SZEGŐ-TYPE ASYMPTOTICS FOR RAY SEQUENCES OF FROBENIUS-PADÉ APPROXIMANTS

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ABSTRACT. Let $\widehat{\sigma}$ be a Cauchy transform of a possibly complex-valued Borel measure σ and $\{p_n\}$ be a system of orthonormal polynomials with respect to a measure μ, $\mathrm{supp}(\mu)\cap\mathrm{supp}(\sigma)=\varnothing$. An (m,n)-th Frobenius-Padé approximant to $\widehat{\sigma}$ is a rational function P/Q, $\deg(P)\leqslant m$, $\deg(Q)\leqslant n$, such that the first $m+n+1$ Fourier coefficients of the linear form $Q\widehat{\sigma}-P$ vanish when the form is developed into a series with respect to the polynomials p_n. We investigate the convergence of the Frobenius-Padé approximants to $\widehat{\sigma}$ along ray sequences $\frac{n}{n+m+1}\to c>0$, $n-1\leqslant m$, when μ and σ are supported on intervals on the real line and their Radon-Nikodym derivatives with respect to the arcsine distribution of the respective interval are holomorphic functions.

1 INTRODUCTION

Representation of functions by means of *series with respect to the Chebyshev polynomials* is a very convenient tool in numerical analysis (see, for example, [1]). Such a series converges in the interior of the largest ellipse into which the function has holomorphic continuation. However, if we need to compute the function beyond the boundary of the maximal ellipse of convergence of the series with respect to the Chebyshev polynomials (or any other orthonormal polynomial sequences) then one has to employ rational rather then polynomial approximation of the orthogonal polynomial expansion (see [2], [3]). The construction of these rational approximants is related to the notion of the generalized *Padé table* [4]. We call them *Padé approximants of an orthogonal expansion*. In this paper we focus on the *Frobenius-Padé approximants*, which are defined by means of a linear system with constant coefficients which are precisely the coefficients of the polynomial expansion of the approximated function (see (2), below). This type of approximants is the most popular in practice due to the ease of their numerical computation.

Let μ be a possibly complex-valued Borel measure supported on an interval $\Delta_\mu\subset\mathbb{R}$. Assume that μ possesses full *orthonormal* system of polynomials that we denote by $\{p_n\}$, i.e.,

$$\int p_n(x)p_m(x)d\mu(x)=\delta_{nm},$$

where δ_{nm} is the usual Kronecker symbol. When μ is a positive measure such a system always exists. For complex measures *orthogonal polynomials of minimal degree* uniquely exist as well, but it might happen that $\deg(p_n)<n$, in which case p_n is orthogonal to itself and therefore cannot be orthonormalized. Given a function $f\in L^1(\mu)$, we can associate to f a series

$$\sum_{i=0}^{\infty}c_i(f;\mu)p_i(x),\quad c_i(f;\mu):=\int f(x)p_i(x)d\mu(x). \tag{1}$$

Key words and phrases. Frobenius-Padé approximants, linear Padé-Chebyshëv approximants, Padé approximants of orthogonal expansions, orthogonal polynomials, Markov-type functions, Riemann-Hilbert problems.

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Definition. *A Frobenius-Padé approximant of type* $(m,n)\in\mathbb{N}^2$ *to* $f\in L^1(\mu)$ *is a rational function* $P_{m,n}/Q_{m,n}$, $\deg(P_{m,n})\leqslant m$, $\deg(Q_{m,n})\leqslant n$, *such that*

$$c_i(Q_{m,n}f-P_{m,n};\mu)=0,\quad i\in\{0,\dots,m+n\}. \tag{2}$$

Frobenius-Padé approximants always exist as finding $Q_{m,n}$ amounts to solving a linear system

$$\begin{pmatrix} c_{m+1}(p_0f;\mu) & \cdots & c_{m+1}(p_nf;\mu)\\ \vdots & \ddots & \vdots\\ c_{m+n}(p_0f;\mu) & \cdots & c_{m+n}(p_nf;\mu)\end{pmatrix}\begin{pmatrix}a_0\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

and letting $Q_{m,n}(x)=\sum_{j=0}^n a_jp_j(x)$ (the system has n equations and $n+1$ unknowns), while $P_{m,n}(x)=\sum_{j=0}^m b_jp_j(x)$ uniquely depends on $Q_{m,n}$ via

$$\begin{pmatrix} c_0(p_0f;\mu) & \cdots & c_0(p_nf;\mu)\\ \vdots & \ddots & \vdots\\ c_m(p_0f;\mu) & \cdots & c_m(p_nf;\mu)\end{pmatrix}\begin{pmatrix}a_0\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix}b_0\\ \vdots\\ b_m\end{pmatrix}.$$

An approximant may not be unique, however, the one corresponding to $Q_{m,n}$ of the smallest degree is. Hence, if $\deg(Q_{m,n})=n$ for all solutions, the approximant is unique.

The main motivation for using Padé approximants of orthogonal expansions is due to their convergence in wider domains than the convergence domains of orthogonal expansions themselves. The problems of convergence of the rows of corresponding tables of the Padé approximants of orthogonal expansions have been investigated by S.P. Suetin [5], [6]. The weak asymptotics and the convergence of the *diagonal* (i.e. type $(n-1,n)$) Padé approximants of orthogonal expansions for Cauchy transforms

$$\widehat{\sigma}(z):=\int\frac{d\sigma(t)}{t-z},\qquad \sigma'(t)>0,\quad t\in\Delta_\sigma\subset\mathbb{R},\quad \Delta_\mu\cap\Delta_\sigma=\varnothing, \tag{3}$$

have been obtained A.A. Gonchar, E.A. Rakhmanov and S.P. Suetin in [7], [8].

In this paper we investigate the *strong asymptotics* and convergence properties of the *ray sequences* (i.e. type $(m,n):n-1\leqslant m$ and $n/(n+m)\to c>0$) of Frobenius-Padé approximants for Cauchy transforms (3) where σ is, generally speaking, a complex-valued Borel measure. To motivate the forthcoming definitions, let us (following [7], [8]) first heuristically describe the asymptotic behavior of the approximants using the formalism of orthogonal polynomials and potential theory.

For the moment, assume that the measures of μ and σ are positive. In this case the linear form

$$R_{m,n}:=Q_{m,n}\widehat{\sigma}-P_{m,n}$$

is real-valued on Δ_μ and is orthogonal to all polynomials of degree at most $m+n$ with respect to μ by (2). Therefore it must have at least $m+n+1$ zeros there. Denote by $V_{m,n}$ the monic polynomial whose zeros are the zeros of $R_{m,n}$ on Δ_μ, $\deg(V_{m,n})\geqslant m+n+1$. The expression $z^kR_{m,n}(z)/V_{m,n}(z)$, $k\leqslant\min\{n-1,m\}$, is holomorphic off Δ_σ and is vanishing at infinity with order at least 2. Then it follows from Cauchy's theorem, Cauchy's integral formula, and (3) that

$$\int\frac{x^kQ_{m,n}(x)}{V_{m,n}(x)}d\sigma(x)=0,\quad k\leqslant\min\{n-1,m\}. \tag{4}$$

When $n-1\leqslant m$, the number of orthogonality conditions above is equal to n and therefore $Q_{m,n}$ must have degree n since $d\sigma(x)/V_{m,n}(x)$ is a real measure of constant sign on Δ_σ. In particular, this implies uniqueness of $Q_{m,n}$ up to a multiplicative factor. On the other hand, similarly to (4), Cauchy integral formula, (3), and orthogonality of $R_{m,n}$ with respect to μ yield that

$$\int\frac{x^kV_{m,n}(x)}{Q_{m,n}(x)}\left(\int\frac{Q^2_{m,n}(t)}{V_{m,n}(t)}\frac{d\sigma(t)}{t-x}\right)d\mu(x)=0,\quad k\in\{0,\dots,m+n\}. \tag{5}$$

Given mutual orthogonality relations (4) and (5), it is well understood [9, 10, 11] which measures describe the limiting behavior of the zeros of $Q_{m,n}$ and $V_{m,n}$. Assuming that $n-1 \leqslant m$ and $n/(n+m) \to c > 0$, let $\tau_{\mu,c}$, $|\tau_{\mu,c}| = 1$, and $\tau_{\sigma,c}$, $|\tau_{\sigma,c}| = c$, be weak* limit points of the counting measures of the zeros of $V_{m,n}$ and $Q_{m,n}$, respectively, normalized by $n+m$. Then the pair $(\tau_{\mu,c}, \tau_{\sigma,c})$ can be uniquely identified as follows [10, 11].

Proposition 1. *Given $c \in (0, 1/2]$, denote by $\mathcal{M}_c$ the following class of pairs of Borel measures:*

$$\mathcal{M}_c := \big\{(\tau_\mu, \tau_\sigma) : \operatorname{supp}(\tau_\nu) \subseteq \Delta_\nu,\ \nu \in \{\mu, \sigma\},\ |\tau_\mu| = 1,\ |\tau_\sigma| = c\big\}.$$

There exists a pair $(\tau_{\mu,c}, \tau_{\sigma,c}) \in \mathcal{M}_c$ such that[1]

$$\left\{ \begin{array}{ll} 2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} = \min_{\Delta_\sigma}\left(2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}}\right) = 3\ell_{\sigma,c} & \text{on} \quad \operatorname{supp}(\tau_{\sigma,c}), \\ 2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}} = \min_{\Delta_\mu}\left(2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}}\right) = 3\ell_{\mu,c} & \text{on} \quad \operatorname{supp}(\tau_{\mu,c}), \end{array} \right. \tag{6}$$

for some constants $\ell_{\mu,c}$ and $\ell_{\sigma,c}$. Moreover, if for some pair $(\tau_\mu, \tau_\sigma) \in \mathcal{M}_c$ relations analogous to (6) are satisfied, then $(\tau_\mu, \tau_\sigma) = (\tau_{\mu,c}, \tau_{\sigma,c})$. In addition, it holds that

$$\operatorname{supp}(\tau_{\mu,c}) = \Delta_\mu \quad \text{and} \quad \operatorname{supp}(\tau_{\sigma,c}) =: \Delta_{\sigma,c} \quad \text{is an interval.}$$

Furthermore, set

$$\left\{ \begin{array}{lll} D^+_{\sigma,c} & := & \big\{z : V^{\tau_{\mu,c}}(z) - 2V^{\tau_{\sigma,c}}(z) + 3\ell_{\sigma,c} > 0\big\}, \\ D^-_{\sigma,c} & := & \big\{z : V^{\tau_{\mu,c}}(z) - 2V^{\tau_{\sigma,c}}(z) + 3\ell_{\sigma,c} < 0\big\}. \end{array} \right. \tag{7}$$

Then $D^+_{\sigma,c} \neq \varnothing$ and $\Delta_{\sigma,c} \subseteq \partial D^+_{\sigma,c}$, $D^-_{\sigma,c_2} \subseteq D^-_{\sigma,c_1}$ when $c_1 \leqslant c_2$, $D^-_{\sigma,c} = \varnothing$ when $c = \frac{1}{2}$ and $\infty \in D^-_{\sigma,c} \neq \varnothing$ otherwise, and $(\Delta_\sigma \setminus \Delta_{\sigma,c}) \subset D^-_{\sigma,c}$, see Figure 1.

The domains $D^+_{\sigma,c}$ and $D^-_{\sigma,c}$ are significant for our analysis as we shall prove that the approximants do converge to $\widehat{\sigma}$ in $D^+_{\sigma,c}$ and diverge to infinity in $D^-_{\sigma,c}$.

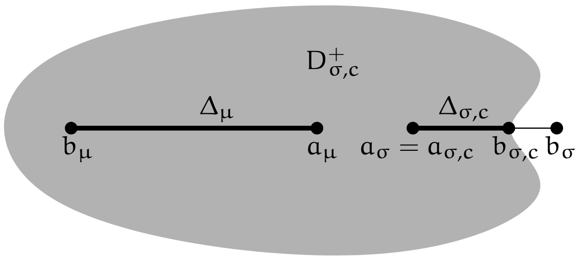

FIGURE 1. Intervals $\Delta_\mu = [b_\mu, a_\mu]$, $\Delta_\sigma = [a_\sigma, b_\sigma]$, and $\Delta_{\sigma,c} = [a_{\sigma,c}, b_{\sigma,c}]$ and the domain $D^+_{\sigma,c}$ (shaded region).

As shown in [12], one can describe the weak asymptotics of the polynomials $Q_{m,n}$ and $V_{m,n}$ using the logarithmic potentials of the measures $\tau_{\sigma,c}$ and $\tau_{\mu,c}$. As we aim at strong (Szegő) asymptotics we shall omit such a description, which was addressed in [8] for the diagonal case $m = n-1$. Let us point out that relations (6) are stated differently in [8]. There, see [8, Equation (2.1)], it shown that there exists a unique probability measure λ, $\operatorname{supp}(\lambda) = \Delta_\mu$, and a constant w such that

$$G^\lambda - 3V^\lambda = w \quad \text{on} \quad \Delta_\mu,$$

where G^λ is the Green potential of λ relative to $\overline{\mathbb{C}} \setminus \Delta_\sigma$. To rewrite the above relation as system (6), recall that $G^\lambda = 0$ on Δ_σ and that $G^\lambda = V^{\lambda - \hat\lambda} - \hat w$ in $\mathbb{C}$ where $\hat w$ is some constant and $\hat\lambda$ is the *balayage* of λ onto Δ_σ, see [25, Theorem II.5.1]. Therefore,

$$\left\{ \begin{array}{lll} 2V^{\hat\lambda/2} - V^\lambda = \hat w & \text{on} & \Delta_\sigma, \\ 2V^\lambda - V^{\hat\lambda/2} = (w + \hat w)/2 & \text{on} & \Delta_\mu. \end{array} \right.$$

The last equations clearly show that $\tau_{\mu,1/2} = \lambda$ and $\tau_{\sigma,1/2} = \hat\lambda/2$.

[1] In what follows, $V^\nu(z) = -\int \log|z-w|d\nu(w)$ is the logarithmic potential of the measure ν.

2 MAIN RESULTS

After the work of J. Nuttall [13], it is well understood that in order to identify strong limits of orthogonal polynomials one needs to replace the potential-theoretic extremal problem with a boundary value problem on a certain Riemann surface. To this end, let $c \in (0, 1/2]$ and $\Delta_{\sigma,c}$ be as in Proposition 1. We define the Riemann surface corresponding to c, say $\mathfrak{R}_c$, through its realization in the following way. Take 3 copies of $\overline{\mathbb{C}}$. Cut one of them along the interval $\Delta_{\sigma,c}$, which henceforth is denoted by $\mathfrak{R}_c^{(0)}$, cut the second one, $\mathfrak{R}_c^{(1)}$, along $\Delta_\mu \cup \Delta_{\sigma,c}$, and the last one, $\mathfrak{R}_c^{(2)}$, along Δ_μ. To finish the construction, glue the banks of the corresponding cuts crosswise, see Figure 2.

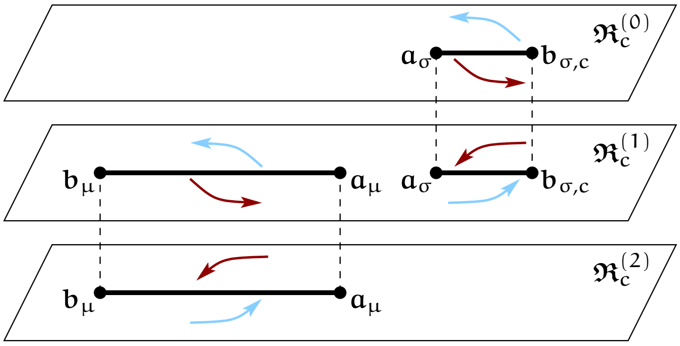

FIGURE 2. Riemann surface $\mathfrak{R}_c$ and its branch points $a_\mu, b_\mu, a_{\sigma,c}, b_{\sigma,c}$.

We denote by π the natural projection from $\mathfrak{R}_c$ to $\overline{\mathbb{C}}$. We shall employ the notation $\boldsymbol{z}$ for a generic point of $\mathfrak{R}_c$ and use the convention $\pi(\boldsymbol{z}) = z$. If we want to specify the sheet of the surface, we write $z^{(i)}$ for a point on $\mathfrak{R}_c^{(i)}$ with $\pi(z^{(i)}) = z$. This notation is well defined everywhere outside of the cycles $\boldsymbol{\Delta}_\mu := \pi^{-1}(\Delta_\mu)$ and $\boldsymbol{\Delta}_{\sigma,c} := \pi^{-1}(\Delta_{\sigma,c})$. Given a function $F(\boldsymbol{z})$ defined on a subset of $\mathfrak{R}_c$, we set $F^{(i)}(z) := F(z^{(i)})$ to be the pull-back from the i-th sheet.

Among all such surfaces, the ones with $c = \frac{n}{n+m}$ are especially important to us. We shall denote them by $\mathfrak{R}_{m,n}$. Observe that any $\mathfrak{R}_c$ has genus 0. Thus, one can arbitrarily prescribe zero/pole multisets of rational functions on them as long as the multisets have the same cardinality. In what follows, we denote by $\Phi_{m,n}$ the rational function on $\mathfrak{R}_{m,n}$ with the divisor[2] $(n+m)\infty^{(2)} - n\infty^{(0)} - m\infty^{(1)}$ and the normalization

$$\Phi_{m,n}^{(0)}(z)\Phi_{m,n}^{(1)}(z)\Phi_{m,n}^{(2)}(z) \equiv 1. \tag{8}$$

Such a normalization is indeed possible since the function $\log\prod_{k=0}^{2}|\Phi_{m,n}^{(k)}|$ extends to a harmonic function on $\mathbb{C}$ which has a well defined limit at infinity. Hence, it is a constant. Therefore, if (8) holds at one point, it holds throughout $\overline{\mathbb{C}}$. It is a simple argument using Schwarz reflection principle, equilibrium relations (6), and the fact that only bounded harmonic function on $\mathfrak{R}_c$ are constants to show that

$$\frac{1}{n+m}\log|\Phi_{m,n}(\boldsymbol{z})| = \begin{cases} V^{-\tau_{\sigma,c}}(z) + \ell_{\mu,c} + 2\ell_{\sigma,c}, & \boldsymbol{z} \in \mathfrak{R}_{m,n}^{(0)}, \\ V^{\tau_{\sigma,c}-\tau_{\mu,c}}(z) + \ell_{\mu,c} - \ell_{\sigma,c}, & \boldsymbol{z} \in \mathfrak{R}_{m,n}^{(1)}, \\ V^{\tau_{\mu,c}}(z) - 2\ell_{\mu,c} - \ell_{\sigma,c}, & \boldsymbol{z} \in \mathfrak{R}_{m,n}^{(2)}, \end{cases} \tag{9}$$

[2] The divisor is a formal expression that describes all the zeros (preceded by positive integer indicated multiplicity) and poles (preceded by negative integer also indicating multiplicity) of the function.

where $c = \frac{n}{n+m}$. Representation (9) is not the only way to understand functions $\Phi_{m,n}$. Define

$$h_{m,n}(z) := \begin{cases} \displaystyle\int \frac{d\tau_{\sigma,c}(x)}{z-x}, & z \in \mathfrak{R}^{(0)}_{m,n}, \\ \displaystyle\int \frac{d(\tau_{\mu,c}-\tau_{\sigma,c})(x)}{z-x}, & z \in \mathfrak{R}^{(1)}_{m,n}, \\ \displaystyle -\int \frac{d\tau_{\mu,c}(x)}{z-x}, & z \in \mathfrak{R}^{(2)}_{m,n}, \end{cases} \tag{10}$$

where, again, $c = \frac{n}{n+m}$. One can readily observe that

$$h_{m,n}(z) = 2\partial_z \left(\frac{1}{n+m} \log|\Phi_{m,n}(\boldsymbol{z})| \right)$$

by (9) and (10), where $2\partial_z := \partial_x - i\partial_y$. As ∂_z-derivative of a harmonic function is holomorphic, $h_{m,n}$ is a rational function on $\mathfrak{R}_{m,n}$. It also follows from the above relation, that $h_{m,n}$ is the logarithmic derivative of $\Phi_{m,n}$ and therefore

$$\Phi_{m,n}(\boldsymbol{z}) = \exp\left\{ (n+m) \int^{\boldsymbol{z}} h_{m,n}(\boldsymbol{x}) dx \right\}, \tag{11}$$

where the initial bound for integration is chosen so (8) holds. Moreover, we also can describe the divisor of $h_{m,n}$.

Proposition 2. *Given $n \leqslant m$, let $c = \frac{n}{n+m}$ and $h_{m,n}$ be defined by (10). Denote the endpoints of Δ_ν by a_ν and b_ν, $\nu \in \{\mu, \sigma\}$, and arrange them so that*

$$\text{either} \quad b_\mu < a_\mu < a_\sigma < b_\sigma \quad \text{or} \quad b_\sigma < a_\sigma < a_\mu < b_\mu.$$

If, using the same convention, we denote the endpoints of $\Delta_{\sigma,c}$ by $a_{\sigma,c}$ and $b_{\sigma,c}$, then $a_{\sigma,c} = a_\sigma$. Moreover, the divisor of $h_{m,n}$ is given by

$$\infty^{(0)} + \infty^{(1)} + \infty^{(2)} + \boldsymbol{z}_{m,n} - \boldsymbol{a}_\mu - \boldsymbol{b}_\mu - \boldsymbol{a}_\sigma - \boldsymbol{b}_{\sigma,c}, \tag{12}$$

where $\boldsymbol{a}_\mu$, $\boldsymbol{b}_\mu$, $\boldsymbol{a}_\sigma$, $\boldsymbol{b}_{\sigma,c}$ are the branch points of $\mathfrak{R}_{m,n}$ with the corresponding projections a_μ, b_μ, a_σ, $b_{\sigma,c}$, and $\boldsymbol{z}_{m,n} \in \mathfrak{R}^{(1)}_{m,n}$ with

$$\pi(\boldsymbol{z}_{m,n}) \in \begin{cases} \left[b_{\sigma,c}, \infty\right) & \text{if} \quad a_\sigma < b_\sigma, \\ \left(-\infty, b_{\sigma,c}\right] & \text{if} \quad b_\sigma < a_\sigma. \end{cases}$$

Furthermore, $\boldsymbol{z}_{m,n} = \boldsymbol{b}_{\sigma,c}$ if and only if $b_{\sigma,c} \in \partial D^-_{\sigma,c}$, see (7) and Figure 1, that is, if and only if the domain $D^-_{\sigma,c}$ touches the interval $\Delta_{\sigma,c}$ (observe also that $b_{\sigma,c} = b_\sigma$ if $b_{\sigma,c} \notin \partial D^-_{\sigma,c}$ since $\Delta_\sigma \setminus \Delta_{\sigma,c} \subset D^-_{\sigma,c}$ by Proposition 1).

We prove Proposition 2 in Section 3.1. We clearly see from Proposition 2 that the function $h_{m,n}$ is algebraic. More precisely, Proposition 2 yields the following.

Corollary 3. *If $b_{\sigma,c} \notin \partial D^-_{\sigma,c}$, in which case $b_{\sigma,c} = b_\sigma$, then $h_{m,n}$ is the solution of the algebraic equation*

$$h^3 - (1-\varkappa)\frac{P_2(z)}{\Pi(z)} h - \varkappa \frac{P_1(z)}{\Pi(z)} = 0, \quad \varkappa = c - c^2, \tag{13}$$

where $c = \frac{n}{n+m}$, $\Pi(z) = (z-a_\mu)(z-b_\mu)(z-a_\sigma)(z-b_\sigma)$, and the polynomials P_j are monic and of degree j, $j = 1, 2$. The three zeros of the polynomials P_1 and P_2 are determined by the three conditions that the discriminant of (13), i.e.,

$$\frac{1}{\Pi^3(z)} \left[\left(\frac{1-\varkappa}{3} P_2(z) \right)^3 - \left(\frac{\varkappa}{2} P_1(z) \right)^2 \Pi(z) \right],$$

has zeros of even multiplicity only and that the Riemann surface of the solution of (13) must be as on Figure 2.

If $b_{\sigma,c} \in \partial D^-_{\sigma,c}$, in which case $\mathbf{z}_{m,n} = \mathbf{b}_{\sigma,c}$, then $h_{m,n}$ is the solution of the algebraic equation

$$h^3 - (1-\varkappa)\frac{\widetilde{P}_1(z)}{\Pi(z)}h - \frac{\varkappa}{\Pi(z)} = 0, \tag{14}$$

where this time $\Pi(z) = (z-a_\mu)(z-b_\mu)(z-a_\sigma)$ and the only zero of the monic polynomial $\widetilde{P}_1$ is such that the discriminant of (14), whose numerator is a cubic polynomial, vanishes at $b_{\sigma,c}$ and has a double zero, and the Riemann surface of (14) must be as on Figure 2.

If we take $a_\mu = a_\sigma =: a$ in (14), then $\widetilde{P}_1(z) = z - a$ and (14) becomes

$$h^3 - \frac{(1-\varkappa)}{(z-b_\mu)(z-a)}h - \frac{\varkappa}{(z-b_\mu)(z-a)^2} = 0. \tag{15}$$

The only zero of the discriminant of (15) is exactly $b_{\sigma,c}$ and is equal to

$$b_{\sigma,c} = \frac{\left(\frac{1-\varkappa}{3}\right)^3 a - \left(\frac{\varkappa}{2}\right)^2 b_\mu}{\left(\frac{1-\varkappa}{3}\right)^3 - \left(\frac{\varkappa}{2}\right)^2}. \tag{16}$$

Explicit expression (15) allows us to numerically compute the boundary $\partial D^-_{\sigma,c}$, which is the trajectory $\Re\left[(h^{(0)}_{m,n}(z) - h^{(1)}_{m,n}(z))dz\right] = 0$ emanating from $b_{\sigma,c}$, see Figure 3.

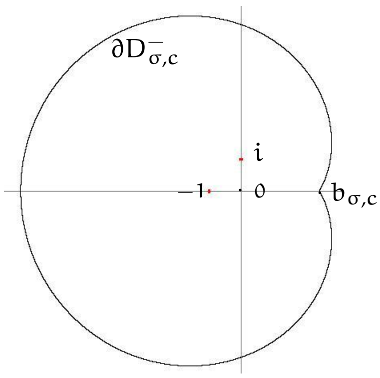

FIGURE 3. The curve $\partial D^-_{\sigma,c}$ numerically computed for parameters $b_\mu = -1$, $a_\mu = a_\sigma = a = 0$, and $c = 1/3$ (in this case $b_{\sigma,c} = 2.43$).

Let us now specify which measures μ and σ we consider. We shall assume that

$$d\nu(x) = \frac{\rho_\nu(x)}{2\pi i}\frac{dx}{w^+_\nu(x)}, \quad \nu \in \{\mu, \sigma\}, \tag{17}$$

where ρ_ν is a non-vanishing and holomorphic function in some neighborhood of Δ_ν and

$$w_\nu(z) := \sqrt{(z-a_\nu)(z-b_\nu)}$$

is the branch holomorphic in $\mathbb{C} \setminus \Delta_\mu$ and normalized so that $w_\nu(z)/z \to 1$ as $z \to \infty$. We define $w_{\sigma,c}(z)$ analogously.

As expected from the classical theory of orthogonal polynomials, we need to introduce an appropriate analog of the Szegő function for the measures μ and σ. This is precisely the content of Proposition 4 below. Its statement is a direct application of [14, Proposition 4] with $\rho_1 = w^+_\sigma/(\rho_\sigma(w^+_{\sigma,c})^2)$ and $\rho_2 = \rho_\mu/(w_{\sigma,c}w^+_\mu)$ (one needs to notice that the labeling of the sheets $\mathfrak{R}^{(0)}$ and $\mathfrak{R}^{(1)}$ is reversed there and the restriction $\alpha_{ij} > -1$ in [14, Eq. (23)] is needed to make functions ρ_i integrable and is not important for [14, Proposition 4] itself).

Proposition 4. *There exists a holomorphic and non-vanishing function on $\mathfrak{R}_c \setminus (\boldsymbol{\Delta}_\mu \cup \boldsymbol{\Delta}_{\sigma,c})$, say S_c, that has continuous traces on $\boldsymbol{\Delta}_\mu \cup \boldsymbol{\Delta}_{\sigma,c} \setminus \{\mathbf{a}_\mu, \mathbf{b}_\mu, \mathbf{a}_\sigma, \mathbf{b}_{\sigma,c}\}$, satisfies*

$$S^{(1)\pm}_c(x) = \begin{cases} S^{(0)\mp}_c(x)(\rho_\sigma w^+_{\sigma,c}/w^+_\sigma)(x), & x \in \Delta^\circ_{\sigma,c}, \\ S^{(2)\mp}_c(x)(w_{\sigma,c}/\rho_\mu)(x), & x \in \Delta^\circ_\mu, \end{cases} \tag{18}$$

where Δ° is the interior of the closed interval Δ, is bounded around $\mathbf{a}_\mu, \mathbf{b}_\mu, \mathbf{a}_\sigma$ as well as $\mathbf{b}_{\sigma,c}$ when $b_{\sigma,c} = b_\sigma$, and behaves like $\left|S_c^{(1)}(z)\right| \sim \left|S_c^{(0)}(z)\right|^{-1} \sim |z - b_{\sigma,c}|^{1/4}$ as $z \to b_{\sigma,c} \neq b_\sigma$. Moreover, it holds that $S_c^{(0)}S_c^{(1)}S_c^{(2)} \equiv 1$.

Now we are ready to state our main result.

Theorem 5. *Let μ and σ be of the form (17) and assume that μ possesses the full system of orthonormal polynomials. Assume further that $n-1 \leqslant m$ and $\frac{n}{n+m} \to c > 0$ as $n \to \infty$. Then for all n large, (m,n)-th Frobenius-Padé approximant $P_{m,n}/Q_{m,n}$ is unique and $\deg(Q_{m,n}) = n$. Moreover, if $K \subset \overline{\mathbb{C}} \setminus \Delta_\sigma$ is closed, then*

$$\left\{\begin{array}{lll} Q_{m,n} = [1+o(1)]\Phi_{m+1,n}^{(0)}S_c^{(0)} & \text{on} & K, \\ Q_{m,n} = [1+o(1)]\Phi_{m+1,n}^{(0)+}S_c^{(0)+} + [1+o(1)]\Phi_{m+1,n}^{(0)-}S_c^{(0)-} & \text{on} & \Delta_{\sigma,c}^\circ, \end{array}\right. \tag{19}$$

where $o(1)$ is uniform on K and locally uniform on $\Delta_{\sigma,c}^\circ$; however, if $\Delta_{\sigma,c} \cap \partial D_{\sigma,c}^- = \varnothing$, then $\Delta_{\sigma,c} = \Delta_\sigma$ and $o(1) = \mathcal{O}(C_{\mu,\sigma}^{-n})$ for some constant $C_{\mu,\sigma} > 1$ with the second equality holding uniformly on Δ_σ. Furthermore, if $K \subset \overline{\mathbb{C}} \setminus (\Delta_\sigma \cup \Delta_\mu)$ is closed, then

$$\left\{\begin{array}{lll} w_{\sigma,c}R_{m,n} = [1+o(1)]\Phi_{m+1,n}^{(1)}S_c^{(1)}, & \text{on} & K, \\ w_{\sigma,c}^{\pm}R_{m,n}^{\pm} = [1+o(1)]\Phi_{m+1,n}^{(1)\pm}S_c^{(1)\pm}, & \text{on} & \Delta_{\sigma,c}^\circ, \\ w_{\sigma,c}R_{m,n} = [1+o(1)]\Phi_{m+1,n}^{(1)+}S_c^{(1)+} + [1+o(1)]\Phi_{m+1,n}^{(1)-}S_c^{(1)-}, & \text{on} & \Delta_\mu^\circ, \end{array}\right. \tag{20}$$

where $R_{m,n} = Q_{m,n}\widehat{\sigma} - P_{m,n}$ and $o(1)$ has the same properties as in (19).

Remark. Polynomial $Q_{m,n}$ is defined up to a multiplicative constant. However, choosing $Q_{m,n}$ uniquely determines $P_{m,n}$, and respectively $R_{m,n}$. Polynomials $Q_{m,n}$ in (19) are normalized so that the leading coefficient is equal to the coefficient of $\Phi_{m+1,n}^{(0)}S_c^{(0)}$ next to z^n when the latter function is developed into a power series at infinity.

Remark. The proof of Theorem 5 follows the framework of Riemann-Hilbert analysis for orthogonal polynomials formulated by Fokas, Its, and Kitaev [15, 16], in which Frobenius-Padé approximants are characterized via a certain Riemann-Hilbert problem whose solution is obtained using a variation of Deift and Zhou steepest descent method [17]. In this realm of ideas it is well understood that one can introduce Fisher-Hartwig singularities into (17). That is, (17) can be replaced by

$$d\nu(x) = \rho_\nu(x)\prod_{i=0}^{I_\nu}|x - x_{i,\nu}|^{\alpha_{i,\nu}}\prod_{i=1}^{I_\nu}\left\{\begin{array}{ll} 1, & x < x_{i,\nu} \\ \beta_{i,\nu}, & x > x_{i,\nu} \end{array}\right\}dx,$$

where ρ_ν is as before, $a_\nu = x_{0,\nu} < x_{1,\nu} < \cdots < x_{I_\nu - 1,\nu} < x_{I_\nu,\nu} = b_\nu$, $\alpha_{i,\nu} > -1$, and $\beta_{i,\nu} \notin (-\infty, 0]$, [18, 19, 20, 21, 14]. Implementing such a modification is rather lengthy as details are very technical and does not provide any additional insight on the behavior of the approximants. Thus, we opted to consider only the measures of the form (17).

Remark. As was noticed in [22], in the case of positive measures μ and σ in (1), (3), the statement of Theorem 5 for the diagonal sequence $(n-1,n)$ follows from the theorem on strong asymptotics of multiple orthogonal polynomials from [23].

Remark. By the definition of the linear forms $R_{m,n}$, (19)–(20), and (9), it holds that the error of approximation by Frobenius-Padé approximants behaves like

$$\left|\widehat{\sigma} - P_{m,n}/Q_{m,n}\right| \sim \left|\Phi_{m+1,n}^{(1)}/\Phi_{m+1,n}^{(0)}\right| = \exp\left\{-(n+m+1)\left(V^{\tau_{\mu,c} - 2\tau_{\sigma,c}} + 3\ell_{\sigma,c}\right)\right\}.$$

Hence, the approximants converge to $\widehat{\sigma}$ uniformly on compact subsets of $D_{\sigma,c}^+$ and diverge uniformly on compact subsets of $D_{\sigma,c}^-$. It also follows from Proposition 1, that they converge locally uniformly in $\mathbb{C} \setminus \Delta_\sigma$ only if $m = n + o(n)$.

3 PROOFS

3.1 *Functions* $h_{m,n}$ *and* $\Phi_{m,n}$

For the proof of Proposition 2, put $c = \frac{n}{n+m}$. To find the divisor of $h_{m,n}$, observe that $h_{m,n}$ is holomorphic everywhere outside of the four branch points of $\mathfrak{R}_{m,n}$ and at each of these points it can have at most a simple pole since $\Phi_{m,n}$ is bounded there. Clearly, $h_{m,n}$ has three simple zeros, one at each $\infty^{(k)}$, $k \in \{0,1,2\}$. If $\boldsymbol{b}_{\sigma,c}$ is not a pole, then the remaining three brach points of $\mathfrak{R}_{m,n}$ must be poles (the number of poles must be equal to the number of zeros) and there cannot be any more poles and/or zeros. In this case we put $\boldsymbol{z}_{m,n} = \boldsymbol{b}_{\sigma,c}$, which verifies (12). If $\boldsymbol{b}_{\sigma,c}$ is a pole of $h_{m,n}$, then

$$h^{(1)}_{m,n}(x) \to -(-1)^\iota\infty \quad \text{as} \quad x \to b_{\sigma,c},\ \mathbb{R} \ni x \notin \Delta_{\sigma,c}, \quad \iota := \begin{cases} 0, & a_\sigma < b_\sigma, \\ 1, & b_\sigma < a_\sigma, \end{cases} \tag{21}$$

by the very definition of $h^{(1)}_{m,n}$ in (10) including the positivity of $\tau_{\sigma,c}$ and our labeling convention for the endpoints of $\Delta_{\sigma,c}$. On the other hand, it holds that

$$h^{(1)}_{m,n}(z) = \frac{|\tau_{\mu,c}| - |\tau_{\sigma,c}|}{z} + \mathcal{O}\left(z^{-2}\right) \quad \text{as} \quad z \to \infty,$$

again, by the very definition of $h_{m,n}$. As $|\tau_{\mu,c}| - |\tau_{\sigma,c}| = 1 - c > 0$, we have that

$$(-1)^\iota h^{(1)}_{m,n}(x) > 0 \quad \text{as} \quad \mathbb{R} \ni x \to (-1)^\iota\infty. \tag{22}$$

Therefore there indeed exists $z_{m,n}$ between $b_{\sigma,c}$ and $(-1)^\iota\infty$ such that $h^{(1)}_{m,n}(z_{m,n}) = 0$. Since $h_{m,n}$ has three more zeros, the rest of the branch points must be poles as claimed.

Assume that $\boldsymbol{b}_{\sigma,c}$ is a pole of $h_{m,n}$ (equivalently $\boldsymbol{b}_{\sigma,c} \neq \boldsymbol{z}_{m,n}$). Since $\mathfrak{R}_{m,n}$ has square root branching at $\boldsymbol{b}_{\sigma,c}$, it follows from (21) and the fact that the sum $h^{(0)}_{m,n} + h^{(1)}_{m,n}$ is holomorphic around $b_{\sigma,c}$ that

$$\begin{cases} h^{(1)}_{m,n}(z) = d_{m,n}\big((-1)^\iota(z - b_{\sigma,c})\big)^{-1/2} + \mathcal{O}(1), \\ h^{(0)}_{m,n}(z) = -d_{m,n}\big((-1)^\iota(z - b_{\sigma,c})\big)^{-1/2} + \mathcal{O}(1), \end{cases} \quad \text{as} \quad z \to b_{\sigma,c},\ z \notin \Delta_{\sigma,c},$$

where $(-1)^\iota d_{m,n} < 0$, and the square root is principal. It further follows from the above asymptotics as well as from (9) and (11) that

$$V^{\tau_{\mu,c} - 2\tau_{\sigma,c}}(z) + 3\ell_{\sigma,c} = \mathrm{Re}\left(\int_{b_{\sigma,c}}^{z} \left(h^{(0)}_{m,n} - h^{(1)}_{m,n}\right)(y)dy\right) > 0$$

as $z \to b_{\sigma,c}$, $z \notin \Delta_{\sigma,c}$. Hence, $b_{\sigma,c} \notin \partial D^-_{\sigma,c}$. On the other hand, if $\boldsymbol{b}_{\sigma,c}$ is not a pole of $h_{m,n}$ (equivalently $\boldsymbol{b}_{\sigma,c} = \boldsymbol{z}_{m,n}$), then

$$\begin{cases} h^{(0)}_{m,n}(z) = h_{m,n}(\boldsymbol{b}_{\sigma,c}) + e_{m,n}\big((-1)^\iota(z - b_{\sigma,c})\big)^{1/2} + \mathcal{O}\big(|z - b_{\sigma,c}|\big), \\ h^{(1)}_{m,n}(z) = h_{m,n}(\boldsymbol{b}_{\sigma,c}) - e_{m,n}\big((-1)^\iota(z - b_{\sigma,c})\big)^{1/2} + \mathcal{O}\big(|z - b_{\sigma,c}|\big), \end{cases}$$

as $z \to b_{\sigma,c}$, $z \notin \Delta_{\sigma,c}$, since $h^{(0)}_{m,n} + h^{(1)}_{m,n}$ is holomorphic around $b_{\sigma,c}$. Moreover, as $h^{(0)}_{m,n}$ satisfies (22) and is monotone between $b_{\sigma,c}$ and $(-1)^\iota\infty$, it holds that $(-1)^\iota e_{m,n} < 0$. Therefore,

$$V^{\tau_{\mu,c} - 2\tau_{\sigma,c}}(x) + 3\ell_{\sigma,c} = \int_{b_{\sigma,c}}^{x} \left(h^{(0)}_{m,n} - h^{(1)}_{m,n}\right)(y)dy < 0$$

for $x \to b_{\sigma,c}$, $\mathbb{R} \ni x \notin \Delta_{\sigma,c}$. In particular, $b_{\sigma,c} \in \partial D^-_{\sigma,c}$. This finishes the proof of the last claim of the proposition. Finally, similar analysis can be used to show that $a_{\sigma,c} \notin \partial D^-_{\sigma,c}$, which, as noted at the end of Proposition 2, implies the equality $a_{\sigma,c} = a_\sigma$.

For the future use let us record several facts. Firstly, it holds that

$$\left|\Phi^{(2)}_{m,n}/\Phi^{(1)}_{m,n}\right| < 1 \quad \text{in} \quad \overline{\mathbb{C}} \setminus \Delta_\mu. \tag{23}$$

Indeed, (23) is equivalent to $V^{\tau_{\sigma,c}} - 2V^{\tau_{\mu,c}} + 3\ell_{\mu,\sigma} > 0$ by (9). The left-hand side of this inequality is superharmonic in $\mathbb{C} \setminus \Delta_\mu$, is identically zero on Δ_μ by (6), and approaches $+\infty$ as $z \to \infty$ since $|\tau_{\sigma,c}| = c < 2 = 2|\tau_{\mu,c}|$. The desired inequality now follows from the minimum principle for superharmonic functions [24, Theorem 2.3.1].

Secondly, let $\{c_n\}$ be a sequence such that $c_n \to c > 0$ as $n \to \infty$, $c_n \leqslant 1/2$. Then

$$\tau_{\mu,c_n} \xrightarrow{*} \tau_{\mu,c} \quad \text{and} \quad \tau_{\sigma,c_n} \xrightarrow{*} \tau_{\sigma,c}, \tag{24}$$

where $\xrightarrow{*}$ stands for the weak* convergence of measures. Indeed, besides (6), the pair $(\tau_{\mu,c}, \tau_{\sigma,c})$ is characterized as the unique minimizers in $\mathcal{M}_c$ of the energy functional

$$J(\tau_\mu, \tau_\sigma) := I(\tau_\mu, \tau_\mu) + I(\tau_\sigma, \tau_\sigma) - I(\tau_\mu, \tau_\sigma),$$

where $I(\nu, \lambda) := -\int \log|z - w| d\nu(z) d\lambda(w)$, see [9, 10]. Let τ_μ and τ_σ be weak* limit points of $\{\tau_{\mu,c_n}\}$ and $\{\tau_{\sigma,c_n}\}$, respectively. Clearly, $(\tau_\mu, \tau_\sigma) \in \mathcal{M}_c$. Then

$$J(\tau_{\mu,c}, \tau_{\sigma,c}) = \lim_{n\to\infty} J\left(\tau_{\mu,c}, \frac{c_n}{c}\tau_{\sigma,c}\right) \geqslant \liminf_{n\to\infty} J(\tau_{\mu,c_n}, \tau_{\sigma,c_n}) \geqslant J(\tau_\mu, \tau_\sigma),$$

where the first inequality follows from the fact that $(\tau_{\mu,c_n}, \tau_{\sigma,c_n})$ is the minimizer of the J-functional in $\mathcal{M}_{c_n}$ and the second inequality is the consequence of the principle of descent [25, Theorem I.6.8], i.e, $\liminf I(\tau_{\nu,c_n}, \tau_{\nu,c_n}) \geqslant I(\tau_\nu, \tau_\nu)$, and the fact that $\Delta_\mu \cap \Delta_\sigma = \varnothing$ (in this case the kernel $\log|z - w|$ is continuous on $\Delta_\mu \times \Delta_\sigma$ and therefore $I(\tau_{\mu,c_n}, \tau_{\sigma,c_n}) \to I(\tau_\mu, \tau_\sigma)$ by weak* convergence of measures). As $(\tau_{\mu,c}, \tau_{\sigma,c})$ is the unique minimizer of the J-functional in $\mathcal{M}_c$, (24) follows.

Finally, let us point out that in the above setting $b_{\sigma,c_n} \to b_{\sigma,c}$, and

$$V^{\tau_{\nu,c_n}} \to V^{\tau_{\nu,c}} \quad \text{locally uniformly in} \quad \overline{\mathbb{C}} \setminus \Delta_{\nu,c}, \quad \nu \in \{\mu, \sigma\}, \tag{25}$$

as $n \to \infty$, which is an immediate consequence of (24).

3.2 *Riemann-Hilbert Problem for Frobenius-Padé Approximants*

Given such a pair of integers (m, n), $n - 1 \leqslant m$, we are interested in finding a 3×3 matrix-valued function $\mathbf{Y}$ that solves the following Riemann-Hilbert Problem (RHP-$\mathbf{Y}$):

(a) $\mathbf{Y}$ is analytic in $\mathbb{C} \setminus (\Delta_\mu \cup \Delta_\sigma)$ and

$$\lim_{z\to\infty} \mathbf{Y}(z) \,\mathsf{diag}\left(z^{-n}, z^{-m-1}, z^{n+m+1}\right) = \mathbf{I},$$

where $\mathsf{diag}(\cdot,\cdot,\cdot)$ is the diagonal matrix and $\mathbf{I}$ is the identity matrix;

(b) $\mathbf{Y}$ has continuous traces on $\Delta_\mu^\circ \cup \Delta_\sigma^\circ$ that satisfy

$$\mathbf{Y}_+ = \mathbf{Y}_- \mathsf{T}_\nu \begin{pmatrix} 1 & \rho_\nu/w_\nu \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \Delta_\nu^\circ, \quad \nu \in \{\mu, \sigma\},$$

where transformations T_μ and T_σ act on 2×2 matrices in the following fashion:

$$\mathsf{T}_\mu \mathbf{A} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & [\mathbf{A}]_{11} & [\mathbf{A}]_{12} \\ 0 & [\mathbf{A}]_{21} & [\mathbf{A}]_{22} \end{pmatrix} \quad \text{and} \quad \mathsf{T}_\sigma \mathbf{A} := \begin{pmatrix} [\mathbf{A}]_{11} & [\mathbf{A}]_{12} & 0 \\ [\mathbf{A}]_{21} & [\mathbf{A}]_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

(c) the entries of $\mathbf{Y}$ are bounded except for the second column around the endpoints of Δ_σ and the third column around the endpoint of Δ_μ where they behave as $\mathcal{O}(|z - e|^{-1/2})$ with e being the corresponding endpoint.

To see how RHP-$\mathbf{Y}$ is connected to Frobenius-Padé approximants, observe that the linear form $R_{m,n}$ is a holomorphic function in $\mathbb{C} \setminus \Delta_\sigma$ with a pole of degree at most m at infinity. Moreover, it follows from Plemelj-Sokhotski formulae [26, Section I.4.2] and (17) that

$$R_{m,n}^+ - R_{m,n}^- = Q_{m,n}\rho_\sigma/w_\sigma^+ \quad \text{on} \quad \Delta_\sigma^\circ. \tag{26}$$

It is also known from the theory of boundary behavior of Cauchy integrals [26, Section I.8] that $R_{m,n}(z) \sim |z-e|^{-1/2}$ as $z \to e \in \{a_\sigma, b_\sigma\}$. As mentioned before, condition (2) implies that $R_{m,n}$ is orthogonal to all polynomials of degree at most $m+n$ with respect to μ, i.e.,

$$\int x^i R_{m,n}(x) d\mu(x) = 0, \quad i \in \{0, \dots, m+n\}. \tag{27}$$

Orthogonality relations (27) imply that the Cauchy transform of $R_{m,n}$ vanishes at infinity with order at least $m+n+2$. That is, the function

$$C_{m,n}(z) := \int \frac{R_{m,n}(x)}{x-z} d\mu(x), \quad z \in \overline{\mathbb{C}} \setminus \Delta_\mu,$$

is a holomorphic function in $\overline{\mathbb{C}} \setminus \Delta_\mu$, has a zero of order at least $m+n+2$ at infinity, and satisfies

$$C^+_{m,n} - C^-_{m,n} = R_{m,n}\rho_\mu / w^+_\mu \quad \text{on} \quad \Delta^\circ_\mu. \tag{28}$$

As in the case of $R_{m,n}$, we can conclude that $C_{m,n}(z) \sim |z-e|^{-1/2}$ as $z \to e \in \{a_\mu, b_\mu\}$.

Lemma 6. *Let $n-1 \leqslant m$. If (m,n)-th Frobenius-Padé approximant is unique and $\deg(Q_{m,n}) = n$, then $R_{m+1,n-1}(z) \sim z^{m+1}$ and $C_{m,n-1}(z) \sim z^{-(n+m+1)}$ as $z \to \infty$ for any $(m+1,n-1)$-st and $(m,n-1)$-st approximants, respectively.*

Proof. Assume to the contrary that there is $(m,n-1)$-st approximant such that $C_{m,n-1}(z) \sim z^{-(n+m+j+1)}$ as $z \to \infty$ for some $j > 0$. It can be readily verified that in this case the corresponding linear form $R_{m,n-1}$ is orthogonal to all polynomials of degree at most $m+n+j$. Then we can conclude from (2) that this $(m,n-1)$-st approximant is also $(m+j_1, n-1+j_2)$-th approximant for any choice of $j_1, j_2 \geqslant 0$, $j_1+j_2 \leqslant j$. By taking $j_1 = 1$ and $j_2 = 0$, we see that there exists $(m+1,n-1)$-st approximant for which $R_{m+1,n-1}(z) \sim z^{m+1-i}$ for some $i > 0$ (recall that $n-1 \leqslant m$). This implies that $\deg(P_{m+1,n-1}) = m+1-i \leqslant m$, and respectively, this Frobenius-Padé approximant also corresponds to the index (m,n) and its denominator has degree at most $n-1$. □

Assuming (m,n)-th approximant is the unique and $\deg(Q_{m,n}) = n$, define

$$\mathbf{Y}_{m,n} := \mathbf{C}_{m,n} \begin{pmatrix} Q_{m,n} & R_{m,n} & C_{m,n} \\ Q_{m+1,n-1} & R_{m+1,n-1} & C_{m+1,n-1} \\ Q_{m,n-1} & R_{m,n-1} & C_{m,n-1} \end{pmatrix}, \tag{29}$$

where $\mathbf{C}_{m,n}$ is a diagonal matrix of constants chosen so that $\mathbf{Y}_{m,n}$ satisfies the normalization at infinity from RHP-Y(a). The choice of $\mathbf{C}_{m,n}$ is always possible due to Lemma 6. Then the following lemma holds.

Lemma 7. *Let $n-1 \leqslant m$. If RHP-Y is solvable, then (m,n)-th, $(m,n-1)$-st, and $(m+1,n-1)$-st Frobenius-Padé approximants are unique, $\deg(Q_{m,n}) = n$, and $\mathbf{Y} = \mathbf{Y}_{m,n}$.*

Proof. Assume that RHP-Y is solvable and $\mathbf{Y}$ is a solution. We consider only the first row as the other ones can be analyzed similarly. It follows from RHP-Y(a,b) that $[\mathbf{Y}]_{11}$ must be polynomial of degree n. Further, all three properties RHP-Y(a,b,c) imply that $[\mathbf{Y}]_{12} = [\mathbf{Y}]_{11}\widehat{\sigma} - P$ for some polynomial P, $\deg(P) \leqslant m$. Analogously, we see that $[\mathbf{Y}]_{13}$ must be a Cauchy transform of $[\mathbf{Y}]_{12}\rho_\mu/w^+_\mu$. The vanishing of $[\mathbf{Y}]_{13}$ at infinity with order at least $m+n+2$ implies that $[\mathbf{Y}]_{12}$ is orthogonal to x^i, $i \in \{0,\dots,m+n\}$, with respect to μ. Therefore, $c_i([\mathbf{Y}]_{11}\widehat{\sigma} - P) = 0$ for such i, and, by definition, $P/[\mathbf{Y}]_{11}$ is an (m,n)-th Frobenius-Padé approximant.

To show uniqueness of the approximants, observe first that the solution of RHP-Y is unique. Indeed, Let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be solutions. As the determinant of the jump matrix in RHP-Y(b) is 1 and $\det(\mathbf{Y}_1)$ can have at most square root singularities at the endpoints of Δ_μ and Δ_σ by RHP-Y(c), $\det(\mathbf{Y}_1)$ is an entire function such that $\det(\mathbf{Y}_1)(\infty) = 1$. Hence, $\det(\mathbf{Y}_1) \equiv 1$ and therefore $\mathbf{Y}_1$ is invertible. Then $\mathbf{Y}_2\mathbf{Y}_1^{-1}$ is an entire matrix-valued function that is equal to $\mathbf{I}$ at infinity. Thus, $\mathbf{Y}_2 = \mathbf{Y}_1$.

Second, observe that if (m,n)-th approximant is unique and $\deg(Q_{m,n}) = n$, then $\mathbf{Y}_{m,n}$ solves RHP-$\mathbf{Y}$. Indeed, the fact that $\mathbf{Y}_{m,n}$ satisfies RHP-$\mathbf{Y}$(a,c) easily follows from the analyticity properties and the behavior at infinity of $Q_{m,n}$, $R_{m,n}$, and $C_{m,n}$, as well as from the choice of $\mathbf{C}_{m,n}$. RHP-$\mathbf{Y}$(b) is an immediate consequence of (26) and (28).

Now, let $\mathbf{Y}$ be the solution. Assume $Q_{m,n}$, $R_{m,n}$, and $C_{m,n}$ correspond to another (m,n)-th approximant. Without loss of generality we can assume that $\deg(Q_{m,n}) = n$ (otherwise we should take $[Y]_{11} - Q_{m,n}$ instead of $Q_{m,n}$). Construct matrix $\mathbf{Y}_1$ by replacing the first row of $\mathbf{Y}$ with $\begin{pmatrix} Q_{m,n} & R_{m,n} & C_{m,n} \end{pmatrix}$. From the first paragraph we know that the second and third rows of $\mathbf{Y}_1$ correspond to $(m+1,n-1)$-st and $(m,n-1)$-st approximants and therefore we deduce from the third paragraph that $\mathbf{Y}_1$ is a solution of RHP-$\mathbf{Y}$. By uniqueness, we get that $\mathbf{Y}_1 = \mathbf{Y}$ and therefore (m,n)-th approximant is unique. Thus, we know from Lemma 6 that any $(m+1,n-1)$-st and $(m,n-1)$-st must satisfy its conclusions. Hence, if they were not unique, we could replace the second and third rows of $\mathbf{Y}$ by the functions coming from other approximants and obtain a solution of RHP-$\mathbf{Y}$ different form $\mathbf{Y}$, which is impossible. □

3.3 *Non-Linear Steepest Descent Analysis in the Case $\Delta_{\sigma,c} \cap \partial D^-_{\sigma,c} = \varnothing$*

Recall that in the considered case $\Delta_{\sigma,c} = \Delta_\sigma$, see Proposition 1. Moreover, it follows from (24) and Proposition 2 that $\Delta_{\sigma,\frac{n}{n+m}} = \Delta_\sigma$ for all n large enough. In particular, we have that $\mathfrak{R}_{m,n} = \mathfrak{R}_c =: \mathfrak{R}$ for all such n and we consider only these indices from now on.

Let Γ_μ and Γ_σ be positively oriented Jordan curves lying exterior to each other and containing Δ_μ and Δ_σ in the respective interiors. We denote by Ω_ν the domain delimited by $\Gamma_\nu \cup \Delta_\nu$, $\nu \in \{\mu,\sigma\}$. We assume that ρ_ν extends holomorphically across Γ_ν, $\nu \in \{\mu,\sigma\}$, and that $\Gamma_\sigma \subset D^+_{\sigma,c}$. Observe that in the considered case $\partial D^+_{\sigma,\frac{n}{n+m}}$ approaches $D^+_{\sigma,c}$ by (25) and therefore $\Gamma_\sigma \subset D^+_{\sigma,\frac{n}{n+m}}$ is uniformly separated from $\partial D^+_{\sigma,\frac{n}{n+m}}$. Define

$$\mathbf{X} = \mathbf{Y} \begin{cases} \mathbf{T}_\nu \begin{pmatrix} 1 & 0 \\ -w_\nu/\rho_\nu & 1 \end{pmatrix}, & \text{in} \quad \Omega_\nu, \quad \nu \in \{\mu,\sigma\}, \\ \mathbf{I}, & \text{otherwise}. \end{cases} \tag{30}$$

It is easy to verify that $\mathbf{X}$ solves the following Riemann-Hilbert problem (RHP-$\mathbf{X}$):

(a) $\mathbf{X}$ is analytic in $\mathbb{C} \setminus (\Delta_\mu \cup \Delta_\sigma \cup \Gamma_\mu \cup \Gamma_\sigma)$ and

$$\lim_{z\to\infty} \mathbf{X}(z)\, \mathrm{diag}\left(z^{-n}, z^{-m-1}, z^{n+m+1}\right) = \mathbf{I};$$

(b) $\mathbf{X}$ has continuous traces on $\Delta^\circ_\mu \cup \Delta^\circ_\sigma \cup \Gamma_\mu \cup \Gamma_\sigma$ that satisfy

$$\mathbf{X}_+ = \mathbf{X}_- \begin{cases} \mathbf{T}_\nu \begin{pmatrix} 0 & \rho_\nu/w^+_\nu \\ -w^+_\nu/\rho_\nu & 0 \end{pmatrix} & \text{on} \quad \Delta^\circ_\nu, \\ \mathbf{T}_\nu \begin{pmatrix} 1 & 0 \\ w_\nu/\rho_\nu & 1 \end{pmatrix} & \text{on} \quad \Gamma_\nu, \end{cases} \quad \nu \in \{\sigma,\mu\};$$

(c) $\mathbf{X}$ satisfies RHP-$\mathbf{Y}$(c).

Then the following lemma can be easily checked.

Lemma 8. *RHP-$\mathbf{X}$ is solvable if and only if RHP-$\mathbf{Y}$ is solvable. When solutions of RHP-$\mathbf{X}$ and RHP-$\mathbf{Y}$ exist, they are unique and connected by (30).*

As typical in the steepest descent analysis of Riemann-Hilbert problems, we ignore the jump of $\mathbf{X}$ on $\Gamma_\mu \cup \Gamma_\sigma$ and look for the following approximation to $\mathbf{X}$ (RHP-$\mathbf{N}$):

(a) $\mathbf{N}$ is analytic in $\mathbb{C} \setminus (\Delta_\mu \cup \Delta_\sigma)$ and

$$\lim_{z\to\infty} \mathbf{N}(z)\, \mathrm{diag}\left(z^{-n}, z^{-m-1}, z^{n+m+1}\right) = \mathbf{I};$$

(b) $\mathbf{N}$ has continuous traces on $\Delta_\mu^\circ \cup \Delta_\sigma^\circ$ that satisfy

$$\mathbf{N}_+ = \mathbf{N}_- \mathsf{T}_\nu \begin{pmatrix} 0 & \rho_\nu / w_\nu^+ \\ -w_\nu^+/\rho_\nu & 0 \end{pmatrix} \quad \text{on} \quad \Delta_\nu^\circ, \quad \nu \in \{\mu, \sigma\};$$

(c) $\mathbf{N}$ satisfies RHP-$\mathbf{Y}$(c).

Let $\Phi_{m,n}$ be as defined before (8), which are rational functions on the same surface $\mathfrak{R}$. Denote by Υ_k, $k \in \{0,1,2\}$, a rational functions on $\mathfrak{R}$ with the divisor $\infty^{(0)} - \infty^{(k)}$, normalized as in (8). Clearly, $\Upsilon_0 \equiv 1$, $\Phi_{m+1,n}\Upsilon_1 = \Phi_{m+2,n-1}$ and $\Phi_{m+1,n}\Upsilon_2 = \Phi_{m+1,n-1}$. Further, let $S := S_c$ be the function granted by Proposition 4, again with respect to $\mathfrak{R}$. Define the constants $\gamma^{(k)}_{m+1,n}$ by

$$\lim_{z\to\infty} z^{i(k)} \gamma^{(k)}_{m+1,n} \Phi^{(k)}_{m+1,n}(z) S^{(k)}(z) \Upsilon^{(k)}_k(z) = 1, \tag{31}$$

where $i(0) = -n$, $i(1) = -m-2$, and $i(2) = n+m$. Then the following lemma holds.

Lemma 9. *A solution of RHP-$\mathbf{N}$ is given by $\mathbf{N} = \mathbf{CMD}$, where*

$$\begin{cases} \mathbf{C} & := & \mathsf{diag}\begin{pmatrix} \gamma^{(0)}_{m+1,n} & \gamma^{(1)}_{m+1,n} & \gamma^{(2)}_{m+1,n} \end{pmatrix}, \\ \mathbf{M} & := & \begin{pmatrix} S^{(0)} & S^{(1)}/w_\sigma & S^{(2)}/w_\mu \\ S^{(0)}\Upsilon_1^{(0)} & S^{(1)}\Upsilon_1^{(1)}/w_\sigma & S^{(2)}\Upsilon_1^{(2)}/w_\mu \\ S^{(0)}\Upsilon_2^{(0)} & S^{(1)}\Upsilon_2^{(1)}/w_\sigma & S^{(2)}\Upsilon_2^{(2)}/w_\mu \end{pmatrix}. \\ \mathbf{D} & := & \mathsf{diag}\begin{pmatrix} \Phi^{(0)}_{m+1,n} & \Phi^{(1)}_{m+1,n} & \Phi^{(2)}_{m+1,n} \end{pmatrix}, \end{cases}$$

Proof. Since S is non-vanishing in the domain of holomorphy, the functions w_ν have simple poles at infinity, and the divisors of $\Phi_{m+1,n}$ and Υ_i are explicitly known, it is trivial to check that $\mathbf{CMD}$ satisfies RHP-$\mathbf{N}$(a). RHP-$\mathbf{N}$(b) follows easily from (18) and the fact that $\Phi^{(1)\pm} = \Phi^{(0)\mp}$ on Δ_σ and $\Phi^{(1)\pm} = \Phi^{(2)\mp}$ on Δ_μ for any rational function Φ on $\mathfrak{R}$. Finally, RHP-$\mathbf{N}$(c) is the consequence of the boundedness of $\Phi_{m,n}$ and S around the endpoints of $\Delta_\mu \cup \Delta_\sigma$ and the choice of w_σ and w_μ. □

It can be readily checked that $\mathsf{det}(\mathbf{N})$ is a holomorphic function in $\overline{\mathbb{C}} \setminus (\Delta_\mu \cup \Delta_\sigma)$ and $\mathsf{det}(\mathbf{N})(\infty) = 1$. In fact, it has no jumps across $\Delta_\mu^\circ \cup \Delta_\sigma^\circ$ and since it is either bounded or behaves like $\mathcal{O}(|z-e|^{-1/2})$ near endpoints of $\Delta_\mu \cup \Delta_\sigma$, those points are in fact removable singularities. Therefore $\mathsf{det}(\mathbf{N})$ is a bounded entire function. That is, $\mathsf{det}(\mathbf{N}) \equiv 1$ as follows from the normalization at infinity. It also follows from (8) that $\mathsf{det}(\mathbf{D}) \equiv 1$. In particular, this means that $\mathsf{det}(\mathbf{M})$ is constant and non-zero in $\overline{\mathbb{C}}$.

To take care of the jumps of $\mathbf{X}$ on $\Gamma_\mu \cup \Gamma_\sigma$, consider the following Riemann-Hilbert Problem (RHP-$\mathbf{Z}$):

(a) $\mathbf{Z}$ is a holomorphic matrix function in $\overline{\mathbb{C}} \setminus (\Gamma_\mu \cup \Gamma_\sigma)$ and $\mathbf{Z}(\infty) = \mathbf{I}$;
(b) $\mathbf{Z}$ has continuous traces on $\Gamma_\mu \cup \Gamma_\sigma$ that satisfy

$$\mathbf{Z}_+ = \mathbf{Z}_- (\mathbf{MD}) \mathsf{T}_\nu \begin{pmatrix} 1 & 0 \\ w_\nu/\rho_\nu & 1 \end{pmatrix} (\mathbf{MD})^{-1} \quad \text{on} \quad \Gamma_\nu, \quad \nu \in \{\mu, \sigma\}.$$

Then the following lemma takes place.

Lemma 10. *The solution of RHP-$\mathbf{Z}$ exists for all n large enough and satisfies*

$$\mathbf{Z} = \mathbf{I} + \mathcal{O}\left(C^{-n}_{\mu,\sigma}\right) \tag{32}$$

for some constant $C_{\mu,\sigma} > 1$, where $\mathcal{O}(\cdot)$ holds uniformly in $\overline{\mathbb{C}}$.

Proof. The jump matrix for $\mathbf{Z}$ on Γ_σ is equal to

$$\mathbf{I} + \frac{w_\sigma}{\rho_\sigma} \frac{\Phi^{(1)}_{m+1,n}}{\Phi^{(0)}_{m+1,n}} \mathbf{M} \mathbf{E}_{21} \mathbf{M}^{-1}. \tag{33}$$

where $\mathbf{E}_{ij}$ is the matrix with all zero entries except for (i,j)-th, which is 1. Since $\mathbf{M}$ is fixed and has constant determinant, it follows from (9), definition of $D^+_{\sigma,\frac{n}{n+m+1}}$ in (7), and the choice of Γ_σ that the jump of $\mathbf{Z}$ on Γ_σ is of the form $\mathbf{I} + \mathcal{O}\big(C^{-n}_{\mu,\sigma}\big)$ for some $C_{\mu,\sigma} > 1$. Similarly, we have that the jump $\mathbf{Z}$ on Γ_μ is equal to

$$\mathbf{I} + \frac{w_\mu}{\rho_\mu}\frac{\Phi^{(2)}_{m+1,n}}{\Phi^{(1)}_{m+1,n}}\mathbf{M}\mathbf{E}_{32}\mathbf{M}^{-1}, \tag{34}$$

which is also of the form $\mathbf{I} + \mathcal{O}\big(C^{-n}_{\mu,\sigma}\big)$ for some properly adjusted $C_{\mu,\sigma} > 1$ by (23). The conclusion of the lemma follows now from the same argument as in [27, Corollary 7.108] with another adjustment of $C_{\mu,\sigma}$. □

Finally, let $\mathbf{Z}$ be a solution of RHP-$\mathbf{Z}$ granted by Lemma 10 and $\mathbf{N} = \mathbf{CMD}$ be as in Lemma 9. Then it can be easily checked that $\mathbf{X} = \mathbf{CZMD}$ solves RHP-$\mathbf{X}$ and therefore

$$\mathbf{Y} = \mathbf{CZMD}\begin{cases} \mathbf{T}_\nu \begin{pmatrix} 1 & 0 \\ w_\mu/\rho_\nu & 1 \end{pmatrix}, & \text{in} \quad \Omega_\nu, \quad \nu \in \{\mu,\sigma\}, \\ \mathbf{I}, & \text{otherwise}. \end{cases} \tag{35}$$

solves RHP-$\mathbf{Y}$.

3.4 *Non-Linear Steepest Descent Analysis in the Case $\Delta_{\sigma,c} \cap \partial D^-_{\sigma,c} \neq \varnothing$*

In the case $\Delta_{\sigma,c} \cap \partial D^-_{\sigma,c} \neq \varnothing$ there no longer exists a Jordan curve $\Gamma_\sigma \subset D^+_{\sigma,c}$ encircling $\Delta_{\sigma,c}$, which prevents us from carrying out the estimate (33). Hence, we shall require the

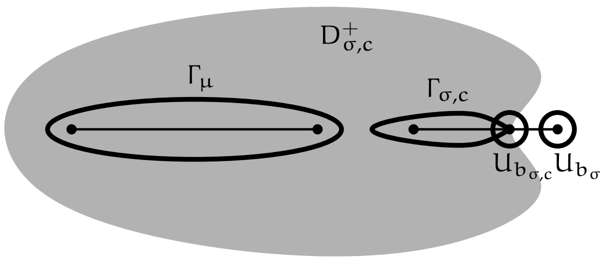

FIGURE 4. Contours Γ_μ, $\Gamma_{\sigma,c}$, and the disks $U_{b_{\sigma,c}}$, U_{b_σ}.

Jordan curve $\Gamma_{\sigma,c}$ to encircle $\Delta_{\sigma,c}$ except for the point $b_{\sigma,c}$, which they have in common. Moreover, given disjoint disks $U_{b_{\sigma,c}}$ and U_{b_σ} centered at $b_{\sigma,c}$ and b_σ, respectively, (unless $b_{\sigma,c} = b_\sigma$ in which case these disks coincide) we also require that $\Gamma_{\sigma,c} \setminus U_{b_{\sigma,c}} \subset D^+_{\sigma,c}$, see Figure 4. To slightly alleviate the notation, let us set

$$b_{m+1,n} := b_{\sigma,\frac{n}{n+m+1}}, \quad \Delta_{m+1,n} := \Delta_{\sigma,\frac{n}{n+m+1}}, \text{and} \quad \Gamma_{m+1,n} := \Gamma_{\sigma,\frac{n}{n+m+1}},$$

where the curves $\Gamma_{m+1,n}$ are selected analogously to $\Gamma_{\sigma,c}$ with the requirement that $\Gamma_{m+1,n} \to \Gamma_{\sigma,c}$ as $n \to \infty$ in Hausdorff metric (recall that $b_{m+1,n} \to b_{\sigma,c}$ as $n \to \infty$, see (25)). We take Γ_μ to be a Jordan curve encircling Δ_μ, which is disjoint from all $\Gamma_{m+1,n}$. As before, we assume that ρ_μ is holomorphic across Γ_μ and ρ_ν is holomorphic across each $\Gamma_{m+1,n}$. We continue to denote by Ω_μ and $\Omega_{m+1,n}$ the domains bounded by $\Gamma_\mu \cup \Delta_\mu$ and $\Gamma_{m+1,n} \cup \Delta_{m+1,n}$, respectively.

In what follows, we shall often refer back to Riemann-Hilbert problems formulated in Section 3.3. For each such reference it is understood that when Δ_σ, Γ_σ, and Ω_σ occur, they should be replaced by $\Delta_{m+1,n}$, $\Gamma_{m+1,n}$, and $\Omega_{m+1,n}$.

Define $\mathbf{X}$ by (30). Then $\mathbf{X}$ satisfies RHP-$\mathbf{X}$(a,c) and RHP-$\mathbf{X}$(b) with an additional jump

$$\mathbf{X}_+ = \mathbf{X}_-\mathbf{T}_\sigma \begin{pmatrix} 1 & \rho_\sigma/w^+_\sigma \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \Delta^\circ_\sigma \setminus \Delta_{m+1,n}.$$

Clearly, Lemma 8 remains valid.

Define $\mathbf{D}$ as in Lemma 9, where $\Phi_{m+1,n}$ is a rational function on $\mathfrak{R}_{m+1,n}$ with the divisor $(n+m+1)\infty^{(2)} - n\infty^{(0)} - (m+1)\infty^{(1)}$ and normalized as in (8). Let $S_{m+1,n}$ be the function on $\mathfrak{R}_{m+1,n}$ granted by Proposition 4 applied with $c = \frac{n}{n+m+1}$ and $\Upsilon_k := \Upsilon_{k;m+1,n}$, $k \in \{0,1,2\}$, be the rational function on $\mathfrak{R}_{m+1,n}$ with the divisor $\infty^{(0)} - \infty^{(k)}$ and the normalization as in (8). Define matrices $\mathbf{M}$ and $\mathbf{C}$ as in Lemma 9 using the above functions. Then $\mathbf{N} = \mathbf{CMD}$ again solves RHP-N and it is still true that $\det(\mathbf{N}) \equiv 1$. Therefore $\det(\mathbf{M})$ is a constant, but in this case it might depend on (m,n). However, observe that an analogous matrix $\mathbf{M} = \mathbf{M}_c$ can be defined on the "limiting" surface $\mathfrak{R}_c$ as well. Moreover, it was shown in [14, Section 7] that

$$S_{m+1,n} \to S_c \quad \text{and} \quad \Upsilon_{k;m+1,n} \to \Upsilon_{k;c} \tag{36}$$

uniformly on $\mathfrak{R}_{c,\delta}$ for any $\delta > 0$, where $\mathfrak{R}_{c,\delta}$ is obtained from $\mathfrak{R}_c$ be removing circular neighborhoods of radius δ around each branch point of $\mathfrak{R}_c$ and functions $S_{m+1,n}$ and $\Upsilon_{k;m+1,n}$ are carried over to $\mathfrak{R}_{c,\delta}$ with the help of natural projections. Hence, $\det(\mathbf{M}_{m+1,n}) \to \det(\mathbf{M}_c)$, in particular, the determinants $\det(\mathbf{M})$ are uniformly bounded away from zero and infinity with m and n.

Let $\mathbf{P}_b$, $b \in \{b_{\sigma,c}, b_\sigma\}$, be a matrix-valued function that solves RHP-X inside of U_b and satisfies

$$\mathbf{P}_b = \mathbf{M}\big(\mathbf{I} + \mathcal{O}(\varepsilon_{m+1,n})\big)\mathbf{D} \tag{37}$$

uniformly on ∂U_b, where $0 < \varepsilon_{m+1,n} \to 0$ as $n \to \infty$. Such matrices do exist. Indeed, when $b_{\sigma,c} \neq b_\sigma$, one can easily check that

$$\mathbf{P}_{b_\sigma} = \mathbf{M}\mathsf{T}_\sigma \begin{pmatrix} 1 & \mathcal{C}_\sigma \Phi^{(0)}_{m+1,n}/\Phi^{(1)}_{m+1,n} \\ 0 & 1 \end{pmatrix} \mathbf{D},$$

where $\mathcal{C}_\sigma(z) := \frac{1}{2\pi i}\int_{\Delta_\sigma} \frac{\rho_\sigma(x)}{x-z}\frac{d\sigma(x)}{w_\sigma^+(x)}$. As the construction of $\mathbf{P}_{b_{\sigma,c}}$ is quite long and is absolutely identical to the one in [14, Sections 9.4 and 9.5], we omit it here. Let us just mention that it is based on the model Riemann-Hilbert problem associated with solutions Panlevé XXXIV equation [14, Section 4.3] (matrix $\mathbf{\Psi}_{0,1}$ when $b_{\sigma,c} \neq b_\sigma$ and $\widetilde{\mathbf{\Psi}}_{-1/2,0}$ otherwise), see also [28, 29].

FIGURE 5. Lens Σ_Z consisting of the curve Γ_μ, the arc $\Gamma_{m+1,n} \setminus U_{b_{\sigma,c}}$, the interval $\Delta_\sigma \setminus (\Delta_{\sigma,c} \cup U_{b_{\sigma,c}} \cup U_{b_\sigma})$, and the circles $\partial U_{b_{\sigma,c}}$, ∂U_{b_σ}.

Let the contour Σ_Z be as depicted on Figure 5. The last matrix-valued function needed to solve RHP-Y is described by the following Riemann-Hilbert problem:

(a) $\mathbf{Z}$ is a holomorphic matrix function in $\overline{\mathbb{C}} \setminus \Sigma_Z$ and $\mathbf{Z}(\infty) = \mathbf{I}$;

(b) $\mathbf{Z}$ has continuous traces on Σ_Z except perhaps at its branching points that satisfy

$$\mathbf{Z}_+ = \mathbf{Z}_-(\mathbf{MD})\mathsf{T}_\nu \begin{pmatrix} 1 & 0 \\ w_\nu/\rho_\nu & 1 \end{pmatrix} (\mathbf{MD})^{-1}$$

on Γ_μ when $\nu = \mu$ and $\Gamma_{m+1,n} \setminus U_{b_{\sigma,c}}$ when $\nu = \sigma$,

$$\mathbf{Z}_+ = \mathbf{Z}_-(\mathbf{MD})\mathsf{T}_\sigma \begin{pmatrix} 1 & \rho_\sigma/w_\sigma^+ \\ 0 & 1 \end{pmatrix} (\mathbf{MD})^{-1}$$

on $\Delta_\sigma \setminus (\Delta_{\sigma,c} \cup U_{b_{\sigma,c}} \cup U_{b_\sigma})$, and $\mathbf{Z}_+ = \mathbf{Z}_- \mathbf{P}_b(\mathbf{MD})^{-1}$, on ∂U_b, $b \in \{b_{\sigma,c}, b_\sigma\}$.

Exactly as in the previous case, the following lemma holds.

Lemma 11. *The solution of RHP-Z exists for all n large enough and satisfies*

$$\mathbf{Z} = \mathbf{I} + \mathcal{O}(\varepsilon_{m+1,n}) \tag{38}$$

where $\mathcal{O}(\cdot)$ holds uniformly in $\overline{\mathbb{C}}$ and $\varepsilon_{m+1,n}$ are the constants from (37).

Proof. Recall that the determinants $\det(\mathbf{M})$ are identically constant as functions of z and that these constants are uniformly separated from zero and infinity with m and n. Hence, the estimates of the size of the jumps on Γ_μ and $\Gamma_{m+1,n}\setminus U_{b_{\sigma,c}}$ are absolutely analogous to (33) and (34). The jumps in this case are geometrically close to the identity where the constant of proportionality depends on how close $\Gamma_{m+1,n}\setminus U_{b_{\sigma,c}}$ is to $\partial D_{\sigma,c}$ (the latter sets are uniformly separated from each other by our construction of $\Gamma_{m+1,n}$). The jump on $\Delta_\sigma\setminus(\Delta_{\sigma,c}\cup U_{b_{\sigma,c}}\cup U_{b_\sigma})$ is also geometrically close to the identity since this interval belongs to $D^-_{\sigma,c}$ where $|\Phi^{(1)}_{m+1,n}|>|\Phi^{(0)}_{m+1,n}|$. Finally, we see that the jump on ∂U_b, $b\in\{b_{\sigma,c},b_\sigma\}$, is $\mathcal{O}(\varepsilon_{m+1,n})$ close to the identity by (37) and the normality of $\mathbf{M}$, see (36). The existence of $\mathbf{Z}$ again follows from [27, Corollary 7.108]. The size of the error is proportional to $\varepsilon_{m+1,n}$ as the latter is of order $1/n$ at best, see [14, Sections 9.4 and 9.5]. □

Altogether, the solution of RHP-X is given by

$$\mathbf{X}=\mathbf{CZ}\begin{cases}\mathbf{P}_b, & \text{in } U_b,\quad b\in\{b_{\sigma,c},b_\sigma\},\\ \mathbf{MD}, & \text{otherwise},\end{cases}\tag{39}$$

and then the solution of RHP-Y is obtained by inverting (30).

3.5 *Asymptotic Analysis*

Below we write $S_{m+1,n}$ and $\Upsilon_{k;m+1,n}$ irrespectively of whether we are in the case of Section 3.3 or Section 3.4. Write $\mathbf{Z}=\mathbf{I}+[\upsilon^{(i,j)}_{m+1,n}]_{i,j=1}^3$, where we know from Lemmas 10 and 11 that

$$|\upsilon^{(i,j)}_{m+1,n}|=\mathcal{O}\big(C^{-n}_{\mu,\sigma}\big)\quad\text{or}\quad|\upsilon^{(i,j)}_{m+1,n}|=\mathcal{O}\big(\varepsilon_{m+1,n}\big)\tag{40}$$

uniformly in $\overline{\mathbb{C}}$ depending on the considered case ($\upsilon^{(i,j)}_{m+1,n}(\infty)=0$ as $\mathbf{Z}(\infty)=\mathbf{I}$). Given any closed set $K\subset\overline{\mathbb{C}}\setminus\Delta_\sigma$, choose Ω_σ or $\Omega_{m+1,n}\cup\bigcup_b U_b$ so that K belongs to the complement of its closure. Then we get from (35) and (39) that $\mathbf{Y}=\mathbf{CZMD}$ on K and therefore

$$[\mathbf{Y}]_{11}=\gamma^{(0)}_{m+1,n}\Phi^{(0)}_{m+1,n}S^{(0)}_{m+1,n}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(0)}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(0)}_{2;m+1,n}\right)$$

on K regardless whether it intersects Ω_μ or not. The first relation in (19) now follows from (29), (36), and (40). On the other hand, we get from (35) and (39) that

$$\begin{aligned}[\mathbf{Y}]_{11}&=\gamma^{(0)}_{m+1,n}\Phi^{(0)\pm}_{m+1,n}S^{(0)\pm}_{m+1,n}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(0)\pm}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(0)\pm}_{2;m+1,n}\right)+\\ &\gamma^{(0)}_{m+1,n}\Phi^{(1)\pm}_{m+1,n}\frac{w^\pm_\sigma}{w^\pm_{\sigma,c}\rho_\sigma}S^{(1)\pm}_{m+1,n}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(1)\pm}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(1)\pm}_{2;m+1,n}\right)\end{aligned}$$

on Δ_σ or $\Delta_{\sigma,c}\setminus U_{b_{\sigma,c}}$, depending on the considered case. The second relation in (19) now follows from (29), (18), (36), (40), and the fact that $\Upsilon^{(0)\pm}_k=\Upsilon^{(1)\mp}_k$ on $\Delta^\circ_{\sigma,c}$.

Now, let $K\subset\overline{\mathbb{C}}\setminus(\Delta_\mu\cup\Delta_\sigma)$. Adjust the set $\Omega_\mu\cup\Omega_\sigma$ or $\Omega_\mu\cup\Omega_{m+1,n}\cup\bigcup_b U_b$ if necessary so that K belongs to the complement of its closure. Then $\mathbf{Y}=\mathbf{CZMD}$ on K and therefore

$$[\mathbf{Y}]_{12}=\frac{\gamma^{(0)}_{m+1,n}}{w_{m+1,n}}\Phi^{(1)}_{m+1,n}S^{(1)}_{m+1,n}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(1)}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(1)}_{2;m+1,n}\right).$$

Even though $\Upsilon^{(1)}_{1;m+1,n}$ has a pole at infinity, the product $\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(1)}_{1;m+1,n}$ is finite satisfies (40) by (36) and the maximum modulus principle. As $w^{-1}_{m+1,n}\to w^{-1}_\sigma$ locally uniformly in $\overline{\mathbb{C}}\setminus\{a_\sigma,b_{\sigma,c}\}$, the first relation in (20) follows from (29), (36), and (40). Observe also that the last equality essentially does not change on Δ_σ or $\Delta_{\sigma,c}\setminus U_{b_{\sigma,c}}$, i.e., we simply need to replace the functions by their boundary values. This yields the second formula in (20). Finally, we get that

$$[\mathbf{Y}]_{12} = \gamma^{(0)}_{m+1,n}\Phi^{(1)\pm}_{m+1,n}\frac{S^{(1)\pm}_{m+1,n}}{w_{\sigma,c}}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(1)\pm}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(1)\pm}_{2;m+1,n}\right)+$$
$$\gamma^{(0)}_{m+1,n}\Phi^{(2)\pm}_{m+1,n}\frac{S^{(2)\pm}_{m+1,n}}{\rho_\mu}\left(1+\upsilon^{(1,1)}_{m+1,n}+\upsilon^{(1,2)}_{m+1,n}\Upsilon^{(2)\pm}_{1;m+1,n}+\upsilon^{(1,3)}_{m+1,n}\Upsilon^{(2)\pm}_{2;m+1,n}\right)$$

on Δ_μ. The third relation in (20) now follow from (29), (18), (36), and (40). This finishes the proof of Theorem 5 since the uniformity of the estimates in the case $b_{\sigma,c}\notin\partial D^-_{\sigma,c}$ follows from the fact $S_{m+1,n}=S_c$ and $\Upsilon_{k;m+1,n}=\Upsilon_k$ for all n large enough and therefore we do not need to use (36).

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