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Valuing guaranteed annuity options using the principle of equivalent utility

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ABSTRACT

This thesis considers the valuation of guaranteed annuity options using an equivalent utility principle from the point of view of the policyholder. In this model I give an explicit form to the value functions involved in the indifference valuation. Also I offer a numerical implementation. For instance, in a setting where interest rates are constant, I find an explicit solution for the indifference problem, where the individual is described by a power (instantaneous) utility function. In this setting, I compare two strategies at the time of conversion, and two strategies at the moment when the policy is purchased. In the former, I assume that if the annuitant does not exercise the option, first she withdraws her policy's accumulated funds, and then seeks to solve a standard Merton's problem, under an infinite time horizon setting. In the latter strategy, I compare the agent's expected utility associated to a policy that embeds a guaranteed annuity option, and a policy that does not embed such an option. In order to accumulate the retirement funds, I assume in both cases a pure premium paid at a constant continuous stream. Regarding the optimal strategy, I am able to derive explicit solutions for a class of problems where finite horizon, bequest motive and power consumption utility are jointly considered.

The present research has as a primary objective to elaborate an utility indifference valuation model for guaranteed annuity options. The literature available until now considers both financial and actuarial approaches that have been used to evaluate and describe the nature of such options. On the contrary, the approach I present is able to embed the theory of the optimal asset allocation toward the end of the life cycle in the valuation of guaranteed annuity options. To my knowledge, the indifference approach I propose, is new and never developed before.

The main results show that the option's indifference value, both at the time when the policy is purchased and at the conversion time, depends on the difference between the guaranteed conversion rate h *and the market interest rate* r. *In line with the literature, at the time of conversion the agent will in general find advantageous to exercise the option. The dependency on* h *and* r *of the equivalent valuation also reveals that in periods characterized by high market interest rates, the value of the* G.A.O. *turns out to be very small. This model remains coherent if we compare the policyholder's point of view and the insurer's point of view, under an economic setting characterized by high interest rates.*

The present model can be extended in order to consider a richer setting, concerning both the accumulation and the decumulation period. These ideas are suggested and described in the course of chapters that follow.

The remainder is organized as follows. After a short introduction on the theory of controlled diffusion processes, chapter 2 recalls the models for human mortality and the concept of longevity risk. In the same chapter the nature of the guaranteed annuity option is described and some preliminary concerns on valuing this kind of rights are highlighted. Chapter 3 proposes the indifference model used for valuing a guaranteed annuity option. In this context two arrangements are outlined. Finally chapter 4 offers the explicit solution for the indifference valuation problem and numerical implementations.

J.E.L. classification. D91; G11; J26.

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CONTENTS

CHAPTER 1

INTRODUCTION

Guaranteed annuity options (G.A.O.) are options available to holders of some insurance policies. After a given period, in which the policyholder accumulates funds paying a single or a regular premium, the agent is given the right to convert the accumulated funds at pre-determined rate. In particular, the guaranteed conversion may concerns an amount of cash, with the option to convert to an annuity; or an amount of annuity, with the option to convert in to cash. In the literature generally just the former option is actually intended as a guaranteed annuity option.

The factors that influence the cost and the risk associated to a guaranteed annuity option concern the structure of long-term interest rates and the future mortality rates. In fact, the survival assumption implicit in the guarantee need to consider the future improvement in mortality. In other words, guaranteed annuity options may incorporate an import risk that can affect the stability and insurer's solvability: they may represent an important and valuable liabilities associated to these guarantees.

The literature over guaranteed annuity options begins with Bolton et al. (1997), where the nature of these options is analyzed, and a first approach to measure their value is proposed. For a stochastic approach on modelling the pioneering approach is offered by Milevsky and Promislow (2001), wherein both mortality and interest rates are considered.

In the present framework I approach the problem of valuing the guaranteed annuity options from the point of view of the insured. In doing that I will use an utility indifference argument. It will be offered a model where different strategies are analyzed both at the moment of conversion and at the initial time. In doing that, the theory of optimal annuitization policies – developed in some contributions offered by Milevsky and Young – is embedded in the indifference model that described in this thesis.

In order to present a self-contained work, in the following sections I recall some well known results on the theory of stochastic optimal control.

1.1 Short notes on the theory of controlled diffusion processes

In chapter 3 it is proposed an indifference valuation for guaranteed annuity options. In order to do that I offer a short review regarding some main results on the theory of controlled diffusion processes. For a comprehensive overview of this theory of stochastic optimal control see Øksendal (2003), Björk (2004) and also to Krylov (1979), Fleming and Rishel (1975), Fleming and Soner (2006) and for a more applied introduction to Chang (2004).

Consider the random process $\{w_s\}$, $s \in \mathrm{R}$, such that

$$w_t = w + \int_0^t b(\alpha_s, w_s)\,\mathrm{d}s + \int_0^t \sigma(\alpha_s, w_s)\,\mathrm{d}B_s \tag{1.1}$$

where b is a n-valued function, defined on $U \times \mathrm{R}^n$, where $U \subset \mathrm{R}^k$, and σ is a $n \times m$ matrix, defined on $U \times \mathrm{R}^n$. The initial value of the process is w and $\{B_s\}$ denotes a m-dimensional Brovnian motion on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}, \mathrm{P})$ and U is the set of admissible controls. Choosing different random processes $\{\alpha_s\}$, with values in U, we obtain various solutions for equation (1.1). In other words we *control* the process $\{w_t\}$. In order to have a well-defined stochastic integral we require that $\{\alpha_s\}$ is a stochastic process, with value in U and which is adapted to the filtration $\{\mathscr{F}_s\}$. For *usual* conditions that we shall assume for the probability space, I address to Protter (2005). Also we require suitable conditions in order that process $\{w_s\}$ exists and is unique. I do not want to enter into details and these property for $\{w_s\}$ are assumed. Finally, it is common to write equation (1.1) in the following *formal* way:

$$\begin{cases} dw_s = b(\alpha_s, w_s)\,ds + \sigma(\alpha_s, w_s)\,dB_s \\ w_0 = w \end{cases} \tag{1.2}$$

Consider a fixed domain $D \subset \mathbb{R}^n$ and let T the first exit time from D for the process $\{w_s\}$:

$$T := \inf\{s > 0 : w_s \notin D, \text{ given } w_0 = w\} \leqslant +\infty$$

and suppose that for all w and $\alpha_s \in U$

$$\mathbb{E}\left[\int_0^T \left|f^{\alpha_s}(w_s)\right| ds + \left|g(w_T)\right| \cdot \chi_{\{T<+\infty\}} \,\middle|\, w_0 = w \right] < +\infty$$

where by χ is denoted the indicator function and $f^{\alpha_s}(w_s) := f(w_s, \alpha_s)$ and g are given functions. Notice that in the definition of function f^{α_s}, we should better write $f^{\{\alpha_s\}}(w_s)$ to the cost, however, of using heavier notation.

Given the dynamics (1.2) and the assumptions above, we are interested in solving the following optimization problem

$$\Phi(w) = \sup_{\{\alpha_s\} \in \mathscr{D}} \mathbb{E}\left[\int_0^T f^{\alpha_s}(w_s)\,ds + g(w_T) \cdot \chi_{\{T<+\infty\}} \,\middle|\, w_0 = w \right] \tag{1.3}$$

where $\mathscr{D}$ is a given family of *admissible* controls, adapted with respect to the filtration $\{\mathscr{F}_s\}$ and with value in U. If a control $\{\alpha_s^*\}$ that solves previous problem exists, given w, it will be called *optimal control*. Function Φ will be called *value function*.

Different kinds of admissible controls can be considered. *Feedback* or *closed loop* controls are those ones measurable with respect to the σ-algebra generated by the process $\{w_s\}$ up to time t, for each time $t \geqslant 0$. However, under some extra conditions, the optimality that can be obtained from a feedback control, it is also obtained considering controls of the form $\alpha_s = \alpha(w_s)$, for some function $\alpha : \mathbb{R}^n \to U$. In other words the value we choose at time s for the control, only depends on the state of the system at that time. These controls are called *Markov* *controls*. In this sense the optimal control law will be denoted by α^*. For more details see Øksendal (2003). For the reasons just mentioned, Markov controls will be considered, where the process $\{\alpha_s\}$ at time s is intended to depend on the value assumed by w_s.

Remark 1.1. Taking $D = \mathrm{R}^n$ we obtain $T = +\infty$. In this sense previous problem 1.3 considers both the so called infinite horizon problems and the finite horizon problems.

Remark 1.2. For $n \geqslant 2$, a time dependent problem can always be written as follow

$$w_s = (t_0 + s, z_{t_0+s})$$

under the initial condition $w_0 = (t_0, z)$, $t_0 \in \mathrm{R}$ and $z \in \mathrm{R}^{n-1}$, where $\{z_r\}$ is a given diffusive process. We are lead to write

$$\int_0^T f^{\alpha_s}(w_s)\,\mathrm{d}s = \int_0^{\widehat{T}-t_0} f^{\alpha_s}(t_0+s, z_{t_0+s})\,\mathrm{d}s = \int_{t_0}^{\widehat{T}} f^{\alpha_{r-t_0}}(r, z_r)\,\mathrm{d}r$$

where the second equality is obtained changing variables: $s = r - t_0$ and $\widehat{T}$ is defined as follows:

$$\widehat{T} := \inf\left\{ r > t_0 : (r, z_r) \notin D, \text{ given } z_{t_0} = z \right\}$$

In this case function Φ will in general show a dependency on both initial time t_0 and the initial value $z_{t_0} = z$. Also notice that if we have $D \subset (t_0, t_1] \times \mathrm{R}^{n-1}$, where t_1 is a real number greater that t_0, it turn out that $\widehat{T} = t_1$, and the optimization problem can be written as

$$\Phi(t_0, z) = \sup_{\{\widetilde{\alpha}_s\} \in \mathscr{D}} \mathbb{E}\left[\int_{t_0}^{t_1} f^{\widetilde{\alpha}_r}(r, z_r)\,\mathrm{d}r + g(w_{t_1}) \,\middle|\, z_{t_0} = z \right] \tag{1.4}$$

where $\widetilde{\alpha}_r := \alpha_{r-t_0}$.

Remark 1.3. Assuming the following dynamics for process $\{z_s\}$:

$$\mathrm{d}z_s = b^z(\alpha_s, z_s)\,\mathrm{d}s + \sigma^z(\alpha_s, z_s)\,\mathrm{d}B_s$$

where b^z is a $(n-1)$-valued function and σ^z is a $(n-1)\times m$ matrix defined on $U \times \mathrm{R}^{n-1}$, the dynamics of process $\{w_s\}$ can be formally written as follows:

$$\mathrm{d}w_s = \begin{bmatrix} 1 \\ b^z(\alpha_s, z_s) \end{bmatrix} \mathrm{d}s + \begin{bmatrix} 0 \cdots 0 \\ \sigma^z(\alpha_s, z_s) \end{bmatrix} \mathrm{d}B_s$$

under the initial condition $w_0 = (t_0, z)$.

1.2 The HJB equation

Without entering into details, the present section present an outline concerning the Hamilton-Jacobi-Bellman equation.

Consider a diffusive process described by the following dynamics:

$$\mathrm{d}w_s = b(\alpha_s, w_s)\,\mathrm{d}s + \sigma(\alpha_s, w_s)\,\mathrm{d}B_s$$

for $v \in U$, the differential operator is defined as follow:

$$(\mathscr{A}^v)(y) = \sum_{i=1}^n b_i(v, y)\frac{\partial}{\partial y_i} + \sum_{i,j=1}^n a_{i,j}(v, y)\frac{\partial^2}{\partial y_i \partial y_j}$$

where $a_{i,j} := \frac{1}{2}\left(\sigma\sigma^T\right)_{i,j}$, the superscript T being the transposing operator.

Consider the optimization problem (1.3):

$$\Phi(y) = \sup_{\{\alpha_s\}\in\mathscr{D}} \mathbb{E}\left[\int_0^T f^{\alpha_s}(w_s)\,\mathrm{d}s + g(w_T)\cdot\chi_{\{T<+\infty\}} \middle| w_0 = y\right]$$

and assume that function Φ is $\mathscr{C}^2$ on D and $\mathscr{C}$ on the closure of D. Under some integrability and regularity conditions – see Øksendal (2003, chap. 11) – if an optimal Marcov control α^* exists, we have

$$0 = \sup_{v\in U}\{f^v(y) + (\mathscr{A}^v\Phi)(y)\}, \quad \text{for all } y \in D \tag{1.5}$$

and

$$\Phi(y) = g(y), \quad \text{for all } y \in \partial D$$

where ∂D, denote the boundary of D. The supremum is obtained if $v = \alpha^*(y)$. Equation (1.5) is called the Hamilton-Jacobi-Bellman (HJB) equation. Notice that we have:

$$f^{\alpha^*(y)} + \left(\mathscr{A}^{\alpha^*(y)}\Phi\right)(y) = 0, \quad \text{for all } y \in D$$

As remarked by Øksendal (2003), the HJB equation states that if an optimal control α^* exists, then its value v at the point y is a point v where the function

$$v \to f^v + (\mathscr{A}^v\Phi)(y), \quad v \in U$$

attains its maximum, which is 0. Moreover, this condition is not just *necessary* but also *sufficient*. In fact, denoting by

$$J^\alpha(y) := \mathbb{E}\left[\int_0^T f^{\alpha_s}(w_s)\,\mathrm{d}s + g(w_T)\cdot\chi_{\{T<+\infty\}}\,\middle|\, w_0 = y\right]$$

it can be proved that if φ is $\mathscr{C}^2$ on D and $\mathscr{C}$ on the closure of D such that, for all $v \in U$

$$f^v + (\mathscr{A}^v\varphi)(y) \leqslant 0, \quad y \in D$$

with boundary values

$$\lim_{t\to T}\varphi(w_t) = g(w_T)\chi_{\{T<+\infty\}}, \quad \text{a.s. } Q^w$$

where Q^w is the probability law of the process $\{w_s\}$ starting at w for $s = 0$, and φ respects some integrability conditions, then

$$\varphi(y) \geqslant J^\alpha(y)$$

for all Marcov controls α and all $y \in D$. Moreover, if for each $y \in D$ we have a law α_0 such that

$$f^{\alpha_0(y)}(y) + \left(\mathscr{A}^{\alpha_0(y)}\varphi\right)(y) = 0$$

then α_0 is a Marcov control such that

$$\varphi(y) = J^{\alpha_0}(y)$$

and, if α_0 is also admissible and respects some integrability conditions, we have $\varphi(y) = \Phi(y)$.

1.2.1 A particular case

In chapter 3 a special form for the HJB equation will be needed, when a discount factor is considered. Following the line of Krylov (1979) consider to solve the following problem:

$$\Phi(w) = \sup_{\{\alpha_s\}\in\mathscr{D}} \mathbb{E}\left[\int_0^{+\infty} e^{-s} f^{\alpha_s}(w_s)\,\mathrm{d}s \,\middle|\, w_0 = w\right]$$

where, for simplicity, an infinite horizon is considered. If function f is bounded, we wish to reduce this problem to some of known problems above. To this end, consider $l \in \mathrm{R}$ and consider a new controlled process

$$l_t = l + t = l + \int_0^t 1\,\mathrm{d}s$$

and consider the new problem

$$\Psi(w, l) = \sup_{\{\alpha_s\}\in\mathscr{D}} \mathbb{E}\left[\int_0^{+\infty} \widetilde{f}^{\alpha_s}(w_s, l_s)\,\mathrm{d}s \,\middle|\, w_0 = w,\, l_0 = l\right]$$

where $\widetilde{f}$ is defined in the obvious way. We have

$$\int_0^{+\infty} \widetilde{f}^{\alpha_s}(w_s, l_s)\,\mathrm{d}s = \int_0^{+\infty} e^{-l_s} f^{\alpha_s}(w_s)\,\mathrm{d}s =$$
$$= \int_0^{+\infty} e^{-(l+t)} f^{\alpha_s}(w_s)\,\mathrm{d}s = e^{-y}\int_0^{+\infty} e^{-t} f^{\alpha_s}(w_s)\,\mathrm{d}s$$

from which we deduce

$$\Psi(w, l) = e^{-l}\Phi(w)$$

Writing the HJB equation for Ψ and rearranging using the previous equality we obtain that the HJB equation for Φ is

$$0 = \sup_{v \in U} \{f^v(y) - \Phi(y) + (\mathscr{A}^v \Phi)(y)\}, \quad \text{for all } y \in \mathrm{R}^n$$

The same reasoning can show that if we consider a problem of the form

$$\Upsilon(w) = \sup_{\{\alpha_s\} \in \mathscr{D}} \mathbb{E}\left[\int_0^{+\infty} \exp\left\{-\int_0^s \delta^{\alpha_r}(w_r)\,\mathrm{d}r\right\} f^{\alpha_s}(w_s)\,\mathrm{d}s \,\middle|\, w_0 = w\right]$$

where $\delta^v(y)$ is a given function of y and v, the HJB equation for Υ will be

$$0 = \sup_{v \in U} \{f^v(y) - \delta^v(y) \cdot \Upsilon(y) + (\mathscr{A}^v \Upsilon)(y)\}, \quad \text{for all } y \in \mathrm{R}^n$$

1.3 Optimal stopping times

Optimal stopping time is a class of problems where one is asked to find, in addition to an optimal control, also a stopping time in order to maximize a given objective function:

$$\Phi(w) = \sup_{\tau, \{\alpha_s\} \in \mathscr{D}} \mathbb{E}\left[\int_0^{\tau} f^{\alpha_s}(w_s)\,\mathrm{d}s + g(w_\tau) \cdot \chi_{\{\tau < +\infty\}} \,\middle|\, w_0 = w\right] \qquad (1.6)$$

the supremum being taken over all stopping times τ for the process $\{w_s\}$. These problems are analyzed by Øksendal (2003, chap. 10), who present the related theory using the concept of *supermeanvlued* functions. A classic approach to that is instead offered by Krylov (1979), where the problem can be reduced to an infinite horizon problem with a discount term, using the concept of randomized stopping times.

It turn out that a sufficiency condition for optimal stopping involve, under some regularity conditions, a combination of variational inequalities. Herein I will not enter into the details concerning this result, rather just notice that in order to find the value function for the optimal stopping problem the following three conditions have to be satisfied:

$$\sup_{v \in U} \{(\mathscr{A}^v \Phi)(y) + f^v(y)\} \leqslant 0$$

$$g(y) - \Phi(y) \leqslant 0$$

$$\sup_{v \in U} \{(\mathscr{A}^v \Phi)(y) + f^v(y)\} = 0 \quad \text{when} \quad g(y) - \Phi(y) < 0$$

for $y \in D$. These conditions can also be written as follows:

$$g(y) - \Phi(y) + \sup_{v \in U} \left[(\mathscr{A}^v \Phi)(y) + f^v(y) + \Phi(y) - g(y)\right]_+ = 0$$

where the subscript + denote the positive part. This condition is used in the important contribution given by Milevsky and Young (2003a and 2007a) that will be reviewed in the course of chapter 3 and that will be a key ingredient for the indifference model proposed in the course of this thesis.

CHAPTER 2

RECALLS ON MODELS OF HUMAN MORTALITY AND LONGEVITY RISK IN LIFE INSURANCE

The present chapter provides a short review of actuarial models of human mortality. Moreover, the concept of *systematic longevity risk* and *guaranteed annuity option* are introduced. In particular, after an overview, where the nature of such options is presented, we emphasize some concerns over their valuation, highlighting the impact of the mortality developments on the liabilities associated to these options.

2.1 The individual's future lifetime

Let T_0 be the future lifetime – measured in number of years – of an individual just born, i.e currently aged $x = 0$. It is assumed that T_0 is a positive real valued random variable and its distribution $F_0(t)$ is continuous with density $f_0(t)$. Its support is supposed to span $[0, +\infty)$ or alternatively the interval $[0, \omega]$, where ω is intended as an "extremal age". For every $t \geqslant 0$ we have:

$$F_0(t) = \mathrm{P}\{T_0 \leqslant t\} = \int_0^t f_0(s)\,\mathrm{d}s$$

and it is common to set ${}_tq_0 := F_0(t)$. Moreover, if Δt is a positive real number, and if f_0 is continuous, we get

$$\begin{aligned} \mathrm{P}\{t < T_0 \leqslant t + \Delta t\} &= F_0(t + \Delta t) - F_0(t) \\ &= F_0'(t) \cdot \Delta t + o(\Delta t) \\ &= f_0(t) \cdot \Delta t + o(\Delta t) \end{aligned}$$

If not specified otherwise, f_0 is assumed to be continuous. By F_0 the so called *survival function* is defined as follow:

$$l_x = \mathrm{P}\{T_0 > x\} = 1 - F_0(x)$$

where it is usual to set ${}_sp_0 := l_s$. Since F_0 is a distribution function, it follow that

$$l_0 = 1; \qquad \lim_{s \to +\infty} l_s = 0$$

For an individual currently aged $x > 0$ it is significant to consider the future remaining lifetime $T_x = T_0 - x$ conditional to $T_0 > x$. The distribution of T_x is

$$F_x(t) := \mathrm{P}\{T_x \leqslant t\} = \mathrm{P}\{T_0 - x \leqslant t \,|\, T_0 > x\} = 1 - \frac{l_{x+t}}{l_x}$$

Also, in actuarial mathematics it is familiar to set

$${}_tp_x := \frac{l_{x+t}}{l_x} = 1 - F_x(t); \quad {}_tq_x := 1 - {}_tp_x = F_x(t) \tag{2.1}$$

The definitions above have as immediate consequence the following relation

$${}_{s+t}p_x = {}_tp_x \cdot {}_sp_{x+t} \tag{2.2}$$

that will turn useful for the valuation of a deferred single premium fixed life annuity.

2.2 The distribution of future lifetime and the force of mortality

From equation (2.1) it is possible to derive an alternative expression for the probability of an individual aged x dying in the interval $[x, x + \Delta x]$, where x and Δx are positive real numbers. Indeed, we have

$$
{}_{\Delta x}q_x = \mathrm{P}\{T_0 \leqslant x + \Delta x \,|\, T_0 > x\} = \\
= \frac{\mathrm{P}\{x < T_0 \leqslant x + \Delta x\}}{\mathrm{P}\{T_0 > x\}} = \frac{F_0(x + \Delta x) - F_0(x)}{1 - F_0(x)}
$$

from which we obtain

$$
{}_{\Delta x}q_x = \frac{f_0(x)}{1 - F_0(x)} \cdot \Delta x + o(\Delta x)
$$

Denoting by

$$
\mu_x = \frac{f_0(x)}{1 - F_0(x)}
$$

the *force of mortality*. Hence, assuming a "small" value for Δx, we get that ${}_{\Delta x}q_x$ is approximatively equal to $\mu_x \cdot \Delta x$, the error being of order Δx:

$$
{}_{\Delta x}q_x = \mu_x \cdot \Delta x + o(\Delta x)
$$

Notice that the probability of dying in the interval $[x, x + \Delta x]$ depends on x and it is approximatively proportional to Δx. We also have

$$
\mu_x = -\frac{l'_x}{l_x} = -\frac{\mathrm{d}}{\mathrm{d}t} \ln l_x \tag{2.3}
$$

The importance of the function μ_x is not just the possibility – as I wrote before – of an alternative way of writing the probability of a person to die. In fact, the assumptions on the process of mortality can be better stated by a convenient choice of a functional form of μ_x. In particular, after making assumptions on the force of mortality μ_x, in what follow, the survival function

l_x is obtained by integration: If the force of mortality is specified, the survival function is the solution of the following Cauchy problem:

$$\begin{cases} l'_x = -\mu_x \cdot l_x \\ l_0 = 1 \end{cases}$$

from which one obtain

$$l_x = \exp\left\{-\int_0^x \mu_s \, ds\right\}$$

As noticed by Milevsky (2006, p. 38), the force of mortality can be thought as an instantaneous rate of death at a certain age. Actually

$${}_tp_x = \frac{l_{x+t}}{l_x} = \exp\left\{-\int_0^{x+t} \mu_s \, ds + \int_0^x \mu_s \, ds\right\}$$

from which we obtain, after setting $s = x + \eta$

$${}_tp_x = \exp\left\{-\int_x^{x+t} \mu_s \, ds\right\} = \exp\left\{-\int_0^t \mu_{x+\eta} \, d\eta\right\}$$

Several analytical models are proposed in order to model human mortality. They are based on a realistic approximation of the remaining lifetime, as described in the next section.

An interesting model in actuarial mathematics is the Gompertz's representation. This model is an excellent description of mortality patterns at adult ages. Herein the main assumption is that increments of the force of mortality can be written as follow:

$$\Delta\mu_x = \beta\mu_x \cdot \Delta x + o(\Delta x)$$

where $o(\Delta x)$ is an error of order Δx. In other words there exists constants $\alpha > 0$ and $\beta > 0$ such that

$$\mu_x = \alpha e^{\beta x}$$

The case of $\beta = 0$ is called Dormoy's model and leads to an exponential law of mortality. The Gompertz's model is generalized by Makeham's force or mortality, where μ_x depends also on a constant γ:

$$\mu_x = \alpha e^{\beta x} + \gamma \tag{2.4}$$

from what, by integration, we get

$$l_x = \exp\left\{-\int_0^x \mu_s \, \mathrm{d}s\right\} = \exp\left\{\frac{\alpha}{\beta}\left(1 - e^{\beta x}\right) - \gamma x\right\}$$

When $\beta > 0$, it is common to rearrange the Gompertz-Makeham's specification (2.4) in the following way

$$\mu_x = \frac{1}{b} e^{(x-m)/b} + \gamma \tag{2.5}$$

where m and b are real numbers: The former represents the modal value of life and the latter the dispersion coefficient. Also, in human mortality it is usually observed $m > b > 0$. Milevsky (2006) notice that m can be seen as a modal value because $\mu_m = \gamma + 1/b$, while we have $\mu_m < \gamma + 1/b$ if $x < m$, and $\mu_m > \gamma + 1/b$ if $x > m$.

According to (2.5), the force of mortality is a constant plus an age dependent exponential curve. The former aims to capture the component of the death rate that is attribuite to accidents, while the exponentially increasing portion reflect natural death causes. This curve increases with age and approach to infinity as $t \to +\infty$. By integrating (2.5), we get that the function l_x has the form

$$l_x = \exp\left\{e^{-m/b}\left(1 - e^{x/b}\right) - \gamma x\right\}$$

Also observe that if $\gamma = 0$, for every $\varepsilon > 0$ we have

$$\lim_{b \to 0}\left(l_{m-\varepsilon} - l_{m+\varepsilon}\right) = 1$$

and Carriere (1994a) points out that this limit suggests that all the mass concentrates about m when b is small. Thus m can also be interpreted as a location parameter when $m > 0$.

Notice that by the definition of the survival function, once l_x is known it is possible to give a formulation of the cumulative density function of T_0. For computational simplicity, I assume the parameter $\gamma = 0$. This is equivalent to say that the Gompertz-Makeham's distribution is considered:

$$G(x) = 1 - \exp\left\{e^{-m/b}\left(1 - e^{x/b}\right)\right\}$$

In other words, I focuss on the following specification for the force of mortality:

$$\mu_x = \frac{1}{b} e^{(x-m)/b}$$

From the specifications above we have:

$${}_t p_x = \exp\left\{e^{(x-m)/b}\left(1 - e^{t/b}\right)\right\}$$

and also

$$p_x := {}_1 p_x = \exp\left\{e^{(x-m)/b}\left(1 - e^{1/b}\right)\right\} \tag{2.6}$$

from which

$$q_x := 1 - p_x = 1 - \exp\left\{e^{(x-m)/b}\left(1 - e^{1/b}\right)\right\} \tag{2.7}$$

Equations (2.6) and (2.7) will be useful when life tables estimations are available, as described in the next section.

2.3 Life tables

In order to define the concept of survival table, in what follows a probabilistic model based on the notion of survival function is considered. I refer to Milevsky (2006) and Pitacco (2002a) for the explanation of matters concerning the estimation of such a table.

Consider a cohort of L_α individuals all aged α years. L_α is called *root* and it supposed to be a positive integer; also α is supposed to be a positive natural number. If we think of a cohort of "homogeneous" enough persons, we can also assume that everyone of those is characterized by the same survival function l. Hence, for the representative individual in this cohort we have:

$${}_t p_\alpha = \frac{l_{t+\alpha}}{l_\alpha}$$

The number of people Y_x that will survive till age x is a random variable that assumes values on the set $\{0, 1, 2, \ldots, L_\alpha\}$, whose expectation is

$$L_x := \mathbb{E}Y_x = L_\alpha \cdot {}_t p_\alpha = L_\alpha \cdot \frac{l_x}{l_\alpha}$$

If now an initial age α is fixed as well as an extremal age ω, the following sequence

$$\left\{L_\alpha, L_{\alpha+1}, \dots, L_x, \dots, L_{\omega-1}\right\}$$

is called *life table*. Statistical methods allow actuaries to estimate life tables. Making the hypothesis that the assumed mortality law will not change over the time, the following relations holds,

$$l_x = \frac{L_x}{L_\alpha} l_\alpha$$

for a positive integer t, it is possible to compute the probability

$${}_t p_x = \frac{l_{x+t}}{l_x} = \frac{L_{x+t}}{L_x}$$

Notice that, in order to have significance, in the expression above t has to be a positive integer. Referring to equations (2.1), it is now clear the advantage of defining p_x and q_x as: $p_x := {}_1 p_x$ and $q_x := {}_1 q_x$. For the positive integer t we get

$${}_t p_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+t-1}$$

Table 2.1 is an example of a life table. It represents the estimation concerning a cohort 100,000 women in 2004, taking $\alpha = 0$, from the province of Ontario, Canada. We remark that every value in table 2.1 has to be thought as an expected value. This table shows some estimations other than L_x. d_x represents the number of deaths between exact age x and $x + 1$:

$$d_x = L_x - L_{x+1}$$

and q_x is the probability of death between exact ages x and $x + 1$. It follow that:

$$q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{L_x - L_{x+1}}{L_x} = \frac{d_x}{L_x}$$

TABLE 2.1: Life table fragment estimated from a cohort of 100,000 individuals (female) from Ontario, Canada. Deaths are given by age and calendar year. Data obtained by Statistics Canada and Canadian Human Mortality Database 2004.

Age	m_x	q_x	L_x	d_x	e_x
0	0.00531	0.00529	100,000	529	82.61
1	0.00033	0.00033	99,471	32	82.05
2	0.00021	0.00021	99,439	21	81.08
3	0.00016	0.00016	99,418	16	80.09
4	0.00021	0.00021	99,402	21	79.11
...	...	...	...	...	...
57	0.00444	0.00443	95239	422	27.75
58	0.00419	0.00418	94816	397	26.87
59	0.00465	0.00464	94420	438	25.98
60	0.00578	0.00577	93982	542	25.10
...	...	...	...	...	...
107	0.61356	0.46952	98	46	1.56
108	0.64498	0.48770	52	25	1.50
109	0.67519	0.50478	27	13	1.45
110	0.70401	1.00000	13	13	1.42

The value e_x is an approximation of the life expectancy at age x or, in other words, an approximation of the expected value of T_x. Suppose that the density $f_x(t)$ exists and it is continuous. Hence we have

$$f_x(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_x(t) = -\frac{l'_{x+t}}{l_x}$$

Then, the life expectancy of an individual now aged x is:

$$\bar{e}_x := \mathbb{E}T_x = \int_0^{+\infty} t \cdot f_x(t)\,\mathrm{d}t = -\frac{1}{l_x}\int_0^{+\infty} t \cdot l'_{x+t}\,\mathrm{d}t = \frac{1}{l_x}\int_0^{+\infty} l_{x+t}\,\mathrm{d}t$$

where the last equality can be proved by integration by parts. For more details we suggest Gerber (1995), Promislow (2006), Pitacco (1989, 2002a). Finally we get

$$\overline{e}_x = \int_0^{+\infty} {}_t p_x \, \mathrm{d}t$$

When only a life table is available, it is just possible to compute approximated values of $\overline{e}_x$. Several formulas are proposed in actuarial mathematics. Among them, herein we recall the *incomplete expectancy life*, denoted by e_x, actually used in the table (2.1):

$$e_x := \frac{1}{l_x} \sum_{i=1}^{\omega-x-1} l_{x+i} = \sum_{i=1}^{\omega-x-1} {}_i p_x$$

where ω is the extremal age:

2.4 Demographical trends on lifetime insurance contracts

Throughout this thesis I use population data instead of insured lives data. I believe that it will be a more appropriate source of analysis for the model proposed in the next chapter. In the first part of this section I present some preliminary facts regarding the mortality experience over the last decades. Then I defer the section 2.5 the importance concerning the so called *longevity risk* and its influence on the annuity contracts. In the second part, I recall the difficulties arising from considering general life-contingent claims in a stochastic – rather than deterministic – mortality risk environment.

2.4.1 Preliminary facts concerning survival trends

If we compare survival tables concerning different periods of estimation, we could notice important trends on mortality. Figure 2.1 and 2.2 show the improving of mortality in terms of the number of survivors as a function of the attained age x. Clearly, it is not possible to value today the "exact" probability of death in t years, for a person now aged x.

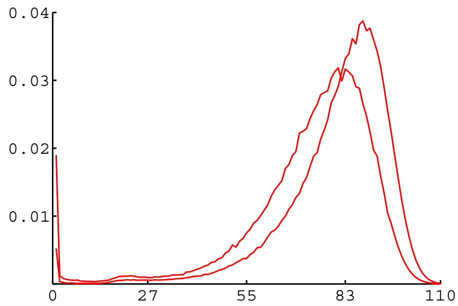


FIGURE 2.1: Empiric death density function, assuming a cohort of 100,000 individuals female and male, for years 1970 and 2004. Range: 0-110 years old.

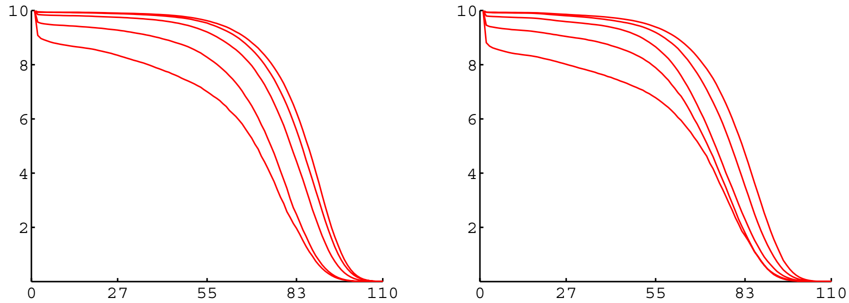
Random mortality patterns may refer to both young and old age. If the latter is concerned, usually it is referred to *longevity risk*. In particular, from plotted data in figure 2.3, one can observe a decrease in mortality rates at adult and old ages and increase in life expectancy. From figure 2.2 it is evident a *rectangularization* of the survival function that implies an increasing concentration around the mode of the curve of deaths; also it can be seen how the *expansion* of the survival function, in the sense that the curve of deaths move towards very old ages.

A review of the literature on the nature and causes of historical changes in longevity is made by Stallard (2006). In his paper he also focus on the use of deterministic and stochastic process models for forecasting the distribution of future survival outcomes for pricing models for longevity bonds for a set of closed cohorts.

A more complete discussion regarding the longevity risk is presented in section 2.5.2, where I focus on life annuity policies and the post-retirement income.

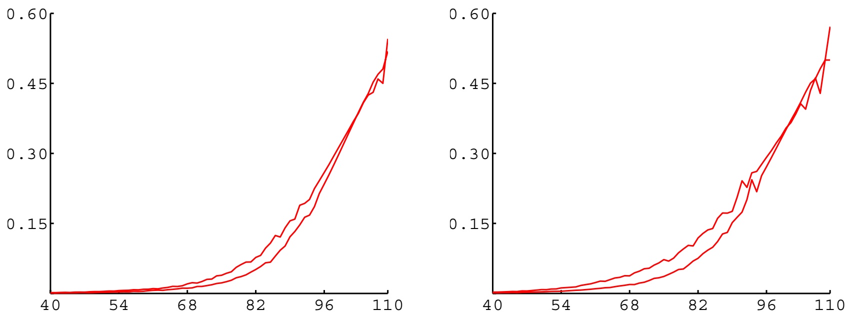
* * *

In order to estimate the Gompertz's parameters I refer to Carriere (1994b). Defining $q_x := {}_1q_x$, and assuming Gompertz's mortality, we have that q_x de-



(a) Females from Ontario (b) Males from Ontario

FIGURE 2.2: Population of Ontario. Number of survivors at exact age x, assuming a cohort of 100,000 individuals, for years 1921, 1940, 1970, 1990 and 2004. Range 0-110 years old.



(a) Females from Ontario (b) Males from Ontario

FIGURE 2.3: Mortality rates q_x on logarithmic scale, $40 \leqslant x \leqslant 110$ for females and males from Ontario, Canada, for years 1970 and 2004. Source: Canadian Human Mortality Database.

pends on m and b by the relation described as follows:

$$q_x(m, b) = 1 - \exp\left\{e^{(x-m)/b}\left(1 - e^{1/b}\right)\right\}$$

Consider now, the (empiric) estimation of a life table based on a cohort of homogeneous individuals and set:

$$\widehat{q}_x = \frac{\widehat{L}_x - \widehat{L}_{x+1}}{\widehat{L}_x}$$

where $\widehat{L}_x$ and $\widehat{L}_{x+1}$ refer to the (empiric) estimations of L_x and L_{x+1}, respectively. Carriere (1994b) suggests that a good way of estimating m and b is to minimize the robust loss function:

$$\min_{m \in \mathrm{R},\, b>0} \sum_x^{\omega} \sqrt{\widehat{L}_x - \widehat{L}_{x+1}} \left| 1 - \frac{q_x(m,b)}{\widehat{q}_x} \right|$$

where x is intended as an adult age and ω as an "extremal" age. In particular we set $x = 40$ and from the available database we have $\omega = 110$.

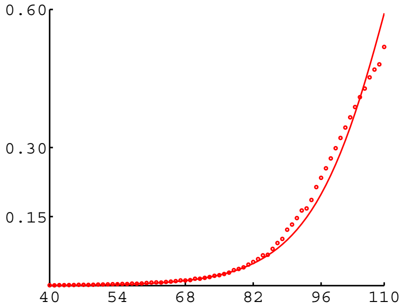
From data available at the Canadian Human Mortality Database, table 4.1 offer the estimation[1] for the Gompertz's force of mortality parameters for years 1970 and 2004, for both females and males. Data refer to the Province of Ontario.

TABLE 2.2: Estimated Gompertz's force of mortality parameters for the province of Ontario, for years 1970 and 2004 both for females and males, conditional on survival to age 40. Source: Canadian Human Mortality Database.

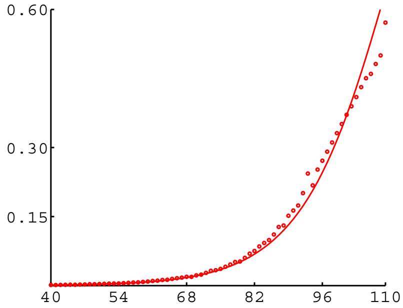
	Female		Male	
Year	m	b	m	b
1970	85.3827	10.4673	78.9549	11.7863
2004	89.7651	9.3109	85.8689	10.1301

From table 4.1 it can be observed that the parameter m increase over years, both for females and males; instead the parameter b, that express the volatility, decrease. This phenomena are coherent with the so called rectangularization of the survival function.

[1]The fitting process has been implemented by using Matlab and employing the function fminnunc and the function fminsearch.



(a) Females from Ontario



(b) Males from Ontario

FIGURE 2.4: Fitted mortality rates $q_x(m, b)$, solid line, on linear scale, $40 \leqslant x \leqslant 110$ for females and males from Ontario, Canada, for year 2004. Source: Canadian Human Mortality Database.

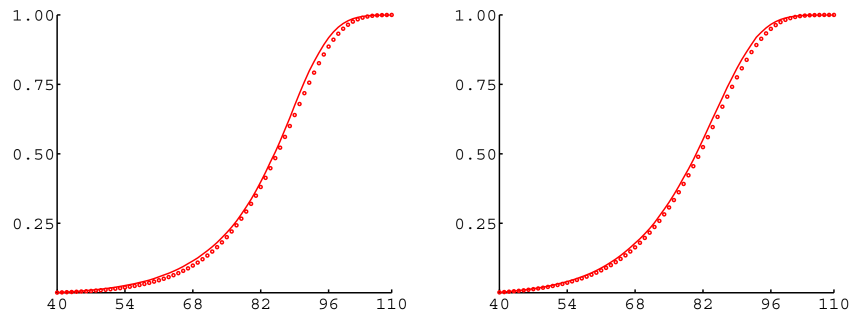
Figure 2.4 show the fitted mortality rates $q_x(m, b)$, for every $40 \leqslant x \leqslant 110$, for females and males from Ontario, Canada, for year 2004. The superimposed lines are obtained by using estimations for m and b presented in table 4.1. By using the same estimation, figure 2.5 show the fitted probability ${}_tq_x$, conditional to survival at age $x = 40$, for every $40 < t \leqslant 110$.

Figure 2.6, finally show the estimated death density at adult aged, for both a female and a male aged 40. The curves are plotted using the parameters given in table 4.1. We will use these estimations in the course of the next chapter, in order to consider a stochastic force of mortality.

2.4.2 Life-contingent claims under stochastic mortality risk

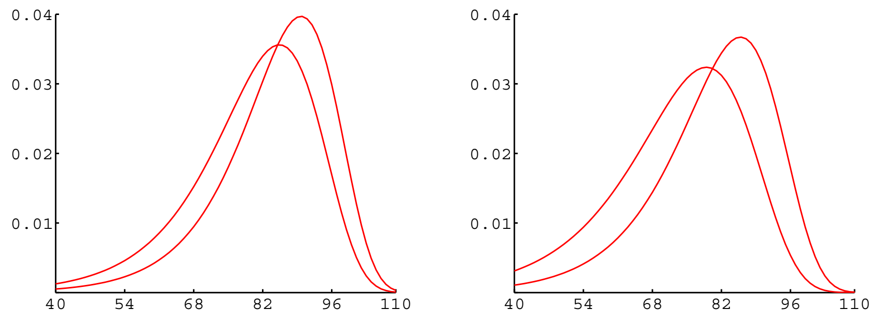
We focus now on life insurance policies. In such a contract the insurer has the obligation to pay a certain lump-sum or a cash-flow stream contingent on the survival or the death of the insured person (the policyholder) or, in some cases, of a group of persons. Then the pay-out of the insurer is a function of the random variable T_x.

The compensation of the insurer – i.e. the obligation of the insured – consists of the payment of a premium. In certain cases the premium may consist of



(a) Females from Ontario (b) Males from Ontario

FIGURE 2.5: Fitted probability $_tq_x$, conditional to survival age $x = 40$, $40 < t \leqslant 110$, for females and males from Ontario, Canada, for year 2004. Source: Canadian Human Mortality Database.



(a) Females from Ontario (b) Males from Ontario

FIGURE 2.6: Estimated death density, conditional to survival age $x = 40$, $40 < t \leqslant 110$, for females and males from Ontario, Canada, for year 2004. Source: Canadian Human Mortality Database.

a payment stream. In this case payments will also depend on the event that the insured is still alive at the moment of the payment: more precisely the payment stream is contingent itself on the random variable T_x.

In the case when the policy is contingent on the survival of the insured, as noticed by Milevsky and Promislow (2001), it is easy to see that insurance companies can be exposed to unanticipated longevity risk. While the chance that any particular insured is healthier than average can be eliminated taking advantage of the law of large numbers, the risk that the insurance company overestimated the population's force of mortality is more subtle: the longevity risk cannot be hedged by appealing to the law of large numbers. As an example, we consider two works proposed by Milevsky, Promislow and Young (2006 and 2005) . This example shows how the law of large numbers breaks down when pricing life-contingent claims under stochastic as opposed to deterministic mortality rates.

Along the lines of Milevsky, Promislow and Young, consider a simple one-period model and think at an insurance contract (endowment policy) which pays $2 if the annuitant survives to the end of the period, and $0 if the person dies. The payoff Y of this liability will be

$$y = \begin{cases} 2 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

where p is the probability that the person will survive till the end of the period. The expected value and the variance of y are

$$\mathbb{E}y = 2p; \qquad \text{var}(y) = 4p(1-p)$$

Consider now N of these policies, with payoffs y_i, $i = 1, 2, \ldots, N$, respectively. If we simplify the problem and we do not consider that the companies might issue these claims on an ongoing basis and have other liabilities, the insurance company's aggregate liability at the end of the period is

$$Y = \sum_{i=1}^{n} y_i$$

whose expected value and variance are:

$$\mathbb{E}Y = 2p \cdot N; \qquad \text{var}(Y) = 4p(1-p) \cdot N$$

The standard deviation per policy is

$$\frac{1}{N} \cdot 2\sqrt{p(1-p) \cdot N}$$

that approach to 0 when $N \to +\infty$.

Now, if the probability parameter p is unknown – if it is a random variable itself – it is not anymore possible to add-up the individual variance terms, but an implicit dependence is created by the common parameter. Consider a common factor $\widetilde{p}$ with a symmetric distribution:

$$\widetilde{p} = \begin{cases} p + \pi & \text{with probability } 0.5 \\ p - \pi & \text{with probability } 0.5 \end{cases}$$

where we are actually assuming that $\widetilde{p}$ is a random variable with expected value $\mathbb{E}\widetilde{p} = p$. Take N liabilities y_i, defined as before, with a common parameter $\widetilde{p}$ instead of p, and consider the aggregate insurance company's exposure Y. The expected value and the variance of Y, when the parameter $\widetilde{p}$ is a symmetric random variable, are:

$$\begin{aligned} &\mathbb{E}Y = 2p \cdot N \\ &\operatorname{var}(Y) = 4Np(1-p) + 4N(N-1)\pi^2 \end{aligned}$$

The standard deviation per policy is

$$\lim_{N \to +\infty} \frac{\sqrt{4Np(1-p) + 4N(N-1)\pi^2}}{N} = 2\pi$$

Moreover if $N = 1$, the variance of the payout is the same it would be in the deterministic case: $4p(1-p) \cdot N$. In other words, as noticed by Milevsky et al., the portfolio aggregation creates the extra risk; in fact an individual policy is not any riskier under a stochastic common factor versus a deterministic parameter.

This example, proposed along the lines of Milevsky, Promislow and Young (2006), emphasize the issue of stochastic mortality (probability, hazard) rates. As we shall see in the next chapter, stochastic mortality rates also matter in relation to the pricing of embedded options in insurance and annuity contracts. Indeed, also for that reason, their fair valuation is still very complex and expose insurance company to a considerable risk that is difficult to hedge.

2.5 Life annuity policies. Longevity risk

From the previous section it is highlighted that life insurance is concerned by the issue of mortality trend. In what follow I present the life annuity policies. In fact, they constitue one of the most important insurance product concerned by longevity risk. See for example the article by Pitacco (2002b). Also, the notion of life annuity policies will be necessary for next chapter, where the assessment of embedded options in life insurance contracts is presented. For the interested reader on insurance contract structures, the references cited above are still suggested. Also see Booth et al. (2005).

2.5.1 Life annuity policies

In a life insurance contract the insurer has the obligation to pay a certain lump-sum or a cash-flow stream contingent on the survival or the death of the insured person (the policyholder) or, in some cases, of a group of persons. The pay-out of the insurer is, then, a function of the random variable T_x.

The compensation of the insurer – i.e. the obligation of the insured – consists in the payment of a sum called premium. In certain cases the premium can be divided into a payment stream. In this case, payments will also depend on the event that the insured is still alive at the moment of the payment: more precisely the payment stream is contingent itself on the random variable T_x.

A contract that, in return of an initial premium, pays regular payments as long as the policyholder is alive, is named *annuity*. For a person aged x, if the annuity consists in a payout of one dollar at the end of every year as long as the insured is alive, the insurance company face on the following random variable

$$Y = v + v^2 + v^3 + \cdots + v^T$$

where $v = 1/(1+r)$ and r is the discount rate and very often it can be thought equal to the the market interest rate. We define the random variable T as follow:

$$T = k \quad \Leftrightarrow \quad k - 1 < T_x \leqslant k, \quad k = 1, 2, \ldots$$

We define the *net premium* of an annuity contract to be the expected value of Y. To this end we recall that

$$\mathrm{P}\{z < T_x \leqslant z + t\} = {}_z p_x \cdot {}_t q_{x+z}$$

hence,

$$\mathbb{E}Y = v^1 \cdot {}_1 p_x \cdot {}_1 q_{x+1} + (v^1 + v^2) \cdot {}_2 p_x \cdot {}_1 q_{x+2} + (v^1 + v^2 + v^3) \cdot {}_3 p_x \cdot {}_1 q_{x+3} + \dots$$

and rearranging,

$$\mathbb{E}Y = \sum_{i=1}^{+\infty} v^i \cdot {}_i p_x = \sum_{i=1}^{+\infty} \frac{{}_i p_x}{(1+r)^i} \tag{2.8}$$

Notice that the previous series is convergent since $v^i \cdot {}_i p_x < v^i$ and $v < 1$, $i = 1, 2, \dots$ Also $\mathbb{E}Y$ is a function of the age x of the annuitant.

In the present work I will consider a continuous-time environment. To this end, it is necessary to generalize the definition of a life annuity policy considering a contract that pay out at a continuous compounded unitary rate per year. In particular, considering continuous compounding, we have

$$\mathbb{E}Y = \int_0^{+\infty} \int_0^t e^{-rs}\, \mathrm{d}s\, \mathrm{d}\big({}_t q_x\big) = \int_0^{+\infty} e^{-rs}\, {}_s p_x\, \mathrm{d}s$$

$\mathbb{E}Y$ takes into account just the annuity payout. For that reason it is called net single premium. In order to arrive at a market price for the annuity policy, it is common to consider – see Milevsky (2001) – a proportional insurance load ϱ. It will contemplate all expenses, taxes, commissions, and distribution fees. Therefore, in a discrete time setting, for a person aged x, the market price a_x, of an annuity that insure an unitary amount for the end of each period, is the value

$$a_x = (1+\varrho) \sum_{i=1}^{+\infty} \frac{{}_i p_x}{(1+r)^i} \tag{2.9}$$

and in a continuous-time setting, the value of an annuity with a constant unitary rate of payment is

$$a_x = (1+\varrho) \int_0^{+\infty} e^{-rs}\, {}_s p_x\, \mathrm{d}s \tag{2.10}$$

Since I will focus just on continuous time, I will not distinguish between the two cases using different symbols. Otherwise, in the specialized literature, it is common to denote by $\overline{a}_x$ the value of an annuity with instantaneous compounding. Also notice that generally, the load ϱ is supposed to depend on the age x of the annuitant. Herein a constant discount rate is assumed over the time. In what follow, the *actuarial present value* of an annuity is obtained setting $\varrho = 0$.

For continuous life insurance annuities an explicit expression for a_x is given by Carriere (1994a). He finds that for Gompertz's mortality

$$a_x = (1 + \varrho) \cdot b \exp\{b\mu_x + r(x - m)\} \cdot \Gamma\left(b\mu_x, -rb\right)$$

where Γ is the left-truncated Gamma function defined as follows:

$$\Gamma(t, \alpha) = \int_t^{+\infty} u^{\alpha-1} e^{-u} \, \mathrm{d}u, \qquad t > 0, \quad \alpha \in \mathrm{R}$$

Consider an individual with a wealth $W > 0$. She can buy a quantity W/a_x of annuity policies. For example, it means that in a continuous-time setting, an agent endowed with an initial wealth W, can assure herself a continuous cash-flow stream at a rate of $H = W/a_x$, for the rest of her life. The so called *conversion rate* is defined as

$$h := \frac{1}{a_x} \tag{2.11}$$

and it represents the extent of the payout stream once an unitary wealth is converted into an immediate life annuity for a person now aged x. It is clear that a given conversion rate h also imply a given technical rate r_h by the relation:

$$\frac{1}{h} = a_x = (1 + \varrho) \int_0^{+\infty} e^{-r_h \cdot s} {}_s p_x \, \mathrm{d}s \tag{2.12}$$

Notice that, as we shall see in the next section, insurance companies often guarantees their policyholders to convert at maturity an accumulated wealth into a life annuity at a fixed rate. For example, typical rate in the UK was to convert a cash value of £1,000 into a £111 annuity per annum, i.e. $h = 1/9$. Hence, looking at the equation above, the insurer actually guarantee the policyholder an option on *both* the future interest rates *and* mortality rates: Improvements in mortality rates and in the longevity risk make insurance companies exposed to a *non-pooling risk*, hard to hedge.

Remark 2.1. Equations (2.10 and 2.12), for a load $\varrho > 0$, can be rewritten as follows:

$$a_x = \int_0^{+\infty} e^{-\tilde{r}s} {}_s p_x \, \mathrm{d}s$$

where we set $\tilde{r} := r - \log(1 + \varrho)$. In this way it is always possible to write the market price of a fixed unitary immediate life annuity, as its actuarial present value, under an convenient discount rate $\tilde{r}$. For this reason next chapter does not consider proportional insurance loads.

2.5.2 Longevity risk

Demographical trends imply a longevity risk for annuity products. Olivieri and Pitacco (2005 and 2001) emphasize how past mortality experience clearly reveals trends in the age pattern of mortality. In many countries, a decrease in mortality rates (in particular at adult and old ages), an overall increase in the most probable age of death (i.e. the Lexis Point), and an increase in the expected lifetime (both at birth and at adult and old ages) are important aspects of such trends. Improvements in medical knowledge and surgery, smoking habits and prevalence of some illnesses affect those tendencies. However, actuarial calculations concerning pensions, life annuities and other living benefits are based on the estimations of survival probabilities extended over a long horizon.

It is comprehensible that accurate methodology for the projections of mortality tables are required: any mortality table cannot lead to a suitable evaluation of futures mortality rates even when they are constantly updated: Even when a projection method is considered, deviations and over -estimations could arise. Those errors can be either non systematic deviations – that imply a *pooling risk* for the insurer, that vanish if a large cohort of individuals are considered – or a more subtle systematic variation – that cannot be eliminated considering a larger collectivity. The latter phenomenon leads to a *non-pooling risk*, whose monetary impact on the insurer cash flows increases if a larger number of policyholders is kept in view.

Mortality trends worldwide and the related longevity risk are analyzed by Rütterman (1999), Macdonald et al. (1998) and Stallard (2006); also Willets (1999) and Willets et al (2004) survey the mortality improvement in the United Kingdom. The main issue coming from those contributions – mostly concerning people from developed countries – is that the mortality trends are improving where both a "rectangularization" and an "expansion" are observed. Moreover the improvements are substantial over the age of 40 (lesser improvements relatively to females). Mortality is also improved for people in their 60s, that usually purchase insurance products with guaranteed options. Causes of death are changed, in fact violent and accidental causes are more typical for younger lives, whilst heart diseases and cancer are dominant for individuals aged 40.

2.5.3 Payout life annuities. Post-retirement incomes

In next chapter we will consider an equivalent valuation of guaranteed annuity options – whose the contract description is introduced in the next section – from the part of the insured. The important model of optimal annuitization (purchasing) policy introduced and developed by Milevsky and Young will be recalled. Even if next chapter focus on annuities that assure a fixed payout, it is worth, in the present section, to make clear some concerns over life payout annuities, referring to the part of the annuitant.

In Chen and Milevsky (2003), the problem of determining an optimal asset allocation mix with payout annuities is considered, emphasizing that the investor needs to make their own decisions on what products should be used to generate income in retirements. We do not intend to go deep on this problem but we just want to end the present section offering a review of costs and benefits concerning payout annuities. In particular, two important risk factors have to be considered: the financial market risk and the longevity risk. Payout annuities reduce the probability of outliving wealth and hedge against the longevity risk. However, the inflation rate erode the payments assured by a fixed payout annuity and investors cannot trade out the fixed payout amount once it is purchased. Instead variable payout annuities offer payments that fluctuate in value depending on some underlying variables, but they may present a financial risk. The contribution offered by Chen and Milevsky, to which the reader is addressed for more details, develop a model for optimally allocation investment assets within and between these two different categories, maximizing a suitably defined objective function. In another framework, Charupat and Milevsky (2002) show, that under some conditions, the optimal asset allocation during the annuity decumulation phase is identical to the accumulation phase, which is the classical Merton solution. In this model authors do not take into account the issues when and how much to annuitize, focusing on the asset allocation within the annuity contract.

2.6 Guaranteed annuity options (G.A.O.s)

We introduce here the concept of *guaranteed annuity options* and we offer a short overview to the so called *implicit options* in life insurance. Instead, in section 2.6.2, we offer a more deep overview about the literature concerning the valuation of these product that influence the life insurance company risk. For the moment we refer to Boot et al. (2005, sec 3.6 and 6.7), O'Brien (2002), Hardy (2003), Gatzert and Schmeiser (2006).

Policyholders may be granted the right to additional benefit, by some contracts, to be taken at their choice. These options generally are of significant value. As mentioned by Gatzert and Schmeiser (2006), participating life insurance contracts are contracts featuring death and survival benefits as well as participation in the return generated by the insurer's investment portofolio. Numerous guaranteed and rights may be contained in these type of contracts. Also these option can be verty valuable and can represent a significant risk to the insurance company. Most common implicit (also called "embedded") option can be divided into *rights* and *guarantees*. I will focus on a life insurance policy embedding a guaranteed annuity option, that is classified as a right. For more details concerning other kind of guarantees and rights we address the reader to Hardy (2003, chpters 1, 6, 12 and 13) and Milevsky (2006, chapter 11)

2.6.1 Introductory overview

In the present section and in the course of the next chapter, I focus on guaranteed annuity options (G.A.O.). Insurance companies often include very long-term guarantees in their products, which can turn out to be very valuable. Guaranteed annuity options are options available to holders of certain policies that are common practice in U.S. tax-sheltered insurance product and in U.K. retirement savings. In particular, see O'Brien (2002), the policyholder pays either a single or a regular premium, securing a guaranteed benefit at a specific age, that may coincide with the retirement age. Then, the guaranteed benefit can consist of either an amount of cash, with an option to convert to an annuity at a guaranteed rate; or an amount of annuity, with an option to take cash as an alternative at a guaranteed rate.

In what follows just G.A.O. that guarantee to convert a certain accumulated amount of cash into an annuity at a guaranteed rate will be considered. These option guarantees that a given (minimum) conversion rate will be applied at the time of conversion if the company's normal conversion rates are found less favorable at that time. In other words, under a guaranteed annuity option, the insurance company guarantees to convert a policyholder's accumulated funds to a life annuity at a fixed rated h, when the policy matures, see Boyle and Hardy (2002 and 2003). Significant change in economic and investment condition, between the time at which the option is purchased and the time at which it is exercised, can lead to a very significant cost to the company: The value of these options is influenced by the interest rates and by the longevity risk which has not been accounted for a long time and only recently and increasing number of contribution is concerned with this issue, as Gatzert and Schmeiser (2006) precise. As remarked by Milevsky and Promislow (2001), the company has essentially granted the policyholder an option on two underlying stochastic variables; future interest rates and future mortality rates.

The rate implicit in the G.A.O.is a function of the interest rate and the hazard (mortality) rate. To understand the effect of the improvements in longevity, consider the following example, where the discount rate is taken as fix to a certain level r. Notice that in the example that follows I just consider the life table estimations for years 1970 and 2004; the purpose of the easy computations that follow is to highlight the strong impact of longevity risk for insurance companies, without using any projection methods. Two different superscript for the probability will highlight the moment which the different estimations are taken into account. For the sake of simplicity I consider just here a discrete time setting.

Example 2.1. Suppose in 1970 a female aged 31, from the province of Ontario, decided to purchase, for a certain premium, a pension plan that will accumulate untill her time of retirement, say 2004, a wealth $W = \$100,000$ (if she will be alive). Also immagine that the insurer, looking at the available life tables at that time, decided to write a G.A.O. that will assure a payout of \$11,000 per annum. Considering a load $\varrho = 7\%$, the insurer actually guaranteed an technical rate of $r = 5.74\%$ that is implicitly given by the following equation

$$100000 = 11000 \cdot 1.07 \cdot \sum_{i=1}^{\infty} \frac{{}_{34+i}p_{31}^{'70}}{(1+r)^i} \tag{2.13}$$

where ${}_{i+n}p_x$ is computed using equations (2.2 and 2.3) and the estimated survival function from the available table of 1970. Also the "extremal" age is set at level $\omega = 110 + 1$ (i.e. it is assumed that all the lives aged 110, will die during the next year). In 2004, at the moment of the conversion, if the individual (if alive) will decide to exercise the option, she will have the right to convert the wealth of \$100,000 for a fixed immediate life annuity of \$11,000 per annum. Now, from the following relation, using the survival table available in 2004, one can compute the technical rate that the insurer is going to guarantee to the individual for the rest of her life, solving for r the following equation:

$$100000 = 11000 \cdot 1.07 \cdot \sum_{i=1}^{\infty} \frac{{}_{i}p_{65}^{'04}}{(1+r)^i} \tag{2.14}$$

that imply $r = 7.94\%$. In other words the improvement in longevity implicity affect the interest rate guaranteed in options to annuitise. In the case that the interest rate in the market is less that 7.94% and the insurer has to make up the difference.

2.6.2 Preliminary concerns on valuing guaranteed annuity options

Ending this review on guaranteed annuity options, we present here some preliminary concerns over these options. In particular an outline of some approaches to value guaranteed annuity options is offered.

The contribution given by Bolton et al (1997) focus on reserving for annuity guarantees. However it is important to recall the situation at that moment. In particular they write: historically many pension contracts issued by life companies contained options to convert the cash proceeds of the policy on retirement into annuities on terms guaranteed in advance. With relative low interest rates and improving mortality, the guarantees may be very valuable. Moreover they notice that up to 1997 no industry wide attempt to analyse the nature of the guarantees and the approaches adopted by companies to reserving for them. In that contest, the Report of the Annuity Guarantees Working Party an analysis was made of the implications of guarantees, and two alternative approaches to measuring the value of the guarantees were considered, concerning the required reserves – under various stochastic investment models – and a marked based approach to hedge guarantees.

In the contribution given by O'Brien (2002) five issues of resolution are proposed where, in particular, the possible investment strategies, the solvency of the insurance companies and the G.A.O. liabilities are taken into account. They conclude remarking that guaranteed annuity options are a significant issue both for policyholder – for whom they provide guarantees, whenever or not this returns out to be valuable at retirement, and it is implied that such policyholder should pay for this benefit – and for some life offices.

Usually risk-neutral valuation models are used for valuing embedded options in life insurance contracts. As recalled by Gatzert and King (2007), there are financial and actuarial approaches to handling embedded options: while the former is concerned with risk-neutral valuation and fair pricing, the former focuses on shortfall risk under an objective real-world measure, which plays an important role in insurance risk management and practice. In particular in their contribution authors analyze the interaction between these two approaches.

The pioneering approach of Milevsky and Promislow (2001) both interest rate risk and mortality risk are taken into account. In particular authors assume that at a given time, the force of mortality, for an individual with a certain age, is viewed as a random variable forward rate, whose expectation is the force of mortality in the classical sense. As recalled by Bacinello (2006), the choice of suitable stochastic models for longevity risk and for the term structure of interest rates is absolutely necessary. In the same line, the framework proposed by Dahl (2004) – reviewed in section 3.5 and concerning the stochastic mortality – views the motality intensity as a stochastic process, which is adapted to some filtration. In particular he focus on a model for mortality intensity such that it is described by a diffusion process characterized by what he call an affine mortality structure. Following this line Ballotta and Haberman (2006), extending the contribution given by Ballotta and Haberman (2003), analyze the behavior of pension contracts with guaranteed annuity options to the case in which mortality risk is incorporated via a stochastic model for the evolution over time of the underlying hazard rate. In particular the find that – considering a stochastic component governed by an Ornstein-Uhlenbeck process – leads to a reduction in the expected value of the guaranteed annuity option, when the valuation formula relates to an expected present value obtained by the methodology of risk-neutral valuation.

A different approach, concerning the pricing and the hedging for policies with guaranteed annuity options, is offered by Wilkie, Waters and Yang (2003) and Pelsser (2003a, 2003b). These approaches focus on modelling the annuity price. In particular Wilkie et al. investigate the feasibility of using option pricing methodology to dynamically hedge a guaranteed annuity option. In Pelsser a market value for with-profit G.A.O., using martingale modelling techniques, is derived and, he shows how to construct a static prelicating portfolio of vanilla swaptions that replicate the with-profit G.A.O.

Finally a recent framework from Biffis and Millossovich (2006), emphasizes that the exercise decision made by the policyholder may not be rational from the insure's point of view.

2.7 Conclusions

Recently, the literature on insurance premiums and mortality risk considers models with a financial market structure and where dynamic trading is allowed. In the next chapter we will recall and focus on some contributions on insurance risk pricing and on optimal annuitization policies. Therein, stochastic financial market models and indifference arguments are considered. To this end, the present chapter review some fundamentals on actuarial theory, that will be used in the field of the models presented in the next chapter. Also, stochastic mortality models together with stochastic financial models have been introduced in Milevsky and Promislow (2001): Liabilities concerning the embedded options in life insurance contracts have to be seen from a more comprehensive point of view that considers, in the same environment, the *financial risk*, the *systematic mortality risk* and the *unsystematic mortality risk*.

CHAPTER 3

THE POLICYHOLDER'S VALUATION MODEL FOR THE GUARANTEED ANNUITY OPTIONS

This chapter proposes an indifference valuation model in order to value guaranteed annuity options, from the policyholder's point of view. Before doing that, in section 3.2, it is offered a review of the literature regarding models on optimal annuitization policy and optimal annuity purchasing, where I recall some important contributions of Milevsky and Young. Some of the conclusions and facts arising from those contributions will be used in the model I propose herein. For instance, two possible arrangement will be considered in order to value guaranteed annuity options. Moreover, the reservation value for these options is analyzed at the time when the insurance policy is purchased and, then, both at the time of conversion and during the accumulation period. In order to do that, a stochastic financial and actuarial market structure is considered. Assuming constant interest and hazard rates, explicit solutions regarding the stochastic problems that follow, are computed and implemented in the field of next chapter. The present chapter ends considering the analytical complications arising in a model where stochastic mortality and interests rates are taken into account.

3.1 Why an indifference model?

The end of the previous chapter review the main approaches followed in pricing the options "embedded" in life insurance contracts. All of them are based on the existence of a risk neutral measure of probability.

The value of the conversion rate of a G.A.O.depends on the assumed interest rate and the assumed mortality rate: At the moment of writing the contract, the insurance company faces to the central problem of defining an accurate mortality base. Once it is done, the company can propose a conversion factor h (see section 2.5.1). Moreover, the decline in long-term interest rates and improvements in mortality rates are factors that cause the liabilities associated with these guarantees. In particular, the mortality risk makes the insurance markets incomplete.

The model proposed below is an *indifference argument* for pricing implicit options in life insurance contracts. Indifference models are built around the investor's attitude toward the risk. They are now very common in the financial literature that concerns incomplete markets with non-traded assets. In a dynamic setting, based on utility maximization criteria and on the concept of certain equivalent, the *indifference pricing* methodology was initially proposed by Hodges and Neuberger (1989), that suggested the concept of the so called *reservation price*. For an overview, I address the reader also to the following contributions and to the related bibliography: Henderson and Hobson (2004), Musiela and Zariphopoulou (2004), Zariphopoulou (2002).

Recently Young and Zariphopoulou (2002) and Young (2003), extended the *priciple of equivalent utility*, formulating, in a dynamic setting, the pricing problem for the insurance risk as a stochastic control problem. This framework connect financial mathematics and actuarial mathematics. The innovative idea is the consideration that both a rational insurer and a rational insured can go in the financial market and trade dynamically. In other words, a stochastic financial market in the standard actuarial models is introduced. Contributions that followed this approach for dynamic insurance risk are proposed by Moore and Young (2003), Jaimungal and Young (2005), Ludkovsk and Young (2006 and 2008), Ma and Yu (2006).

In what follows a indifference based model for valuing guaranteed annuity options in life insurance contracts is offered. This argument will be applied to G.A.O.s and we are willing to consider analogous models for other kinds of “embedded”options, for future research. To my knowledge an indifference method for valuing guaranteed annuity options is new and never developed before. Moreover, I believe that such a new approach gives new prospectives that should taken into account in order to describe such an option. In fact, the advantage of implementing an indifference is to consider the theory of optimal asset allocation toward the end of the life cycle, beside the mathematical models regarding guaranteed annuity options. This ways offer a larger sight over the nature of such options. For this reason, in the course of next section I review the theory of optimal annuitization policies.

3.2 A review of optimal annuitization policies

For the purpose to introduce an indifference model for the evaluation of implicit options in life insurance contracts, it is necessary to spend some words and to review some important contributions offered by: Milevsky (1998, 2001), Milevsky and Young (2002, 2003a, 2007a, 2007b), Milevsky, Moore and Young (2006). and also, Blake, Cairns and Dowd (2002). Considering an individual during the retirement years, these papers focus on the question when and if the agent will proceed to purchase a life annuity, by paying a non refundable lump sum to an insurance company in exchange for a lifelong consumption stream that cannot be outlived. In particular, referring to the definition given by Milevsky and Young (2007a), in what follow it is reviewed both the institutional *all-or-nothing* arrangement – where the annuitization can take place just at one distinct point of time – and also an *open-market* structure – where individuals can annuitise a fraction of their wealth at distinct points in time, locating a general optimal annuity purchasing policy. However I will mostly focus on the first setting.

Consider an agent at time T when her retirement begins. At this instant in time she face her wealth $W_T > 0$. Also assume that the individual has the opportunity to invest in a riskless asset whose price at time $s > T$, for some

$r \geqslant 0$, is described by the following dynamics:

$$\begin{cases} d\xi_s = r\xi_s \, ds \\ \xi_T = \xi > 0 \end{cases} \tag{3.1}$$

She can also trade dynamically in a risky asset whose price at time $s > T$ obey to the following dynamics

$$\begin{cases} dS_s = \mu S_s \, ds + \sigma S_s \, dB_s \\ S_T = S > 0 \end{cases} \tag{3.2}$$

where $0 < r < \mu$, $\sigma > 0$ and where $\{B_s\}$ is a standard Bownian motion in the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}, \mathrm{P})$, satisfying the *usual hypotheses* as defined by Protter (2005).

Let W_s be the wealth at time s of the inverstor and let π_s the amount of the investment in the risky asset. It is assumed that the agent can consume at a instantaneous rate c_s, self-financing herself; henceforth the amount allocated in the riskless asset will be $W_s - \pi_s$. The investor's wealth dynamics, if he or she does not purchase any annuity at time T, will be

$$\begin{aligned} dW_s &= r\left(W_s - \pi_s\right) ds + \pi_s \left(\mu \, ds + \sigma \, dB_s\right) - c_s \, ds \\ &= \left[r W_s + (\mu - r)\pi_s - c_s\right] ds + \sigma \pi_s \, dB_s \end{aligned} \tag{3.3}$$

under the initial condition $W_T = w_T > 0$. Strictly speaking, notice that we should denote de dependence of both $\{W_s\}$ and its associated differential operator, on the control laws $\{c_s\}$ and $\{\pi_s\}$.

Assume that the processes $\{c_s\}$ and $\{\pi_s\}$ are admissible in the sense that they are adapted to $\{\mathscr{F}_s\}$, square integrable, the equation above has an unique solution and also that $c_s \geqslant 0$, $\forall s \geqslant T$, see, for example, Björk (2004) and Øksendal (2003). Also notice, as remarked in the papers cited above, that herein only a simple geometric Brownian motion is considered and a constant risk-free rate. The latter assumption will be removed in section 3.5, however, for a bibliography concerning richer models, see for example Milevsky and Young (2003a).

3.2.1 The institutional all-or-nothing arrangement

Consider an individual aged x at time 0. In a market where just the all-or-nothing arrangement is allowed, at some point in time τ, the individual is asked to annuitise all his or her wealth W_τ in a lump sum. Then, following the line of Milevsky and Young (2007a), the associated value function of this problem is

$$U(w_T, T) := \sup_{\substack{\{c_s, \pi_s, \tau\} \\ c_s \geqslant 0, \forall s \geqslant T}} \mathbb{E}\left[\int_T^\tau e^{-r(s-T)} {}_{s-T}p^S_{x+T} \cdot u(c_s)\, \mathrm{d}s + \right.$$
$$\left. + \int_\tau^{+\infty} e^{-r(s-T)} {}_{s-T}p^S_{x+T} \cdot u(\overline{c})\, \mathrm{d}s \,\middle|\, W_T = w_T \right] \tag{3.4}$$

where the superscript to the survival probability denote that we consider the annuitant's subjective evaluation of mortality.

Once the individual decide to purchase the life annuity at time τ, having the current wealth W_τ, $\overline{c}$ denote the instantaneous consumption stream rate paid by the insurer. In particular we have

$$\overline{c} := \frac{W_\tau}{a^O_{x+\tau}}$$

where $a^O_{x+\tau}$ denote the actuarial present value (net of any insurance loading: $\varrho = 0$) of a life annuity that pays continuously a constant unitary rate, to an individual who is aged $x + \tau$ at the time of purchase (see equation 2.10). I recall – see remark 2.1 – that there is no loose of generality assuming $\varrho = 0$. Finally, the superscript O denote that this value is computed employing an objective hazard rate to calculate the survival probabilities. We shall use the superscript S if the individual's subjective hazard rate is applied.

Notice that in this model it is assumed that the individual discounts consumption at the riskless rate r. However, denoting by λ the instantaneous force of mortality implied by p,

$$e^{-r(s-T)} {}_{s-T}p^S_{x+T} = \exp\left\{ -\int_0^{s-T} r + \lambda^S_{x+T+\vartheta}\, \mathrm{d}\vartheta \right\} \tag{3.5}$$

if in (3.4) we want to use a subjective discount rate, say υ, this is equivalent to considering a different subjective force of mortality defined as

$$\widetilde{\lambda}^S := \lambda^S + (\upsilon - r)$$

The second integral in the expectation above can be rearranged as follow:

$$u(\overline{c}) \cdot \int_{\tau}^{+\infty} e^{-r(s-T)} {}_{s-T}p^S_{x+T} \, \mathrm{d}s = u(\overline{c}) e^{\tau - T} \cdot \int_0^{+\infty} e^{-rz} {}_{z+(\tau-T)}p^S_{x+T} \, \mathrm{d}z$$

the second term being obtained changing variable $s = z + \tau$. Now, considering relation (2.2) we get

$${}_{z+(\tau-T)}p^S_{x+T} = {}_{\tau-T}p^S_{x+T} \cdot {}_z p^S_{(x+T)+(\tau-T)}$$

that led us to write, remembering equation (2.10) with $\varrho = 0$:

$$U(w_T, T) := \sup_{\substack{\{c_s, \pi_s, \tau\} \\ c_s \geqslant 0, \forall s \geqslant T}} \mathbb{E}\left[\int_T^{\tau} e^{-r(s-T)} {}_{s-T}p^S_{x+T} \cdot u(c_s) \, \mathrm{d}s + \right.$$
$$\left. + e^{-r(\tau-T)} u(\overline{c}) \cdot {}_{\tau-T}p^S_{x+T} \cdot a^S_{x+\tau} \,\middle|\, W_T = w_T \right] \quad (3.6)$$

where $\overline{c}$ is computed, as shown before, using the objective probability seen by the insurer, while the last factor p^S is computed on the basis of the annuitant's subjective assessment for her mortality.

Problems (3.4 and 3.6) belong to an important class of stochastic optimal control problems. The value fucntion U require to choose both an optimal consumption and investment policy $\{c_s, \pi_s\}$, but also the optimal random stopping time τ. For a general presentation of such a maximum problem and the related solution techniques using Hamilton Jacobi Bellman equation, see Krylov (1979) and Øksendal (2003, Chap. 10.4 and Chap. 11). Setting

$$g(W_\tau, \tau; T) = e^{-r(\tau-T)} u(\overline{c}) \cdot {}_{\tau-T}p^S_{x+T} \cdot a^S_{x+\tau}$$

and recalling that the discount factor can be written as in (3.5), the value function can be finally written as

$$U(y,s) = \sup_{\substack{\{c_\eta,\pi_\eta,\tau\} \\ c_\eta \geqslant 0, \forall \eta \geqslant s}} \mathbb{E}\left[\int_s^\tau u(c_\eta) \cdot \exp\left\{ -\int_0^{\eta - s} r + \lambda^S_{x+s+\vartheta} \, \mathrm{d}\vartheta \right\} \mathrm{d}\eta + \right.$$

$$\left. + \; g(W_\tau, \tau; s) \;\middle|\; W_s = y \right] \quad (3.7)$$

Let $s \geqslant T$ be a fixed point in time and y a fixed point in the state space and, for the dynamics (3.3), consider the initial condition $W_s = y$. the differential operator is defined as

$$\left(\mathscr{A}_1^{c,\pi}\right)(y,s) := \left[ry + (\mu - r)\pi - c\right] \frac{\partial}{\partial y} + \frac{1}{2}\sigma^2\pi^2 \frac{\partial^2}{\partial y^2}$$

where $c := c_s$, $\pi := \pi_s$. Indicating with U_s, U_y and U_{yy} the partial derivatives of U with respect to the first and the second variable, provided that they exists continuous, the following three conditions have to be satisfied:

$$\sup_{c \geqslant 0, \pi} \left\{ U_s(y,s) + \left(\mathscr{A}_1^{c,\pi} U\right)(y,s) + u(c) - \left(r + \lambda^S_{x+s}\right) \cdot U(y,s) \right\} \leqslant 0 \quad (3.8)$$

$$g(y,s;s) - U(y,s) \leqslant 0 \quad (3.9)$$

inequality (3.8) being satisfied with equality whenever (3.9) is strict (3.10)

From relations above we get

$$U_s(y,s) + \sup_{c \geqslant 0, \pi} \left\{ \left[ry + (\mu - r)\pi - c\right] U_y(y,s) + \frac{1}{2}\sigma^2\pi^2 U_{yy}(y,s) + \right.$$

$$\left. + \; u(c) - \left(r + \lambda^S_{x+T}\right) \cdot U(y,s) \right\} \leqslant 0 \quad (3.11)$$

and, recalling that in the Bellman equation (3.11) we value U at the starting time s, i.e. the initial time T is assumed to be s, the condition $g(y,s;s) - U(y,s) \leqslant 0$ is

$$U(y,s) \geqslant u(\overline{c}) \cdot a^S_{x+s}$$

where a strict inequality has to imply the equality in (3.11). This problem has been solved by Milevsky and Young (2003a and 2007a). In particular, using conditions above, the authors show that, in the case in which the utility function exhibits constant relative risk aversion

$$-\frac{u''(c)}{u'(c)} \cdot c = \gamma := \text{constant}$$

solving problem (3.6) is equivalent to assuming that the optimal stopping annuitization time is some fixed time in the future τ. Based on that value of τ, one finds the optimal consumption and investment policies. Finally, one finds the optimal value $\tau \geqslant T$. Moreover, the authors show that the feature of constant relative risk aversion utility that drive this result is that wealth factors out of the value function; therefore, the stopping time is deterministic. In particular, in follows we consider an utility function u given by

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1 \tag{3.12}$$

and, for $\gamma = 1$, u is the logarithmic function. Looking for a solution of the form $U(y,s) = 1/(1-\gamma)y^{1-\gamma}\alpha^{\gamma}(t)$ authors obtain the optimal consumption and investment policies, given in a feedback form by

$$C^*_s = c^*\left(W^*_{1,s}, s\right) = \frac{W^*_{1,s}}{\alpha(s)}; \qquad \Pi^*_s = \pi^*\left(W^*_{1,s}, s\right) = \frac{\mu - r}{\sigma^2\gamma} W^*_{1,s} \tag{3.13}$$

where, setting $\delta_1 := r + \frac{1}{2}(\mu - r)^2/(\gamma\sigma^2)$, for $s < \tau$, function α is given by

$$\alpha(s) = \left[a^S_{x+\tau} \Big/ \left(a^O_{x+\tau}\right)^{1-\gamma}\right]^{1/\gamma} e^{-d_1(\tau-s)} \left({}_{\tau-s}p^S_{x+s}\right)^{1/\gamma} + \\ + \int_s^{\tau} e^{-d_1(\eta-s)} \left({}_{\eta-s}p^S_{x+s}\right)^{1/\gamma} \mathrm{d}\eta \tag{3.14}$$

and for $s \geqslant \tau$, we have

$$\alpha(s) = \left[a^S_{x+\tau} \Big/ \left(a^O_{x+\tau}\right)^{1-\gamma}\right]^{1/\gamma} \tag{3.15}$$

where $d_1 := r/\gamma - \delta_1(1-\gamma)/\gamma$.

Now, the authors also show that differentiating $\widetilde{U}(\tau) = U(w, T; \tau)$ with respect to τ, lead us to find the optimal time τ^* of annuitization. In particular we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U} \propto \left[\frac{\gamma}{1-\gamma}\left(\frac{a^S_{x+\tau}}{a^O_{x+\tau}}\right)^{-(1-\gamma)/\gamma} - \frac{1}{1-\gamma} + \frac{a^S_{x+\tau}}{a^O_{x+\tau}}\right] + a^S_{x+\tau}\left[\delta_1 - \left(r + \lambda^O_{x+\tau}\right)\right]$$

therefore, if the right sight is negative for all $\tau \geqslant T$, then it is optimal to annuitize immediately, getting $U(w, T) = \widetilde{U}(0)$. On the contrary, if thre exists a value $\tau^* > T$ such that the right sight of the previous expression is positive for all $T \leqslant \tau < \tau^*$ and is negative for all $\tau > \tau^*$, the it is optimal to annuitize at time τ^*, having $U(w, T) = \widetilde{U}(\tau^*)$. Herein the artifact of CRRA utility is that the decision to annuitize is indipendent of one's wealth. Also observe that in the particular case that the subjective and the objective force of mortality are equal, $\lambda := \lambda^O = \lambda^S$, expression above reduce to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U} \propto \delta_1 - \left(r + \lambda_{x+\tau}\right)$$

then – if the force of mortality is increasing with respect to τ – then either $\delta_1 \leqslant \left(r + \lambda_{x+\tau}\right)$, i.e. it is optimal to annuitize at time T, or $\delta_1 > \left(r + \lambda_{x+\tau}\right)$, from which it follows that there exists a time $\tau^* \in (T, +\infty]$ where it is optimal to annuitize. In other words, the optimal time to purchase a fixed immediate life annuity is when

$$\lambda_{x+\tau^*} > \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2$$

3.2.2 *The life cycle puzzle*

The process of annuitization, which require to pay to an insurance company a nonrefundable lump sum provide a *longevity insurance*. In particular a consumer that will purchase a life annuity, instead of creating his or her own consumption stream, will never run out of money. Empirically, however, it has been observed that most agents are reluctant to purchase actively life annuities. In Milvesky (2001 and 1998) the references and a more detailed explanation concerning this debate can be found. In fact, using his words, this phenomena is especially puzzling within the paradigm of the Ando and Modigliani (1963) life cycle hypothesis, or Yaari (1965), under which individuals would seek to smooth their lifetime consumption by annuitizing wealth. Life annuities can "smooth" and "guarantee" consumption for the rest of one's natural life.

3.2.3 The open-market structure

In an unconstrained market structure the assumption is that the annuitant consider a more general annuity purchasing process $\{\Psi_s\}$, instead of assuming that the individual will annuitize all his or her wealth in a lump sum at some point in time τ . For instance, the individual is allowed: *i.* to possess pre-existing annuities; *ii.* to annuitise only a portion of her wealth at a given time; *iii.* to buy annuities more than once in lump sums or even continuously; *iv.* to consume something other than the annuity income after annuitization.

Such a model has been proposed by Milevsky and Young (2003a, 2003b, 2007a), where they define Ψ_s as the non-negative annuity income rate at time s after any annuity purchases at that time. They also assume that $\{\Psi_s\}$ is right-continuous with left limits. In particular it is noticed that the source of this income could be previous annuity purchases or a pre-existing annuity, such a social security or a pension income. If we assume that at any point in time $s \geqslant T$ the individual can purchase an annuity at the price a^O_{x+s} per dollar of annuity income, the dynamics of the wealth process is given by:

$$dZ_s = \left[rZ_{s-} + (\mu - r)\pi_s + \Psi_{s-} - c_s \right] \mathrm{d}s - a^O_{x+s}\, \mathrm{d}\Psi_s + \sigma \pi_s\, \mathrm{d}B_s \qquad (3.16)$$

under the initial condition $Z_{T-} = z > 0$. The negative sign used for Z_{s-} and Ψ_{s-}, denote the left-hand limit of those quantities before any annuity purchase.

In this open-market structure the annuitant is supposed to maximize the expected utility of discounted lifetime consumption as well as bequest, over admissible $\{c_s, \pi_s, \Psi_s\}$. In particular, admissible control $\{\Psi_s\}$ are those that are non-negative and non-decreasing. The latter property can be interpreted as requiring the irreversibility of the annuity purchases. The optimization problem is then expressed by the following value function

$$U^z(z,\psi,T) := \sup_{\substack{\{c_s,\pi_s\}\\ c_s\geqslant 0,\forall s\geqslant T}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)}\,{}_{s-T}p^S_{x+T}\cdot u\left(c_s\right)\mathrm{d}s + \right.$$

$$\left. + \; e^{-r(\Theta-T)}v\left(Z_\Theta\right)\middle| Z_T = z;\, \Psi_T = \psi\right] \quad (3.17)$$

where Θ is the random time of the individual's death and v is a strictly increasing, concave function of bequest. The same remark which has been made above, concerning a subjective discount rate, is still good. Notice that U can be written in a more useful form where the bequest function is considered in the integral. The way of doing that is similar to what is done in section 3.3.2.

It is proved by Milevsky and Young that the value function U^z is jointly concave in z and ψ, it is strictly increasing with respect to both z and ψ, and it is continuous on $\overline{D} := \{(y,a,s) : y \geqslant 0, a \geqslant 0, s \geqslant 0\}$. Moreover they show that U is a constrained viscosity solution on $\overline{D}$of the following HJB equation

$$0 = \min\left\{\left(r+\lambda^S\right)U - U_s - (sy+a)\,U_y - \max_\pi\left[\tfrac{1}{2}\sigma^s\pi^2 U_{yy} + (\mu - r)U_y\right] + \right.$$

$$\left. - \max_{c\geqslant 0}\left[-c\,U_y + u(c)\right] - \lambda_{x+s}v(y); \quad a^O_{x+s}U_y - U_a\right\} \quad (3.18)$$

For specialized results – concerning the solutions of previous equation and the equivalence of the optimal annuity purchasing problem and the optimal consumption and investment problem, in the presence of proportional transaction costs – I remaind to the frameworks cited before. Just recall that the main results pointed out by the authors is that an utility-maximizing investor will initially acquire a base amount of annuity income and then will annuitise additional amounts if and when their wealth-to-income ratio exceeds a certain level. Also, individuals will annuitise a part of their wealth as soon as they have the opportunity to do so, having as an effect that, as they become older, more annuities are purchased.

3.3 Indifference valuation for the guaranteed annuity option I

The present section propose the indifference valuation model for the guaranteed annuity options. Herein I assume an institutional all-or-nothing arrangement and deterministic hazard (mortality plus interest) rates. In the course of next chapter I present the analytical results and the closed formulas related to the indifference model presented herein. Also, sections 3.4 relaxes some of the hypotheses that are assumed in the course of the present arrangement, that we are willing to consider for future research.

Recall that the assumed financial market follows the lines of Merton (1969, 1971, 1992) and it can be generalized – at cost of less analytical tractability – by the contributions provided, for example, by Trigeorgis (1993), Kim and Omberg (1996), Koo (1998), Sørensen (1999) and Wachter (2002).

Next, we shall consider an agent that holds a life insurance product written at time $t_0 \geqslant 0$. It is assumed that that this policy may embed a guarantee annuity option that gives the right to convert, at time of maturity $T > t$, some policyholder's accumulated funds to an immediate life annuity for a fixed conversion rate h. I also refer to the period $T - t_0$ as the *accumulation period*.

The pricing model for guaranteed annuity options is developed in two steps. First, I introduce the main considerations and the analysis of the annuitant's options at time T. The second step moves the valuation model at time $t \in [t_0, T)$. A paradigm in order to provide the time T and t_0 valuation for the guarantee annuity option is provided in section 3.3.4.

3.3.1 Main considerations

Motivation: Why the need of an equivalent utility argument?

An indifference argument for the valuation of implicit options at time $t_0 \leqslant t \leqslant T$, offers a specific advantage in approximating the extent of liabilities concerning a insurance policy. A guaranteed annuity option can be thought as an option defined on the future interest rates and the future mortality rates. *However, I believe that the choice to exercise such an option should also depend on the optimal asset allocation choice toward the end of the life cycle, based on the policyholder sub-*

jective assessment of her future survival probability. In fact, as suggested by Boyle and Hardy (2003). The choice between to exercise the option or not – that actually imply a longevity plus interest risk for the insurer – *can also depend on the individual preferences on whether to annuitise all the accumulated funds immediately at time T and on her subjective expectancy for her future life time*. Clearly the insurer exposition shall also depends on this attitude. The facts mentioned above may not be seen if we consider an analysis of a guarantee annuity option at time of conversion, on arguments concerning just the interest rate and the force of mortality.

Relevant matters at time T and t_0.

The indifference approach for guaranteed annuity options proposed in the present section is based on two steps. Essentially they are motivated by the following remarks:

I. In order to simplify the analysis, I make the assumption that the right represented by the option can be exercised just at time T and not over a given period of time. Thus, the first consideration concerns the nature of the G.A.O. at time T: *An individual can exercise such an option just at that time otherwise this right will be destroyed.*

II. A second remark is necessary to give us a way to formalize our model at time t. We can ask: *How much would the annuitant wish to pay, at time t_0, or in general at time $t \in [t_0, T)$, in order to get such an option in her plan?*

III. Once, at time t_0, the individual take the decision to purchase a policy, we can ask: *How much wealth the annuitant wish to accumulate during period $[t_0, T)$?*

Remark 3.1. Point III. highlights that at time t_0, when an agent choose to purchase a policy, she has to take a decision not only over the typology of the contract that she wants to buy, but also with respect to the extent of the accumulated funds that she will accumulate up to time T. The second choice indirectly act on the importance of the premium that she will pay from t_0 to T.

Remark 3.2. In the sense of point II. the evaluation of the embedded G.A.O. is made considering also a more difficult matter: the event that the option will be not exercised at time T.

3.3.2 Valuation at time of conversion

At time T consider an annuitant, currently aged $x+T$, that hold an insurance policy embedding a G.A.O. to convert some accumulated funds A. Hence, the agent must decide whether or not to exercise it. If we suppose that the annuitant is also currently endowed by a wealth $w>0$, other than the accumulated funds A, if she decide to *do not* exercise the option, we assume that at time T she withdraws the accumulated funds A and seeks to solve a standard Merton's problem given by:

$$U(w_T+A,T):=\sup_{\{c_s,\pi_s\}}\mathbb{E}\left[\int_T^{+\infty}e^{-r(s-T)}\,{}_{s-T}p^S_{x+T}\cdot\right.$$
$$\left.\cdot\,u(c_s)\,\mathrm{d}s\,\middle|\,W_T=w_T+A\right] \quad (3.19)$$

under the dynamics (3.3) for the wealth:

$$\begin{cases}\mathrm{d}W_s=\left[rW_s+(\mu-r)\pi_s-c_s\right]\mathrm{d}s+\sigma\pi_s\,\mathrm{d}B_s\\ W_T=w+A\end{cases} \quad (3.3')$$

***Assumption* 3.1.** The previous analysis, regarding the function U, considers an agent that holds a policy embedding a guaranteed annuity option. We need to extend our analysis also to a situation in which the annuitant holds a policy with no guaranteed annuity option embedded in it. It is natural to assume that the same value function U will represent the expected reward arising from the only strategy the agent can pursuit, if at time t_0 she purchased a plan with no guaranteed annuity option. In this case, in fact, at time T the agent has not a right to convert her accumulated funds.

In the case the individual *do* decide to exercise the G.A.O., she will receive a continuous cash-flow stream at a constant rate

$$H := A \cdot h \tag{3.20}$$

where h is the guaranteed rate (see equation 2.12), where the survival probability is determined considering the objective mortality assessment from the insurer point of view (we shall denote this measure by p^O). At time T, once funds A are converted for purchasing an immediate life annuity, the annuitant will remain with the wealth w and will receive a continuous cash-flow at a rate $H_s = H > 0$ per annum. This income will affect her consumption stream as follows

$$\begin{aligned} \mathrm{d}W_s &= r\left(W_s - \pi_s\right)\mathrm{d}s + \pi_s\left(\mu\,\mathrm{d}s + \sigma\,\mathrm{d}B_s\right) + \left(H - c_s\right)\mathrm{d}s \\ &= \left[r\,W_s + (\mu - r)\pi_s + H - c_s\right]\mathrm{d}s + \sigma\,\pi_s\,\mathrm{d}B_s \end{aligned} \tag{3.21}$$

under the initial condition $W_T = w > 0$, $H_T = H > 0$. Notice that the same notation is used to intend a different wealth dynamics. Finally, also notice that the G.A.O. is written just on funds A, therefore, leaving the agent, at time T, with the positive wealth w.

***Assumption* 3.2.** we assume that the agent will not purchase any annuity other than the one that she has already got at time T exercising the G.A.O..

Under the previous assumption and remarks, the problem that the agent will seek to solve is described by the following value function

$$V(w_T, T) :=\ \sup_{\{c_s,\pi_s\}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)}\ {}_{s-T}p^S_{x+T}\ \cdot\ u(c_s)\,\mathrm{d}s\ \middle|\ W_T = w_T\right] \tag{3.22}$$

under the dynamics (3.21).

Remark 3.3. A more general model – that I am willing to consider for future research – is proposed in section 3.4. The hypothesis I make in the present section are inspired to a “specific” all-or-nothing idea: In order to come to an indifference valuation for the guaranteed annuity option, the annuitization is considered just if the individual decide to exercise such option. Even restrictive, the strategy described by U, represent, however, an easy way to consider the scenario described by the assumption 3.1.

Remark 3.4. Value function (3.19) differ from (3.4) recalled in previous section. In fact, in order to consider an indifference valuation for the guaranteed annuity option, two comparable value functions are needed. For instance, in the course of the present arrangement, in order to come to explicit formulas and to take into account assumptions 3.1 and 3.2, value function U need to be reduced to a standard Merton's problem.

We assume the control processes $\{c_s\}$ and $\{\pi_s\}$ are admissible, in the sense that they are both progressively measurable with respect to $\{\mathscr{F}_s\}_{s \geqslant T}$, where $\mathscr{F}_s$ is the augmentation of $\sigma\left(B_t : T \leqslant t \leqslant s\right)$. Also, the following conditions hold a.s. for every $s \geqslant T$:

$$c_s \geqslant 0 \quad \text{and} \quad \int_T^s c_t \,\mathrm{d}t < \infty$$

$$\int_T^s \pi_t^2 \,\mathrm{d}t < \infty$$

Considering value function U and V, at time T, a rational individual will exercise the G.A.O. to convert the accumulated funds w, whenever the following inequality holds:

$$U(w + A, T) \leqslant V(w, T) \tag{3.23}$$

in which the dependency of V on A is indirectly given by the rate H in the equation (3.21). The previous inequality will be a fundamental part for the statement of the model at time t, for the evaluation of the G.A.O. during the accumulation period as follow in the next section.

3.3.3 Valuation at the beginning of the accumulation period

The goal of the present approach is to propose an indifference approach for valuing guarantee annuity options at every point in time $t_0 \leqslant t \leqslant T$. In fact, considering time t, it is important to provide an evaluation method for valuing the the implicit option still embedded in life insurance products during the accumulation period.

Consider an individual that face the opportunity to purchase an insurance product at time $t_0 < T$. To this end, assume that the agent is required to pay an instantaneous premium at a constant rate $P > 0$, for the accumulation period $[t_0, T)$. Also assume that the individual, aged $x + t_0$, at time t_0, is endowed by an initial wealth $w_0 > 0$. Moved by consideration II. we can ask: *If the insurance product does not provide any guarantee to convert the accumulated funds at time T, how much would the annuitant wish to pay, at time t_0, in order to embed a* G.A.O. *in her life insurance contract?*

Remark 3.5. In some insurance contracts, it can make sense to consider also a non-constant positive process $\{P_s\}$. In this case a tax-shelter plan generally embeds some other right other than just the G.A.O. In fact options like *Paid-up* or *Resumption* or again the *Dynamic premium adjustment*, that are very common in participating life insurance contracts, may allow the annuitant to customize a more performant plan, acting on the control $\{P_s\}$.

At time t_0 the annuitant is asked to make different choices: the typology of the contract that she wants to purchase (whether or not including the G.A.O.), and, second, the extent of the accumulated funds that the agent wants to realize at time T. The second choice actually define the plan the annuitant will finally purchase and the extent of the premium P. Now, since the actuarial value (with no loads) of the accumulated funds at time T is

$$A_T := \int_{t_0}^{T} e^{r(T-s)} P \, \mathrm{d}s$$

choosing an insurance contract that will assure a sum A_T is equivalent to choose a value for P. Actually this is the case when insurers offer to policyholders different policies at different prices.

Remark 3.6. For the sake of simplicity, I do not consider investment guarantees for the accumulation period and, in particular equity-indexed annuities or variable annuities (V.A.s) with guaranteed minimum maturity benefits. For more details concerning investment guarantees, see Hardy (2003, chapters 1, 6 and 13) and also Milevsky and Posner (2006, chapter 11).

In order to answer the above question, we can make the following

***Assumption* 3.3.** The individual is required to pay a lump sum L at time t_0, if she wants to embed a G.A.O. in her plan.

Remark 3.7. The previous assumption is just formal and it does not affect the generality and applicability of this indifference. In fact, we could assume that the agent face the decision to purchase a plan that embed a G.A.O. or not, under the condition that if she will choose to hold a guarantee option she has to pay a constant premium at a instantaneous rate, say P_2, while if she will opt for not include any implicit option in her contract, she will required to pay a rate, say P_1. As assumed before, both P_1 and P_2 are positive and constant in time, even if the rate P_2 at any point $t \in [t_o, T)$ will consist in a part expressing the accumulation process, say $\widetilde{P}_2(t)$, and a part that express the additional cost for the G.A.O., say $l(t)$. If, now, the annuitant wants to be assured for a final amount A_T, we need

$$\int_{t_0}^{T} e^{r(T-s)} P_1 \, \mathrm{d}s = \int_{t_0}^{T} e^{r(T-s)} \widetilde{P}_2(t) \, \mathrm{d}s$$

where the premium to be paid $P_2 = \widetilde{P}_2(t) + l(t)$ is constant, but just the part $\widetilde{P}_2(t)$ is worth to accumulate the funds A_T. Therefore $l(t)$ actually express the extravalue that the agent is willing to pay, if she decide to purchase a plan embedding a G.A.O. In other words we can assume that the present actuarial value of the option implicit in her plan is

$$\int_{t_0}^{T} e^{-r(s-t_0)} l(t) \, \mathrm{d}s$$

and, setting L to be equal to the integral above, it makes clear that, in the course of the present context, assumption 3.3 is just a formal hypothesis.

Previous hypothesis lead us to present a model where, for period $[t_0, T)$, the same wealth controlled dynamics can be considered independently of having embed the G.A.O. in the insurance product. In fact – keeping in mind that the annuitant, at any point in $[t_0, T)$ can dynamically trade in the financial market – her controlled wealth dynamics will be

$$\begin{aligned} \mathrm{d}W_s &= r\left(W_s - \pi_s\right)\mathrm{d}s + \pi_s\left(\mu\,\mathrm{d}s + \sigma\,\mathrm{d}B_s\right) - \left(c_s + P\right)\mathrm{d}s \\ &= \left[r\,W_s + (\mu - r)\pi_s - (P + c_s)\right]\mathrm{d}s + \sigma\pi_s\,\mathrm{d}B_s \end{aligned} \tag{3.24}$$

where no *labor income* is considered. In particular, for a richer model where a stochastic labor income is specified, one can follow the lines of Koo (1998). Making the hypothesis that the agen receive a stochastic labor income at a rate ζ_s, at time s, previous equation can be rewritten as follow

$$\begin{aligned} \mathrm{d}W_s &= \mathrm{d}\left(W_s - \pi_s\right) + \mathrm{d}\pi_s + \zeta_s\,\mathrm{d}s - \left(c_s + P\right)\mathrm{d}s \\ &= \left[r\,W_s + (\mu - r)\pi_s + \zeta_s - P - c_s\right]\mathrm{d}s + \sigma\pi_s\,\mathrm{d}B_s \end{aligned} \tag{3.25}$$

$\{\zeta_s\}$ being a diffusive process defined by the dynamics

$$\mathrm{d}\zeta_s = \nu\zeta_s\,\mathrm{d}s + \varsigma\zeta_s\,\mathrm{d}\overline{B}_s; \qquad \zeta_{t_0} = \zeta_0 > 0 \tag{3.26}$$

where $\nu \geqslant 0$ and $\varsigma \geqslant 0$ and $\{\overline{B}_s\}$ is a standard Brownian motion defined on the probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_s\}, \mathrm{P}\right)$. Also, it is supposed to be instantaneously correlated with $\{B_s\}$, by a constant coefficient δ. As remarked by Koo, the geometric Brownian motion assumption for the income process means that the shocks to the income growth rates are all permanents.

Coming back to the initial condition for process $\{W_s\}$, it will depend on the choice to incorporate the option in the insurance contract. Precisely, if the annuitant will not opt for having a G.A.O. in her plan, she will seek to maximize the expected utility till time T. Then – since our agent does not hold a policy that include any guarantee option – after that point she will have just the opportunity to optimize the time of annuitization. In other words a rational agent will face to the following maximization problem:

$$\mathcal{U}(w_0, \zeta_{t_0}, t_0) := \sup_{\substack{\{c_s, \pi_s\} \\ c_s \geqslant 0, \forall s \geqslant t_0}} \mathbb{E}\left[\int_{t_0}^{T} e^{-r(s-t_0)}\,{}_{s-t_0}p^S_{x+t_0} \cdot u(c_s)\,\mathrm{d}s + \right.$$
$$\left. + \, e^{-r(T-t_0)}\,{}_{T-t_0}p^S_{x+t_0}\, U(A_T + W_T, T) \,\middle|\, W_{t_0} = w_0;\ \zeta_{t_0} = \zeta_0 \right] \quad (3.27)$$

where $\{W_s\}$ follows dynamics (3.25).

Remark 3.8. Function U is valued at $A_T + W_T$. In fact at time T, the annuitant will see the current wealth W_T, but also the accumulated amount A_T. From that time, the decision when and if to annuitise has to concern this total initial wealth.

Remark 3.9. After time T the wealth controlled dynamics, that the agent must consider, is given by equation (3.3). However, that constraint is already considered in value function U. Hence in solving the previous problem it will be necessary to consider just dynamics (3.24).

Remark 3.10. Function $\mathcal{U}$ could also work to afford a different problems that are not take into account in the present framework: The choice of the contract. In fact, if we consider the optimization also with respect to process $\{P\}$, it would be equivalent to say that the agent is also asked at time t_0 to take a decision on the extent of the accumulated funds at time T. Now, since every different premium to pay characterize a different policy, in the present framework we work with a specific insurance product, take as given, and that the agent is just asked to take a decision on whether or not embedding a guaranteed annuity option.

If the individual will opt, at time t_0, to incorporate a guaranteed annuity option, for a given conversion rate h, she has immediately to pay the lump sum L, leaving him or her wih an initial wealth $w_0 - L$. Moreover, at time T, she will have the possibility to chose between exercising the guarantee option or not to exercise it. Therefore she face now the following problem

$$\begin{aligned}\mathcal{V}\left(w_0 - L, \zeta_0, t_0\right) := \sup_{\substack{\{c_s, \pi_s\} \\ c_s \geqslant 0, \forall s \geqslant t_0}} \mathbb{E}\Bigg[\int_{t_0}^{T} e^{-r(s-t_0)}\,{}_{s-t_0}p^{S}_{x+t_0} \cdot u(c_s)\,\mathrm{d}s + \\ + e^{-r(T-t_0)}\,{}_{T-t_0}p^{S}_{x+t_0} \times \\ \times \max\left\{U(A_T + W_T, T), V(W_T, T)\right\} \Bigg| W_{t_0} = w_0 - L;\, \zeta_{t_0} = \zeta_0\Bigg]\end{aligned} \tag{3.28}$$

where $\{W_s\}$ follows the dynamics (3.25). Regarding the period beginning at time t_0 and ending at time time T, we assume that the control processes $\{c_s\}$ and $\{\pi_s\}$ are still admissible. For instance we still require that they are both progressively measurable with respect to $\{\mathscr{F}_s\}_{t_0 \leqslant s < T}$, where $\mathscr{F}_s$ is the augmentation of $\sigma\left(B_t : t_0 \leqslant t \leqslant s\right)$. Yet, the following conditions hold a.s.: we have $c_s \geqslant 0$, for every $t_0 \leqslant s < T$, and

$$\int_{t_0}^{T} c_s\,\mathrm{d}s < \infty, \qquad \int_{t_0}^{T} \pi_s^2\,\mathrm{d}s < \infty$$

Remark 3.11. Value function V is valued on W_T while U is evaluated on the sum $A_T + W_T$. In fact, if the annuitant will opt to exercise the G.A.O., all the accumulated funds A_T will be immediately converted in a life long insurance annuity, whose the actuarial value is a_x^O, given the implicit rate r_h implied by h. Therefore, the effective initial wealth that the agent will face at time T, will be the current value of W_T.

At time t_0 the decision maker will opt to embed the G.A.O., paying a lump sum L_{t_0}, as long as the following relation will hold

$$\mathcal{U}\left(w_0, \zeta_0, t_0\right) \leqslant \mathcal{V}\left(w_0 - L_{t_0}, \zeta_0, t_0\right)$$

3.3.4 *The indifference valuation for the* G.A.O.

Consider an agent who, at time t_0, compares the two expected rewards arising from the value functions $\mathcal{U}$ and $\mathcal{V}$. To this end, for a given initial wealth w_0, consider

$$L_0^* := \sup\left\{L_0 : \mathcal{U}(w_0, \zeta_{t_0}, t_0) \leqslant \mathcal{V}(w_0 - L_0, \zeta_{t_0}, t_0),\ w_0 - L_0 > 0\right\}$$

We say that L_0^* is the indifference price for the guaranteed annuity option, if the following equality holds

$$\mathcal{U}(w_0, \zeta_{t_0}, t_0) = \mathcal{V}(w_0 - L_0^*, \zeta_{t_0}, t_0)$$

Otherwise we say that L_0^* is the maximum sum that the agent is willing to pay in order to embed the guaranteed annuity option in her policy.

It also makes sense to define an indifference price for the guaranteed annuity option, considering the time of conversion T. We define

$$L_T^* := \sup\left\{L_T : U(w_T + A, T) \leqslant V(w_T - L_T, T),\ w_T - L_T > 0\right\}$$

Similarly, we say that L_T^* is the indifference price for the guaranteed annuity option, if the following equality holds

$$U(w_T + A, T) = V(w_T - L_T^*, T)$$

3.3.5 *Valuation during the accumulation period*

In order to value a guaranteed annuity option at a fixed point $t_0 < t < T$, we need to make clear some preliminary facts. Since the option has to be referred to the same extent for the accumulation funds, set P_t as the real number such that for a given A_T the following relation holds:

$$A_T = \int_t^T e^{r(T-s)} P_t \, \mathrm{d}s$$

In other words we may think at P_t as the constant instantaneous premium rate that has to be paid in order to accumulate the wealth A_T at time T, if no load is considered. Now, set $P_0 := P$ and, at each point in time t, consider the controlled dynamics

$$\mathrm{d}\widetilde{W}_s = \left[r\widetilde{W}_s + (\mu - r)\pi_s + \zeta_s - P_t - c_s \right] \mathrm{d}s + \sigma \pi_s \, \mathrm{d}B_s; \quad W_t = w_t > 0 \quad (3.29)$$

the process $\{\zeta_s\}$ being specified by the equation (3.26), under the initial condition that the process $\{\zeta_s\}$ is valued $\zeta_t > 0$ at time t.

The problem now is to value a guaranteed annuity option for a person aged $x + t$, characterized by the same utility function and the same subjective assessment for her mortality intensity. In this sense we want to answer to the question: *Is it possible to determine a "fair" (in some sense) value for a guaranteed option that assure the same conversion rate at time T?*

Remark 3.12. The valuation at time t consider a differ controlled dynamics for the individual's wealth. Strictly speaking, even if the G.A.O. written at time t assure the same conversion rate, the insurance product written at time t as to be considered different from another one written at a different time in $[t_0, T)$.

Assumptions above are consistent, however, with the following "forced" interpretation: Suppose to have an insurance product that embed an option, and assume to be possible to exchange this contract with someone else (characterized by the same utility function and the same subjective judgement for her subjective probability), having in return, in any case, the accumulated funds up to time t plus the "current value" of the G.A.O. In such a (abstract but useful) circumstance, the embedded option should be anyways valued considering (3.29) and the same indifference method shown before. In this sense we are lead to define the *reservation value for the* G.A.O. *at time* $t_0 \leqslant t < T$ in the same fashion of section 3.3.4: the maximum amount, if there exists, L^*_t such that

$$\mathscr{U}(w_t, \zeta_t, t) = \mathscr{V}(w_t - L^*_t, \zeta_t, t) \tag{3.30}$$

where $\mathscr{U}$ and $\mathscr{V}$ are computed under the wealth controlled dynamics constraint $\{\widetilde{W}_s\}$ and the controlled dynamics (3.29).

3.4 Indifference valuation for the guaranteed annuity option II

The indifference model to value guaranteed annuity options, can be generalized in a different market arrangement. Inspired by the open-market structure proposed by Milevesky and Young (2003b, 2007a), I propose in the present section a more rich setting that I am willing to consider for future research. For instance I assume that, after the retirement, the agent is allowed to purchase more annuities than once, even continuosly. In order to focus just on a plan where a G.A.O. is embed, it is necessary to consider the following

***Assumption* 3.4.** The premium to be paid at an instantaneous rate P is referred only to the accumulation plan that embed the option we want to value and, in particular, I do not consider any annuity income during the period $[t_0, T)$ where the agent accumulate funds A available at T.

Notice that the difference in such a new context is given by the analysis of the options at time T of conversion. In fact the valuing process at time $t_0 < T$ and at any point in the interval $[t_0, T)$, is similar to those ones proposed in previous section 3.3.

As we did for the first arrangement type, I begin to analyze the options for the decision maker at the time T of conversion. Recall that if we consider an open-market structure the wealth dynamics is given by equation (3.16):

$$dZ_s = \left[rZ_{s-} + (\mu - r)\pi_s + \Psi_{s-} - c_s \right] \mathrm{d}s - a^O_{x+s}\, \mathrm{d}\Psi_s + \sigma \pi_s \, \mathrm{d}B_s \qquad (3.16')$$

where the negative sign used for Z_{s-} and Ψ_{s-}, denote the left-hand limit of those quantities before any annuity purchase. The initial condition, however, depends on the decision of the annuitant. As we did in section 3.3.2, we can suppose that the agent is currently (time T) endowed by a wealth w, other than the accumulated funds A.

The relevant matter now is that if the annuitant decides to *do not* exercise the G.A.O., she is not anymore required to annuitise all her wealth at a optimal point in time $\tau^* \geqslant T$. In fact, in this new setting, she can determine an optimal annuity purchasing strategy. In this sense now a bequest function at the death time is required either if she does not exercise the G.A.O. or if she decides to exercise it. In particular, if the agent decides to *do not* exercise the option he will seeks to solve the following optimization:

$$U^z(w+A,0,T) := \sup_{\substack{\{c_s,\pi_s\} \\ c_s \geqslant 0, \forall s \geqslant T}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)}\,{}_{s-T}p^S_{x+T} \cdot u\left(c_s\right) \mathrm{d}s + \right.$$

$$\left. + \; e^{-r(\Theta-T)} v\left(Z_\Theta\right) \middle| Z_T = w+A; \Psi_T = 0 \right] \quad (3.31)$$

where accordingly with previous assumption, we shall suppose no annuity income till time T. On the contrary, if the annuitant decides to exercise the G.A.O., funds A will be converted into an annuity that pays an immediate instantaneous rate $H = A \cdot h$, but also she can purchase other annuities using the current wealth at each time $s \geqslant T$. In other words, the annuitant will face the value function associated to the following problem:

$$V^z(w,H,T) := \sup_{\substack{\{c_s,\pi_s\} \\ c_s \geqslant 0, \forall s \geqslant T}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)}\,{}_{s-T}p^S_{x+T} \cdot u\left(c_s\right) \mathrm{d}s + \right.$$

$$\left. + \; e^{-r(\Theta-T)} v\left(Z_\Theta\right) \middle| Z_T = w; \Psi_T = H \right] \quad (3.32)$$

Comparing utility in the present market structure, a rational decision maker will exercise the guaranteed annuity option as long as the following relation holds

$$U^z(w+A,0,T) \leqslant V^z(w,H,T)$$

3.5 Stochastic mortality and stochastic interest rates

Stochastic models for longevity risk and interest rates is necessary for a comprehensive analysis of the concerns around guaranteed annuity options. For instance, the matter of mortality risk is analyzed in a flourishing branch of Actuarial Mathematics. Herein an overview of the literature concerning stochastic mortality and dynamical survival models is presented. While the latter refer to actuarial projecting techniques for survival tables, the former concentrate on the stochastic modelling for the intensity of mortality.

3.5.1 Stochastic frameworks for mortality modelling

Dynamic survival models and projecting methods for longevity risk

In chapter 2 we concerned with demographical trends in lifetime insurance contracts and with longevity risk. It is also clear that in valuing guaranteed annuity options, a suitable stochastic mortality model is required. Projecting mortality tables including a forecast for future mortality rates is what we call dynamic survival models. These models represent a big issue in actuarial mathematics, when life annuities and other living benefits are considered. For a survey concerning survival models in a dynamic context see Pitacco (2004), for a survey in this subject and the methods used in order to projects mortality tables. Also the following contributions are suggested: Brouhns, Denuit and Vermunt (2002), Di Lorenzo and Sibillo, Haberman and Russolillo (2005), Lee (2000), Marocco and Pitacco (1997), Olivieri (2001), Olivieri and Pitacco (2003, 2005, 2001), Olivieri and Pitacco a, Olivieri and Pitacco b, Pitacco (2002b and 2004), Marceau and Gaillardetz (1999).

Stochastic models for mortality intensity

The tool of stochastic processes can be applied to model the evolution of the mortality intensity. The approach introduced by Dahl (2004) allow to capture both time dependency and uncertainty of the future development or mortality intensity. In particular the mortality intensity is modelled by a fairly general diffusion model, including the mean reverting brownian Gompertz model proposed by Milevsky and Promislow (2001) (also see Milevsky Promislow and Young 2005). In particular in the latter approach, it is important to highlight that stochastic (interest plus mortality) hazard rates are considered. The reader is finally addressed the following contributions given by Cairns, Blake and Dowd (2005 and 2006), Schrager (2006).

3.5.2 Valuation for G.A.O.*s with stochastic mortality*

Stochastic mortality rates represent an important concern over the so called *longevity risk*. Moreover, liabilities afferent to guaranteed annuity options depends on the variations of interests rates and mortality rates over the time. In this sense a richer model has to take into account. Herein a sketch for stochastic models for mortality intensity is recalled.

The debate over the stochastic mortality is very prolific and the literature concerning this problem is huge. The contribution by Dahl (2004), propose to model the mortality intensity by a fairly general diffusion process, which include the mean reverting model proposed by Milevsky and Promislow (2001). Precisely the author consider a P dynamics for the mortality intensity given by

$$d\lambda_{x+s} = \alpha^{\lambda}\left(s, \mu_{x+s}\right) ds + \sigma^{\lambda}\left(s, \lambda_{x+s}\right) d\widetilde{W}_s \tag{3.33}$$

where α^{λ} and σ^{λ} are non-negative and $\{\widetilde{W}_s\}$ is a standar Wiener process with respect to the same filtration $\{\mathscr{F}_s\}$, defined above, for $s \geqslant t_0$. $\{\widetilde{W}_s\}$ is assumed uncorrelated with $\{W_s\}$. In order to embed stochastic mortality in our reservation model, we can assume an initial condition for the dynamics (3.33) : $\lambda_{x+t_0} = \overline{\lambda}_{x+t_0}$, where the number $\overline{\lambda}_{x+t_0}$ can be thought as the mortality intensity for a person aged $x + t_0$, estimated at time t_0. Then in order to value a G.A.O. at time t_0, the same reasoning proposed above, remembering that expectations relatives to functions $\mathscr{U}$, $\mathscr{V}$, U and V, have also to be conditionate with respect to the initial condition $\lambda_{x+t_0} = \overline{\lambda}_{x+t_0}$.

Previous approach present relevant analytical difficulties, since the value function will depend also on the initial condition assumed for the process $\{\lambda_{x+s}\}$. Then, another way to include in our model a stochastic mortality is to compare different scenarios for different survival probabilities.

3.6 Conclusions

An indifference valuation model for guaranteed annuity option is proposed in the course of the present chapter. This model refers to the point of view of an agent who is willing to purchase an insurance policy embedding a guaranteed annuity option. In order to come to a reservation value, we have considered the indifference valuation at the time of conversion and, then, we have also given a valuation for the G.A.O. at any time during the accumulation period. For the case named as "first arrangement" we can obtain explicit solutions that are presented and implemented in the course of next chapter.

CHAPTER 4

MAIN RESULTS AND IMPLEMENTATION

Referring to the first arrangement presented in the course of previous chapter, herein analytical results and explicit solutions are found and numerical implementations are presented. For instance, the HJB approach is considered and the related partial differential equations (PDEs) are specified and explicit solved. In order to find value function $\mathcal{U}$ and $\mathcal{V}$, the explicit solution for a class of stochastic problems is found, where finite horizon, bequest motive and power consumption utility are jointly considered.

In the present context, if applied to the symbol of a function, the subscript s, y and yy will denote its partial derivative with respect to the related variable. Also all regularity conditions are assumed: See Øksendal (2003, Chap. 10 and Chap. 11). Finally, in order to get a closed form for all value functions defined above, labor income is not considered in the present chapter.

4.1 Main results

4.1.1 *The inequality* $U(w_T + A, T) \leqslant V(w_T, T)$

Consider a wealth w_T at time T. In 4.2.1 and 4.2.2 it is shown that, if the policyholder is characterized by a constant relative risk averse utility from consumption (4.3) and the technical assumption 4.1 hold, the value function U and the value function V are given in a closed form by:

$$U(w_T + A, T) = \frac{1}{1-\gamma}(w_T + A)^{1-\gamma} \cdot \varphi^{\gamma}(T) \tag{4.1}$$

$$V(w_T, T) = \frac{1}{1-\gamma}\left(w_T + \frac{H}{r}\right)^{1-\gamma} \cdot \varphi^{\gamma}(T) \tag{4.2}$$

where φ is an opportune function of time, given by (4.7), that turns to be the same for both value functions U and V. Notice that for every $\gamma > 0$, $\gamma \neq 1$, we have

$$U(w_T + A, T) \leqslant V(w_T, T) \quad \Leftrightarrow \quad r \leqslant h$$

From an economic point of view, previous inequality tells us that, at time of conversion T (that may coincide with her retirement), the policyholder will find convenient to exercise the guaranteed annuity option se e soltanto se the guaranteed rate h is greater than the current interest rate r. Moreover, recalling that

$$1/h = \int_T^{+\infty} e^{-r_h(s-T)} \, {}_{s-T}p^{O}_{x+T} \, \mathrm{d}s =: \overline{a}^{(h)}_{x+T}$$

where p^O denote the objetive mortality assessment from the insurer's point of view, the previous inequality can be also written in the following way:

$$U(w_T + A, T) \leqslant V(w_T, T) \quad \Leftrightarrow \quad \overline{a}^{(h)}_{x+T} \leqslant 1/r$$

Previous relation is very interesting. It says that, in order to come to a decision, the policyholder actually compares the minimum between the actuarial cost of buying a per dollar-guaranteed life long annuity (assured by the insurance company), whose the present value is given by a guaranteed implicit rate r_h, and the cost of a per dollar-life long annuity, whose the present value is given employing the market interest rate r.

4.1.2 A closed form for value functions $\mathcal{U}$ and $\mathcal{V}$

By section 3.3.3, we know that the value function $\mathcal{V}$ at time T needs to be equal to

$$G(w_T, T) := \max\left\{U(w_T + A, T);\ V(w_T, T)\right\}$$

where w_T is the agent's wealth at time T. However, notice that, by the previous section, an explicit expression for the value function $\mathscr{V}$ can be obtained. In fact G can be written as follows:

$$G(w_T, T) = \begin{cases} U(w_T + A, T), & \text{if } r \geqslant h \\ V(w_T, T), & \text{if } r < h \end{cases}$$

Using this fact, in section 4.3 it is shown that a closed form, for the value function $\mathscr{V}$, can be found giving an explicit solution for a class of stochastic problems. Using the result presented in the course of section 4.3 and the previous characterization of G, we arrive at the following expression for the value function $\mathscr{V}$:

$$\mathscr{V}(w_0, t_0) = \begin{cases} \mathscr{U}(w_0, t_0) & \text{if } r \geqslant h \\ \frac{1}{1-\gamma}\left(w_0 - \widehat{\xi}_{\mathscr{V}}(t_0)\right)^{1-\gamma} \psi^{\gamma}(t_0) & \text{if } r < h \end{cases}$$

where $\widehat{\xi}_{\mathscr{V}}$ is given by (4.18) and ψ is defined by (4.16). Yet, section 4.3, also gives a way to find a closed form for the value function $\mathscr{U}$:

$$\mathscr{U}(w_0, t_0) = \frac{1}{1-\gamma}\left(w_0 - \widehat{\xi}_{\mathscr{U}}(t_0)\right)^{1-\gamma} \psi^{\gamma}(t_0)$$

where $\widehat{\xi}_{\mathscr{U}}$ is given by (4.17).

4.1.3 The indifference value for the G.A.O.

If the indifference price exists, recall section 3.3.4, it is straightforward to deduce that it is given by

$$L_0^* = \left(\frac{H}{r} - A\right) e^{-r(T-t0)}$$

Moreover, notice that, if at time T, an indifference price exists, it can be computed as follows:

$$L_T^* = \frac{H}{r} - A$$

From an economic point of view, it represents the difference between the present value at time T of a perpetuity that pays a continuous stream at a rate H per year, and the value (at time T) of the accumulated funds A. Recall that a life long annuity that will pay a stream at a continuous rate H per year, is what the guaranteed annuity option assures at time T. It is interesting to see that, if both indifference prices L_0^* and L_T^* exist, they are tied by the following relation:

$$L_0^* = e^{-r(T-t_0)} \cdot L_T^*$$

4.2 Computing value functions U and V

Explicit forms for the value functions U, V, $\mathcal{U}$ and $\mathcal{V}$ are found, assuming a constant relative risk aversion (CRRA) utility from the consumption, i.e.

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1 \tag{4.3}$$

***Assumption* 4.1.** The following optimization problems turn to be well-posed if $r > 0$ and

$$r > (1-\gamma)\delta \tag{4.4}$$

where $\delta := r + 1/(2\gamma) \cdot (\mu - r)^2/\sigma^2$, coherently to the hypothesis assumed also in Karatzas, Lehoczky, Sethi, and Shreve (1986).

4.2.1 The value function U

Next, we construct a value function that measure the utility since time $t \geqslant T$, for a generic initial wealth $y > 0$:

$$\widetilde{U}(y, t) := \sup_{\{c_s, \pi_s\}} \mathbb{E}_{y,t} \left[\int_t^{+\infty} e^{-r(s-t)}\ {}_{s-t}p^S_{x+t} \cdot u(c_s)\,\mathrm{d}s \right]$$

where $\mathbb{E}_{y,t}$ denotes the expectation conditioned on $W_t = y$, given the dynamics (3.3’). The associated differential operator is given by

$$(\mathscr{A}^{c,\pi})(y, t) := [ry + (\mu - r)\pi - c]\,\tfrac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2\pi^2\,\tfrac{\partial^2}{\partial y^2}$$

see, for example, Øksendal(2003), Krylov(1979) and Björk(2004). Notice that the discounting factor and the subjective probability can be rewritten as follows:

$$e^{-r(s-t)}\,{}_{s-t}p^S_{x+t} = \exp\left\{-\int_t^s r + \lambda^S_{x+\eta}\,\mathrm{d}\eta\right\}$$

where λ^S denotes the subjective force of mortality. Therefore , the HJB equation associated to value function $\widetilde{U}$ is

$$0 = \sup_{c\geqslant 0,\pi}\left\{\widetilde{U}_t(y,t) - (r+\lambda^S_{x+t})\widetilde{U}(y,t) + u(c) + \left(\mathscr{A}^{c,\pi}\widetilde{U}\right)(y,t)\right\}$$

Assuming a CRRA consumption utility, as given by (4.3), the previous equation leads to the following partial differential equation

$$\widetilde{U}_s - (r+\lambda^S_{x+t})\widetilde{U} + \frac{\gamma}{1-\gamma}\widetilde{U}_y^{(\gamma-1)/\gamma} + ry\widetilde{U}_y - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\frac{\widetilde{U}_y^2}{\widetilde{U}_{yy}} = 0$$

under the boundary condition

$$\lim_{t\to+\infty}\widetilde{U}(y,t) = 0$$

a.s. with respect to the law of process $\{W_s\}$ defined by the stochastic differential equation (3.3'). In equation (4.2.1) variables (y,s) are suppressed.

In order to solve the previous partial differential equation, we try a solution of the form

$$\widetilde{U}(y,t) = \frac{1}{1-\gamma}y^{1-\gamma}\cdot\beta^{\gamma}(t) \tag{4.5}$$

Taking derivatives and plugging into equation (4.2.1), considering $\widetilde{U}$ of the form (4.5), we obtain that β solves the following ordinary differential equation

$$\beta'(t) + \left[\frac{(1-\gamma)\delta - (r+\lambda^S_{x+t})}{\gamma}\right]\beta(t) = -1 \tag{4.6}$$

where $\delta := r + 1/(2\gamma) \cdot (\mu - r)^2/\sigma^2$. Taking limits up to infinity and considering the boundary condition, we find that the previous ordinary differential equation is solved by the following function:

$$\varphi(t) = \int_t^{+\infty} e^{-b(s-t)} \cdot \left({}_{s-t}p^S_{x+t} \right)^{1/\gamma} \mathrm{d}s \tag{4.7}$$

where $b := -[(1-\gamma)\delta - r]/\gamma$.

Remark 4.1. It is easy to see that, under the assumption 4.1 the integral above is convergent. In fact:

$$\begin{aligned} |\varphi(t)| &= \left| \int_t^{+\infty} e^{-b(s-t)} \cdot \left({}_{s-t}p^S_{x+t} \right)^{1/\gamma} \mathrm{d}s \right| \\ &\leqslant \int_t^{+\infty} \left| e^{-b(s-t)} \right| \cdot \left| {}_{s-t}p^S_{x+t} \right|^{1/\gamma} \mathrm{d}s \\ &\leqslant \int_t^{+\infty} e^{-b(s-t)} \mathrm{d}s < +\infty \end{aligned}$$

Therefore, by the verification theorem, we have found the value function U.

Given the optimal controlled wealth $\{W_t^*\}$, the optimal consumption and investment policies are given in feedback form by

$$C_t^* = W_t^*/\varphi(t), \qquad \Pi_t^* = \frac{\mu - r}{\gamma\sigma^2} W_t^*$$

4.2.2 *The value function V*

In order to find the value function V, consider the following value function $\widetilde{V}$, starting at time $t \geqslant T$, for an initial wealth $y > 0$:

$$\widetilde{V}(y, t) := \sup_{\{c_s, \pi_s\}} \mathbb{E}_{y,t} \left[\int_t^{+\infty} e^{-r(s-t)} {}_{s-t}p^S_{x+t} \cdot u(c_s) \, \mathrm{d}s \right]$$

subject to the dynamics (3.21). In this case the differential operator is given by

$$(\mathscr{A}^{c,\pi})(y, t) := [ry + (\mu - r)\pi + H - c] \tfrac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2\pi^2 \tfrac{\partial^2}{\partial y^2}$$

Henceforth, the HJB equation associated to $\widetilde{V}$ is

$$0 = \sup_{c,\pi} \left\{ \widetilde{V}_s(y,t) - (r + \lambda^S_{x+t})\widetilde{V}(y,t) + u(c) + \left(\mathscr{A}^{c,\pi}\widetilde{V}\right)(y,t) \right\}$$

The equation above leads to the following partial differential equation for the value function $\widetilde{V}$:

$$\widetilde{V}_s - (r + \lambda^S_{x+t})\widetilde{V} + (ry + H)\widetilde{V}_y + \\ + \frac{\gamma}{1-\gamma}\widetilde{V}_y^{(\gamma-1)/\gamma} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\widetilde{V}^2}{\widetilde{V}_{yy}} = 0 \quad (4.8)$$

under the boundary condition

$$\lim_{t\to+\infty} \widetilde{V}(y,t) = 0$$

Given the assumption regarding the interest rates, in section **??**, in order to solve (4.8) I consider a technique similar the one proposed in Koo(1998). For instance, consider the following form for $\widetilde{V}$:

$$\widetilde{V}(y,t) = \frac{1}{1-\gamma}(y + H/r)^{1-\gamma}\beta_1^{\gamma}(t)$$

where β_1 is a function of time. Taking derivatives and after rearranging expressions, it is straightforward to show that, once again, $\beta_1(t)$ satisfies the ordinary differential equation (4.6). Therefore, value function V is characterized by φ.

Remark 4.2. Since, in the present model we require $r > 0$ and H is a positive constant, the well-posedness of the solution is assured by Koo (1998, Condition B.).

Given the optimal controlled wealth $\{W_t^*\}$, the optimal consumption and investment policies are given in feedback form by

$$C_t^* = \frac{W_t^* + H/r}{\varphi(t)}, \qquad \Pi_t^* = \frac{\mu - r}{\gamma\sigma^2}\left(W_t^* + H/r\right)$$

4.3 Computing value functions $\mathscr{U}$ and $\mathscr{V}$

In order to give a closed form for value functions $\mathscr{U}$ and $\mathscr{V}$, consider the class of stochastic problems given by

$$\mathscr{G}(w_0, t_0) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r+\lambda^S_{x+\eta}\, \mathrm{d}\eta} \cdot u(c_s)\, \mathrm{d}s + \right.$$
$$\left. + \; e^{-r(T-t_0)}\; {}_{T-t_0}p^S_{x+t_0} \cdot \frac{(W_T-\Delta)^{1-\gamma}}{1-\gamma} \;\middle|\; W_{t_0} = w_0 \right] \quad (4.9)$$

subject to the following dynamics for process $\{W_s\}$:

$$\begin{cases} \mathrm{d}W_s = \left[r\,W_s + (\mu - r)\pi_s - c_s - \Delta_1\right] \mathrm{d}s + \sigma\,\pi_s\, \mathrm{d}B_s \\ W_{t_0} = w_0 \end{cases} \quad (4.10)$$

where $\Delta < 0$ and $\Delta_1 > 0$. The previous problem is difficult because of two reasons: first, the horizon time T is finite and the function

$$\widetilde{g}(W_T, t_0; T) = e^{-r(T-t_0)}\; {}_{T-t_0}p^S_{x+t_0} \cdot \frac{(W_T-\Delta)^{1-\gamma}}{1-\gamma}$$

that acts as a bequest function, is not null, being associated to a finite-time horizon, and also different from the CRRA utility of the consumption $u(c)$. Second, another complication arises from the drift term associated to the dynamics of the process $\{W_t\}$ since this contains the constant Δ_1.

Kingston and Thorp (2005) provide a technique of solution for a different class of problems considering a finite time horizon, bequest function, and the presence of a consumption floor. A similar technique may be applied to problem (4.9, 4.10). To this end, construct a value function that measure the remaining utility since time $t \geqslant t_0$, given a positive wealth y:

$$\widetilde{\mathscr{G}}(y, t) := \sup_{\{c_s, \pi_s\}} \mathbb{E}_{y,t}\left[\int_{t}^{T} e^{-\int_{t}^{s} r+\lambda^S_{x+\eta}\, \mathrm{d}\eta} \cdot u(c_s)\, \mathrm{d}s + \right.$$
$$\left. + \; e^{-r(T-t)}\; {}_{T-t}p^S_{x+t} \cdot \frac{(W_T-\Delta)^{1-\gamma}}{1-\gamma} \right] \quad (4.11)$$

The differential operator associated to dynamics (4.10) is given by:

$$(\mathscr{A}^{c,\pi})(y,t) := \left[ry + (\mu - r)\pi - c - \Delta_1\right] \tfrac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2\pi^2 \tfrac{\partial^2}{\partial y^2}$$

Henceforth, value function $\widetilde{\mathscr{G}}(y,t)$ needs to satisfy the following HJB equation:

$$0 = \sup_{c \geqslant 0, \pi} \left\{ \widetilde{\mathscr{G}}_t(y,t) - (r + \lambda_{x+t})\widetilde{\mathscr{G}}(y,t) + u(c) + \left(\mathscr{A}^{c,p}\,\widetilde{\mathscr{G}}\right)(y,t) \right\}$$

that leads to the following partial differential equation

$$\widetilde{\mathscr{G}}_t - (r + \lambda_{x+t})\widetilde{\mathscr{G}} + (ry - \Delta_1)\widetilde{\mathscr{G}}_y + \\ + \frac{\gamma}{1-\gamma}\widetilde{\mathscr{G}}_y^{(\gamma-1)/\gamma} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\widetilde{\mathscr{G}}_y^2}{\widetilde{\mathscr{G}}_{yy}} = 0 \quad (4.12)$$

under the terminal condition

$$\widetilde{\mathscr{G}}(y,T) = \widetilde{g}(y,T;T) \quad (4.13)$$

Motivated by the contributions of Kingston and Thorp (2005) and Koo(1998), and by the result recalled in previous section 4.2.2, consider the change of variables:

$$\widehat{\xi}(t) := \frac{\Delta_1}{r}\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)} \quad (4.14)$$

$$\xi(t) := y - \widehat{\xi}(t) \quad (4.15)$$

that leads to

$$y = \xi + \widehat{\xi} = \xi + \frac{\Delta_1}{r}\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)}$$

Thus

$$\widetilde{g}(y,T;T) = \frac{1}{1-\gamma}\xi^{1-\gamma}(T)$$

Under this change of variable guess a solution, for the equation (4.12), of the following form:

$$\widetilde{\mathscr{G}}(y,t) = \frac{1}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma}(t)$$

for which, taking derivatives:

$$\widetilde{\mathscr{G}}_t = \xi^{-\gamma}\left(\Delta_1 e^{r(t-T)} - r\Delta e^{r(t-T)}\right)\alpha^{\gamma} + \frac{\gamma}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma-1}\alpha'$$

$$\widetilde{\mathscr{G}}_y = \xi^{-\gamma}\alpha^{\gamma}$$

$$\widetilde{\mathscr{G}}_{yy} = -\gamma\xi^{-\gamma-1}\alpha^{\gamma}$$

and plugging into equation (4.12) we arrive to

$$\begin{aligned}
0 = &\ \xi^{-\gamma}\left(\Delta_1 e^{r(t-T)} - r\Delta e^{r(t-T)}\right)\alpha^{\gamma} + \frac{\gamma}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma-1}\alpha' \\
&- (r+\lambda_{x+t})\frac{1}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma} + \frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\xi^{1-\gamma}\alpha^{\gamma} \\
&+ \frac{\gamma}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma} + (ry-\Delta_1)\xi^{-\gamma}\alpha^{\gamma}
\end{aligned}$$

Notice that previous equality can be simplified having care that

$$\begin{aligned}
\xi^{-\gamma}\left(\Delta_1 e^{r(t-T)} - r\Delta e^{r(t-T)}\right)\alpha^{\gamma} + (ry-\Delta_1)\xi^{-\gamma}\alpha^{\gamma} = & \\
&= r\cdot\xi^{-\gamma}\left[y - \left(\frac{\Delta_1}{r}\left(1-e^{r(t-T)}\right) + \Delta e^{r(t-T)}\right)\right]\alpha^{\gamma} \\
&= r\cdot\xi^{-\gamma}\left(y-\widehat{\xi}\right)\alpha^{\gamma} \\
&= r\cdot\xi^{1-\gamma}\alpha^{\gamma}
\end{aligned}$$

that finally leads to write the following ordinary differential equation:

$$\alpha'(t) + \left[\frac{(1-\gamma)\delta - (r+\lambda^S_{x+t})}{\gamma}\right]\alpha(t) = -1$$

Under the condition (4.13) the solution of the previous ordinary differential equation is

$$\psi(t) = e^{-b(T-t)} \cdot \left({}_{T-t}p^{S}_{x+t}\right)^{1/\gamma} + \int_t^T e^{-b(s-t)} \cdot \left({}_{s-t}p^{S}_{x+t}\right)^{1/\gamma} \mathrm{d}s \tag{4.16}$$

Therefore, the value function $\mathcal{G}$ is given by the following expression:

$$\mathcal{G}(w_0, t_0) = \frac{1}{1-\gamma}\left[y - \widehat{\xi}(t_0)\right]^{1-\gamma} \cdot \psi^{\gamma}(t_0)$$

Remark 4.3. In order to have a well-posed solution we need $y - \widehat{\xi}(t) > 0$, that is $\widehat{\xi}(t) \leqslant 0$, for every $t_0 \leqslant t < T$. It is straightforward to see that this condition is assured by the assumptions on Δ and Δ_1, i.e.: $\Delta < 0$ and $\Delta_1 > 0$.

Remark 4.4. The well-posedness of the solution is also assured by

$$|\psi(t)| \leqslant \left|e^{-b(T-t)} \cdot \left({}_{T-t}p^{S}_{x+t}\right)^{1/\gamma}\right| + \left|\int_t^T e^{-b(s-t)} \cdot \left({}_{s-t}p^{S}_{x+t}\right)^{1/\gamma} \mathrm{d}s\right|$$

that, by assumption 4.1 and remark 4.1, assure $|\psi(t)| < +\infty$.

Given the optimal controlled wealth $\{W_t^*\}$, the optimal consumption and investment policies are given in feedback form by

$$\begin{aligned} C_t^* &= \frac{\xi(t)}{\psi(t)} = \frac{1}{\psi(t)} \cdot \left(W_t^* - \Delta_1/r\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)}\right) \\ \Pi_t^* &= \frac{\mu - r}{\gamma\sigma^2}\xi(t) = \frac{\mu - r}{\gamma\sigma^2} \cdot \left(W_t^* - \Delta_1/r\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)}\right) \end{aligned}$$

Under this result, a closed form for value functions $\mathcal{U}$ and $\mathcal{V}$ can be found. In fact, notice that with respect to the value function $\mathcal{U}$ we need to have $\Delta = -A < 0$ and $\Delta_1 = P > 0$. Instead, for $r < h$, in order to find the value function $\mathcal{V}$ we need $\Delta = -H/r < 0$ and again $\Delta_1 = P > 0$. In both cases, the two problems can be solved considering two different change of variables:

$$\widehat{\xi}_{\mathscr{U}}(t) := \frac{P}{r}\left(1 - e^{r(t-T)}\right) - Ae^{r(t-T)} \tag{4.17}$$

$$\widehat{\xi}_{\mathscr{V}}(t) := \frac{P}{r}\left(1 - e^{r(t-T)}\right) - \frac{H}{r}e^{r(t-T)} \tag{4.18}$$

4.4 Numerical examples and insights

Consider $t_0 = 0$ and at this time, a female aged $x = 35$ who is willing to purchase a policy. Also, suppose that this plan will accumulate, until time $T := 30$ (i.e. when the annuitant will be aged $x + T = 65$) to an amount $A :=$ \$350,000. In order to be concrete, we can think that T may coincide with her retirement time and that the purchase takes place in 1970. In this context, the G.A.O.(if the agent decides to embed such an option in her policy) could be exercised in 2005. I would like to stress that these calendar dates are not necessary to implement a numerical experiment. However they give a stronger economic meaning for a contract designed as follows: assume that the agent is asked to decide whether to include a guaranteed annuity option assuring a conversion rate $h := 1/9$ (very common in 1980's and 1970's), implying an assured cash flow stream at the nominal rate $H \approx 38.89$ per year. Notice that, in this situation, if we refer to survival tables available in 1970 (see table 4.1), the implicit discount rate is $r_h \approx 0.0754$ and, from the point of view of an insurance company in the 1970's, such an option was considered to be far in the money at the conversion time.

Under previous hypothesis, the value functions $\mathscr{U}$ and $\mathscr{V}$ are plotted in figure 4.1, where it is assumed a Gompertz's mortality specification. I estimate parameters ς and m, minimizing a loss function using the method proposed by Carriere (1994b). I refer to the Human Mortality Database for the province of Ontario, Canada, for a female and a male both aged 35 in year 1970 or in 2004. The results of our estimations are summarized in table 4.1,

For some values of the market interest rate r, table 4.2 shows the premium P and the equivalent valuation L_0^*, for the policy considered in the present example. Figure 4.2 depicts the dependency of L_0^* on both the guaranteed conversion rate h and the interest rate r. As expected, the greater the interest rate, the

TABLE 4.1: Estimated female and male Gompertz's parameters for the province of Ontario, Canada, conditional on survival to age 35. Source: Canadian Human Mortality Database available for year 1970 and 2004.

	Female		Male	
Year	m	ς	m	ς
1970	85.3758	10.5098	79.1089	11.5890
2004	89.7615	9.3216	85.8651	10.1379

lower the agent indifference price for the option. In fact, the market interest rate is seen to be more beneficial than the guaranteed rate. Also, the analysis remains consistent with respect to h: the lower the guaranteed rate, the lower the agent's indifference price.

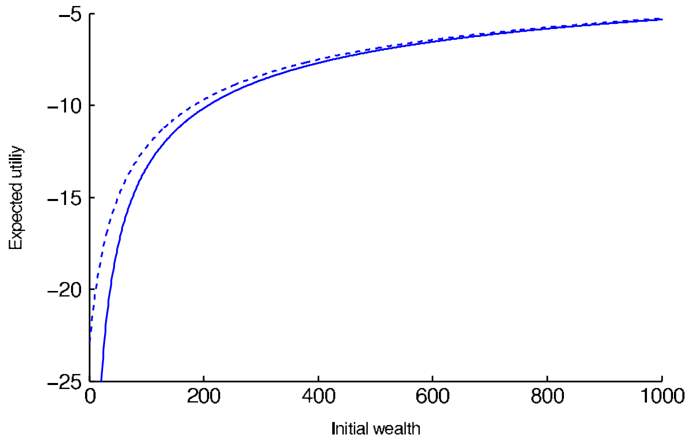
Depending on r, $0 < r < \mu$, table 4.2, shows the nominal instantaneous rate for the premium P (that the policyholder needs to pay to in order to accumulate $A = \$350,000$) and the indifference valuation L_0^* for the G.A.O.. Notice that it is not immediately possible to compare L_0^* and P since the former denotes a lump sum, while the latter refers to a nominal instantaneous rate to be converted infinitely many times per year.

In order to better understand the meaning of P and L_0^*, it can be useful to think of an auxiliary problem. This problem is independent of the previous

TABLE 4.2: Premium and indifference valuation associated to the policy, depending on the current interest rate.

r	P	L_0^*	p_{12}	l_{12}	Total
0.035	\$6,594	\$266,342	\$550	\$419	\$969
0.050	\$5,026	\$ 95,450	\$420	\$115	\$535
0.085	\$2,519	\$ 8,395	\$211	\$ 5	\$216

FIGURE 4.1: Value function $\mathscr{U}$ (solid) and value function $\mathscr{V}$ (dashed), for an individual characterized by $\gamma = 1.4$, that observes a financial market described by $r = 0.07$, $\mu = 0.08$, $\sigma = 0.12$. The value of r and μ are taken large enough to simulate the 1970's financial market. In this setting $L_0^* = 25,171$. The price is given for a G.A.O.exercisable in 2005, for a female in year 1970, from the province of Ontario, assuming a (subjective) mortality specification given by the survival table available in 1970, see table 4.1.



indifference model, but will offer a way to validate the previous results. To do this, consider a premium to be payed, in a real-world, for a pension or an insurance plan. Generally they are payed monthly. We can ask two questions. First, which is the extent p_{12} of a monthly annuity whose the future value, after 30 years, is exactly A. Second, which is the monthly annuity l_{12} necessary to amortize, after 30 years, the lump-sum L_0^* payed at $t_0 = 0$.

Remark 4.5. Previous considerations turn out to be useful from an intuitive point of view. However, I need to stress, and to make clear, that the agent's indifference valuation model is based on the lump sum L_0^* (if it exists) to pay at time t_0, and on a premium payed at the instantaneous force P. For these reasons, I am aware that l_{12} and p_{12} cannot be thought of as a part of the indifference model presented above: l_{12} and p_{12} must be considered independent from the strategies analyzed in our model. Therefore, I suggest the reader takes this monthly arrangement at face value. It just represents a practical way to compare L_0^* and P, inspired by a concrete pension market system.

In order to compute l_{12}, consider a horizon of $T \times 12$ months. Thus l_{12} is given by the following relation:

$$L_0^* = l_{12} \cdot a_{\overline{T \times 12}|i_{12}}$$

where $i_{12} := e^{r/12} - 1$ is the effective interest rate compounded monthly with respect to e^r, and where in general we define

$$a_{\overline{n}|i} := \frac{1 - (1+i)^{-n}}{i}$$

as the present value of an annuity that pays one dollar for n periods, discounted by the effective interest rate i compounded each period. Similarly, define p_{12} such that

$$A = p_{12} \cdot s_{\overline{T \times 12}|i_{12}}$$

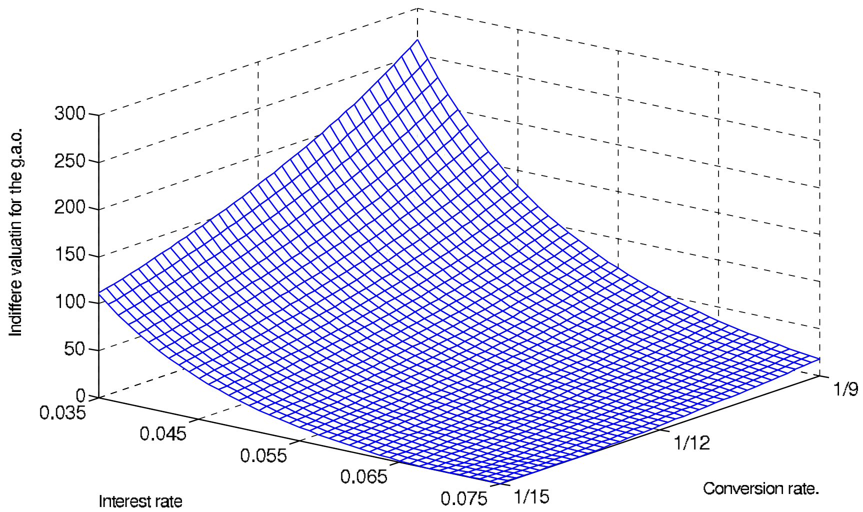
where

$$s_{\overline{n}|i} := \frac{(1+i)^n - 1}{i} = (1+i)^n \cdot a_{\overline{n}|i}$$

represents the future value after n periods, of an annuity that pays one dollar per period, under an effective interest rate i compounded each period.

Coming back to table 4.2 it is interesting to see that for $r = 0.035$, a monthly cash flow of \$550 and a monthly stream of \$419 equivalently amortize L_0^*. Setting $r = 0.085$, a similar situation it is observed for a monthly premium of \$ 211 and a monthly stream of only \$5. These intuitive results - keeping in mind the remark 4.5 - are consistent with the literature concerning the guaranteed annuity option: As mentioned by Boyle and Hardy(2003), these guarantees were popular in U.K. retirement savings contracts issued in the 1970's and 1980's, when long-term interest rates were high. The same authors also write that at that time, the options were very far out of the money and insurance companies

FIGURE 4.2: Indifference price L_0^* depending on the guaranteed conversion rate h and the market interest rate r. The valuation is given for a G.A.O.exercisable in 2005, for a female in year 1970, from the province of Ontario, assuming a (subjective) mortality specification given by the survival table available in 1970, see table 4.1.



apparently assumed that interest rates would remain high and thus the guarantees would never become active. As a result, from the indifference model discussed in the present paper, when the interest rate is very high - as was the case in the 1970's and 1980's - the guaranteed annuity option's value, given by the policyholder, is very small. Interestingly, in the same period, empirically it was observed that a very small valuation was also given by insurers.

These facts are proved by the extremely low value of $L_0^* = \$8,395$ (over $T - t_0 = 30$ years), against the yearly nominal premium $P = \$2,519$. This is better seen in terms of the auxiliary "monthly valuation problem": The evaluation L_0^* can be amortized by a monthly cash flow of \$5, against a monthly equivalent premium of \$211. Moreover, p_{12} and l_{12} by construction are homogeneous quantities. Their sum gives an idea of the equivalent monthly value associated to the policy the agent is willing to buy at time t_0. This sum is showed in the last column of table 4.2. It is interesting to note the huge difference between the total value corresponding to $r = 0.035$ compared to $r = 0.085$.

4.5 Conclusions

In the course of this chapter, in a setting where interest and hazard rates are constant, an explicit solution for the indifference problem is found, where power consumption utility is assumed. The indifference price for the guaranteed annuity option, both at the time when the policy is purchased and at the conversion time, depends on the difference between the guaranteed conversion rate h and the market interest rate r. This fact lead us to find an explicit solution for a class of problems where bequest motives and finite time-horizon are jointly considered, together with the assumption of a power utility from consumption. The dependency on h and r of the equivalent valuation also reveals that in periods characterized by high market interest rates, the value of the G.A.O.turns out to be very small. Our model remains coherent if we compare the policyholder's point of view and the insurer's point of view, under an economic setting characterized by high interest rates. Finally, with regards to numerical experiments, an auxiliary problem is considered, in which it is possible to compare the pure premium asked by the insurance company and the indifference price for the embedded option.

CONCLUSIONS AND FUTURE RESEARCH

The model I propose and implement in the course of this thesis uses the principle of equivalent utility in order to value guaranteed annuity options embedded in life insurance policy, from a policyholder's point of view. For constant relative risk aversion utility functions, an explicit solution for the reservation problem is found under a specific institutional arrangement. For instance, two strategies at the time of conversion, and two strategies at the moment when the policy is purchased are analyzed. For the former it is assumed that, if the annuitant does not exercise the option, she first withdraws her accumulated funds and then she seeks to solve a standard Merton's problem under an infinite time horizon case. At the purchasing time, the agent's expected utility, associated to a policy embedding a guaranteed annuity option, and the expected reward given by a policy that does not embed such an option are considered. It is shown that the option's indifference value, both at the time when the policy is purchased and at the conversion time, depends on the difference between the guaranteed conversion rate h and the market interest rate r. This fact also lead us to find an explicit solution for a class of problems where bequest motives and finite time-horizon are jointly considered, together with the assumption of a power utility from consumption. In the course of some numerical experiments, the pure premium, asked by the insurance company, and the indifference price for the embedded option are compared under specific assumptions regarding the level of the interest rates and survival scenarios.

Future researches, that I am willing to consider, can be developed from the model presented in this thesis. In fact a more general institutional arrangements can be considered, where the agent is allowed to purchase more than one annuity. Also labor income can be considered during the accumulation period. Finally, stochastic interest rates and mortality rates, as well as stock dynamics, can be can be developed in a richer setting. For instance, an unrestricted market – as defined by Milevsky and Young (2007a) – and stochastic interest rates and stochastic labor income, are worth to be considered in order to develop a more comprehensive and rich model. To this end, I recall the work proposed Koo (1998). Finally, in the present framework, the longevity risk is considered by comparing different scenarios, given by the survival tables available in 1970 and in 2004. A more general stochastic approach – as proposed by Dahl (2004) – can be taken into account.

The future research in this field has to consider the analytical complications arising from including other stochastic components to the present model. However, it would be interesting and stimulating to consider, even at a first glance, a comparison between the standard risk neutral methods used until now for valuing guaranteed annuity options, and the indifference method that I propose in this thesis.

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