University of Pisa
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Linear response theory for complex systems

Master degree thesis

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To my family, to my friends, for their unfailing support.

Contents

Preface

We shall present in this work a proposal for modeling perturbation on complex systems.

In the chapter I we shall introduce the main mathematical tools and definitions of our model: Renewal processes. Our attempt to model the perturbation of complex systems will be limited to those for which a renewal perspective is allowed.

In chapter II we shall show, with the help of numerical simulations, that these systems exhibit a non stationary, thus non ergodic, behavior.

In chapter III we shall present our proposal for a perturbation theory of complex systems. Since Kubo's fluctuation-dissipation theorem:

$$\langle A(t)\rangle_{\text{pert}} - \langle A(t)\rangle_{\text{unpert}} = \varepsilon \int_0^t \chi(t-s)B(s)\mathrm{d}s$$

where $\chi(t-s) = \frac{\mathrm{d}}{\mathrm{d}s}C(t-s)$ holds for stationary processes, we have to extend it in order to use it for complex renewal systems.

In those cases perturbation can act either on the event generating operator (thus perturbing the leading process without affecting the event occurrence time) or on the global interaction (then perturbing our waiting time distribution). The first approach, which we refer to as "phenomenological" gives $\chi(t,s) = \frac{\mathrm{d}}{\mathrm{d}s}C(t,s)$; the second one, which we call "dynamic" gives $\chi(t,s) = -\frac{\mathrm{d}}{\mathrm{d}t}C(t,s)$. In the stationary case, both prescriptions lead to Kubo theorem again.

We assert that the "dynamical" approach is the one which better describes our processes and then extend this theory to non dichotomous processes. In this case, besides the linear response term, a new term appears. If the perturbation is harmonic, $A\cos(\omega t + \varphi)$, the linear response theory leads to a response of the following form:

$$\langle A(t)\rangle_{\text{pert}} - \langle A(t)\rangle_{\text{unpert}} = \varepsilon BR(t)\cos(\omega t + \Phi)$$

with $R(t) \sim 1/t^{2-\mu}$ and B and φ depending on the peculiar characteristics of the system .

We then illustrate an experimental result on Liquid crystals dynamics that confirms our theory.

1 Complex systems power laws and subordination theory

Answering a question like "what is complexity science?" is still a very hard task: complexity science is a very recent discipline and, in spite of an exponentially increasing number of results, it still lacks the support of a unifying theory, accepted by the majority of the scientists working in this field. The large variety of systems studied, the diversity of behaviors which are usually labeled as " complex" and its interdisciplinary status have made it to be a very fast changing discipline in which many different approaches, of which each one has its pros and cons, coexists: there is not yet a commonly recognized foundations even if some typical behaviors are recognized.

As a discipline, complexity science suffers on account of difficulty of defining what a complex system is. A typical heuristic reply to this question may be a negative one: a system is *complex* if it is *neither* a completely deterministic one *nor* a completely stochastic one. Complex systems stands in a certain way between Newtonian physics (that is physics of large scales) and statistical and quantum physics (physics of small scales).

Although this definition is correct, it is too vague unsatisfactory and we would like to elucidate some specific behavior of complex systems

1.1 Characteristics of complex systems

This said, we would like to be able to give a more "positive" definition of complex system, and we would like to be able to give some property that we would label as "bookmarks" of complexity behavior (see [42])

We want to identify three different types of behavior that can characterize a complex system:

Chaos As usually defined a *chaotic system* is a *causal* system with unpredictable evolution. This is historically the first example of complex system.

Non linearity Non linear systems, that is systems whose outputs are not proportional to their inputs, are another class of systems that exhibit *complex* dynamics (e.g. limiting circles, bifurcations, period doubling)

Self-organization and cooperation Complex systems like neural networks, scale free complex networks, cellular automata, decision-making networks show some typical dynamics characterized by different form of self-organization, by the birth and death of coherent structures (patterns), and power law behaviors.

We shall be mainly interested in strongly cooperative systems whose evolution can be characterized by a renewal process.

We shall not consider in our discussion, any specific model, but rather we shall propose a generalized theory that may be applied whenever an event driven renewal description is plausible.

We then shall refer to some recent experimental evidences on liquid crystals [54, 6] which prove that an event driven description is possible in this case, and so we shall confront our theoretical proposals with some experimental results.

1.1.1 Power laws

In complex systems the emergence of power law is ubiquitous. Power Laws have been found to govern the occurrence time of large earthquakes [46, 37], to model financial markets behavior [32], rains [50] and many others.

Brain dynamics too seems to undergo an event power law distribution. Similarities between Omori's law for earthquakes and epileptic seizures distributions have been found [48],

Many complex networks too, like World Wide Web [9] (see figure 1.1), Social Networks, human dynamics (e.g. electronic correspondence [8] and traditional one [45]) exhibit a complex topology characterized by a power law distribution of the degrees of nodes. Those networks, called scale-free complex networks [2, 10], lighten the nature of power law emergence in complex system.

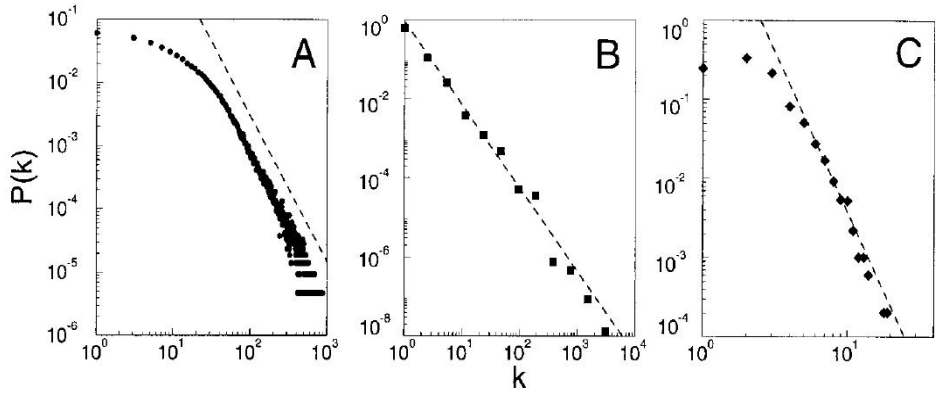


Figure 1.1: A figure taken from [?] which shows the emergence of a power law distribution for real networks. The distribution function of connectivity for various large networks. (A) Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$ (B) WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). (C) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$ The dashed lines have slopes (A) $\mu = 2.3$ (B) $\mu = 2.1$ and (C) $\mu = 4$

While in fact purely random traditional complex network models (i.e. Erdős - Rényi graphs [17], Watts-Strogartz small world [64]) are characterized by degree distribution

that is mainly poissonian, real-world complex networks, which have actually a very strong cooperative behavior, are actually better modeled by scale free model.

Power laws are in fact able to correctly model what has been called *sporadicity*. Moreover with respect to poissonian power laws they allow rare events to occur with a higher and not negligible probability.

We consider it the only parameter which governs the *universality class* of system driven by a power law distribution, and experimental observation are able to determine the exponent μ.

Many complex physical systems have shown to exhibit a power law decay from a non-equilibrium: a recent example is provided by liquid crystals [54]. Then we are interested to analyze power law event driven processes as a model from which extrapolate theoretical predictions.

Let us point here that power laws are asymptotic. They can be useful to model a universal long term behavior but not the transient one which is strongly dependent on the microscopic details.

In order to facilitate calculations we have to make some assumptions on the form of the distribution we shall use. These assumptions will introduce biases that should not affect the asymptotic behavior of our results. The assumption done will influence the choice of these parameter without afflicting the asymptotic behavior.

In this chapter we shall present some functions that exhibits an asymptotic power law behavior. We shall use these function our calculations.

Mittag-Leffler Function derivative

Mittag-Leffler function [65] has frequently been considered to extend the concept of exponential. To understand it we have digress a little and give a rapid introduction to Fractional derivation.

There are many ways we could extend the concept of derivation. Liouville's guess on exponential function (i.e. $D^\alpha e^{ax} = a^\alpha \mathrm{e}^{ax}$),has been historically the first attempt to extend the concept of Derivative but soon Liouville was confronted to the problems of this definition (it was not a coherent definition). More then a century took to mathematicians to give a coherent theory.

A fractional derivative cannot be a local operator. For the derivative operator defined over a L^2 space this is obvious, since it has to can be constructed by the mean of infinite series of operator. But in general this is not obvious unless we use one of the many forms in which fractional Derivatives may be expressed, the Riemann-Liouville form.

For $q < 0$ we set [39] and [65]

$$ {}_a\mathrm{D}_t^{(q)} X(t) = \frac{1}{\Gamma(-q)} \int_a^t \frac{X(\xi)}{(t-\xi)^{q+1}} \,\mathrm{d}\xi \tag{1.1} $$

and we extend this definition to $\alpha = q + n$

$$ {}_a\mathrm{D}_t^{(\alpha)} X(t) = {}_a\mathrm{D}_t^{(q+n)} X(t) = -\frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_a\mathrm{D}_t^{(q-n)} X(t). \tag{1.2} $$

Here the non locality of this operator is clear.

We want to introduce Mittag-Leffler function as a generalization of exponential functions (see [52]). Since we know that the exponential function is the solution of the equation ordinary kinetic equation

$$\mathrm{D}_t X_i(t) = c_i X_i(t) \tag{1.3}$$

Integrating (notice that integration according to 1.1 is nothing but ${}_0\mathrm{D}_t^{(-1)}$ operator) we have:

$$X_i(t) - X_i(0) = c_i \, {}_0\mathrm{D}_t^{(-1)} X_i(t) \tag{1.4}$$

We can thus generalize this equation dropping indices and letting ${}_0\mathrm{D}_t^{(-1)} \to {}_0\mathrm{D}_t^{(-\nu)}$ that is

$$X_\nu(t) - X_\nu(0) = c^\nu \, {}_0\mathrm{D}_t^{(-\nu)} X_i(t) \tag{1.5}$$

This equation can be solved and we obtain

$$X_\nu(t) = X_\nu(0) c^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{\nu k}}{\Gamma(\nu k + 1)} = X_\nu c^\nu(0) E_\nu(c^\nu t^\nu) \tag{1.6}$$

We call the function :

$$E_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t)^k}{\Gamma(\nu k + 1)} \tag{1.7}$$

the *Mittag-Leffler function*.

As this derivation shows, *Mittag-Leffler function* is a kind of interpolating function between exponential law and power law.

Let us consider now the function $\psi_{ML}(t) = -\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha(\lambda^\alpha t^\alpha)$. If we consider its Laplace transform we obtain

$$\hat{\psi}_{ML}(s) = \frac{1}{1 - \lambda^\alpha s^\alpha} \text{ with } \alpha \in [0, 2] \tag{1.8}$$

Bochner's theorem assures us that $\psi_{ML}(t)$ is actually a probability density function. Using Tauberian theorem ([66], cap V) for Laplace transform we have ($\mu = \alpha + 1$):

$$\hat{\psi}_{ML}(s) \sim 1 + \lambda^\alpha s^\alpha \tag{1.9}$$

and so for $t \to \infty$ we have

$$\psi_{ML}(t) \sim \frac{1}{\Gamma(\mu + 1) t^\mu} \tag{1.10}$$

obtaining an asymptotic power law.

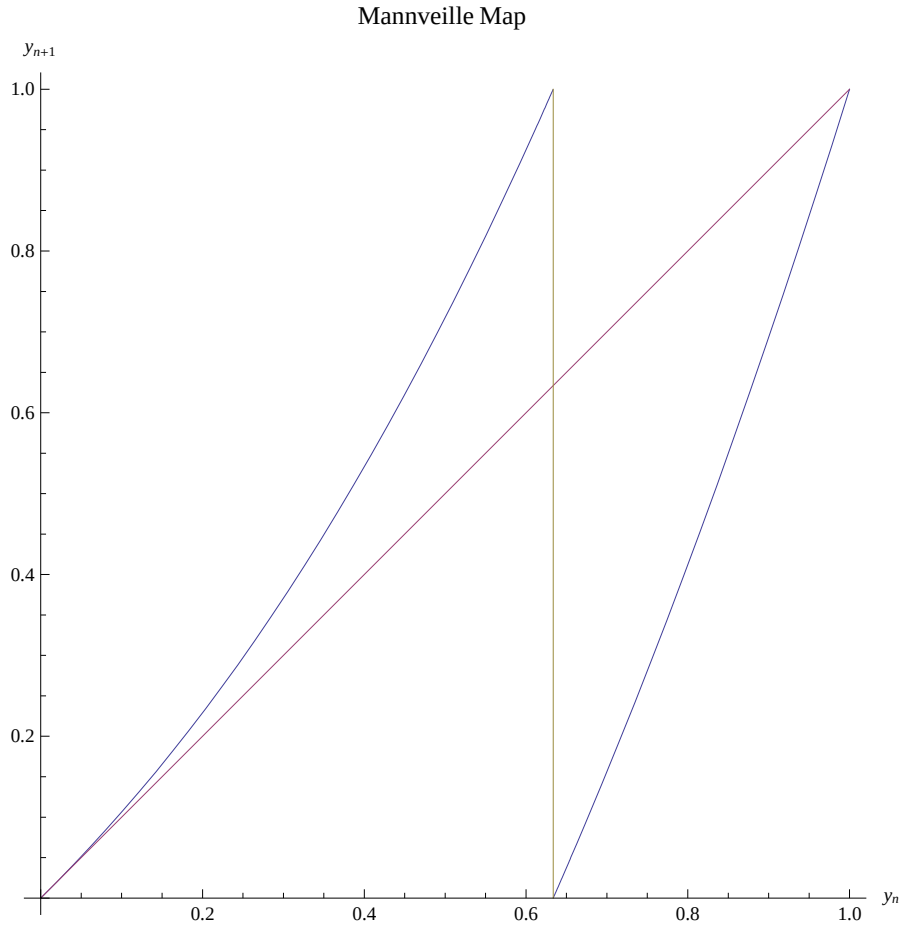


Figure 1.2: Manneville map for $z = 2.2$ $\alpha = 1$

Manneville's Map and Manneville's distribution

In an article of 1980 [35] Paul Manneville proposed a model for intermittent turbulence, which we shall call Manneville's map

$$y_{n+1} = M(y_n) = y_n + \alpha y_n^z \pmod 1 \tag{1.11}$$

with $z > 1$. This function is plotted in Figure 1.2.

As Gaspard and Wang found in 1998 [21] Manneville map dynamic has a very distinctive behavior

$$1 \leq z \leq \frac{3}{2} \qquad \text{normal dynamics (Gaussian Fluctuations} \tag{1.12}$$

$$\frac{3}{2} \leq z \leq 2 \qquad \text{transient anomalous dynamics} \tag{1.13}$$

$$2 \leq z \qquad \text{anomalous dynamics (Lévy fluctuations)} \tag{1.14}$$

As we see from figure 1.3 for $z = 2.2$ dynamics of Manneville model is characterized by a certain form of clustering: long period *laminar phases* interrupted by chaotic burst.

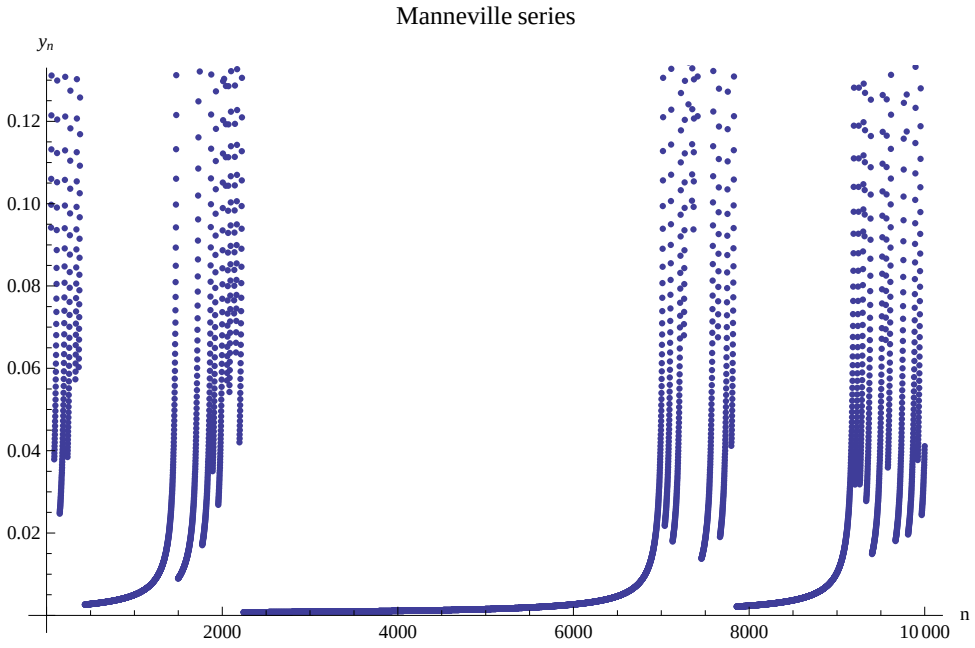


Figure 1.3: Manneville series for $z = 2.2$ $a = 1$ $x_0 = 0.2$

After establishing the intermittent nature of y we aim to calculate its probability density function [5], to do that we have to take a continuous time limit, by example considering the differential equation:

$$y' = \alpha y^z. \tag{1.15}$$

the solutions of this equation is given by:

$$\alpha(\tau_0 - \tau) = \int_{y_0}^{y} y^z \, \mathrm{d}y = \frac{1}{1-z}\left(\frac{1}{y^{z-1}} - \frac{1}{y_0^{z-1}}\right). \tag{1.16}$$

Thus, the time distance between two consecutive jumps (that is, by the structure of 1.15, we set $t_0 = 0$ and $y = 1$) is given by

$$\alpha\tau = \frac{1}{1-z}\left(\frac{1}{y_0^{z-1}} - 1\right). \tag{1.17}$$

by inverstion of this equation we get:

$$y_0 = \zeta(\tau) = \left(\frac{1}{(1-z)\alpha\tau + 1}\right)^{1/z-1}. \tag{1.18}$$

Since $y_0 \sim U(0,1)$ we have

$$\psi_M(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Prob}(\tau < t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Prob}(y_0 < \zeta(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{(1-z)t+1}\right)^{\frac{1}{z-1}} = \frac{(\mu-1)T^{\mu-1}}{(T+\tau)^{\mu}} \tag{1.19}$$

where we have set $\mu = \frac{z}{z-1}$ and $T = \frac{\alpha}{(z-1)}$.

We can apply Gaspard and Weng analysis [21] to this case and obtain:

$$\mu \geq 3 \qquad \text{normal dynamics (Gaussian Fluctuations, finite mean and variance)} \tag{1.20}$$

$$2 \leq \mu \leq 3 \qquad \text{transient anomalous dynamics (finite mean, variance not defined)} \tag{1.21}$$

$$1 \leq \mu \leq 2 \qquad \text{anomalous dynamics (Lévy fluctuations, mean and variance not defined)} \tag{1.22}$$

Thus, Manneville's intermittency is governed by a power law. We shall see that Laplace Transform (see [66]) of probability density functions is of fundamental importance for our theory but in this case a closed form is not available. We have, in fact,

$$\hat{\psi}_M(s) = \int_0^{\infty} \mathrm{e}^{-st}\frac{(\mu-1)T^{\mu-1}}{(T+\tau)^{\mu}}\,\mathrm{d}t = (\mu-1)T^{\mu-1}s^{\mu-2}\mathrm{e}^{-sT}\Gamma(2-\mu, sT) \tag{1.23}$$

where $\Gamma(x,\alpha) = \int_{\alpha}^{\infty} \mathrm{e}^t t^{x-1}\,\mathrm{d}t$ is the upper incomplete Gamma function [1]. $\psi_M(t)$ is a probability density function $\hat{(\psi)}_M(0) = 1$ we can compute his asymptotic behavior for $s \to 0$ and obtain

If $1 < \mu < 2$ we can expand the function and obtain:

$$\hat{\psi}_M(s) \sim 1 - \Gamma(2-\mu)(sT)^{\mu-1} \tag{1.24}$$

IF $2 < \mu < 3$ we obtain instead:

$$\hat{\psi}_M(s) \sim 1 + \frac{sT}{2-\mu} - \Gamma(2-\mu)(sT)^{\mu-1} = 1 + s\langle t\rangle - \Gamma(2-\mu)(sT)^{\mu-1} \tag{1.25}$$

Lévy function

Lévy's distribution is another function which can be used as asymptotic power law.

Lévy introduced his distribution looking for [19, 22]

Definition 1.1.1 (infinite divisible distributions)**.** Let $\varphi_X(t)$ be the characteristic function of a probability distribution $f(t)$ (i.e $\varphi_X(t) = \mathbf{E}_\mu(e^{i\omega X}$ where $X \sim \mu$). A probability function is said to be *infinitively divisible* if for any n there exists a probability measure ν whose characteristic function $\lambda_n(t)$ satisfies

$$\varphi_X(t) = (\lambda_n(t))^n.$$

Among all the infinite divisible distributions a particular class is of wide interest. To explore this point let us use

Definition 1.1.2 (Stable distribution)**.** A distribution is stable if it is stable under convolution that is if, for any a_1,a_2,b_1, b_2 there exist $a, b \in \mathbb{R}$, so that its characteristic function $\varphi_X(t)$ satisfies:

$$\varphi_{aX+b}(t) = \varphi_{a_1X+b_1}(t)\varphi_{a_2X+b_2}(t)$$

If two random variables X_1 and X_2 are distributed according to a stable distribution the their sum (rescaled and translated) is also distributed with the same distribution.

The most widely known stable distribution is the Gaussian distribution. Levy and Kinchine have shown that the only possible attractors of probability distribution are stable distribution and Levy has given a canonical representation theorem

Theorem 1.1.1 (Lévy-Kintchine representation theorem)**.** *The most general form for the characteristic function $L_{\alpha,\beta}(k)$ of a stable is given by*

$$\ln L_{\alpha,\beta}(k) = i\gamma k - c|k|^\alpha \left(1 - \beta \frac{k}{|k|\omega(k,\alpha)}\right) \tag{1.26}$$

where

$$\omega(k,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) \ \textit{if } \alpha \neq 1 \\ \frac{2}{\pi}\ln|k| \ \textit{if } \alpha = 1 \end{cases} \tag{1.27}$$

where γ is arbitrary, $c > 0$, $\alpha \in [0, 2]$, $-1 < beta < 1$

Since γ and c are scale factors, the do not contribute to the shape of the distribution. α and β instead determines the shape Lévy distribution. The first exponent is called the characteristic exponent since it governs the asymptotic behavior of the distribution, we have in fact

for $0 < \mu < 2$ The (bilateral) Laplace transform of the distribution can be calculated since we have the characteristic function and expanding near origin we have

$$L_{\alpha,\beta}(is) = \hat{\psi}_L(s) \sim 1 - |k|^\alpha - \beta|k|^{\alpha-1}\frac{k}{\omega(k,\alpha)}$$

we get that

$$\psi_L(t) \to \pm\frac{1}{|x|^{1+\alpha}} \text{ for } x \to \pm\infty \tag{1.28}$$

for $\alpha = 2$ the function becomes a Gaussian distribution.

The β is called skewness parameter since it control the symmetry of the function. For $\beta = 0$ we obtain symmetric Lévy function, for $\beta = -1$ The distribution in concentrated in the half line $[\gamma, \infty]$

1.2 Mesoscopic phenomena and stochastic processes

Phenomena usually studied by physicists have well defined physical scales. Newtonian dynamics, classical (equilibrium) statistical physics and general relativity investigates macroscopical phenomena in which the fluctuations can be neglected, quantum physics investigates microscopical phenomena in which quantum fluctuations are not negligible any more. Recent advances in technologies (e. g. molecules tracking) have enabled scientists to investigate phenomena whose typical scale are not large enough that fluctuations due to microscopical dynamic can be totally neglected and still not small enough that a complete quantum mechanical treatment can be set up. For these phenomena the name of *mesoscopic phenomena* has been proposed.

The natural framework in which those systems are studied is that of stochastic processes. Stochastic processes can be seen as a kind of "microscopical phenomenological description" of the systems, that is, we take into account the microscopical dynamics through a fluctuating variable which describes it "phenomenologically".

This approach is obviously not new to physics: it has been, indeed, widely used in those fields which *ante tempora* studied phenomena we could today define as *mesoscopic* (e.g. non equilibrium statistical physics, Brownian motions etc.). We want to point, nonetheless, that recent advances in experimental techniques have enabled us to study an extremely rich variety of new systems and phenomena which cannot be interpreted from within the usual perspective applied to those disciplines. New "interpretational" paradigm are needed to describe these new fundamental phenomena (and some, like self-organized criticality have yet been provided and have manifestated a powerful exegetic strength).

1.2.1 Stochastic processes: some definitions

Usual mathematical description of probability is quite cumbersome, under certain aspects. Here we shall limit ourselves to state some definition a property needed further. As a reference books we have mainly used [44] and [49] (and also [18], [19]).

Definition 1.2.1 (Stochastic Process)**.** Let $L^1(X, \mu, \mathscr{B})$ a probability space, a *stochastic (or random) process* is collection of stochastic variables $\{X_t\}_{t\in T}$ parameterized over a set T and assuming values in $\mathbb{R}^n$

If $T = \mathbb{R}$ then we shall call the process a *continuous time stochastic process*. If $T = \mathbb{N}$ then we shall call the process a *discrete time stochastic process*

This definition enables us to translate every concept we already have on stochastic variables to stochastic processes

Definition 1.2.2 (Finite dimensional distributions). Given a *a stochastic process* $\{X_t\}_{t\in T}$ over the probability $L^1(X,\mu,\mathscr{B})$, for any finite dimensional set of indexes $\{t_1,\dots,t_k\}$ we define we define the *finite dimensional distributions of the process* the sets of measures $\{\mu_{t_1,\dots,t_k}(F_1\times\cdots\times F_k)\}$ over $\mathbb{R}^{nk}$ defined by

$$\mu_{t_1,\dots,t_k}(F_1\times\cdots\times F_k)=\text{Prob}(X_{t_1}\in F_1\cap X_{t_2}\in F_2\cap\cdots\cap X_{t_n}\in F_n) \tag{1.29}$$

The mathematical definition of our process allows us to interpret them in tree different ways: either as random variables (i.e. measurable functions over our probability space $X_t : X\to\mathbb{R}^n$) or as functions defined over the set $T\times X$ (i.e. instead of interpreting it like $X_t(F)$ we look at it as $X:(t,F)\in T\times X\to\mathbb{R}^n$). A third possibility is to interpret them as model functions for physics problems.

Definition 1.2.3 (Path). For any Fixed $F\in\mathscr{B}$ we call the function

$$f_F(t)=X(t,F) \tag{1.30}$$

a *path* of our process.

As usually done for any set of variables, we can define some statistical properties for our processes. Two statistical properties are particularly important :

Definition 1.2.4 (Mean). Let $\{X_t\}_{t\in T}$ be a process on the probability space $L^1(X,\mu,\mathscr{B})$, and let μ_t the 1-dimensional distribution as defined in definition 1.29. We call the function

$$\mu(t)=\mathbf{E}(X_t)=\int X_t\,\mathrm{d}\mu_t \tag{1.31}$$

the *mean* of the process

and

Definition 1.2.5 (Autocorrelation). Let $\{X_t\}_{t\in T}$ be a process on the probability space $L^1(X,\mu,\mathscr{B})$ and let $\mu_{t,s}$ the 2-dimensional distribution as defined in definition 1.29 we define the function

$$C(t,s)=\mathbf{E}(X_tX_s)=\int X_tX_s\,\mathrm{d}\mu_{t,s} \tag{1.32}$$

the *autocorrelation* of the process.

Among the great variety of processes a particular class of continuous time processes are very important and are characterized by Markov Property

Definition 1.2.6 (Markov Processes). Let $\{X_t\}_{t\in T}$ be a process on the probability space $L^1(X,\mu,\mathscr{B})$, We say that the process is a *Markov process* if, for any set of indexes $\{t_1,\dots,t_{k-1},s\}\in\mathbb{R}^k_+$ so that $t_1<t_2<\cdots<t_{k-1}<s$ the process has the *Markov Property*

$$\text{Prob}(X_s\in F_s\mid X_{t_{k-1}}\in F_{t_{k-1}}\cup\cdots\cup X_{t_1}\in F_{t_1})=\text{Prob}(X_s\in F_s\mid X_{t_{k-1}}\in F_{t_{k-1}}) \tag{1.33}$$

Using the definition of conditional probability and definition of finite dimensional distribution we get the previous property translates

$$\mu_{s,t_{k-1},\dots,t_1}(G \times F_{k-1} \times \cdots \times F_1) = \frac{\mu_{s,t_{k-1}}(G \times F_{k-1}) \cdot \mu_{t_{k-1},t_{k-2}}(F_{k-1} \times F_{k-2}) \cdot \ldots \cdot \mu_{t_2,t_1}(F_2 \times F_1)}{\mu_{t_{k-1}}(F_{k-1}) \cdot \ldots \cdot \mu_{t_2}(F_2)} \tag{1.34}$$

that is the 1 and 2 dimensional distributions totally determines the process.

If $\mu_{t,s}(F,G) = \mu_{t-s,0}(F,G)$ the Markov process will be called *time homogenous*, otherwise *time inhomogeneous*.

Since we are interested in modeling physical systems we content ourselves to choosing $\mathbb{R}$ as basic space . We, moreover, will assume that the probabilities involved can be expressed in terms of their probability density functions (to be true with a slight abuse of notation we shall consider among those densities also the tempered distributions like Dirac's $\delta(t)$).

In the following, if nothing is otherwise expressed, we shall indicate the processes simply by X_t or even $X(t)$ where no confusion is possible and the probability density function of our process will be simply denoted like $p(x,t) = \mathrm{Prob}(X_t \in [x, x + \mathrm{d}x])$. Conditional probability analogously will be written like $p(x,t \mid yt')$.

Markov Chains

Among all Markov processes an important class shall be studied more accurately

Definition 1.2.7 (Markov Chain)**.** A Markov process $\{X_t\}$ over a countable (of finite) subset of $\mathbb{R}$ is called a *Markov Chain*.

Since the possible outcomes the stochastic process can give are countable we shall use a more comfortable notation. We label each possible outcome which we shall refer to as *state*, with an integer i and denote the probability of the state i to occur at time t with $\pi_i(t)$.

We will call the *transition probability* $W_{ij}(t,s)$ the conditional probability $\mathrm{Prob}(i,t \mid j,s)$. Obviously $W_{ij}(t,t) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol.

If Markov property holds we can write

$$\pi_i(t) = \sum_j W_{ij}(t,s)\pi_j(s) \tag{1.35}$$

It straightforward to derivate some important properties like

Proposition 1.2.1 (Chapman-Kolmogorov-Smoluchovski)**.** *For all i, j and for all $s < u < t$ the transition probabilities satisfies*

$$W_{ij}t,s = \sum_k W_{ik}(t,u)W_{kj}(u,s) \tag{1.36}$$

We shall show that under certain regularity hypothesis a differential system of equation can be obtained.

Let us assume that $W_{ij}(t + \mathrm{d}t \mid t) = \delta_{ij} + K_{ij}(t)\,\mathrm{d}t + o(\mathrm{d}t^2)$ and $\pi_i(t + \mathrm{d}t) = \pi_i(t) + \frac{\mathrm{d}}{\mathrm{d}t}\pi(t)\,\mathrm{d}t + o(\mathrm{d}t^2)$. We can thus write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_i(t)\,\mathrm{d}t = \sum_j K_{ij}(t)\,\mathrm{d}t\pi_j(t). \tag{1.37}$$

If the series of $K_{ij}p_j(t)$ still converge we can take limits and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_i(t) = \sum_j K_{ij}(t)\pi_j(t), \tag{1.38}$$

which we shall call *time inhomogeneous Master equation*.

If the Markov Chain is *time homogenous* the transition probability will satisfy $W_{ij}(t, s) = W_{ij}(t - s, 0)$ and thus $W_{ij}(t + \mathrm{d}t, t) = W_{ij}(\mathrm{d}t, 0)$.

If $W_{ij}(\mathrm{d}t, 0) \to \delta_{ij}$ at least linearly in $\mathrm{d}t$ we can take the limit for $\mathrm{d}t \to 0$ and we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_i(t) = \sum_j K_{ij}\pi_j(t) \tag{1.39}$$

where $K_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}W_{ij}(t)\big|_0$. Equation 1.39 is usually known in physics as *master equation*.

Since $\pi_i(t)$ are probability, we have to request that $\sum_i \pi_i(t) = 1$ for every time. If $\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \pi_i(t) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\pi_i(t)$ we have that $\sum_i \sum_j K_{ij}\pi_j(t) = 0$. If we change the order of summation[1] we obtain that the K_{ij} satisfy the request

$$\sum_i K_{ij} = 0. \tag{1.40}$$

Under these conditions ($K_i i = -\sum_{i \neq j} Kij$) we thus restate the master equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_i(t) = \sum_{j \neq i} K_{ij}\pi_j(t) - K_{ij}\pi_i(t). \tag{1.41}$$

The physical interpretation of master equation is now clear. If we interpret the p_i as the occupation probability of a state i, $K_{ij}\pi_j(t)$ measures the occupation growth of state i due to particles that leave the state j to go to the state i, $K_{ij}\pi_i(t)$ measures, instead, the decrease of occupation of state i due to particles that leave state i to go to state j.

A Markov chain is said to have reached *equilibrium* if its probability distribution is time independent.

If our Markov chain satisfies a master equation an equilibrium exists if $\sum_j K_{ij}\pi_j(t) = 0$. If the states are infinite we cannot establish a priori if an equilibrium exists. If we are dealing with a finite state homogenous Markov Chain the existence of equilibrium is guaranteed. We have, in fact, that K_{ij} are the entries of a matrix K and the π_i can be thought as the elements of a vector $\boldsymbol{\pi}$. Since equation 1.40 says that one raw it a linear combination of the other ones, we have that Ker(K) contain at leat one possible equilibrium solution.

[1] the hypothesis we have made are trivially true for finite dimensional Markov chains

Discrete time Markov Chain

The previous discussion cannot be carried out for discrete time Markov Chains since limits are not allowed. This is not a big deal. We shall repeat our discussion to obtain a similar result.

Probability transition will now depend on to discrete indices $W_{ij}(n,m)$ with $W_{ij}(n,n) = \delta_{ij}$. In this case Markov Property translates nicely since in the discrete case there is a "last step before", that is :

$$\pi_i(n) = \sum W_{ij}(n,k)\pi_j(k) = \sum W_{ij}(n,n-1)\pi_j(n-1). \tag{1.42}$$

We can define $K_{ij}(n-1) = W_{ij}(n,n-1) - \delta_{ij}$ and restate the previous condition as

$$\pi_i(n) = \sum_j (\delta_{ij} + K_{ij}(n-1))\pi_j(n-1), \tag{1.43}$$

which is the discrete analog of inhomogeneous Master equation.

If the process it finite dimensional we can adopt a vector form that is a matrix $(\mathrm{K})_{ij} = K_{ij}$ and a vector $\boldsymbol{\pi}(n)$ whose components are $\pi_i(n)$. The *discrete time master equation* then become:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) + \mathrm{K}(n)\boldsymbol{\pi}(n-1). \tag{1.44}$$

Thus

$$\boldsymbol{\pi}(n) = \Pi(n)\boldsymbol{\pi}(0) \tag{1.45}$$

where $\Pi(n) = \prod_j (\mathrm{I} + \mathrm{K}(j))$ is called the *propagator* of the system.

For time homogenous discrete Markov Chain we have $\Pi(n) = \Pi(0)^n$.

Coin tossing

Fair Coin tossing may be seen as a Markov Chain. There are only two states, we denote them +1 and -1, and at each step the system can move to the state +1 with probability 1/2 or in the state −1 with probability 1/2.

The most general form the operator K fitting the constraint $\sum_i K_{ij} = 0$ is

$$\mathrm{K} = \begin{pmatrix} a & b \\ -a & -b \end{pmatrix}. \tag{1.46}$$

In order for that the equilibrium to be $(1/2, 1/2)$ we must set:

$$\mathrm{K} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}. \tag{1.47}$$

We notice that $\mathrm{K}^2 + \mathrm{K} = 0$. We refer to this throughout as a dichotomous process.

Dice throwing

A generalization of previous problem is that of a Markov chain in which the system has k states and at each time moves to one of those state with a given probability π_k. We call $\boldsymbol{\pi}(n)$ the vector of probability at time n, $\boldsymbol{\pi}_{eq}$ the steady distribution and K the transition matrix previously defined, we have

$$\boldsymbol{\pi}(1) = (\mathrm{I} + \mathrm{K})\boldsymbol{\pi}(0) = \boldsymbol{\pi}_{eq} \tag{1.48}$$

and

$$\boldsymbol{\pi}(2) = (\mathrm{I} + \mathrm{K})^2\boldsymbol{\pi}(0) = \boldsymbol{\pi}_{eq} \tag{1.49}$$

thus yielding the matricial equations:

$$\mathrm{K}^2 + \mathrm{K} = 0 \tag{1.50}$$

and

$$\mathrm{K}\,\boldsymbol{\pi}_{eq} = 0 \tag{1.51}$$

We shall refer to this throughout as multichotomous process.

Random Walk

Another important example of discrete Markov Chain is random walk. In this case we chose our transition probability to be constant and have

$$W_{ij} = p\delta_{ij-1} + q\delta i, j+1 \text{ with } p + q = 1 \tag{1.52}$$

The master equation then reads

$$\pi_i(n) = p\pi_{i-1}(n-1) + q\pi_{i+1}(n-1) \tag{1.53}$$

the solution is easily obtainable and has a typical Bernoulli distribution

$$\pi_i(n) = \binom{n}{n-2i} p^i q^{n-2i} \tag{1.54}$$

1.3 Subordination theory and renewal processes

As we have earlier pointed out, we are mainly interested in systems which exhibit complex behavior characterized by the presence of abrupt transition (which we call "events") between two or more state, with a power law distribution density of the time distance between two consecutive events.

A benchmark characteristic of those system it *ageing*, that is the system maintains a *memory* of the moment of preparation.

The theoretical frame which better suits the description of these systems is that of subordination theory of renewal systems. A substantial treatment of this topic in advanced probability theory can be found in Feller's work([18]and [19]) and a general review on renewal theory can be found in Cox' work [15]

We shall first illustrate Blinking quantum dots behavior as a prototype of the systems we are interested in.

Blinking quantum dots: a prototypical system

Quantum dots (nanocrystals of semiconductors) are intensively studied since they seem to promise great applications like light emissive diodes, solid state lighting, lasers. Investigation on their nature has pointed out some characteristic behavior: fluorescence intermittency [43].

This intermittency, subsequently called *blinking*, still has a non completely understood microscopical origin (even if many interpretation have been advanced) and constitutes a major problem to be solved to be able to use semiconductors nanocrystals at their best.[2]

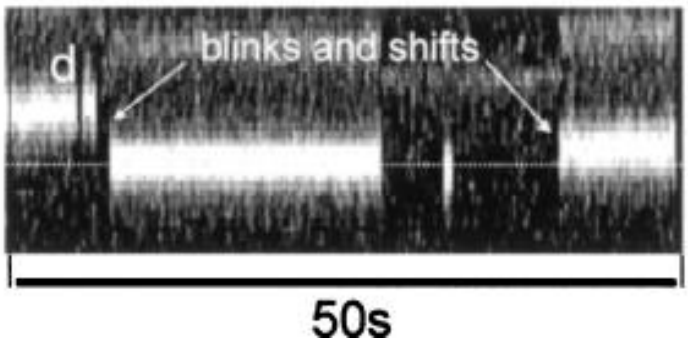


Figure 1.4: A figure taken from [41] which shows the typical blinking behavior of quantum dots

The behavior of the blinking is complex; that is, it cannot be described by a poissonian waiting time distribution. As shown in [13] this typical behavior cannot be interpreted as a consequence of slow modulation of parameters, since non ageing is possible within the framework of this theory. Renewal subordination theory instead has, as a benchmark, that of showing ageing.

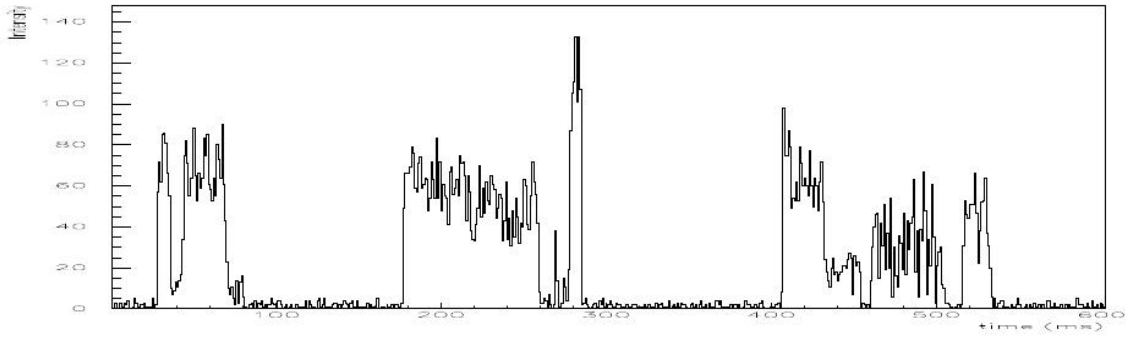


Figure 1.5: A figure taken from [13] which shows the typical intensity over time behavior of quantum dots

Figure 1.5 shows a typical intensity over time fluctuation of blinking quantum dots. Analysis of those data shows that the permanence time is a random variable which is roughly

[2] recently a possible solution to this problem has been proposed [20] but still it does not unveil the nature of this behavior

distributed like $\sim \frac{1}{t^{\mu}}$.

To model these systems we imagine that the interaction among units has the effect of creating abrupt transitions from one state to another. This is equivalent to assume that the process is modeled by a coin tossing Markov chain. The time between two tossings, due to complex interactions, is not constant any more but is distributed according to a power law.

This is the basic idea of subordination theory and we shall analyze it now.

1.3.1 Subordination theory

We start with a definition

Definition 1.3.1 (Subordinated process)**.** Let $\{X_n\}$ be a discrete time stochastic process defined over $\mathbb{R}$, and $\{T_n\}_{n>0}$ a discrete time stochastic process defined over $\mathbb{R}^+$. We defined the subordinated process of X_n to T_n the continuous time process $\xi(t)$ defines

$$\xi(t) = \begin{cases} X_0 \text{ if } t < T_1 \\ X_n \text{ if } T_n < t < T_{n+1} \end{cases} \tag{1.55}$$

We shall call $\{X_n\}$ the *leading process* and $\{T_n\}_{n>0}$ the *subordination generating process*.

We want to point out that this is a *mathematical model*. The only physical process is the result of subordination $\xi(t)$ and both the leading process and the subordination generating process are phenomenological description of collective interaction.

Surely in certain cases we can give the leading process a microscopical interpretation, like the modeling of shocks in a ideal gas if we accept Boltzmann's Stosszahlansatz. The waiting time distribution, in this case, will have to be inferred from the statistical properties of ideal gases.

Complex phenomena do not allow, usually, such a simple interpretation in terms of local microscopical vs. global macroscopical behavior, since both processes involved in subordination structure emerge from cooperative global interactions. The distinction is rather made in term of the effects both processes give rise to (i.e. the distinction we make is an *a posteriori* phenomenological one that enables us to propose a model).

Another fact that should be stressed is that subordination is a key mechanism to explore cooperative systems which could also not be simple fundamental physical systems. Studies in neural and social network have shown to exhibit this characteristic behavior (which happens to be tunable moreover).

Independent Increment Processes and renewal processes

The previous definition is rather general. We could choose as *subordination generating process* an arbitrary one. The first simplification we shall make is that of considering :

Definition 1.3.2 (Independent increment processes)**.** A stochastic process $\{X_t\}$ is said to be an *independent increment process* if for any $s < t < w < u$ $X_t - X_s$ and $X_u - X_w$ are independent variables.

The previous condition implies that the probability density function of the interval depends only on the time difference (i.e. $\text{Prob}(X_t - X_s \in [x, x + \mathrm{d}x]]) = f(x, t - s)$). Among all independent increment processes we are particularly interested in *renewal processes*

Definition 1.3.3 (Renewal process). A discrete time independent increment process $\{T_n\}$ defined on $\mathbb{R}^+$ is called a *renewal process*

The reason why this kind of processes are called renewal processes will be clarified in next section. We first point out the most important property of those systems.

A renewal process $\{T_n\}$ is totally determined by only one distribution

$$\psi(t) = \text{Prob}(T_{n+1} - T_n \in [t, t + \mathrm{d}t]) = \text{Prob}(T_1 - T_0 \in [t, t + \mathrm{d}t]) = f(t, 1) \tag{1.56}$$

which in the contest of subordination theory we shall call *waiting time distribution*.

The independent increment process condition enables us, in fact, to obtain all the other conditions immediately. If we define the events

$$A(t, n) = \{T_n - T_0 \in [t, t + \mathrm{d}t]\} \tag{1.57}$$

by the independent increment hypothesis those events are independent. In particular the event $A(t, n)$ can be split recursively in this way:

$$A(t, n) = \bigcup_{t'} \Big[A(t - t', 1) \cap A(t', n - 1) \Big], \tag{1.58}$$

that is, we consider the probability of having the n-th element in $[t, \mathrm{d}t]$ as probability to find the $n - 1$-th in $t' < t$ and that last interval has the length $t - t'$ (obviously this works because $T_n \in \mathbb{R}^+$) we immediately write

$$\psi_n(t) = \text{Prob}(A(t, n)) = \text{Prob}(\bigcup_{t'} [A(t - t', 1) \cap A(t', n - 1)]) = \int_0^t \psi_{n-1}(t')\psi(t - t')\,\mathrm{d}t'. \tag{1.59}$$

If we consider the Laplace transform (for the mean properties look Appendix) of the waiting time distribution $\hat{\psi}(u)$ the Laplace transform of $\psi_n(t)$

$$\hat{\psi_n}(u) = \hat{\psi}^n(u). \tag{1.60}$$

Renewal hypothesis: waiting time distribution as renewal failure time distribution

To clarify why discrete time positive valued independent increments processes are called *renewal processes* we have to think waiting time distribution as a failure time distribution. In its classical Monograph on Renewal Theory ([15]) Cox gives a simple but insightful description of what a renewal process is.

Renewal theory is originally linked to the study of probabilistic problems connected with the failure and replacement of components. typical terminology could sound a little weird to a scientist's ear, but we shall use it, for now, to let reader to easily find it in specialistic literature.

Let us then think to a robotized assembly line. It will work efficiently if all its components are working. But even the best constructed robot will endure soon or later some failure problems (due to wear e.g.). We now think that every time a robot fails it is immediately and completely restored in a perfectly working state. This is called the *renewal hypothesis*

For simplicity sake (and we are actually interested in these kinds of mechanism) we consider a single robot line.

We can model the failure probability of these robots as real positive random variable T called *failure time*. This failure time give rise to a failure time distribution $f(t)$. The probability for the system not to break if called *survival probability* and has the obvious expression

$$\Psi(t) = \text{Prob}(T > t) = \int_t^\infty f(t')\,\mathrm{d}t'. \tag{1.61}$$

We can construct a discrete time process setting $\{T_n\}$ is the time of the n-th failure and renewal. This is, by construction, a discrete time positive values independent interval process as previously defined and now the reason why it's called *renewal process* is clear.

Renewal hypothesis make us able to give a nice description of failure time distribution $f(t)$

Let us consider a key property of renewal processes called *failure rate*:

$$g(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}(T \in [t, t+\Delta t] | t < T)}{\Delta t}. \tag{1.62}$$

Since $\text{Prob}(T \in [t, t+\Delta t] | t < T) = \frac{\text{Prob}((T \in [t, t+\Delta t]) \cap T > t)}{\text{Prob}\, T > t}$ we get

$$g(t) = \frac{f(t)}{\Psi(t)}. \tag{1.63}$$

By definition $\Psi(t) = -\frac{\mathrm{d}}{\mathrm{d}t} f(t)$. Therefore:

$$g(t) = -\frac{\Psi'(t)}{\Psi(t)} = -\frac{\mathrm{d}}{\mathrm{d}t} \log \Psi(t) \tag{1.64}$$

integrating equation 1.64 and noticing that $\Psi(0) = 1$, we obtain

$$\Psi(t) = \exp(-\int_0^t g(t')\,\mathrm{d}t'). \tag{1.65}$$

Therefore, a renewal-process occurrence time is completely characterized by its failure rate. Equation 1.65 enables us to make some analysis on $g(t)$.

- $g(t) = 0$ in this the mean failure rate is constant and we obtain $\Psi(t) = \exp(-gt)$ and subsequently $f(t) = g\exp(-gt)$: this is the case of *Poissonian* failure time distribution. This is the typical " failure" mechanism in traditional physics (e.g. radioactive decay, usual statistical physics phenomena etc.)

- $g(t) \sim At^\alpha$ **with** $\alpha > 0$ In this case we get a probability whose queues are super exponentially depressed $\sim exp{-}Bt^{\alpha+1}/(\alpha+1)$

- $g(t) \sim A * t^{\alpha}$ **with** $1 < \alpha < 0$ In this case we get sub exponential distribution which asymptotically give rise to what are called *stretched exponentials distribution* $\Psi(t) \sim exp(-At^{\gamma}/\gamma)$ with $\alpha + 1 = \gamma \in [0, 1]$

- $g(t) \sim A/t$ In this case we get power laws in fact $\Psi \sim \exp 8 \log(t^{-A})) = \frac{1}{t^A}$

- $g(t) \sim t^{\alpha}$ **with** $\alpha < -1$ In this case we find the construction is impossible since it would lead to immortality that is the $f(t)$ is not normalized to 1

This analysis shows than that the power laws are a limiting case of failure time distributions that seem to correctly model *sporadicity*.

We notice, moreover, that $g(t)$ is not constant so the failure rate, that is the probability of decaying, changes over time: the system is ageing in the sense that from an estimation $g*$ of $g(t)$ we can get an estimation of the age (i.e. the time elapsed since last failure) of the system $g^{-1}(g*)$

Specific choice of $g(t)$ let us to derive the power laws we have already presented.

If $g(t) = \frac{r_0}{1+r_1 t}$ we obtain $\Psi(t) = (r_1 + t)^{-r_0/r_1}$. If we call $\mu = 1 - \frac{r_0}{r_1}$ and $T = r_1$ we obtain back Manneville's distribution 2.60.

More complicated (i.e. non analytical) choices lead to Lévy and Mittag-Leffler distributions.

The rate of event per unit time

Let us consider the random variable

$$N(t) = \#\text{ events occurred in } [0, t] \tag{1.66}$$

we may ask what is the mean number of event. This calculation is easily carried out if we notice that the probability of having n events before time t that is $B(n, t) = \{n \text{ events have occurred before time } t\}$ can be split using using 1.58

$$\text{Prob}(B(n, t)) = \text{Prob}(\bigcup_{t'}(A(n, t') \cap A(0, t')) = \int_0^t \psi_n(t - t')\Psi(t')\,\mathrm{d}t' \tag{1.67}$$

and thus the mean is easily written out:

$$H(t) = \mathbf{E}(N(t)) \sum_n n\,\text{Prob}(B(n, t)) = \sum_{n=0}^{\infty} \int_0^t n\psi_n(t - t')\Psi(t')\,\mathrm{d}t' \tag{1.68}$$

Using Laplace transform we have :

$$H(u) = \frac{1 - \hat{\psi}(t)}{u} \sum_{n=0}^{\infty} \int_0^t n\hat{\psi}^n = \frac{1 - \hat{\psi}(u)}{u}\psi(u)\frac{\mathrm{d}}{\mathrm{d}\hat{\psi}(u)} \sum_{n=1}^{\infty} \hat{\psi}^n(u) = \frac{1}{u}\frac{\hat{\psi}(u)}{1 - \hat{\psi}(u)} \tag{1.69}$$

and thus

$$H(t) = \int_0^t \sum_{n=0}^{\infty} \psi_n(t')\,\mathrm{d}t'. \tag{1.70}$$

We can now define a crucial quantity for our renewal processes *mean rate of events* :

$$R(t) = \frac{\mathrm{d}}{\mathrm{d}t}H(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{E}(N(t)) = \sum_{n=1}^{\infty}\psi_n(t) \tag{1.71}$$

Another way to understand what $R(t)$ is, can be that of considering the event $E = \{$ an event occur at time $t\,\}$. In can be easily be split into an union of independent event that is

$$\mathrm{Prob}(E) = \mathrm{Prob}(\bigcup_n A(n,t))\,\mathrm{d}t = \sum_{n=1}^{\infty}\psi_n(t)\,\mathrm{d}t = R(t)\,\mathrm{d}t. \tag{1.72}$$

For Manneville power law, using Tauberian Theorem and asymptotic expansion 1.24 and 1.25 we write for $1 < \mu < 2$

$$R(u) \sim \frac{(uT)^{1-\mu}}{\Gamma(2-\mu)} \tag{1.73}$$

and thus

$$R(t) \sim \frac{1}{T^{\mu-1}\Gamma(2-\mu)\Gamma(\mu-1)}. \tag{1.74}$$

and for $2 < \mu < 3$

$$R(u) \sim \frac{1}{\langle t\rangle u} + \frac{(uT)^{1-\mu}}{\Gamma(2-\mu)} \tag{1.75}$$

and thus

$$R(t) \sim \frac{1}{\langle\tau\rangle}[1 + \frac{T^{\mu-2}}{3-\mu}\frac{1}{t^{\mu-2}}]. \tag{1.76}$$

Subordinated renewal processes

We can now completely analyze subordinate renewal processes. Our analysis is based on the seminal works of Montroll and Weiss on Continuous Time Random Walk (CTRW) [40] .

Our end is to obtain the the distribution of the subordinated process $\pi(\xi, t)$.

The key idea is to consider that according to our definition, the to processes are independent. Since we know by hypothesis the distribution of the leading process $p(x, n)$ and our waiting time distribution we have everything. In fact the pdf

$$\begin{aligned} p(\xi,t)\,\mathrm{d}\xi = \mathrm{Prob}[&(X_0 \in [\xi, \xi+\mathrm{d}\xi] \cap \text{ no event occurred until } t) \cup \cdots \\ &\cup (X_n \in [\xi, \xi+\mathrm{d}\xi] \cap \text{ exactly } n \text{ events occurred before time} t \cup \cdots] \end{aligned} \tag{1.77}$$

Since, by independence we can write

$$p(\xi,t)\,\mathrm{d}\xi = \mathrm{Prob}(\bigcup_n (B(n,t) \cap X_n \in [\xi, \xi+\mathrm{d}\xi) = \sum_n \mathrm{Prob}(B(n,t))\pi(\xi,n)\,\mathrm{d}\xi \tag{1.78}$$

Now we have all the pieces of information needed to write (which sometimes known as Montroll-Weiss equation)

$$p(\xi,t) = \sum_{n=0}^{\infty}\int_0^t \psi_n(t-t')\Psi(t')\pi(\xi,n)\,\mathrm{d}t'. \tag{1.79}$$

This is the most general form Montroll-Weiss equation can take unless we make some other hypothesis on our system.

Generalized master equation

If $\{X_n\}$ is a finite time homogenous discrete Markov Chain, adopting our shortcut notation [3] we can write:

$$\mathbf{p}(t) = \sum_{n=0}^{\infty} \int_0^t \psi_n(t-t')\Psi(t')\Pi(0)^n \, \mathrm{d}t' \boldsymbol{\pi}(0). \tag{1.80}$$

Taking the Laplace transform of both sides we write:

$$\hat{\mathbf{p}}(u) = \frac{1-\hat{\psi}(u)}{u} \sum_{n=0}^{\infty} (\psi(u)\Pi(0))^n \, \mathrm{d}t' \boldsymbol{\pi}(0). \tag{1.81}$$

Since both $|\hat{\psi}|$ and $\|\Pi(0)\|$ are less than 1 we can sum the geometrical series and considering that $\mathbf{p}(0) = \boldsymbol{\pi}(0)$, we have:

$$\hat{\mathbf{p}}(u) = \frac{1-\hat{\psi}(u)}{u} \frac{1}{1-\hat{\psi}(u)\Pi(0)} \mathbf{p}(0). \tag{1.82}$$

Defining $\mathrm{K} = \Pi(0) - I$ and rearranging we obtain:

$$u\hat{\mathbf{p}}(u) - \mathbf{p}(0) = \frac{u\hat{\psi}}{1-\hat{\psi}(u)} \mathrm{K}\, \hat{\mathbf{p}}(u). \tag{1.83}$$

Transforming back we obtain the *Generalized Master Equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \int_0^t \Phi(t-t') K \mathbf{p}(t') \, \mathrm{d}t', \tag{1.84}$$

where the quantity $\Phi(t)$ is called *memory kernel* and is defined by its Laplace transform:

$$\hat{\Phi}(u) = \frac{u\hat{\psi}(u)}{1-\hat{\psi}(u)}. \tag{1.85}$$

We thus see that subordination induces a loss of Markoviantity, that is, it introduces memory in the process.

Ageing

Renewal processes by derived by subordination are characterized by ageing. To see it let us suppose a renewal system is prepared at time 0, and our observation starts at time s. Obviously the first occurrence time is no more governed by our waiting time distribution. We have in fact (a graphical sketch can be found in picture 1.6) to find the distribution the event $O = \{$ the first observable event occur at time t given the system observation started at times$\}$.

As usually we can split, for any $(0 < t' < s < t$ this event as follows:

$$O = \bigcup_{t'} \bigcup_{n} (A(n,t') \cap A(1, t-t')) \tag{1.86}$$

[3] we label states by j and consider the probability vector $\boldsymbol{\pi}(n)$ of $\pi(j,n)$ and the vector $\mathbf{p}(t)$ of the probability $\mathrm{p}(i,t)$

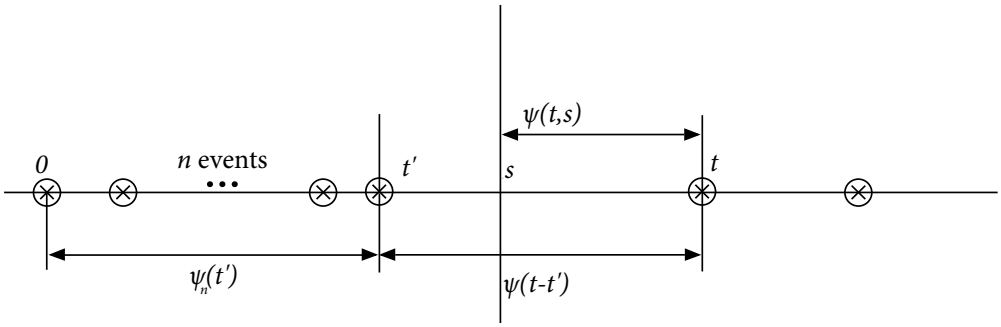


Figure 1.6: A visual sketch of aged waiting time calculation

that is, by disjunction and independence we can write the *waiting time distribution of age s*:

$$\psi(t,s) = \mathrm{Prob}(O) = \sum_{n=0}^{\infty} \int_0^s \psi_n(t')\psi(t-t')\,\mathrm{d}t' = \psi(t) + \int_0^s R(q)\psi(t-q)\,\mathrm{d}q. \tag{1.87}$$

We can associate the *survival probability of age s* integrating :

$$\Psi(t,s) = \int_t^{\infty} \sum_{n=0}^{\infty} \int_0^s \psi_n(t')\psi(t''-t')\,\mathrm{d}t'\,\mathrm{d}t''. \tag{1.88}$$

Changing the order of integration and using formula 1.71 we obtain:

$$\Psi(t,s) = \Psi(t) + \int_0^s R(t')\Psi(t-t')\,\mathrm{d}t' \tag{1.89}$$

We can obtain directly this prescription considering the stochastic failure rate that is :

$$r(t) = g(t-t_i) \tag{1.90}$$

In a certain way $r(t)$ represents the failure rate of the entire process see figure 1.7
Remembering the definition of g we can write:

$$\begin{aligned} \Psi(t,s) &= \left\langle \int_0^s \delta(q) + R(q)\mathrm{e}^{-\int_q^s r(\tau)\,\mathrm{d}\tau} \right\rangle = \\ \int_0^s (\delta(q) + R(q))\mathrm{e}^{-\int_q^s g(\tau)\,\mathrm{d}\tau}\,\mathrm{d}q &= \Psi(s) + \int_0^s R(q)\Psi(t-q)\,\mathrm{d}q \end{aligned} \tag{1.91}$$

We want to point that for poissonian processes we have

$$g(t) = r(t) = R(t) = \frac{1}{\langle t \rangle} \tag{1.92}$$

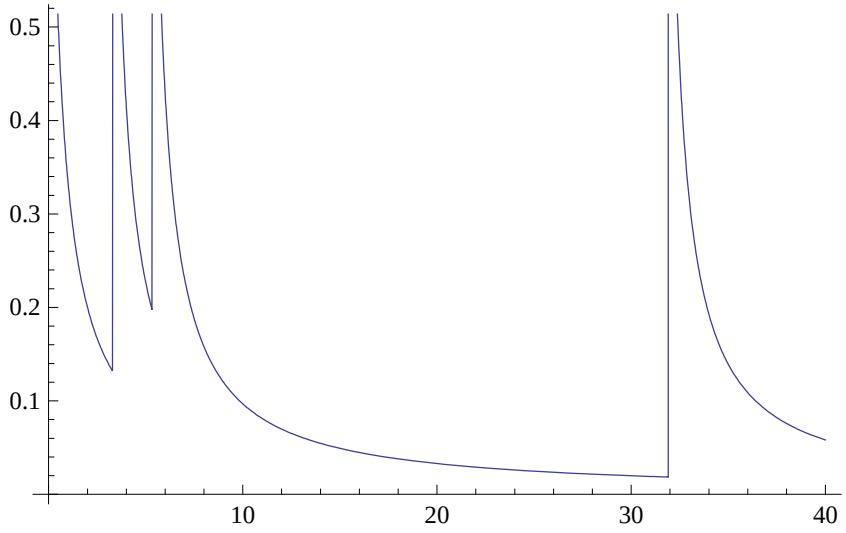


Figure 1.7: A simple example of $r(t)$ corresponding to equation $g(t) = \frac{1}{1+2t}$

2 Ergodicity, ergodicity breaking and non stationarity

The aim of this chapter is to introduce the reader to the concept of ergodicity as it has been conceived by physicists and mathematicians and to analyze some physical phenomena which exhibit an " ergodicity breaking'. It will be shown, in particular, that complex systems are likely to be considered "non ergodic systems"

2.1 Boltzmann's Ergodic hypothesis

Theory of "irreversibility' had always been a hard problem to deal with for physicists of XIX century. Clausius law, which had been proved by experiments, posed ha complicated problem. How can irreversibility arise from fundamental microscopical laws, which are time invariant?

It was not until the end of the century that a solution appeared, thanks to the work of Boltzmann.

Ludwig Boltzmann had yet began to organize his theories about irreversibility while building his kinetic theory. His H functional seemed, then, to provide a good mathematical instrument to show that irreversibility could be outputted by his kinetic theory but still he wasn't able to link his "phenomenological' theory to microscopical fundamental laws.

During the '70 and '80 of XIX century Boltzmann in his papers proposed his *Ergodic Hypothesis* as the foundations of his, then innovative, theory of *irreversibility*. The usual form under which ergodic hypothesis is stated nowadays is to be ascribed to Ehrenfest, who in a review of 1911 [59], stated it :

Boltzmann-Ehrenfest's Ergodic hypothesis A dynamical system during his evolution will take *all* the microscopical configurations compatibles with a given macroscopic state (i.e. a single trajectory will cover the whole phase space during his evolution)

To be true Boltzmann never stated his hypothesis this way, but he limited himself in assuming a "uniform probability" of phase space.

Conservative systems' evolution is known to follow Liouville's equation

$$\partial_t \rho = \mathscr{L} \rho \tag{2.1}$$

where $\rho(p_i, q_i) \prod_i \mathrm{d}^d p_i \, \mathrm{d}^d q_i$ is the measure on the phase space (MPS).

Liouville's theorem warrants us that time evolution preserves phase space measure. Normalizing MPS we get a probability space. Thus for conservative systems, this probability is

invariant under time evolution. This mean that we can "safely" consider *temporal means* of a variable f:

$$\lim_{t\to\infty} 1/t \int_0^t f(x(t))\,\mathrm{d}t \tag{2.2}$$

Under Boltzmann-Gibbs frame, we "introduce" a measure of our ignorance of the effective initial conditions of the system by defining a new space, the *Gibbs ensemble*, which is nothing but the set of infinite copies of the given dynamical system at fixed time, each one of which is the time evolute of one of all the possible compatible initial conditions. We associate a probability measure to each phase space configuration in the usual frequency limit way and use it to calculate averages.

Ergodic hypothesis is, roughly speaking, nothing but the assumption that MPS and Gibbs measure are the same that is, temporal mean and Gibbs average are the same.

As stated earlier, Boltzmann-Ehrenfest Hypothesis is proved to be false. The original Boltzmann hypothesis as been weakened and stated in a more "realistic" way:

Ergodic hypothesis (weak form) The set of values taken by a dynamical system is dense in the set of *all* the microscopical configurations compatible with a given macroscopic state.

Under this form, which has enabled mathematician to state and prove ergodic theorems, Ergodic hypothesis has been proved to hold for some dynamical systems but it is still not clear why it should be true for all. Moreover if warrants the existence of time and ensemble averages and their equivalence it has been shown that for an arbitrary observable the time needed to reach equilibrium is exponential in the number of elements of the system.

Most authors (i.e. Landau [30]) tend to diminish the importance of this hypothesis as a foundational hypothesis of Statistical physics and in recent years many examples of ergodicity breaking has been shown to exist.

We will show that the complex systems of our interest are non ergodic.

2.2 Mathematical theory of ergodicity and Brickhoff theorem

Mathematicians have tried to establish a well founded theory of ergodicity, during the XX century and have succeeded in establishing very powerful results, which are linked to mathematical theory of dynamical systems. Before correctly stating the main, and most known result of this theory, we have to give some preliminary definitions (see also [51]).

As we have shown in the previous chapter, Boltzmann's ergodic hypothesis allows us to associate to any system a probability space $L^1(X, \mathscr{B}, \mathrm{d}\mu)$ which describes how certain microscopical configurations lead to a given macroscopical configuration. In this theory the macroscopical value of a *dynamical variable* A is calculated as the mean $\langle A\rangle = \int A\,\mathrm{d}\mu$.

Traditionally a statistical dynamical system is described mathematically by a flow from a metric space to another

Definition 2.2.1. Flow Let X be a metric space, we define a flow over X a collection of maps $\{T_t : T_t : X \to X\}$ indexed over a given set I such that:

i. $T_t T_s = T_{t+s}$

ii. $T_0 = 1$

Generically mathematicians call *ergodic* any asymptotic property of a dynamical system expressed by a flow. To find any connection with the main problem of statistical physics we confine ourselves to consider *temporal means* of dynamical variable f (i.e. a L^1 function of dynamical variables)

The mathematical ergodic theory aims to analyze the temporal mean

$$\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(T_t x)\, \mathrm{d}t \tag{2.3}$$

and its relation with *spacial mean*

$$\langle f \rangle = \int_0^\infty f\, \mathrm{d}\mu \tag{2.4}$$

One of the most fundamental question of mathematical theory of ergodicity is to assess when the temporal mean of f is equal to its space mean.

2.2.1 Invariant Measure

Before we continuing our discussion we have to consider some definitions

Definition 2.2.2 (Invariant measure)**.** Let $L^1(X, \mathscr{B}, \mathrm{d}\mu)$ be a probability space. Then a measure is said to be *invariant* with respect to the flow $T : X \to X$ if $\mu(A) = \mu(T^{-1}A)$

An obvious characterization of invariant measure is the following :

Lemma 2.2.1. *A map T preserves μ if and only if $\int f\, \mathrm{d}\mu = \int T \circ f\, \mathrm{d}\mu$ for all in $L^1(X, (B), \mu)$*

A trivial generalization of the previous definitions can be obtained for flows

Definition 2.2.3 (Invariant measure)**.** Let $L^1(X, \mathscr{B}, \mathrm{d}\mu)$ be a probability space. Then a measure is said to be *invariant* with respect to the flow $T_t$for t in I if $\mu(A) = \mu(T_t^{-1}A)$ for all t in I

From now on we shall consider $I = \mathbb{N}$ and so $T_n = T^n$. Obviously if a temporal mean exists we can confine ourselves to consider discrete flows. In this case invariance for flows is simply T invariance.

Let us state one of the most fundamental results of ergodic theory:

Theorem 2.2.1 (Poincaré Recurrence Theorem)**.** *Let $T : X \to X$ be a measurable transformation on a probability space $L^1(X, \mathscr{B}, \mu)$ preserving μ. Let $A \in \mathscr{B}$ so that $\mu(A) > 0$; then for almost all points $x \in A$ the orbit $\{T^n x\}_{n\geq 0}$ returns to A infinitely many often*

Proof. Let us define the set

$$F = \{x \in A : T^n x \notin A, n > 0\} \tag{2.5}$$

First we note that $T^{-n}A \cap T^{-m}A = \varnothing$, for $n > m$. Where it not, we would have for $w \in T^{-n}A \cap T^{-m}A$, $T^m w \in F$ and $T^{n-m}(T^m w) \in A$ contradicting our hypothesis. We can thus write

$$\sum_n \mu(T^{-n}F) = \mu(\cup_n T^{-n}F) \leq 1 \tag{2.6}$$

but μ is T-invariant and so equation 2.6 can hold only if $\mu(F) = 0$ □

2.2.2 Ergodic measures and Birkhoff's theorem ergodic and invariant version

A stronger property is needed to establish Birkhoff theorem

Definition 2.2.4 (Ergodic measure)**.** Let $L^1(X, \mathscr{B}, \mathrm{d}\mu)$ be a probability space. Then a measure is said to be *ergodic* with respect to $T : X \to X$ if for every set $B \in \mathscr{B}$ with $B = T^{-1}B$, $\mu(B) = 0$ o $\mu(B) = 1$

As previously we can characterize ergodic measure in a simple way

Lemma 2.2.2. *A map T is ergodic with respect to μ if and only if for every $f \in L^1(X, \mathscr{B}, \mu)$, $f = T \circ f$ implies f be constant.*

Now we can state the first version of Birkhoff theorem

Theorem 2.2.2 (Birkhoff theorem)**.** *Let $f \in L^1(X, \mathscr{B}, \mu)$. If μ is ergodic then*

$$\lim_{N\to\infty} \frac{1}{T} \sum_{n=0}^{N} f(T^n x) = \int f \, \mathrm{d}\mu \tag{2.7}$$

for almost every x in X

This demonstration is quite technical and not very significant on a pysical point of view. Assuming without loss of generality that $\int f \, \mathrm{d}\mu = 0$, if it is not so we can substitute f with $f - \int f \, \mathrm{d}\mu$. The main idea of this demonstration is to show that the set defined:

$$E_\varepsilon(f) = \{x \in X : \limsup_{N\to\infty} \frac{1}{N} \mid \sum_{n=0}^{N-1} f(T^n x) \mid \geq \varepsilon\} \tag{2.8}$$

has null measure (i.e. $\mu(E_\varepsilon(f) = 0)$.

We first prove two sublemmas

sublemma 2.2.2.1. $\mu(E_{2\varepsilon}(f)) \leq \frac{\inf|f|\mathrm{d}\mu}{\varepsilon}$

Proof. Defining $f = f_+ - f_-$ where $f_+(x) = \max(f(x), 0)$ and $f_- = \max(-f(x), 0)$. Obviously $| f | = f_+ + f_-$. Now we define

$$E_\varepsilon^M(f_+) = \{x \in X : \exists 1 \leq N \leq M, \sum_{n=0}^{N-1} f_+(T^n x) \geq \varepsilon N\} \tag{2.9}$$

and

$$E_\varepsilon^M(f_-) = \{x \in X : \exists 1 \leq N \leq M, \sum_{n=0}^{N-1} f_-(T^n x) \geq \varepsilon N\} \tag{2.10}$$

for $M \geq 1$.

If we consider that:

$$\sum_{n=0}^{P-1} f_+(T^n x) \geq \varepsilon \sum_{j=0}^{P-M} \chi_{E_\varepsilon^M(f_+)}(T^j x) \tag{2.11}$$

and

$$\sum_{n=0}^{P-1} f_-(T^n x) \geq \varepsilon \sum_{j=0}^{P-M} \chi_{E_\varepsilon^M(f_-)}(T^j x) \tag{2.12}$$

where we have bounded f from below by o or ε. Thus, integrating both sides of 2.11 and 2.12,we write:

$$\int \sum_{n=0}^{P-1} f_+(T^n x)\, \mathrm{d}\mu(x) = P \int f_+ \, \mathrm{d}\mu \geq \varepsilon(P - M)\mu(E_\varepsilon^M(f_+)) \tag{2.13}$$

and analogously:

$$\int \sum_{n=0}^{P-1} f_-(T^n x)\, \mathrm{d}\mu(x) = P \int f_- \, \mathrm{d}\mu \geq \varepsilon(P - M)\mu(E_\varepsilon^M(f_-)) \tag{2.14}$$

for all $M \geq 1$.

When $P \to \infty$ we have:

$$\int f_\pm \, \mathrm{d}\mu \geq \varepsilon\mu(E_\varepsilon^M(f_\pm)) \tag{2.15}$$

and thus

$$\mu(E_{2\varepsilon}(f) \leq \limsup_{M\to\infty} \mu(E_\varepsilon^M(f_+)) + \limsup_{M\to\infty} \mu(E_\varepsilon^M(f_-)) \leq \int f_+ \, \mathrm{d}\mu + \int f_- \, \mathrm{d}\mu. \tag{2.16}$$

□

Now we need to be able to control the size of the higher bound and to do this we can we prove this second lemma

sublemma 2.2.2.2. *If $\int f \, \mathrm{d}\mu = 0$, then, for every $\delta \geq 0$ there exists a function $h \in L^\infty(X, \mathscr{B}, \mu)$ for which $\int | f - (hT - h) | \, \mathrm{d}mu < \delta$*

Proof. Let S be defined by:

$$S = \{h \circ T - h : h \in h \in L^\infty(X, \mathscr{B}, \mu)\} \tag{2.17}$$

and the B_0

$$B_0 = \{f \in L^1(X, \mathscr{B}, \mu) : \int f \, \mathrm{d}\mu = 0\}. \tag{2.18}$$

. We first show that S is dense in B_0. Hann Banach theorem guarantees us we only need to show that every null functional on S is also a null functional on B_0 .

As known for every functional $\alpha(f)$ defined on $L^1(X, \mathscr{B}, \mu)$, there exists a function $k \in L^\infty(X, \mathscr{B}, \mu)$ so that $\alpha(f) = \int f \cdot k \, \mathrm{d}\mu$ Now let us suppose that α vanishes on S thus $\int (h \circ T - h) \cdot k \, \mathrm{d}\mu = 0$ if $h = k$ we have $k \cdot (kT)k = \int k^2 \, \mathrm{d}\mu$

We can then write:

$$\int (k \circ T - k)^2 \, \mathrm{d}\mu = \int (k \circ T)^2 \, \mathrm{d}\mu + \int k^2 \, \mathrm{d}\mu - 2 \int (k \circ T) k \, \mathrm{d}\mu = 2\left(\int k^2 \, \mathrm{d}\mu - \int (k \circ T) \cdot k \, \mathrm{d}\mu\right) = 0 \tag{2.19}$$

We have that $k = k \circ T$ and so k must be constant by ergodicity hypothesis. We can thus write $0 = k \int f \, \mathrm{d}\mu = \int f k \, \mathrm{d}\mu = \alpha(f)$ which proves the lemma. □

We can now proceed to prove Birkhoff theorem.

Birkhoff's theorem proof. As earlier done, we consider without loss of generality $f \in B_0$. Let $delta > 0$. Using sublemma 2.2.2.2 and choose h so that $\int \mid f - (hT - h) \mid \mathrm{d}\mu \leq \delta$. $E_\varepsilon(f) = E_\varepsilon([f - (hT - h)] + (hT - h)) \subset E_{\varepsilon/2}(f - (hT - h)) + E_{\varepsilon/2}(hT - h))$ and so:

$$\mu(E_\varepsilon(f)) \leq \mu(E_{\varepsilon/2}(f - (hT - h))) + \mu(E_{\varepsilon/2}(hT - h))). \tag{2.20}$$

But $\forall x \in X$ we can write

$$\frac{1}{N} \left| \sum_{n=0}^{N-1} (hT - h)(T^n x) \right| = \frac{1}{N} \left| h(T^N x) - h(x) \right| \leq \frac{2\|h\|_\infty}{N} \tag{2.21}$$

and so $\mu(E_{\varepsilon/2}(hT - h)) = 0$.

Using 2.2.2.1 we have

$$\mu(E_{\varepsilon/2}(f - (hT - h))) \leq \frac{\int |f - (hT - h) \, \mathrm{d}\mu|}{\varepsilon/4} \leq \frac{4\delta}{\varepsilon}$$

and thus $\mu(E_{\varepsilon/2}(f - (hT - h))) = 0$ which proves the result. □

2.3 Ergodicity of time series

It is a well known fact that Dynamical Systems like those considered in the previous section are in fact Markov Chains (see [53]).

In a certain way the Markov Chain perspective is nothing but a microscopical phenomenological description of the effect of global dynamic of the system. Under this perspective we wonder how ergodicity is espressed in stochastic Process.

In the previous sections we have seen that ergodicity is roughly equivalent to say that the temporal means equal statistic means. Thus a single process we can express ergodicity as follows (see [49]):

Definition 2.3.1 (Strict ergodic process)**.** A stochastic process is *ergodic* if *all* his statistical means can be calculated trough a single realization of the process

as above we can confine ourselves to considering a weaker form of ergodicity that is

Definition 2.3.2 (Wide sense ergodic process)**.** A stochastic process is *ergodic in the wide sense* if if holds:

$$\bar{X}_t = \lim_{T->\infty} \frac{1}{2T} \int_{-T}^{T} X(t')\,\mathrm{d}t' = \mathbf{E}[X(t)] \tag{2.22}$$

and

$$\mathscr{R}_{XX}(\tau) = \lim_{T->\infty} \frac{1}{2T} \int_{-T}^{T} X(t')X(t'+\tau)\,\mathrm{d}t' = \mathbf{E}[X_t X t+\tau] \tag{2.23}$$

It is natural to wonder what is the equivalent concept of invariance in the language of stochastic processes. In this case too, little work is needed to translate concept:

Definition 2.3.3 ((Strictly) Stationary processes)**.** A random process $\{X_t\}$ is called a *(Strictly) Stationary process* if his cumulative distributions

$$F_{X_{t_1}\dots X_{t_n}}(x_{t_1}\dots x_{t_n}) = F_{X_{t_1+\tau}\dots X_{t_n+\tau}}(x_{t_1}\dots x_{t_n}), \tag{2.24}$$

for all $t_i, \tau \in R$

Usually weaker form of stationarity is required to get useful results, that is only the first and the second moment are stationary:

Definition 2.3.4 (Wide sense Stationary processes)**.** A random process $\{X_t\}$ is called a *(Weak) Stationary process* if its mean

$$\mathbf{E}[X_t] = \mathbf{E}[X_{t+\tau}] = \mu \tag{2.25}$$

and its auto covariance (or autocorrelation)

$$\mathbf{E}[X_t X_{t+\tau}] = \mathbf{E}[X_0 X_\tau] = C(\tau) \tag{2.26}$$

For all $t, \tau \in R$

In a stationary process, thus, we can begin an observation at any time and we shall still be able to access to all the information on the process.

As shown in the previous chapter, ergodicity is a stronger property than invariance: the same holds for ergodic and stationary processes.

Proposition 2.3.1. *Ergodicity in the wide sense implies stationarity in the wide sense*

Proof. The proof is almost trivial. In fact the limit exists equation 2.22 reads

$$\mathbf{E}[X(t)] = \lim_{T->\infty} \frac{1}{2T} \int_{-T}^{T} X(t')\,\mathrm{d}t' = \mu \tag{2.27}$$

and 2.23 reads

$$\mathbf{E}[X_t X t + \tau] = \lim_{T->\infty} \frac{1}{2T} \int_{-T}^{T} X(t')X(t'+\tau)\,\mathrm{d}t' = \mathscr{R}_{XX}(\tau). \tag{2.28}$$

□

Thus proving that a process (or a dynamical system) *is not stationary is the same as showing that it is not ergodic.*

But stationarity does not imply ergodicity. To see it let U a random variable with mean μ. Let us consider the process defined as follows:

$$X_t = \begin{cases} U & \text{if } t = 0 \\ X_0 & \text{if } t > 0 \end{cases}. \tag{2.29}$$

By construction this is a stationary process but it is clearly non ergodic. In fact $\langle X_t \rangle = \mu$ but $\bar{X}_t = U$.

2.4 Ergodicity breaking

When coming to "Ergodicity breaking " many physicists think to usual critical phenomena. Second phase transition have, in fact, have provided very rich experimental ground upon which physicists have built a very well founded theory (see [34]). Typically, in those systems, ergodicity breaking is explained as a consequence of *spontaneous symmetry breaking* at a certain critical temperature T_c (e.g. Curie Law for magnetization).

Similar but slightly different systems are those which undergo critical dynamics. In this case the system is thought to be in a non equilibrium state and expected to regress to equilibrium during his time evolution. In a totally ergodic system regression to equilibrium should occur with a precise an fixed "mean regression time" which is nothing but the "time correlation length".

When the system is near a critical point this happens to be false and the more the system is near the critical point, the more the "time correlation length" of the system grows: system exhibit what is called *critical slowing down*.

Yet the simple and rough Van Hove model [61] had shown it , and the models further proposed by Kawazaki in the late sixties [26] and to the work of Höhenberg and Halperin [23] [25] who managed to give a Renormalization Group description of critical dynamics have confirmed it.

All these theories have shown that the typical characteristic behavior of a system near a critical point satisfies what is called *dynamical scaling hypothesis*, that is, the typical time behaves like:

$$\tau_k = \xi^z \mathrm{f}(k\xi) \tag{2.30}$$

where k is the (Fourier) mode index ξ is the spatial correlation length and z is the so called *dynamic exponent* which is quite a universal property of many similar systems.

When approaching critical point $\xi \to \infty$ (i.e. the global properties of the system determine at a great extent his singular behavior) and thus system dynamics slows down i.e. the system take a very long time to explore the entire phase space:e otherwise saying ergodicity breaks.

A simple toy model, which can be considered a first model for some thresholds problems, enables us to show that non ergodic behavior may occur even in simple models if we are dealing with "events".

2.4.1 Toward a event driven ergodicity breaking: recurrence time for discrete random walk

Even some simple systems can exhibit non ergodic behavior. As a classical example let us consider a discrete time, one dimensional random walk over a Lattice of step length a with constant transition probability $p = \frac{1}{2}$. We will denote the value of the walk with X_n. In this case writing down a recurrence equation for probability is trivial:

$$\mathrm{Prob}(X_n = ka) = \frac{1}{2}P((k-1)a, n-1) + \frac{1}{2}P((k-1)a, n-1) \tag{2.31}$$

With very simple calculation we get that :

$$\mathrm{Prob}(X_n = (n-k)a) = \binom{n}{n-k}\frac{1}{2^n} \tag{2.32}$$

And so we can define an origin recurrence time probability (only even period can admit return to origin)

$$\mathrm{Prob}(X_{2n} = 0) = \binom{2n}{n}\frac{1}{2^{2n}} \tag{2.33}$$

by Stirling approximation we obtain that.

$$u_{2n} = \mathrm{Prob}(X_{2n} = 0) \sim \frac{1}{\sqrt{2\pi n}}. \tag{2.34}$$

It is easy to prove (see [18] cap. III) that:

$$\mathrm{Prob}(\mathrm{Prob}\{X_0 > 0) \cup (X_1 > 0) \cup \ldots \cup (X_{2n-1} > 0) \cup (X > 0)\} = \frac{1}{2}u_{2n} \tag{2.35}$$

and more interestingly that the probability of first return is:

$$\begin{gathered} f_{2n} = \mathrm{Prob}\{(X_0 \neq 0) \cup (X_1 \neq 0) \cup \ldots \cup (X_{2n-1} \neq 0) \cup (X_{2n} = 0)\} = \\ u_{2n-2} - u_{2n} = \frac{1}{2n-1}u_{2n}. \end{gathered} \tag{2.36}$$

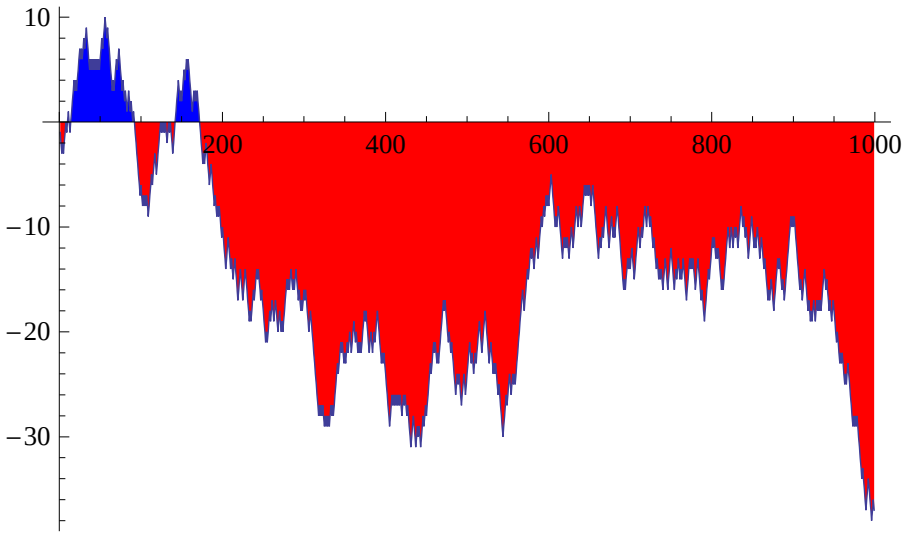


Figure 2.1: A single realization of random walk

Intuition suggests that a single random walk should be equally distributed between positive and negative zone but Figure 2.1 shows our intuition is likely to be wrong.

A precise statement of this is possible, since, in this case, we are able to calculate the *last visit probability* that is the probability $\alpha_{2k,2n}$ the up to an including epoch $2n$ the last visit to the origin happen at time $2k$. An easy calculation shows that:

$$\alpha_{2k,2n} = u_{2n}u_{2n-2k}. \tag{2.37}$$

We can then state this interesting theorem

Theorem 2.4.1 (Discrete Arc Sin law for sojourn time). *The probability that up to an epoch $2n$ the system is $2k$ times on the positive side and $2n-2k$ times on the negative time is $a2k, 2n$*

and as a corollary we find

Corollary 2.4.1. *The probability that the systems stays xn times on the positive side and $(1-x)n$ times on the negative side tend to $\frac{2}{\pi}\arcsin\sqrt{x}$.*

This last result was found also in Brownian motion by Lévy [31] and for the positive partial sums of mutually independent variables by Erdős and Kac [16]

Let us now consider the stochastic process constructed in the following way :

$$\xi_n = \text{sign}(X_n). \tag{2.38}$$

Theorem 2.4.1 suggests that this process *cannot be ergodic*.

Its statistic mean is clearly zero since the problem is symmetric.

We can, in fact, associate to any trajectory $T(X_n) = \{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$ a reflected one $\tilde{T}(X_n) = \{X_1 = -x_1, X_2 = -x_2, \ldots, X_n = -x_n\}$ whose probability is exactly the same a thus the corresponding trajectories for ξ will have exactly the same probability to occur. This obviously implies that $\langle\xi\rangle = 0$

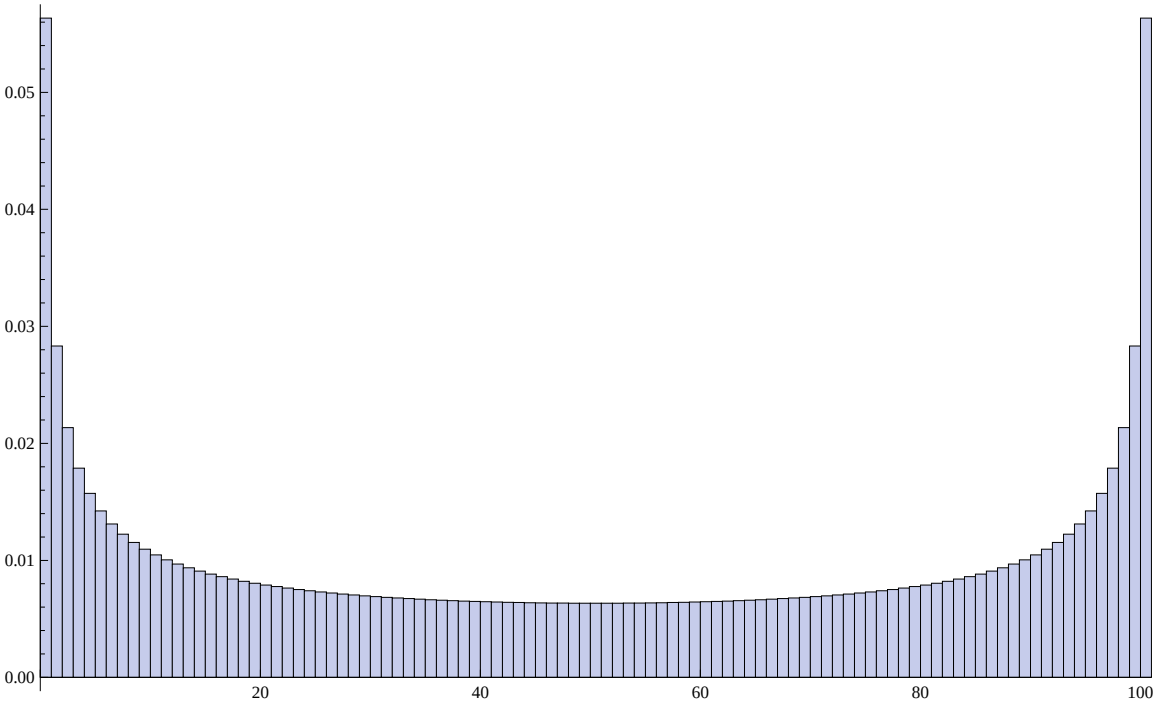


Figure 2.2: sojourn time distribution for $n = 100$

To see that we first define the stochastic variables

$$\pi_n = \{k \text{ if } X_2 n \text{ stays on the positive side for } k \text{ steps}\} \tag{2.39}$$

and

$$\nu_n = \{h \text{ if } X_2 n \text{ stays on the negative side for } h \text{ steps}\} \tag{2.40}$$

The *temporal mean* of ξ_n over a realization of length 2n can then be written as

$$\bar{\xi}(n) = \frac{\pi_n - \nu_n}{n}. \tag{2.41}$$

for $n \gg 0$, setting

$$X = \frac{\pi_n}{n} \tag{2.42}$$

we write for most trajectories

$$\frac{\nu_n}{n} \sim 1 - X \tag{2.43}$$

and thus

$$\bar{\xi}(n) \sim 2X - 1. \tag{2.44}$$

If the process were ergodic we would have

$$\text{Prob}(\bar{\xi}(n) = 0) \to 1 \text{ as } n \to \infty, \tag{2.45}$$

but we actually get that using 2.4.1

$$\text{Prob}(\bar{\xi}(n) = 0) \to \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \tag{2.46}$$

2.4.2 Ergodicity breaking in sub diffusive system

A very interesting example of complex system where ergodicity breaks has been described by He et al. in a recent work [24].

In this article, the authors analyze mRNA molecules and lipid granules diffusion in *Escherichia Coli* bacterium[1]

Since the internal structure of bacteria is usually extremely complicated (many different types of almost macroscopical molecules are present, irregular membrane structure, etc.) a complete microscopical description is not possible. Recent single particle tracking techniques, nevertheless, have enabled biologists to obtain very precise experimental data on the characteristics of diffusion in those systems which have shown a certain regularity and universality in diffusive motion and a clear deviation with respect of Fick's Law.

Mean square Deviation :

$$\bar{\delta}^2(\Delta, t) = \frac{\int_0^{t-\Delta} (x[t' + \Delta] - x[t'])^2 \, \mathrm{d}t'}{t - \Delta} \tag{2.47}$$

on a single time series can be calculated for these systems.

Observations a sub diffusive behavior, i. e.

$$\bar{\delta}^2 \sim \bar{D}_\alpha \Delta^\alpha \text{ with } \alpha \approx 0.75 \tag{2.48}$$

If the system were ergodic, $\bar{\delta}^2$ should trend to to the ensemble variance $\langle x^2(t) \rangle$. Using Continuous Time Random Walk (CTRW) method He et al. have found that the mean square deviation fluctuate and he was able to give an exact form of MSD distribution using CTRW formalism.

In particular He called $f(\delta x)$ the probability density function of jumps and considered a waiting time distribution of the form:

$$\psi(\tau) \sim A \frac{\tau^{-1-\alpha}}{\Gamma(-\alpha)} \text{ with } \alpha \in [0, 1]. \tag{2.49}$$

If the jumps δx_i the system makes in the time interval $[t, t + \Delta]$ are i.i.d. random variables — we are thus considering an unbiased — we have $\langle \delta x_i \rangle = \langle \delta x \rangle = 0$ and $\langle \delta x_i \delta x_j \rangle = \langle \delta x \rangle \delta_{ij}$) and we can write

$$\langle x[t + \Delta] - x[t] \rangle = \langle \sum_0^{n(t,\Delta)} \delta x_i \rangle = \langle \delta x \rangle \langle n(t, \Delta) \rangle = 0, \tag{2.50}$$

where $n(t, \Delta)$ is the number of jumps in the interval $[t, t + \Delta]$ and thereby yielding

$$\langle (x[t + \Delta] - x[t])^2 \rangle \langle \sum_0^{n(t,\Delta)} \delta x_i \sum_0^{n(t,\Delta)} \delta x_i \rangle = \langle \delta x^2 \rangle \langle n(t, \Delta) \rangle. \tag{2.51}$$

In a sub diffusive CTRW we can estimate the number of jumps by

$$\langle n(t, \Delta) \rangle = \langle n(0, t + \Delta) \rangle - \langle n(0, t) \rangle \text{ where } \langle n(0, t) \rangle \sim \frac{t^\alpha}{A\Gamma(1 + \alpha)}, \tag{2.52}$$

[1] *Escherichia Coli* is a bacterium species usually used as general model for bacteria in biological literature

giving

$$\langle \bar{\delta}^2 \rangle = \frac{\langle \delta x^2 \rangle}{A\Gamma(1+\alpha)} \frac{t^{1+\alpha} - \Delta^{1+\alpha} - (t-\Delta)^{1+\alpha}}{(1+\alpha)(t-\Delta)}. \tag{2.53}$$

We thus see that for $\alpha \neq 1$ the MSD do not converge to ensemble variance $\langle x^2(t) \rangle = \frac{2D_\alpha t^\alpha}{\Gamma(1+\alpha)}$. For $\Delta \ll t$ we have

$$\bar{\delta}^2 \sim 2D_\alpha \frac{\Delta}{\Gamma(1+\alpha)t^{1-\alpha}} = D(t)\Delta. \tag{2.54}$$

This biological system shows then a non ergodic behavior which is not linked to any kind of symmetry breaking. This system is, in fact, always in a non ergodic a so non equilibrium state.

The system we aim to describe are in a similar situation, as we shall see, and exhibits interesting properties.

2.4.3 Subordinated renewal processes

Until now we have dealt with quite general concepts. Let us specify to the Subordinated renewal process we have introduced in chapter II.

Let us consider a discrete time renewal stationary process X_n that is:

$$\text{Prob}(X_n \in [x, x + \mathrm{d}x]) = \text{Prob}(X_0 \in [x, x + \mathrm{d}x]) = \pi(x)\,\mathrm{d}x \tag{2.55}$$

we then subordinate it with a waiting time distribution $\psi(t)$ that is, we first consider the discrete time process T_n defined by

$$\text{Prob}(T_n \in [t, t + \mathrm{d}x]) = P(T_0 \in [t, t + \mathrm{d}t]) = \psi(t)\,\mathrm{d}t \tag{2.56}$$

and then the continuous time process:

$$\xi(t) = X_k \text{ if } t \in [T_k, T_{k+1}]. \tag{2.57}$$

As we have seen in chapter XXX we are mainly interested in processes in which the waiting time distribution is asymptotically a power law with index μ. Let us consider (for numerical purposes) the Manneville distribution

$$\psi(t) = \frac{(\mu-1)T^{\mu-1}}{(t+T)^\mu}. \tag{2.58}$$

We want to study how does our process autocorrelation behave. In chapter III it is shown (3.44) that the correlation function is

$$C(t,s) = \mathbf{E}(\xi(t)\xi(s)) = \Psi(t,s) \tag{2.59}$$

where $\Psi(t,s)$ is the waiting time distribution of age $s < t$. While the correlation is analytical is hard to plot it since it is defined in terms n-th term of the convoluted waiting time., but its structure shows that it cannot be stationary.

For this reason we have made some numerical simulations to investigate better the nature of this process.

If we use Manneville's waiting time distribution 2.60 we can write analytically the Survivor probability

$$\Psi(t) = \left(\frac{T}{(t+T)}\right)^{\mu-1}. \tag{2.60}$$

It is a well known fact that, for a random variable $t \sim \psi(t)$, the random variable $k = \Psi(t) \sim U(0,1)$ where $U(0,1)$ indicates the uniform distribution over the interval $[0,1]$.

We then generate our simulated time series of length L, $\xi[n\tau]$, where τ is our sampling time, in the following way:

i. At time zero we assign the +1 or −1 to with probability 0.5 to $\xi[0]$

ii. we then extract a time with the following prescription (here $u \sim U(0,1)$:

$$t^* = T\left[\frac{1}{u^{\frac{\mu}{1-\mu}}} - 1\right] \tag{2.61}$$

iii. then for every n so that $n\tau < t^*$ we set $\xi[n\tau] = \xi[(n-1)\tau]$ and return to the first step unless our time series has length L.

We simulate N time series$\xi^{[i]}$ to simulate our statistical ensemble and build an estimator

$$\mathscr{C}(n\tau, k\tau) = \sum_{i=1}^{N} \frac{\xi^{[i]}[n\tau]\xi^{[i]}[k\tau]}{(N-1)} \tag{2.62}$$

of autocorrelation $C(t,s)$

All these figures show that our system has a non stationary correlation function which takes into account that in those systems become more and more correlated (since the number of events per unit time become smaller and smaller). To see it we have to think that a stationary correlation matrix would have a band structure.

In figure we have plotted $\mathscr{C}(s, s+\tau)$ for different value of s. In fact a stationary correlation function would have showed a band structure, that is the elements $\mathscr{C}(n\tau, (n+x)\tau)$ would have been the same for all n.

As we find with our figures, there are three different possible regimes possible:

Case: $1 < \mu < 2$

In this case, both mean and variance of our waiting time are not defined. We can still use Lévy theorem in this case, but, as we can also see in Figure.., the events are rare and clustered (that is the probability of having N events in our series is) We know that in this case the event rate $R(t) \sim \frac{1}{t^{2-\mu}}$ and our simulation confirms this. We see form figure 2.3 that the system is strongly non stationary.

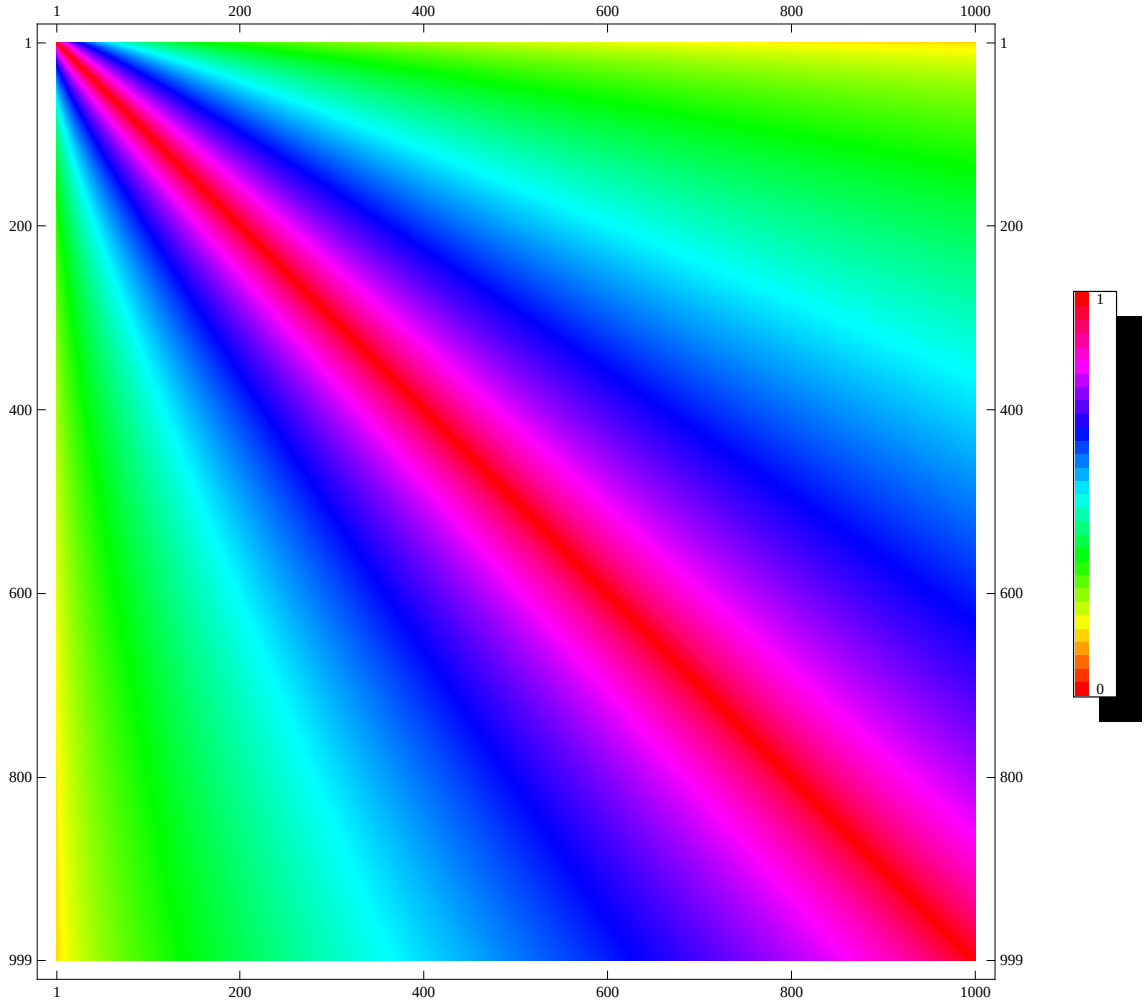


Figure 2.3: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 20$, $\mu = 1.5$ and for a sampling time $\tau = 0.5$ arbitrary time unit. Each pixel corresponds to a value of the matrix according to the color scale of the legend. Number of elements of the Gibbs ensemble $N = 5'000'000$. The leading process was dichotomous. A simple fit over the fist row ($\sim \Psi(t)$) gives $T = 18.99 \pm 0.04$ and $\mu = 1.4924 \pm 0.0003$

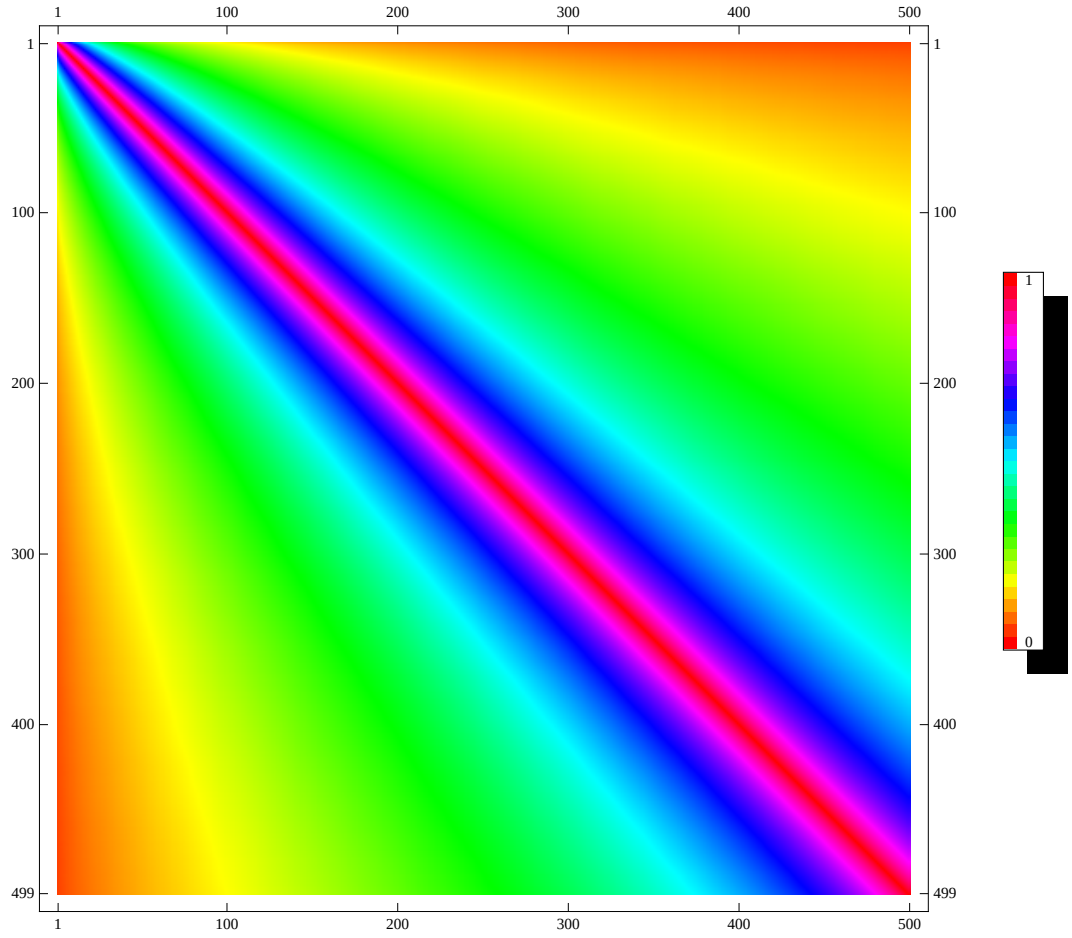


Figure 2.4: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 1.9$ and for a sampling time $\tau = 1$ arbitrary time unit. Each pixel corresponds to a value of the matrix according to the color scale of the legend. Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

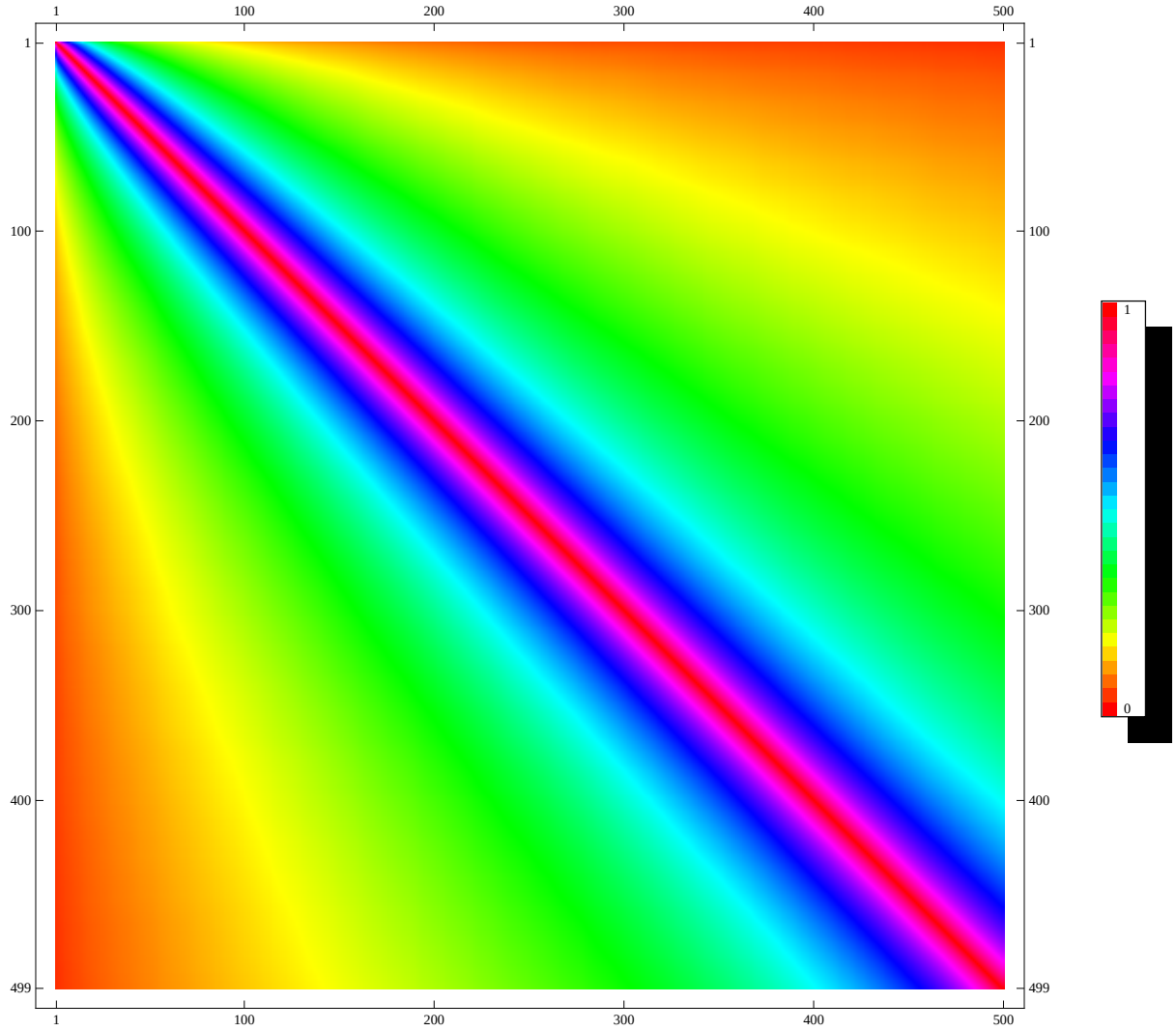


Figure 2.5: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 2$ and for a sampling time $\tau = 1$ arbitrary time unit.Each pixel corresponds to a value of the matrix according to the color scale of the legend.Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

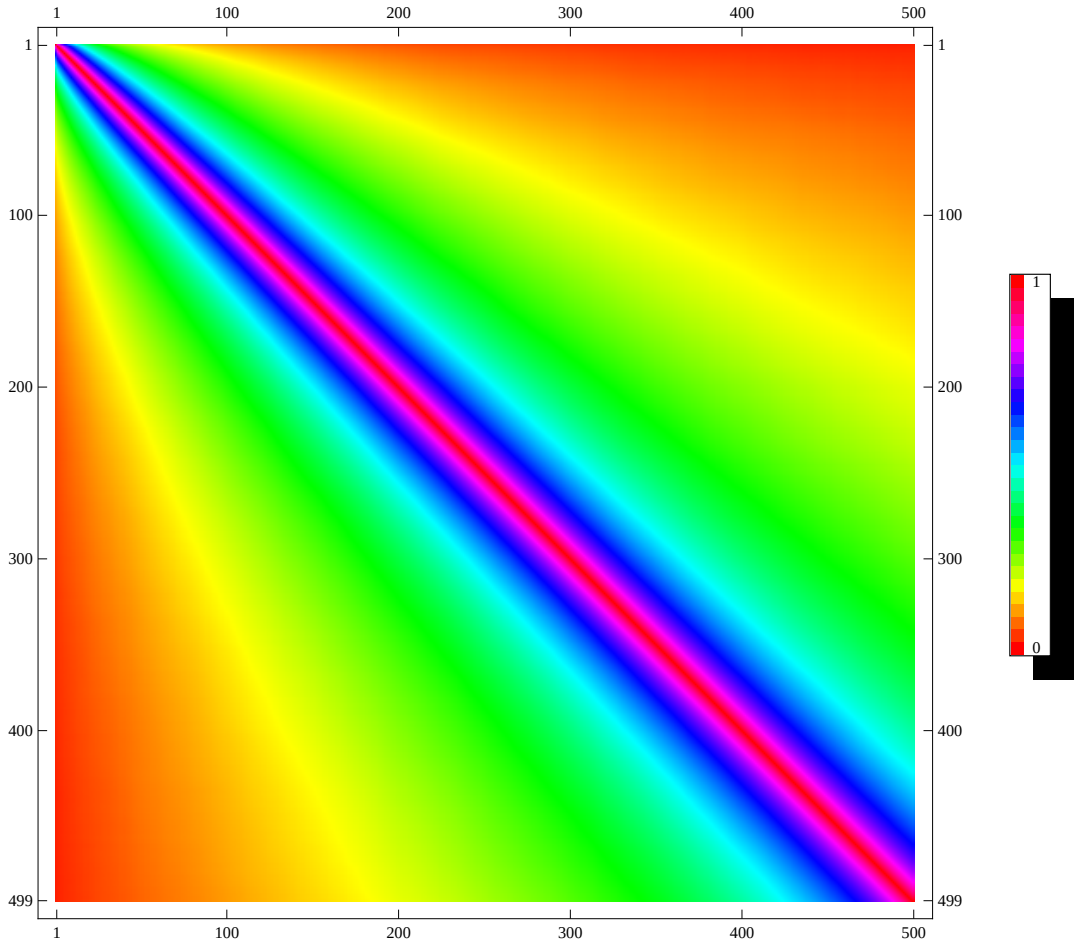


Figure 2.6: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 2.1$ and for a sampling time $\tau = 1$ arbitrary time unit.Each pixel corresponds to a value of the matrix according to the color scale of the legend.Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

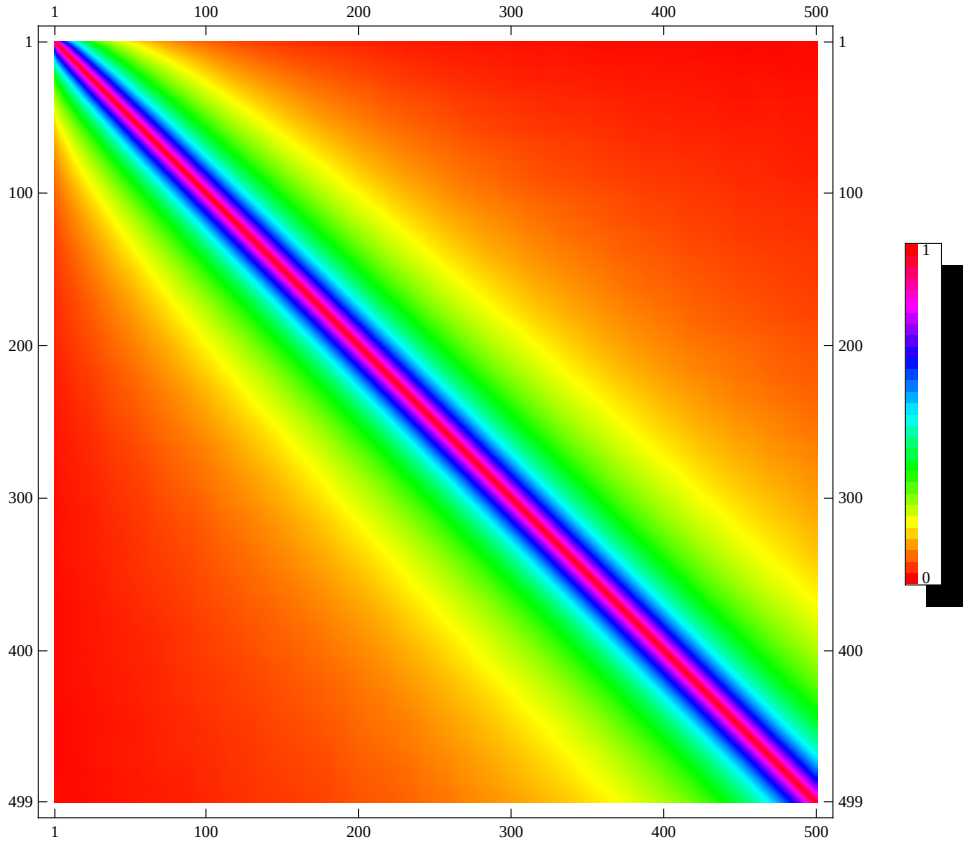


Figure 2.7: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 2.9$ and for a sampling time $\tau = 1$ arbitrary time unit.Each pixel corresponds to a value of the matrix according to the color scale of the legend.Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

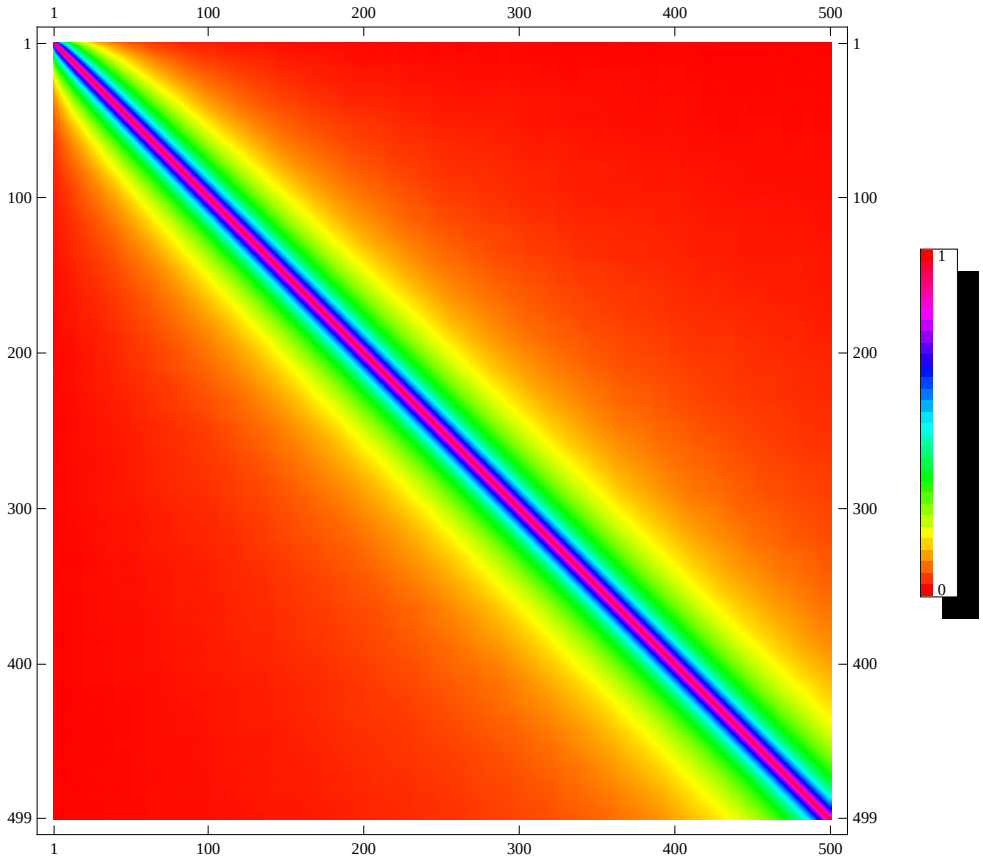


Figure 2.8: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 3$ and for a sampling time $\tau = 1$ arbitrary time unit.Each pixel corresponds to a value of the matrix according to the color scale of the legend.Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

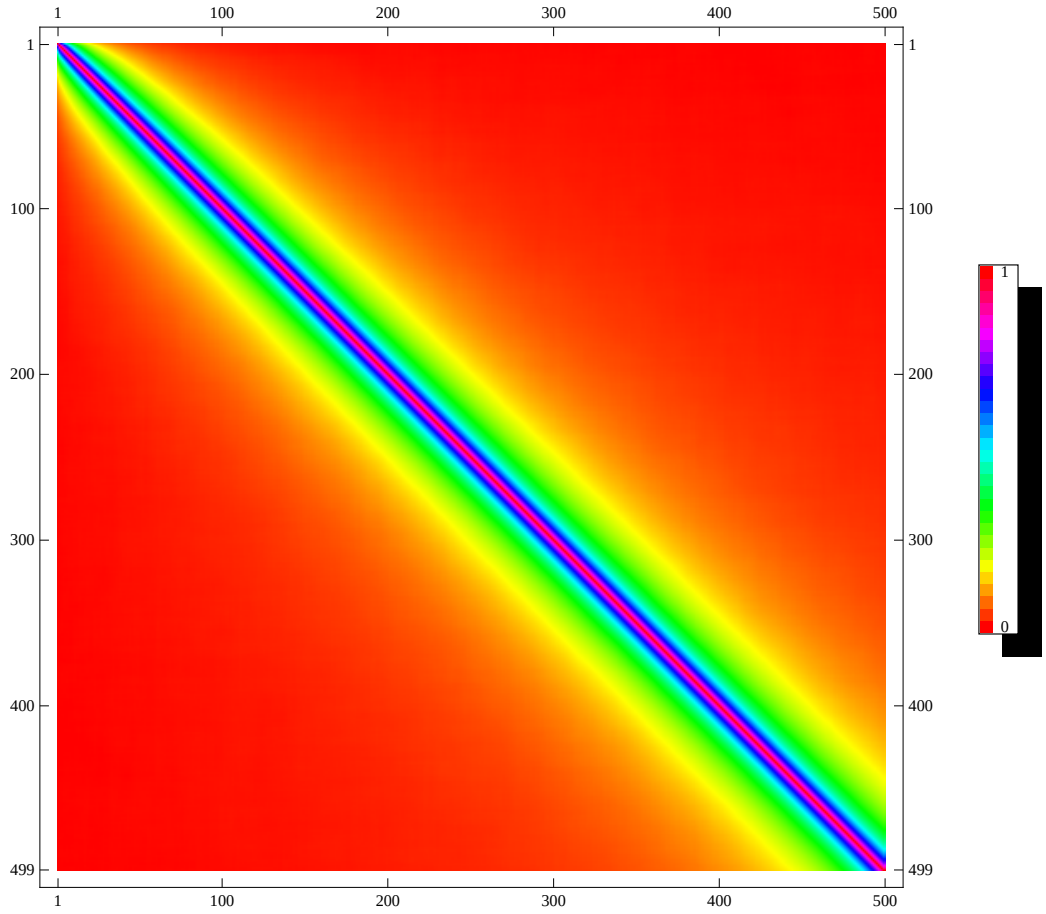


Figure 2.9: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with $T = 15$, $\mu = 3.1$ and for a sampling time $\tau = 1$ arbitrary time unit.Each pixel corresponds to a value of the matrix according to the color scale of the legend.Number of elements of the Gibbs ensemble $N = 1'000'000$. The leading process was dichotomous.

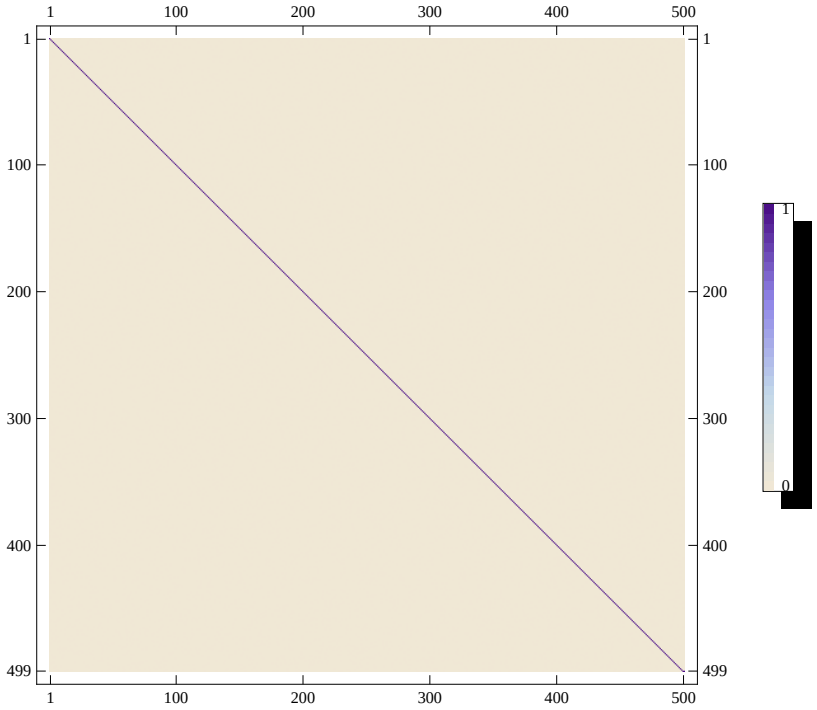


Figure 2.10: A graphic representation of the symmetrized correlation matrix $\mathscr{C}(n\tau, k\tau)$ with a poissonian waiting time distribution and $N = 1'000'000$

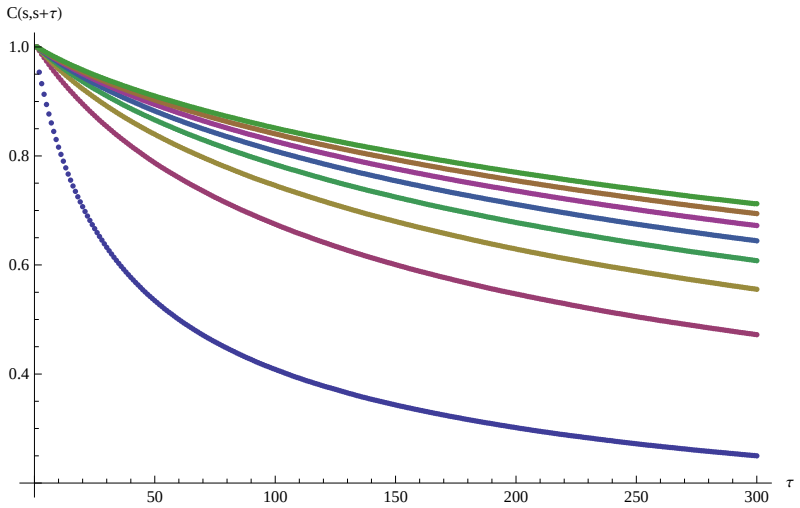


Figure 2.11: A graphic representation of the the aged correlation matrix $C_{h\tau}(k\tau) = \mathscr{C}(h\tau, (h+k)\tau)$ from the matrix of the same data of figure 2.3. $s = \{0, 100, 200, 300, 400, 500, 600, 700\}$

Case: $2 < \mu < 3$

In this case, we are in a intermediate regimen. As we see from figures 2.4, 2.6 and **??**, the fact that now the mean is defined is not sufficient for the system to be stationary.

If on the one hand theory says us the system should reach the stationarity our simulations show it does not.

The explanation is readily available since it is true the system reaches stationarity in a very slow way, that it like $\frac{1}{t^{\mu-2}}$ and thus less than hyperbolically.

Case: $3 < \mu$

Here the system reaches rapidly the stationary condition thus behaves exactly like a normal poissonian/Gaussian system, since in this range the variance is finite and thus the central limit theorem holds.

3 Linear Response Theory

Traditional Linear Response Theory [28] has been one of the most powerful working tools in non equilibrium statistical physics and especially in condensed matter problems. Van Kampen [62] criticized it on the basis that microscopic linearity, which is assumed in LRT, is quite different from Macroscopic linearity. Van Kampen correctly noted that to observe linear microscopic response (ie of individual particle trajectories) over macroscopic time (seconds, minutes or even hours), requires external fields which are orders of magnitude smaller than those for which linear macroscopic behavior is actually observed. Therefore, so the argument goes, the theoretical justification of, the Green-Kubo relations for linear transport coefficients, is suspect.

Nevertheless, the experimental work done over a time span of about 52 years did not reveal any breakdown of the theory. The search for the theoretical explanation of why the LRT work in spite of the hard criticism of Van Kampen is so extended that we limit ourselves to refer the reader to the book of Carolyn van Vliet [63] and the paper of Bianucci Mannella and Grigolini [14]. Both explanations seem to fail when we move from the systems originally studied by Kubo and van Kampen to the complex processes that are the current subject of an extended research work.

In short, for Carolyn van Vliet, the violation of the LRT cannot be experimentally revealed because of the fast decay of the correlation functions of the variable perturbed by external stimuli. The research work on anomalous diffusion and related processes [38,58,27], however, revealed the existence of complex phenomena, characterized by extremely slow correlation functions, and by the ensuing ergodicity breakdown, thereby making it natural to conclude that in these conditions the LRT is violated [55].

The authors of Ref. [14] noticed that the condition of strong chaos to which van Kampen refers generates the breakdown of the LRT at the level of single trajectories, but at the same time, thanks to chaos, makes the Gibbs ensemble of trajectories fulfill the LRT predictions.

The density of a set of independent trajectories, each of which violates the linear response condition, does obey the LRT of Kubo. Both these explanations fail when we come to study the dynamic of complex networks. As we have pointed out in first chapter mesoscopic phenomena e and systems with strong self-organization seem to exhibit a predilection for power laws and sporadicity; so, in this case, Kubo Theory do not hold because the system are not ergodic (not either stationary).

We will show in this chapter that for those systems a generalization of LRT can be provided. And since renewal non poissonian event driven systems are in fact non ergodic, from one single realization of the system it is impossible to derive the statistical properties of the system. Even if the correlation function is not stationary we shall adopt also in this case the Gibbs perspective. To do this, we shall propose our Ansatz and show that, with our proposal, LRT can be extended also to the dynamic of complex networks.

We first shall introduce traditional linear response theory by Kubo and stochastic resonance, we shall then proceed to extend this theory to non stationary subordinated renewal processes.

3.1 Traditional linear Response theory

Traditional Linear response theory interested in studying the response of a system at equilibrium to a external perturbation. Usual exposition begin with static response theory then dynamic theory. Very rich and insightful exposition can be found in Kubo's classical exposition [29]. We shall follow here a more synthetic exposition due to Zwanzig [67] applied to classical systems.

3.1.1 Static response

Let $\mathscr{H}_0(X)$ be the unperturbed Hamiltonian over the a set of dynamical variable and let E be and external field. We shall model the coupling of two systems by a coupling variable $M(X)$ thought to be a certain function of dynamic variables.

Let's consider the perturbed Hamiltonian $\mathscr{H}_{\mathrm{P}}(X) = \mathscr{H}_0(X) - \varepsilon M(X)E$.

Equilibrium distribution is given by canonical distribution $\rho_{\mathrm{eq}} = Z^{-1}\exp(\beta\mathscr{H}_0(X))$. Where Z is the Partition Function (i.e. $Z = \int exp\beta\mathscr{H}(X)\,\mathrm{d}X$ in quantum version we have the trace) and $\beta = (kT)^{-1}$.

Since we want to push our theory up to the first order in ε we have to perturb ρ_{P} up to the first order in ε.

Noticing that:

$$\mathrm{e}^{\beta(\mathscr{H}_0(X)-\varepsilon M(X)E)} \approx \exp(\beta\mathscr{H}_0(X))[1-\varepsilon\beta M(X)E + o(\varepsilon^2)] \tag{3.1}$$

and that:

$$Z \approx \int (\exp(\beta\mathscr{H}_0(X))[1-\varepsilon\beta M(X)E + o(\varepsilon^2))]\,\mathrm{d}X = Q_0 + \varepsilon\langle(\rangle M)E + o(\varepsilon^2) \tag{3.2}$$

We obtain

$$\rho_{\mathrm{P}} = \rho_{\mathrm{eq}}[1-\varepsilon\beta(M-\langle M\rangle)E + o(\varepsilon^2)). \tag{3.3}$$

If A is dynamic variable of the system we can write:

$$\langle A\rangle_{\mathrm{P}} = \langle A\rangle_{\mathrm{eq}} - \varepsilon\beta\int(\rho_{\mathrm{eq}}A(X)M(X)E\,\mathrm{d}X + o(\varepsilon^2) \approx \langle A\rangle_{\mathrm{eq}} - \varepsilon\chi_{AM}E. \tag{3.4}$$

3.1.2 Dynamic Linear Response

If the perturbation is not a static one we have to use a little more sophisticated language. As we know from Liouville's theorem the phase space density a constant of motion and thus the dynamic of the system is given by the Liouville equation

$$\partial_t\rho(t) = \mathscr{L}\rho. \tag{3.5}$$

where

$$\mathscr{L}\rho = \{\rho, \mathscr{H}\} = \sum_i \frac{\partial \rho}{\partial p_i}\frac{\partial \mathscr{H}}{\partial q} - \frac{\partial \rho}{\partial q_i}\frac{\partial \mathscr{H}}{\partial p_i}. \tag{3.6}$$

With our choice for the Hamiltonian we have:

$$\mathscr{L} = \mathscr{L}_0 - \varepsilon\mathscr{L}_1 + o(\varepsilon^2) = \{\cdot, \mathscr{H}_o\} - \varepsilon\{\cdot, M\}E(t) \tag{3.7}$$

We want to solve the equation perturbatively that is $\rho = \rho_0 + \varepsilon\rho_1 + o(\varepsilon^2)$. We thus obtain the equations:

$$\begin{cases} \partial_t \rho_0 = \mathscr{L}_0 \rho_o \\ \partial_t \rho_1 = \mathscr{L}_0 \rho_1 - \mathscr{L}_1 \rho_0 \end{cases} \tag{3.8}$$

Solving the first equation ($\rho_0(0) = \rho_{eq}$) formally we obtain:

$$\rho_0(t) = \exp \mathscr{L}_0 t \rho_{eq} = \rho_e q \tag{3.9}$$

Since

$$\mathscr{L}_1 \rho_e q = \{\frac{\mathrm{e}^{\beta\mathscr{H}_0}}{Z}, M\}E(t) = \rho_{eq}\{\mathscr{H}_0, M\}E(t) = -\beta\rho_{eq}(\frac{\mathrm{d}}{\mathrm{d}t}M)E(t)\,\mathrm{d}s \tag{3.10}$$

The second equation can be written down ($\rho_1(0) = 0$)

$$\rho_1(t) = +\beta \int_0^t \exp -\mathscr{L}_0(t-s)(\frac{\mathrm{d}}{\mathrm{d}s}M)E(s)\rho_{eq} \tag{3.11}$$

we can then write for a generic variable

$$\langle A\rangle_{\mathrm{P}}(t) - \langle A\rangle_{\mathrm{eq}}(t) = -\varepsilon\beta \int_0^t \langle A \exp -\mathscr{L}_0(t-s)(\frac{\mathrm{d}}{\mathrm{d}s}M)\rangle E(s)\,\mathrm{d}s = -\varepsilon\beta \int_0^t \chi_{AM}(t-s)E(s)\,\mathrm{d}s \tag{3.12}$$

where

If $\langle A\rangle = 0$ and $\langle M\rangle = 0$ we can write down Kubo's formula [28]:

$$\langle A\rangle_{\mathrm{P}} = -\varepsilon\beta \int_0^t \chi_{AM}(t-s)E(s)\,\mathrm{d}s \tag{3.13}$$

3.1.3 Velocity Autocorrelation Function

If we consider the Langevin Equation for a particle subjected To a stochastic and Variable Force we have

$$\dot{v}(t)_{\mathrm{eq}} = -\gamma v + f(t) + \xi(t) \tag{3.14}$$

If $f(t)$ is a white noise and $\langle v\rangle_{\mathrm{eq}} = 0$ we obtain:

$$\langle v\rangle(t) = \int_0^t \mathrm{e}^{-(\gamma(t-s)}\xi(t) \tag{3.15}$$

since the equilibrium autocorrelation of velocities for :

$$\langle v(t)v(s)\rangle = \int_0^t \int_0^s \mathrm{e}^{-(\gamma(t-u)}\mathrm{e}^{-(\gamma(s-v)}\langle f(u)f(v)\rangle \,\mathrm{d}u\,\mathrm{d}v = \mathrm{e}^{-\gamma|t-s|}\frac{2D}{\gamma} \tag{3.16}$$

using equipartition of energy we obtain

$$\langle v\rangle(t) = \beta \int_0^t \chi_{xv}(t-s)\xi(s)\,\mathrm{d}s \tag{3.17}$$

that is the same that considering the system coupled with the external perturbation $\mathscr{H} = -x\xi(t)$ where $\dot{x} = v$. Since the system is linear the linear response is exact.

If we consider system in which also a potential is present we can write Langevin equation

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\gamma v + F(t) - \frac{\mathrm{d}}{\mathrm{d}x}V(x) + \xi(t) \end{cases} \tag{3.18}$$

For $V(x) = \frac{1}{2}\omega^2 x^2$ and with the Smoluchowski approximation $\dot{v} \approx 0$ we can write the equation becomes

$$\dot{v} = -\frac{\omega^2}{\gamma}x + \frac{F(t)}{\gamma} + \frac{\xi}{\gamma} = -\Gamma x + G(t) + \Xi(t) \tag{3.19}$$

formally equivalent to equation 3.14. In this case:

$$C(t,s) = \langle x(t)x(s)\rangle = -\gamma\Gamma\mathrm{e}^{-\gamma(t-s)} \tag{3.20}$$

And thus

$$\langle v\rangle(t) = \beta \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}C(t,s)\xi(s)\,\mathrm{d}s \tag{3.21}$$

Since the system is stationary the response function can be written as $\frac{\mathrm{d}}{\mathrm{d}s}C(t,s) = \frac{\mathrm{d}}{\mathrm{d}s}C(t,s) = -\frac{\mathrm{d}}{\mathrm{d}s}C(t-s) = -\frac{\mathrm{d}}{\mathrm{d}t}C(t,s)$.

When trying to extend LRT to non stationary systems those two prescriptions are no more equivalent and correspond to make two profoundly different hypothesis.

3.2 Stochastic Resonance

We want to proceed to illustrate what is, in fact, another form of linear response theory, stochastic resonance. As its name specify its *resonant character* is usually stressed but we want, instead, point that RS is actually a specific form of LRT for a class of stochastic systems.

Let us consider a stochastic system with a potential. A typical example of this kind of systems is that of a particle subject to a potential V(x), a dumping and a white noise random force of intensity $2D$ whose Langevin equations are:

$$\begin{cases} \dot{v} = -\gamma v - \nabla V(x) + F(t) \\ \dot{x} = v \end{cases} \tag{3.22}$$

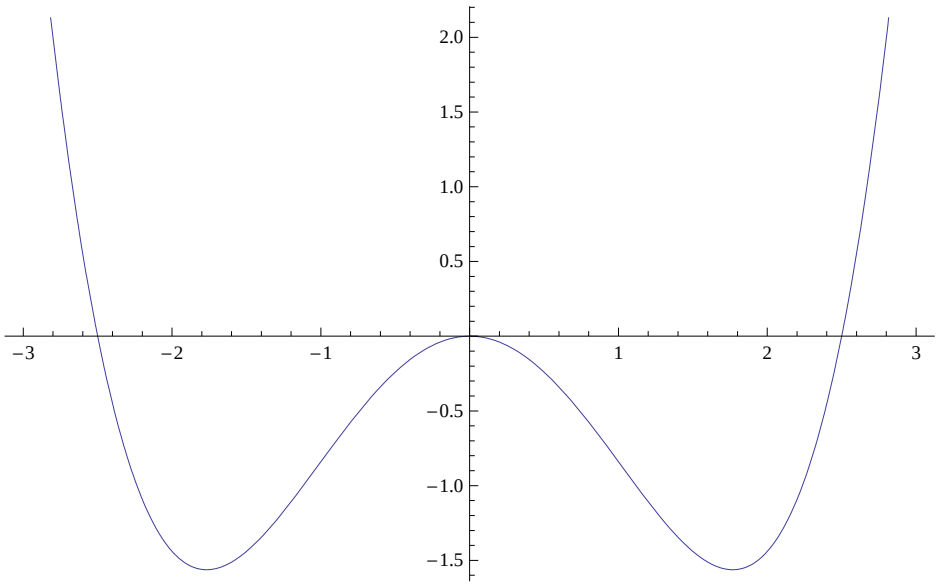


Figure 3.1: A plot of $V(x) = -x^2 + 0.16x^4$

The transitional moments are easily calculated in this case, since by integration we get:

$$\langle x \rangle = \langle v \rangle \Delta t \tag{3.23}$$

and

$$\langle v \rangle = \gamma \langle v - \nabla V(x) \rangle + \int_0^t \langle F(t) \rangle \, \mathrm{d}t. \tag{3.24}$$

We derive from them the first two transitional moments $a_x = v$ and $a_v = -\gamma v - \nabla V(x)$. Because of 3.23 the coupled transitional moments are null and since:

$$\langle v^2 \rangle = \gamma \langle v - \nabla V(x) \rangle^2 \Delta t^2 + \int_0^t \int_0^s \langle F(t)F(s) \rangle \, \mathrm{d}t \, \mathrm{d}s \tag{3.25}$$

we obtain $a_{vv} = 2D$. We can write *Fokker-Planck equation* for the process:

$$\partial_t p(x, v, t) = \partial_x [v p(x, v, t)] + \partial_v [(-\gamma v - \nabla V(x)) p(x, v, t)] - D \partial_x^2 p(x, v, t), \tag{3.26}$$

where $p(x, v, t) = P(x, v, t | x_0, v_0, 0)$.

Solving the previous equation is a hard task. It can be made easier if me make Smoluchowski approximation (that is $\dot{v} = 0$) in this case the system's equation simplifies to

$$\dot{x} = \frac{1}{\gamma} \nabla V(x) - \frac{1}{\gamma} F(t) \tag{3.27}$$

and subsequently *Fokker-Planck* equation becomes

$$\partial_t \sigma(x, t) = \partial_x \left(-\frac{1}{\gamma} \nabla V(x) \sigma(x, t) \right) + \frac{1}{\gamma} \partial_x^2 \sigma(x, t) \tag{3.28}$$

equilibrium distribution of the previous equation can be analytical found.

If we choose a double well potential 3.1 $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}\lambda^2 x^4$ our system is bistable, that is there are two stable states, which we label with +1 and −1. Because of the stochastic force the system will stay in a state for a few time than go to the other and so on. In practice the system will act as a two state system [36]. It can then be described by a Master Equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}p_+(t) = r_- p_+(t) - r_+ p_-(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}p_+(t) = -r_- p_+(t) + r_+ p_-(t) \end{cases}, \tag{3.29}$$

where the quantity $r_\pm$ are called *Kramers rates*.

They can be derived by the equilibrium solution of equation in the case of double well potential ($x_\pm$ are the minima points) obtaining:

$$r_\pm = \frac{\left(V''(x_\pm)V''(0)\right)^{\frac{1}{2}}}{2\pi\gamma}\mathrm{e}^{-\gamma\frac{V(0)-V(x_{pm})}{2D}}. \tag{3.30}$$

If we perturb this system with a small perturbation $\varepsilon g(x)\xi(t)$, that is, we consider the equation

$$\dot{x} = \frac{1}{\gamma}\nabla V(x) + \frac{1}{\gamma}F(t) + \varepsilon g(x)\xi(t), \tag{3.31}$$

this can be seen as taking a time dependent potential $W(x,t) = V(x) + \varepsilon h(x)\xi(t)$ where $h'(x) = g(x)$. If $g = 1$ and $\varepsilon << 1$ so that the minima point can be considered fixed, we write Kramers rate

$$r_\pm = r_\pm^0 \mathrm{e}^{\gamma\frac{\mp x_\pm \xi(t)}{D}} \tag{3.32}$$

By expanding up to first terms in ε , for a symmetric double well (i.e. $r_\pm^0 = r_0$ and $x_\pm = \lambda$), we obtain [1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = r_0\begin{pmatrix} -\frac{1-\lambda\varepsilon\xi(t)}{2} & +\frac{1+\lambda\varepsilon\xi(t)}{2} \\ +\frac{1+\lambda\varepsilon\xi(t)}{2} & -\frac{1-\lambda\varepsilon\xi(t)}{2} \end{pmatrix}\mathbf{p}(t). \tag{3.33}$$

Moving to the the variable $\Pi(t) = p+(t) - p-(t)$ (the sum equals 1) the equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = -r_0\Pi(t) + \varepsilon r_0\lambda\xi(t) \tag{3.34}$$

thereby yieldin to the solution

$$\Pi(t) = \mathrm{e}^{-r_0 t}\Pi(0) + \varepsilon r_0\lambda\int_0^t \mathrm{e}^{-r_0(t-s)}\xi(s). \tag{3.35}$$

From Pi we then recover the probabilities

$$p_+(t) = \frac{1+\Pi(t)}{2} \text{ and } p_-(t) = \frac{1-\Pi(t)}{2}, \tag{3.36}$$

[1] using a vector notation $\mathbf{p} = (p_+(t), p_-(t))$

Thus any observable average becomes

$$\langle A\rangle_{\mathrm{P}} = \frac{A(+)+A(-)}{2} + \frac{A(+)-A(-)}{2}\Pi(t). \tag{3.37}$$

The response becomes then

$$\langle A\rangle_{\mathrm{P}} - \langle A\rangle_{\mathrm{eq}} = \varepsilon r_0 \lambda \frac{A(+)-A(-)}{2} \int_0^t \mathrm{e}^{-r_0(t-s)}\xi(s)\,\mathrm{d}s. \tag{3.38}$$

We notice that:

$$\frac{A(+)-A(-)}{2} = \mathbf{A}^T \mathrm{O}\,\mathbf{p}_{\mathrm{eq}} = m_A. \tag{3.39}$$

For an harmonic perturbation $\xi(t) = A\cos(\omega t)$ the system response becomes:

$$\langle A\rangle_{\mathrm{P}} - \langle A\rangle_{\mathrm{eq}} = -\varepsilon\lambda A \frac{r_0}{\sqrt{r_0^2+\omega^2}}\cos(\omega t+\varphi), \tag{3.40}$$

with $\varphi = \arctan\left(\frac{r_0}{\omega}\right)$.

3.3 Linear Response theory for complex systems

This section is devoted to the illustration of the generalization of LRT to complex event driven systems

3.3.1 Event Driven Systems

Our extention to LRT theory aims to show that for event driven systems a LRT framework can be provided even if stationarity does not hold.

An event driven (complex) system is a system which can be described by an event driven renewal process as defined in the first chapter. As it has been shown there, those systems can be described by a subordinated process which characterized by a leading process X_n and a subordination generating process characterized by a waiting time distribution.

We want to point here that the theory which shall be discussed further is applicable whenever such a description is allowed.

It is, as Kubo's theory for ordinary statistical physics, a framework in which the details have to be provided case by case. (i.e. what is perturbing what defines the fine details of the theory but its structure remains the same).

A more detailed discussion of what event Driven systems are is fundamental to prevent any misunderstanding.

An "event", is an abrupt transition of the system from one state to another. Obviously this is an idealization but this idealization is enough to capture some of the fundamental properties FL those system. An event driven system is a system whose characteristic dynamics can be described as due to events, that is, to these abrupt transitions. For power law waiting time distributions this characteristic is extremely important. In this case , in fact, for $\mu < 0$

the system characteristics are mainly determined by those events. It has been shown, in fact, that the power spectra of event driven systems with a power law distribution have the characteristic power spectrum $P(f) \sim 1/f^{\eta}$ for $f \to 0$ where $\eta = 3 - mu$ [33]. Renewal processes with a power law distribution are, in this sense, special since the sole presence of events determine at least one of the fundamental characteristics of power spectra: that is we could refer to the events in this case as "crucial event".

Stochastic resonance, in McNamara and Wiesenfeld approach, can be thought of, as we shall see, as a poissonian event driven system. The "events" in this case are the crossings of the potential barrier which make the state of the systems to undergo a transition between two stable states, while the waiting time distribution represent the time distribution between two of those events. The action of the perturbation can then be seen either as modifying the transition rates (and we shall refer to it as *phenomenological approach*) or by perturbing the time event occurrence. In this case the result is the same, an in poissonian case this is always true since the quantity which determine the renewal process $g(t) = r(t) = R(t)$ coincides. When we depart from poissonian statistics. This equivalence breaks.

For complex systems a microscopical approach typically is not available. When it is available it can be extremely complicated ore simply useless (e.g. the sub diffusion of mesoscopic molecules in the *E. Coli* bacterium already proposed [24]) or liquid crystals experiments [54]. In many cases like in the decision making networks [60] it is simply unavailable.

Perturbing a complex system

When we apply a *small perturbation* to the system, like an oscillatory potential in the liquid crystal experiment [54], we are perturbing the global propertied of the system.

It makes then sense to wonder how a complex system respond to an external perturbation.

We shall remember that subordination theory is a mathematical *phenomenological* picture of a complex phenomena as a whole. We cannot identify the leading process with a microscopical process and the waiting time distribution as a global properties but *both* are the expression of global cooperative structure of this systems.

In this frame we have two possibilities to model the action of the perturbation of the subordinated process.

We can assume that the net effect of the perturbation on the system is that of modifying the operator K and leaving the waiting time distribution unperturbed (Phenomenological approach) or we can make the hypothesis that the perturbation acts on the *entire laminar regions* letting the waiting time distribution to be perturbed. Those approaches lead to two different results.

Preliminary results on the liquid crystals confirm that for this specific system the dynamical approach is the correct one.

We notice that, when perturbations act on complex systems, they modify all their global properties. The dynamical approach better models the net effect of these perturbations on the element of the systems since it take into account also the effect of this perturbation on the occurring time (which is a fundamental expression of global properties of the system).

Liquid Crystals

In a recent article [54] it has been shown that Liquid crystals in weak turbulence exhibit a non exponential relaxation to equilibrium.

In the dynamic elecrohydrodynamic convection regime, in which the authors have made their experimental works, the light tranmission is mediated by the creation and destruction of defects leading to a increase or decrease of light trasmittivity.

The liquid crystals where placed between two transparent glass plates coated with iridium to allow electrical conductivity. Then an alterate electrical difference of potential was placed between the two plates.

Near a threshold voltage $V_0 \approx 20\text{V}$ the convective instability appears visualized as a regular pattern of light vs. darkness stripes. Turbulence near threshold can be modeled by a Ginsburg-Landau theory.

The authors define an order parameter $\varepsilon = \frac{V-V_0}{V_0}$ and show that for $0 < \varepsilon < \varepsilon_1 = 0.21 \pm 0.01$, liquid crystals presented spacial inhomogeneity and did not equilibrate. In this case a Ginsburg Landau theory is no more effective and the system exhibits and *inhomogeneous* extremely complex dynamic of defects (of birth, death, oscillatory or drifting).

In this regime liquid Crystals can be modeled by a renewal non stationary event driven process $\xi_s(t)$ *experimentally* compatible with the assumption that the permanence in the on (+ state and off (−) state is determined by a Manneville distribution

$$\psi_{\pm}(t) = \frac{(\mu-1)T_{\pm}^{\mu-1}}{(T_{\pm}+t)^{\mu}} \tag{3.41}$$

with the same μ parameter. In the limiting case of weak turbulence the two values $T_+ \to T_-$

Since $\langle \xi_s(i) \rangle$ is proportional to the number of defects of the sample it can be *experimentally* measured through a measure of the total light intensity transmitted by the sample.

The external voltage, in this case, acts as the perturbation ξ_{P} to the liquid crystal system.

For this system a perturbation experiment has been done and the results agrees with the predictions of the dynamical LRT.

3.3.2 The Onsager Principle

Onsager's principle [47] is one of the key relations for traditional non equilibrium physics.

Onsager's principle The regression of system out of equilibrium is proportional to his unperturbed correlation function. That is for a system observable ξ_t

$$\frac{\langle \xi(t) \rangle_{\text{P}}}{\langle \xi(0) \rangle} \sim \frac{\langle \xi(t)\xi(0) \rangle_{\text{eq}}}{\langle \xi^2(0) \rangle_{\text{eq}}} \tag{3.42}$$

This principle is true if the system we are dealing with admits an *equilibrium correlation function* that is if the correlation function of the system is stationary. As we have seen the systems we are interested in are non stationary one. They exhibits different properties like

ageing which are obviously non compatible with Onsager's principle stated the way we have stated it.

Let us notice that Onsager's principle can be used an experimental definition of correlation function of a system, and has been extensively used under this perspective, since the Gibbs averages can be measured (in classical Ising models these are Magnetizations).

Is has been shown that Onsager's principle can be extended also to this case [3]). We want to find what the non stationary correlation function in this case is.

Let $p(t,s)$ be the probability that the system is in the same laminar region between at time t and s, $t > s$. In two different laminar regions the system will take values from two different variables that are clearly independent and their mean, therefore, is the product of the means of each variable, if the system is in the same laminar region instead we have to consider its variance. Clearly, then:

$$\langle \xi(t)\xi(s)\rangle = \langle p(t,s)\xi_i^2 + (1-p(t,s))\xi_i\xi_j\rangle = \mathbf{E}(\xi^2)\langle p(t,s)\rangle. \tag{3.43}$$

But the probability for the system to stay in the same laminar region can be calculated. In we can split the event $C = \{$The system at t is in the same laminar region than it was at $s\}$ can be split as the union of the events: $C(n,t,s) = \{$ The system at $t$is in the n-th laminar region and so it wa We can then write:

$$\begin{aligned}\langle \xi(t)\xi(s)\rangle &= \mathrm{Prob}(\bigcup_n C(n,t,s)) = \sum_{n=0}^{\infty}\int_0^s \psi_n(t')\Psi(t-t')\,\mathrm{d}t' \\ &= \Psi(t) + \int_0^s R(t')\Psi(t-t')\,\mathrm{d}t' = \Psi(t,s).\end{aligned} \tag{3.44}$$

3.3.3 Linear Response theory for event driven Poissonian processes

Before extending our theory to non stationary processes, we want to focus on poissonian dichotomous systems. In fact analyzing those systems will lead us to make a crucial Ansatz needed in order to be able to state a consistent linear response theory for complex non stationary systems.

The terms of the problem

Let us consider a system whose variable of interest can take two values +1 and −1. The transition between this two states is governed by the Master equation of coin tossing that is

$$\mathbf{p}(n) = \mathbf{p}(n) + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\mathbf{p}(n). \tag{3.45}$$

We want this process to be a poissonian event driven process.

Using the renewal formalism this implies we set a time dependent failure rate $g(t)$ to be a constant g_0. The waiting time distribution is then the exponential one

$$\psi(t) = g_0 \mathrm{e}^{-g_0 t} \tag{3.46}$$

In this simple case we can write down the Generalize Master equation noting that

$$\hat{\Phi} = g_0 \tag{3.47}$$

the GME becomes then[2]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = -g_0 \int_0^t \delta(t-t') \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{p}(t')\,\mathrm{d}t' = g_0 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{p}(t') \tag{3.48}$$

There are two possibilities of perturbing this equation. We could think the perturbation act on the transition rate between the two states, in this case we will call the *phenomenological* perturbation theory, or we could figure that the perturbation act on the system in order to change the perturbation rate, approach we shall call *dynamic*.

Phenomenological Linear Response theory

In this case the perturbation acts on the probabilities, that is we are rigging the coin toss. In practice we let $p \to p + \frac{\varepsilon}{2}\xi_{\mathrm{P}}(t)$.The master equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = g_0 \begin{pmatrix} \frac{1-\varepsilon\xi_{\mathrm{P}}(t)}{2} & \frac{-1+\varepsilon\xi_{\mathrm{P}}(t)}{2} \\ \frac{-1+\varepsilon\xi_{\mathrm{P}}(t)}{2} & \frac{1-\varepsilon\xi_{\mathrm{P}}(t)}{2} \end{pmatrix} \mathbf{p}(t) \tag{3.49}$$

which is formally similar to the equation 3.33 of Stochastic Resonance. We obtain then exactly the same result for an observable variable A:

$$\langle A\rangle_{\mathrm{P}} - \langle A\rangle_{\mathrm{eq}} = \varepsilon\lambda \frac{A(+) - A(-)}{2} \int_0^t g_0 \mathrm{e}^{-g_0(t-s)}\xi(s)\,\mathrm{d}s. \tag{3.50}$$

If we calculate the equilibrium correlation function in this case we get:

$$C(t,s) = \mathbf{E}\,\xi(t)\xi(s) = p_+(t)p_+(s) + p_-(t)p_-(s) - p_+(t)p_-(s) - p_-(t)p_+(s) \tag{3.51}$$

were $p_\pm(t)$ using the usual trick of writing $P(t) = p_+(t) - p_-(t)$ the equation then becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = -g_0 P(t) \tag{3.52}$$

ant thus, remembering that $p_+ + p_- = 1$, and taking into account that $P(t) = \exp(-g_0 t)P(0)$ we obtain:

$$C(t,s) = P(0)\mathrm{e}^{-g_0(t-s)} = C(t-s) \tag{3.53}$$

thus we have $\chi(t,s) = -\frac{\mathrm{d}}{\mathrm{d}t}C(t-s) = \frac{\mathrm{d}}{\mathrm{d}(}s)C(t-s)$

[2]ageing is not issue for poissonian processes since $\psi(t,s) = \psi(t)$

Dynamical Linear Response

In this case we have to perturb the waiting time distribution $\psi(t)$.

In our renewal perspective the time dependent failure rate $g(t)$ totally determines the waiting time distribution, perturbing our waiting time distribution is the same that perturbing our time dependent failure rate.

For a Poissonian waiting time distribution $g(t) = g_0$. We make then a fundamental Ansatz that is that our perturbation acts on the time dependent failure rate only in the following way:

$$g_{\mathrm{P}}(t) = g_0 - g_0 \varepsilon \xi_{\mathrm{P}}(t) + o(\varepsilon^2) \tag{3.54}$$

that is for our system master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = (g_0 - g_0\varepsilon\xi_{\mathrm{P}}(t)) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{p}(t) = g_0 \begin{pmatrix} \frac{1-\varepsilon\xi_{\mathrm{P}}(t)}{2} & \frac{-1+\varepsilon\xi_{\mathrm{P}}(t)}{2} \\ \frac{-1+\varepsilon\xi_{\mathrm{P}}(t)}{2} & \frac{1-\varepsilon\xi_{\mathrm{P}}(t)}{2} \end{pmatrix} \mathbf{p}(t) \tag{3.55}$$

that look formally exactly the same that equation 3.49. This could be expected since we are in a stationary condition and then Kubo theorem hold. But this discussion has taught us how to perturb waiting time distributions, at least in Poissonian case. We notice, in fact, that, according to renewal formulas, we have:

$$\Psi_{\mathrm{P}}(t-s) = \mathrm{e}^{-\int_s^t g(t)} = \mathrm{e}^{-g_0 t - s}\mathrm{e}^{-g_0\varepsilon\int_s^t \xi(t')\,\mathrm{d}t'} = \Psi\left(t - s - \varepsilon\int_s^t \xi(t')\,\mathrm{d}t'\right) \tag{3.56}$$

3.3.4 Non poissonian event driven linear response theory

Until now we have only stated known results in a fancier way, pointing out some extremely important aspects.

Perturbing Generalized Master Equation: Sokolov's *phenomenological* approach

We first want to outline phenomenological linear response theory proposed by Sokolov [56]. As we have seen a non poissonian, infinitely young, system satisfies the generalized master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \int_0^t \Phi(t-t')\,\mathrm{K}\,\mathbf{p}(t') \tag{3.57}$$

where the Laplace transform of the memory kernel is given by

$$\Phi(u) = \frac{u\hat{\psi}(u)}{1-\psi(\hat{u})} \tag{3.58}$$

Using an approach equivalent to Sokolov proposal we can think our perturbation acts on the operator K. We can figure the perturbation acts on K changing it into $\mathrm{K}(t)$. In the following way:

$$\mathrm{K}(t) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \varepsilon\frac{1}{2}\xi_{\mathrm{P}}(t)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{3.59}$$

Solving the integro-differential system is not an easy task in general but if we consider only the 2 state system we can do it.

In fact since we are dealing with probabilities we have $p_+ + p_- = 1$ and thus introducing the quantity $\Pi(t) = p_+(t) - p_-(t)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = -\int_0^t \Phi(t-t')\Pi(t')\,\mathrm{d}t' - \varepsilon\xi_{\mathrm{P}}\int_0^t \Phi(t-t')\,\mathrm{d}t' \tag{3.60}$$

Since

$$A(t) = \int_0^t \Phi(t-t') = -\int_t^0 \Phi(s)\,\mathrm{d}s = \int_0^t \Phi(s)\,\mathrm{d}s \tag{3.61}$$

and

$$\hat{A}(u) = \frac{1}{u}\frac{u\hat{\psi}(u)}{1-\psi\hat{(u)}} = \sum_1^\infty \hat{\psi}^n = \hat{R}(u) \tag{3.62}$$

We can write

$$u\hat{\Pi(u)} - \Pi(0) = \frac{u\hat{\psi}(u)}{1-\hat{\psi}(u)}\hat{\Pi(u)} - \mathscr{L}\left[\varepsilon\xi_{\mathrm{P}}R\right](t) \tag{3.63}$$

that is

$$\hat{\Pi(u)} = \frac{1-\hat{\psi}(u)}{u}\left(\mathscr{L}\left[\varepsilon\xi_{\mathrm{P}}R\right](u) - \Pi(0)\right) \tag{3.64}$$

which leads, when starting from equilibrium $\Pi(0) = 0$, and remembering that $\langle\xi\rangle(t) = \Pi(t)$, to a response

$$\langle\xi\rangle_{\mathrm{P}} = -\varepsilon\int_0^t \Psi(t-t')R(t')\xi_{\mathrm{P}}(t') \tag{3.65}$$

We notice that in this case

$$\chi(t,t') = -\frac{\mathrm{d}}{\mathrm{d}t'}C(t,t') = -R(t)\Psi(t-t') \tag{3.66}$$

This choice while simple, do not take into account that the perturbation could act on the system during the *entire laminar region* causing the system to undergo a crucial event sooner or later than when it would have occurred in an unperturbed system.

For this reason we want to go further a perturb the waiting time distribution too. To do this we have to make an Ansatz on how to perturb our waiting time distribution.

The dynamic perturbation ansatz

We have now to make a new *Ansatz* which will enable us to carry on a consistent dynamical pertubation approach.

As we have seen in the previous sections, perturbational approach in Poissonian case can be carried out trivially. We have seen that thought as a dynamical perturbation it leads to equation 3.55

We are then naturally led to extend this perturbative structure also to non poissonian event driven systems that is:

Dynamic approach Ansatz

$$\Psi_{\mathrm{P}}(\tau|s) = \Psi\left(\tau - \mathrm{O}\,\varepsilon \int_s^{s+\tau} \xi(t')\,\mathrm{d}t'\right) \tag{3.67}$$

Where the operator O take into account the effects of the perturbation on probabilities of the leading process. We can think this ansatz in a fancy way.

We can think to expand our survivor probability in terms of exponential functions. According to Beck prescriptions [12] we write [3]:

$$\Psi(t-s) = \int_0^\infty \Pi(g)\mathrm{e}^{-g(t-s)}\,\mathrm{d}g \tag{3.68}$$

and then we think that these exponential functions are perturbed as in the poissonian case:

$$\Psi_{\mathrm{P}}(\tau|s) = \int_0^\infty \int_0^\infty \Pi(g)\mathrm{e}^{-g_0(\tau)+g_0\varepsilon\,\mathrm{O}\int_s^{s+\tau}\xi(t')\,\mathrm{d}t'}\,\mathrm{d}g = \Psi\left(\tau - \varepsilon\,\mathrm{O}\int_s^{s+\tau}\xi(t')\right) \tag{3.69}$$

In this way we perturb the probability without affecting the statistics.

By the definition of $\Psi_{\mathrm{P}}(\tau|s)$ we can infer the form of the perturbation as it should act on the waiting time distribution. We have in fact:

$$\begin{aligned}\psi_{\mathrm{P}}(\tau|s) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi_{\mathrm{P}}(\tau) &= \left[1+\mathrm{O}\,\varepsilon\xi(\tau)\right]\psi\left(\tau - \mathrm{O}\,\varepsilon\int_s^{s+\tau}\xi(t')\,\mathrm{d}t'\right)\\ &\approx \psi(\tau) + \varepsilon\,\mathrm{O}\left[\psi(\tau)\xi(\tau) + \psi'(\tau)\int_s^{s+\tau}\xi(t')\,\mathrm{d}t'\right] + o(\varepsilon^2)\end{aligned} \tag{3.70}$$

The presence of the integral term characterize our perturbation theory in that it is not time invariant since the way the perturbation acts depends on explicitely of the age of our system s.

If we expand our perturbation up to the first order in ε we obtain:

$$\Psi_{\mathrm{P}}(t-s|s) \approx -\varepsilon\,\mathrm{O}\,\psi(t-s)\int_s^t \xi_{\mathrm{P}}(q)\,\mathrm{d}q = \varepsilon\Psi_{\mathrm{P}}^{(1)}(t-s|s)\,\mathrm{O} \tag{3.71}$$

and for the perturbed waiting time distribution:

$$\psi_{\mathrm{P}}(t-s|s) \approx \varepsilon\,\mathrm{O}\left[\psi(t-s)\xi(t-s) + \psi'(t-s)\int_s^t \xi(t')\,\mathrm{d}t'\right] = \varepsilon\psi^{(1)}(t-s|s)\,\mathrm{O} \tag{3.72}$$

The meaning of operator O

Until now we have not said anything about the operator O we have inserted in our Ansatz. At that point we have inserted our operator as a simple add on. We want now try to specify its form at least in the dichotomous case. If **A** is an observable of the system (i.e $\mathbf{A} = \{-1, 1\} \to \mathbb{R}$) we define our operator matrix by:

$$m_A = \langle \mathbf{A} \rangle = \mathbf{A}^T\,\mathrm{O}\,\mathbf{p}_{\mathrm{eq}} \tag{3.73}$$

[3] Ψ is in practice the inverse Laplace transform of $\Pi(g)$

That is how would the system respond if it were a discrete time process described by our leading process.

Our operator O acts then on the system

$$\mathrm{O}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix} \tag{3.74}$$

that is

$$(\mathrm{I}+\mathrm{K})\,\mathrm{O}\,\mathbf{p}_{\mathrm{eq}} = 0. \tag{3.75}$$

In general the equilibrium probability vector will be the 0 eigenvector of the matrix K. The perturbational operator O then will be the operator that applied to equilibrium probability state will give the eigenvector corresponding to the first nonzero eigenvalue ("the first excited state"). Since for multicothomous system $\mathrm{K}^2 + \mathrm{K} = 0$ it is straightforward to see $(1+\mathrm{K})\,\mathrm{O}\,\mathbf{p}_{\mathrm{eq}} = 0$.

3.3.5 The Fluctuation-Dissipation theorem

We are now able to state the main result of this thesis.

We are interested in expanding our probability vector in terms of ε

$$\mathbf{p}(t) = \sum_{n=0}^{\infty} \mathbf{p}^{(n)}(t)\varepsilon^n \tag{3.76}$$

Let us call the the Montroll Weiss evolution operator:

$$\mathbf{p}(t) = T_{\mathrm{K}}(t-s|s)\mathbf{p}(s) = \sum_{n=0}^{\infty}\int_0^{t-s} \Psi(t-s-t')\psi_n(t')(1+\mathrm{K})^n\mathbf{p}(s), \tag{3.77}$$

which governs the evolution of the system from from time s to time t provided that at time s an event has occurred.

By definition, on the equilibrium state, we have:

$$\mathbf{p}_{\mathrm{eq}}(t) = T_{\mathrm{K}}(t-s|s)\mathbf{p}_{eq}(s). \tag{3.78}$$

The unperturbed evolution, under our hypothesis that the system is a renewal system, is obviously governed by $T_{\mathrm{K}}(t-s|s)$.

In doing our calculations we have to remember that, in order for the perturbation to act, we have that a least one event occur after preparation time.

We shall see that under the dynamical perturbational frame this leads to :

$$\mathbf{p}^{(1)}(t) = \mathbf{L}(t) + \mathbf{E}(t) \tag{3.79}$$

where:

$$\mathbf{L}(t) = \int_0^t (\delta(s) + R(s))\Psi_{\mathrm{P}}(t-s|s)\mathbf{p}_{\mathrm{eq}}\,\mathrm{d}s \tag{3.80}$$

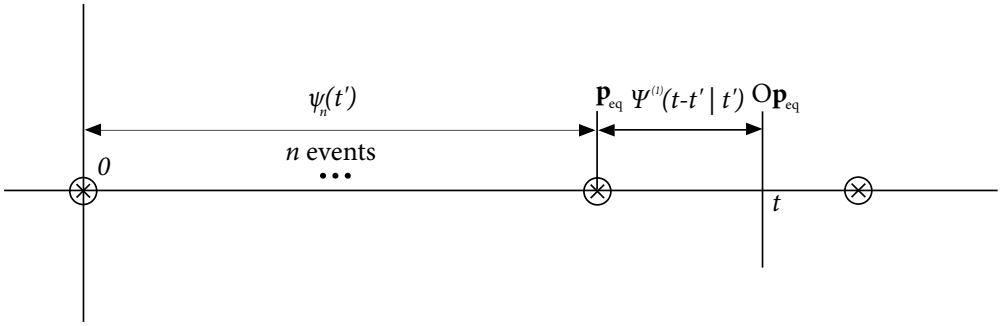


Figure 3.2: One of the terms of the $\mathbf{L}(t)$ contribution

and:

$$\mathbf{E}(t) = \int_0^t \int_0^s \psi^{(1)}(s-q|q) R(q)\, \mathrm{T_K}(t-s)(\mathrm{I}+\mathrm{K})\mathbf{p}_{eq}\, \mathrm{d}q\, \mathrm{d}s. \tag{3.81}$$

To understand where those contributions come from we have to make these considerations.

Since we are interested in the linear term to the perturbational series we have to consider only terms in which only one event as been affected by pertubation .

Taking into account the first order contribution we can see they arise from two different elements of the Gibbs ensemble:

1. Those in which the last event occured at time $s < t$ and the perturbation is acting in the last laminar region 3.2. This contribution will be denoted as $\mathbf{L}(t)$
2. Those in which the effect of perturbation occurred in a previous laminar region 3.85. This contribution will be denoted as $\mathbf{E}(t)$.

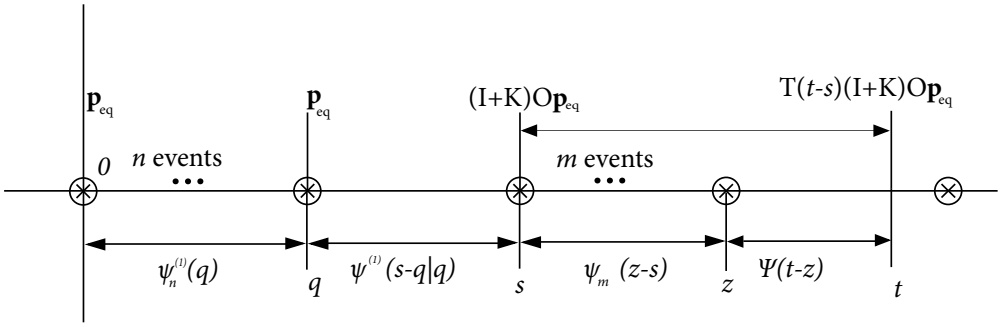


Figure 3.3: One of the infinite terms of the $\mathbf{E}(t)$ contribution

Calculation of $\mathbf{L}(t)$

$\mathbf{L}(t)$ take into account the contribution of the elements of Gibbs ensemble in which the last laminar region is influenced by the perturbation.

For these elements, since we are starting from an equilibrium distribution, the system probability distribution will be $\mathbf{p}_{\text{eq}}$ until the last event occur and only during the last event it will shift to the first excitation $\mathrm{O}\,\mathbf{p}_{\text{eq}}$. That is:

$$\begin{aligned}\mathbf{L}(t) = \mathrm{O}\,\mathbf{p}(t)\,\mathrm{Prob}\{(\bigcup_{q} &\text{ last event occurred at time } q \\ &\text{last laminar region is perturbed and lasts until } t)\}\end{aligned} \tag{3.82}$$

and thus by independence and remembering that our ansatz prescribes that we have to use the perturbed survivor probability of equation 3.71 for last laminar region: we obtain:

$$\mathbf{L}(t) = \int_0^t (\delta(s) + R(s))\Psi_{\mathrm{P}}^{(1)}(t-s|s)\,\mathrm{O}\,\mathbf{p}_{\text{eq}}\,\mathrm{d}s. \tag{3.83}$$

using formula 3.71 we obtain:

$$\mathbf{L}(t) = \int_0^t (\delta(s) + R(s))\psi(t-s)\int_s^t \xi_{\mathrm{P}}(q)\,\mathrm{d}q\,\mathrm{O}\,\mathbf{p}_{\text{eq}}\,\mathrm{d}s \tag{3.84}$$

Calculation of $\mathbf{E}(t)$

The second contribution is more difficult to obtain. As we have said, we have to consider in this contribution only those elements of Gibbs in which only one laminar region is perturbed. Since we are dealing with renewal systems, the evolution after the perturbation has acted, is given by Montroll-Weiss evolution operator applied to the perturbed probability vector. Because we are staring from our equilibrium distribution, up to the laminar region in which the perturbation acts the system will remain at equilibrium

Accordingly to the sketch figure 3.3 we can write then:

$$\mathbf{E}(t) = \int_0^t \int_0^s \psi^{(1)}(s-q|q)R(q)\,\mathrm{T}_{\mathrm{K}}(t-s)(\mathrm{I}+\mathrm{K})\,\mathrm{O}\,\mathbf{p}_{\text{eq}}\,\mathrm{d}q\,\mathrm{d}s. \tag{3.85}$$

3.3.6 Dichotomous non-poissonian case

For dichotomous non poissonian event driven systems a LRT had yet been provided with a slightly different formalism here [7]. This allows us to test our theory.

Since we are starting on equilibrium and $(\mathrm{I}+\mathrm{K})\,\mathrm{O}\,\mathbf{p}_{\text{eq}} = 0$ because the operator $\mathrm{O}\,\mathbf{p}_{\text{eq}}$ is antisymmetric and operator $\mathrm{I}+\mathrm{K}$ is symmetric.

Therefore only the linear contribution is present:

$$\mathbf{L}(t) = \varepsilon\,\mathrm{O}\int_0^t (\delta(s) + R(s))\psi(t-s)\int_s^t \xi_{\mathrm{P}}(q)\,\mathrm{d}q\,\mathrm{d}s \tag{3.86}$$

To obtain a linear response form we have to do some calculations:

Some calculations

In this subsection we prove the important relationship

$$\int_0^t \mathrm{d}s \Xi(s) \left(\int_0^s Y(q)\,\mathrm{d}q \right) = \int_0^t \mathrm{d}s Y(s) \int_s^t \mathrm{d}q \Xi(q). \tag{3.87}$$

To prove this relation we set

$$\Xi(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \int_s^t \Xi(q)\,\mathrm{d}q = -\frac{\mathrm{d}}{\mathrm{d}s} A(s) \tag{3.88}$$

By plugging it into the left-hand term of equation 3.87) we obtain

$$\begin{aligned} \int_0^t \mathrm{d}s \Xi(s) \left(\int_0^s Y(q)\,\mathrm{d}q \right) &= -\int_s^t \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_s^t \Xi(\tau)\,\mathrm{d}\tau \right) \left(\int_0^s Y(q)\,\mathrm{d}q \right) \\ &= -\left[\int_s^t \Xi(s)\,\mathrm{d}s \cdot \int_0^s Y(q)\,\mathrm{d}q \right]_0^t + \int_0^t Y(s) \int_s^t \Xi(q)\,\mathrm{d}q\,\mathrm{d}s. \end{aligned} \tag{3.89}$$

On the other hand

$$\left[\int_s^t \mathrm{d}q \Xi(q) \cdot \int_0^s Y(q)\,\mathrm{d}q \right]_0^t = 0, \tag{3.90}$$

thereby yielding Eq. (3.87).

The dynamic Linear Response Theory

We are now able to calculate the Linear Response for a dynamical quantity under our dynamical prescription.

$$\langle \mathbf{A} \rangle_{\mathrm{P}}(t) = \varepsilon \mathbf{A}^T \mathbf{L}(t) = \varepsilon m_A \int_0^t (\delta(s) + R(s)) \psi(t-s) \int_s^t \xi_{\mathrm{P}}(q)\,\mathrm{d}q\,\mathrm{d}s \tag{3.91}$$

We set $\Xi(q) = \xi_P(q)$ and

$$Y(s) = \int_0^s [\delta(q) + R(q)]\,\psi(t-q)\,\mathrm{d}q. \tag{3.92}$$

Thus, using Eq.(3.87) we write Eq. (3.91) under the form

$$\langle \mathbf{A} \rangle(t) = \varepsilon m_A \int_0^t \xi_{\mathrm{P}}(s) \int_0^s (\delta(s) + R(s)) \psi(t-s)\,\mathrm{d}s\,\mathrm{d}t \tag{3.93}$$

Differentiating the correlation function found in equation 3.44 we can obtain:

$$\psi(t,s) = \frac{\mathrm{d}}{\mathrm{d}t} C(t,s) = \int_0^s (\delta(s) + R(s)) \psi(t-s)\,\mathrm{d}s \tag{3.94}$$

and thus equation 3.93 reads

$$\langle \mathbf{A} \rangle(t) = \varepsilon m_A \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} C(t,s) \xi_{\mathrm{P}}(s)\,\mathrm{d}s \tag{3.95}$$

Which is the statement of *dynamic linear response*.

3.3.7 Response to an harmonic perturbation

In this section we want to derive how the system respond to an harmonic perturbation. Let us begin by calling:

$$Q(t) = \frac{\langle A \rangle(t)}{\varepsilon m_A} \tag{3.96}$$

To be able to do our calculation correctly we will consider first $\xi_{\mathrm{P}}(t) = \exp(\mathrm{i}\omega t)$.

We have then to calculate

$$Q(t) = \int_0^t \psi(t,s)\exp(i\omega t)\,\mathrm{d}s \tag{3.97}$$

Let us make the Laplace transform of Eq. (3.97). We apply the method of integration by parts, yielding

$$\hat{Q}(u) = \int_0^\infty \mathrm{d}t \exp(-ut) \int_0^t \psi(t,s)\exp\left(-i\omega s\right) = \int_0^\infty \mathrm{d}t \exp(-ut) \int_0^t \left(\frac{1}{-i\omega}\right)\psi(t,s)\frac{\mathrm{dexp}\left(-i\omega s\right)}{\mathrm{d}s}. \tag{3.98}$$

Thus, we obtain

$$\hat{Q}(u) = \left(\frac{i}{\omega}\right)\int_0^\infty \mathrm{d}t \exp\left(-ut\right)\left\{\left[\psi(t,s)\exp\left(-i\omega s\right)\right]_0^t - \int_0^t \mathrm{d}s \exp\left(-i\omega s\right)\frac{d\left(\psi(t,s)\right)}{\mathrm{d}s}\right\}. \tag{3.99}$$

Note that, as prescribed by the dynamic LRT, $\psi(t,s)$ is the probability density of an event occurrence at time t given the condition that the waiting process begins at time s. Using the corresponding survival probability $\Psi(t,s)$ we write $\psi(t,s)$ as

$$\psi(t,s) = \psi(t) + \int_0^s R(q)\psi(t-q). \tag{3.100}$$

Here, to establish a closer connection with the calculations done in Ref. [11] we consider also the preparation event occurring at $t = 0$. Using the definition of $\psi(t,s)$ given by Eq. (3.100), we get

$$\hat{Q}(u) = \frac{i}{\omega}\int_0^\infty \exp(-ut)\left[\psi(t,t)\exp(-i\omega t) - \psi(t) - \int_0^t \exp(-i\omega s)\sum_{n=1}^\infty \psi_n(s)\psi(t-s)\,\mathrm{d}s\right]\mathrm{d}t \tag{3.101}$$

Using again the definition of Eq. (3.100), we have

$$\hat{Q}(u) = \left(\frac{i}{\omega}\right)\int_0^\infty \mathrm{d}t \exp(-ut)\left\{\exp(-i\omega t)\left[\psi(t) + \sum_{n=1}^\infty \psi_n(s)\psi(t-s)\right] - \psi(t) - \int_0^t \mathrm{d}s exp(-i\omega s)\sum_{n=1}^\infty \psi_n(s)\psi(t-s)\right\} \tag{3.102}$$

Taking into account that we are dealing with uncorrelated events ($\hat{\psi}_n(u+i\omega) = (\hat{\psi}(u+i\omega))^n$) we write the Laplace transform of Eq. (3.102) as follows:

$$\hat{Q}(u) = \left(\frac{i}{\omega}\right)\left\{\hat{\psi}(u+i\omega) + \sum_{n=1}^{\infty}\left[\hat{\psi}(u+i\omega)\right]^n \psi(u+i\omega) - \hat{\psi}(u) - \sum_{n=1}^{\infty}\left[\hat{\psi}(u+i\omega)\right]^n \psi(u+i\omega)\right\}, \tag{3.103}$$

which yields

$$\hat{Q}(u) = \left(\frac{i}{\omega}\right)\left\{\left[\hat{\psi}(u+i\omega) - \hat{\psi}(u)\right] + \sum_{n=1}^{\infty}\left[\hat{\psi}(u+i\omega)\right]^n \left[\psi(u+i\omega) - \hat{\psi}(u)\right]\right\}. \tag{3.104}$$

By summation of the geometric series, we obtain the important result

$$\hat{Q}(u) = \left(\frac{i}{\omega}\right)\frac{\hat{\psi}(u+i\omega) - \hat{\psi}(u)}{1 - \hat{\psi}(u+i\omega)}. \tag{3.105}$$

With the help of Eq. (3.105) we now evaluate the system's response to a harmonic perturbation. Let us define

$$\Pi(t) \equiv \varepsilon Re(Q(t)) \tag{3.106}$$

Thus, the linear response to a harmonic perturbation is obtained by anti-Laplace transforming:

$$\hat{\Pi}(u) = \varepsilon Re\left[\frac{i}{\omega}\frac{\psi(u) - \hat{\psi}(u+i\omega)}{\left[1 - \hat{\psi}(u+i\omega)\right]}\right], \tag{3.107}$$

We use as density distribution the negative derivative of a Mittag-Leffler, namely

$$\psi(t) = -\frac{d}{dt}E_\alpha\left[-\left(\frac{t}{T}\right)^\alpha\right], \quad \alpha = \mu - 1. \tag{3.108}$$

Its Laplace transform is

$$\psi(u) = \frac{1}{1 + (Tu)^\alpha}. \tag{3.109}$$

Plugging Eq. (3.109) into Eq. (3.107), after a little algebra we obtain

$$\hat{Q}(u) = \varepsilon Re\left[\frac{i}{\omega}\left(1 - \frac{u^\alpha}{(u+i\omega)^\alpha}\right)\right]\frac{1}{1 + (Tu)^\alpha}. \tag{3.110}$$

As we see Eq. (3.110) has a correction factor with respect to the expression for the linear response [11]. This factor could be neglected for $u \to 0$. As we shall see this extra factor affects the amplitude and the phase as well. It is possible write an exact expression for the inverse Laplace transform of Eq. (3.110). Using the result of [11] we can write

$$\Pi(t) = \varepsilon(\mu-1)Re\left\{\int_0^t F[2-\mu, 2, i\omega(t-t')]\exp[i\omega(t-t')]\frac{d}{dt'}E_\alpha\left[-\left(\frac{t'}{T}\right)^\alpha\right]dt'\right\} \tag{3.111}$$

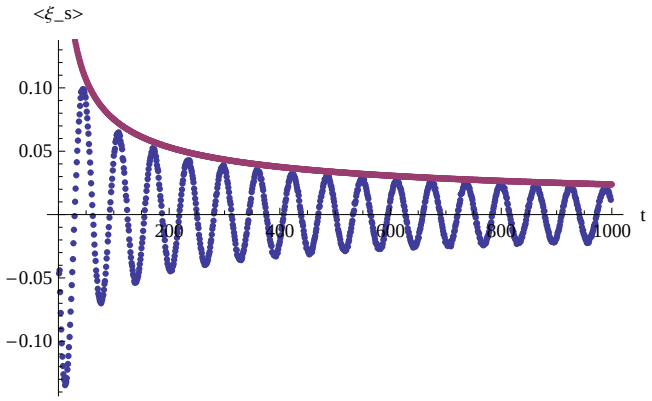


Figure 3.4: Dynamical Linear Response to an harmonic perturbation $\cos(0.1t)$ calculated using the correlation function $\mathscr{C}$ of figure 2.3($T = 20, \mu = 1.5$). The Purple dots represent $\frac{1}{\Gamma(\mu)}\frac{1}{t^{\mu-2}}$

The asymptotic expression for $\Pi(t)$ is

$$\Pi(t) \approx \varepsilon Re\left\{\frac{\exp\left[i\left(\omega t+\frac{\pi}{2}\mu\right)\right]}{\Gamma[\mu-1](\omega t)^{2-\mu}}\frac{1}{1+(i\omega T)^{\mu-1}}\right\} = \varepsilon\,|z|\,\frac{\cos\left[\omega t+\frac{\pi}{2}\mu+\varphi\right]}{\Gamma[\mu-1](\omega t)^{2-\mu}} \tag{3.112}$$

where we defined

$$\frac{1}{1+(i\omega T)^{\mu-1}} = |z|\,\exp[i\varphi].$$

Clearly for $\omega T \ll 1$ we obtain the result of ([11]):

$$\Pi(t) = \frac{\varepsilon}{\Gamma(\mu-1)}\frac{cos\left(\frac{\pi\mu}{2}+\omega t\right)}{(\omega t)^{2-\mu}}. \tag{3.113}$$

3.3.8 Phenomenological Response to harmonic perturbation: The "Freud effect"

With calculations of the same kind as those used to derive Eq. (3.105) we obtain [4]

$$\hat{Q}(u) = -\frac{1-\hat{\psi}(u)}{u}Re\left\{\frac{\hat{\psi}(u+i\omega)}{[1-\hat{\psi}(u+i\omega)]}\right\}. \tag{3.114}$$

It is shown [4]that this formula yields

$$\Pi(t) \approx \varepsilon\left(\frac{T}{T+t}\right)^{\mu-1} - \frac{\varepsilon}{\Gamma(2-\mu)(\omega t)^{\mu-1}}\sin\frac{\pi\mu}{2} + \frac{\varepsilon}{\Gamma(\mu-1)}\frac{cos\left(\frac{\pi\mu}{2}+\omega t\right)}{(\omega t)^{2-\mu}}, \tag{3.115}$$

where the last term is identical to that of Eq. (3.113). Thus, we see that the adoption of the phenomenological theory generates a slow non-oscillatory regression to equilibrium, which depends on the sign of perturbation at the time when the perturbation is applied.

This characteristic is known as "Freud Effect" (as shown in figure 3.5)

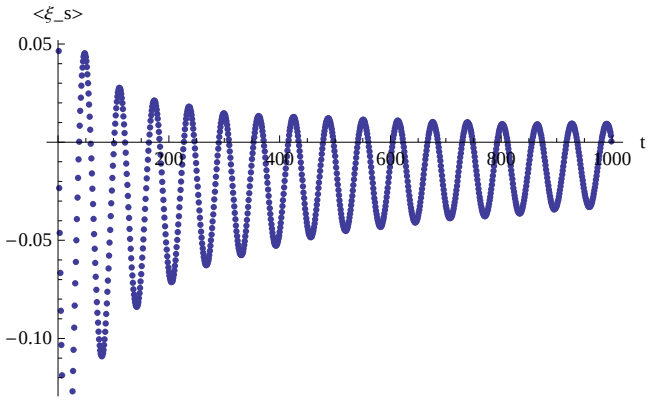


Figure 3.5: Phenomenological Linear Response for an harmonic perturbation $\cos(0.1t)$ calculated using the correlation function $\mathscr{C}$ of figure 2.3

3.3.9 Liquid Crystals experiment

We want to describe here an experiment on liquid crystals that could be interpreted trough our theoretical approach. Since it has been proved in recent paper [54] that these systems exhibit an event-driven power law relaxation we are led to analyze what happens when the system is perturbed by an harmonic perturbation.

The prediction of our dynamical linear response theory for harmonic perturbations provides a response of the form:

$$\Pi(t) = \frac{\varepsilon}{\Gamma(\mu-1)} \frac{cos\left(\frac{\pi\mu}{2} + \omega t\right)}{(\omega t)^{2-\mu}} = CR(t)\cos(\omega t + \varphi) \tag{3.116}$$

In an recently accepted article [6], the same authors have verified an experimental quenching of harmonic stimuli, that is the response to an external harmonic perturbation is characterized by an effective decay of a factor:

$$\langle \xi \rangle_S \sim CR(t)cos(\omega t + \varphi), \tag{3.117}$$

where C and φ are fitting parameters while μ is calculated from the regression to equilibrium.

We want to illustrate better the experimental results included in this article in order to interpret them in the dynamic LRT perspective.

The experimental setup is the same used in [54]. In this case the potential difference applied was modulated with an harmonic perturbation.

preparation The system if first prepared by applying a potential difference of $V = 24 \pm 0.1V$ which bring the system to a fully developed turbulence in order to destroy preexistent structures ($\mu > 3$) and then put it to voltage $V_c = 20.4 \pm 0.1V$ which correspond to a defect mediated turbulence $\mu \sim 1.5$. Each defect it modeled as a dichotomous non stationary renewal process $l(t)$.

perturbation Once the system has been prepared, (a time of the order of $20s$ is awaited), a small voltage perturbation is applied to the system. To test the presence of Freud effect

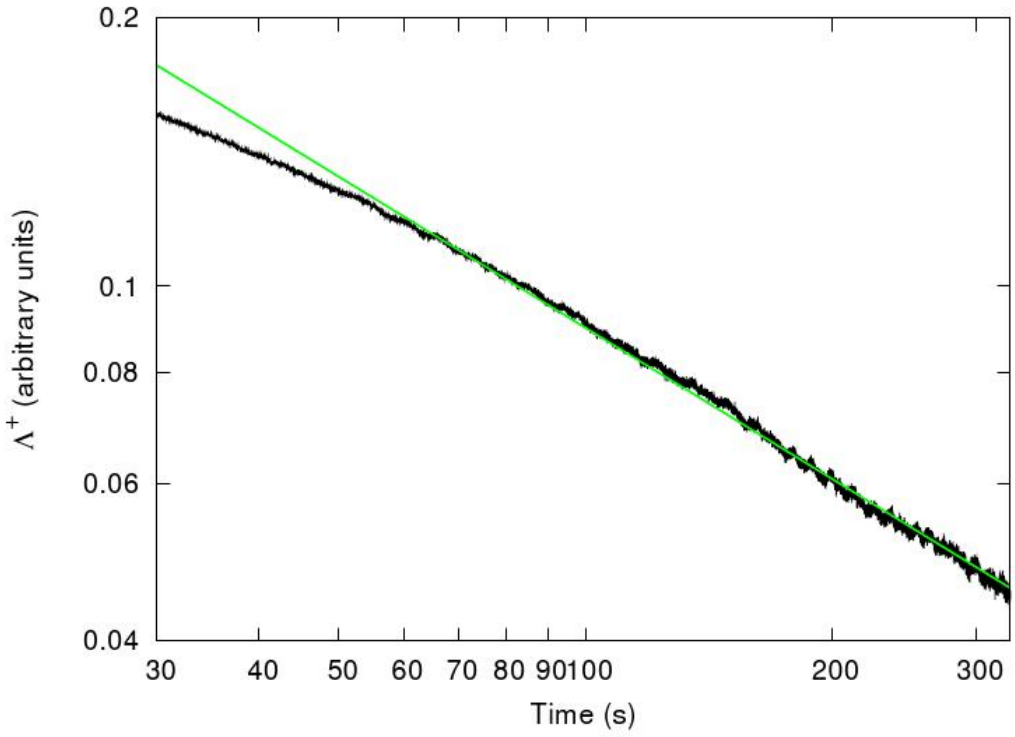


Figure 3.6: Here $S(t) = \Lambda_+(t)$ and the green line represent $t^{\mu-1}$ in log log plot.

the author apply first a voltage perturbation $\xi_P = \varepsilon \cos(\omega t)$ then a perturbation $\xi_+ = -\varepsilon \cos(\omega t)$ where $\varepsilon = \Delta V/(V_1 - V_c)$ is a small volt amplitude $< 0,05$.

The statistical mean of $\langle l \rangle(t)$ is proportional to the number of defects and so light intensity transmitted by the sample which can be experimentally measured. . We shall denote with $\langle l \rangle_+(t)$ the response to the perturbation ξ_+ and with $\langle l \rangle_+(t)$ the response to the perturbation ξ_-. In order to analyze the data the quantities

$$S(t) = \frac{\langle l \rangle_+(t) + \langle l \rangle_+(t)}{2} \tag{3.118}$$

and

$$D(t) = \frac{\langle l \rangle_-(t) + \langle l \rangle_+(t)}{2} \tag{3.119}$$

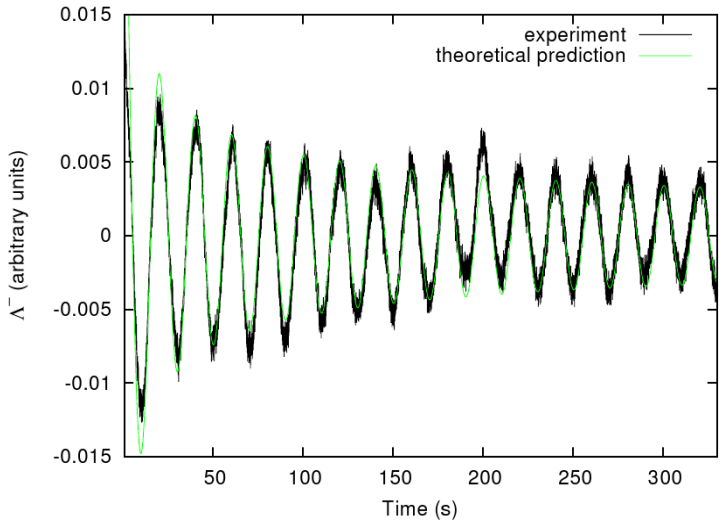


Figure 3.7: Here $D(t) = \Lambda_+(t)$ and the green line represent the prediction of equation

Figure 3.6 show the free regression to equilibrium. Since the process is prepared a time zero in highly out of equilibrium state at time zero there is a free regression to equilibrium which is proportional the the survivor probability and which is $\sim t^{1-\mu}$ this provide a good way to estimate μ and gives $\mu = 1.56 \pm 0.01$. $D(t)$ instead in shown in figure 3.7 and exhibit an extremely good agreement with our theoretical proposal. No Freud effect appears then we are led to conclude that Sokolov Phenomenological proposal cannot be applied in this case.

The authors have also controlled if the response were actually linear and the (positive) answer is given in figure 3.8. Using perturbation of different intensity it was showed that the intensity scaling of the perturbation was linear.

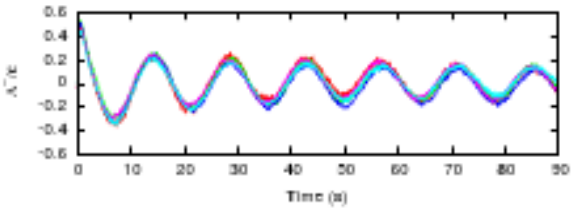


Figure 3.8: Here = $\Lambda_{-}(t)/\varepsilon = D(t)/\varepsilon$ with $\omega/2\pi = 0.07$Hz and $espilon = 0.025$ (red line), 0.031 (green), 0.037 (blue), 0.044 (purple), 0.050 (cyan)

3.4 Further consideration and conclusions

We want here point some arguments which need some further investigation. For dichotomous/multicotonomous system The correction to linear response theory is not zero anymore.

3.4.1 Fokker-Planck equation

If we consider, by example, discrete time Fokker Planck equation

$$p(y, n+1) = (\mathrm{I} + \mathrm{L_{FP}})p(y, n) = p(y, n) + (\gamma\partial_y y + D\partial_y^2)p(y, n) \tag{3.120}$$

Deriving from a microscopical discrete time Langevin

$$y(n+1) = -\gamma y(n) + f(n) \tag{3.121}$$

In this case the operator O, that is the operator that give rise to the first excitation is $\mathrm{O} = -\partial_y$ [4].

Since $p_{\mathrm{eq}}(y)$ defined by equation $\mathrm{L_{FP}}\, p_{\mathrm{eq}}(y) = 0$ and is a known quantity $p_{\mathrm{eq}} = \mathrm{e}^{-\frac{\gamma y^2}{2D}}$
In this case we easily see that:

$$(\mathrm{I} + \mathrm{L_{FP}})\,\mathrm{O}\, p_{\mathrm{eq}}(y, n) = \gamma\,\mathrm{O}\, p_{\mathrm{eq}} = \gamma\frac{\gamma}{D}\mathrm{e}^{-\frac{\gamma y^2}{2*D}} \tag{3.124}$$

In this case $L(t)$ does not vanish any more. It is an open question to see whether this contribution can neglected, and thus, wether LRT is still valid or not.

[4] We can easily calculate it in the continuous time case since, in this case, perturbing Langevin equation:

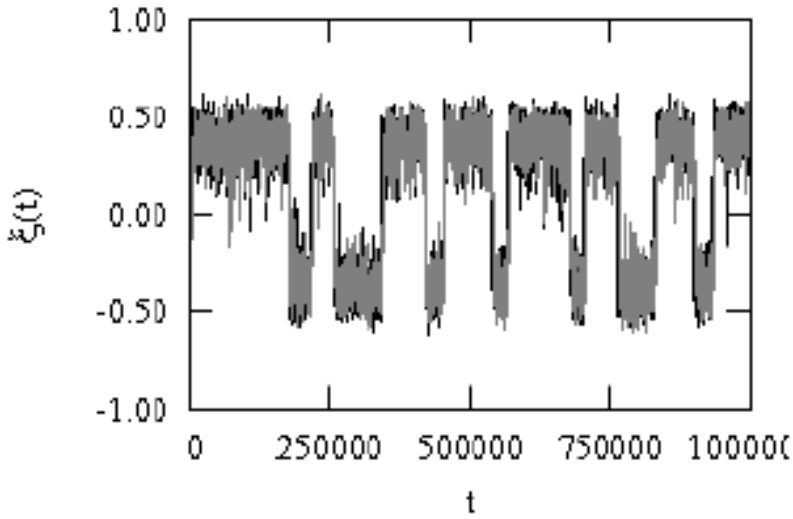
$$\dot{y} = -\gamma y + f(t) + \varepsilon\xi_P(t) \tag{3.122}$$

Leading to a Fokker-Planck equation:

$$\partial_t p(y, t) = (\gamma\partial_y y + D\partial_y^2 - \varepsilon\xi(t) - \partial_y)p(y, t) \tag{3.123}$$

3.4.2 Complexity Matching

Another very interesting question to be studied is what happens when the system is perturbed by a statistic signal. Numerical simulations have been done in [60] in which a complex network which is known to have a renewal non poissonian event driven dynamics is coupled with another system with the same characteristics.



(b)Introducing the coupling between network K_P (black line) and network K_S (gray line) leads to the synchronization. The coupling constant of network P is K_P=1.06. The coupling constant of network S is K_S=1.03.

Figure 3.9: An example of the response of a network of stochastic oscillator in a complex regime (renewal non poissonian regime) when weakly coupled with another complex networks of stochastic oscillators. Image taken from [60]

The system in this cases show a perfect response 3.9 which do not decay. This behavior has been called *complexity matching* since both the perturbing and the perturbed system are complex system. In a very "vivid" way complexity matching can be thought as a communication from one complex system to another.

The precise statements of this particular behavior is beyond the purposes of this thesis, but represents one of the most interesting questions of complexity science

3.4.3 Conclusions

Perturbational approaches are for physicists extremely natural. They take into account that small stimuli cannot produce significant changes in the dynamical properties of systems, that is, with an vivid metaphor, a fly cannot move an elephant (but still annoy him).

Kubo's theory extends this approach to statistical system. Though the derivation of Kubo's prescriptions proceeds from the microscopical picture of phenomena provided by Liouville's equations, what is actually perturbed linearily is the entire Gibbs system. Liouville equation

simply provide a sort of transition from microscopic to macroscopic world. This means that a linear response theory is acceptable even if the single trajectories do not respond linearily.

When we consider ergodic systems this transitions from the microscopic to the macroscopic is not straightforward.

Complex systems of our interest, however, are non stationary, and thus non ergodic. They exhibit a complex inhomogeneous dynamic which can effectively be modeled within a non poissonian renewal perspective. This means that the global statistical properties are well defined and can be measured, but their value cannot be inferred from microscopical description of their dynamics. Does a perturbative approach make sense in this case? This is a hard question to answer but the results of this thesis seem to suggest that, for a vast class of systems, the answer is yes.

Obviously, since the microscopical description either is not available or is too complex to be useful, a linear response theory of complex system lacks the usual, clear connection between perturbation and perturbated variable, which is expressed by the coupling Hamiltonian. It is not hard to figure that the perturbation acts on a complex system by perturbing his global properties which characterize the complex dynamics. But, expressing precisely what is being perturbed by what, is a hard (if not impossible) task.

We have taken into account our ignorance on the effective interaction by proposing a "phenomenal" ansatz, what we have called the dynamic perturbation ansatz, on the way Gibbs ensemble is perturbed by this perturbation applied to complex system. Carrying out calculation we have obtained an resonable result 3.95. A different Ansatz proposed from Sokolov lead to another result 3.65.

At least on the theoretical point of view, both proposals are equally plausible. There are attempts to model *phenomenologically* systems whose microscopical description is unknown or useless.

Predictions of those theories, nevertheless, are different. Sokolov's *phenomenological* approach exhibits what he calls "Freud Effect". Our dynamical approach does not; Neither do the experimental results on the liquid crystals. We are then led to think that, at least for liquid crystals, dynamic paradigm is the one which better models the structure of the response.

Liquid crystals experiment, and our theory too, show that a decaying response can appear *as a result of Linear Theory* without the need to invoke the "death of Linear Response theory" as some authors did [55, 57]. We are then led to ask:" Is Linear response theory universal?".

The presence of a correcting term to linear response seems to indicate that this is false for general systems and general responses forms but there are evidences that, at least for harmonic perturbations, the structure

$$\langle \xi_s \rangle = CR(t)\cos(\omega t + \varphi) \tag{3.125}$$

seem to persist.

We notice by the way that for harmonic perturbation the structure of equation 3.125 is a natural generalization of stochastic resonance response which takes into account the ageing of the system, that is, the fact the (infinitely lasting) regression to equilibrium induces a progressive diminution of the number of events per unit time (which are the indication of the non equilibrium state of the system).

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