic curve for f in p.*

2.2.5 Robust parabolic curves

As we have seen we have deduced the existence of parabolic curves proving the existence of such curves for the desingularized map and then projecting it to the map. A natural question arises: can all parabolic curves be obtained in this way? As noticed by Abate and Tovena [5] the curves found by this construction have some additional properties. So we introduce a new concept.

Definition 2.24. *A* ***robust parabolic curve*** *is a parabolic curve that satisfies the following conditions:*

1. *we can blow-up φ at level h for any $h \geq 1$,*
2. *there is a formal power series $Q \in (\mathbb{C}[[x]])^2$ such that for every $h \geq 1$ there is $r_h > 0$ such that $\varphi - Q_h = O(\zeta^{h+1})$ in Δ_{r_h}, where Q_h denotes the truncation at degree h of Q.*

Essentially when we say condition (1) we mean that the strict transform of the parabolic curve is also a parabolic curve. Now we make more precise such idea [5].

Let $\varphi : \Delta \to M$ be a parabolic curve for f at p. If there exists $v \in \mathbb{P}(T_pM)$ such that $\tilde{\varphi} = \pi^{-1} \circ \varphi$ is a parabolic curve at v for $\tilde{f}$ (where $\pi : \tilde{M} \to M$ is the blow-up of M at p and $\tilde{f}$ is the blow-up of f), then we say that φ is tangent to v at p and $\tilde{\varphi}$ is the strict transform of φ.

We say that we can blow-up at level 1 a parabolic curve φ if there exists $r_0 > 0$ such that $\varphi \mid_{\Delta_{r_0}}$ is tangent to some direction $v \in \mathbb{P}(T_pM)$, where $\Delta_{r_0} \subset \Delta$ is a disc tangent to $\partial\Delta$ in the origin. Let φ^1 denote the strict transform of $\varphi \mid_{\Delta_{r_0}}$; if we can blow-up φ^1 at level 1, we say that we can blow-up φ at level 2, and we denote by φ^2 the parabolic curve so obtained. In an inductive way we can say that we can blow-up φ at level h if we can blow-up φ^{h-1} at level 1.

All the parabolic curves found by the techniques described in this chapter are robust but it is known that not all parabolic curves are robust [5]. In the last chapter we analyze in more detail this class of parabolic curves giving a more geometric interpretation.

Chapter 3

Parabolic curves for maps tangent to the identity near singular curves

3.1 Introduction

In Chapter one we have seen that if $\mathcal{F}$ is a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$ singular at zero then at least one separatrix passes through it [18].

A natural question is whether the knowledge of this separatrix S allows to infer the existence of another separatrix. There are essentially two types of results, one of local and the other of global flavor. The first kind of result is essentially a re-formulation of Camacho-Sad theorem [20] which says that if S is non singular and $\operatorname{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists another separatrix through p. The second type of result requires global conditions on S, like S compact (but possibly singular), globally and locally irreducible and $S \cdot S < 0$ to provide the existence of another separatrix at some point of S [50].

The aim of this chapter is to prove a result of local nature when S is possibly singular, using the index defined by Suwa [55]. We prove:

Theorem 3.1 ([27]). *Let M be a complex two dimensional manifold, $\mathcal{F}$ a holomorphic foliation on same open subset of M, $S \subset M$ a possibly singular curve locally irreducible at a point $p \in M$, such that it is a separatrix for $\mathcal{F}$ at p. If Ind$(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists (at least) another separatrix for $\mathcal{F}$ at p.*

In chapter two we have seen that Abate, Bracci and Tovena [3], [11], [12] found a way to translate results about foliations to holomorphic diffeomorphisms. The proof of Theorem 3.1 respects their dictionary and so the results about the existence of separatrices for foliations can be translated in results about the existence of parabolic curves for diffeomorphisms. So we get the following result in discrete dynamics:

Theorem 3.2 ([27]). *Let M be a two dimensional complex manifold, $f : M \longrightarrow M$ a holomorphic map such that Fix$(f) = S$ with S a locally irreducible, possibly singular curve at a point $p \in M$. Assume that f is tangential on S and Ind$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$. Then there exists (at least) a parabolic curve for f at p.*

Theorem 3.2 has been proved by Abate [2] in case S is non singular and by Bracci in [11] in case S is a generalized cusp, i.e. of the form $\{x^m = y^n\}$.

As we can see these two results give conditions that guarantee the existence of separatrices or parabolic curves. In [29] and in Chapter five we pursue an analysis of the converse problem, i.e., to find an upper bound for the number of separatrices and parabolic curves.

This chapter is devoted to the proof of the two previous results. The idea is to prove that, in both cases, the point is an appropriate singularity for which we know the existence of a separatrix [20] and a parabolic curve [11]. We have seen that the index shows strong similarities in the continuous and discrete case (this will be analyzed with much more attention in chapter four). In particular index theorems have the same expression and the indices have the same behavior under blows-up. In the proof we will use only these two properties and so we only provide the proof of Theorem 3.1 because the proof of Theorem 3.2 is formally the same.

3.2 Proof of the result

In order to get Theorem 3.1 we need to prove that the point p is an appropriate singularity.

We know that the theorem of resolution of singularities curves [37] ensures that after a finite number of blow-ups we have the geometric structure required for the existence of (C_1) or (C_2) points. In order to prove our result we need to analyze the Camacho-Sad-Suwa index (C.S.S. index for short) under this process. The behavior of the index is strongly related to the evolution of the geometric structure under blow-up. We can divide the proof in two steps:

1. study of the geometric structure under the resolution of singularities,

2. study of the C.S.S. index under this process.

3.2.1 Geometric structure under blow-up

In order to get step one we give the following definition:

Definition 3.3. *Let M be a two dimensional complex manifold and $S_1, \cdots,$ $S_n \subset M$ given complex curves. We say that a point p is a* ***double intersection point*** *if p belongs to exactly two distinct curves among $S_1, \cdots, S_n$. If instead p belongs to exactly three of them it is called a* ***triple intersection point****.*

Remark 3.4. *In the study of curve desingularization the set of curves that we find is formed by the strict transform of the curve* S *and many exceptional divisors obtained by successive blow-ups. Because of the structure of the blow-up process we can only have double and triple intersection points (see [37]). A triple intersection point belongs to the strict transform of* S *and to two exceptional divisors. To distinguish these two* $\mathbb{C}\,\mathbb{P}(1)$ *we will call* old exceptional divisor *the strict transform of a given exceptional divisor. Instead we will call* new exceptional divisor *the exceptional divisor produced by the last blow-up.*

Now we can describe the geometric evolution under blow-up. Note that the only intersection point that can be triple is the one made up by the strict transform of S. We will prove the following:

Proposition 3.5. *Let* S *be a singular curve and let* p *be a singularity of* S*. The resolution process of* S *in* p *is related to the behavior of the multiplicity of* S *in* p *in the following way:*

- *If we blow-up a singularity and the multiplicity does not reduce we have two cases:*

    1. *if we are in a double intersection point at the next blow-up we find another double intersection,*
    2. *if we are in a triple intersection point at the next blow-up we can find either a double intersection or a triple intersection point. More precisely we find a double intersection point if the tangent cone to the curve does not coincide with any exceptional divisor, while we find a triple intersection point if the tangent cone coincides with one of the two exceptional divisors and the new triple*

intersection point belongs to the strict transform of the old exceptional divisor.

- *If we blow-up a singularity and the multiplicity reduces we have two cases:*

  1. *if we are in a double intersection point at the next blow-up we find a triple intersection point,*
  2. *if we are in a triple intersection point at the next blow-up we find a triple intersection point that belongs to the strict transform of the new exceptional divisor.*

Remark 3.6. *In the previous proposition we have used improperly the expression "the tangent cone coincides with one of the two exceptional divisors" to mean that the tangent cone of S in p coincides with the tangent space of D in p.*

In order to get Proposition 3.5 we need some elementary lemmas.

Lemma 3.7. *Let M be a two dimensional complex manifold, S an analytic irreducible curve on M and $p \in S$ a singularity of S. Blow-up M in p and let $\hat{S}$ be the strict transform of S, D the exceptional divisor and $\hat{p} := \hat{S} \cap D$. The multiplicity of $\hat{S}$ in $\hat{p}$ is strictly smaller than the multiplicity of S in p if and only if D coincides with the tangent cone of $\hat{S}$ in $\hat{p}$.*

Proof. We can assume that $p = (0,0)$ and $S = \{l(x,y) = 0\}$ with $l(x,y) = y^m + l_{m+1}(x,y) + \cdots$. Blow-up in p and using the chart such that the projection becomes $\pi(u,v) = (u,uv)$ we have: $\hat{S} = \{\hat{l}(u,v) = 0\}$, with $l(u,v) = v^m + ul_{m+1}(1,v) + \cdots = v^m + uq_{k-1} + \cdots$ and $D = \{u = 0\}$. The multiplicity of $\hat{S}$ in $(0,0)$ is strictly less then m if and only if $k < m$ and then if and only if the tangent cone is $\{uq_{k-1}(u,v) = 0\}$ and so if and only if D

is included in the tangent cone. Because S is irreducible this can happen if and only if $q_{k-1}(u,v) = u^{k-1}$, i.e. if and only if D is the tangent cone. □

Lemma 3.8. *Let M be a two dimensional complex manifold, S an analytic irreducible curve on M and $p \in S$ a singularity of S. Blow-up M in p and let $\hat{S}$ be the strict transform of S, D the exceptional divisor and $\hat{p} := \hat{S} \cap D$. The exceptional divisor D is the tangent cone of $\hat{S}$ in $\hat{p}$ if and only if blowing-up in $\hat{p}$ we get a triple intersection point.*

Proof. Let $\hat{D}$ be the strict transform of D and D_1 the new exceptional divisor. Now $\hat{D}$ intersects D_1 in the point corresponding to the tangent of D in p, so $\hat{D} \cap \hat{\hat{S}} \neq \emptyset$ if and only if D and $\hat{S}$ have the same tangent in p. So we get a triple intersection point if and only if the tangent cone of $\hat{S}$ coincides with D. □

Using the previous two lemmas we obtain the following:

Lemma 3.9. *Let $S \subset M$ be an analytic irreducible curve of multiplicity m in the singular point p. Suppose that after a finite number of blows-up the strict transform of S, $\tilde{S}$, intersects the exceptional divisor in a point $\tilde{p}$ and denote by D the irreducible component of the exceptional divisor containing $\tilde{p}$, i.e. $\tilde{p}$ is a double intersection point. Blow-up in $\tilde{p}$ and let D_1 be the new exceptional divisor and $\hat{S}$ the strict transform of $\tilde{S}$. If the multiplicity of $\hat{S}$ in $\hat{p} := D_1 \cap \hat{S}$ is equal to the multiplicity of $\tilde{S}$ in $\tilde{p}$ then at the following blow-up we find again a double intersection point.*

By Lemma 3.8 we also get:

Lemma 3.10. *Let $S \subset M$ be an analytic irreducible curve of multiplicity m in the singular point p. Suppose that after a finite number of blows-up we have a triple intersection point. At the following blow-up we have two cases:*

1. *if the tangent cone in the singularity contains one of the two exceptional divisors then at the next blow-up we find again a triple intersection point,*

2. *if the tangent cone in the singularity does not contain any of the two exceptional divisors then at the next blow-up we find a double intersection point.*

Remark 3.11. *Observe that the method of proof used in Lemma 3.10 does not give information on which of the exceptional divisors goes to create the new triple intersection. To get this information we need some more computations. Let $\hat{S}$ be the strict transform of S after some blow-ups and suppose to have a triple intersection point. We can assume that $p = (0,0)$ and $\hat{S} = \{\hat{l}(u,v) = 0\}$ with $\hat{l}(u,v) = v^m + u^{k_1}[q_{k_2-k_1}(u,v) + \cdots]$, and $D_1 = \{v = 0\}$, $\hat{D} = \{u = 0\}$ where D_1 is the new exceptional divisor and $\hat{D}$ is the old one (according to Remark 3.4). Let examine the various cases:*

1. *If $m > k_2$ then the tangent cone is $\{u^{k_1}q_{k_2-k_1}(u,v) = 0\}$ and by the irreducibility of S is $\{cu^{k_2} = 0\}$ with $c \neq 0$ and so it contains an exceptional divisor, $\hat{D}$. Blow-up again $(0,0)$ and using the chart such that the projection is $\pi(x,y) = (xy,y)$ we have:*

$$\hat{l}(xy,y) = y^m + cx^{k_2}y^{k_2} + x^{k_1}y^{k_2+1}[q_{k_2-k_1+1} + \cdots]$$

and because $m > k_2$

$$\hat{\hat{l}}(x,y) = y^{m_1-k_2} + cx^{k_2} + x^{k_1}y[q_{k_2-k_1+1}(x,1) + \cdots]$$

with $D_2 = \{y = 0\}$ e $\hat{\hat{D}} = \{x = 0\}$. So $(0,0)$ is a triple intersection point made up by D_2, $\hat{\hat{S}}$, $\hat{\hat{D}}$. If instead we use the other chart we find only a double intersection points.

2. *If* $m_1 < k_2$ *we proceed in the same way obtaining a triple intersection point made by* D_2, $\hat{D}_1$ *and* $\hat{\hat{S}}$.

3. *If* $m_1 = k_2$ *the tangent cone is given by* $\{v^{m_1} + u^{k_1} q_{k_2 - k_1}(u, v) = 0\}$ *and by the irreducibility of the curve it is* $\{(v + cu)^{m_1} = 0\}$ *with* $c \neq 0$ *and it does not contain any exceptional divisor. So by Lemma 3.10 at the next blow-up we find only double intersection points.*

3.2.2 C.S.S. index under blow-up

Now we can proceed in order to get step two by studying the behavior of the index in a general resolution process via blow-up. The upshot is to prove that in the resolution process we necessarily find a (C_1) or (C_2) point ,i.e., p is an appropriate singularity and then Theorem 3.1 holds.

The intent is to analyze the C.S.S. index in all possible geometric evolutions (see Proposition 3.5).

Remark 3.12. *In the analysis we will omit the case in which at some blow-up we find a dicritical point. In fact in this case the goal is obtained by Proposition 1.31.*

We will consider resolution processes only at a combinatoric level in a sense that will be specified later.

Thanks to Proposition 3.5 the structure of a resolution process of a singular point p is completely described by the behavior of the multiplicity of the strict transform at the intersection with the exceptional divisor. We can then consider a sequence of blow-ups only as a sequence of positive numbers (representing the evolution of the multiplicity) and forgetting any type of geometric obstruction.

Definition 3.13. *A* ***process*** *is an ordinate list of the form:*

$$P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\}$$

where $k, \alpha_i, m_i \in \mathbb{N}$ *and* $m > m_1 \geq \cdots \geq m_n$. *We associate to P, from a purely formal point of view, a blow-up sequence for a curve S where the blows-up are made at the beginning at the point p and then at the intersection point of the strict transform of the curve and the exceptional divisor. The blow-up sequence satisfies the following rules:*

- *from the first to the* $k - th$ *blow-up we find only double intersection points and the curve multiplicity is constantly equal to* m,

- *from the* $(k + 1) - th$ *to the* $(k + \alpha_1) - th$ *blow-up we find a triple intersection point and the multiplicity of the strict transform of S is constantly equal to* $m_1 < m$,

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

- *from the* $(k+\alpha_1+\cdots+\alpha_{n-1}+1)-th$ *to the* $(k+\alpha_1+\cdots\alpha_n)-th$ *blow-up we find a triple intersection point and the multiplicity is constantly equal to* $m_n \leq m_{n-1}$.

Remark 3.14. *At the end of P the curve S is not desingularized, in fact we have triple points and this type of point are not admitted in the desingularized curve.*

Now, according to Proposition 3.5 we start analyzing all the possible cases.

3.2.3 Case of double intersection

It corresponds to a process $P = \{(k, m)\}$, i.e. we start with multiplicity m and we remain with this multiplicity for k blows-up finding only double points.

Let S be the separatrix of the holomorphic foliation $\mathcal{F}$ through the point p. Let blow-up in p and denote by $\hat{S}$ the strict transform of S with D the exceptional divisor.

By proposition 1.34 $\operatorname{Ind}(\tilde{\mathcal{F}}, \hat{S}, \tilde{p}) = \operatorname{Ind}(\mathcal{F}, S, p) - m^2$ and remembering that $\operatorname{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ we have that $\operatorname{Ind}(\tilde{\mathcal{F}}, \hat{S}, \tilde{p}) \notin \mathbb{Q}_{\geq -m^2}$.

If there are not (C_1) points on $D \setminus \{\tilde{p}\}$ the index theorem implies $\operatorname{Ind}(\tilde{\mathcal{F}}, D, \tilde{p}) \in \mathbb{Q}_{\leq -1}$.

Let blow-up, in $\tilde{p}$, the foliation and denote, again, with $\tilde{\mathcal{F}}$ the saturated foliation, with $\hat{\hat{S}}$ the strict transform of $\hat{S}$ (that still has multiplicity m), with $\hat{D}$ the strict transform of D and with D the new exceptional divisor. Let be $q_1 := \hat{D} \cap D$ and $q_0 := \hat{\hat{S}} \cap D$. By proposition 1.34

$$\operatorname{Ind}(\tilde{\mathcal{F}}, \hat{\hat{S}}, q_0) \notin \mathbb{Q}_{\geq -2m^2}$$
$$\operatorname{Ind}(\tilde{\mathcal{F}}, \hat{D}, q_1) \in \mathbb{Q}_{\leq -2}.$$

If there are not (C_1) points on $D \setminus \{q_0, q_1\}$ we have, by index theorem:

$$\operatorname{Ind}(\tilde{\mathcal{F}}, D, q_0) + \operatorname{Ind}(\tilde{\mathcal{F}}, D, q_1) \in \mathbb{Q}_{\leq -1}.$$

If $\operatorname{Ind}(\tilde{\mathcal{F}}, D, q_1) \notin \mathbb{Q}_{-\frac{1}{2}}$ then q_1 is a (C_2) point; if $\operatorname{Ind}(\tilde{\mathcal{F}}, D, q_1) \in \mathbb{Q}_{-\frac{1}{2}}$ then $\operatorname{Ind}(\tilde{\mathcal{F}}, D, q_0) \in \mathbb{Q}_{-\frac{1}{2}}$.

Iterating this type of reasoning, if we do not find (C_1) or (C_2) points in the total transform then at the $k-$th blow-up the indices are of type

$$\begin{aligned} &\operatorname{Ind}(\tilde{\mathcal{F}}, D, q) \in \mathbb{Q}_{\leq -\frac{1}{k}} \\ &\operatorname{Ind}(\tilde{\mathcal{F}}, \hat{S}, q) \notin \mathbb{Q}_{\geq -km^2}. \end{aligned} \tag{3.1}$$

where $q := \hat{S} \cap D$.

3.2.4 Case of triple intersection

We consider now a slightly more complicated process, $P = \{(k,m),(1,m_1),\cdots,(1,m_n)\}$. Let us suppose not to find (C_1) or (C_2) points during P.

We denote by S the strict transform of the curve, $\mathcal{F}$ the saturated foliation, D_1, D_2 the two exceptional divisors that intersect S at the last triple intersection point q.

Proposition 3.15. *In this situation at the last blow-up of P, if we have not found (C_1) or (C_2) points, we can find $x, y \in \mathbb{N}$ and $a, b \in \mathbb{N} \cup \{0\}$ such that the indices are:*

$$\begin{aligned} Ind(\mathcal{F}, S, q) &\notin \mathbb{Q}_{\geq -km^2 - m_1^2 - \cdots - m_n^2} \\ Ind(\mathcal{F}, D_1, q) &\in \mathbb{Q}_{\leq -\frac{x}{y}} \\ Ind(\mathcal{F}, D_2, q) &\in \mathbb{Q}_{\leq -\frac{yk+a}{xk+b}}. \end{aligned} \tag{3.2}$$

Proof. At the k-th blow-up the indices are of type (3.1). Let blow-up again. If some point of the new exceptional divisor D_1 is of type (C_1), then p is an appropriate singularity and we have the assertion. Otherwise $\mathrm{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\geq 0}$ $\forall p \in D_1 \setminus \{q\}$ and then by the Index Theorem 1.33:

$$\mathrm{Ind}(\mathcal{F}, D_1, q) \in \mathbb{Q}_{\leq -1}.$$

Then by Proposition 1.34 and observing that D has multiplicity one:

$$\begin{aligned} \mathrm{Ind}(\mathcal{F}, \hat{S}, q) &\notin \mathbb{Q}_{\geq -km^2 - m_1^2} \\ \mathrm{Ind}(\mathcal{F}, \hat{D}_2, q) &\in \mathbb{Q}_{\leq -\frac{k+1}{k}}. \end{aligned}$$

Proceeding by induction on n we can assume the assertion true for n and we prove it for $n+1$. We have to analyze separately two different cases that can occur blowing-up:

1. the new triple point is made by $\{\hat{S}, \hat{D}_2, D\}$;

2. the new triple is made by $\{\hat{S}, \hat{D}_1, D\}$,

where D is the new exceptional divisor and D_1 and D_2 are the ones of the n blows-up whose indices satisfy (3.2) by inductive hypothesis. We consider only the case (1) because the others are similar. By Proposition 1.34 the indices are of type:

$$\begin{aligned}
\mathrm{Ind}(\mathcal{F}, \hat{S}, q_1) =& \ \mathrm{Ind}(\mathcal{F}, S, p) - m_1^2 - \cdots - m_n^2 - m_{n+1}^2 \\
\mathrm{Ind}(\mathcal{F}, \hat{D}_2, q_1) \in& \ \mathbb{Q}_{\leq -\frac{(x+y)k+(a+b)}{xk+b}} \\
\mathrm{Ind}(\mathcal{F}, \hat{D}_1, q_0) \in& \ \mathbb{Q}_{\leq -\frac{x+y}{y}},
\end{aligned}$$

where q_1 is the new triple point and $q_0 := \hat{D}_1 \cap D$. If there are not (C_1) points on $D \setminus \{q_0, q_1\}$ then by Index Theorem q_0 is a (C_2) point or $\mathrm{Ind}(\mathcal{F}, D, q_1) \in \mathbb{Q}_{\leq -\frac{x}{x+y}}$. In the last case the indices satisfy:

$$\begin{aligned}
\mathrm{Ind}(\mathcal{F}, \hat{S}, q_1) =& \ \mathrm{Ind}(\mathcal{F}, S, p) - m_1^2 - \cdots - m_n^2 - m_{n+1}^2 \\
\mathrm{Ind}(\mathcal{F}, \hat{D}_2, q_1) \in& \ \mathbb{Q}_{\leq -\frac{(x+y)k+(a+b)}{xk+b}} \\
\mathrm{Ind}(\mathcal{F}, D, q_1) \in& \ \mathbb{Q}_{\leq -\frac{x}{x+y}}.
\end{aligned} \tag{3.3}$$

and then the assertion follows putting $y' = x+y, x' = x, a' = a+b, b' = b$. $\square$

Remark 3.16. *A general process can always be written in the form $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\}$ with $m_i \neq m_j$ if $i \neq j$. The coefficients (x, y, a, b) that occur in Proposition 3.15 depend only on the α_i and to the order in which they appear but not on the multiplicities m_i and the coefficient k.*

We list now some simple properties of the index under a process that will be useful later:

Lemma 3.17. *In* (3.2) *it follows that $xa - yb = 1$.*

Proof. We proceed by induction on the number of blows-up and argue as in the proof of Proposition 3.15 □

With the same arguments we can also prove:

Lemma 3.18. *Let consider a process* $P = \{(k, m), (1, m_1), \cdots, (1, m_n)\}$ *and denote by* S, D_1, D_2 *the curves that create the triple intersection point. Then if* (x, y, a, b) *are the coefficients that appear in the indices* (3.2) *we have, according to Remark 3.4:*

if $x > y$ *then* D_2 *is the new exceptional divisor and* D_1 *is the old one,*

if $x \leq y$ *then* D_1 *is the new exceptional divisor and* D_2 *is the old one.*

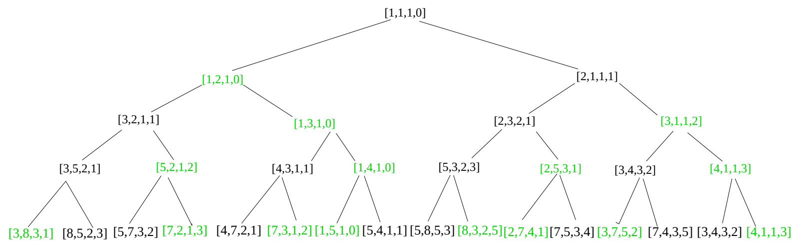
Using Lemma 3.18 and Remark 3.11 we can easily prove:

Lemma 3.19. *If we blow-up a triple intersection point and the multiplicity decreases then the coefficients* (x', y', a', b') *of the indices of the new triple are such that:*

if $x > y$ *then* $x' = x, y' = x + y$,

if $x \leq y$ *then* $x' = x + y, y' = y$.

In the analysis of the $C.S.S.$ index in the triple intersection case the knowledge of the index is equivalent to the knowledge of the coefficients (x, y, a, b). According to Remark 3.11 the reduction of of the multiplicity creates different coefficients. In the next subsections we are going to investigate these cases. To make clearer the possible evolutions of the coefficients we report in the figure the coefficients (x, y, a, b) that can appear in the first five blows-up in triple intersection. We indicate in black the coefficients related to a decrease of multiplicity and in grey the others.



3.2.5 Transition from triple intersection with multiplicity lowering to triple with constant multiplicity

We now consider a process of type $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_{n-1}, m_{n-1}), (\alpha_n, m_n)\}$ with $m_i \neq m_j$ if $i \neq j$. We want to relate the coefficients of the last blow-up with the ones obtained at the first lowering of multiplicity $m_{n-1} \to m_n$, i.e., we want to relate the last indices of the process $\{(k, m), (\alpha_1, m_1), \cdots, (\alpha_{n-1}, m_{n-1}), (1, m_n)\}$ to the last ones of P.

Proposition 3.20. *Suppose to have indices of type:*

$$\begin{aligned} Ind(\mathcal{F}, S, p) &\notin \ \mathbb{Q}_{\geq -km^2 - \alpha_1 m_1^2 - \cdots - \alpha_{n-1} m_{n-1}^2 - m_n^2} \\ Ind(\mathcal{F}, D_1, p) &\in \ \mathbb{Q}_{\leq -\frac{x}{y}} \\ Ind(\mathcal{F}, D_2, p) &\in \ \mathbb{Q}_{\leq -\frac{yk+a}{xk+b}} \end{aligned} \tag{3.4}$$

with $m_i \neq m_j$ if $i \neq j$, i.e., n is the number of multiplicity lowering. The indices at the end of the process P are of type:

if $x > y$

$$\begin{aligned} Ind(\mathcal{F}, S, p) &\notin \ \mathbb{Q}_{\geq -km^2 - \alpha_1 m_1^2 - \cdots - \alpha_{n-1} m_{n-1}^2 - \alpha_n m_n^2} \\ Ind(\mathcal{F}, D_1, p) &\in \ \mathbb{Q}_{\leq -\frac{x + (\alpha_n - 1)y}{y}} \\ Ind(\mathcal{F}, D_2, p) &\in \ \mathbb{Q}_{\leq -\frac{yk+a}{(x + (\alpha_n - 1)y)k + ((\alpha_n - 1)a + b)}} \end{aligned} \tag{3.5}$$

if $x \leq y$

$$\begin{aligned} Ind(\mathcal{F}, S, p) \notin\ & \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_n m_n^2} \\ Ind(\mathcal{F}, D_1, p) \in\ & \mathbb{Q}_{\leq -\frac{((\alpha_n-1)x+y)k+(a+(\alpha_n-1)b)}{xk+y}} \\ Ind(\mathcal{F}, D_2, p) \in\ & \mathbb{Q}_{\leq -\frac{x}{(\alpha_n-1)x+y}}. \end{aligned} \tag{3.6}$$

Proof. By Proposition 3.5 blowing-up with constant multiplicity we know that the new triple point is made up by the curve, the new exceptional divisor and the strict transform of the old one (see Remark 3.4). We have to analyze separately the case in which the old exceptional divisor is D_1 or D_2. This distinction can be made in terms of $x > y$ or $x \leq y$ thanks to Lemma 3.18. Suppose, for instance, $x > y$ in the indices (3.4), then we conclude that the old exceptional divisor is D_1. Now blowing-up again and using Proposition 1.34, the Index theorem and the assumption of non existence of (C_1) or (C_2) points we can prove the result for $\alpha_n = 1, 2$. Then proceeding by induction and repeating the same argument for the case $x \geq y$ we have the assertion. □

3.2.6 Transition from triple to double intersection

Suppose that, after k blows-up in double intersection and a finite number of blows-up in triple intersection, we return to double intersection. Let consider the generic indices of the triple (3.2) and write the index along S in the form:

$$\mathrm{Ind}(\mathcal{F}, S, p) \notin\ \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_n m_n^2},$$

with $m_i \neq m_j$ if $i \neq j$. Using Lemma 3.17 we obtain that the indices in the double point are:

$$\begin{aligned} \mathrm{Ind}(\mathcal{F}, S, q) \notin\ & \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_n m_n^2-m_n^2} \\ \mathrm{Ind}(\mathcal{F}, D, q) \in\ & \mathbb{Q}_{\leq -\frac{1}{(x+y)^2k+(x+y)(a+b)}}. \end{aligned} \tag{3.7}$$

3.2.7 Estimate of the term km^2

We estimate the term $-km^2 - \alpha_1 m_1^2 - \cdots - \alpha_n m_n^2 - m_n^2$ showing that, if the curve is resolved, then q is a point of type (C_2); otherwise we obtain indices of the form (3.1) and so we can use again the results found in the previous sections in order to get a desingularization. In this subsection we estimate the term km^2.

Proposition 3.21. *If we indicate with $(x_i^j, y_i^j, a_i^j, b_i^j)$ the coefficients that appear in the indices of the triple intersection point at the $j-$th blow-up with multiplicity m_i then, if $n \geq 2$:*

$$\begin{aligned} m &= x_{n-1}^{\alpha_{n-1}} m_{n-1} + y_{n-1}^{\alpha_{n-1}} m_n \text{ if } x_{n-1}^{\alpha_{n-1}} \geq y_{n-1}^{\alpha_{n-1}}, & (3.8)\\ m &= y_{n-1}^{\alpha_{n-1}} m_{n-1} + x_{n-1}^{\alpha_{n-1}} m_n \text{ if } y_{n-1}^{\alpha_{n-1}} \geq x_{n-1}^{\alpha_{n-1}}. & (3.9)\end{aligned}$$

Proof. We proceed by induction on the number n of changes of multiplicity. For $n = 2$ the indices are of the form:

$$\begin{aligned} \mathrm{Ind}(\mathcal{F}, S, p) &\notin \mathbb{Q}_{\geq -km^2 - \alpha_1 m_1^2 - \alpha_2 m_2^2}, \\ \mathrm{Ind}(\mathcal{F}, D_1, p) &\in \mathbb{Q}_{\leq -\frac{\alpha_1\alpha_2+1}{\alpha_1}}, \\ \mathrm{Ind}(\mathcal{F}, D_2, p) &\in \mathbb{Q}_{\leq -\frac{\alpha_1 k+1}{(\alpha_1\alpha_2+1)k+\alpha_2}}. \end{aligned}$$

The indices we find at the α_1-th blow-up with multiplicity m_1 are:

$$\begin{aligned} \mathrm{Ind}(\mathcal{F}, S, p) &\notin \mathbb{Q}_{\geq -km^2 - \alpha_1 m_1^2}, \\ \mathrm{Ind}(\mathcal{F}, D_2, p) &\in \mathbb{Q}_{\leq -\frac{\alpha_1 k+1}{k}}, \\ \mathrm{Ind}(\mathcal{F}, D_1, p) &\in \mathbb{Q}_{\leq -\frac{1}{\alpha_1}}. \end{aligned} \tag{3.10}$$

Because we make α_1 blows-up with multiplicity m_1 and because the curve is irreducible by the Enriques-Chisini theorem ([13] pg. 516) we have:

$$m_2 = m - \alpha_1 m_1$$

and then the assertion. We prove the inductive step. The index along S is:

$$\mathrm{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_n m_n^2-\alpha_{n+1}m_{n+1}^2}$$

We consider the case $x_{n-1}^{\alpha_{n-1}} \geq y_{n-1}^{\alpha_{n-1}}$ (the other is similar). Because we make α_n blows-up with multiplicity m_n we have:

$$m_{n+1} = m_{n-1} - \alpha_n m_n \text{ and then } m_{n-1} = \alpha_n m_n + m_{n+1}.$$

By inductive hypothesis and the above relation we find an expression of m in terms of m_n and m_{n+1}. Now we have to prove that this expression is the one of the statement. Using Lemma 3.19 we have that $x_n^1 \leq y_n^1$ and for Proposition 3.20 the indices at the α_n−th blow-up with multiplicity m_n are:

$$\begin{aligned}
&\mathrm{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_{n-1}m_{n-1}^2-\alpha_n m_n^2},\\
&\mathrm{Ind}(\mathcal{F}, D_2, p) \in \mathbb{Q}_{\leq -\frac{(y_n^1+(\alpha_n-1)x_n^1)k+(a_n^1+(\alpha_n-1)b_n^1)}{x_n^1 k+b_n^1}},\\
&\mathrm{Ind}(\mathcal{F}, D_1, p) \in \mathbb{Q}_{\leq -\frac{x_n^1}{(\alpha_n-1)x_n^1+y_n^1}}.
\end{aligned}$$

Clearly $x_n^{\alpha_n} \leq y_n^{\alpha_n}$ and so computing the expression $y_n^{\alpha_n} m_n + x_n^{\alpha_n} m_{n+1}$, using the above form of the coefficients and Lemma 3.19 we get the assertion. □

Proposition 3.22. *When in the resolution process we return in double intersection the indices:*

$$\begin{aligned}
&Ind(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -km^2-\alpha_1 m_1^2-\cdots-\alpha_n m_n^2-m_n^2},\\
&Ind(\mathcal{F}, D, q_0) \in \mathbb{Q}_{\leq -\frac{1}{(x_n^{\alpha_n}+y_n^{\alpha_n})^2 k+(x_n^{\alpha_n}+y_n^{\alpha_n})(a_n^{\alpha_n}+b_n^{\alpha_n})}},
\end{aligned} \tag{3.11}$$

satisfy

$$m \geq (x_n^{\alpha_n} + y_n^{\alpha_n})m_n.$$

Proof. It follows directly from the previous proposition and from Lemma 3.19. □

3.2.8 Estimate of the terms $km^2 + \alpha_1 m_1^2 \cdots + \alpha_n m_n^2 + m_n^2$

Proposition 3.23. *The indices at the return in double intersection* (3.11), *with* $n \geq 2$, *satisfy:*

$$\alpha_1 m_1^2 + \cdots \alpha_n m_n^2 + m_n^2 \geq (x_n^{\alpha_n} + y_n^{\alpha_n})(a_n^{\alpha_n} + b_n^{\alpha_n}) m_n^2.$$

Before proving this statement we consider the following one:

Proposition 3.24. *Let* $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\}$ *be a process and let denote by* (x, y, a, b) *the coefficients of the indices that appear at the last blow-up described by* P. *We associate to* P *the process* $\bar{P} = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n)\}$ *and we denote by* $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ *the coefficients of the indices that appear at the last blow-up described by* $\bar{P}$. *Then:*

$$b = \bar{y} \qquad x = \alpha_1 \bar{y} + \bar{a}$$
$$a = \bar{x} \qquad y = \alpha_1 \bar{x} + \bar{b}$$

Proof. We proceed by induction on the number n of multiplicities decreases. By a direct computation the proposition is true for $n = 2$. Suppose the assertion true for n and let prove it for $n + 1$.
Let consider the two processes $P' = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\}$ and $\bar{P}' = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\}$ with respectively end coefficients (x', y', a', b') and $(\bar{x}', \bar{y}', \bar{a}', \bar{b}')$.
Let now construct the following two processes $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\}$, $\bar{P} = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_n, m_n)\}$ with end coefficients (x, y, a, b) and $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$. Starting by coefficients (x, y, a, b) we get (x', y', a', b') after one blow-up with multiplicity decrease and other $\alpha_{n+1} - 1$ blows-up with constant multiplicity m_{n+1}. By Propositions 3.5 and 3.20:

$$(x', y', a', b') = (x, y + \alpha_n x, a + \alpha_n b, b) \text{ if } x > y,$$
$$(x', y', a', b') = (x + \alpha_n y, y, a, \alpha_n a + b) \text{ if } x \leq y.$$

Similarly we get:

$$(\bar{x}', \bar{y}', \bar{a}', \bar{b}') = (\bar{x}, \bar{y} + \alpha_n \bar{x}, \bar{a} + \alpha_n \bar{b}, \bar{b}) \text{ if } \bar{x} > \bar{y},$$
$$(\bar{x}', \bar{y}', \bar{a}', \bar{b}') = (\bar{x} + \alpha_n \bar{y}, \bar{y}, \bar{a}, \alpha_n \bar{a} + \bar{b}) \text{ if } \bar{x} \leq \bar{y}.$$

The processes P e P' differs only on one multiplicity decrease. Propositions 3.5 and 3.20 say that x and y relations invert only when a multiplicity decrease occurs. Then we can conclude that $\bar{x} > \bar{y}$ if and only if $x \leq y$. If, for instance, $x > y$, by inductive hypothesis:

$b' = b = \bar{y} = \bar{y}'$,

$a' = a + \alpha_n b = \bar{x} + \alpha_n \bar{y} = \bar{x}'$,

$x' = x = \alpha_1 \bar{y} + \bar{a} = \alpha_1 \bar{y}' + \bar{a}'$,

$y' = y + \alpha_n x = \alpha_1 \bar{x} + \bar{b} + \alpha_n \bar{y} + \alpha_n \bar{a} = \alpha_1(\bar{x} + \alpha_n \bar{y}) + (\bar{b} + \alpha_n \bar{a}) = \alpha_1 \bar{x}' + \bar{b}'$.

and then the assertion. □

Now we can prove Proposition 3.23.

Proof. Let proceed by induction on the number of changes of multiplicity. If $n = 2$ the structure of the indices can be easily computed to obtain the assertion. Let prove the inductive step. Let $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n), (\alpha_{n+1}, m_{n+1})\}$ be a generic process. Thanks to the inductive step applied on the process $\bar{P} = \{(k, m), (\alpha_2, m_2), \cdots, (\alpha_{n+1}, m_{n+1})\}$ we have:

$$\alpha_1 m_1^2 + \cdots \alpha_n m_n^2 + \alpha_{n+1} m_{n+1}^2 \geq \alpha_1 m_1^2 + (\bar{x} + \bar{y})(\bar{a} + \bar{b}) m_{n+1}^2.$$

In order to estimate $\alpha_1 m_1^2$ we consider the process $P' = \{(\alpha_1, m_1), (\alpha_2, m_2), \cdots, (\alpha_{n+1}, m_{n+1})\}$ and thanks to Remark 3.16 and Proposition 3.22 we have:

$$m_1^2 \geq (\bar{x} + \bar{y})^2 m_{n+1}^2.$$

Then:

$$\begin{aligned}\alpha_1 m_1^2 + \cdots \alpha_n m_n^2 + \alpha_{n+1} m_{n+1}^2 &\geq \alpha_1(\bar{x}+\bar{y})^2 m_{n+1}^2 + (\bar{x}+\bar{y})(\bar{a}+\bar{b}) m_{n+1}^2 \\ &= (\bar{x}+\bar{y})(\alpha_1\bar{x}+\alpha_1\bar{y}+\bar{a}+\bar{b}) m_{n+1}^2.\end{aligned}$$

We conclude thanks to Proposition 3.24. □

Remark 3.25. *The estimate of km^2 and of the remaining terms are valid only if $n \geq 2$. The case $n = 1$ can be easily proved using equation* (3.10)*, Section 3.2.6 and observing that because of the $\alpha_1 + 1$ blows-up $m \geq (\alpha_1 + 1)m_1$.*

3.2.9 Proof of the Theorem

All the previous separately considered particular cases can now be glued together to get Theorem 3.1. We have observed that in the resolution process we can have only double or triple intersection points and so we studied the index in these cases.

The triple point case presents two different subcases, linked to the multiplicity of the curve: it can decrease or not. This information is extremely useful for the study of the index evolution because it identifies the right exceptional divisor that will occur in the next triple point. Now we observe that if at the end of a process $P = \{(k, m), (\alpha_1, m_1), \cdots, (\alpha_n, m_n)\}$ we find a double point and the curve is desingularized we are in the geometric conditions of a (C_2) point. The indices are the ones given by equation (3.7) and by Propositions 3.22 and 3.23 we can estimate them in such a way they became:

$$\begin{aligned}&\mathrm{Ind}(\mathcal{F}, S, p) \notin \mathbb{Q}_{\geq -[(x+y)k^2+(x+y)(a+b)]} \\ &\mathrm{Ind}(\mathcal{F}, D, p) \in \mathbb{Q}_{\leq \frac{1}{(x+y)^2k+(x+y)(a+b)}}\end{aligned}$$

and so p is a (C_2) point. Otherwise we are not in the right geometric conditions, i.e. the resolution is not ended, but the same propositions give indices:

$$\begin{aligned} \mathrm{Ind}(\mathcal{F}, S, p) &\notin \mathbb{Q}_{\geq -[(x+y)k^2+(x+y)(a+b)]m_n^2} \\ \mathrm{Ind}(\mathcal{F}, D, p) &\in \mathbb{Q}_{\leq \frac{1}{(x+y)^2k+(x+y)(a+b)}} \end{aligned}$$

and so we have indices exactly of the form of the ones associated to a process $P = \{hm^2\}$ and then we can apply all the previous argument to the new process which is starting. Such process ends after a finite number of blows-up by the theorem of resolution of singularities [37].

3.3 Applications

The demonstrative method used to get Theorem 3.1 allows us to generalize to the case in which we start with more than one separatrix:

Proposition 3.26. *Let M be a two dimensional complex manifold, $\mathcal{F}$ a holomorphic foliation on M and $S_0, S_1, \cdots, S_n$ separatrices of $\mathcal{F}$ passing through a point $p \in M$. Let assume that $S_1, \cdots, S_n$ are non singular and transverse each other and to S_0. If, besides, the indices are of the following form:*

$$\begin{aligned} Ind(\mathcal{F}, S_0, p) &\notin \mathbb{Q}_{\geq -m^2} \\ Ind(\mathcal{F}, S_i, p) &\in \mathbb{Q}_{\leq -(2n-1)} \quad \forall i \geq 1, \end{aligned}$$

then another separatrix through p exists.

Proof. We prove that p is an appropriate singularity. We observe that after the first blow-up, if we have not finished, we have the same indices found in the study made to prove Theorem 3.1 and so we conclude with the same argument. □

We show briefly that Theorem 3.1 includes as particular cases the classical results in discrete and continuous dynamics.

Corollary 3.27 ([18]). *Let M be a two dimensional complex manifold, $\mathcal{F}$ a holomorphic foliation on M and $p \in M$ a singularity of $\mathcal{F}$. Then a separatrix of $\mathcal{F}$ for p exists.*

Proof. We blow-up M in p. If the exceptional divisor is not a separatrix for the saturated foliation, then p is dicritical and we conclude. Otherwise using the index theorem (see [55]) and remembering that $D \cdot D = -1$, we obtain the existence of a singularity $\tilde{p}$ of the saturated foliation $\tilde{\mathcal{F}}$ such that $\mathrm{Ind}(\tilde{\mathcal{F}}, D, \tilde{p}) \notin \mathbb{Q}^+ \cup \{0\}$ and then by Theorem 3.1 we have the existence of another separatrix for $\tilde{p}$ that projects in a separatrix for $\mathcal{F}$ in p. □

With similar arguments we also have:

Corollary 3.28 ([50]). *Let M be a two dimensional complex manifold, $\mathcal{F}$ a holomorphic foliation on M. Let $S \subset M$ be a compact, globally and locally irreducible curve with $S \cdot S < 0$. If S is a separatrix for $\mathcal{F}$ then there exists a point $p \in S$ for which passes another separatrix for $\mathcal{F}$.*

Remark 3.29. *Analogously to what done for Theorem 3.2 we can obtain, in local discrete dynamics, a similar result to Proposition 3.26 and find as particular cases results of Abate [2] and Bracci [11].*

Proposition 3.30. *Let M be a two dimensional complex manifold, $f : M \longrightarrow M$ a holomorphic map on M with $Fix(f) = S_0 \cup, S_1, \cdots, \cup S_n$ with $S_0, \cdots, S_n$ analytic curves passing through the same point $p \in M$. Let suppose that $S_1, \cdots, S_n$ are non singular and transverse each other and to S_0. If, besides, the indices are of the following form:*

$$\begin{aligned} &Ind(f, S_0, p) \notin \mathbb{Q}_{\geq -m^2} \\ &Ind(f, S_i, p) \in \mathbb{Q}_{\leq -(2n-1)} \quad \forall i \geq 1, \end{aligned}$$

then there exists a parabolic curve for f through p.

Corollary 3.31 ([2]). *Let M be a two dimensional complex manifold, $f : M \longrightarrow M$ a holomorphic map on M and $p \in M$ an isolated singularity of f such that $df_p = Id$. Then there exists a parabolic curve for f through p.*

Corollary 3.32 ([50]). *Let M be a two dimensional complex manifold, $f : M \longrightarrow M$ a holomorphic map on M. Let $S \subset M$ be a compact, globally and locally irreducible curve with $S \cdot S < 0$. If $f\mid_S = Id$ and f is tangential along S then there exists a point $p \in S$ for which passes a parabolic curve for f.*

Chapter 4

Transversely formal vector fields

4.1 Introduction

We have just noticed the strict relation between discrete and continuous dynamics. Abate, Bracci and Tovena [2], [3], [11] show that this link is well expressed by the presence of indices, in both cases, that have the same properties (index theorem) and the same behavior under blow-up. This analogy, as seen in the previous chapter, is expressed also by the existence of separatrices and parabolic curves under the same assumptions.

The aim of this chapter is to analyze more in detail this relationship. The more intuitive construction to link maps to vector fields is to associate to a map tangent to the identity F in $\mathbb{C}^2$ a formal vector field X such that

$$\exp(X) = F.$$

In this way the vector field associated to the map is not, generally, holomorphic but only formal. Thus, in order to know if this construction is useful we have to see if the vector field preserves the dynamical properties we are interested in for our study. In particular, that means, to know if the vector field admits an index theorem. The right assumption to take in discrete dynamics is to consider a germ of diffeomorphism in $\mathbb{C}^2$ with a smooth curve of fixed points. The case of an isolated singularity can be reduced to this setting exploiting a blow-up of the map. So the general setting is: F a germ of diffeomorphism of $(\mathbb{C}^2, 0)$ tangent to the identity with a smooth curve of fixed points and X a formal vector field such that $\exp(X) = F$.

The first thing we want to analyze is the general setting in which the C.S.S. index lives i.e. the minimum geometric setting that assures Camacho-Sad-Suwa results. In order to explore this problem we observe that if we blow-up a formal vector field the blown-up vector field admits a very good *geometric* structure on the exceptional divisor: the vector field is transversely formal on it [43], [44] i.e. if the exceptional divisor is $\{y = 0\}$ then the vector field belongs to

$$\mathbb{C}\{x\}[[y]] \oplus \mathbb{C}\{x\}[[y]].$$

This is an additional structure that has the formal vector field and we will see that it is sufficient in order to guarantee the existence of an index theorem [49].

Theorem 4.1. *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ with $Fix(F) = S$, where S is a smooth holomorphic curve, and let suppose F is tangential on S. Then the formal vector field X such that $\exp(X) = F$ is transversely formal along the separatrix S.*

We observe that the standard hypothesis in discrete dynamics of tangentiality of the map [2], [3], [11] assumes an easily geometric interpretation in this construction: it means that S is a separatrix for the reduced vector field X. This intuitive method for studying discrete dynamics allows to read from a more geometric point of view the *dictionary* discrete/continuous dynamics.

This it will useful in Chapter five [29] to convert in discrete dynamics other results of continuous one.

4.2 Singularities of formal vector fields

In this section we traduce in the formal category all the definitions we have used in the study of holomorphic vector fields. First of all, we will make precise the notion of singularity of a formal vector field. In the holomorphic category, we have seen that a vector field

$$X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

is singular at the origin if $A(0,0) = B(0,0) = 0$, i.e., the power series expressions of A and B have not terms of degree zero. This last interpretation is valid even in the formal category:

Definition 4.2. *Let consider a formal vector field,*

$$X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

where $A, B \in \mathbb{C}[[x,y]]$. *Let call the* complexity parameter *of the vector field* X *the quantity*

$$r(X) = min\{subdeg(A), subdeg(B)\},$$

where $subdeg(\cdot)$ *is the smaller degree of the monomial appearing in the series expression of* $\cdot$. *We say that the origin* $(0,0)$ *is a singularity of* X *if* $r(X) \geq 1$.

4.3 Transversely formal vector fields

Let consider a complex surface M and let $S \subset M$ be a complex non singular curve.

Let consider the following exact sequence of sheaves that defines the normal sheaf $\mathcal{N}_S$:

$$0 \to \Theta_{M,S} \to \Theta_M \to \mathcal{N}_S \to 0, \tag{4.1}$$

where $\Theta_{M,S}$ is the sheaf of holomorphic vector fields tangent to S, Θ_M is the sheaf of holomorphic vector fields tangent to M and $\mathcal{N}_S$ is the sheaf of holomorphic vector fields normal to S.

Remark 4.3. *Note that the exact sequence 4.1 of locally free sheaves is the one associated to the holomorphic sections of the corresponding vector bundles.*

Let consider

$$\hat{\mathcal{O}}_M := \lim_{\leftarrow} \frac{\mathcal{O}_M}{\mathcal{I}_S^n}, \tag{4.2}$$

where $\mathcal{I}_S := \{f \in \mathcal{O}_M \mid f \mid_S \equiv 0\}$.

Remark 4.4. *In local coordinates, if $S = \{y = 0\}$, then*

$$\hat{\mathcal{O}}_M \cong \mathbb{C}\{x\}[[y]].$$

Tensoring by (4.2) the sequence (4.1), by Proposition (10.14) of [7] (pg. 109), we get the following exact sequence:

$$0 \to \hat{\Theta}_{M,S} \to \hat{\Theta}_M \to \hat{\mathcal{N}}_S \to 0$$

where:

$$\hat{\Theta}_{M,S} := \Theta_{M,S} \otimes \hat{\mathcal{O}}_M, \qquad \hat{\Theta}_M := \Theta_M \otimes \hat{\mathcal{O}}_M, \qquad \hat{\mathcal{N}}_S := \mathcal{N}_S \otimes \hat{\mathcal{O}}_M.$$

We can read the previous objects as:

- $\hat{\Theta}_{M,S}$ the transversely formal sheaf of tangent vector fields to S,

- $\hat{\Theta}_M$ the transversely formal sheaf of vector fields,

- $\hat{\mathcal{N}_S}$ the sheaf of transversely formal normal vector fields to S.

Remark 4.5. *The previous sheaves generally are not locally free and so they are not associated to vector bundles but we can see them as the transversely formal extension of the holomorphic ones.*

After this construction is natural to give the following definition:

Definition 4.6. *A transversely formal vector field tangent to S is a section of the sheaf $\hat{\Theta}_{M,S}$.*

4.4 Index Theorem for transversely formal vector fields

Let consider a section $\hat{X}$ of $\hat{\Theta}_{M,S}$. According to the Camacho-Sad index theory [18] we are interested only on the liner part of $\hat{X}$ along S, so let consider the following sequence of sheaves:

$$0 \to \hat{\Theta}_{M,S} \otimes \mathcal{I}_S \to \hat{\Theta}_{M,S} \to \hat{\mathcal{N}}^1_{M,S} \to 0,$$

where,

$$\hat{\mathcal{N}}^1_{M,S} := \frac{\hat{\Theta}_{M,S}}{\hat{\Theta}_{M,S} \otimes \mathcal{I}_S}.$$

Remark 4.7. *Let observe that a section of $\hat{\mathcal{N}}^1_{M,S}$ is of the form:*

$$a(x)\frac{\partial}{\partial x} + b(x)y\frac{\partial}{\partial y},$$

where $a(x), b(x) \in \mathbb{C}\{x\}$.

Now we have the following isomorphism:

$$\hat{\mathcal{N}}^1_{M,S} \cong \mathcal{N}^1_{M,S},$$

where $\mathcal{N}^1_{M,S}$ is the holomorphic analogous of $\hat{\mathcal{N}}^1_{M,S}$.

The previous isomorphism says that we can see $\hat{\mathcal{N}}^1_{M,S}$ as the sheaf of sections of a holomorphic bundle.

Lemma 4.8. *If $\hat{X}$ is a transversely formal vector field on M then it defines a global section of $\hat{\mathcal{N}}^1_{M,S}$.*

Proof. Let $\hat{X}$ be a global transversely formal vector field. In the (non empty) intersections of the two local charts $U_\alpha \cap U_\beta \neq \emptyset$ we have:

$$X_\beta = \phi^*_{\beta\alpha}(X_\alpha),$$

where the $\phi_{\beta\alpha}$'s are the transition functions. If we choose coordinate adapted to S, i.e. such that $S \cap U_\alpha = \{y_\alpha = 0\}$ for all α, then $\phi_{\beta\alpha}$ is of the form:

$$\phi_{\beta\alpha}(x_\beta, y_\beta) = ({\phi^1}_{\beta\alpha}(x_\beta, y_\beta), {\phi^2}_{\beta\alpha}(x_\beta, y_\beta)) = (\sum_{i\geq h} \nu_i(x_\beta) y^i_\beta, \sum_{i\geq k} \mu_i(x_\beta) y^i_\beta).$$

with $h \geq 0, k \geq 1$ and $h < k$. We have to prove that, denoting by $[X_\alpha] \in \hat{\mathcal{N}}^1_{M,S}$ the class of X_α, then

$$[\phi^*_{\beta\alpha} X_\alpha] = [X_\beta].$$

So it is sufficient to prove that $\phi_{\beta\alpha}$ transforms terms of order k with respect to y_β in terms at least of order k with respect to y_α.

For this, it is enough to observe that the inverse of $\phi_{\beta\alpha}$, $\phi_{\alpha\beta}$, has the same order with respect to y_α. In fact:

$$\frac{\partial}{\partial x_\alpha} = \frac{{\phi^1}_{\alpha\beta}}{\partial x_\alpha}\frac{\partial}{\partial x_\beta} + \frac{\partial {\phi^2}_{\alpha\beta}}{\partial x_\alpha}\frac{\partial}{\partial y_\beta} = o(y^h)\frac{\partial}{\partial x_\beta} + o(y^k)\frac{\partial}{\partial y_\beta},$$

and analogously:

$$\frac{\partial}{\partial y_\alpha} = o(y^{h-1})\frac{\partial}{\partial x_\beta} + o(y^{k-1})\frac{\partial}{\partial y_\beta}.$$

□

Definition 4.9. *Let $X \in \hat{\mathcal{O}}_{M,S}$ be a transversely formal vector field on a complex surface M with S a non singular curve. If, in local coordinates, $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ and $S = \{y = 0\}$ then X is **reduced** on S if X has only isolated singularities on $\{y = 0\}$.*

We can now state and prove the announced Index Theorem:

Theorem 4.10. *Let $\hat{X}$ be a transversely formal vector field on a complex manifold of dimension two tangent to a compact, connected non singular curve $S \subset M$. Then for every $p \in S$ there exists an index $Ind(\hat{X}, S, p) \in \mathbb{C}$ such that:*

$$\sum_{p \in S} Ind(\hat{X}, S, p) = S \cdot S.$$

Proof. The vector field $\hat{X}$ has only a finite number of singularities on S. Let $\Sigma := \{p_1, \cdots, p_n\}$ be the singular set on S and let be $V := S - \Sigma$. For every $i = 1, \cdots, n$ let be U_i a coordinate neighborhood such that $U_i \cap \Sigma \cap S = \{p_i\}$. By Lemma (2.5) of [10] it exists a basic connection ∇ (see Definition (3.24) of [10]) for $N^1_{M,S} |_{S-\Sigma}$. Let be W_i be a simply connected open set in S such that $\bar{W}_i \subset U_i \cap S$. On each $U_i \cap S$ let ∇_i be a connection for ${N^1}_{M,S} |_{U_i \cap S}$. Let ψ be a C^∞ function on S such that ψ has support in $\cup_i(U_i \cap S)$ and $\psi |_{W_i} \equiv 1$ for $i = 1, \cdots, n$. Let $\nabla_1 := \psi \sum \nabla_i + (1 - \psi)\nabla$ a connection for ${N^1}_{M,S}$ and K_1 its curvature. By Proposition (3.27) of [10] we get that K_1 localizes around the singularities and so we get the index theorem by the following relations:

$$S \cdot S = \frac{-1}{2\pi i}\int_S K_1,$$

and

$$Ind(F, S, p) := \frac{-1}{2\pi i} \int_{V_p} K_1,$$

where V_p is a simply connected open set in S containing p such that $V_p \cap \cup_i (U_i \cap S) = \emptyset$ if $p \notin \Sigma$ and $V_p = U_i \cap S$ if $p = p_i$.

Let observe that in this way we have localized near singularities the first Chern class $c_1(N^1_{M,S})$. To conclude we have to observe that $c_1(N^1_{M,S}) = c_1(N_S) = S \cdot S$. The last equality follow from the fact that:

$$N^1_{S,M} \mid_S = N_{S,M} \mid_S,$$

and then:

$$\int_S K_1 = \int_S K = S \cdot S,$$

where K is the curvature of $N_{S,M}$. □

4.5 Flows of formal vector fields

The aim of this section is to find a definition of the flow of a formal vector field that in the convergent case coincides with the usual one. In order to get this goal we consider the following power series expression of the flow of a convergent vector field.

Proposition 4.11. *Let X be a germ of holomorphic vector field in $(\mathbb{C}^2, 0)$. Then its flow can be written has:*

$$F^t(x, y) = \left(x + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.x, y + \sum_{n=1}^{\infty} \frac{t^n}{n!} X^n.y \right), \tag{4.3}$$

where $X^n.x$ is defined by X applied to $X^{n-1}.x$ and $X.x$ is the application of X to x.

This proposition allows to give the following definition:

Definition 4.12. *Let be $\hat{X}$ a formal vector fields in* $(\mathbb{C}^2, 0)$ *i.e.*

$$\hat{X} = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \tag{4.4}$$

with $a(x,y), b(x,y) \in \mathbb{C}[[x,y]]$. *The **formal flow** of $\hat{X}$ is defined by the formal map* (4.3).

Remark 4.13. *If $\hat{X}$ is a formal vector fields in* $(\mathbb{C}^2, 0)$ *then its time one map is given by:*

$$\exp(\hat{X}) = \left(x + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{X}^n.x, y + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{X}^n.y \right). \tag{4.5}$$

4.6 Formal vector fields associated to holomorphic maps

Let F be a holomorphic germ of diffeomorphism at $(\mathbb{C}^2, 0)$ tangent to the identity at $(0,0)$, i.e.:

$$F(x,y) = \left(x + \sum_{i+j>1} A_{i,j}x^i y^j, y + \sum_{i+j>1} B_{i,j}x^i y^j \right). \tag{4.6}$$

By equation (4.5) we can associate to F a formal vector field $\hat{X}$ such that $\exp(\hat{X}) = F$. The vector field $\hat{X}$, determined by comparing the two series. It is easy to see that it has the form:

$$\hat{X} = \sum_{i+j>1} a_{i,j}x^i y^j \frac{\partial}{\partial x} + \sum_{i+j>1} b_{i,j}x^i y^j \frac{\partial}{\partial y}. \tag{4.7}$$

If $\{y = 0\}$ is a curve of fixed points for F then F has the form:

$$F(x,y) = (x + y^\mu F_1(x,y), y + y^\mu F_2(x,y)).$$

In this case the vector field is of the form (4.7) with the additional condition $j \geq 1$.

Then if $\{y=0\}$ is a curve of fixed points for F then we can write the associated formal vector field in the form:

$$\hat{X} = \sum_{j\geq 1} a_j(x) y^j \frac{\partial}{\partial x} + \sum_{j\geq 1} b_j(x) y^j \frac{\partial}{\partial y}. \tag{4.8}$$

4.7 Transversely formal vector fields associated to holomorphic maps

In this section we investigate in more detail the structure of $\hat{X}$ in case the map F has a non singular curve of fixed points. We will prove that, under *generic* hypothesis for F, the vector field has a transverse formal structure, i.e., in the expression (4.8) the power series $a_j(x)$ and $b_j(x)$ are convergent.

To prove this we proceed with a method of majorant series due to Cauchy.

Proposition 4.14. *If $X^n.$ is the operator defined before we have:*

$$X^n.y = (\sum_{i_0\geq n} \sum_{i_1+\cdots+i_n=i_0} b_{i_1,1}\cdots b_{i_n,1} x^{i_0})y + T_n(x,y),$$

where $T_n(x,y)$ contains only terms that are of order at least two in y.

Proof. We proceed by induction on n. For $n=1$ we have:

$$X.y = \sum_{i+j>1 j\geq 1} b_{i,j} x^i y^j = \sum_{i_0\geq 1} b_{i_0,1} x^{i_0} y + \sum_{i_0+j>1, j\geq 2} b_{i_0,1} x^{i_0} y^j,$$

and we have the assertion with:

$$T_1(x,y) := \sum_{i_0\geq 1, j\geq 2} b_{i_0,1} x^{i_0} y^j.$$

Suppose the assertion true for n and let go to prove it for $n+1$.

$$X^{n+1}.y = X.(X^n.y) =$$

$$(\sum_{i+j>1} a_{i,j} x^i y^j)[\sum_{i_0\geq n} \sum_{i_1+\cdots+i_n=i_0} b_{i_1,1}\cdots b_{i_n,1} i_0 x^{i_0-1} y + \frac{\partial}{\partial x} T_n(x,y)] +$$

$$+ (\sum_{i+j>1} b_{i,j} x^i y^j)[\sum_{i_0\geq n} \sum_{i_1+\cdots+i_n=i_0} b_{i_1,1}\cdots b_{i_n,1} x^{i_0} + \frac{\partial}{\partial y} T_n(x,y)] =$$

$$
\begin{aligned}
&\sum_{i+j>1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} i_0 a_{i,j} b_{i_1,1}\cdots b_{i_n,1} x^{i_0+i-1} y^{j+1} + O(y^2)\\
&+\sum_{i+j>1} a_{i,j} x^i y^j \frac{\partial}{\partial x} T_n(x,y) + O(y^2)\\
&+\sum_{i+j>1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} b_{i,j} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y^j +\\
&+\sum_{i+j>1} b_{i,j} x_i y^j \frac{\partial}{\partial y} T_n(x,y) O(y^2) =\\
&=\sum_{i+j>1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} b_{i,j} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y^j + O(y^2).
\end{aligned}
$$

We now observe that:

$$
\begin{aligned}
&\sum_{i+j>1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} b_{i,j} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y^j =\\
&\sum_{i\geq 1}\sum_{i_1\geq n}\sum_{i_0+\cdots+i_n=i_0} b_{i,1} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y +\\
&+\sum_{j\geq 2,\ i+j>1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} b_{i,j} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y^j.
\end{aligned}
$$

Then the part of degree one in y is:

$$
\begin{aligned}
&\sum_{i\geq 1}\sum_{i_0\geq n}\sum_{i_1+\cdots+i_n=i_0} b_{i,1} b_{i_1,1}\cdots b_{i_n,1} x^{i+i_0} y =\\
&=\sum_{j_0\geq n+1}\sum_{i_1+\cdots+i_n+i_{n+1}=j_0} b_{i_1,1}\cdots b_{i_n,1} b_{i_{n+1},1} x^{j_0} y.
\end{aligned}
$$

Where the last equality is obtained putting $j_0 = i + i_0$. □

Now we can get the following:

Corollary 4.15. *Using the notations of* (4.6) *and of* (4.7)*:*

$$
B_{i_0,1} = \sum_{k=1}^{i_0} \frac{1}{k!} \sum_{i_1+\cdots+i_k=i_0} b_{i_1,1}\cdots b_{i_k,1} \forall i_0 \in \mathbb{N}. \tag{4.9}
$$

Proof. We have to consider the terms of

$$\sum_{n\geq 1} \frac{1}{n!} \hat{X}^n.y$$

that have degree one with respect to y and degree i_0 with respect to x. By the previous Proposition the terms of the required degree are:

$$\sum_{i_1+\cdots+i_n=i_0} b_{i_1,1}\cdots b_{i_n,1}.$$

By proposition 4.14, the terms in $\hat{X}^n.y$ with degree one with respect to y are at least of degree n with respect to x and so we can conclude. □

In order to prove the convergence of the series

$$b_1(x) := \sum_{n\geq 1} b_{n,1}x^n$$

we have to solve the system of infinite equations found in Corollary (4.15). After, we have to prove the convergence of the series by a majorant argument.

Remark 4.16. *We are going to study only terms that appear in equation* (4.9) *and so we will write simply B_n instead of $B_{n,1}$ and b_n instead of $b_{n,1}$.*

Proposition 4.17. *By inversion of the system of equations* (4.9) *we obtain:*

$$b_n = \sum_{\substack{i_1+\cdots+i_n=n \\ 0\leq i_j\leq n}} a_{i_1\cdots i_n} B_{i_1}\cdots B_{i_n},$$

where $B_0 = 1$.

Proof. Let proceed by induction on n. For $n = 1$ it is true. Let prove it for n. By equation (4.9) we have:

$$B_n = b_n + \sum_{k=2}^{n} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} b_{i_1}\cdots b_{i_k}.$$

Because in the last sum $i_j < n$ for every $j \in \{1, \cdots, k\}$ we can apply the inductive hypothesis to every b_{i_j} that appear in the last sum.

Then:

$$b_{i_j} = \sum_{h_1^{i_j} + \cdots h_{i_j}^{i_j} = i_j} a_{h_1^{i_j} \cdots h_{i_j}^{i_j}} B_{h_1^{i_j}} \cdots B_{h_n^{i_j}} \qquad \forall i_j < n.$$

So:

$$\begin{aligned} b_n =& B_n - \sum_{k=2}^{n} \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \left(\sum_{h_1^{i_1} + \cdots + h_{i_1}^{i_1} = i_1} a_{h_1^{i_1} \cdots h_{i_1}^{i_1}} B_{h_1^{i_1}} \cdots B_{h_{i_1}^{i_1}} \right) \cdots \\ & \cdots \left(\sum_{h_1^{i_k} + \cdots + h_{i_k}^{i_k} = i_k} a_{h_1^{i_k} \cdots h_{i_k}^{i_k}} B_{h_1^{i_k}} \cdots B_{h_{i_k}^{i_k}} \right) = \\ =& B_n - \sum_{k=2}^{n} \frac{1}{k!} \sum_{\substack{h_1^{i_1} + \cdots + h_{i_1}^{i_1} + \cdots + h_1^{i_k} + \cdots h_{i_k}^{i_k} = \\ = i_1 + \cdots + i_k}} a_{h_1^{i_1} \cdots h_{i_1}^{i_1}} \cdots a_{h_1^{i_k} \cdots h_{i_k}^{i_k}} B_{h_1^{i_1}} \cdots B_{h_{i_k}^{i_k}} \\ =& B_n - \sum_{h_1 + \cdots + h_n = n} a_{h_1 \cdots h_n} B_{h_1} \cdots B_{h_n}. \end{aligned}$$

□

Now we can proceed by a majorant argument. For this we need the following lemma that can be proved easily by induction:

Lemma 4.18. *For every $n \geq 1$ we have the following inequality:*

$$\binom{2n-1}{n-1} \leq 5^{n-1}$$

Proposition 4.19. *Every b_n can be written in the form:*

$$b_n = \sum_{k=1}^{n} \sum_{i_1 + \cdots i_k = n} a_{i_1 \cdots i_k} B_{i_1} \cdots B_{i_k} \tag{4.10}$$

and this expression is made by less than 2^{n-1} *terms and the coefficients* $a_{i_1 \cdots i_k}$ *satisfy:*

$$\mid a_{i_1 \cdots i_k} \mid \leq 5^{n-1}.$$

Proof. We easily get the expression (4.10) by using Proposition 4.17 (and making the sum only with the $i_j > 0$). Now let us count the number of terms that appears in expression (4.10). This number is obviously less than the number of terms that we have if all the $a_{i_1 \cdots i_k}$ are different from zero. Then:

$$\sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} 1 = \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{h=0}^{n-1} \binom{n-1}{h} = 2^{n-1},$$

where the first equality is due to the solution to the linear Waring problem (see [46] pg. 124) and the last one is due to the binomial formula (see [32], pg. 52).

Let us prove the second statement of the proposition. Let proceed by induction on n. For $n = 1$ we only have:

$$b_1 = B_1$$

and so the statement is true. Let suppose it is true for $i < n$ and let go to prove it for n. As we have seen in the proof of Proposition 4.17:

$$b_n = B_n - \sum_{k=2}^{n} \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} b_{i_1} \cdots b_{i_k}.$$

Obviously the b_{i_j} that appear in the previous equation are such that $i_j < n$ and so we can apply the inductive hypothesis. So we can substitute at b_{i_j} its expression and then we have to sum similar terms. To sum similar terms we consider the field $\mathbb{C}$ as a non commutative one. Before summing the similar terms a generic terms is of type:

$$a_{h_1^{i_1} \cdots h_{i_1}^{i_1}} \cdots a_{h_1^{i_k} \cdots h_{i_k}^{i_k}} B_{h_1^{i_1}} \cdots B_{h_{i_k}^{i_k}}$$

with $h_1^{i_j} + \cdots + h_{i_j}^{i_j} = i_j$ and $i_1 + \cdots i_k = n$. By inductive hypothesis the coefficient of each such term has modulus less or equal to 5^{n-h}.

So to conclude it is sufficient to prove that the number of the similar terms, in the non commutative field $\mathbb{C}$, are less or equal to 5^{h-1}. To prove this it is sufficient to observe that the similar terms are less or equal to the number of solutions in nonnegative integers of the linear Waring equation:

$$i_1 + \cdots + i_h = h.$$

By [46] (pg.124) this number is $\binom{2h-1}{h-1}$. We get assertion by Lemma 4.18.

□

At this point we are ready to prove the convergence of the series:

Proposition 4.20. *The series:*

$$\sum_{n \geq 1} b_n x^n$$

is convergent.

Proof. We proceed constructing a convergent majorant series. By equation (4.10) and the Cauchy estimates ([22], pg. 135) for derivatives of the power convergent series

$$\sum_{i,j} B_{i,j} x^i y^j$$

we have:

$$| b_n | \leq max \mid a_{i_1 \cdots i_n} \mid \{\text{number of terms of the equation (4.10)}\} \frac{M^n}{R^{2n}},$$

where (R, R) is the polyradius of convergence of the series and we assume, without loss of generality, $R \leq 1$ and $M \geq R$. Then by Proposition 4.19 we have:

$$| b_n | \leq 5^{n-1} 2^{n-1} \frac{M^n}{R^{2n}} = 10^{n-1} \frac{M^n}{R^{2n}}.$$

So the convergent majorant series is given by coefficients:

$$c_n := 10^{n-1}\frac{M^n}{R^{2n}}.$$

□

Now we start to prove the convergence of all the $a_j(x)$ and the $b_j(x)$ $\forall j \geq 1$. We proceed by induction on j. For $j = 1$ it is true by Proposition 4.20. We have only to prove the inductive step.

To simplify the notation we will denote by $(X^k.x)_j$ the part of $X^k.x$ of order j with respect to y.

Remark 4.21. *With the preceding notation we have:*

$$A_j(x) := \sum_i A_{i,j}x^i = \sum_{k\geq 1}\frac{1}{k!}(X^k.x)_j,$$
$$B_j(x) := \sum_i B_{i,j}x^i = \sum_{k\geq 1}\frac{1}{k!}(X^k.y) - j.$$

Proposition 4.22. *Using notations of* (4.8) *for every $j_0 \in \mathbb{N}$ the following recursive relation holds:*

$$(X^k.x)_{j_0} = \sum_{i=1}^{j_0-1} a_i(x)\frac{\partial}{\partial x}(X^{k-1}.x)_{j_0-i} + \sum_{i=1}^{j_0} b_i(x)(j_0-i+1)(X^{k-1}.x)_{j_0-i+1}.$$

Proof. We observe that:

$$X^k.x = \sum_{i\geq 1} a_i(x)y^i\frac{\partial}{\partial x}(X^{k-1}.x) + \sum_{i\geq 1} b_i(x)y^i\frac{\partial}{\partial y}(x^{k-1}.x).$$

The terms of degree j_0 with respect to y are:

$$\sum_{i=1}^{j_0-1} a_i(x)(\frac{\partial}{\partial x}X^{k-1}.x)_{j_0-i} + \sum_{i=1}^{j_0} b_i(x)(\frac{\partial}{\partial y}X^{k-1}.x)_{j_0-i} =$$
$$\sum_{i=1}^{j_0-1} a_i(x)\frac{\partial}{\partial x}(X^{k-1}.x)_{j_0-1} + \sum_{i=1}^{j_0} b_i(x)(j_0-i+1)(X^{k-1}.x)_{j_0-i+1}.$$

Where we have used the fact that the derivation $\frac{\partial}{\partial x}$ does not change the degree with respect to y while the derivation $\frac{\partial}{\partial y}$ takes y^i in iy^{i-1}. □

Remark 4.23. *Let observe that the part:*

$$b_{j_0}(x)(X^{k-1}.x)_1 + b_1(x)j_0(X^{k-1}.x)_{j_0}$$

is the only part that contains the terms $b_{j_0}(x)$ and $a_{j_0}(x)$.

Let decompose $(X^k.x)_{j_0}$ in the following way:

$$(X^k.x)_{j_0} = P_k(x)a_{j_0}(x) + Q_k(x)b_{j_0}(x) + R_k(x).$$

Let go to study this decomposition.

Proposition 4.24.

$$P_k(x) = j_0^{k-1}b_1(x)^{k-1}.$$

Proof. Let proceed by induction on k. For $k = 1$ we have:

$$X.x = \sum_{j\geq 1} a_j(x)y^j,$$

and so the part of degree j_0 is simply $a_{j_0}(x)$ according to the assertion. Let go to prove the inductive step. By Proposition 4.22 the only part where we find $a_{j_0}(x)$ is:

$$b_1(x)j_0(X^{k-1}.x)_{j_0}.$$

By inductive hypothesis we can conclude:

$$P_k(x) = b_1(x)j_0(j_0b_1(x))^{k-2} = (j_0b_1(x))^{k-1}.$$

□

Lemma 4.25. *With the preceding notations the following relation holds:*

$$(X^k.x)_1 = b_1(x)^{k-1}a_1(x).$$

Proof. Let proceed by induction on k. For $k = 1$ it is true because $(X.x)_1 = a_1(x)$. Let go to prove the inductive step. For inductive hypothesis we have that $X^{k-1}.x = b_1(x)^{k-2}a_1(x)y + O(y^2)$ where $O(y^2)$is an expression of order bigger than two in y.

$$X^k.x = X.(X^{k-1}.x) = \sum_{i\geq 1} a_i(x)y^i[\frac{\partial}{\partial x}(b_1(x)^{k-2}a_1(x))y + \frac{\partial}{\partial x}O(y^2)] + \sum_{i\geq 1} b_i(x)y^i[b_1(x)^{k-2}a_1(x) + \frac{\partial}{\partial y}O(y^2)].$$

We conclude observing that $\frac{\partial}{\partial x}(b_1(x)^{k-2}a_1(x))y + \frac{\partial}{\partial x}O(y^2)$ and $\frac{\partial}{\partial y}O(y^2)$ have degree bigger than one in y. □

Proposition 4.26. *With the preceding notations the following relation holds:*

$$Q_k(x) = c_k b_1(x)^{k-2}a_1(x)$$

for every $k \geq 1$. Where the sequence $\{c_k\}$ is defined by recurrence by:

$$\begin{cases} c_1 = 0 \\ c_k = j_0 c_{k-1} + 1 \end{cases}$$

Proof. Let proceed by induction on k. For $k = 1$ we have that $(X.x)_{j_0} = a_{j_0}(x)$ and then $c_1 = 0$. Let go to prove the inductive hypothesis. By Remark 4.23 the part containing $b_{j_0}(x)$ is:

$$(X^{k-1}.x)_1 b_{j_0}(x) + b_1(x) j_0 (X^{k-1}.x)_{j_0}.$$

By inductive hypothesis and Lemma 4.25 we have that the part with $b_{j_0}(x)$ is given by:

$$\begin{aligned}
&(X^{k-1}.x)_1 + b_1(x) j_0 c_{k-1} b_1(x)^{k-3} a_1(x) = \\
&b_1(x)^{k-2}a_1(x) + j_0 b_1(x)^{k-2} a_1(x) c_{k-1} = \\
&(1 + j_0 c_{k-1}) b_1(x)^{k-2} a_1(x) = c_k b_1(x)^{k-2} a_1(x).
\end{aligned}$$

□

The sequence $\{c_k\}$ is monotone and so admits a limit, then we have:

Lemma 4.27. *The following series have non zero radius of convergence:*

1. $\sum_{k\geq 1} \frac{j_0^{k-1}}{k!} z^{k-1}$,
2. $\sum_{k\geq 2} \frac{c_k}{k!} z^{k-2}$.

By this lemma and by Propositions 4.24 and 4.26 we get:

Corollary 4.28. *With the previous notations the following series are convergent:*

1. $P(x) := \sum_{k\geq 1} \frac{P_k(x)}{k!}$,
2. $Q(x) := \sum_{k\geq 2} \frac{Q_k(x)}{k!}$.

Proposition 4.29. *Let fix $j_0 \in \mathbb{N}$, then the series*

$$R(x) := \sum_{k\geq 1} \frac{R_k(x)}{k!},$$

is convergent.

Proof. We can easily prove by induction on k that:

$$\begin{cases} R_2(x) = \displaystyle\sum_{i=1}^{j_0-1} a_i(x) \frac{\partial}{\partial x} a_{j_0-i}(x) + \sum_{i=2}^{j_0-1} (j_0 - i + 1) b_i(x) a_{j_0-i+1}(x), \\ R_k(x) = \displaystyle\sum_{i=1}^{j_0-1} a_i(x) \frac{\partial}{\partial x} (X^{k-1}.x)_{j_0-i} + \sum_{i=2}^{j_0-1} (j_0 - i + 1) b_i(x) (X^{k-1}.x)_{j_0-i+1} + \\ \qquad + j_0 b_1(x) R_{k-1}(x) \end{cases}$$

Now we can prove that the the number of terms that appear in R_k are at most

$$(k-1)(2j_0 - 3).$$

Let proceed by induction on k.

For $k = 2$ we have $j_0 - 1 + j_0 - 2 = 2j_0 - 3 = (2-1)(2j_0 - 3)$ terms. Then by the previous relations for R_k and by the inductive hypothesis we have at most $j_0 - 1 + j_0 - 2 + (k-2)(2j_0 - 3) = (k-1)(2j_0 - 3)$ terms.

Now let observe, by induction on k, that in R_k we find expressions homogenous of degree k in $a_i, \frac{\partial}{\partial x} a_i, b_i, \frac{\partial}{\partial x} b_i$ for $i = 1, \cdots, j_0$.

Let K be a compact set and let

$$M := \max_{i=1,\cdots,j_0} \{\max_K \{| a_i |\}, \max_K \{| \frac{\partial}{\partial x} a_i |\}, \max_K \{| b_i |\}, \max_K \{| \frac{\partial}{\partial x} b_i |\}\}.$$

So we have that:

$$| R_k(x) | \leq (k-1)(2j_0 - 3) M^k.$$

Then the series $| R(x) |$ is majorized by the convergent series:

$$\sum \frac{(k-1)(2j_0 - 3) M^k}{k!}$$

and then is convergent. □

Using the notations previously introduced we have:

$$A_{j_0}(x) = \sum_{k \geq 1} \frac{(X^k.x)_{j_0}}{k!} = P(x) a_{j_0}(x) + Q(x) b_{j_0}(x) + R(x).$$

If we define

$$\tilde{A}_{j_0}(x) := A_{j_0}(x) - R(x)$$

we have:

$$\tilde{A}_{j_0}(x) = x^s \tilde{P}(x) a_{j_0}(x) + Q(x) b_{j_0}(x),$$

where:

$$P(x) =: x^s \tilde{P}(x) \text{ with } \tilde{P}(0) \neq 0.$$

So that:

$$x^s a_{j_0}(x) = \frac{\tilde{A}_{j_0}(x) - Q(x) b_{j_0}(x)}{\tilde{P}(x)}. \tag{4.11}$$

For the symmetry of the problem we have also the following relation:

$$B_{j_0}(x) = \sum_{k\geq 1} \frac{(X^k.y)_{j_0}}{k!} = P'(x)a_{j_0}(x) + Q'(x)b_{j_0}(x) + R'(x), \tag{4.12}$$

with $P'(x), Q'(x), R'(x)$ convergent.

Multiplying the (4.12) by x^s and substituting the expression (4.11) we get:

$$B_{j_0}(x) = \frac{P'(x)\tilde{A}_{j_0}(x)}{\tilde{P}(x)} - \frac{P'(x)Q(x)}{\tilde{P}(x)}b_{j_0}(x) + x^s Q'(x)b_{j_0}(x) + x^s R'(x).$$

Because all the terms are convergent except $b_{j_0}(x)$ then even $b_{j_0}(x)$ is convergent and so even $a_{j_0}(x)$.

In this way we proved the inductive step and so following it holds:

Proposition 4.30. *According with equation* (4.8) *all the coefficients $a_j(x)$ and $b_j(x)$ are convergent, i.e. $\hat{X}$ is a transversely formal vector field.*

4.8 Index Theorem for maps tangent to the identity

Let us consider a germ of holomorphic diffeomorphism tangent to the identity with a smooth curve of fixed points. In the previous section we proved that the corresponding vector field $\hat{X}$ is transversely formal with respect to S. Such a vector field does not admit a discrete singular set along S. So we have to consider the reduced associated vector field, i.e., we have to divide $\hat{X}$ by the highest power of an expression of S that divide the vector field. We denote the reduced vector field always with $\hat{X}$.

After this procedure the new vector field can not belong to $\hat{\Theta}_{M,S}$. To assure that this does not happen we have to impose some additional conditions on the behavior of F near S.

With this assumption we have:

Proposition 4.31. *Let F be a germ of holomorphic diffeomorphism in* $(\mathbb{C}^2, 0)$ *with $Fix(F) = S$, where S is a germ of irreducible smooth curve. If F is tangential on S at some, hence every, point $p \in S$ then the reduced associated transversely formal vector field $\hat{X}$ is in $\hat{\Theta}_{M,S}$.*

Proof. In Section 4.7 we have seen that the first not zero component of $\hat{X}$ is the one corresponding to the first power of y that appear in the corresponding component of F. By this remark we get the assertion. □

In this situation we can use the previous results in order to get the following:

Theorem 4.32. *Let F be a germ of a holomorphic diffeomorphism of a complex manifold of dimension two. Assume $Fix(F) = S$, where S is a non singular compact connected curve and suppose that F is tangential on S. For every $p \in S$ there exists an index $Ind(F, S, p) \in \mathbb{C}$ such that:*

$$\sum_{p \in S} Ind(F, S, p) = S \cdot S.$$

4.9 Reduction of singularities for transversely formal vector fields

The aim of this section is to observe that Seidenberg's reduction theorem is true even in the transversely formal category. To make this more precise let fix some notations.

Remark 4.33. *The definition of isolated singularities and the ones of this section are given in the formal category but are valid even in the transverse formal one.*

As we have seen in Section 1.3 the reduction of singularity is an algorithmic strategy for decreasing the complexity parameter of the vector field X. By this process we can assumes that singularities are of a special kind:

Definition 4.34. *Let X be a formal vector field with $r(X) = 1$. Let λ_1 and λ_2 be the eigenvalues of the linear part of X. The origin is a **reduced singularity** for X if one of the following conditions holds:*

(∗1) $\lambda_1 \neq 0, \lambda_2 \neq 0$ *and* $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+ \cup \{0\}$

(∗2) $\lambda_1 \neq 0, \lambda_2 = 0$ *or* $\lambda_1 = 0, \lambda_2 \neq 0$.

In formal category a blow-up is simply a formal transformation of type

$$\begin{cases} x = u \\ y = uv. \end{cases}$$

Theorem 4.35 ([51])**.** *Let $X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ be a formal vector field with $(A,B) = 1$ and let the origin be a singularity for X. Then after a finite number of blow-ups we get a vector field with only reduced singularities.*

In case of transversely formal vector fields we observe that the transverse formal structure of X is invariant under blows-up.

Proposition 4.36. *Let $X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ be a transversely formal vector field with $(A,B) = 1$ and let the origin be a singularity for the vector field. Then after a finite number of blow-ups we get a transversely formal equation with only reduced singularities.*

Now we briefly recall what we know about the solutions through non singular points.

Definition 4.37. *Let* $X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ *be a formal vector field. An integral curve for* X *through the origin is a couple of formal series* $x(t), y(t) \in \mathbb{C}[[t]]$ *both not constantly zero and such that:*

$$A(x(t),y(t))(\sum ic_i t^{i-1}) = B(x(t),y(t))(\sum jd_j t^{j-1}),$$

where $x(t) = \sum d_j t^j$ *and* $y(t) = \sum c_i t^i$*.*

Theorem 4.38 ([51])**.** *Let* $X := A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ *be a non singular formal vector field. If the complexity parameter* $r(X) = 0$ *then only one solution passes through the origin. If* $r(X) = 1$ *and the origin is a reduced singularity then at least two solutions pass through the origin.*

4.9.1 Existence of formal solutions

We want to get existence of formal solutions through points of a smooth invariant curve of a transversely formal vector field. The idea is to use Camacho-Sad type index techniques for transversely formal vector fields. We recall the ingredients of the proof of existence of separatrices through singular points used in [18]:

1. an index theorem,
2. a reduction of singularities process,
3. existence of solutions for reduced singularities
4. combinatorics to deduce from the existence of solutions for the reduced equations the existence of solutions for the original equation.

Obviously in our case we need some modifications caused from the existence of an invariant curve through the singularity. As we have just made in the holomorphic category the right modification is given by J. Cano's definition of (C_1) and (C_2) points [20].

Definition 4.39. *Let X be a transversely formal vector field on the separatrix S.*

- *the point $p \in S$ is of type (C_1) if S is not singular at p and*
$$Ind(X, S, p) \notin \mathbb{Q}^+ \cup \{0\}.$$
- *the point $p \in S$ is of type (C_2) if S has two irreducible non singular branches S_0, S_1 that intersect transversally in p and it exists a number $r > 0$ such that:*
$$Ind(X, S_0, p) \in \mathbb{Q}_{\leq -\frac{1}{r}} := \{x \in \mathbb{Q} \mid x \leq -\frac{1}{r}\}$$
$$Ind(X, S_1, p) \notin \mathbb{Q}_{\geq -r} := \{x \in \mathbb{Q} \mid x \geq -r\}.$$

As usual, we introduce even in the transverse formal category the definition of appropriate singularity.

Definition 4.40. *Let X be a transversely formal vector field on the separatrix S. The point $p \in S$ is an* ***appropriate singularity*** *for X if after a finite number of blow-ups there exists a point of type (C_1) or (C_2) on the total transform of S.*

We have just constructed the first three ingredients. For the last one we just observe that the combinatoric part is independent from the convergence of the object under investigation, so it is also valid in our context.

We can then state the following:

Theorem 4.41. *Let*
$$A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$$
be a transversely formal vector field with an invariant smooth curve S and let $p \in S$ be an appropriate singularity of the vector field. Then there exists another formal separatrix through p.

4.10 Existence of parabolic curves

If $F = \exp(X)$ then, as we have seen, the terms of lowest degree of F and X are exactly the same with the the same coefficients. This guarantees that the type of singularity of F and X are the same, i.e., the existence of $(*1)$ and $(*2)$ singularities for one of them implies the existence of such a point for the other.

This property is preserved even under blows-up as the following classical lemma (see Proposition 4.2.4 pg. 267 of [6]) assures:

Lemma 4.42. *Let F be a map tangent to the identity in $(\mathbb{C}^2, 0)$ and set $\tilde{F}$ the blow-up of F. Let X be the vector field associated to F i.e., $\exp(X) = F$. We have that:*

$$\tilde{F} = exp(\pi^* X),$$

where $\pi^* X$ is the pull-back of X by the blow-up map π.

Remark 4.43. *This lemma says that the relationship between maps and vector fields is preserved under blow-up if, instead of saturating the vector field, we divide it only by the first power of a local expression of the exceptional divisor.*

According with the strategy described in the previous section we recover even for maps the existence of parabolic curves:

Theorem 4.44. *Let $F \in Diff(\mathbb{C}^2, 0)$ be a germ of holomorphic map such that $Fix(F) = S$ where S is a non singular complex curve. Let suppose F is tangential on S. If $p \in S$ is an appropriate singularity of F then at least one parabolic curve passes through p.*

Proof. Let consider the vector field X such that $\exp(X) = F$. Then X has a separatrix S passing through an appropriate singularity p. The resolution of singularities and the combinatorics of Cano ([20]) assures the existence of a $(*1)$ point for the reduced vector field $\tilde{X}$. By the previous lemma this implies the existence of a $(*1)$ point for the reduced map $\tilde{F}$. So by Proposition 2.19 we have the existence of at least one parabolic curve for F. $\square$

Chapter 5

Upper-bound for the number of robust parabolic curves of a class of maps tangent to identity

5.1 Introduction

In dimension two Hakim [36] and Abate [2] proved that if f is a holomorphic map tangent to the identity in $\mathbb{C}^2$ and $\nu(f)$ is the degree of the first non vanishing jet of $f - Id$ then there exist $\nu(f) - 1$ robust parabolic curves (RP curves for short). The set of the exponential of holomorphic vector fields (of order greater than or equal to two), $\Phi_{\geq 2}(\mathbb{C}^2, 0)$, is dense in the space of germs of maps tangent to the identity.

In this chapter we give an upper-bound for the number of robust parabolic curves of $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$.

Theorem 5.1. *Let $f = (f_1, f_2) \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a non-dicritical holomorphic map. Set $\eta(f) := \max\{ord(f_1 - Id), ord(f_2 - Id)\}$ and $\mu(f)$ the Milnor* *number of f. Then the number of RP curves is at most*

$$(\mu(f) + 1)(\eta^2(f) - \eta(f)).$$

In the dicritical case in [11] Bracci proved that f is dicritical if and only if it has infinite parabolic curves. Here we show that the parabolic curves at a dicritical point are indeed robust ones:

Proposition 5.2. *Let $f = (f_1, f_2) \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a dicritical holomorphic map. Then there exist infinitely many RP curves.*

In case f is the exponential of a holomorphic vector field we notice a very strict relation between the dynamics of the map and the dynamics of the vector field. So, if the diffeomorphism f is such that there exists a vector field X such that $\exp(X) = f$, then it turns out that the RP curves are "geometrically" determined by the fact they lay in a separatrix of X. We can use a result of Corral and Fernandez Sanchez [24] concerning the upper-bound of the number of separatrices of X to estimate the number of RP curves .

5.2 Robust parabolic curves

In this section we analyze the relationship between the separatrices of the vector field associated to f and the robust parabolic curves (see section 2.2.5).

According to the relation between f and X the geometric meaning of Definition 2.24 is clarified by the following proposition:

Proposition 5.3. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a holomorphic map and let X be a vector field such that $\exp(X) = f$. Let φ be a robust parabolic curve. Then φ is contained in a formal separatrix of X. Conversely in every formal separatrix of X there exists at least one RP curve for f.*

Proof. Let be $p \in \varphi(\Delta)$. Since

$$\exp(X) = f$$

then the orbit $\{f^n(p)\}$ is contained in a separatrix, S, of X. We have only to prove that every orbit generated by a generic point $q \in \varphi(\Delta)$ stays inside S. By contradiction we can find two orbits that converge to zero living in two different separatrices , say S_1 and S_2 Let $l_1(x, y)$ and $l_2(x, y)$ be (respectively) the local expressions of S_1 and S_2 and let h be the order of the first non zero jet of $l_1 - l_2$. If we blow-up the vector field h times then, by property (1) of the definition of RP curves, we have that the two orbits converge to zero with two different directions and this contradicts property (2) of Definition 2.24.

Let prove the converse. Let S be a separatrix and let $y = \varphi(x) = x^{\frac{p}{q}} + \cdots$ be its expression in Puiseux series.

Remark 5.4. *We can suppose p and q co-prime and that $\frac{p}{q} \geq 1$. Indeed, if this is not the case we can choose the parametrization of the separatrix in the form $x = \psi(y)$, which satisfies the required condition.*

Let now make the following change of variables:

$$\begin{cases} u =& x \\ v =& y - \varphi(x) \end{cases}$$

The vector field in the new coordinates is:

$$\begin{cases} \dot{u} =& A(u, v + \varphi(u)) \\ \dot{v} =& B(u, v + \varphi(u)) - \dot{\varphi}(u)A(u, v + \varphi(u)) \end{cases}$$

If we compute the exponential of this new vector field restricted to the separatrix $\{v = 0\}$we find:

$$\exp(A(u, \varphi(u))\frac{\partial}{\partial u})$$

Let us make the change of variables:

$$u = z^q$$

and then the first component of the vector field is:

$$\dot{z} = \frac{A(z^q, \varphi(z^q))}{qz^{q-1}}. \tag{5.1}$$

By Remark 5.4 the left-hand side of (5.1) is expressed as a power series. Now if we take the exponential of this new vector field we find a map tangent to the identity conjugated to the original one given by $(z, w) \mapsto (z + z^h + \cdots, w)$. By the Leau-Fatou Theorem ([21]) we get the assertion. $\square$

As a consequence of this last result we easily prove the existence of RP curves for map in $\Phi_{\geq 2}(\mathbb{C}^2, 0)$ [2],[36], [5].

Proposition 5.5. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a holomorphic map tangent to the identity in $\mathbb{C}^2$. Then there exists at least one RP curve.*

5.3 Local invariants

In this section we briefly recall the local invariants we need in this study. Let $\mathcal{F}$ be a foliation on a complex surface M and let $p \in M$ a point and let $\omega = a(x, y)dx + b(x, y)dy$ be a representant for the foliation $\mathcal{F}$.

Definition 5.6. *The Milnor number of $\mathcal{F}$ in p is:*

$$\mu_p(\mathcal{F}) := dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{x, y\}}{a(x, y)\mathbb{C}\{x, y\} + b(x, y)\mathbb{C}\{x, y\}}\right).$$

Remark 5.7. *Let observe that $\mu_p(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is not singular in p.*

The analogous of the Milnor number in case of maps is the intersection number. To give a rigorous definition of such invariant we need the following result:

Definition 5.8. *Let $X \subset (\mathbb{C}^2, 0)$ be a germ of analytic irreducible curve of equation $f(x, y) = 0$. A parametrization γ is called* primitive *if for every parametrization μ there exists a function u such that $\mu = \gamma \circ u$.*

Remark 5.9. *By Puiseux theorem such a parametrization always exists.*

Definition 5.10. *Let f_1, f_2 be two germs of maps. Suppose f_2 irreducible and let γ be a primitive parametrization of $f_2 = 0$. We define the intersection number of f in 0:*

$$I(f; 0) := \nu_0(f_1 \circ \gamma),$$

where $\nu_0(f_1 \circ \gamma)$ is the multiplicity of $f_1 \circ \gamma$ in zero.

If f_2 is not irreducible and $f_2 = g_1^{\alpha_1} \cdots g_k^{\alpha_k}$ then

$$I(f_1, f_2; 0) := \sum_{i=1}^{k} \alpha_i I(f_1, g_i; 0).$$

Now we can express the relationship between these two invariants.

Proposition 5.11. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a holomorphic map tangent to the identity in $\mathbb{C}^2$ and let X be the associated vector field. Then the Milnor number of X, at the origin, is equal to the intersection multiplicity of $f - Id$.*

Proof. Let observe that, if

$$X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y},$$

then the Milnor number of X is equal to the intersection multiplicity at the origin [9]. Now observe that the intersection multiplicity of two functions

$g(x, y)$ and $h(x, y)$ depends only on the first non zero jet of g and on the lowest exponent of the Puiseux parametrization of h [33]. The Newton-Puiseux polygon shows that the lowest exponent of the parametrization depends only on the part of the polygon determined by the first non zero jet of the function [23]. This concludes the proof because the lowest non zero jets of $f - Id$, A and B are the same. $\square$

5.4 Non-dicritical case

Proposition 5.3 shows that the RP curves live inside the separatrices of the associated vector field. The idea is to estimate the number of separatrices of the vector field and then the number of RP curves inside a separatrix. In [24] Corral and Fernandez Sanchez find the optimal estimates of the number of separatrices by means of the Milnor number of X [9].

Proposition 5.12 ([24])**.** *Let X be a holomorphic vector field in $\mathbb{C}^2$, singular at the origin. Let S be the curve determined by all the separatrices passing trough the origin. Let $r_0(S)$ be the number of the irreducible components of S. Then:*

$$r_0(S) \leq \mu_0(X) + 1, \tag{5.2}$$

where μ_0 is the Milnor number of X at the origin.

The proof of this proposition can be found in [24]. In order to express the previous estimation in terms of invariants of f we introduce the intersection multiplicity [2].

Now we can start with the proof of the estimates of the number of RP curves that are contained in a separatrix. Proceeding as in Proposition 5.3, we can conjugate the restriction of f to the separatrix to a map of the kind $(z, w) \mapsto (z + z^h + \cdots, w)$.

We have now to estimate the exponent h. An easy computation shows that the exponent h is the lowest degree of the expression of $\frac{A(z^q,\varphi(z^q))}{qz^{q-1}}$. The same computation proves that the order of $A(x,y)$ is the same as the order, ν_1 of $f_1 - Id$. Then

$$\frac{A_{\nu_1}(z^q,\varphi(z^q))}{qz^{q-1}} = \sum_{i+j=\nu_1} z^{qi}\varphi(z^q)^j z^{1-q},$$

so the lowest degree is:

$$qi + pj + 1 - j, \tag{5.3}$$

for $0 \le i, j \le \nu_1$. Let us maximize the quantity (5.3). According to the cases $p, q > 0$ and $p, q < 0$ and by the assumption $\frac{p}{q} \ge 1$ we have that:

$$qi + pj + 1 - j \le \nu_1 p + 1 - q \le \nu_1 p,$$

where the last inequality holds because $q \ge 1$. By Remark 5.4 the number of RP curves in S is bounded from above by:

$$\max\{\nu_1, \nu_2\}p.$$

This estimate depends on p and q and then on the particular separatrix. It is possible to improve this result removing the dependence on the separatrix in the following way. Since:

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}$$

and replacing $y = \varphi(x) = x^k + \cdots$, we find that:

$$k = \frac{i_1 - i_1 + 1}{j_2 - j_1 + 1},$$

with $i_1 + j_1 = \nu_1$ and $i_2 + j_2 = \nu_2$. So we easily find:

$$k \le \frac{\nu_2 - 1}{\nu_1 - 1}.$$

Then $p \le (\nu_2 - 1)$ and $q \le (\nu_1 - 1)$ because p and q are prime each other.

This proves Theorem 5.1.

5.5 Dicritical case

We know that in case of dicritical singularities we have, for maps, the existence of infinite parabolic curves and, for vector fields, the existence of infinite separatrices. This suggests the following:

Proposition 5.13. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a map tangent to the identity in $\mathbb{C}^2$ and let X be the vector field such that $exp(X) = f$. Then f is dicritical at 0 if and only if X is dicritical at 0.*

Proof. If X is dicritical then by Proposition 1.31 there exists infinitely many separatrices and then, by Proposition 5.3, f admits infinitely many RP curves. So f, by Theorem 5.1, has to be dicritical. Let us prove the converse. Let X be not dicritical and let prove that f is not dicritical. Let denote by $\tilde{f}$ and $\tilde{X}$ (respectively) the blow-up of the map f and of the field X. We have to prove that if the exceptional divisor D is invariant by $\tilde{X}$ then $\tilde{f}$ is tangential on D. We can suppose that:

$$\tilde{X} = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

and $D = \{l(x,y) = x = 0\}$. The invariance of D is equivalent to the fact that x divides $A(x,y)$. Let be

$$T = \max\{s \in \mathbb{N} \mid x^s \mid A(x,y)\},$$

i.e. $A(x,y) = x^T a(x,y)$. So we have:

$$X = x^T a(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

with $x \nmid B(x,y)$ i.e. $B(0,y) = y^k + \cdots$. Now let be $\bar{X} := \pi^*(X)$ and observe that this field has the following structure:

$$\bar{X} = x^\alpha a(x,y)\frac{\partial}{\partial x} + x^\beta B(x,y)\frac{\partial}{\partial y},$$

where $a(x,y)$ and $B(x,y)$ are the previous ones and $\alpha > \beta$. By Lemma 4.42 we know that $\exp(\bar{X}) = \tilde{f}$ and so we can reconstruct the map by formula (4.5).

We find $\bar{X}^j.x = x^\alpha(\cdots)$ for all j. On the other hand when we compute $\bar{X}^i.y$, we find a structure of the type $\bar{X}^i.y = x^\alpha(\cdots) + x^{i\beta}(\cdots)$. Then the lowest power of x appears in the term $\bar{X}.y$ and it is $x^\beta y^k$. So the order of $\tilde{f}$ on D is $\min(\alpha,\beta) = \beta$ and then

$$\frac{l \circ \tilde{f} - l}{l^T} = \frac{\tilde{f}_1 - x}{x^\beta} = x^{\alpha-\beta}(\cdots) \equiv 0 \mod \mathcal{I}(S)_p.$$

□

In this setting we have

Proposition 5.14. *Let $f \in \Phi_{\geq 2}(\mathbb{C}^2, 0)$ be a dicritical holomorphic map tangent to the identity in $\mathbb{C}^2$. Then there exist infinitely many RP curves.*

Proof. Since f is dicritical then, by Proposition 5.13, X is dicritical. By Proposition1.31 there exist infinite separatrices and then by Proposition 5.3 we get the assertion. □

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