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LOCAL ENERGY METHODS FOR FREE BOUNDARY PROBLEMS

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Introduction

In the last decades, the study of free boundary problems has been a very active subject of research occurring in a variety of applied sciences. What these problems have in common is their formulation in terms of suitably posed initial and boundary value problems for nonlinear partial differential equations.
Free boundary problems (FBP) constitute a mathematical research topic characterized by the occurrence of frontiers, whose location is a priori unknown. Free boundaries separate space-time regions with different properties. As an example, we can consider the solidification of water, where the location of the boundary between water and ice is not fixed, and changes during the process. Free boundaries occur naturally in the mathematical formulation of a great variety of scientific and technological processes, e.g. in material processing (steel casting, crystal and dendritic growth, etc.), in biology (population dynamics, growth of bacteria), in combustion theory, in reaction-diffusion problems, in electrochemistry or in fluid flow through porous media.
These types of FBPs have received little attention from mathematicians until fifty years ago: in the sixties-seventies, the modern approach to the theory of nonlinear partial differential equations brought new insight and new methods to their mathematical study. Over the past two decades, the great development in this sector has established it as an important interdisciplinary area, covering topics in applied mathematical modelling and in pure mathematics, such as partial and ordinary differential equations, numerical and functional analysis, calculus of variations and optimal control, differential geometry. The study of such problems raises profound issues in nonlinear analysis, as well as causing new and difficult questions to emerge in numerical analysis and scientific computation.
In the present work we deal with some spatial localization properties, displayed by the solutions of very general second order elliptic and parabolic equations, where the free boundary arises in a natural manner, as the boundary of the support of such solutions. These topics have been extensively

treated in the scientific literature, with a wide range of approaches and techniques (see [13] for a very complete survey). Several results have been found by using classical tools, like comparison principles, sub-super solutions methods (see [6], and [7]): they consist in the construction of suitable sub-super solutions, that display the kind of behaviour which one wants to investigate, and try to show that the eventual solution of the originary problem must possess the same properties. Another useful approach, driven by the same philosophy, was introduced by B.H. Gilding and R. Kersner in [8], and consists in trying to find solutions of *travelling wave* type, for some class of reaction-convection-diffusion processes; in this way the finite speed of propagation is proved, and other interesting properties can be investigated.
The merit of this kind of techniques lies in the fact that they give results with a global character, combined with some quantitative information on the free boundary of the solutions (see [20] for a result on the rate of shrinking of the support), although it presents serious technical difficulties, in the construction of explicit sub-supersolution, or even more it may fail at all if it is not possible to apply comparisons theorems; to avoid these difficulties, an innovative technique was proposed in a 1981 paper of S.N. Antonsev [4], and successively developed by other authors (see [9], and [10]). This approach is known in the literature as local energy method, and consists in deriving, starting from the weak formulation of the problem, a suitable ordinary differential inequality for some *energy functions*, whose choice depends on the problem one have to deal with. Carrying out a qualitative study of such inequality, one can obtain direct information about the solution. This technique, due to its simplicity, just needs very general hypotheses and can be applied to a large class of PDEs problems, for example to higher order equations or systems. Although this is a local technique, and so it cannot provide global information on the solutions, nevertheless it seems to work very well in the study of suitable *spatial localization properties*.
In this work we try to give a presentation of such *local energy methods*, proving some results coming from the direct application of this technique.
To expose such results we start with an heuristic explanation of the method, applying it to a quite simple, one dimensional model, and avoiding to justify all the calculations in a rigorous manner, with the aim to emphasize the philosophy of the method. Next we consider a more general class of quasilinear second order equations under minimal hypotheses, apt to guarantee at least the local existence of the solutions. this is the content of the first chapter.

In the second chapter we study the parabolic case: here an opportune implementation of the method allows to obtain some structural sufficient hypotheses under which solutions can display suitable behaviours, strictly

related to the degenerate character of the diffusion operator in the evolution problem. One of these is the *finite speed of propagation*: given an initial condition with bounded support, the boundary of the support expands with finite speed. This phenomenon, that typically characterizes hyperbolic problems, shows the nonlinear and degenerate character of the class of equations we deal with, so different if compared, for example, with the property displayed by the solutions of the classic heat equation. Other types of behaviours are: the *waiting time property*, that represents an extreme case of the above property and it consists in a temporary local stopping of the boundary of the support; the *shrinking support* property, an exactly inverse phenomenon to that of the finite sped of propagation, which can be explained by the overcoming effect of the absorption term present in the PDE with respect to the diffusion term effect; lastly, the formation of *dead cores*, regions on which, starting from an opportune instant, the solution vanishes. In each of these cases we stress the peculiar local character nature of this technique that allows to ignore global information on the initial conditions.
Finally in the third chapter we give an example of implementation of the method in the case of a PDE system arising from the modelization of a fluid under warming, and characterized by internal convective motions, known in the literature as Boussinesq problem. It derives from the coupling of a Navier-Stokes equation for the velocity field of the fluid, with a diffusion absorption equation for the variable temperature, that presents, unlike the precedent cases, a transport term.

Chapter 1

Non linear stationary problem

1.1 Setting of the problem

The study of the steady-state of many different problems governed by a nonlinear diffusion, in presence of an absorption term, leads to the equation

$$-\triangle\,\varphi(u) + C(u) = f(x) \quad \text{in } \Omega \tag{1.1.1}$$

$$u = h(x) \quad \text{on } \partial\Omega \tag{1.1.2}$$

where Ω is an open bounded set, φ and C are real continuous and nondecreasing functions such that $\varphi(0) = C(0) = 0$ and f and h are given functions. Equations (1.1.1) is sometimes written in divergence form

$$-div(k(u)\nabla u) + C(u) = f(x)$$

where the function k is any primitive of φ. Some typical choices of φ in the applications are the following:
Flows through porous media (slow diffusion problems).
Via Darcy's law equation (1.1.1) holds for φ satisfying the additional assumption $\varphi'(0) = 0$ and $\varphi'(u) > 0$ if $u \neq 0$ as, for instance, $\varphi(u) = |u|^{m-1}u,\ m > 1$ (see [18]). This same type of φ also occurs for nonlinear heat conduction when the thermal conductivity depends on the temperature.
Plasma physics (fast diffusion problems).
Now the natural assumption on φ are $\varphi'(0) = +\infty$ and $\varphi'(u) > 0$ if $u \neq 0$ as for instance $\varphi(u) = |u|^{m-1}u$ with $0 < m < 1$.
Stefan-like problems. Equation (1.1.1) is also related to the classic two phase Stefan problem, as well as to its generalizations (see [20]). Mainly, φ is taken such that $\varphi([0, a]) = 0$ and $\varphi'(u) > 0$ for $u \notin (0, a)$, for some $a > 0$.
In all the preceding examples, solutions, if existing, can display very particular features; one of theses, and the one we are interested to, is the appearance

of a subset of the domain Ω, where u vanishes
Under what conditions does this vanishing set appear? What geometric properties does it possess? and how the form of the PDE problems is related to these properties? .
To deal with these questions, let us start by introducing the following:

Definition 1.1.1 (Free boundary). *Let*

$$N(u) \equiv \{x \in \bar{\Omega} : u = 0\}$$

$$S(u) \equiv \{x \in \Omega \bar{:} \, u \neq 0\}.$$

The free boundary is given by

$$F(u) \equiv \partial S(u) \cap \partial N(u)$$

where u is the solution of the nonlinear PDE under consideration

The study of the properties of the free boundary will give the answers to the questions above. Before proceeding further it seems interesting to examine what may be the reasons for the existence of the free boundary $F(u)$. Recall that in the case of linear equations the solution of an elliptic equation such as (1.1.1) with $\varphi(u) = u$ and $C(u) = u$, satisfying a Dirichlet condition corresponding to data $f \geq 0$ and $h \geq 0$, verifies $u > 0$ on Ω. This well known fact can be proved in many different ways: strong maximum principle, Harnack inequality, unique continuation property, and so on. Thus, in some sense, the existence of the free boundary is a nonlinear typical phenomenon. However, it doesn't appear in all nonlinear equations. One of the keystone in order this happens is the property of the diffusion operator ($- \triangle \varphi(u)$ in the case of (1.1.1)), of being *non uniformly elliptic*, i.e. *degenerate*, in the sense that it looses its elliptic character around the set $\{u = 0\}$ or $\{\nabla u = 0\}$.
However this degenerate character is only a necessary condition for the existence of $F(u)$, so results a very interesting task, to develop an approach that can give some sufficent conditions.

1.2 Localized solutions of nonlinear stationary problems

In order to introduce and develop a local energy method as a tool for the study of nonlinear stationary problems which gives rise to free boundaries, we first of all need the concept of localized solution.

Definition 1.2.1. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $u : \Omega \longrightarrow \mathbb{R}$ be a function satisfying (at least in a weak sense) a given stationary partial differential equation in Ω. We say that u is a localized solution if it vanishes on an open part of Ω, i.e.the set (supp u)$\cap\Omega$ is strictly contained in Ω.*

A special class of localized solutions, widely studied in the literature corresponds to the case where Ω is unbounded and the supports of solutions are bounded (and therefore compact).
Localized solutions occur in problems where the influence of data (such as for instance, boundary conditions and/or some source terms) on the behavior of solutions is restricted to the points of Ω close enough to the support of the data (the boundary and/or the support of the source terms). Several examples of equations of this kind are furnished by mathematical models of fluid mechanics.

1.2.1 A heuristic explanation of the method

The main ideas of the energy method that we want to develop here can be explained considering heuristically some very simple nonlinear stationary problems.
Consider the one-dimensional equation

$$-\frac{d}{dx}(|u_x|^{p-2}u_x) + |u|^{q-1}u = f(x) \quad in\ \Omega \tag{1.2.1}$$

where

$$\Omega = (L_1, L_2), \quad -\infty \leq L_1 < L_2 \leq +\infty. \tag{1.2.2}$$

The exponents are assumed to satisfy the inequalities $1 < p < \infty$ and $q > 0$. Note that in the special case $p = 2$ and $q = 1$ the corresponding equation becomes linear. For the sake of simplicity we assume about the "source" term $f(x)$ that

$$f \in L^\infty(\Omega) \cap L^1(\Omega).$$

The occurrence of localized solutions is due to the key assumption on the structure of equation

$$q < p - 1. \tag{1.2.3}$$

The explicit solutions are not easy to obtain, however, if the source term $f(x)$ is neither radially simmetric, nor zero or if the boundary conditions are not of Dirichlet type. One of the merits of the energy method is that the associated boundary conditions play a secondary role and are only needed to obtain global estimates to the solutions.

We begin with one of the simplest cases corresponding to the situation when Ω is bounded and $f = 0$. To be precise, we assume that

$$\Omega = (-L, L) \quad \text{with some} \ \ L > 0, \quad f = 0. \tag{1.2.4}$$

Let u be an arbitrary solution of (1.2.1) which we assume to be regular just to simplify matters; we postpone a more rigorous approach to subsequent sections.
The energy method starts by defining what we call local energy functions. Of course, these functions happen to be very simple in our present one dimensional case, but later we will extend this idea to the solutions of multi-dimensional equations.
We obtain the local energy functions by multiplying the equation by the unknown u. In the special case of equation (1.2.1), and under assumption (1.2.3), a natural choice of these auxiliary energy functions is the following: given $\rho \in (0, L)$ we define

$$E(\rho) = \int_{-\rho}^{\rho} |u_x(x)|^p dx \ \text{ and } \ b(\rho) = \int_{-\rho}^{\rho} |u(x)|^{q+1} dx. \tag{1.2.5}$$

These functions can be viewed as the diffusion and absorption energies generated by the solution u in the interval $(-\rho, \rho)$. The total energy function is then defined by

$$T(\rho) = E(\rho) + b(\rho) \quad \rho \in (0, L). \tag{1.2.6}$$

Note that the domain of integration in the definition of the local energy functions, which we call the local energy set, is determined by assumption (1.2.4) on Ω. The choice of the local energy set plays an important role. In fact, differents choices of this set lead to completely different estimates on the supports of the localized solutions.
The second step of the energy method consists in deriving a differential inequality for some of the energy functions. This can be formally done by multiplying equation 1.2.1 by u and integrating the results by parts over the local energy set. In our case , making use of (1.2.3) and (1.2.4), we obtain from (1.2.1)

$$T(\rho) = E(\rho) + b(\rho) = |u_x(x)|^{p-2} u_x(x) u(x) \mid_{x=-\rho}^{x=\rho} = I(\rho). \tag{1.2.7}$$

The information on the boundary behavior of the local energy set will be expressed in terms of the local (and global) energy functions. First of all, we assume that the global energy is bounded, i.e.,

$$I(L) < \infty \tag{1.2.8}$$

Since

$$\begin{aligned} \frac{dE(\rho)}{d\rho} &= |u_x(\rho)|^p + |u_x(-\rho)|^p, \\ \frac{db(\rho)}{d\rho} &= |u(\rho)|^{q+1} + |u(-\rho)|^{q+1}, \end{aligned} \tag{1.2.9}$$

then

$$\begin{aligned} I(\rho) &= \big||u_x(\rho)|^{p-2}u_x(\rho)u(\rho) - |u_x(-\rho)|^{p-2}u_x(-\rho)u(-\rho)\big| \\ &\le |u_x(\rho)|^{p-1}|u(\rho)| + |u_x(-\rho)|^{p-1}|u(-\rho)| \\ &\le (|u_x(\rho)|^{p-1} + |u_x(-\rho)|^{p-1})(|u(\rho)| + |u(-\rho)|) \\ &\le 2^{\frac{1}{p}}(|u_x(\rho)|^p + |u_x(-\rho)|^p)^{\frac{p-1}{p}}(|u(\rho)| + |u(-\rho)|) \\ &= 2^{\frac{1}{p}}\left(\frac{dE(\rho)}{d\rho}\right)^{\frac{p-1}{p}}(|u(\rho)| + |u(-\rho)|), \end{aligned} \tag{1.2.10}$$

where the last inequality follows from the concavity relation:

$$\left[\frac{|u_x(\rho)|^p + |u_x(-\rho)|^p}{2}\right]^{\frac{p-1}{p}} \ge \frac{|u_x(\rho)|^{p-1} + |u_x(-\rho)|^{p-1}}{2}$$

and the structural hypothesis $p > 1$.
There are now two possible ways of estimating the boundary values $|u(\pm\rho)|$ in (1.2.10). The first one is typical for the one dimensional case. It consists in using again comvexity. We have

$$I(\rho) \le 2^{\frac{1}{p}}\left(\frac{dE(\rho)}{d\rho}\right)^{\frac{p-1}{p}}\left(\frac{db(\rho)}{d\rho}\right)^{\frac{1}{q+1}} 2^{\frac{q}{q+1}} \le C\left(\frac{dI(\rho)}{d\rho}\right)^{\frac{1}{q+1}+\frac{p-1}{p}} \tag{1.2.11}$$

The second way of estimating $I(\rho)$ is more general and can also be applied in the multidimensional case. It is based on the non trivial interpolation-trace inequality [see C]

$$|u(x)| \le C(\|u_x\|_{L^p(B_\rho)} + \rho^{-\delta}\|u\|_{L^{q+1}(B_\rho)})^\theta \|u\|^{(1-\theta)}_{L^{q+1}(B_\rho)}, \tag{1.2.12}$$

where $x \in B_\rho = (-\rho, \rho)$, $C = C(p, q)$, and

$$\begin{aligned} \theta &= \frac{p}{p + (p-1)(q+1)} \in (0, 1), \\ \delta &= \frac{1}{q+1} + \frac{p-1}{p} = \frac{1}{\theta(q+1)}. \end{aligned} \tag{1.2.13}$$

In the one dimensional case inequality (1.2.12) follows from the formula

$$|u(z)|^{\frac{1-\theta}{\theta}}u(z) = \frac{1}{\theta}\int_0^z |u(y)|^{\frac{1-\theta)}{\theta}}u'(y)dy + C_0 \tag{1.2.14}$$

with

$$C_0 = |u(0)|^{\frac{(1-\theta)}{\theta}}u(0).$$

this formula is a byproduct of the fundamental theorem of calculus. In order to get (1.2.12) we first integrate (1.2.14) with respect to z over the interval $(-\rho, \rho)$:
so we have:

$$C_0 = \frac{1}{2l}\left[\int_{-\rho}^{\rho} |u|^{\frac{1-\theta}{\theta}}udz - \frac{1}{\theta}\int_{-\rho}^{\rho}\int_0^z |u|^{\frac{1-\theta}{\theta}}u'dydz\right]$$

now by taking absolute values from both terms of (1.2.14) we can write:

$$|u(z)|^{\frac{1}{\theta}} \leq |C_0| + \frac{1}{\theta}\left[\int_0^z |u(y)|^{\frac{1-\theta}{\theta}}|u'(y)|dy\right]$$

$$\leq |C_0| + \frac{1}{\theta}\left(\int_{-\rho}^{\rho} |u(y)|^{\frac{1-\theta}{\theta}p'}dy\right)^{\frac{1}{p'}}\left(\int_{-\rho}^{\rho} |u'(y)^p dy\right)^{\frac{1}{p}}$$

$$= |C_0| + \frac{1}{\theta}\|u\|^{\frac{1-\theta}{\theta}}_{L^{q+1}(B(\rho))}\|u'\|_{L^p(B(\rho))}.$$

where we have applied Holder inequality with $\theta = \frac{p}{p+(p-1)(q+1)}$, from which $\frac{1-\theta}{\theta}p' = q+1$. Then

$$|u(z)| \leq \left[|C_0| + \frac{1}{\theta}\|u\|^{\frac{1-\theta}{\theta}}_{L^{q+1}(B(\rho))}\|u'\|_{L^p(B(\rho))}\right]^{\theta} \leq |C_0|^{\theta} + \frac{1}{\theta^{\theta}}\|u\|^{1-\theta}_{L^{q+1}(B(\rho))}\|u'\|^{\theta}_{L^p(B(\rho))}$$

and finally, gathering the above espressions:

$$|C_0|^{\theta} \leq\ c\rho^{-\theta}\left[\int_{-\rho}^{\rho} |u|^{\frac{1}{\theta}}dz + \frac{1}{\theta}\int_{-\rho}^{\rho} 2c|u|^{\frac{1-\theta}{\theta}}|u'|dz\right]^{\theta}.$$

Now observing that

$$\frac{1}{\theta} = \frac{p+(p-1)(q+1)}{p} = 1 + q + 1 - \frac{q+1}{p} < q+1$$

being $q+1>p$ we can conclude:

$$|C_0|^\theta \leq c\rho^{-\theta}\left[\int_{-\rho}^{\rho}|u|^{q+1}dz\right]^{\frac{1}{q+1}} + c\left[\int_{-\rho}^{\rho}|u|^{q+1}dz\right]^{\frac{1-\theta}{q+1}}\left[\int_{-\rho}^{\rho}|u'|^p dy\right]^{\frac{\theta}{p}}$$

that is (1.2.12).
Let us return to estimating $|I(\rho)|$ in (1.2.10). Using (1.2.9), (1.2.12) we get the inequality

$$\begin{aligned} I(\rho) &\leq C\left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}}(E^{\frac{1}{p}}+\rho^{-\delta}b^{\frac{1}{(q+1)}})^\theta b^{\frac{1-\theta}{q+1}} \\ &\leq C_1\rho^{-\delta\theta}\left(\frac{dE}{d\rho}\right)^{\frac{(p-1)}{p}}(E^{\frac{1}{p}}+b^{\frac{1}{p}}b^{\frac{1}{q+1}-\frac{1}{p}})^\theta b^{\frac{1-\theta}{q+1}} \\ &\leq C_1\rho^{-\delta\theta}\left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}}(E^{\frac{1}{p}}+b^{\frac{1}{p}})^\theta b^{\frac{1-\theta}{1+q}} \\ &\leq C_1\rho^{-\delta\theta}\left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}}(E+b)^{\frac{\theta}{p}} b^{\frac{1-\theta}{q+1}}, \end{aligned} \tag{1.2.15}$$

where

$$\begin{aligned} C_1 &= C(p,r)\max_{\rho\in(0,L)}(1,\rho^{+\delta\theta})\max_{\rho\in(0,L)}\left(1,b(\rho)^{\frac{(p-q-1)\theta}{p(q+1)}}\right) \\ &\leq C(p,r)\max\left(1,L^{\delta\theta}\right)\max\left(1,T(L)^{\frac{(p-q-1)\theta}{p(q+1)}}\right)\leq C_2 \end{aligned} \tag{1.2.16}$$

with some $C_2 = C_2(p,q,L,T(L))$.
Let us now apply the estimates (1.2.11), (1.2.12) to derive a differential inequality for the local energy functions. In the case of (1.2.11) we easily obtain from (1.2.7) that

$$CT^\nu(\rho) \leq \frac{dI(\rho)}{d\rho} \tag{1.2.17}$$

with an appropriate constant $C = C(p,q)$ and the exponent

$$\nu = \frac{p(q+1)}{p+(q+1)(p-1)}. \tag{1.2.18}$$

If we use (1.2.15), we have from (1.2.7)

$$E(\rho)+b(\rho) \leq C_2\rho^{-\delta\theta}\left(\frac{dE(\rho)}{d\rho}\right)^{\frac{(p-1)}{p}}(E+b)^{\frac{(1-\theta)}{(q+1)}+\frac{\theta}{p}} \tag{1.2.19}$$

and then

$$E^{\nu}(\rho) \leq (E(\rho)+b(\rho))^{\nu} \leq C_3\rho^{-\alpha}\left(\frac{dE(\rho)}{d\rho}\right), \tag{1.2.20}$$

where now

$$C_3 = C_2^{\frac{p}{(p-1)}}, \quad \alpha = \frac{\delta\theta p}{p-1} = \frac{p}{(q+1)(p-1)} \tag{1.2.21}$$

$$\nu = \frac{p}{p-1}\left(1-\frac{\theta}{p}-\frac{1-\theta}{q+1}\right) = \frac{p(q+1)}{p+(q+1)(p-1)}. \tag{1.2.22}$$

The third and last step of the energy method is to apply the derived differential inequalities to the study of the localization property of the solution u. We use here the fundamental assumption 1.2.3 which implies that the exponent ν, given by (1.2.17), satisfies the inclusion $\nu \in (0,1)$. Then formally rewriting (1.2.16) (resp (1.2.18)) as

$$Cd\rho \leq \frac{dT}{T^{\nu}} \quad \left(resp.\rho^{\alpha}d\rho \leq C_3\frac{dE}{E^{\nu}}\right),$$

and integrating over an interval $(\rho,\rho_1) \subset [0,L]$, we get

$$T^{1-\nu}(\rho) \leq T^{1-\nu}(\rho_1) - \frac{1-\nu}{C}(\rho_1-\rho) \tag{1.2.23}$$

and correspondingly,

$$E^{1-\nu}(\rho) \leq E^{1-\nu}(\rho_1) - \frac{1-\nu}{C_3(1+\alpha)}(\rho_1^{1+\alpha}-\rho^{1+\alpha}) \tag{1.2.24}$$

Since the energy functions are non negative, we conclude that

$$T(\rho)=0 \;\; (resp.E(\rho)=0) \quad \text{and so } u(x)=0 \; for \; |x|\leq\rho \tag{1.2.25}$$

with an arbitrary ρ such that

$$\rho \leq \rho_1 - \frac{c}{1-\nu}T^{1-\nu}(\rho_1). \tag{1.2.26}$$

Correspondingly

$$\rho^{1+\alpha} \leq \rho_1^{1+\alpha} - \frac{C_3(1+\alpha)}{1-\nu}E^{1-\nu}(\rho_1). \tag{1.2.27}$$

In order to get a nonempty conclusion, we have to be sure of the positiveness of the right-hand side of inequalities (1.2.26),(1.2.27). This leads to a suitable balance between the size of the domain Ω and the global energy. In particular, this is true if (1.2.8) is replaced by the stronger assumption

$$L \geq \frac{C}{1-\nu}T^{1-\nu}(L) \qquad \left(resp.L^{1+\alpha} \geq \frac{C_3(1+\alpha)}{1-\nu}E^{1-\nu}(L)\right). \tag{1.2.28}$$

1.2.2 An example of localized solutions.

Let us present a simple localized solution of equation (1.2.1). It is not difficult to check that the function

$$u(x) = u_0 \left(1 - \frac{x}{x_0}\right)_+^{\frac{p}{p-q-1}} \qquad (v_+ = \max(0, v)), \tag{1.2.29}$$

where

$$x_0 = |u_0|^{\frac{(p-q-1)}{p}} \frac{p}{p-q-1} \left[\frac{(p-1)(q+1)}{p}\right]^{\frac{1}{p}} \tag{1.2.30}$$

is a solution of equation 1.2.1 in the domain $\Omega = (0, \infty)$ with $f = 0$. Observe that *supp* $u = (0, x_0)$. Due to (1.2.30), the size of *supp* u depends on the quantity of u_0 and the localization effect is absent if $p = q+1$. We also point out that the function $u(x)$ defined by (1.2.29) is a solution of the following free-boundary problem: to find a nonnegative function $u(x)$ and a finite number x_0, $0 < x_0 < \infty$, such that u satisfies 1.2.1 on $(0, x_0)$ and

$$u(0) = u_0, \quad u(x_0) = 0, \quad |u_x(x_0)|^p u_x(x_0) = 0.$$

If we consider the boundary conditions associated with 1.2.1 in the bounded domain $\Omega = (0, 1)$,

$$u(0) = 0, \quad u(1) = u_1, \tag{1.2.31}$$

then the condition

$$1 - x_0 = |u_1|^{\frac{(p-q-1)}{p}} \frac{p}{p-q-1} \left(\frac{(p-1)(q+1)}{p}\right)^{\frac{1}{p}} < 1 \tag{1.2.32}$$

implies that the function

$$u(x) = u_1 \left(\frac{x - x_0}{1 - x_0}\right)_+^{\frac{p}{p-q-1}}, \ (supp\ u \in (x_0, 1)), \tag{1.2.33}$$

is a solution of problem 1.2.1, (1.2.31) localized in the interval $[x_0, 1]$. Let us note once again that if either $x_0 = 1$, or $p = q + 1$, the localization effect fails. Due to (1.2.32), it is easy to see that here the localization effect only appears for small values of u_1.

Let us consider now boundary conditions of the form

$$u(0) = u_0, \quad u(\infty) = 0 \tag{1.2.34}$$

and the equation (1.2.1) with the "source"

$$f(x) = \epsilon \left(1 - \frac{x}{x_0}\right)_+^{\frac{qp}{p-q-1}}, \quad 0 < x_0 < \infty. \tag{1.2.35}$$

(ϵ is termed the "intensity" of the source $f(x)$). Let us try to find a solution of problem 1.2.1, (1.2.34), (1.2.35) in the form (1.2.29). A simple computation shows that such a solution does exist if the constant u_0, x_0, ϵ are subject to the relation

$$|u_0|^{q-1}u_0\left[1-u_0^{p-q-1}x_0^{-p}\left(\frac{p}{p-q-1}\right)^p\frac{(p-1)(q+1)}{p}\right]=\epsilon. \qquad (1.2.36)$$

The following assertions hold:

1. given $x_0\in(0,\infty)$, there exist $(\epsilon,u_0),(\epsilon>0,\ u_0>0)$ such that (1.2.36) is true;

2. given $u_0>0$, there exist $(x_0,\epsilon),(x_0>0,\ \epsilon>0)$ such that (1.2.36) holds;

3. given (ϵ,u_0) such that $(\frac{\epsilon}{u_0})^q<1$, there exists $x_0\in(0,\infty)$ satisfying (1.2.36).

Thus, under an appropriate choice of u_0, ϵ and x_0 the solution to problem (1.2.34), (1.2.35) vanishes once the "source" $f(x)$ is "switched off".

1.2.3 General local theorems

Let us consider the class of nonlinear second order elliptic equations of the form

$$-div\ A(x,u,\nabla u)+B(x,u,\nabla u)+C(x,u)=f(x) \qquad (1.2.37)$$

where

$$\nabla u=\left(\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_N}\right) \qquad A=(A_1,...,A_N),$$

$$divA=\sum_{i=1}^{N}\frac{d}{dx_i}A_i(x,u,\nabla u)=\sum_{i=1}^{N}\left(\frac{\partial A_i}{\partial x_i}+\frac{\partial A_i}{\partial u}\frac{\partial u}{\partial x_i}+\frac{\partial A_i}{\partial u_{x_k}}\frac{\partial^2 u}{\partial x_k x_i}\right).$$

We will consider equation (1.2.37) in an open domain $\Omega\subseteq\mathbb{R}^N, N\geq 1$. To simplify matters, we always assume that Ω is connected.
The vector valued function A and the scalar function B,C are assumed to satisfy the following structural conditions:

$$|A(x,r,q)|\leq C_1|q|^{p-1}, \qquad (1.2.38)$$

$$C_2|q|^p\leq A(x,r,q)\cdot q, \qquad (1.2.39)$$

$$|B(x,r,q)|\leq C_3|r|^{\alpha}|q|^{\beta}, \qquad (1.2.40)$$

$$C_4|r|^{\sigma+1} \leq C(x,r)r. \tag{1.2.41}$$

The conditions on $(x,r,q) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and the positive constants p, α, β,σ,$C_1,..C_4$ will be specified later on. Equation (1.2.37) occurs in a number of problems of fluid mechanics.
There already exists a wide literature devoted to the study of the problem of existence of weak solutions of the class (1.2.37) ([3],[14]). Here our task is the study of qualitative properties of weak solutions.

Definition 1.2.2. *Let $f \in L^1_{loc}(\Omega)$. A locally integrable function u is said to be a weak solution of* (1.2.37) *if*

1. $u \in W^{1,p}_{loc}(\Omega)$;
2. $B(\cdot,u,\nabla u), A_i(\cdot,u,\nabla u) \in L^1_{loc}(\Omega), i = 1,...N$;
3. $C(\cdot,u) \in L^1_{loc}(\Omega)$;
4. for any test function $\varphi \in C_0^\infty(\Omega)$, the equality

$$\int_\Omega \{A(x,u,\nabla u)\cdot\nabla\varphi + B(x,u,\nabla u)\varphi + C(x,u)\varphi\}dx = \int_\Omega f\varphi dx \tag{1.2.42}$$

holds.
We introduce the following class of sets:

$$B_\rho(x_0) = \left\{x \in \mathbb{R}^N : |x-x_0| < \rho\right\}, \qquad S_\rho(x_0) = \partial B_\rho(x_0),$$

they represent the *energy sets*, and then define.

$$E(\rho,u) \equiv E(\rho) = \int_{B_\rho(x_0)} A(x,u,\nabla u)\cdot\nabla u dx,$$

$$b(\rho,u) \equiv b(\rho) = \int_{B_\rho(x_0)} |u|^{\sigma+1}dx = \|u\|^{\sigma+1}_{L^{\sigma+1}(B_\rho)}.$$

Then

$$C_2\|\nabla u\|^p_{L^p(B_\rho)} \leq E \leq C_1\|\nabla u\|^p_{L^p(B_\rho)}.$$

We call E and b the energy functions associated to the solution u of equation (1.2.36).
Passing to spherical cordinates (r,ω) with the origin at the point x_0, and assuming that $E(\rho,u) < \infty$, we get the equality

$$E(\rho) = \int_0^\rho \int_{S_1} A(x_0 + r\omega, u, \nabla u)\cdot\nabla u r^{N-1}d\omega dr = \int_0^\rho \left(\int_{S_r} A\cdot\nabla u dS\right)dr.$$

It follows from the last equality that $E(\rho)$ is an absolutely continuous function. Hence, it has a derivative a.e. $E'(\rho) = \frac{dE(\rho)}{d\rho}$ and, due to (1.2.38),(1.2.39), it possesses the following properties:

$$E'(\rho) = \int_{S_\rho} A \cdot \nabla u dS \ a.e.\ \rho \in (0, \rho_0),$$

$$C_2 \|\nabla u(\rho, \cdot)\|^p_{L^p(S_\rho)} \leq E'(\rho) \leq C_1 \|\nabla u(\rho, \cdot)\|^p_{L^p(S_\rho)} \quad a.e.\ in\ [0, \rho_0] \quad (1.2.43)$$

where $\rho_0 = dist(x_0, \partial\Omega)$, and $E' \in L^1(0, \rho_0)$. We are now in a position to formulate the main result on the local vanishing property of weak solutions to equations (1.2.37).

Theorem 1.2.1. *Let u be a weak solution of equation* (1.2.37). *Assume that conditions* (1.2.38),(1.2.39),(1.2.40) *hold with*

$$C_2 > 0, \quad C_4 > 0, \quad 0 \leq \sigma < p - 1, \quad \alpha = \sigma - \beta\frac{\sigma}{p}, \quad \beta \in [0, p]$$

and, additionally, that either

$$C_3 < C_4 \ \ if\ \beta = 0 \ \ (\ respectively,\ \ C_3 < C_2\ if\ \beta = p),$$

or

$$C_3 < \left(C_4 \frac{p}{p - \beta} C_2 \frac{p}{\beta} \right) \quad if\ 0 < \beta < p. \quad (1.2.44)$$

Assume that $f(x) \equiv 0$ in Ω. Then, given an arbitrary point $x_0 \in \Omega$, we have

$$u(x) = 0\ a.e.\ in\ B_{\rho_1}(x_0)$$

with ρ_1 given by the expression

$$\rho_1^\nu = \rho_0^\nu - C \min_{\frac{\sigma+1}{p} < \tau \leq 1} \left\{ \frac{E^\gamma(\rho_0)}{\tau p - \sigma - 1} \max(1, \rho_0^{\nu - 1} \max(b^\mu(\rho_0), b^\eta(\rho_0)) \right\}, \quad (1.2.45)$$

where $C = C(C_1, C_2, C_3, C_4, N, p, \sigma, \beta)$ is a constant, $\tau \in (0, 1)$ and

$$\rho_0 = dist(x_0, \partial\Omega),$$

$$\nu = \frac{\kappa}{(p-1)(\sigma+1)}, \quad \gamma = \frac{\tau p - 1 - \sigma}{\kappa}, \quad \mu = \frac{p(1-\tau)}{\kappa},$$

$$\eta = \left(\frac{p - \sigma - 1}{(p-1)(1+\sigma)} + \frac{\tau p - 1 - \sigma}{\kappa} \right), \quad \kappa = N(p - \sigma - 1) + p(1 + \sigma) > 0.$$

Remark 1.2.1. *Observe that, if $\rho_1 = 0$, the above statement offers no information on the vanishing set of u. However if the total energy*

$$D(u) = E(\rho_0) + b(\rho_0)$$

is small enough, then always $\rho_1 > 0$ and we can conclude that

$$u(x) = 0 \ a.e.\ x \in \Omega$$

Proof. The proof of the theorem is split into several steps
Step 1

Lemma 1.2.1. *Under the above assumptions,*

$$A(\cdot, u, \nabla u) \cdot \nabla u,\ \ B(\cdot, u, \nabla u)u,\ \ |u|^{\sigma+1}, \in L^1(B_{\rho_0}(x_0)),$$

and for almost all $\rho \in (0, \rho_0)$ the inequality

$$\int_{B_\rho(x_0)} [A(x, u, \nabla u) \cdot \nabla u + B(x, u, \nabla u)u]dx + C_4 \int_{B_\rho(x_0)} |u|^{\sigma+1}dx$$
$$\leq - \int_{S_\rho(x_0)} A(x, u, \nabla u)u \cdot \nu ds := I(\rho) \tag{1.2.46}$$

holds, where $\nu = (\nu_1, ...\nu_N)$ is the unit outer normal vector to $S_\rho(x_0)$. Moreover, $I \in L^1(0, \rho_0)$.

Proof. Inequality (1.2.46) and the inclusion $I \in L^1(0, \rho_0)$ follow from (1.2.38), (1.2.39), (1.2.40), definition 1.2.2 and the embedding $W^{1,p}(\Omega) \subset L^m(\Omega)$ with some m. More precisely we have that

$$\|u\|_{L^m(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \tag{1.2.47}$$

for every $m \geq 1$ satisfying the conditions

$$m \leq \frac{Np}{N-p}\ if\ N > p, \quad m \geq 1\ arbitrary\ \ if\ N = p, \quad m = \infty\ \ if\ \ p > N.$$

Applying Holder inequality, inequality (1.2.47) and conditions (1.2.38)-(1.2.41), we have, since $\beta + \alpha < p$,

$$\|A \cdot \nabla u\|_{L^1(B_{\rho_0})} \leq C_1\|\nabla u\|^p_{L^p(B_{\rho_0})} \leq C_1\|u\|^p_{W^{1,p}(B_{\rho_0})},$$

$$\|u\|_{L^{\sigma+1}(B_{\rho_0})} \leq C\|u\|_{W^{1,p}(B_{\rho_0})},$$

$$\|B \cdot u\|_{L_1(B_{\rho_0})} \leq C_3\|\nabla u\|_{L^p(B^\beta_{\rho_0})} \cdot \|u\|_{L^p(B^{\alpha+1}_{\rho_0})} \leq C\|u\|_{W^{1,p}(B^{p+1}_{\rho_0})}, \tag{1.2.48}$$

$$\|Au\|_{L_1(B_{\rho_0})} \leq C_1\|\nabla u\|_{L^p(B^{p-1}_{\rho_0})}\|u\|_{Lp(B_{\rho_0})}C_1\|u\|_{W^{1,p}(B^p_{\rho_0})},$$

$$\|I\|_{L^1(0,\rho_0)} \leq \textstyle\int_{B_{\rho_0}} |A||u|dx \leq C_1\|u\|^p_{W^{1,p}(B_{\rho_0})}.$$

To start with the energy method, we need to multiply by the same solution u equation (1.2.37), and localize the the weake formulation (A.0.2) on the energy domain defined above, of course ,if u is only a weake solution, we are not allowed to proceed in this way. To this aim we have to introduce some suitables cut-off functions, so we define: $T_k(u) = \min(k, |u|) sign\ u$ with $k \in \mathbb{N}$, and

$$\psi_n(r) = \begin{cases} 1 & if\ r \in [0, \rho - \frac{1}{n}], \\ n(\rho - r) & if\ r \in [\rho - \frac{1}{n}, \rho], \\ 0 & if\ r \in [\rho, \rho_0],\ n \in \mathbb{N}. \end{cases} \tag{1.2.49}$$

According to results of Stampacchia [A] the cutoff function:

$$\varphi_{n,k}(x) \equiv \psi_n(|x - x_0|) T_k(u(x))$$

belongs to $W_0^{1,p}(B_\rho) \cap L^\infty(B_\rho)$. The set $C_0^\infty(B_\rho)$ is dense in $W_0^{1,p}(B_\rho) \cap L^\infty(B_\rho)$. Hence $\varphi_{n,k}(x)$ can be taken as a test function in (A.0.2). Letting in (A.0.2) $\varphi = \varphi_{n,k}$, we have:

$$\int_{B_{\rho_0}} \psi_n \{A(x, u, \nabla u) \cdot \nabla T_k(u) + B(x, u, \nabla u) T_k(u) + C(x, u) T_k(u)\} dx$$
$$= - \int_{B_{\rho_0}} A(x, u, \nabla u) T_k(u) \cdot \nabla \psi_n dx.$$

Passing to the limit as $k \to \infty$ and taking into account (1.2.48), we arrive at the inequality

$$\int_{B_{\rho_0}} \psi_n \left\{ A(x, u, \nabla u) \cdot \nabla u + B(x, u, \nabla u) u + C_4 |u|^{\sigma+1} \right\} dx$$
$$\leq - \int_{B_{\rho_0}} A(x, u, \nabla u) u \cdot \nabla \psi_n dx.$$

Introducing spherical coordinates (r, ω) and recalling the properties of ψ_n we have

$$- \int_{B_{\rho_0}} uA \cdot \nabla \psi_n dx = n \int_{\rho - \frac{1}{n} < |x - x_0| < \rho} uA(x, u, \nabla u) \cdot \frac{x - x_0}{|x - x_0|} dx$$
$$= n \int_{\rho - \frac{1}{n}}^{\rho} \left(\int_{S_r} A(r\omega, u, \nabla u) u \cdot \nu r^{N-1} d\omega \right) dr \equiv - \int_{\rho - \frac{1}{n}}^{\rho} nI(r) dr.$$

Since $I \in L^1(0, \rho_0)$, it follows from Lebesgue theorem that for a.e.$\rho \in (0, \rho_0)$

$$\lim_{n \to \infty} \int_{B_{\rho_0}} uA \cdot \nabla \psi_n dx = -I(\rho) = \int_{S_\rho} uA(x, u, \nabla u) \cdot \nu dS, \tag{1.2.50}$$

which proves (1.2.46). □

Step 2 Let us prove the existence of a constant $C_5 = C_5(C_2, C_3, C_4, p, \sigma) > 0$ such that

$$C_5(E(\rho) + b(\rho)) \leq E(\rho) + C_4 b(\rho) + \int_{B_\rho} B(x, u, \nabla u) u dx. \qquad (1.2.51)$$

First of all , if $\beta = 0$ we have that

$$\int_{B_\rho} |Bu| dx \leq C_3 \int_{B_\rho} |u|^{\sigma+1} dx = C_3 b(\rho) \ \ and\ C_5 = \min(1, C_4 - C_3).$$

If $\beta = p$, then

$$\int_{B_\rho} |Bu| dx \leq C_3 \int_{B_\rho} |\nabla u|^p dx \leq \frac{C_2}{C_3} E(\rho) \ \ and\ C_5 = \min\left(C_4, 1 - \frac{C_3}{C_2}\right) > 0.$$

Next, let us assume that $0 < \beta < p$. Applying Young inequality,

$$ab \leq \frac{\epsilon^\tau}{\tau} a^\tau + \frac{\tau - 1}{\tau} \epsilon^{-\frac{\tau}{\tau-1}} b^{\frac{\tau}{\tau-1}} \quad \forall a, b \geq 0, \epsilon > 0,\ \tau > 1,$$

we get

$$|u|^{\alpha+1} |\nabla u|^\beta \leq \frac{\epsilon^\tau}{\tau} |u|^{\tau(\alpha+1)} + \frac{\tau - 1}{\tau} \epsilon^{-\frac{\tau}{(\tau-1)}} |\nabla u|^{\frac{\beta\tau}{(\tau-1)}}$$

Letting here $\tau = \frac{(\sigma+1)}{(\alpha+1)}$ and respectively, $\beta\tau = p(\tau - 1)$, we arrive at the estimate

$$\left| \int_{B_\rho} B(x, u, \nabla u) u dx \right| \leq \epsilon^{\frac{p}{p-\beta}} C_3 \frac{p-\beta}{p} b(\rho) + \frac{\beta C_3}{C_2 p} \epsilon^{-\frac{p}{\beta}} E(\rho). \qquad (1.2.52)$$

Since C_3 satisfies (1.2.44), there exists $\epsilon > 0$ depending on p, β, C_2, C_4 such that

$$\epsilon^{\frac{p}{p-\beta}} C_3 \frac{p-\beta}{p} < C_4 \ and\ \frac{\beta}{C_2 p} C_3 \epsilon^{-\frac{p}{\beta}} < 1 :$$

indeed , it is sufficient to find an $\epsilon > 0$ such that

$$C_3 \leq \left(C_4 \frac{p}{p-\beta} \frac{C_2 p}{\beta} \right) \epsilon^{\frac{p}{\beta} - \frac{p}{p-\beta}}$$

with $\epsilon^{\frac{p}{\beta} - \frac{p}{p-\beta}} > 1$, which is always possible, being $\beta \in [0, p]$.

If we now set

$$C_5 = \min\left(C_4 - \epsilon^{\frac{p-\beta}{p}} C_2 \frac{p-\beta}{p}, 1 - \frac{\beta C_3}{C_2 p} \epsilon^{-\frac{p}{\beta}} \right),$$

(1.2.51) becomes a byproduct of (1.2.52).
Step 3
It follows from (1.2.46) and (1.2.51) that

$$C_5(E(\rho)+b(\rho)) \le -\int_{S_\rho} A(x,u,\nabla u)u\cdot ndS := I(\rho). \tag{1.2.53}$$

Whit the help of (1.2.38)-(1.2.40) and (1.2.43), the right-hand side of (1.2.53) can be estimated in the following way:

$$\begin{aligned}|I(\rho)| &\le C_1\left(\int_{S_\rho}|\nabla u|^p dS\right)^{\frac{(p-1)}{p}}\cdot\left(\int_{S_\rho}|u|^p dS\right)^{\frac{1}{p}}\\ &\le C_1 C_2^{\frac{(1-p)}{p}}\left(\frac{dE}{d\rho}\right)^{\frac{(p-1)}{p}}\|u\|_{L^p(S_\rho)}.\end{aligned} \tag{1.2.54}$$

To proceed with the estimations , we have now to deal with the term $\|u\|_{L^p(S_\rho)}$; here we state a very usefull inequality arising from the theory of the Sobolev embeddings, the proof is given in the appendix [C].

Proposition 1.2.1 (interpolation-trace inequality)**.** *For any $u\in W^{1,q+1}(B_\rho(x_0))$ there exists a constant C such that:*

$$\|u\|_{L^p(S_p)} \le C\left(\|\nabla u\|_{L^p(B_\rho)}+\rho^\delta\|u\|_{L^{1+\sigma}(B_\rho)}\right)^\theta\|u\|^{1-\theta}_{L^{1+\sigma}(B_\rho)}, \tag{1.2.55}$$

where

$$\begin{gathered}\delta=-\tfrac{k}{p(1+\sigma)},\qquad \theta=\tfrac{N(p-\sigma-1)+\sigma+1}{k}\in(0,1),\\ k=N(p-\sigma-1)+p(\sigma+1),\qquad C=C(N,\sigma,p).\end{gathered} \tag{1.2.56}$$

Inequality (1.2.55) is true for any $u\in W^{1,p}(B_\rho)$ In terms of the energy functions $E(\rho)$ and $b(\rho)$ inequality (1.2.55) takes the form

$$\|u\|_{L^p(S_\rho)} \le \bar{C}\left(E^{\frac{1}{\rho}}+\rho^\delta b^{\frac{1}{(\sigma+1)}}\right)^\theta b^{\frac{(1-\theta)}{(1-\sigma)}} \tag{1.2.57}$$

with $\bar{C}=\bar{C}(N,\sigma,p,C_2,C_4)$. Gathering (1.2.53),(1.2.54), (1.2.57), we get the inequality

$$(E+b)\le K\left(\frac{dE}{d\rho}\right)^{\frac{(p-1)}{p}}\left(E^{\frac{1}{p}}+\rho^\delta b^{\frac{1}{(\sigma+1)}}\right)^\theta b^{\frac{(1-\theta)}{(1-\sigma)}} \tag{1.2.58}$$

with $K = K(C_1, C_2, C_5, N, \sigma, p, \beta)$. It is easy to verify that for $0 \le \tau \le 1$, $\rho \le \rho_0$ the relations

$$\begin{aligned} &E^{\frac{1}{p}} b^{\frac{(1-\theta)}{\theta(1+\sigma)}} + \rho^{\delta} b^{\frac{1}{\theta(\sigma+1)}} = \\ &E^{\frac{1}{p}} b^{\frac{\tau(1-\theta)}{\theta(\sigma+1)}} b^{\frac{(1-\tau)(1-\theta)}{\theta(\sigma+1)}} + \rho^{\delta} b^{\frac{1}{p}+\frac{\tau(1-\theta)}{\theta(\sigma+1)}} b^{\frac{1}{\theta(\sigma+1)}-\frac{\tau(1-\theta)}{\theta(\sigma+1)}\frac{-1}{p}} \end{aligned} \tag{1.2.59}$$

hold.
Applying Young inequality we have:

$$E^{\frac{1}{p}} b^{\frac{(1-\theta)}{\theta(\sigma+1)}} + \rho^{\delta} b^{\frac{1}{\theta(\sigma+1)}}$$

$$\le 2\rho^{\delta} K_0^{\frac{1}{\theta}} \max(1, \rho_0^{-\delta})(E(\rho) + b(\rho))^{\frac{1}{p}+\frac{\tau(1-\theta)}{\theta(\sigma+1)}}$$

where

$$K_0 = \max(b(\rho_0)^{\frac{(1-\tau)(1-\theta)}{(\sigma+1)}}, b(\rho_0)^{\frac{(1-\tau)(1-\theta)}{(\sigma+1)}-\frac{\theta}{p}}),$$

It follows from (1.2.57) (1.2.58) that

$$E + b \le K \left(\frac{dE}{d\rho}\right)^{\frac{(p-1)}{p}} 2\rho^{\delta\theta} K_0 \max(1, \rho_0^{-\delta\theta}) \, (E+b)^{\left(\frac{\theta}{p}+\frac{\tau(1-\theta)}{(\sigma+1)}\right)} \tag{1.2.60}$$

that is, since $\frac{\theta}{p} + \frac{\tau(1-\theta)}{(\sigma+1)} < 1$,

$$(E+b)^{1-\frac{\theta}{p}-\frac{\tau(1-\theta)}{\sigma+1)}} \le 2K\rho^{\delta\theta} K_0 \max(1, \rho_0^{-\delta\theta}) \left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}}$$

Now raising both members to the exponent $\frac{p}{p-1}$ and putting for simplicity

$$K_1 = (2KK_0 \max(1, \rho_0^{-\delta\theta}))^{\frac{p}{p-1}}$$

we obtain the following differential inequality

$$K_1 \rho^{\delta\theta(\frac{p}{p-1})} \frac{dE}{d\rho} \ge E^{1+\frac{1-\theta}{p-1}-\frac{\tau(1-\theta)p}{(p-1)(\sigma+1)}} \tag{1.2.61}$$

Integration of (1.2.61) will give us information on the shape of the zero-set of the solution u. So let us proceed to conclude the proof of the theorem, integrating the above differential inequality between ρ_1 and ρ_0 with $\rho_1 \in (0, \rho_0)$. This gives us the following relation (if $\tau p - \sigma - 1 > 0$):

$$\begin{aligned} &\frac{K_1(\sigma+1)(p-1)}{(1-\theta)(\tau p - \sigma - 1)} \times \{E(\rho_0)^{\frac{(1-\theta)(\tau p-\sigma-1)}{(p-1)(\sigma+1)}} - E(\rho_1)^{\frac{(1-\theta)(\tau p-\sigma-1)}{(p-1)(\sigma+1)}}\} \ge \\ &\frac{p-1}{p-1-\delta\theta p}(\rho_0^{1-\delta\theta(1+\frac{1}{p-1})} - \rho_1^{1-\delta\theta(1+\frac{1}{p-1})}) \end{aligned} \tag{1.2.62}$$

This leads us to state that if:

$$\rho_1^{1-\delta\theta(1+\frac{1}{p-1})} = \rho_0^{1-\delta\theta(1+\frac{1}{p-1}))} \tag{1.2.63}$$

$$- \frac{K_1(\sigma+1)(p-1-\delta\theta p)}{(1-\theta)/(\tau p-\sigma-1)} E(\rho_0)^{\frac{(1-\theta)(\tau p-\sigma-1)}{((p-1)(\sigma+1))}},$$

then $E(\rho_1) = 0$ and $E(\rho) = 0$ for $\rho \leq \rho_1$. Remembering that (1.2.58) implies $b(\rho) = 0$ this means $u(x) = 0$ a.e in $B_\rho(x_0)$ for $\rho \leq \rho_1$. Since ν does not depend on the parameter τ, (1.2.63) remains true if its right hand side attains the minimum value as a function of τ proves the theorem. Finally, if we compute the exponents substituting the value of θ, we have:

$$1 - \delta\theta\left(1 + \frac{1}{p-1}\right) = \frac{\kappa}{(p-1)(\sigma+1)},$$

$$\frac{(1-\theta)(\tau p-\sigma-1)}{(p-1)(\sigma+1)} = \frac{\tau p-\sigma-1}{\kappa}.$$

□

Once having fullfilled this result a natural question than arises: what we can say if the source term is not zero but it has a localized support?
We could expect some relations between the way the source term vanishes, i.e. its behaviour near the boundary of support, and the respective boundary zero set of the solution. The task we set ourselves now is to apply the same machinery of above to problem (1.2.37) with a prescribed right-hand side $f(x) \neq 0$, to deduce an analogous differential inequality and, dealing with it, to find some reasonable hypotheses on the behaviour of such $f(x)$.
This leads us to state the following:

Theorem 1.2.2. *Let u be a weak solution of equation* (1.2.37) *and let $f \in L^1_{loc}(\Omega)$ such that $f = 0$ in $B_{\rho_1}(x_0)$ with some $0 < \rho_1 < \rho_0 = dist(x_0, \partial supp f)$. Assume the fulfillment of the hypotheses of Theorem 1.2.1 and, additionally, let the following condition be true:*

$$I(\rho) := \int_{\rho_1}^{\rho} (\tau - \rho_1)^{-\frac{1}{1-\mu}} F(\tau) d\tau < \infty \tag{1.2.64}$$

with

$$\mu = \frac{(\sigma+1)(p-\theta) - p\tau(1-\theta)}{(p-1)(\sigma+1)} \in (0,1),$$

$$\theta = \frac{N(p-\sigma-1)+\sigma+1}{N(p-\sigma-1)+p(\sigma+1)} \in (0,1),$$

and

$$F(\rho) := M\|f\|_{L^{\frac{(\sigma+1)}{\sigma}}(B_\rho)}^{\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{\sigma(p-1)}}$$

where M is a suitable constant and $\tau \in (0,1)$. Then there exist positive constants I_*, E_*, such that once

$$I(\rho_0, f) \leq I_*, \quad E(\rho_0, u) \leq E_*, \tag{1.2.65}$$

any weak solution of equation (1.2.37) *possesses the property*

$$u(x) \equiv 0 \quad in\ B_{\rho_1}(x_0).$$

Remark 1.2.2. *Theorem 1.2.2 asserts that if the right-hand side f vanishes fast enough (the admissibile rate of vanishing is controlled by the condition of convergence of the integral $I(\rho_0; f)$) then the boundaries of supports of f and that of the solution u may have common parts or even coincide.*

Proof. Given a weak solution $u(x)$ of equation (1.2.37), the inequality

$$\int_{B_\rho(x_0)} \left\{A(x,u,\nabla u)\cdot\nabla u + B(x,u,\nabla u)u + C_4|u|^{\sigma+1}\right\} dx \tag{1.2.66}$$
$$\leq -\int_{S_\rho(x_0)} A(x,u,\nabla u)u\cdot n dS + \int_{B_\rho(x_0)} f(x)u(x)dx$$

holds.
To obtain this inequality we apply the arguments used in the proof of lemma 1.2.1. Applying Holder and Young inequalities, we have that

$$\left|\int_{B_\rho(x_0)} fudx\right| \leq \|f\|_{L^{\frac{(\sigma+1)}{\sigma}}_{(B_\rho(x_0))}} b^{\frac{1}{(\sigma+1)}}(\rho) \tag{1.2.67}$$
$$\leq \epsilon b(\rho) + \frac{\sigma}{\sigma+1}(\epsilon(\sigma+1))^{-\frac{1}{\sigma}}\|f\|^{\frac{(\sigma+1)}{\sigma}}_{L^{(\sigma+1)}_{B_\rho(x_0)}}.$$

To estimate the resting terms of (1.2.65), we make use of (1.2.53),(1.2.54), (1.2.59), (1.2.61). An appropriate choice of ϵ in (1.2.67) leads to the following generalization of inequality (1.2.60):

$$E + b \leq K_1\left(\frac{dE}{d\rho}\right)^{\frac{(p-1)}{p}} \rho^{\delta\theta}(E+b)^{\left(\frac{\theta}{p}+\frac{\tau(1-\theta)}{(\sigma+1)}\right)} + M\|f\|^{\frac{(\sigma+1)}{\sigma}}_{L^{(\sigma+1)}(B_\rho(x_0))} \tag{1.2.68}$$
$$:= I_1 + I_2,$$

where only the constant K was changed. We estimate the term I_1 by the Young inequality with $p = \frac{p(\sigma+1)}{\theta(\sigma+1)+p\tau(1-\theta)}$:

$$I_1 \le c_1(\epsilon)(E+b) + c_2(\epsilon)\left(\tilde{K}_1\rho^{\delta\theta}\right)^{\frac{p(\sigma+1)}{(\sigma+1)(p-\theta)-p\tau(1-\theta)}} \frac{dE}{d\rho}^{\frac{(p-1)(\sigma+1)}{(\sigma+1)(p-\theta)-p\tau(1-\theta)}},$$

and so, choosing ϵ such that $c_1(\epsilon) = \frac{1}{2}$ and defining a suitable constant $\tilde{K}_1$ we can write

$$I_1 \le \frac{1}{2}(E+B) + \left(\tilde{K}_1\rho^{\delta\theta}\frac{dE}{d\rho}\right)^{\frac{(p-1)(\sigma+1)}{(\sigma+1)(p-\theta)-p\tau(1-\theta)}}. \tag{1.2.69}$$

Gathering (1.2.68), (1.2.69), and raising both sides of the obtained inequality to the power $\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{(p-1)(\sigma+1)}$, we finally get

$$E^{\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{(p-1)(\sigma+1)}} \le (E+b)^{\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{(p-1)(\sigma+1)}} \le \tilde{K}_1\rho^{\delta\theta}\frac{dE}{d\rho} + K_2\|f\|_{L^{\frac{(\sigma+1)}{\sigma}}(B_\rho)}^{\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{\sigma(p-1)}}.$$

whence

$$E^\mu \le \Lambda\frac{dE}{d\rho} + F(\rho) \quad for\ \rho \in (\rho_1, \rho_0) \tag{1.2.70}$$

with

$$\Lambda = \tilde{K}_1\rho_0^{\delta\theta},$$

$$F(\rho) := K_2\|f\|_{L^{\frac{(\sigma+1)}{\sigma}}(B_\rho)}^{\frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{\sigma(p-1)}},$$

$$K_2 = K_2(M,\tau,\theta),$$

$$\mu = \frac{(\sigma+1)(p-\theta)-p\tau(1-\theta)}{(p-1)(\sigma+1)}.$$

So we reduced our problem to the qualitative study of a quite different differential inequality from above. Now we can't proceed with a direct integration like in the previous case, but we use an appropriate technical lemma. Here we only state it, the proof being left to the appendix [B].

Lemma 1.2.2. *Let $Y \in W^{1,1}_{loc}(0,R)$, with $Y \ge 0$ and $Y' \ge 0$. Assume that the inequality*

$$\Lambda Y'(\rho) + F(\rho) \ge Y(\rho)^{1-\mu} \quad for\ a.e\ \rho \in (R_0, R) \tag{1.2.71}$$

holds with some $R_0 \in (0, R)$*, where* $\mu \in (0,1)$*,* $\Lambda > 0$*,* $F(t) \geq 0$*.*
Assume that the integral

$$\Gamma(\rho) = \int_{R_0}^{\rho} (\tau - R_0)^{-\frac{1}{\mu}} F(\tau) d\tau$$

is convergent. Then the function $Y(\rho)$ *admits the estimate*

$$Y(R_0) \leq G(\rho) \equiv Y(R) - (\rho - R_0)^{\frac{1}{\mu}} \left(\left(\frac{\mu}{\Lambda}\right)^{\frac{1}{\mu}} - \frac{\Gamma(\rho)}{\Lambda} \right) \tag{1.2.72}$$

for every $\rho \in (R_0, R)$*, and* $Y(R_0) = 0$ *if there exists* $\rho^* \in (R_0, R)$ *such that* $G(\rho^*) = 0$*.*

Observe that this lemma suggests us to introduce a so called "*flatness hypothesis*" on the behaviour of the source term f near the boundary of its support.
Let's now proceed with the completion of proof of theorem 1.2.2. First of all we note that it is very easy to check that the differential inequality (1.2.70) satisfies the hypothesis of the above lemma, then putting in our case $R_0 = \rho_1$, $R = \rho_0$ the following estimate holds:

$$E(\rho_1) \leq E(\rho_0) - (\rho - \rho_1)^{\frac{1}{1-\mu}} \left(\left(\frac{1-\mu}{\Lambda}\right)^{\frac{1}{1-\mu}} - \frac{I(\rho)}{\Lambda} \right) := \Theta(\rho)$$

where $\Gamma(\rho) = K_2 I(\rho)$, and $I(\rho)$ is like in (1.2.64). Now to make the lemma work we only have to suppose that

$$I(\rho_0) < \Lambda \left(\frac{1-\mu}{\Lambda}\right)^{\frac{1}{1-\mu}} = I^*$$

and

$$E(\rho_0) < (\rho_0 - \rho_1)^{\frac{1}{1-\mu}} \left(\left(\frac{1-\mu}{\Lambda}\right)^{\frac{1}{1-\mu}} - \frac{I(\rho_0)}{\Lambda} \right)$$

since this implies that $\Theta(\rho_1) > 0 > \Theta(\rho_0)$. Hence there is $\rho^* \in (\rho_1, \rho_0)$ such that $\Theta(\rho^*) = 0$, and then, the lemma says that $E(\rho_1) = 0$ and this proves the theorem. $\square$

Remark 1.2.3. *The assertion of theorem 1.2.2 can be interpreted as follows: if the source* f *vanishes in a ball* B_{ρ_1}*, and* (1.2.64) *is fulfilled, then every weak solution* u *of equation* (1.2.37) *in a ball* B_{ρ_0} *vanishes in the ball* B_{ρ_1}*, provided that the energy* $E(\rho_0)$ *is sufficiently small.*

Chapter 2

Space and time localization in nonlinear evolution problems

2.1 Heuristic approach

In this chapter the local energy method is applied to study the space and time localization of weak solutions for nonlinear degenerate parabolic equations. Let us describe the types of localization we are going to study. Let $u(x,t)$ be a real valued function of $N+1$ variables $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^N$. It is assumed that $u(x,t)$ is defined in a cilinder $Q = \Omega \times (0,T)$ for some open set $\Omega \subseteq \mathbb{R}^N$, $T > 0$, and that for every $t \in [0,T]$ it is a measurable functions of the variable $x \in \Omega$ and

$$u \in C\left([0,T]; L^1_{loc}(\Omega)\right).$$

Let us take $x_0 \in \Omega$ and assume that

$$u(x,0) = 0 \ a.e. \ \text{ in a ball } \ B_{\rho_0}(x_0) \subset \Omega.$$

Definition 2.1.1. *(i) A function $\rho(t) : [0,T^*) \mapsto [0,\infty)$, with $\rho(0) \le \rho_0$ is called a "rate at the point x_0" if*

$$\forall t \in [0,t^*), \quad u(x,t) = 0 \ a.e. \ in \ B_{\rho(t)}(x_0).$$

(ii)A function $u(x,t)$ is said to possess the property of "finite speed of propagation" (from nonzero disturbances) if for some $x_0 \in \Omega$ and $t^* > 0$ there exists a strictly positive rate at the point x_0.
(iii) Let $u(x,t)$ be defined on $Q = \Omega \times [0,\infty)$. We say that $u(x,t)$ possesses the property of "stable localization" if it has the property of finite speed of propagation with a rate $\rho(t)$ defined on the whole $[0,\infty)$ such that

$$\liminf_{t\to\infty} \rho(t) > 0$$

(iv) Given $x_0 \in \Omega$, let

$$\rho_0 = \sup\{\rho > 0 : u(x,0) = 0 \text{ } a.e. \text{ } in B_\rho(x_0) \subset \Omega\}$$

We say that $u(x,t)$ possesses the "generalized waiting time property" if for some $t^* > 0$ the function $\rho(t) = \rho_0$ is a rate at the point x_0 on the interval $[0, t^*]$.
Let $x_* \in \overline{supp\ u(x,0)}$. The instant

$$t_w(x_*) = \sup_{x_0 \in \Omega}\{t^* : |x_* - x_0| = \rho_0 \text{ and } \rho_0 \text{ is a rate on } (0,t^*) \text{ at the point } x_0\}$$

is called the waiting time at the point x_*.

This definition needs several comments. First of all, it is clear that the rate is not unique: given a rate $\rho(t)$ at the point $x_0 \in \Omega$, any positive function $\delta(t) \in (0, \rho(t)]$ is also a rate at the point x_0. The optimal rate on an interval $(0, t^*)$ can be defined by

$$\eta(t) = \sup\{\rho(t) : \rho(t) \text{ is a rate at the point } x_0, \ t \in (0,t^*)\}. \qquad (2.1.1)$$

For functions of two variables, the function $\eta(t)$ defined in this way coincide with the traditional definition of the free boundary or interface occurring in nonnegative solutions of parabolic equations (by free boundary or interface we mean a curve in the (x,t)-plain separating the region where the solution is positive or zero).
Definition 2.1.1 says, in essence, that a function possesses the property of finite speed of propagation of disturbances if the "zero caverns“ take time to disappear. This is what is guaranted by the existence of a rate. The instant speed of propagation don't need to be finite, however. Let $N = 1$: given the optimal rate of propagation $\eta(t)$, we can introduce the functions

$$V^+(t) = \limsup_{\Delta t \to 0} \frac{\eta(t+\Delta t) - \eta(t)}{\Delta t}, \quad V^-(t) = \liminf_{\Delta t \to 0} \frac{\eta(t+\Delta t) - \eta(t)}{\Delta t}$$

which are of dimension LT^{-1} (lenght/time) and can be interpreted as upper and lower bounds for the instant velocity of propagation of nonzero disturbances (the existence of $\eta'(t)$ is not assumed here). It is possible to give several exemples of explicit solutions to nonlinear parabolic equations which possess the property of finite speed of propagation in the sense of Definition 2.1.1 but with instant velocity infinite at certain points.
It is important to stress, that finite speed of propagation is not displayed by the solutions of linear parabolic problems, but is typical for solutions of

linear hyperbolic equations. In the simplest case, where $u(x,t)$ is a solution of the Cauchy problem for the linear hyperbolic equation

$$cu_{tt} - u_{xx} = 0$$

u is given by d'Alembert formula, and in the above notation we have $V^+(t) = V^-(t) = c$ for all $t > 0$.
For the sake of convenience, we have introduced the concept of finite speed of propagation of disturbances, with respect to the zero-level of $u(x,t)$. Considering the function $u(x,t) - s$ with $s \neq 0$, we can extend these concepts in a natural way, to define the finite speed of propagation of disturbances with respect to $s-$level.
The next definition describes the situation when the function $u(x,t)$ admits a strictly increasing rate $\rho(t)$ at a point x_0.

Definition 2.1.2. *Let $x_0 \in \Omega$ be a given point, and*

$$\rho_0 = \sup\{\rho > 0 : u(x,0) = 0 \text{ a.e. in } B_\rho(x_0) \subset \Omega\}.$$

1. *Let $\rho_0 > 0$. We say that $u(x,t)$ possesses the property of "support shrinking" at x_0 if on some interval $(0,t^*) \neq \emptyset$ there exists a monotone increasing rate $\rho(t)$ at the point x_0 such that $\rho(0) = \rho_0$*

2. *Let $\rho_0 = 0$. The function $u(x,t)$ is said to possess the property of a dead core formation at x_0 if on some interval $(0,t^*) \neq \emptyset$ it admits a strictly positive rate $\rho(t)$ at the point x_0.*

2.1.1 Simple examples

We present here a few simple examples of explicit solutions to nonlinear parabolic equations that admit explicit formulas.
The first example is furnished by the self similar solution for the so-called "porous medium equation"

$$u_t = \Delta u^m, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \tag{2.1.2}$$

with the parameter of nonlinearity $m > 1$. This name is due to one of the most natural interpretations of this equation. If we describe the motion of a polytropic gas with density u, pressure $p = \lambda u^{m-1}$, and velocity $v = -\kappa \nabla p$ through a porous medium, equation (2.1.2) expresses the mass balance law of the motion (up to a constant which we scale out to unit).
Equation (2.1.2) admits the class of explicit "self-similar" solutions constructed in [15]

$$U(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right), \tag{2.1.3}$$

where

$$f(\xi) = \left[A - B|\xi|^2\right]_+^{\frac{1}{m-1}}$$

and the constants B,α,β are defined as

$$\alpha = \frac{1}{m - 1 - \frac{2}{N}}, \quad \beta = \frac{\alpha}{N}, \quad B = \frac{m-1}{m}\beta.$$

The constant A is arbitrary. It distinguishes a concrete solution of this family. The evolution of the space profile of $u(x, t+\epsilon)$, $\epsilon > 0$, is shown in figure.

The solution takes the Dirac mass as initial datum

$$U(\cdot, t) \to M\delta(x) \;\; as\ t \to 0+,$$

where the constant M depends on A and can be found from the relation

$$\int_{\mathbb{R}^N} U(x,t)dx = M \;\; for\ all\ t > 0.$$

The interface between the regions where the solution $U(x, t)$ is positive or is equal identically to zero is given by the exact formula

$$|x| = \sqrt{A/B}t^\beta \tag{2.1.4}$$

The velocity of propagation of disturbances from the initial data, is equal to $v_n = \beta\sqrt{A/B}t^{\beta-1}$, where v_n is the derivative in the direction of the outer unit normal to the surface (B.0.2). Obviously, the velocity v_n is infinite at the instant $t = 0$.

2.1.2 The energy methods

A preliminary approach. Once an equation is proven to possess a class of exact solutions (or sub-/supersolutions) which display one of the localization properties, a tipical argument which allows one to extend this property to every admissible solution is to apply the maximum principle for parabolic equations and to compare the solution with an exact solution (or sub-/super solution) through the input data. In so doing one has to impose certains restrictions on both the structure of the equation under study, and the input data. This strategy meets several difficulties, most of which can be avoided by the application of the local energy method. The idea consists in describing

the evolution of the null set of the solution using only nonlinearity of the equation and a threshold value of the "total energy" of the solution. This value accumulates all the information required to perform a local study of a "zero-cavern" in the space-time domain.
Let us present the scheme of the method by considering the one-dimensional equation

$$\left(u|u|^{\gamma-1}\right)_t - u_{xx} = f(x,t), \quad \gamma \in (0,1). \tag{2.1.5}$$

Equation (2.1.5) is considered in the rectangle

$$Q = (-L, L) \times (0, T) \ \ \textit{with}\ 0 < L < \infty,\ T > 0,$$

and is endowed with the initial condition

$$u(x,0) = u_0(x), \quad x \in (-L, L). \tag{2.1.6}$$

The solution of problem (2.1.5)-(2.1.6) is understood in the following sense.

Definition 2.1.3. *Let $f \in L^2(Q)$, and $u_0 \in L^{\gamma+1}(-L,L)$, a measurable function $u(x,t)$ is called* local weak solution *of problem* (2.1.5),(2.1.6) *if*

1. $u \in V(Q) = L^\infty(0,T; L^{1+\gamma}(-L,L)) \cap L^2(0,T; W^{1,2}(-L,L))$;

2. $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^{1+\gamma}(-L,L)} = 0$;

3. *for every test function $\varphi \in C^\infty(0,T; C_0^\infty(-L,L))$, vanishing at $t = T$,the integral identity*

$$\iint_Q \left\{u|u|^{\gamma-1}\varphi_t - u_x\varphi_x + f\varphi\right\} dxdt + \int_{-L}^{L} u_0|u_0|^{\gamma-1}\varphi(x,0)dx = 0$$

holds.

The application of the method can be conventionally divided into three steps:

1. choice of the appropriate local energy functions;

2. derivation of the differential inequality for the local energy functions;

3. analysis of the differential inequality and interpretation of results.

Given a local weak solution $u(x,t)$ of problem (2.1.5) (2.1.6), let us define the *local energy functions*

$$
\begin{aligned}
E(\rho,t) &= \int_0^t \int_{B_\rho} |u_x|^2 dx d\tau \equiv \int_0^t \|u_x(\cdot,\tau)\|^2_{L^2(B_\rho)} d\tau, \\
b(\rho,t) &= \int_{B_\rho} |u|^{1+\gamma} dx \equiv \|u(.,t)\|^{1+\gamma}_{L^{1+\gamma}(B_\rho)}, \\
\bar{b}(\rho,t) &= \text{ess sup}_{0\le\tau\le t} b(\rho,\tau),
\end{aligned}
\tag{2.1.7}
$$

where B_ρ is the interval $(-\rho,\rho)$. According to these definitions, the *local energy set* is a cylinder of radius ρ and height t; ρ is viewed as the independent variable while t serves as a parameter. This simple choice is convenient to reveal the property of finite speed of propagation.
We shall consider only those solutions to problem (2.1.5),(2.1.6), that satisfy the condition

$$
D(u) \equiv D(u,L,T) \equiv \bar{b}(L,T) + E(L,T) \le D_0 < \infty. \tag{2.1.8}
$$

The quantity $D(u)$ is called *total energy* of the solution $u(x,t)$ in the domain Q. The constant D_0 (the upper estimate on the total energy) absorbs all the global information on the data of the problem under study.
The derivatives of $E(\rho,t)$ in ρ and t are given by the formulas

$$
\begin{aligned}
\frac{\partial E(\rho,t)}{\partial \rho} &= \int_0^t \left(|u_x(\rho,t)|^2 + |u_x(-\rho,\tau)|^2\right) d\tau, \\
\frac{\partial E(\rho,t)}{\partial t} &= \int_{-\rho}^{\rho} |u(x,t)|^2 dx, \\
\frac{\partial^2 E(\rho,t)}{\partial \rho \partial t} &= (|u_x(\rho,t)|^2 + |u_x(-\rho,t)|^2).
\end{aligned}
$$

Let us multiply equation (2.1.5) by $u(x,t)$ and then formally integrate the resulting equality over the domain $B_\rho \times (0,t) \subset Q$: this leads to the energy relation

$$
\frac{\gamma}{1+\gamma} b(\rho,\tau)\,\Big|_{\tau=0}^{\tau=t} + E(\rho,t) = I(f) + I(\rho), \tag{2.1.9}
$$

where

$$
I(f) = \int_0^t \int_{B_\rho} f u dx d\tau, \quad I(\rho) = \int_0^t u_x u d\tau\,\Big|_{x=\rho}^{x=-\rho}.
$$

We postpone the rigorous justification of relation (2.1.9) in a second time.

Finite speed of propagation of disturbances. To simplify matters, we first assume that equation (2.1.5) is homogeneous:

$$f(x,t) = 0 \tag{2.1.10}$$

Let

$$u_0(x) = 0, \quad x \in B_{\rho_0} \subset (-L, L).$$

Assumption (2.1.10) allows us to rewrite (2.1.9) in the following form: for every $\rho \leq \rho_0$,

$$\frac{\gamma}{1+\gamma} b(\rho,\tau) \mid_{\tau=0}^{\tau=t} + E(\rho,t) = I(\rho) \tag{2.1.11}$$

Note that $b(\rho,0) = 0$ if $\rho \leq \rho_0$. The term $I(\rho)$ on the right-hand side of (2.1.11) can be estimated by making use of the interpolation-trace inequality, in the same way we did for the estimation in stationary problems. So we can write the following chain of inequalities (where we use (1.2.12) with the exponents $q = \gamma$, $p = 2$, and (1.2.14)):

$$\begin{aligned}
|I(\rho)| &\leq \int_0^t |u_x(\rho)u(\rho) - u_x(-\rho)u(-\rho)| d\tau \qquad (2.1.12)\\
&\leq \int_0^t [|u_x(\rho)u(\rho)| + |u_x(-\rho)u(-\rho)|] d\tau \\
&\leq \int_0^t (|u_x(\rho)| + |u_x(-\rho)|)(|u(\rho)| + |u(-\rho)|) d\tau \\
&\leq 2^{\frac{1}{2}} \int_0^t (|u_x(\rho)|^2 + |u_x(-\rho)|^2)^{\frac{1}{2}} (|u(\rho)| + |u(-\rho)|) d\tau \\
&\leq C \int_0^t \left(\frac{\partial^2 E}{\partial\rho\partial\tau}\right)^{\frac{1}{2}} \left(E_\tau^{\frac{1}{2}} + \rho^{-\delta} b^{\frac{1}{1+\gamma}}\right)^{\theta} b^{\frac{(1-\theta)}{(1+\gamma)}} d\tau \\
&\leq C_1 \rho^{-\delta\theta} \bar{b}^{\frac{(1-\theta)}{(1+\gamma)}} \int_0^t \left(\frac{\partial^2 E}{\partial\rho\partial\tau}\right)^{\frac{1}{2}} (E_\tau + b)^{\frac{\theta}{2}} d\tau
\end{aligned}$$

where

$$\theta = \frac{2}{3+\gamma}, \qquad \delta = \frac{1}{1+\gamma} + \frac{1}{2} = \frac{1}{\theta(1+\gamma)},$$

$$C_1 = C \max\left(\rho_0^{\delta\theta}, \bar{b}(\rho_0, T)^{\frac{\theta(2-\gamma-1)}{2(1+\gamma)}}\right), \quad C = C(\gamma).$$

Using Holder inequality and the formulas

$$\int_0^t E_\tau(\rho,\tau) d\tau = E(\rho,\tau), \quad \int_0^t E_{\rho\tau}(\rho,\tau) d\tau = E_\rho(\rho,\tau),$$

in (2.1.12) we arrive at the estimate

$$
\begin{aligned}
|I(\rho)| &\leq C_1\rho^{-\delta\theta}\bar{b}^{\frac{(1-\theta)}{(1+\gamma)}}\left[\int_0^t \frac{\partial^2 E}{\partial\rho\partial\tau}d\tau\right]^{\frac{1}{2}}\left[\int_0^t (E_\tau+b)^\theta d\tau\right]^{\frac{1}{2}}\\
&= C_1\rho^{-\delta\theta}\bar{b}^{\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}\left(\int_0^t (\frac{\partial E}{\partial\tau}+b)^\theta d\tau\right)^{\frac{1}{2}}\\
&\leq C_1\rho^{-\delta\theta}\bar{b}^{\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}\left(\int_0^t \left(\frac{\partial E}{\partial\tau}+b\right)d\tau\right)^{\frac{\theta}{2}} t^{\frac{(1-\theta)}{2}} \qquad (2.1.13)\\
&\leq C_1\rho^{-\delta\theta}\bar{b}^{\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}\left(E+\int_0^t b d\tau\right)^{\frac{\theta}{2}} t^{\frac{(1-\theta)}{2}}\\
&\leq C_2\rho^{-\delta\theta}t^{\frac{(1-\theta)}{2}}(E+\bar{b})^{\frac{\theta}{2}+\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}
\end{aligned}
$$

with $C_2 = C_1\max(1,T)^{\frac{\theta}{2}}$.Returning to (2.1.11) we get the inequality

$$
\frac{\gamma}{1+\gamma}b(\rho,t)+E(\rho,t) \leq C_2\rho^{-\delta\theta}t^{\frac{(1-\theta)}{2}}(E+\bar{b})^{\frac{\theta}{2}+\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}.
$$

Since the right-hand side of this inequality is nondecreasing in t, the inequality holds if we replace $b(\rho,t)$ by $\bar{b}(\rho,t)$, and C_2 by $2C_2$. So

$$
\frac{\gamma}{1+\gamma}\bar{b}(\rho,t)+E(\rho,t) \leq 2C_2\rho^{-\delta\theta}t^{\frac{(1-\theta)}{2}}(E+\bar{b})^{\frac{\theta}{2}+\frac{(1-\theta)}{1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}. \qquad (2.1.14)
$$

From this we arrive at

$$
\begin{aligned}
\frac{\gamma}{1+\gamma}\bar{b}(\rho,t)+E(\rho,t) &\leq \bar{b}(\rho,t)+E(\rho,t)\\
&\leq 2C_2\rho^{-\delta\theta}t^{\frac{(1-\theta)}{2}}(E+\bar{b})^{\frac{\theta}{2}+\frac{(1-\theta)}{(1+\gamma)}}\left(\frac{\partial E}{\partial\rho}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence simplifying and taking the square of both sides

$$
E^\nu(\rho,t) \leq (E+\bar{b})^\nu \leq C_3\rho^{-\alpha}t^\beta\frac{\partial E}{\partial\rho} \qquad (2.1.15)
$$

with the parameters

$$
\begin{aligned}
\alpha &= 2\delta\rho, \quad \beta = 1-\theta,\\
C_3 &= (2C_2)^2,\\
\nu &= 2\left(1-\frac{\theta}{2}-\frac{1-\theta}{1+\gamma}\right) = \frac{2(1+\gamma)}{3+\gamma}.
\end{aligned}
$$

Inequality (2.1.15) is an ordinary differential inequality for the function $E(\rho, t)$ that we consider depending on t as a parameter. Integrating (2.1.15) leads to the estimate

$$\begin{aligned} E^{1-\nu}(\rho,t) &\leq E^{1-\nu}(\rho_0,t) - \frac{1-\nu}{C_3(1+\alpha)} t^{-\beta} \left(\rho_0^{1+\alpha} - \rho^{1+\alpha}\right) \\ &< D_0^{1-\nu} - \frac{1-\nu}{C_3(1+\alpha)} t^{-\beta} \left(\rho_0^{1+\alpha} - \rho^{1+\alpha}\right) \end{aligned}$$

where the upper bound for the *total energy* D_0 is like in (2.1.8). Since $E(\rho, t)$ is a nondecreasing nonnegative function of ρ, we conclude that

$$E(\rho,t) = b(\rho,t) = 0 \;\; for\ all\ \rho \in (0, \rho(t))$$

with

$$\rho^{1+\alpha}(t) = \rho_0^{1+\alpha} - \frac{C_3(1+\alpha)t^{\beta}}{1-\nu} D_0^{1-\nu},$$

that is,

$$u(x,t) = 0 \;\; for\ a.\ a.\ |x| \leq \rho(t).$$

So we can conclude that

$$\rho(0) = \rho_0\ and\ \rho(t) > 0\ for\ 0 < t < t^* = \left(\frac{1-\nu}{C_3(1+\alpha)} \rho_0^{1+\alpha} D_0^{\nu-1}\right)^{\frac{1}{\beta}}$$

and then to state the following

Proposition 2.1.1. *Let $u(x,t)$ be a weak solution of problem* (2.1.5)-(2.1.6) *in the sens of definition 2.1.3. Let $u(x,0) = 0$ in $B_{\rho_0} \subset (-L, L)$. If $D_0 < \infty$, there always esists $t^* > 0$ such that the solution $u(x,t)$ has a nonzero rate $\rho(t)$ at the point $x = 0$. An admissibile rate is given by the formula*

$$\rho(t) = \left(\rho_0^{1+\alpha} - \frac{C_3(1+\alpha)t_*^{\beta}}{1-\nu} D_0^{1-\nu}\right)^{\frac{1}{(1+\alpha)}}.$$

2.2 General second order equations

2.2.1 Finite speed of propagation

In this section, we study the property of finite speed of propagation of disturbances from the initial data for weak solutions of second order parabolic equations (for an exsistence result see [14]). The energy estimates derived here will serve the analytic framework for all further considerations of this

chapter.
We consider the parabolic equation

$$\frac{\partial \psi(x,u)}{\partial t} - divA(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) = f(x,t), \quad (2.2.1)$$

where

$$Du = \nabla u, \quad A = (A_1, ..., A_N), \quad divA = \sum_{i=1}^{N} \frac{d}{dx_i} A_i(x,t,u,Du).$$

Equation (2.2.1) is considered in a cilinder $Q = \Omega \times (0,T),\ T \in \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^N,\ N \geq 1$. It is assumed that the coefficients of equations (2.2.1) satisfy the structural conditions, for all $(t,r,q) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$,

$$|A(x,t,r,q)| \leq C_1 |q|^{p-1}, \quad (2.2.2)$$

$$C_2 |q|^p \leq A(x,r,t,q) \cdot q \quad (2.2.3)$$

$$|B(x,t,r,q)| \leq C_3 |r|^{\alpha} |q|^{\beta}, \quad (2.2.4)$$

$$C_4 |r|^{1+\sigma} \leq C(x,t,r) r, \quad (2.2.5)$$

$$C_5 |r|^{\gamma+1} \leq G(x,r) \leq C_6 |r|^{\gamma+1}, \quad (2.2.6)$$

$$G(x,r) = \psi(x,r) r - \int_0^r \psi(x,\tau) d\tau \equiv \psi(x,r) r - j(r) \quad (2.2.7)$$

Here $C_1 \cdots, C_6, p, \alpha, \beta, \sigma, \gamma, \kappa$ are positive constants which will be specified later on.
With respect to the function $\psi(x,r)$ we assume the following:

1. $\psi(x,r)$ is a Caratheodory function (measurable in x for all $r \in \mathbb{R}$ and continuous in r for almost all $x \in \Omega$,)

2. $\psi(x,r)$ is nondecreasing in r for almost all $x \in \Omega$,

3. $\psi(x,r)$ satisfies for all $(r,s) \in \mathbb{R} \times \mathbb{R}$ and for a.a. $x \in \Omega$:

4. $\psi(x,0) = 0$,

$$|C(x,t,r)| \leq C_0 |r|^{\gamma} \leq \psi(x,r) sign\ r,$$

with some constant $C_0 > 0$.

We will consider the weak solutions of equation (2.2.1) satisfying the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \quad (2.2.8)$$

Definition 2.2.1. *A measurable in Q function $u(x,t)$ is called a weak solution of problems* (2.2.1),(2.2.8) *if*

1. $u \in L^\infty(0,T;L^{\gamma+1}_{loc}) \cap L^p(0,T;W^{1,p}_{loc})$;
2. $A(\cdot,\cdot,u,Du), B(\cdot,\cdot,u,Du), C(\cdot,\cdot,u) \in L^1(Q)$;
3. $\liminf_{t\to 0} G(x,u(\cdot,t)) = G(x,u_0) \in L^1(\Omega)$, *where G is the function defined in (2.2.7);*
4. *for every test function*
$$\varphi \in L^\infty\left(0,T,W_0^{1,p}(\Omega)\right) \cap W^{1,2}(0,T;L^\infty(\Omega))$$
with $\varphi = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces, the identity
$$\int_Q \{\psi(x,u)\varphi_t - A\cdot D\varphi - B\varphi - C\varphi\}dxdt - \int_\Omega \psi(x,u)\varphi dx\,|_{t=0}^{t=T} = \tag{2.2.9}$$
$$= -\int_Q f\varphi dxdt$$
holds.

Let us introduce the following energy functions
$$E(\rho,t) = \int_0^t \int_{B_\rho} Ax,\tau,u,Du)\cdot Dudxd\tau,$$
$$C(\rho,t) = \int_0^t \int_{B_\rho} |u(x,\tau)|^{1+\sigma}dxd\tau, \tag{2.2.10}$$
$$b(\rho,\tau)\int_{B_\rho} |u(x,\tau)|^{1+\sigma}dxd\tau,$$
$$\bar b(\rho,\tau) = \sup\,\mathrm{ess}_{0\le\tau\le t} b(\rho,\tau), \quad B_\rho = \{x\in\Omega : |x-x_0|<\rho\}.$$
and the total energy
$$D(u) \equiv D(u,\Omega,T) = \bar b(\Omega,T) + E(\Omega,T)$$
that we will always suppose finite.
The following equalities hold:
$$\frac{\partial E(\rho,t)}{\partial\rho} = \int_0^t \int_{S_\rho(x_0)} A(x,t,u,Du)\cdot DudSdt, \quad S_\rho = \partial B_\rho,$$
$$\frac{\partial E(\rho,t)}{\partial t} = \int_{B_\rho(x_0)} A(x,t,u,Du)\cdot Dudx, \tag{2.2.11}$$
$$\frac{\partial^2 E(\rho,t)}{\partial\rho\partial t} = \int_{S_\rho(x_0)} A\cdot DudS,$$

with the estimates

$$
\begin{aligned}
&C_2\int_0^t\int_{S_\rho(x_0)}|Du|^p dSd\tau \le \frac{\partial E(\rho,t)}{\partial\rho} \le C_1\int_0^t\int_{S_\rho(x_0}|Du|^p dSd\tau,\\
&C_5 b(\rho,t)\le \int_{B_\rho(x_0)} G(u(\cdot,t))dx \le C_6 b(\rho,t),\\
&C_2\|Du\|^p_{L^p((0,t)\times B_\rho(x_0))}\le E(\rho,t)\le C_1\|Du\|^p_{L^p((0,t)\times B_\rho(x_0))}
\end{aligned}
\tag{2.2.12}
$$

we will assume that

$$u_0(x)=0 \text{ for } \ x\in B_{\rho_0}(x_0) \text{ with } \ \rho_0\in(0,dist(x_0,\partial\Omega)). \tag{2.2.13}$$

Our goal is to draw some information about the time evolution of this peculiar zero set, in order to get a qualitative idea about how the support of solution u moves in proximity of a point x_0 of its domain. Application of the local energy method, allows us to find suitable hypothesis on the structural constants, arising in (2.2.2)-(2.2.7), that are related to the character of, respectively, the cumulative, the diffusive and the absorption terms. So let us state the following:

Theorem 2.2.1 (Finite speed of propagation)**.** *Assume that conditions (2.2.2)-(2.2.7),* (2.2.13)*, and conditions* $(1)-(2)-(3)$ *on* ψ *are fulfilled. Let the structural constants satisfy the inequalities*

$$C_2>0,\ C_3\ge 0,\ C_4\ge 0,\ C_5>0$$

and

$$1+\gamma<p,\ 0\le\beta\le p,\ \alpha=\gamma-\frac{(1+\gamma)\beta}{p}.$$

Then there exist T^* *and* D^* *such that every weak solution of problem* (2.2.1)*, with* $f(x,t)\equiv 0$ *in* $B_{\rho_0}(x_0)\times(0,T^*)$*, possesses the finite speed of propagation property:*

$$u(x,t)=0 \ for\ x\in B_{\rho(t)}(x_0),\quad 0\le t\le T^*,$$

with $\rho(t)$ *given by the formula*

$$\rho^\nu(t)=\rho_0-Ct^\lambda\min_{\frac{(\gamma+1)}{p}<\tau\le 1}\left(E^\epsilon(\rho_0,\tau)M(\rho_0,\tau)\right), \tag{2.2.14}$$

$$M=\frac{1}{\tau p-1-\gamma}\max(1,\rho_0^{\nu-1})\max(b^\mu(\rho_0,t),b^{\eta(\rho_0,t)}),$$

with some constant $C = C(C_1, C_2, C_3, C_5, N, p, \gamma, \beta, T)$ *and*

$$\nu = \frac{\gamma\kappa}{(p-1)(\gamma+1)}, \quad \lambda = \frac{\gamma+1}{\gamma\kappa}, \quad \mu = \frac{p(1-\tau)}{\gamma\kappa},$$
$$\eta = \frac{p-1-\gamma}{(p-1)(\gamma+1)} - \frac{1+\gamma-\tau p}{\gamma\kappa}, \quad \epsilon = \frac{p(1-\tau)}{\gamma\kappa},$$
$$\kappa = \frac{N(p-1-\gamma)}{\gamma} + \frac{p(1+\gamma)}{\gamma}.$$

Note that since $\rho(t)$ is a monotone decreasing function with $\rho(0) > 0$, the set $B_{\rho(t)}(x_0)$ is nonempty for small t.

Remark 2.2.1. *We stress that, the above theorem give sufficent hypotheses, to ensure ourselves the solutions of the problem* (2.2.1)-(2.2.8)*, dysplay a property of finite speed of propagation of their support, in particular the fundamental assumption we make is* $\gamma < p-1$ *say us that the process of diffusion modelled by the P.D.E. take place in a slow manner, so we can speak of slow diffusion condition.*

Proof. We will split the proof of this theorem into two steps. The first step is to derive the energy relation valid for every weak solution $u(x,t)$ of problem (2.2.1),(2.2.13). The second step is to study the properties of the functions satisfying the ordinary nonlinear differential inequality which follows from the energy relation.
Let us start with a lemma:

Lemma 2.2.1. *Under the hypotheses of theorem 2.2.1, we have*

$$A(\cdot,\cdot,u,Du)\cdot Du,\ B(\cdot,\cdot,u,Du)u,\ C(\cdot,\cdot,u)u,\ u|\mathrm{A}(t,x,u,Du)| \in L^1(0,T;B_{\rho_0}(x_0))$$

and for almost all $\rho \in (0,\rho_0)$ *and* $t \in (0,T^*) \subset (0,T)$*, the integration-by-parts inequality*

$$\begin{aligned}
&\int_{B_\rho(x_0)} G(x,u(x,\tau))dx \,|_{\tau=0}^{\tau=t} + \\
&\int_0^t \int_{B_\rho(x_0)} (A(x,\tau,u,Du)\cdot Du + B(x,\tau,u,Du)u + C(x,\tau,u)u + fu)dxd\tau \\
&\leq \int_0^t \int_{S_\rho(x_0)} A(x,\tau,u,Du)\cdot \nu u dS d\tau \equiv I
\end{aligned} \tag{2.2.15}$$

holds, where ν *is the unit outer normal vector to* $S_\rho(x_0)$.

Proof. From hypotheses is clear that $A(t,x,u,Du)\cdot Du$, $B(t,x,u,Du)u$ and $u|A(\cdot,\cdot,u,Du)|$ are locally integrable in $\mathbb{R}^+\times\Omega$.
To obtain the above integral inequality one would be tempted to multiply both members of 2.2.1 to u and then integrate over an appropiate energy set. However a rigorous manner to do this is to choose an appropriate sequence of test functions approximating u and supported in the local energy set, that in our case will take the following form:

$$P(\rho,t)=\{(x,\tau)\in Q : |x-x_0|\le\rho,\ \rho\le\rho_0,\ \tau\in(0,t)\}$$

So we introduce the following cut-off functions

$$\varphi(x,\tau)\equiv\varphi_{n,l,\kappa,h}(x,\tau)=\zeta_n(|x-x_0|)\chi_\kappa(\tau)\frac{1}{h}\int_\tau^{\tau+h}T_l(u(x,s))ds,$$

where $h\in(0,T^*-t)$, and

$$T_l(u)=\min(|u|,l)\mathrm{sign}\ u,\quad l\in\mathbb{N},$$

$$\chi_\kappa(\tau)=\begin{cases}1 & if\quad \tau\in[0,\tau-\frac{1}{\kappa}],\\ \kappa(t-\tau) & if\ \tau\in[\tau-\frac{1}{\kappa},t],\\ 0 & if\ \tau\in[t,T^*],\kappa\in\mathbb{N},\end{cases}\tag{2.2.16}$$

$$\zeta_n(r)=\begin{cases}1 & if\ r\in[0,\rho-\frac{1}{n}],\\ n(\rho-r) & if\ r\in[\rho-\frac{1}{n},\rho],\\ 0 & if\ r\in[\rho,\rho_0],n\in\mathbb{N}.\end{cases}\tag{2.2.17}$$

The functions $\varphi(x,\tau)$ are admissible test functions and their supports coincide with the energy set.
Substituting φ into 2.2.9, we get

$$\begin{aligned}&\int_0^T\int_{B_{\rho_0}(x_0)}(A(x,\tau,u,Du)\cdot D\varphi+B(x,t,u,Du)\varphi+C(x,t,u)\varphi)dxd\tau\\ &=\int_0^T\int_{B_{\rho_0}(x_0)}\psi(u(x,\tau))\frac{\partial\varphi}{\partial\tau}dxd\tau+\int_{B_{\rho_0}(x_0)}\psi(u(x,0))\varphi(x,0)dx.\end{aligned}\tag{2.2.18}$$

Now we proceed with passing to the limit in the above integral relation.
First of all we compute the term with the time derivative by splitting the integral domain:

$$\lim_{k\to\infty}\int_0^T\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\frac{\partial\varphi}{\partial\tau}dxd\tau$$

$$=\lim_{k\to\infty}\int_0^{t-\frac{1}{k}}\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\frac{\partial}{\partial\tau}\left(\zeta_n(|x-x_0|)\frac{1}{h}\int_\tau^{\tau+h}T_l(u(x,s))ds\right)dxd\tau$$

$$+\lim_{k\to\infty}\int_{t-\frac{1}{k}}^{t}\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\frac{\partial}{\partial\tau}\left(\zeta_n(|x-x_0|)k(t-\tau)\frac{1}{h}\int_\tau^{\tau+h}T_l(u(x,s))ds\right)dxd\tau.$$

and so sending $\kappa\to\infty$, the first term becomes :

$$\int_0^t\int_{B_{\rho_0}(x_0)}\psi(u(x,\tau))\frac{\partial\tilde{\varphi}}{\partial\tau}dxd\tau$$

where

$$\tilde{\varphi}=\tilde{\varphi}_{n,l,h}(x,\tau)=\zeta_n(|x-x_0|)\frac{1}{h}\int_\tau^{\tau+h}T_l(u(x,s))ds.$$

while for the second one, it is sufficient to observe that the function

$$\tau\to\psi(u(\tau,.))\frac{1}{h}\int_\tau^{\tau+h}T_l(u(s,.))ds$$

belongs to $L^\infty(0,T^*,L^1(B_{\rho_0}(x_0)))$, so we can suppose that t is one of its Lebesgue points. Hence computing the time derivative we have:

$$\lim_{k\to\infty}\int_{t-\frac{1}{k}}^{t}\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\frac{\partial}{\partial\tau}\left(\zeta_n(|x-x_0|)k(t-\tau)\frac{1}{h}\int_\tau^{\tau+h}T_l(u(x,s))ds\right)dxd\tau$$
$$=-\lim_{k\to\infty}\int_{t-\frac{1}{k}}^{t}\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))k\zeta_n(|x-x_0|)\frac{1}{h}\int_\tau^{\tau+h}T_l(u)dsdxd\tau$$
$$+\lim_{k\to\infty}\int_{t-\frac{1}{k}}^{t}\int_{B_{\rho_0}(x_0)}\psi(u)\zeta_n(|x-x_0|)k(t-\tau)\frac{1}{h}(T_l(u(\tau+h))-T_l(u(\tau))dxd\tau$$

so passing to the limit and using the property of Lebesgue points we get

$$-\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\tilde{\varphi}(t,x)dx\ +\ 0$$

due to the fact that $k(t-\tau)=0$ for $\tau=t$. Hence gathering all we have:

$$\begin{aligned}&\lim_{k\to\infty}\int_0^T\int_{B_{\rho_0}(x_0)}\psi(u(\tau,x))\frac{\partial\varphi}{\partial\tau}dxd\tau\\&=\int_0^t\int_{B_{\rho_0}(x_0)}\psi(u(x,\tau))\frac{\partial\tilde{\varphi}}{\partial\tau}dxd\tau\ -\int_{B_{\rho_0}(x_0)}\psi(u(t,x))\tilde{\varphi}(t,x)dx\end{aligned}$$

So (2.2.18) becomes:

$$\begin{aligned}&\int_0^t\int_{B_\rho(x_0)}(A(x,\tau,u,Du)\cdot D\tilde{\varphi}+B(x,t,u,Du)\tilde{\varphi}+C(x,t,u)\tilde{\varphi})dxd\tau\\&=\int_0^t\int_{B_{\rho_0}(x_0)}\psi(u(x,\tau))\frac{\partial\tilde{\varphi}}{\partial\tau}dxd\tau-\left[\int_{B_\rho(x_0)}\psi(u(\tau,x))\tilde{\varphi}(\tau,x)dx\right]_{\tau=0}^{\tau=t}.\end{aligned}$$

Let us remember now that we have defined the function $j(u)=\int_0^u\psi(x,\tau)d\tau$ Since j is convex and increasing in $\mathbb{R}^+$(decreasing in $\mathbb{R}^-$), Hence it is straightforward to verify , using the definition of T_l, that

$$\begin{aligned}&j'(u)(T_l(u(x,\tau+h))-T_l(u(x,\tau))=\psi(u(x,\tau))(T_l(u(x,\tau+h))-T_l(u(x,\tau))\\&\le j(T_l(u(x,\tau+h)))-j(T_l(u(x,\tau))).\end{aligned}\tag{2.2.19}$$

So computing $\frac{\partial\tilde{\varphi}}{\partial\tau}$ and using (2.2.19) we have:

$$\begin{aligned}&\int_0^t\int_{B_{\rho_0}}\psi(u(x,\tau))\frac{\partial\tilde{\varphi}}{\partial\tau}dxd\tau\\&\le\frac{1}{h}\int_t^{t+h}\int_{B_{\rho_0}(x_0)}j(T_lu)\zeta_ndxd\tau-\frac{1}{h}\int_0^h\int_{B_{\rho_o}(x_0)}j(T_lu)\zeta_ndxd\tau\end{aligned}\tag{2.2.20}$$

where $\zeta_n(\cdot)=\zeta(|\cdot-x_0|)$. Next, using the following hypothesis on j:

$$j(u(\cdot,t))\in L^\infty(0,T;L^1(\Omega)),\ \text{and}\ \ \liminf_{t\to0}j(u(\cdot,t))=j(u_0(\cdot))\in L^1(\Omega),$$

the right-hand side of (2.2.20) converges to

$$\int_{B_{\rho_0}(x_0)}j(T_lu(t,x))\zeta_n(x)dx-\int_{B_{\rho_0}(x_0)}j(T_lu_0(x))\zeta_n(x)dx,\tag{2.2.21}$$

as h goes to 0, for almost all t. If we set

$$\varphi_{n,l}(x,\tau)=\zeta_n(|x-x_0|)T_l(u(x,\tau))\tag{2.2.22}$$

we deduce with Lebesgue and Fatou's theorems (as $h \to 0$)

$$\begin{aligned}&\int_0^t \int_{B_{\rho_0}(x_0)} (A(x,\tau,u,Du)\cdot D\varphi_{n,l} + B(x,\tau,u,Du)\varphi_{n,l} + C(x,\tau,u)\varphi_{n,l} dx d\tau \\ &\leq \left[\int_{B_{\rho_0}(x_0)} (j(T_l(u(x,\tau))) - \psi(u(x,\tau))T_l(u(x,\tau)))\zeta_n(|x-x_0|)dx \right]_{\tau=0}^{\tau=t}.\end{aligned} \tag{2.2.23}$$

So computing $D\varphi_n$, from hypotheses as l goes to $+\infty$ (2.2.23) converges to

$$\begin{aligned}&\int_0^t \int_{B_{\rho_0}(x_0)} \zeta_n\{A(x,\tau,u,Du) + B(x,\tau,u) + C(x,\tau,u)u\} dx d\tau \leq \\ &\leq -\left[\int_{B_{\rho_0}(x_0)} G(x,u(x,\tau))\zeta_n(|x-x_0|)dx \right]_{\tau-0}^{\tau=t} - \int_0^t \int_{B_\rho(x_0)} uA(x,\tau,u,Du)\cdot D\zeta_n dx d\tau\end{aligned}$$

but

$$-\int_0^t \int_{B_{\rho_0}} uA(x,\tau,u,Du)\cdot D\zeta_n dx d\tau \equiv n \int_0^t \int_{\rho-\frac{1}{n}<|x-x_0|<\rho} uA(x,\tau,u,Du)\cdot \frac{x-x_0}{|x-x_0|} dx d\tau.$$

Using spherical cordinates (r,ω) with center x_0, and computing $D\zeta_n$ we have

$$\begin{aligned}&n\int_0^t \int_{\rho-\frac{1}{n}<|x-x_0|<\rho} uA(x,\tau,u,Du)\frac{x-x_0}{|x-x_0|} dx d\tau \\ &= n\int_0^t \int_{\rho-\frac{1}{n}}^{\rho} \int_{S^{N-1}} uA(x,\tau,u,Du)\cdot \nu r^{N-1} d\omega dr d\tau\end{aligned}$$

where $x = r\omega$. Computing $D\zeta_n$ and using differentiation under the integral sign we have that for almost all $\rho \in (0,\rho_0)$

$$\begin{aligned}&\lim_{n\to\infty} -\int_0^t \int_{B_\rho(x_0)} uA(x,\tau,u,Du)\cdot D\zeta_n dx d\tau = \\ &\int_0^t \int_{S_\rho(x_0)} uA(x,\tau,u,Du)\nu ds d\tau\end{aligned} \tag{2.2.24}$$

and finally we have:

$$\begin{aligned}&\int_0^t \int_{S_\rho(x_0)} (A(x\tau,u,Du)\cdot Du + B(x,\tau,u,Du)u + C(x,u,Du)u) dx d\tau \\ &\leq \int_0^t \int_{S_\rho(x_0)} uA(x,\tau,u,Du)\cdot \nu ds d\tau + \left[\int_{B_\rho(x_0)} G(x,u(x,\tau))dx \right]_{\tau=t}^{\tau=0},\end{aligned} \tag{2.2.25}$$

that is (2.2.15). □

Let us proceed now to derive from the *energy relation* (2.2.15) an ordinary differential inequality for the energy function E.
The way we do it is to provide some estimation to every term of the energy inequality just proved.
Fix some $T, 0 < t \leq T < T^*$; we firstly observe:

$$\int_{B_\rho(x_0)} G(x, u(x,0))dx = 0 \quad for\ \rho \leq \rho_0,$$

and

$$C_5 b(\rho, t) \leq \int_{B_\rho(x_0)} G(x, u(x,t))dx; \tag{2.2.26}$$

such relations follow from the hypothesis on the vanishing set of the initial data $u_0(x)$, and from (2.2.7)
Now we apply (2.2.5) and Young's inequality to obtain

$$|u|^{\alpha+1}|Du|^{\beta} \leq \frac{\epsilon^{\tau}}{\tau}|u|^{\tau(\alpha+1)} + \frac{(\tau-1)}{\tau}\epsilon^{-\frac{\tau}{\tau-1}}|Du|^{\frac{\beta\tau}{(\tau-1)}}.$$

If we choose $\tau = \frac{\gamma+1}{\alpha+1}$ then $\frac{\beta\tau}{(\tau-1)} = p$ (this is a consequence of having chosen in the hypotheses $\alpha = \gamma - \frac{\beta(1+\gamma)}{p}$) and rewrite $\tau = \frac{p}{p-\beta}$ and $\frac{\tau}{\tau-1} = \frac{p}{\beta}$, we have:

$$\begin{aligned}
&\left|\int_0^t \int_{B_\rho(x_0)} B(x,\tau,u,Du)dxd\tau\right| \leq C_3 \int_0^t \int_{B_\rho(x_0)} |u|^{\alpha+1}|Du|^{\beta}dxd\tau \\
&\leq C_3\frac{(p-\beta)}{p}\epsilon^{\frac{p}{p-\beta}}\int_0^t \int_{B_\rho(x_0)} |u|^{\gamma+1}dxd\tau + \frac{\beta}{p}C_3\epsilon^{-\frac{p}{\beta}}\int_0^t \int_{B_\rho(x_0)} |Du|^{p}dxd\tau \\
&\leq C_3\frac{(p-\beta)}{p}\epsilon^{\frac{p}{p-\beta}}t\bar{b}(\rho,t) + \frac{\beta C_3}{pC_2}\epsilon^{-\frac{p}{\beta}}E(\rho,t),
\end{aligned} \tag{2.2.27}$$

From Holder inequality and (2.2.15) we have:

$$\begin{aligned}
|I| &\leq \int_0^t \int_{S_\rho(x_0)} |u||A|dSd\tau \\
&\leq C_1\left(\int_0^t \int_{S\rho(x_0)} |Du|^p dsd\tau\right)^{\frac{(p-1)}{p}} \left(\int_0^t \int_{S_\rho(x_0)} |u|^p dSd\tau\right)^{\frac{1}{p}} \\
&\leq C_1 C_2^{-\frac{(p-1)}{p}} \left(\frac{\partial E}{\partial\rho}\right)^{\frac{(p-1)}{p}} \left(\int_0^t \int_{S_\rho(x_0)} |u|^p dSd\tau\right)^{\frac{1}{p}}
\end{aligned} \tag{2.2.28}$$

Gathering (2.2.26),(2.2.28) we come to the inequality:

$$
\begin{aligned}
&C_5 b(\rho,t)+E(\rho,t)+C_4 C(\rho,t)\\
&\le \epsilon^{\frac{p}{p-\beta}} C_3 \frac{(p-\beta)}{p} t\bar{b}(\rho,t)+\frac{\beta C_3}{pC_2}\epsilon^{-\frac{p}{\beta}}E(\rho,t)\\
&+C_1 C_2^{-\frac{(p-1)}{p}}\left(\frac{\partial E}{\partial \rho}\right)^{\frac{(p-1)}{p}}\left(\int_0^t\int_{S_\rho}|u|^p dS d\tau\right)^{\frac{1}{p}}.
\end{aligned}
\tag{2.2.29}
$$

Since the right-hand side of the last inequality is nondecreasing in t, we can replace $b(\rho,t)$ by $\bar{b}(\rho,t)$ and C_5 by $\frac{C_5}{2}$.
Since $t<T^*$, one may take in (2.2.27) ϵ so that

$$
\epsilon^{\frac{p}{p-\beta}} C_3 t\frac{p-\beta}{p}<\frac{C_5}{2} \quad and \quad \frac{\beta C_3}{pC_2}\epsilon^{-\frac{p}{\beta}}<1. \tag{2.2.30}
$$

If we set

$$
K=\min\left(\frac{C_5}{2}-\epsilon^{\frac{p}{p-\beta}} C_3 t\frac{p-\beta}{p},\quad 1-\frac{\beta C_3}{pC_2}\epsilon^{-\frac{p}{\beta}}\right),
$$

(2.2.29) yelds the inequality

$$
\bar{b}(\rho,t)+E(\rho,t)\le K_1\left(\frac{\partial E}{\partial \rho}\right)^{\frac{(p-1)}{p}}\left(\int_0^t\int_{S_\rho}|u|^p dS d\tau\right)^{\frac{1}{p}} \tag{2.2.31}
$$

with $K_1=\frac{C_1}{K}C_2^{-\frac{(p-1)}{p}}$. To estimate the integral on the right-hand side we proceedy by utilizing the *interpolation-trace* inequality just introduced in the first chapter, [see app. C]:
if $\gamma\le p-1$ we have

$$
\|u\|_{L^p(S_\rho)}\le C(p,q)(\|Du\|_{L^p(B_\rho)}+\rho^\delta\|u\|_{L^q(B_\rho)})^\theta(\|u\|_{L^q(B_\rho)})^{1-\theta} \tag{2.2.32}
$$

with the exponents

$$
\theta=\frac{N(p-q)+q}{N(p-q)+pq},\quad \delta=-\frac{N(p-q)+qp}{pq},\quad q=1+\gamma.
$$

Let us raise both sides of (2.2.32) to the power p and then integrate over the interval $\tau \in (0,t)$. Applying Holder's inequality with $p = \frac{1}{\theta}$ we have that

$$\int_0^t \int_{S_\rho(x_0)} |u|^p dS d\tau$$
$$\leq C^p \int_0^t \left[\left(\|Du\|_{L^p(B_\rho(x_0))} + \rho^\delta \|u\|_{L^{\gamma+1}(B_\rho(x_0))} \right)^{\theta p} \left(\|u\|_{L^{\gamma+1}(B_\rho(x_0))} \right)^{(1-\theta)p} \right] d\tau$$

$$\leq C^p \left[\int_0^t \left(\|Du\|_{L^p(B_\rho(x_0))} + \rho^\delta \|u\|_{L^{\gamma+1}(B_\rho(x_0))} \right)^p d\tau \right]^\theta \times \left[\int_0^t \left(\|u\|_{L^{\gamma+1}(B_\rho(x_0))} \right)^p d\tau \right]^{(1-\theta)}$$

$$\leq C^p \left[\int_0^t \left(\|Du\|_{L^p(B_\rho(x_0))} + \rho^\delta \|u\|_{L^{\gamma+1}(B_\rho(x_0))} \right)^p d\tau \right]^\theta \times \bar{b}^{\frac{p(1-\theta)}{\gamma+1}} t^{(1-\theta)}$$

Now making use of the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ we can write:

$$\int_0^t \int_{S_\rho} |u|^p dS d\tau \leq C^p 2^{p\theta} \bar{b}^{\frac{p(1-\theta)}{\gamma+1}} t^{1-\theta} \left[\int_0^t \left(\int_{B_\rho} |Du|^p dx + \rho^{\delta p} \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{\frac{p}{\gamma+1}} \right) d\tau \right]^\theta$$

and raising both sides to the exponent $\frac{1}{p}$ we obtain:

$$\left(\int_0^t \int_{S_\rho} |u|^p d\Gamma d\tau \right)^{\frac{1}{p}} \leq C 2^\theta \bar{b}^{\frac{(1-\theta)}{1+\gamma}} t^{\frac{(1-\theta)}{p}} \left[\int_0^t \left(\int_{B_\rho} |Du|^p dx \right) + \rho^{\delta p} \int_0^t b(\tau)^{\frac{p}{\gamma+1}} d\tau \right]^{\frac{\theta}{p}}$$

and finally

$$\left(\int_0^t \int_{S_\rho} |u|^p dS d\tau \right)^{\frac{1}{p}} \leq 2C t^{\frac{(1-\theta)}{p}} \bar{b}^{\frac{(1-\theta)}{(1+\gamma)}} \left(\|Du\|_{L^p((0,t)\times(B_\rho))} + \rho^\delta t^{\frac{1}{p}} \bar{b}^{\frac{1}{1+\gamma}} \right)^\theta \tag{2.2.33}$$

Using (2.2.11), from (2.2.31) and (2.2.33) we obtain

$$\bar{b}(\rho,t) + E(\rho,t) \leq K_2 \left(\frac{\partial E}{\partial \rho} \right)^{\frac{p-1}{p}} t^{\frac{(1-\theta)}{p}} \bar{b}^{\frac{(1-\theta)}{(1+\gamma)}} \left(E^{\frac{1}{p}} + \rho^\delta t^{\frac{1}{p}} \bar{b}^{\frac{1}{(1+\gamma)}} \right)^\theta \tag{2.2.34}$$

with a constant $K_2 = K_2(K_1, C, p, q)$. Like in the stationary case, we will make use of the following identity, which is valid for every $\tau \in (0,1)$, and $\kappa = \frac{1}{\theta(1+\gamma)}$:

$$E^{\frac{1}{p}} \bar{b}^{(1-\theta)\kappa} + \rho^\delta t^{\frac{1}{p}} \bar{b}^\kappa = E^{\frac{1}{p}} \bar{b}^{\tau(1-\theta)\kappa} \bar{b}^{(1-\tau)(1-\theta)\kappa} + \rho^\delta t^{\frac{1}{p}} \bar{b}^{\frac{1}{p}+\tau(1-\theta)\kappa} \bar{b}^{\kappa-\frac{1}{p}-\tau(1-\theta)\kappa}.$$

Applying Young inequality we obtain:

$$\begin{aligned}
&E^{\frac{1}{p}}\bar{b}^{(1-\theta)\kappa} + \rho^{\delta} t^{\frac{1}{p}} \bar{b}^{\kappa} \\
&\leq 2\rho^{\delta} K_0^{\frac{1}{\theta}} \max(1, \delta_0^{-\delta})(E(t,\rho) + b(t,\rho))^{\frac{1}{p}+\frac{\tau(1-\theta)}{\gamma+1}}
\end{aligned}$$

where the constant K_0 stands for :

$$K_0 = \max(1, T^{\frac{\theta}{p}}) \max\left(b(t,\rho_0)^{\frac{(1-\tau)(1-\theta)}{\gamma+1}}, b(t,\rho_0)^{\frac{(1-\tau)(1-\theta)}{\gamma+1}-\frac{\theta}{p}}\right).$$

So we have

$$\bar{b}(t,\rho)^{\frac{(1-\theta)}{(1+\gamma)}} \left(E^{\frac{1}{p}}(t,\rho) + \rho^{\delta} t^{\frac{1}{p}} \bar{b}(t,\rho)^{\frac{1}{(1+\gamma)}}\right)^{\theta} \leq K_3 \rho^{\delta\theta} \left(\bar{b} + E\right)^{\frac{\theta}{p}+\frac{\tau(1-\theta)}{(1+\gamma)}} \tag{2.2.35}$$

with the constant K_3 given by the formula

$$K_3 = 2K_0^{\frac{1}{\theta}} \max(1, \rho_0^{-\delta}).$$

Then, gathering the above expression with (2.2.34) we are allowed to write

$$\bar{b}(t,\rho) + E(t,\rho) \leq K_2 K_3 t^{\frac{(1-\theta)}{p}} \rho^{\delta\theta} \left(\frac{\partial E(t,\rho)}{\partial \rho}\right)^{\frac{(p-1)}{p}} \left(\bar{b}(t,\rho) + E(t,\rho)\right)^{\frac{\theta}{p}+\frac{\tau(1-\theta)}{(1+\gamma)}}. \tag{2.2.36}$$

Now, putting $\mu = \frac{\theta}{p} + \frac{\tau(1-\theta)}{\gamma+1}$, we observe that if τ is in $(\frac{\gamma+1}{p}, 1)$ and supposing $\gamma < p-1$ (that is the slow diffusion condition), we have $\mu < 1$ and so we can apply Young inequality to the (2.2.36) with the exponent $\frac{1}{\mu}$, and we can rewrite the above expression in the form:

$$E(t,\rho) + \bar{b}(t,\rho) \leq C_1(\epsilon)\left(E(t,\rho) + \bar{b}(t,\rho)\right) + \left(K_2 K_3 t^{\frac{(1-\theta)}{p}} \rho^{\delta\theta}\right)^{\frac{1}{1-\mu}} \left(\frac{\partial E(t,\rho)}{\partial \rho}\right)^{\frac{p-1}{p(1-\mu)}}$$

and then, by recalling all the constants

$$E(t,\rho) + \bar{b}(t,\rho) \leq \tilde{K} \left(t^{\frac{(1-\theta)}{p}} \rho^{\delta\theta}\right)^{\frac{1}{1-\mu}} \left(\frac{\partial E(t,\rho)}{\partial \rho}\right)^{\frac{p-1}{p(1-\mu)}}$$

Finally raising both sides of the last inequality to the power $\frac{p(1-\mu)}{(p-1)}$ we transform (2.2.36) into

$$\left(\bar{b} + E\right)^{\frac{p(1-\mu)}{p-1}} \leq K t^{\frac{1-\theta}{p-1}} \rho^{\frac{\delta\theta p}{p-1}} \frac{\partial E(t,\rho)}{\partial \rho}, \tag{2.2.37}$$

i.e.

$$E^{1-\epsilon} \leq (\bar{b}+E)^{1-\epsilon} \leq Kt^{\lambda}\rho^{1-\nu}\frac{\partial E}{\partial \rho} \tag{2.2.38}$$

with the exponents

$$\epsilon = 1 - \frac{p}{p-1}\left(1-\frac{\theta}{p}-\frac{\tau(1-\theta)}{1+\gamma}\right), \quad \lambda = \frac{1-\theta}{p-1}, \quad \nu = 1-\frac{\delta\theta p}{p-1},$$

$$\mu = \frac{(1-\tau)(1-\theta)}{\gamma+1}, \quad \eta = \frac{(1-\tau)(1-\theta)}{\gamma+1} - \theta p,$$

that coincide with the constants defined in (2.2.14) if we compute then by substituting the values of θ expressed in (2.2.16). Integrating (2.2.38) on the interval (ρ, ρ_0), and considering t like a parameter we come to the estimate for the energy function $E(\rho, t)$

$$E^{\epsilon}(\rho, t) \leq E^{\epsilon}(\rho_0, t) - \frac{\epsilon}{\nu}\left(Kt^{\lambda})^{-1}(\rho_0^{\nu}-\rho^{\nu}\right), \tag{2.2.39}$$

Now recalling the hypothesis of boundedness of the total energy $D(u, \Omega, T)$, if we put $D(u, \Omega, T) < D^*$, we have:

$$E^{\epsilon}(\rho, t) \leq D^{*\epsilon} - \frac{\epsilon}{\nu}\left(Kt^{\lambda})^{-1}(\rho_0^{\nu}-\rho^{\nu}\right), \tag{2.2.40}$$

Now as in the example considered in the heuristic approach, we can conclude, since $E(\rho, t)$ is a nondecreasing function of ρ, that $E(\rho, t) = 0$ for all $\rho \in (0, \rho(t))$.
Then from (2.2.34) $b(\rho, t) = 0$ for all $\rho \in (0, \rho(t))$ too, where the function $\rho(t)$ takes the form

$$\rho^{\nu}(t) = \rho_0^{\nu} - \frac{\nu}{\epsilon}Kt^{\lambda}D*^{\epsilon}(\rho_0, t). \tag{2.2.41}$$

Since ν is independent of τ, we can take the minimum in τ of the right-hand side of (2.2.41). and so obtain (2.2.14). Finally we conclude that

$$u(x, t) = 0 \text{ for a.a. } x \in B_{\rho(t)}$$

and

$$\rho(t) > 0 \text{ for } 0 < t < T^* = \left(\min_{\frac{\gamma+1}{p}\leq\tau\leq 1}\left(\frac{\epsilon}{\nu K D^{*\epsilon}}\right)\right)^{\frac{1}{\lambda}} \rho_0^{\frac{\nu}{\lambda}}.$$

Notice that 2.2.41 gives us the evolution of the zero-set at the zero istant $B_{\rho_0}(x_0)$ for successive values of t. This completes the proof. □

2.3 The waiting time effect

In this section we study the generalized waiting time effect in weak solutions to equation (2.2.1) understood in the sense of Definition 2.1.2.

We again assume that conditions (2.2.2)-(2.2.7), (1)-(3) are fulfilled and that the energy functions possess properties (2.2.10), (2.2.11). Let

$$u_0(x) \equiv 0 \quad x \in B_{\rho_0}(x_0) \quad \text{for some } x_0 \in \Omega \text{ and } \rho_0 > 0 \tag{2.3.1}$$

and

$$f(x,t) \equiv 0 \quad (x,t) \in P(T, \rho_0) = B_{\rho_0}(x_0) \times (0,T). \tag{2.3.2}$$

It is assumed that the data $u_0(x)$ and $f(x,t)$ are sufficiently "flat" near the boundaries of their supports. These conditions are formulated as follows: for

$$F \equiv F(u_0, f, \rho) = \left(\|u_0\|^{\gamma+1}_{L^{1+\gamma}(B_\rho(x_0))} + \|f(\cdot,t)\|^{\frac{q}{q-1}}_{L^{q/q-1}(B_\rho(x_0))} \right) \tag{2.3.3}$$

we put

$$I(\rho) := \int_{\rho_0}^{\rho} (s-\rho_0)^{-\frac{1}{1-\nu}} F^{\nu} ds < \infty, \quad B_{\rho_0} \subset B_{\rho_*},\, \overline{B}_{\rho_*} \subset \Omega \tag{2.3.4}$$

where

$$\begin{cases} \theta = \dfrac{N(p-\gamma-1)+\gamma+1}{N(p-\gamma-1)+p(\gamma+1)} \in (0,1) \\ \mu = \dfrac{\theta}{p} + \dfrac{\tau(1-\theta)}{1+\gamma} & \text{if} \quad C_4 = 0 \\ q = 1+\gamma, \\ \nu = \dfrac{p(1-\mu)}{p-1} \end{cases}$$

and

$$q = 1+\sigma, \qquad \text{if} \quad C_4 > 0.$$

We assume also that $\nu \in (0,1)$, i.e. $p < \frac{1}{\mu}$, and the global energy of the solution $u(x,t)$,

$$D(u,\, \Omega,\, T) = \operatorname*{ess\,sup}_{0 \le \tau \le t} \int_{\Omega} |u(\cdot,t)|^{1+\gamma} dx + \int_0^T \int_{\Omega} \left(|Du(x,t)|^p + C_4 |u(x,t)|^{1+\sigma} \right) dxdt,$$

finite.

Our goal now is to proof the following :

Theorem 2.3.1 (The waiting time property). *Let $u(x,t)$ be a weak solution of problem* (2.2.1), (2.2.8) *and* u_0, f *satisfy* (2.3.1), (2.3.2) *and* (2.3.4). *Let the global energy be bounded: $D(u,\Omega,T) \leq D_0 < \infty$. Then under the conditions of Theorem* 2.2.1, *there exists a positive constant $t^* \leq T^*$ depending only on the constants in* (2.2.2) – (2.2.7), N, ρ_0, $dist(x_0, \partial\Omega)$ *and* $D(u,\Omega,T)$, *such that*

$$u(x,t) = 0 \quad in \quad B_{\rho_0}(x_0) \times (0,t^*);$$

Proof. To prove the theorem we will make use of the local energy method like in the previous case but with a substantial difference: our task now is to prove that the local energy function starts at zero on a cylinder based in B_{ρ_0} and with heigth t^*; to make this we will localize now the weak formulation of problem 2.2.1 on different "local energy sets". Properly we need to deal with the following class of cylinders:

$$P(\rho,t) = \{(x,\tau) \in Q : |x - x_0| \leq \rho \ : \ \rho \geq \rho_0; \ \tau \in (0,t), \ t \in (0,T)\}$$

Proceeding similarly to the previous case we start with the analogous of the integral inequality proved in lemma 2.2.1 :

$$\int_0^t \int_{B_\rho(x_0)} (A(x,\tau,u,Du) + B(x,\tau,u,Du)u + C(x,u,Du)u)dxd\tau$$
$$\leq \int_0^t \int_{S_\rho(x_0)} uA(x,\tau,u,Du) \cdot \nu dsd\tau$$
$$+ \left[\int_{B_\rho(x_0)} G(x,u(x,\tau))dx \right]_{\tau=t}^{\tau=0} + \int_0^t \int_{B_\rho(x_0)} fudxd\tau$$

We want to manipulate it providing upper estimation of its term.
The same argument used in the previous section works to obtain upper estimation of the terms in the above inequality; the only difference arise for the last three terms.
So let us start to provide a suitable estimation for

$$\left| \int_0^t \int_{S_\rho(x_0)} uA(x,\tau,u,Du) \cdot \nu dsd\tau \right|$$

Since we are working now on "cylinders domains" with radius $\rho > \rho_0$, proceeding as in the previous case we are led to write:

$$\left| \int_0^t \int_{S_\rho(x_0)} uA(x,\tau,u,Du) \cdot \nu dsd\tau \right|$$
$$\leq Lt^{\frac{1-\theta}{p}} \rho_0^{\delta\theta} \left(\frac{\partial E}{\partial \rho} \right)^{\frac{p-1}{p}} (\bar{b} + E)^{\frac{\theta}{p} + \frac{\tau(1-\theta)}{(1+\gamma)}} \tag{2.3.5}$$

Here δ, θ, τ are defined respectively in (2.2.32), and (2.2.1), and the constant L is determined by calculation similar to the previous section. Note that unlike to the previous case there is now no dipendence from the ρ variable. Now proceed with estimate of the following therms:

$$\left|\int_0^t \int_{B_\rho} fudxd\tau\right|$$

We will distinguish two cases: case a) when $C_4 = 0$, and case b) when $C_4 > 0$. In the first we proceed with the following estimate via Holder and Young inequality both with exponent $p = \gamma + 1$:

$$\begin{aligned}
&\left|\int_0^t \int_{B_\rho} fudxd\tau\right| \\
&\leq \left(\int_0^t \int_{B_\rho} |u|^{\gamma+1} dxd\tau\right)^{\frac{1}{\gamma+1}} \times \left(\int_0^t \int_{B_\rho} |f|^{\frac{\gamma+1}{\gamma}} dxd\tau\right)^{\frac{\gamma}{\gamma+}} \\
&\leq \frac{\delta^{\gamma+1}}{\gamma+1} \int_0^t \int_{B_\rho} |u|^{\gamma+1} dxd\tau + \frac{\delta^{-\frac{\gamma+1}{\gamma}}\gamma}{\gamma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\gamma+1}{\gamma}} dxd\tau \\
&\leq \frac{\delta^{\gamma+1}}{\gamma+1} t\bar{b}(\rho) + \frac{\delta^{-\frac{\gamma+1}{\gamma}}\gamma}{\gamma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\gamma+1}{\gamma}} dxd\tau
\end{aligned} \tag{2.3.6}$$

where $\delta > 0$. Gathering (2.3.5) and (2.3.6) for each weak solution of problem (2.2.1), (2.2.8) the following inequality holds:

$$\begin{aligned}
&i_1 + i_2 + i_3 = \\
&C_5 b(\rho, t) + E(\rho, t) + C_4(\rho, t) \\
&\leq \epsilon^{\frac{p}{p-\beta}} C_3 \frac{(p-\beta)}{p} t\bar{b}(\rho, t) + \frac{\beta C_3}{pC_2} \epsilon^{-\frac{p}{\beta}} E(\rho, t) \\
&+ Lt^{\frac{1-\theta}{p}} \rho_0^{\delta\theta} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{p-1}{p}} (\bar{b} + E)^{\frac{\theta}{p} + \frac{\tau(1-\theta)}{1+\gamma}} + \left|\int_{B_\rho} |u_0|^{\gamma+1} dx\right| \\
&+ \frac{\delta^{\gamma+1}}{\gamma+1} t\bar{b}(\rho) + \frac{\delta^{-\frac{\gamma+1}{\gamma}}\gamma}{\gamma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\gamma+1}{\gamma}} dxd\tau + C_6 \|u_0\|^{\gamma+1}_{L^{\gamma+1}(B_\rho(x_0)} \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}$$

So with a suitable choice of parameter ϵ, δ, and replacing b by $\bar{b}$, in 2.3.7 which is always possible since $b(t)$ is non decreasing, and $t < T$, we obtain the inequality:

$$\bar{b} + E \le K_1 t^{\frac{(1-\theta)}{p}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{p-1}{p}} (\bar{b}+E)^{\frac{\theta}{p}+\frac{\tau(1-\theta)}{1+\gamma}} + K_2 F(\rho)$$

where we use the fact that

$$\frac{\delta^{\gamma+1}}{\gamma+1} t\bar{b}(\rho) \le \frac{\delta^{\gamma+1}}{\gamma+1} T\bar{b}(\rho)$$

and the coefficents K_1,K_2 are suitably calculated like in the above section.

Now putting $\mu = \frac{\theta}{p} + \frac{\tau(1-\theta)}{1+\gamma} < 1$, we make use of Young's inequality with $p = \frac{1}{\mu}$ and this leads to the following inequality:

$$\bar{b} + E$$

$$\le \mu \epsilon^{\frac{1}{\mu}} (\bar{b}+E) + (1-\mu)\epsilon^{\frac{1}{\mu-1}} \left[K_1 t^{\frac{1-\theta}{p}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{p-1}{p}} \right]^{\frac{1}{1-\mu}} + K_2 F(\rho)$$

Another time we make a suitable choice of parameter $\epsilon > 0$ and rising both sides to the power $\nu = \frac{p(1-\mu)}{p-1}$ we come to the final inequality:

$$E^{\nu} \le (\bar{b}+E)^{\nu} \le \tilde{K}_1 t^{\frac{1-\theta}{p-1}} \frac{\partial E}{\partial \rho} + \tilde{K}_2 F(\rho)^{\nu} \tag{2.3.7}$$

with $\rho \ge \rho_0$.

Now let condition (2.3.4) holds with $q = 1 + \gamma$. Then we use Lemma B.0.2, whence

$$\Lambda = \tilde{K}_1(t)^{(1-\theta)/(p-1)} \equiv \Lambda_0(t)^{(1-\theta)/(p-1)}, \quad F = \tilde{K}_2 F(\rho)^{\nu}$$

$$\rho_0 = R_0, \quad \rho_1 = R$$

and

$$\mu = 1 - \nu, \quad \Gamma(\rho) = I(\rho)$$

for an arbitry value of $t > 0$. The energy function $E(\rho, t)$ obeys the estimate

$$E(\rho_0, t) \le G(s) \equiv E(\rho_1, t) - (s-\rho_0)^{\frac{1}{1-\nu}} (t)^{\frac{1-\theta}{(1-p)(1-\nu)}} \left(\left(\frac{1-\nu}{\Lambda_0}\right)^{\frac{1}{1-\nu}} - \frac{I(s)t^{1-\nu}}{\Lambda_0} \right), \tag{2.3.8}$$

with $s \in (\rho_0, \rho_1)$, $0 < \theta < 1$, $0 < \nu < 1$, and $I(s)$ like in (2.3.4).
Let us now show that there exist $t^* > 0$, such that for some s^*, $G(s_*) = 0$: it is sufficient to observe that $G(\rho_0) > 0$ while

$$G(s^*) \to -\infty$$

as $t \to 0$. Correspondingly, we have that there exists t^* sufficently small, such that

$$G(s^*) = 0$$

for some $s^* \in (\rho_0, \rho_1)$, and so we have

$$E(\rho_0, t^*) = 0$$

. This completes the proof in the case a).

Let us pass to the case b) i.e. when $C_4 > 0$.
Now the terms I_1, I_3, on the right-hand side of (2.3.7) can be estimated in the same way, while the terms I_4 will be estimated in the following way:

$$\begin{aligned} &\left| \int_0^t \int_{B_\rho} f u dx d\tau \right| \\ &\leq \frac{\delta^{\sigma+1}}{\sigma+1} \int_0^t \int_{B_\rho} |u|^{\sigma+1} dx d\tau + \frac{\delta^{-\frac{\sigma+1}{\sigma}} \sigma}{\sigma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\sigma+1}{\sigma}} dx d\tau \\ &\leq \frac{\delta^{\sigma+1}}{\sigma+1} C(\rho, t) + \frac{\delta^{-\frac{\sigma+1}{\sigma}} \sigma}{\sigma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\sigma+1}{\sigma}} dx d\tau; \end{aligned} \tag{2.3.9}$$

this leads us to the following integral inequality:

$$\begin{aligned} & i_1 + i_2 + i_3 = \\ & C_5 b(\rho, t) + E(\rho, t) + C_4(\rho, t) \\ & \leq \epsilon^{\frac{p}{p-\beta}} C_3 \frac{(p-\beta)}{p} t \bar{b}(\rho, t) + \frac{\beta C_3}{p C_2} \epsilon^{-\frac{p}{\beta}} E(\rho, t) \\ & + L t^{\frac{1-\theta}{p}} \rho_0^{\delta\theta} \left(\frac{\partial E}{\partial \rho} \right)^{\frac{p-1}{p}} (\bar{b} + E)^{\frac{\theta}{p} + \frac{\tau(1-\theta)}{1+\gamma}} + \left| \int_{B_\rho} |u_0|^{\gamma+1} dx \right| \\ & + \frac{\delta^{\sigma+1}}{\sigma+1} C(\rho, t) + \frac{\delta^{-\frac{\sigma+1}{\sigma}} \sigma}{\sigma+1} \int_0^t \int_{B_\rho} |f|^{\frac{\sigma+1}{\sigma}} dx d\tau \\ & = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now proceeding with analogous steps to the previous case, we finally arrive to the desired differential inequality:

$$E^\nu \leq (\bar{b} + E + C)^\nu \leq M_1 t^{\frac{1-\theta}{p-1}} \frac{\partial E}{\partial \rho} + M_2 F(\rho)^\nu \tag{2.3.10}$$

and we can conclude with the same results.

□

2.4 Shrinking of supports and formation of a dead core

In this section we deal with the effects of support shrinking and a dead core formation which are intrinsic for weak solutions of problem (2.2.1), (2.2.8). These effects are due to presence in equation (2.2.1) of the term responsible for "strong absorption".
To simplify matters but without loss of generality we will deal with the case $C_3 = 0$. Infact, this kind of behaviour displayed by the solution depend on the way the inertial effect of the cumulative term $\frac{\partial\psi(u)}{\partial t}$ is contrasted by the absorption effect of the term $C(t, x, u)$.

We assume that conditions (2.2.2),(2.2.3),(2.2.5),(2.2.6),(2.2.7) are fulfilled (now with $C_4 > 0$) and, in addition,

$$\sigma < \gamma, \quad 1 + \sigma \leq \frac{\gamma p}{p-1}, \tag{2.4.1}$$

The results we obtain below may be simplified according to the following description. Let $u(x,t)$ be a local weak solution of the model equation 2.2.1

i If γ, σ, and p satisfy relation (2.4.1)(that is "strong absorption property"), the initial support shrinks, i.e., the inclusion below is strict:

$$\textit{supp}\; u(\cdot, t) \subset\subset \textit{supp}\; u(\cdot, 0) \quad \text{for } t > 0 \text{ small enough;}$$

ii under the assumptions of item (i) on the exponents, but without any assumption on the initial function, a null-set with nonempty interior i.e. *a dead core*, appears

$$\exists t^* > 0 : \quad \forall t > t^* \quad \overline{\Omega} \setminus \{\textit{supp}\; v(\cdot, t)\} \neq \emptyset.$$

To achive ouar goal we shall use the energy functions defined on domains of special form. Let us introduce the following notation:
given $T > 0$, $t \in [0, T)$, $x_0 \in \Omega$, $\rho \geq 0$, and nonnegative parameters ϑ and υ, we will deal with the following particular sets:

a)

$$P(\rho,t) \equiv P(\rho;\vartheta) = \{(x,\tau) \in Q: \ |x-x_0| < \rho(\tau) \equiv \rho + \vartheta\tau, \ ;\tau \in (0,t)\}.$$

and

b)

$$P(t) \equiv P(t,\vartheta,\upsilon) = \{(x,\tau) \in Q: \ |x-x_0| < \rho(\tau) \equiv \vartheta(\tau-t)^{\upsilon}, \ ;\tau \in (t,T)\}.$$

The shape of $P(\rho)$ and respectively $P(t)$ is determined by the choice of the parameters ϑ, υ, ρ, t.
In case a) we have a truncated cone centered at the point $x_0 \in \Omega$ with base $B_\rho(x_0) := \{x \in \Omega : |x-x_0| < \rho\}$ on the plane $s = 0$ and of height t.
In the case b), instead, we have a paraboloid with vertex in the plane $s = t$.

To simplify the notation, we will omit the arguments of P wherever possible. Treating separately the cases a), b), we specially indicate which of the parameters are essential and which are not.

Choosing $P(t)$ and $P(\rho,t)$ respectively for *the local energy sets* , we define the local energy functions associated with any local weak solutions of problem (2.2.1)

For the case of truncated cones

$$\begin{aligned} E(\rho,t) &:= \int_{P(\rho,t)} |\nabla u(x,\tau)|^p \, dxd\tau, \\ C(\rho,t) &:= \int_{P(\rho,t)} |u(x,\tau)|^{\sigma+1} \, dxd\tau, \\ b(t) &:= \text{ess sup}_{s\in(0,t)} \int_{|x-x_0|<\rho+\vartheta\tau} |u(x,s)|^{\gamma+1} \, dx, \\ \bar{b}(T) &:= \text{ess sup}_{s\in(0,T)} \int_{|x-x_0|<\rho+\vartheta\tau} |u(x,s)|^{\gamma+1} \, dx, \end{aligned} \tag{2.4.2}$$

and analogously for the case of the paraboloids

$$\begin{aligned} E(t) &:= \int_{P(t)} |\nabla u(x,\tau)|^p \, dxd\tau, \\ C(t) &:= \int_{P(t)} |u(x,\tau)|^{\sigma+1} \, dxd\tau, \\ b(t) &:= \text{ess sup}_{s\in(t,T)} \int_{\{|x-x_0|<\rho+\vartheta(\tau-t)\}} |u(x,s)|^{\gamma+1} \, dx, \end{aligned} \tag{2.4.3}$$

This choice of the sets is explained by convenience of having at disposal the domains of variable form but in fact depending only on variable ρ and a parameter t in cases a) and on a single variable t in the case b).

Let us pass to the precise statement of our results. The needed global information about the solution under study is formulated in terms of *the global energy function*

$$D(u(\cdot,\cdot)) := b(T,\Omega) + \int_Q \left(|\nabla u|^p + |u|^{\sigma+1}\right) dxdt, \tag{2.4.4}$$

where

$$b(T,\Omega) := \text{sup ess}_{t\in(0,T)} \int_\Omega |u(x,t)|^{\gamma+1} dx,$$

and that we will suppose finite.
It will be assumed that the data u_0 and f are "flat" enough near the boundary of their supports. Suppose that

$$u_0 \equiv 0 \quad \text{in} \quad B_{\rho_0}(x_0) \quad \text{for some } x_0 \in \Omega \text{ and } \rho_0 > 0 \tag{2.4.5}$$

and

$$f \equiv 0 \quad \text{in the truncated cone} \quad P = P(\rho_0, \vartheta). \tag{2.4.6}$$

Then *the flatness condition* is stated by claiming convergence of the auxiliary integral

$$I = \int_{\rho_0}^{\rho_1} (\rho-\rho_0)^{-\frac{1}{1-\nu}} \left(\|u_0\|^{\gamma+1}_{L^{\gamma+1}(B_\rho(x_0))} + \|f\|^{(1+\sigma)/\sigma}_{L^{(1+\sigma)/\sigma}(P(0,\rho))}\right)^{\nu} d\rho, \tag{2.4.7}$$

where $\rho_1 \in (\rho_0, +\infty)$,

$$\nu = \frac{p(1-\mu)}{p-1} \tag{2.4.8}$$

$$\mu = \frac{\tilde{\theta}}{p} + \frac{(1-\tilde{\theta})}{qr} \in (0,1),$$

$$q = \frac{\gamma-\sigma}{\gamma-r+1}$$

with some

$$r \in [1+\sigma, 1+\gamma] \quad \text{and } \tilde{\theta} = \frac{pN-(1+\sigma)(N-1)}{(N+1)p-N(1+\sigma)}$$

and where we assume that $\nu \in (0,1)$, i.e. $p < \frac{1}{\mu}$.

Theorem 2.4.1 (Shrinking of support). *Assume that conditions* (2.2.2), (2.2.3), (2.2.5), (2.2.6), 2.2.7), *hold and that* (2.4.1) *is true. Let* u_0 *satisfy* (2.4.5). *Assume that*

$$f \equiv 0 \quad \textit{in the truncated cone } P \equiv P(\rho_0, T) \tag{2.4.9}$$

for some $\vartheta > 0$ *and let* (2.4.7) *be true. Then there exist positive constants* M *and* t^* *such that each weak solution of problem* (2.2.1) *with the global energy satisfying the inequality* $D(u) \leq M$ *possesses the property*

$$u(x,t) \equiv 0 \quad \textit{in } P(\rho_0, t^*).$$

The next result applies to the case when the initial function need not vanish, namely, when the parameter ρ_0 in the conditions of Theorems 2.4.1 equals zero. Assuming $f \equiv 0$ (which is convenient but not necessary) we show how the strong absorption term causes the formation of the null-set of the solution.

Theorem 2.4.2 (Dead core formation). *Let us assume that conditions* (2.2.2), (2.2.3), (2.2.5), (2.2.6), 2.2.7), *hold,* (2.4.1), *are fulfilled, and that* $\gamma < p - 1$. *Let* $f \equiv 0$. *Then there exist positive constants* t^*, *and* $\upsilon \in (0,1)$ *such that any weak solution of problem* (2.2.1) *possesses the property*

$$u(x,t) \equiv 0 \quad \textit{in } P(t^*, \vartheta, \upsilon).$$

The proof of Theorem 2.4.1 is splitted into several steps.

2.4.1 The energy relation. Integration-by-parts formula.

Firsly we want to localize the weak formulation of the problem 2.2.1; we have to distinguish between case a) and b).
Case a)
We will deal with weak solutions of (2.2.1) such that for every test function

$$\varphi \in L^\infty\left(0, T, W^{1,p}(\Omega)\right) \cap W^{1,2}(0, T; L^\infty(\Omega)),$$

with $\varphi = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces, and $\varphi(\cdot, T) \equiv 0$ in Ω, the identity

$$\int_Q \{\psi(x,u)\varphi_t - A \cdot D\varphi - C\varphi\} dxdt - \int_\Omega \psi(x,u)\varphi dx \ \ |_{t=0} = -\int_Q f\varphi dxdt \tag{2.4.10}$$

holds.
We need to introduce some suitable cut-off functions. Given $x_0 \in \Omega$, $t \in [0,T]$, $\vartheta \geq 0$ and $\upsilon \in (0,1)$, we define:

$$\zeta(x,\tau) := \psi_\varepsilon\left(|x-x_0|,\tau\right)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m\left(u(x,s)\right)ds, \quad h>0,$$

where

$$T_m\left(u(x,t)\right) := sign\, u(x,s)\min\left\{m,|u(x,s)|\right\}, \quad m \in \mathbb{N},$$

and

$$\xi_k(\tau) := \begin{cases} 1 & \text{if } \tau \in \left[0, t-\frac{1}{k}\right], \\ k(t-\tau) & \text{for } \tau \in \left[t-\frac{1}{k}, t\right], \end{cases}$$

$$\psi_\varepsilon\left(|x-x_0|,\tau\right) := \begin{cases} 1 & \text{if } d > \varepsilon, \\ \frac{1}{\varepsilon}d & \text{if } d < \varepsilon, \\ 0 & \text{ if } (x,\tau) \in Q \setminus P(\rho), \end{cases}$$

Here $d = dist((x,\tau), \partial_l P(\rho,t))$, $\quad \varepsilon > 0$ and in what follows $\partial_l P$ denotes the lateral boundary of P i.e.

$$\partial_l P(\rho,t) = \{(x,\tau): \ |x-x_0| = \rho + \vartheta\tau, \tau \in (0,T)\}.$$

Case b)
We will deal with weak solutions of 2.2.1 such that for every test function

$$\varphi \in L^\infty\left(0,T,W^{1,p}(\Omega)\right) \cap W^{1,2}(0,T;L^\infty(\Omega)),$$

with $\varphi = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces, and $\varphi(\cdot,0) = \varphi(\cdot,T) \equiv 0$ in Ω, the identity

$$\int_Q \{\psi(x,u)\varphi_t - A\cdot D\varphi - C\varphi\}dxdt = -\int_Q f\varphi dxdt \tag{2.4.11}$$

holds.
Now the appropiate cut-off function are given by:

$$\zeta(x,\tau) := \psi_\varepsilon\left(|x-x_0|,\tau\right)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m\left(u(x,s)\right)ds, \quad h>0,$$

where

$$T_m\left(u(x,t)\right) := sign\, u(x,s)\min\left\{m,|u(x,s)|\right\}, \quad m \in \mathbb{N},$$

and

$$\xi_k(\tau) := \begin{cases} 1 & \text{if } \tau \in \left[t + \frac{1}{k}, T - \frac{1}{k}\right], \\ k(T-\tau) & \text{for } \tau \in \left[T - \frac{1}{k}, T\right], \\ k(\tau - t) & \text{if } \tau \in \left[t, t + \frac{1}{k}\right] \\ 0 & \text{if } \tau \in [0,t], \quad k \in \mathbb{N}, \end{cases}$$

$$\psi_\varepsilon\left(|x-x_0|, \tau\right) := \begin{cases} 1 & \text{if } d > \varepsilon, \\ \frac{1}{\varepsilon} d & \text{if } d < \varepsilon, \\ 0 & \text{if } (x,\tau) \in Q \setminus P(t), \end{cases}$$

Here $d = dist((x,\tau), \partial_l P(t)), \quad \varepsilon > 0$ and in what follows $\partial_l P$ denotes the lateral boundary of P i.e.

$$\partial_l P(t) = \{(x,\tau) : \ |x-x_0| = \vartheta(\tau - t)^v, \ \tau \in (t,T)\}.$$

By construction we have $supp\, \zeta(x,\tau)$ coincide with $P(\rho,t)$ and $P(t)$, respectively. It is easy to verify that for every natural m, k and positive real numbers h, ε

$$\zeta, \frac{\partial \zeta}{\partial t} \in L^\infty\left((0,T)\times\Omega\right), \quad \frac{\partial \zeta}{\partial x_i} \in L^p\left((0,T)\times\Omega\right).$$

These properties allow one to substitute $\zeta(x,\tau)$ into the respective integral identities (2.4.10),(2.4.11) as a test function. We will only consider the case b), the other being very similar. Firstly we consider:

$$\begin{aligned} &\int_0^T \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\right] dx d\tau - \\ &- \int_0^T \int_{B_{R(\tau,t)}} (A, \nabla[\psi_\epsilon(|x-x_0|,\tau)]\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds] dx d\tau - \\ &- \int_0^T \int_{B_{R(\tau,t)}} C(\psi_\epsilon(|x-x_0|,\tau)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds) dx d\tau + \\ &\int_0^T \int_{B_{R(\tau,t)}} f(\psi_\epsilon(|x-x_0|,\tau)\xi_k(\tau)\frac{1}{h}\int_\tau^{\theta+h} T_m(u(x,s))ds) dx d\tau \ = \ 0, \end{aligned} \tag{2.4.12}$$

where we recall $R(\tau,t) = |x-x_0| = \vartheta(\tau-t)^v$.
Let us proceed now to deduce as in the previous section, an energy relation in integral form. For this we need to provide some suitable estimates of the above terms.
We start with the term containing the time derivative; splitting the integral domain we have:

$$\int_0^T \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\right] dxd\tau \ =$$

$$\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)k(\tau-t)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\right] dxd\tau \ +$$

$$\int_{t+\frac{1}{k}}^{T-\frac{1}{k}} \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\right] dxd\tau \ +$$

$$\int_{T-\frac{1}{k}}^{T} \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)k(T-\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\right] dxd\theta \tag{2.4.13}$$

For the first term of (2.4.13), computing the derivative we have:

$$\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \{\psi(u) \frac{\partial}{\partial \tau}\left[\psi_\epsilon(|x-x_0|,\tau)k(\tau-t)\right]\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\} dxd\tau \ +$$

$$\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \{\psi(u) \left[\psi_\epsilon(|x-x_0|,\tau)k(\tau-t)\right]\frac{\partial}{\partial \tau}\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\} dxd\tau \ =$$

$$\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \{\psi(u) \frac{\partial}{\partial \tau}\psi_\epsilon(|x-x_0|,\tau)k(\tau-t)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\} dxd\tau \ +$$

$$\frac{1}{\frac{1}{k}}\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \{\psi(u)\psi_\epsilon(|x-x_0|,\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\} dxd\tau \ +$$

$$\int_t^{t+\frac{1}{k}} \int_{B_{R(\tau,t)}} \{\psi(u) \left[\psi_\epsilon(|x-x_0|,\tau)k(\tau-t)\right]\frac{\partial}{\partial \tau}\frac{1}{h}\int_\tau^{\tau+h} T_m(u(x,s))ds\} dxd\tau$$

Now we observe that from the structural hypothesis the following functions:

$$\psi(u(x,\tau))\frac{\partial}{\partial\tau}\psi_\epsilon(\tau,\cdot)\frac{1}{h}\int_\tau^{\tau+h}T_m(u)ds,$$

$$\psi(u)\psi_\epsilon(x,\tau)\frac{1}{h}\int_\tau^{\tau+h}T_m(u)ds,$$

$$\psi(u)\psi_\epsilon(x,\tau)\frac{\partial}{\partial\tau}\frac{1}{h}\int_\tau^{\tau+h}T_m(u)ds,$$

belong to $L^\infty(0,T,L^1(\Omega))$ and this allows us to suppose that t is one of their Lebesgue points; hence, passing to the limit when k goes to ∞ it holds

$$\frac{1}{\frac{1}{k}}\int_t^{t+\frac{1}{k}}\int_{B_{R(\tau,t)}}\{\psi(u)\psi_\epsilon(|x-x_0|,\tau)\frac{1}{h}\int_\tau^{\tau+h}T_m(u(x,s))ds\}dxd\tau =$$

$$\int_\Omega \psi(u(t,x)\tilde{\zeta}(x,t)dx.$$

The same argument works for the other terms and so we can write:

$$\int_0^T\int_{B_{R(\tau,t)}}\psi(u)\frac{\partial}{\partial\tau}\left[\psi_\epsilon(|x-x_0|,\tau)\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h}T_m(u(x,s))ds\right]dxd\tau \;=$$

$$\int_t^T\int_{B_{R(\tau,t)}}\psi(u)\frac{\partial}{\partial\tau}\tilde{\zeta}(x,\tau)dxd\tau\;+\int_{P(t,\tau)\cap(\tau=t)}\psi(u)\tilde{\zeta}(x,\tau)dx\;-$$

$$-\int_{P(t,\tau)\cap(\tau=T)}\psi(u)\tilde{\zeta}(x,\tau)dx$$

where

$$\tilde{\zeta}=\psi_\epsilon(|x-x_0|,\tau)\frac{1}{h}\int_\tau^{\tau+h}T_m(u)ds$$

Returning to (2.4.12), this procedure leads to the equality:

$$\int_t^T \int_{B_{R(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau} \tilde{\zeta} dx d\tau \; - \left[\int_{B_{R(\tau,t)}} \psi(u) \tilde{\zeta} dx d\tau \right]_{\tau=t}^{\tau=T}$$

$$- \int_t^T \int_{B_{R(\tau,t)}} (A, \nabla \tilde{\zeta}) dx d\tau \; - \int_t^T \int_{B_{R(\tau,t)}} C \tilde{\zeta} dx d\tau \; + \tag{2.4.14}$$

$$+ \int_t^T \int_{B_{R(\tau,t)}} f \tilde{\zeta} dx d\tau \; = \; 0$$

Now we will focus our attemption on the first term of (2.4.14); first of all we split the spatial domain in a suitable way:

$$\int_t^T \int_{B_{\mathbb{R}(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau} \tilde{\zeta} dx d\tau \; =$$

$$\int_t^T \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \psi(u) \frac{1}{h} (T_m u(\tau+h) - T_m u(\tau)) dx d\tau \; +$$

$$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u) \frac{\partial}{\partial \tau} \left[\frac{1}{\epsilon} (R(\tau,t) - |x-x_0|) \frac{1}{h} \int_\tau^{\tau+h} T_m(u) ds \right] dx d\tau$$

then computing term by term with the respective derivatives we have:

$$\int_t^T \int_{B_{\mathbb{R}(\tau,t)}} \psi(u) \frac{\partial}{\partial \tau} \tilde{\zeta} dx d\tau \; =$$

$$\int_t^T \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \psi(u) \frac{1}{h} (T_m u(\tau+h) - T_m u(\tau)) dx d\tau \; +$$

$$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u) \frac{1}{\epsilon} (\frac{\partial}{\partial \tau} R(\tau,t)) \frac{1}{h} \int_\tau^{\tau+h} T_m(u) ds \; dx d\tau \; +$$

$$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u) \frac{1}{\epsilon} (R(\tau,t) - |x-x_0|) \frac{1}{h} (T_m(u(\tau+h)) - T_m(u(\tau))) \; dx d\tau$$

The same reasoning made in (2.2.20) of chapter two, allows us to write:

$$\int_t^T \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \psi(u)\frac{1}{h}(T_m u(\tau+h) - T_m u(\tau))dxd\tau \ \leq$$

$$\int_t^T \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \frac{1}{h}(j(T_m u(\tau+h)) - j(T_m u(\tau)))dxd\tau \ =$$

$$\int_T^{T+h} \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \frac{1}{h}(j(T_m u(\tau)))dxd\tau - \int_t^{t+h} \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \frac{1}{h}(j(T_m u(\tau)))dxd\tau$$

So passing to the limit when $h \to 0$ for almost all t we have

$$\int_t^T \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} \psi(u)\frac{1}{h}(T_m u(\tau+h) - T_m u(\tau))dxd\tau \ \leq$$
$$\int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} j(T_m u(T))dx \quad - \int_{\{|x-x_0|<R(\tau,t)-\epsilon\}} j(T_m u(t))dx$$

and so:

$$\int_t^T \int_{B_{R(\tau,t)}} \psi(u)\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau \leq$$
$$\int_{|x-x_0|<R(\tau,t)-\epsilon} j(T_m u(\tau)) \mid_{\tau=t}^{\tau=T} +$$
$$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u)\frac{1}{\epsilon}(\frac{\partial}{\partial\tau}R(\tau,t))T_m(u)dxd\tau \ +$$

$$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u)\frac{1}{\epsilon}(R(\tau,t) - |x-x_0|)\frac{1}{h}(T_m(u(\tau+h)) - T_m(u(\tau)))\ dxd\tau$$

Let us set now make some remarks:

- Rewriting in spherical coordinates the term

  $$\int_t^T \int_{\{R(\tau,t)-\epsilon<|x-x_0|<R(\tau,t)\}} \psi(u)\frac{1}{\epsilon}(R(\tau){-}|x{-}x_0|)\frac{1}{h}(T_m(u(\tau{+}h)){-}T_m(u(\tau))dsdxd\tau$$

  we have

  $$\int_t^T \int_{R(\tau,t)-\epsilon}^{R(\tau,t)} \rho^{N-1}d\rho \int_{S^{N-1}} \psi(u)\frac{1}{\epsilon}(R(\tau,t){-}\rho)\frac{1}{h}(T_m(u(\tau{+}h)){-}T_m(u(\tau))d\omega d\tau$$

and so, having

$$\frac{1}{\epsilon}(R(\tau,t)-\rho)<1$$

by the definition of the cut-off functions, we can conclude that the entire integral goes to zero when ϵ goes to 0. we can conclude that the entire integral goes to zero when ϵ goes to 0.

- the following equality holds

$$\frac{\partial}{\partial\tau}(R(\tau,t)) \;=\; -n_\tau\frac{\sqrt{\vartheta^2\upsilon^2+(\tau-t)^{2(1-\upsilon)}}}{(\tau-t)^{(1-\upsilon)}}\leq -n_\tau$$

  where the last term stands for the component of the outer normal vector to $\partial_l P$, parallel to the time axes.

Hence, taking the limits when ϵ goes to zero and argument, for the other terms in the same way that in the previous section, we are allowed to rewrite (2.4.14) in the form:

$$\begin{aligned}
&-\left[\int_{B_{R(\tau,t)}}(\psi(u)T_m(u)-j(T_m(u)))dx\right]_{\tau=t}^{\tau=T}\\
&-\int_t^T\int_{B_{R(\tau,t)}}(\overrightarrow{A},\nabla T_m(u))dxd\tau+\int_t^T\int_{\partial B_{R(\tau,t)}}(\overrightarrow{A}\cdot n_x T_m(u))dxd\tau\\
&-\int_t^T\int_{\partial B_{R(\tau,t)}}(\psi(u)\cdot n_\tau T_m(u))dxd\tau\\
&-\int_t^T\int_{B_{R(\tau,t)}}Cudxd\tau+\int_t^T\int_{B_{\rho(\tau)}}fudxd\tau \;\geq\; 0
\end{aligned}$$

Finnally, from the local integrability of the function under the integral signs, following from the hypothesis of theorem 2.2.1,we can take the limit when m goes to ∞ and for the Lebesgue's dominate's convergences theorem, we have:

$$\begin{aligned}
& i_1 + i_2 + i_3 \\
& \quad =: \int_{P(t)\cap\{\tau=T\}} G(u)dx + \int_{P(t)} A \cdot Du\, dxd\tau \\
& \qquad + \int_{P(t)} C\, udxd\tau \\
& \quad \leq \int_{\partial_l P(t)} n_x, \cdot A\, u\, d\Gamma d\tau - \int_{\partial_l P(t)} n_\tau \psi(u) ud\Gamma d\tau \\
& \qquad + \int_{P(t)\cap\{\tau=t\}} G(u)dx + \int_P uf dxd\tau \\
& \quad := j_1 + j_2 + j_3 + j_4.
\end{aligned} \tag{2.4.15}$$

Here $d\Gamma$ is the differential form on the hypersurface $\partial_l P(t) \cap \{\tau = const\}$, n_x and n_τ are the components of the unit normal vector to $\partial_l P$.

The same approach leads to the equivalent differential inequality for the case of the truncated cone domain. So we have too:

$$\begin{aligned}
& i_1 + i_2 + i_3 \\
& \quad =: \int_{P(\rho,t)\cap\{\tau=T\}} G(u)dx + \int_{P(\rho,t)} A \cdot Du\, dxd\tau \\
& \qquad + \int_{P(\rho,t)} C\, udxd\tau \\
& \quad \leq \int_{\partial_l P(\rho,t)} n_x, \cdot A\, u\, d\Gamma d\tau - \int_{\partial_l P(\rho,t)} n_\tau \psi(u) ud\Gamma d\tau \\
& \qquad + \int_{P(\rho,t)\cap\{\tau=0\}} G(u)dx + \int_P uf dxd\tau \\
& \quad := j_1 + j_2 + j_3 + j_4.
\end{aligned} \tag{2.4.16}$$

2.4.2 Differential inequalities.

We begin with the most complicated case b) where the domain P is a paraboloid determined by the parameters $\upsilon \in (0,1)$, $\vartheta > 0$, and $t \in (0,T)$:

$$P = P(t) = \{(x,\tau) : \; |x - x_0| \equiv \rho(\tau) \leq \vartheta(\tau - t)^\upsilon, \;\; \tau \in (t,T)\}.$$

We assume that $f \equiv 0$ and that P does not touch the initial plane $\{t = 0\}$. These assumptions simplify the basic energy inequality (2.4.16), so we have:

$$i_1 + i_2 + i_3 \leq j_1 + j_2.$$

Let us estimate the term j_1. We start with the following simple observation

Let $(\rho, \boldsymbol{\omega})$, $\rho > 0$, $\omega \in \partial B_1$, be the spherical coordinate system in $\mathbb{R}^N$. Given an arbitrary function $F(x,t)$, we use the notation $x = (\rho, \boldsymbol{\omega})$ and $F(x,t) = \Phi(\rho, \boldsymbol{\omega}, t)$. There holds the equality

$$I(t) := \int_P F(x,\tau)dxd\tau \equiv \int_t^T d\tau \int_0^{R(\tau,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho, \boldsymbol{\omega}, \tau)|J|d\boldsymbol{\omega},$$

where J is the Jacobian determinant and, due to the definition of P, $\rho(\tau, t) = \vartheta(\tau - t)^v$. It is easy to check that:

$$\begin{aligned} \frac{dI(t)}{dt} &= - \left. \int_0^{\rho(\tau,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho, \boldsymbol{\omega}, \tau)|J|d\boldsymbol{\omega} \right|_{\tau=t} \\ &\quad + \int_t^T \rho_t \rho^{N-1} d\tau \int_{\partial B_1} \Phi(\rho, \boldsymbol{\omega}, t)|J|d\boldsymbol{\omega} \\ &= \int_{\partial_l P} \rho_t F(x,\tau) d\Gamma d\tau. \end{aligned} \tag{2.4.17}$$

Viewing the energy function E as a function of t, with the use of (2.2.3), and Hölder's inequality, we obtain the relation

$$\begin{aligned} &\left| \int_{\partial_l P} \boldsymbol{n}_x \cdot \mathbf{A}\, u\, d\Gamma d\tau \right| \leq C_2 \int_{\partial_l P} |\boldsymbol{n}_x| |\nabla u|^{p-1} |u| d\Gamma d\tau \\ &= C_2 \int_{\partial_l P} |n_x| |\nabla u|^{p-1} |u| |\rho_t|^{\frac{p-1}{p}} |\rho_t|^{\frac{1-p}{p}} \leq \\ &\leq C_2 \left(\int_{\partial_l P} |\rho_t| |\nabla u|^p d\Gamma d\tau \right)^{(p-1)/p} \left(\int_{\partial_l P} \frac{|\boldsymbol{n}_x|^p}{|\rho_t|^{p-1}} |u|^p d\Gamma d\tau \right)^{1/p} \\ &= C_2 \left(-\frac{dE}{dt} \right)^{(p-1)/p} \left(\int_t^T \frac{|\boldsymbol{n}_x|^p}{|\rho_t|^{p-1}} \left(\int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma \right) d\tau \right)^{1/p}. \end{aligned} \tag{2.4.18}$$

To estimate the right-hand side of (2.4.18) we use the interpolation inequality: if $\sigma \le p-1$, then $\forall\, v \in W^{1,p}(B_\rho)$

$$\|v\|_{p,S_\rho} \le L_0 \left(\|\nabla v\|_{p,B_\rho} + \rho^\delta \|v\|_{\sigma+1,B_\rho}\right)^{\tilde\theta} \cdot \left(\|v\|_{r,B_\rho}\right)^{1-\tilde\theta} \tag{2.4.19}$$

with a universal constant $L_0 > 0$ not depending on $v(x)$ and the exponents

$$r \in [1+\sigma, 1+\gamma], \quad \tilde\theta = \frac{pN - (1+\sigma)(N-1)}{(N+1)p - N(1+\sigma)}, \quad \delta = -\left(1 + \frac{p-1-\sigma}{p(1+\sigma)} N\right).$$

Observe that here we just utilize the assumption $\gamma > \sigma$ that is properly the *strong absorption property.* Let us introduce the notation

$$E_*(t,\rho) := \int_{B_\rho} |\nabla u|^p dx, \quad C_*(t,\rho) := \int_{B_\rho} |u|^{\sigma+1} dx,$$

so that

$$E = \int_t^T E_*(\tau, \rho(\tau,t)) d\tau, \quad C = \int_t^T C_*(\tau, \rho(\tau,t)) d\tau,$$

and make use of Hölder's inequality

$$\left(\int_{B_\rho} |u|^r dx\right)^{1/r} \le \left(\int_{B_\rho} |u|^{1+\sigma} dx\right)^{1/qr} \cdot \left(\int_{B_\rho} |u|^{\gamma+1} dx\right)^{(q-1)/qr},$$

where

$$q = \frac{\gamma - \sigma}{\gamma - r + 1}.$$

Then, by virtue of (2.4.19),

$$\int_{\partial B_{\rho(\tau,t)}} |u|^p dx \le L_0^p \left((\int_{B_{\rho(\tau,t)}} |\nabla u|^p dx)^{\frac{1}{p}} + \rho^\delta \left(\int_{B_{\rho(\tau,t)}} |u|^{\sigma+1} dx \right)^{1/(\sigma+1)} \right)^{\tilde{\theta} p}$$

$$\times \left(\int_{B_{\rho(\tau,t)}} |u|^r dx \right)^{p(1-\tilde{\theta})/r}$$

$$\le \tilde{L}_0^p \max(1, \rho(T)^{-\delta\tilde{\theta}p}) \rho(\tau,t)^{\delta\tilde{\theta}p} \left(\int_{B_{\rho(\tau,t)}} |\nabla u|^p dx + (\int_{B_{\rho(\tau,t)}} |u|^{\sigma+1})^{\frac{1}{\sigma+1}} dx \right)^{\tilde{\theta}}$$

$$\times \left(\int_{B_{\rho(\tau,t)}} |u|^{\sigma+1} dx \right)^{p(1-\tilde{\theta})/qr} \left(\int_{B_{\rho(\tau,t)}} |u|^{\gamma+1} dx \right)^{p(q-1)(1-\tilde{\theta})/qr}$$

$$\le \tilde{L}_0^p \max(1, \rho(T)^{-\delta\tilde{\theta}p}) \rho(\tau,t)^{\delta\tilde{\theta}p} \max \left(1, \left(\int_{B_{\rho(\tau,t)}} |u|^{\sigma+1} dx \right)^{\frac{p}{\sigma+1}-1} \right)^{\tilde{\theta}}$$

$$\times \left(\int_{B_{\rho,t}} |\nabla u|^p dx + \int_{B_{\rho,t}} |u|^{\sigma+1} dx \right)^{\tilde{\theta}} \left(\int_{B_{\rho(\tau,t)}} |u|^{\sigma+1} dx \right)^{p(1-\tilde{\theta})/qr} \left(\int_{B_{\rho(\tau,t)}} |u|^{\gamma+1} dx \right)^{\frac{p(q-1)(1-\tilde{\theta})}{qr}}$$

$$\le K(\tau) \left(E_* + C_* \right)^{\tilde{\theta}} C_*^{(1-\tilde{\theta})p/qr} b^{(q-1)(1-\tilde{\theta})p/qr}$$

$$\le K(\tau) \left(E_* + C_* \right)^{\tilde{\theta}+(1-\tilde{\theta})p/qr} b^{(q-1)(1-\tilde{\theta})p/qr}, \tag{2.4.20}$$

with

$$K(\tau) = L_0^p \rho(\tau)^{\delta\tilde{\theta}p} \max(1, \rho(T)^{-\delta p\tilde{\theta}}) \max \left(1, \left(ess \sup_{(t,T)} \int_{B_{\rho(\tau)}} |u|^{\sigma+1} dx \right)^{\frac{p}{\sigma+1}-1} \right)^{\tilde{\theta}}$$

and

$$K(\tau) \le K_0 \rho(\tau)^{\delta\tilde{\theta}p}.$$

Let us assume that

$$\mu = \tilde{\theta} + p\frac{1-\tilde{\theta}}{qr} < 1, \tag{2.4.21}$$

Returning to (2.4.18) and applying once again Hölder's inequality with the exponent μ, we have from (2.4.20)

$$
\begin{aligned}
|j_1| &\le L\left(-\frac{dE}{dt}\right)^{(p-1)/p} \\
&\quad\times\left(\int_t^T \frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}} K(\tau)\,(E_*+C_*)^{\mu}\, b^{(q-1)(1-\tilde{\theta})p/qr} d\tau\right)^{1/p} \\
&\le L\left(-\frac{dE}{dt}\right)^{(p-1)/p} \bar{b}^{(q-1)(1-\tilde{\theta})/qr} \\
&\quad\times\left(\int_t^T (E_*+C_*)\,d\tau\right)^{\frac{\mu}{p}}\left(\int_t^T\left(\rho(\tau)^{\delta\tilde{\theta}p}\frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}}\right)^{\frac{1}{1-\mu}} d\tau\right)^{\frac{1-\mu}{p}} \\
&\le L\Lambda(t)\left(-\frac{d\,(E+C)}{dt}\right)^{(p-1)/p}\cdot\bar{b}^{(q-1)(1-\tilde{\theta})/qr}\,(E+C)^{\frac{\tilde{\theta}}{p}+\frac{1-\tilde{\theta}}{qr}}
\end{aligned}
\tag{2.4.22}
$$

for a suitable positive constant L. To obtain (2.4.22) we have assumed (2.4.21) and

$$
\frac{p}{\sigma+1} > 1, \tag{2.4.23}
$$

$$
\Lambda(t) := \left(\int_t^T\left(\frac{\rho(\tau)^{\delta\tilde{\theta}p}}{|\rho_t|^{p-1}}\right)^{\frac{1}{1-\mu}} d\tau\right)^{\frac{1-\mu}{p}} < \infty. \tag{2.4.24}
$$

Inequality (2.4.21) is fulfilled if $p < qr$, which is true under an extra restriction on r. We have:

$$
p < qr \quad\Leftrightarrow\quad (\gamma-\sigma)r > p(\gamma-r+1) \quad\Leftrightarrow\quad r > \frac{1+\gamma}{1+\frac{\gamma-\sigma}{p}}.
$$

The last inequality must be consistent with the previous choice of r: $r \in [1+\sigma, 1+\gamma]$. Thus, we have to claim first

$$
\gamma+1 > \frac{\gamma+1}{1+\frac{\gamma-\sigma}{p}} = \frac{p(\gamma+1)}{p+\gamma-\sigma} \quad\Leftrightarrow\quad p+\gamma-\sigma > p \quad\Leftrightarrow\quad \gamma > \sigma,
$$

which is true because of (2.4.1), and then to choose

$$
r \in \left[\frac{p(\gamma+1)}{p+\gamma-\sigma}, \gamma+1\right].
$$

Inequality (2.4.23) coincides with (2.4.1). To satisfy (2.4.24) one only has to choose $\nu \in (0,1)$ because the condition of convergence of the integral $\Lambda(t)$ has the form

$$(1-\nu)(p-1) > -(1-\tilde{\theta})\left(1-\frac{p}{qr}\right).$$

So, we have obtained an estimation of the following type :

$$\begin{aligned} |j_1| \leq L_1\, \Lambda(t)\, D(u)^{(q-1)(1-\tilde{\theta})}/qr \\ \times L\Lambda(t) D(u)^{(q-1)(1-\tilde{\theta})/qr}\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p} (E+C)^{1-\kappa} \end{aligned} \tag{2.4.25}$$

where L_1 is a universal positive constant, $D(u)$ is *the total energy* of the solution under investigation, and $\kappa := 1-\frac{\tilde{\theta}}{p}-\frac{1-\tilde{\theta}}{qr} \in (0,1)$.

Let us proceed now with the estimation of j_2. Using (2.2.7) we have:

$$|j_2| \leq C_5 \int_{\partial_l P} |u|^{1+\gamma} d\Gamma d\theta. \tag{2.4.26}$$

We apply then the interpolation inequality

$$\forall\, v \in W^{1,p}(B_\rho) \qquad \|v\|_{\gamma+1,\partial B_\rho} \leq L_0 \left(\|\nabla v\|_{p,B_\rho} + \rho^{\delta}\|v\|_{\sigma+1,B_\rho}\right)^{s} \cdot \|v\|_{r,B_\rho}^{1-s} \tag{2.4.27}$$

with a universal positive constant $L_0 > 0$ and the exponents

$$s = \frac{(\gamma+1)N - r(N-1)}{(N+r)p - Nr} \cdot \frac{p}{\gamma+1}, \qquad r \in [1+\sigma, 1+\gamma],$$

and with the constant δ from (2.4.19). By analogy with the previous estimate, we can write

$$\int_{\partial B_\rho} |u|^{\gamma+1} d\Gamma$$

$$\leq L_0^{1+\gamma} \left(\left(\int_{B_\rho} |\nabla u|^p dx \right)^{\frac{1}{p}} + \rho^\delta \left(\int_{B_\rho} |u|^{\sigma+1} dx \right)^{\frac{1}{\sigma+1}} \right)^{s(\gamma+1)} \left(\int_{B_\rho} |u|^r dx \right)^{\frac{(1-s)(\gamma+1)}{r}}$$

$$\leq L_0^{\gamma+1} \max(1, \rho(T)^{-\delta})^{s(\gamma+1)} \rho^{\delta s(\gamma+1)} \left(\left(\int_{B_\rho} |\nabla u|^p \right)^{\frac{1}{p}} + \left(\int_{B_\rho} |u|^{\sigma+1} \right)^{\frac{p}{p(\sigma+1)}} \right)^{s(\gamma+1)}$$

$$\times \left(\left(\int_{B_\rho} |u|^{\sigma+1} dx \right)^{1/qr} \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{(q-1)/qr} \right)^{(1-s)(\gamma+1)}$$

$$\leq L_0^{\gamma+1} \max(1, \rho(T)^{-\delta})^{s(\gamma+1)} \rho^{\delta s(\gamma+1)} \left(\left(\int_{B_\rho} |\nabla u|^p \right) + \left(\int_{B_\rho} |u|^{\sigma+1} \right)^{\frac{p}{(\sigma+1)}} \right)^{\frac{s(\gamma+1)}{p}}$$

$$\times \left(\left(\int_{B_\rho} |u|^{\sigma+1} dx \right)^{1/qr} \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{(q-1)/qr} \right)^{(1-s)(\gamma+1)}$$

$$\leq L_0^{\gamma+1} \max(1, \rho(T)^{-\delta})^{s(\gamma+1)} \rho^{\delta s(\gamma+1)}$$

$$\times \left(\left(\int_{B_\rho} |\nabla u|^p \right) + \left(\int_{B_\rho} |u|^{\sigma+1} \right) \left(\int_{B_\rho} |y|^{\sigma+1} \right)^{\frac{p}{\sigma+1}-1} \right)^{\frac{s(\gamma+1)}{p}}$$

$$\times \left(\left(\int_{B_\rho} |u|^{\sigma+1} dx \right)^{1/qr} \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{(q-1)/qr} \right)^{(1-s)(\gamma+1)} \tag{2.4.28}$$

$$\leq L_0^{\gamma+1}\max(1,\rho(T)^{-\delta})^{s(\gamma+1)}\rho^{\delta s(\gamma+1)}\max\left(1,\left(\int_{B_\rho}|u|^{\sigma+1}\right)^{\frac{p}{\sigma+1}-1}\right)^{\frac{s(\gamma+1)}{p}}$$

$$\times\left(\left(\int_{B_\rho}|\nabla u|^p\right)+\left(\int_{B_\rho}|u|^{\sigma+1}\right)\right)^{\frac{s(\gamma+1)}{p}}$$
$$\times\left(\left(\int_{B_\rho}|u|^{\sigma+1}dx\right)^{1/qr}\left(\int_{B_\rho}|u|^{\gamma+1}dx\right)^{(q-1)/qr}\right)^{(1-s)(\gamma+1)}$$

Observe that if we put now

$$\bar{L}:=L_0^{\gamma+1}\max(1,\rho(T)^{-\delta})^{s(\gamma+1)}\max\left(1,\left(\int_{B_\rho}|u|^{\sigma+1}\right)^{\frac{p}{\sigma+1}-1}\right)^{\frac{s(\gamma+1)}{p}}$$

we have

$$\bar{L}\leq L_0^{\gamma+1}\max(1,\rho(T)^{\delta})^{s(\gamma+1)}\max\left(1,D(u)^{\frac{p}{\sigma+1}-1}\right)^{\frac{s(\gamma+1)}{p}}=L,$$

where $D(u)$ represents the "total energy"that we always suppose finite (it is a very reasonable hypothesis when we are dealing with a model arising from some physical phenomenon for example).
Now suppose that

$$\eta=\frac{s(\gamma+1)}{p}+\frac{(1-s)(\gamma+1)}{qr}<1,\qquad \pi=\frac{(q-1)(1-s)(\gamma+1)}{qr},\qquad \eta+\pi\geq 1.$$

Then, gathering (2.4.26), (2.4.28), we come to the inequality

$$\begin{aligned}|j_2|&=\left|\int_t^T d\tau\int_{\partial B_{\rho(\tau)}}|u|^{\gamma+1}d\Gamma\right|\\&\leq Lb(T)^\pi\left(\int_t^T\rho(\tau)^{\delta s(\gamma+1)}(E_*+C_*)^\eta|n_\tau|d\tau\right)\end{aligned}\tag{2.4.29}$$

With application of Hölder's inequality with the exponent $p=\frac{1}{\eta}$ we can conclude

$$|j_2|\leq L(E+C)^\eta(b(T,\Omega))^\pi\left(\int_t^T(\rho(\tau)^{\delta s(\gamma+1)})^{\frac{1}{1-\eta}}d\tau\right)^{1-\eta}.$$

So we can write

$$
\begin{aligned}
|j_2| &\le L(E+C)(E+C)^{\eta-1}(b(T,\Omega))^{\pi}\left(\int_t^T \rho(\tau)^{\frac{\delta s(\gamma+1)}{1-\eta}}\right)^{1-\eta} \\
&\le L(E+C)D(u)^{\iota}\left(\int_t^T \rho(\tau)^{\frac{\delta s(\gamma+1)}{1-\eta}}\right)^{1-\eta}
\end{aligned}
\tag{2.4.30}
$$

with the exponent

$$\iota := \eta + \pi - 1$$

and

$$\tilde{K} := \max(1,\rho^{\delta})^{s(\gamma+1)} \max\left(1, \left(\int_{B_\rho} |u|^{\sigma+1}\right)^{\frac{p}{\sigma+1}-1}\right)^{\frac{s(\gamma+1)}{p}}.$$

To perform this estimate we have assumed that $\eta < 1$ and $\eta + \pi \ge 1$. The first of these inequalities is a simplified version of (2.4.21). As for the second one, a direct computation shows that it coincides with the second condition of (2.4.1).

We now turn to estimating the left-hand side of (2.4.16). By (2.2.2)-(2.2.7) we have at once that

$$C_2 E + C_4 C + C_5 \int_{P\cap t=T} |u|^{\gamma+1} dx \le i_1 + i_2 + i_3 \tag{2.4.31}$$

$$K\left(\int_{P\cap\{t=T\}} |u|^{1+\gamma} dx + E + C\right) \le i_1 + i_2 + i_3, \tag{2.4.32}$$

where

$$K = \min\left(C_2,\, C_4 C_5\right) > 0.$$

Since the right-hand side of (2.4.16) is an increasing function of t, we may always replace the first term on the left-hand side of (2.1) by $b(T)$. Now, the condition for the convergence of the integral in (2.4.30) is:

$$\rho(\tau)^{\frac{\delta s(\gamma+1)}{1-\eta}} = \theta(\tau - t)^{\frac{\upsilon\delta s(\gamma+1)}{1-\eta}} \in L^1(t,T),$$

i.e.

$$\upsilon\delta s(\gamma+1) > \eta - 1,$$

relay that $\delta < 0$ Hence relay that $\delta < 0$ it is satysfyed by a suitable choice of the parameter $\nu \in (0,1)$ sufficently small,
assuming then $D(u)$ so small that

$$L(E+C)D(u)^{\iota}\left(\int_t^T \rho(\tau)^{\frac{\delta s(\gamma+1)}{1-\eta}}\right)^{1-\eta} < \frac{K}{2},$$

we arrive at the inequality

$$
\begin{aligned}
E + C \leq E + C + \int_{P \cap t = T} |u|^{\gamma+1} dx \leq L_1\, \Lambda(t)\, D(u)^{(q-1)(1-\tilde{\theta})/qr} \\
\times (E + C)^{1-\kappa} \left(-\frac{d(E+C)}{dt} \right)^{(p-1)/p},
\end{aligned}
\tag{2.4.33}
$$

whence we obtain the desired differential inequality for the energy function $Y(t) := E + C$:

$$
Y^{kp/(p-1)}(t) \leq c(t)\, (-Y'(t))\,, \tag{2.4.34}
$$

where

$$
c(t) = \left(L_1\, (D(u))^{(q-1)(1-\tilde{\theta})/qr}\, \Lambda(t) \right)^{p/(p-1)}, \quad L_1 = const > 0,
$$

and κ is like in (2.4.25).
Note that $c(t) \to 0$ as $t \to T$. Moreover, the exponent $\kappa \frac{p}{p-1}$ belongs to the interval $(0, 1)$: indeed, the inequality $\kappa \frac{p}{p-1} < 1$ is equivalent to $qr < p$ which is equivalent to our basic assumption $p > \lambda + 1$.

We pass now to the consideration of domains of the type a).
The differential inequality for the energy function $E + C$ is now derived in much the same way that in the case a) but with certain simplification due to the choice of the domain $P(\rho, t)$. Let

$$
P(\rho, t) = \{(x, t) : \; |x - x_0| < \rho + \vartheta\tau, \; \tau \in (0, T)\}\,, \quad \rho \geq \rho_0 > 0.
$$

The unit outer normal to $\partial_l P$ has the form

$$
n = \frac{1}{\sqrt{1 + \vartheta^2}} (1, -\vartheta)
$$

and if the energy function $Y := E + C$ is considered as a function of ρ, we have:

$$
\begin{aligned}
&\frac{dY(\rho)}{d\rho} = \\
&\quad = \frac{d}{d\rho}\left\{\int_0^T d\theta \int_0^{\rho+\vartheta\theta} \tau^{N-1} d\tau \int_{\partial B_1} |J| \left(|\nabla u|^p + |u|^{\sigma+1}\right)\big|_{x=(\tau,\omega)} d\omega\right\} \\
&\quad = \int_0^T d\theta \int_{\partial B_1} \left\{(\rho+\vartheta\theta)^{N-1}|J| \left(|\nabla u|^p + |u|^{\sigma+1}\right)\big|_{x=(\rho+\vartheta\theta,\omega)}\right\} d\omega \\
&= \int_{\partial_l P(\rho,t)} \left(|\nabla u|^p + |u|^{\sigma+1}\right) d\Gamma d\theta
\end{aligned} \tag{2.4.35}
$$

To estimate the term j_1 in (2.4.16), a very similar argument like that of the above case works, so we have:

$$
\left|\int_{\partial_l P(\rho,t)} n_x \cdot A\, u\, d\Gamma d\tau\right| \leq \frac{C_2}{\sqrt{1+\vartheta^2}} \int_{\partial_l P(\rho,t)} |\nabla u|^{p-1}|u| d\Gamma d\tau \leq
$$

$$
\leq \frac{C_2}{\sqrt{1+\vartheta^2}} \left(\int_{\partial_l P(\rho,t)} |\nabla u|^p d\Gamma d\tau\right)^{(p-1)/p} \left(\int_{\partial_l P} |u|^p d\Gamma d\tau\right)^{1/p} \tag{2.4.36}
$$

$$
= \frac{C_2}{\sqrt{1+\vartheta^2}} \left(\frac{dE}{d\rho}\right)^{(p-1)/p} \left(\int_0^t d\tau \int_{\partial B_{R(\rho,t)}} |u|^p dx d\tau\right)^{1/p}.
$$

The *strong absorption* assumption $\sigma < \gamma$ allow us to apply the interpolation inequality (2.4.19), so we arrive at:

$$
\begin{aligned}
|j_1| \quad &\leq \quad \frac{C_2}{\sqrt{1+\vartheta^2}} \left(\frac{dE}{d\rho}\right)^{(p-1)/p} \max(1, \rho_0^{\delta\tilde{\theta}})(\bar{b})^{(q-1)(1-\tilde{\theta})/qr} \\
&\times \quad \left(\int_0^T (E_* + C_*)^{\tilde{\theta}+p(1-\tilde{\theta})/qr} d\tau\right)^{1/p}.
\end{aligned}
$$

Now let r be such that

$$
\eta = \tilde{\theta} + \frac{(1-\tilde{\theta})p}{qr} < 1.
$$

Such choice is always possible, since

$$
\tilde{\theta} + \frac{(1-\tilde{\theta})p}{qr} < 1 \iff r > \frac{p(\gamma+1)}{p+\gamma-\sigma},
$$

and the last inequality is compatible with the conditions $p > 1+\sigma$, $\gamma > \sigma$ stated in the hypothesis of the theorem, and the starting choice of r:

$$r \in [1+\sigma, 1+\alpha].$$

Applying Hölder's inequality with exponent $\frac{1}{\eta} > 1$ and putting

$$\mu = \frac{\tilde{\theta}}{p} + \frac{(1-\tilde{\theta})}{qr} = \frac{\eta}{p} < 1,$$

the estimate for j_1 then takes the form

$$|j_1| \leq \frac{C_2}{\sqrt{1+\vartheta^2}} \left(\frac{dE}{d\rho}\right)^{(p-1)/p} \max(1, \rho_0^{\delta\tilde{\theta}})(\bar{b})^{(q-1)(1-\tilde{\theta})/qr} t^{\frac{1}{p}-\mu}(E+C)^{\mu}.$$

The estimate for j_2 is the same that of the case a). The only difference is that now we need not claim the smallness of T. The value of the coefficient in the estimate for j_2 is controlled now by the choice of ϑ, since $n_\tau = -\vartheta/\sqrt{1+\vartheta^2}$. So we have:

$$|j_2| \leq (E+C)D(u)^{\iota}\tilde{L}\frac{\vartheta}{\sqrt{1+\vartheta^2}}, \tag{2.4.37}$$

where $\tilde{L}$ is a constant properly calculated. Due to (2.4.5) we have:

$$|j_3| = \left|\int_{|x-x_0|>\rho} G(u(x,0)dx\right| \leq C_5 \int_{|x-x_0|>\rho} |u(x,0)|^{\gamma+1}dx \tag{2.4.38}$$

for $\rho > \rho_0$ At last, we estimate j_4 remembering the hypotheses (2.4.9) and applying Young inequality with $p = \sigma + 1$

$$|j_4| \leq \int_0^T \int_{B_\rho} |uf|dxd\tau$$

$$\leq \frac{1}{\sigma+1}\int_0^T \int_{B_\rho} |u|^{\sigma+1}dxd\tau + \frac{\sigma}{\sigma+1}\int_0^T \int_{B_\rho} |f|^{\frac{\sigma+1}{\sigma}}dxd\tau$$

$$\leq \frac{1}{\sigma+1}C(\rho) + \frac{\sigma}{\sigma+1}\int_{P(\rho,\vartheta)} |f|^{(\sigma+1)/\sigma}dxd\tau.$$

The left-hand side of (2.4.16) is estimated in the same way that in the case a). Gathering these estimates, we arrive to the inequality

$$E + C \leq |j_1| + |j_2| + |j_4| \leq K_1 \left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}} b(T)^{\frac{(q-1)(1-\tilde{\theta}}{qr}} t^{\frac{1}{p}-\mu}(E+C)^{\mu} +$$

$$+ K_2 C(\rho) + K_3 \int_{|x-x_0|>\rho} |u(x,0)|^{\gamma+1} dx + K_4 \frac{\sigma}{\sigma+1} \int_0^T \int_{B(\rho)} |f|^{\frac{\sigma+1}{\sigma}} dx d\tau$$

Applying Young inequality with exponent $\frac{1}{\mu} > 1$ on the term

$$K_1 \left(\frac{dE}{d\rho}\right)^{\frac{p-1}{p}} b(T)^{\frac{(q-1)(1-\tilde{\theta}}{qr}} t^{\frac{1}{p}-\mu}(E+C)^{\mu}$$

and recalling the constants K_i, we have again:

$$(E+C)^{\nu} \leq t^{\frac{1-\mu p}{p-1}} M \frac{d(E+C)}{d\rho} + \tilde{M} F(\rho)^{\nu},$$

where

$$F(\rho) = \|u_0\|^{\gamma+1}_{L^{\gamma+1}(B_\rho(x_0))} + \|f\|^{(\sigma+1)/\sigma}_{L^{(\sigma+1)/\sigma}(P(\rho,\vartheta))},$$

and

$$\nu = \frac{p(1-\mu)}{p-1}.$$

And so putting $Y = (E+C)$

$$Y^{\nu} \leq M \frac{dY}{d\rho} + \tilde{M} F(\rho)^{\nu} \tag{2.4.39}$$

Observe that under the hypotheses of theorem (2.4.1) we can ensure that $\nu < 1$.

2.4.3 -Analysis of the differential inequalities.

Let us provide now to deduce from (2.4.34) the *dead core* formation property for the solution of problem (2.2.1); this will allow us to conclude the proof of theorem 2.4.2. We have only to observe that T is taken so as to provide the inclusion $P \subset Q$, and the coefficient $c(t)$ in inequality (2.4.39) can be estimated from above by $M := c(0)$. More precisely

$$\begin{aligned} c(t) &= \left(L_1 \left(D(u)\right)^{(q-1)(1-\tilde{\theta})/qr} \Lambda(t)\right)^{p/(p-1)} \\ &\leq \left(L_1 \left(M^*\right)^{(q-1)(1-\tilde{\theta})/qr} \Lambda(0)\right)^{p/(p-1)} \leq M, \end{aligned} \tag{2.4.40}$$

Where we are supposing that the *total energy* function is finite and then bounded from above by a suitable constant M^*.

So we can write (2.4.34) in the form:

$$Y^{kp/(p-1)}(t) \leq M\left(-Y(t)\right)' \tag{2.4.41}$$

Let us now introduce the function

$$z(T-t) := Y(t), \quad (z(t) \in [0, D(u)]).$$

It satisfies the following differential inequality

$$z^{\kappa p/(p-1)}(t) \leq M z'(t) \quad \text{as } t \in (0,T), \quad z(0) = 0, \tag{2.4.42}$$

Let us integrate inequality (2.4.42) over the interval $t \in (t, T) \subset (0, T)$. Then

$$z^{1-\mu}(t) \leq z^{1-\mu}(T) - \frac{1-\mu}{M}(T-t),$$

where

$$\mu = \frac{p\kappa}{p-1}, \quad M = \left(L_1 \Lambda(0) M_*^{(q-1)(1-\theta)/qr}\right)^{p/(p-1)} \equiv M_0 M_*^{\mu_0},$$

$$\mu_0 = \frac{p}{p-1}\frac{(q-1)(1-\theta)}{qr}.$$

Now observing that

$$z^{1-\mu}(T) \leq M^{*1-\mu}$$

it follows then

$$z^{1-\mu}(t) \leq \frac{1-\mu}{M_0 M_*^{\mu_0}}\left(\frac{M_0}{1-\mu} M_*^{1-\mu+\mu_0} - T + t\right)$$

and, hence, $z(t) = 0$ if

$$t \geq t^* \equiv \left(T - \frac{M_0}{1-\mu} M_*^{1+\mu_0-\mu}\right), \quad 0 < t^* < T. \tag{2.4.43}$$

Therefore, for every $0 < T < \infty$ and M_* satisfying the condition

$$M_* < \left(\frac{1-\mu}{M_0} T\right)^{1/(1+\mu_0-\mu)}$$

we have: $u(x,t) \equiv 0$ in $P(t^*, 0)$ if $D(u(x,t)) \leq M_*$, and t^* is given by (2.4.43).

Conversely, if $u(x,t)$ is a weak solution of equation (2.2.1), and

$$\sup_{0<t<\infty} D(u(.,t)) \le M^* = const < \infty,$$

there always exists $t^* < \infty$ such that $u(x,t) \equiv 0$ in $P(t^*,0)$. This concludes the proof ot theorem 2.3.1.

Let us proceed now to deduce from (2.4.39) the *shrinking of support* property for the solution of (2.2.1) this will allow us to conclude the proof of theorem 2.4.1.
First of all we observe that equation (2.4.39) satisfies all hypotheses of Lemma B.0.2 of chapter one.
So remembering the notation in the previous lemma and by putting

$$\tilde{M}F(\rho)^\nu = F(\rho), \quad 1-\nu = \mu, \quad R_0 = \rho_0, \;\; R = \rho_1 > \rho_0,$$

$$\Gamma(\rho) = \int_{\rho_0}^{\rho} (\tau - \rho_0)^{-\frac{1}{1-\nu}} F(\rho) < \infty$$

and

$$\Lambda = Mt^{\frac{1-\mu p}{p-1}},$$

the following inequality holds:

$$Y(\rho_0) \le H(\rho) = Y(\rho_1) - (\rho-\rho_0)^{\frac{1}{1-\nu}} t^{-\frac{1-\mu p}{(p-1)(1-\nu)}} \left(\left(\frac{1-\nu}{M}\right)^{\frac{1}{1-\nu}} - \frac{\Gamma(\rho)t^{1-\nu}}{M} \right)$$

$$\le D(u) - (\rho-\rho_0)^{\frac{1}{1-\nu}} t^{-\frac{1-\mu p}{(p-1)(1-\nu)}} \left(\left(\frac{1-\nu}{M}\right)^{\frac{1}{1-\nu}} - \frac{\Gamma(\rho)t^{1-\nu}}{M} \right),$$

where $D(u)$ is the total energy and $\Gamma(\rho)$ stands for

$$\int_{\rho_0}^{\rho} (\tau-\rho_0)^{-\frac{1}{1-\nu}} \tilde{M} \left(\|u_0\|^{\gamma+1}_{L^{\gamma+1}(B_\rho(x_0))} + \|f\|^{(\sigma+1)/\sigma}_{L^{(\sigma+1)/\sigma}(P(\rho,\vartheta))} \right)^\nu \qquad (2.4.44)$$

So we have only to show that we can find $\rho^* \in (\rho_0, \rho_1)$ such that $H(\rho^*) = 0$. So we just have to observe that $H(\rho_0) > 0$, and, noting that from the hypothesis of the theorem $p < \frac{1}{\mu}$ we have that $-\frac{1-\mu p}{(p-1)(1-\nu)} < 0$ and then, the function $H(\rho)$ satisfies

$$H(\rho) \to -\infty$$

when $t \to 0$, for each $\rho > \rho_0$.
This allows us to conclude that there exists a t^*, sufficently small, such that

for some $\rho^* \in (\rho_0, \rho_1)$, we have $H(\rho^*) = 0$.
From the lemma we have then:

$$Y(\rho_0) = 0,$$

and the proof of Theorems 2.4.1, 2.4.2 is thus completed.

Chapter 3

Applications

3.1 Spatial localizations of solutions of a Boussinesq type system with nonlinear thermal diffusion.

3.1.1 The model

The Boussinesq system of hydrodynamics equations arises from a zero order approximation to the coupling between the Navier-Stokes equations and the thermodinamic equation. The presence of density gradients in a fluid leads to the conversion of gravitational potential energy into motion through the action of buoyant forces. Density differences are induced, for instance, by gradients of temperature arising by nonuniform heating of the fluid. In the Boussinesq approximation of a large class of flow problems, thermodinamical coefficients such as viscosity, specific heat and thermal conductivity, can be assumed constant, leading to a coupled system with linear second order operators in the Navier-Stokes and heat equations. However, there are some fluids like lubricants or some plasma flow for which this is no longer an accurate assumption. In this situation the following system of equations must be considered in the unknown (W, θ):

$$\begin{aligned}
&\frac{\partial}{\partial t}W + (W\cdot\nabla)W - div(\mu(\theta)D(W)) + \nabla p = F(\theta),\\
&divW = 0, \qquad\qquad (3.1.1)\\
&\frac{\partial}{\partial t}\psi(\theta) + div(W\psi(\theta)) - \triangle\varphi(\theta) = 0
\end{aligned}$$

where W is the velocity field of the fluid, θ the temperature, p the pressure, $\mu(\theta)$ the viscosity of the fluid, $F(\theta)$ the buoyancy force, $D(W) := \nabla W +$

∇W^T,

$$\psi(\theta) := \int_{\theta_0}^{\theta} G(s)ds \ \ and \ \ \varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)ds,$$

with $C(\theta)$ and $\kappa(\theta)$ being the specific heat and thermal conductivity of the fluid respectively.
This problem was studied from several authors, and a result of existence is in [11]. Allowed by this results we introduce now the notion of weak solution where we follow the usual variational approach, already introduced by Leray [16], based on the consideration of the notion of weak solution for the Navier-Stokes equations holding in divergence free functional spaces (denoted with the subindex σ). More precisely, we consider the following problem:

$$\begin{aligned}
&\frac{\partial}{\partial t}W + (W\cdot\nabla)W - div(\mu(\theta)D(W)) + \nabla p = F(\theta) \ \text{ in } Q_T := \Omega\times(0,T),\\
&\frac{\partial}{\partial t}\psi(\theta) + div(W\psi(\theta)) - \triangle\varphi(\theta) = 0 \ \text{ in } Q_T,\\
&W = 0 \ \text{ and } \varphi(\theta) = \varphi_D \text{ in } \Sigma_T := \partial\Omega\times(0,T),\\
&W(x,0) = W_0(x) \ \text{ and } \theta(x,0) = \theta_0(x) \ \text{ for } x\in\Omega,
\end{aligned} \tag{3.1.2}$$

with the following structural hypotheses:

$$M_1|\zeta|^p \le \nabla\varphi(\theta)\cdot\zeta \le M_2|\zeta|^p$$

$$M_3|s|^{\gamma+1} \le G(x,s) \le M_4|s|^{\gamma+1}$$

$$G(x,s) = \psi(x,s)s - \int_0^s \psi(x,\tau)d\tau$$

forall $(x,t,s,\zeta)\in\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^n$ and for some constants $M_i,\gamma,>0$ and $p>1$,
with $\Omega\subset\mathbb{R}^N$ $(N=2,3)$ a bounded domain and T arbitrarily fixed. Besides, we assume the following regularity on the auxiliary data: $W_0\in L^2_\sigma(\Omega)$, $\theta_0\in\ L^\infty(\Omega)$, $\theta_0\ge 0$ and $\varphi_D\in L^2(0,T,H^1(\Omega))\cap H^1(0,T,L^2(\Omega))$.

Definition 3.1.1. *The pair (W,θ) is said to be a weak solution of equation (3.1.2) if:*

1. $W\in L^2(0,T;W^{1,2}_\sigma(\Omega))\cap L^\infty(0,T;L^2_\sigma(\Omega))\cap H^1(0,T,H^{-1}(\Omega))$,
   $\varphi\in\varphi_D + L^2(0,T;H^1_0(\Omega))$ *and* $\theta\in L^\infty(Q_T)$.

2. $W(0) = W_0$ *and for any test function* $\xi \in W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)$ *it holds*

$$\int_\Omega (W_t \cdot \xi + (W \cdot \nabla) W \cdot \xi + \mu(\theta) D(W) \cdot \nabla \xi) dx = \int_\Omega F(\theta) \cdot \xi dx \quad a.e.t \in (0,T), \tag{3.1.3}$$

3. $\psi_t(\theta) \in L^2(0,T,H^{-1}(\Omega))$ *and for any test functions* $\zeta \in L^2(0,T,H^1_0(\Omega))$, *it holds*

$$\begin{aligned} &\int_{Q_T} \{\psi(x,\theta)\zeta_t - \nabla\varphi(\theta) \cdot \nabla\zeta + \psi(\theta) W \cdot \nabla\zeta\} dx d\tau \\ &- \int_\Omega \psi(x,\theta)\zeta dx \,|_{t=0}^{t=T} = 0 \end{aligned} \tag{3.1.4}$$

In [11]. Galiano and Diaz, proved that there exists a weak solution of (3.1.2) provided:

$$\varphi \in C([0,\infty)) \cap C^1((0,\infty)), \quad \varphi(0) = 0, \ \varphi \text{ nondecreasing}$$

$$F \in C^{0,1}_{loc}([0,\infty); \mathbb{R}^N),$$

$$\mu \in C^{0,1}_{loc}([0,\infty)) \ \text{ and } 0 < m_0 \le \mu(s) \le m_1 \ \ \forall \ s \in [0,\infty)$$

and in addition φ^{-1} is Holder continuous of exponent α whenever $\mu' \neq 0$ and $F' \neq 0$. Moreover, if $N = 2$, μ is constant and either $\varphi^{-1} \in C^{0,1}_{loc}([0,\infty))$ or

$$\varphi'(0) = 0, \quad \varphi'(s) > 0 \ \text{ and } \ \varphi''(s) > 0 \text{ if } s > 0 \text{ and } \nabla\theta \in L^2(Q_T)$$

then the solution is unique. In the sequel we shall show that, under suitable assumptions on the data, the spatial localization property of finite speed of propagation, holds for the temperature component of any weak solution.

3.2 Spatial localization

Now we present here a result concerning the existence of spatial localization of free boundaries (boundary of sets $\{\theta = 0\}$) for problem (3.1.2), more precisely the finite speed of propagation. It is well known that solutions to porous medium type equations exhibit the property of finite speed of propagation (compactly supported solutions) when the initial data vanishes in some part of the domain. The usual way to show this property relies in the existence of a comparison principle for the problem and the use of special subsolutions that already enjoy the property. However, this method is very sensible to perturbations in the problem and it fails when subsolutions are

hard to find (for instance when coefficients depend on space and time variables or when the space dimension is greater than one) or when simply, a comparison principle does not hold (systems of equations, in general). In these situations more general methods (although less accurate) must be considered.
To deal with problem (3.1.2), we introduce a variation of the techniques presented in the preceeding chapters, intended to handle the transport term present in the heat equation; intuitively this term should not affect the existence and properties of the free boundary but only its location, moving it in the direction of the velocity field. This approach improves previous results when a transport term is present in the equation [9]. Until to start we need the following regular hypothesis on the vectorial field W
Hypothesis H1

$$W \in L^\infty_\sigma(Q_T)$$

.

3.2.1 The local energy method approach.

Like widely explained in the previous chapter, the first step to the application of the energy methods is the choice of suitable energy domains on which one localizes the weak formulation of our PDE. As we are interested to the finite speed of propagation property, the appropriate choice should be to take cylindrical domains of the form:

$$P(\rho, t) = \{(x, \tau) \in Q_T : |x - x_0| \leq \rho,\ \tau \in (0, t)\}$$

with $\rho \in (0, \rho_0),\ \rho_0 = \operatorname{dist}(x_0, \partial\Omega)$, $\tau \in (0, T)$ and x_0 arbitrary in Ω. However such a choice now fails, because in the present case we have to keep into account the contribution of the transport term. So let us consider energy domains of the form

$$P(\rho, t) = \{(x, \tau) \in Q_T : x \in B_{R(\rho,\tau)(x_0)},\ \tau \in (0, t)\},$$

with $R(\rho, \tau) = \rho - \tau u$, $u = \|W\|_{L^\infty(Q_T)}$, $\rho \in I := (tu + \epsilon, \rho_0)$ for $\epsilon > 0$ and $\rho_0 \leq \operatorname{dist}(x_0, \partial\Omega)$, and $t < t_1 = \frac{(\rho_0 - \epsilon)}{u}$. They represent truncated cones with the upper base shrinking in time. We introduce the time section of $P_\rho(s)$

$$P_\rho(s) := \{x \in \mathbb{R}^n : (x, s) \in P(\rho, t)\} \subset \mathbb{R}^N, \quad s \in (0, t),$$

so that $P(\rho, t) = \cup_{s \in (0,t)} P_\rho(s)$, and the lateral boundary of $P(\rho, t)$,

$$\partial_l P(\rho, t) := \{(x, \tau) \in \mathbb{R}^N \times \mathbb{R}_+ : x \in \partial(B_{R(\rho,\tau)(x_0)}),\ \tau \in (0, t)\};$$

the parabolic boundary of $P(\rho,t)$ is given by $P_\rho(0)\cup P_\rho(t)\cup\partial_l P(\rho,t)$.
We stress moreover that the outer normal vector to $\partial_l P(\rho,t)$ has the form

$$n=\frac{1}{\sqrt{1+u^2}}(1,u) \tag{3.2.1}$$

and the following relation holds

$$\begin{aligned}
&\frac{d}{d\rho}\int_{P(\rho,t)} g(x,\tau)dxd\tau = \\
&=\frac{d}{d\rho}\left\{\int_0^t d\tau\int_0^{\rho-\tau u} r^{N-1}dr\int_{\partial B_1} g(x_0+r\omega)d\omega\right\} \\
&=\int_0^t d\tau\int_{\partial B_1}\left\{(\rho-\tau u)^{N-1}g(x_0+(\rho-\tau)\omega,\tau)\right\}d\omega \\
&=\int_{\partial_l P} gd\Gamma d\tau,
\end{aligned} \tag{3.2.2}$$

with Γ the element of surface on $\partial_l P(\rho,\tau)$
Now we are ready to state and proof the following theorem:

Theorem 3.2.1. *Suppose that $\gamma<1$ and that $\theta_0\equiv 0$ in B_{ρ_0}. Then there exist a $t^*>0$ and a nonnegative function $r(\tau)$ defined in $(0,t^*)$, with $r(0)=\rho_0$, such that for any wake solution (W,θ) of 3.1.2 the function θ satisfies*

$$\theta(x,t)\equiv 0 \ \ a.e. \ in \ \ \{(x,\tau): x\in B_{r(\tau)},\ \tau\in(0,t^*)\}.$$

Proof. First of all we define a suitable *energy function* on the energy sets defined above:

$$E(\rho,t):=\int_{P(\rho,t)}|\nabla\theta(x,\tau)|^p dxd\tau,\ \ b(\rho,t):=\text{ess sup}_{\tau\in(0,t)}\int_{P_\rho(t)}|\theta(x,\tau)|^{\gamma+1}dx, \tag{3.2.3}$$

as well as the *global total energy*,

$$D(\theta)=\int_{Q_T}|\nabla\theta|^p dxd\tau+\text{ess sup}_{\tau\in(0,T)}\int_\Omega|\theta(x,\tau)|^{\gamma+1}dx,$$

that we will always suppose finite.
To localize the problem we will make use of the following *cut-off* functions.
Given $x_0\in\Omega$, $t\in[0,T]$, $\vartheta\geq 0$ and $\upsilon\in(0,1)$, we define:

$$\zeta(x,\tau):=\psi_\varepsilon(|x-x_0|,\tau)\,\xi_k(\tau)\frac{1}{h}\int_\tau^{\tau+h}T_m(\theta(x,s))\,ds,\quad h>0,$$

where

$$T_m\left(\theta(x,\tau)\right) := sign\,\theta(x,\tau)\min\left\{m, |\theta(x,\tau)|\right\}, \quad m \in \mathbb{N},$$

and

$$\xi_k(\tau) := \begin{cases} 1 & \text{if } \tau \in \left[0, t-\frac{1}{k}\right], \\ k(t-\tau) & \text{for } \tau \in \left[t-\frac{1}{k}, t\right], \quad k \in N^+, \end{cases}$$

$$\psi_\varepsilon\left(|x-x_0|,\tau\right) := \begin{cases} 1 & \text{if } d > \varepsilon, \\ \frac{1}{\varepsilon}d & \text{if } d < \varepsilon, \\ 0 & \text{if } (x,\tau) \in Q_T \setminus P(\rho,t), \quad \epsilon > 0. \end{cases}$$

Here $d = dist((x,\tau), \partial_l P(\rho,t))$. Let us go now to substitute this functions in (3.1.4) and to proceed with a limit process: this will give us a localized version of the weak formulation from which we will start to deduce a differential inequality like exaustively stressed in the previous chapter. So we have:

$$\begin{aligned} &\int_0^t \int_\Omega \nabla\varphi(\theta)\nabla\zeta dxd\tau - \int_0^t \int_\Omega \psi(\theta)W\cdot\nabla\zeta dxd\tau = \\ &\int_0^t \int_\Omega \psi(\theta)\frac{\partial}{\partial t}\zeta dxd\tau + \int_\Omega \psi(\theta)\zeta dx\,|_{\tau=0} \end{aligned} \tag{3.2.4}$$

We start, analyzing the first term at the right member, by splitting the time domain in a suitable way:

$$\begin{aligned} &\lim_{k\to\infty}\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\zeta dxd\tau = \\ &\lim_{k\to\infty}\int_0^{t-\frac{1}{k}} \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\left(\psi_\varepsilon(x,\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(\theta(x,s))ds\right)dxd\tau + \\ &\lim_{k\to\infty}\int_{t-\frac{1}{k}}^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\left(\psi_\varepsilon(x,\tau)k(t-\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(\theta(x,s))ds\right)dxd\tau. \end{aligned} \tag{3.2.5}$$

Taking the limit and observing that we can always suppose that $\tau = t$ is a Lebesgue point for the function

$$\tau \longrightarrow \psi(\theta(x,\tau))\frac{1}{h}\int_\tau^{\tau+h} T_m(\theta(x,s))ds,$$

we have

$$\begin{aligned} &\lim_{k\to\infty}\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\zeta dxd\tau = \\ &\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta} dxd\tau - \int_{B_{R(\rho,t)}(x_0)\cap\{\tau=t\}} \psi(\theta(x,\tau))\tilde{\zeta} dx, \end{aligned} \tag{3.2.6}$$

where
$$\tilde{\zeta}(x,\tau) = \psi_\varepsilon(x,\tau)\frac{1}{h}\int_\tau^{\tau+h} T_m(\theta(x,s))ds$$
So, gathering this with (3.2.4) we obtain:
$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \left[\nabla\varphi(\theta(x,\tau))\cdot\nabla\tilde{\zeta} - \psi(\theta(x,\tau))W\cdot\nabla\tilde{\zeta}\right] dxd\tau =$$
$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau - \left[\int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\tilde{\zeta}dx\right]_{\tau=0}^{\tau=t}. \tag{3.2.7}$$
we turn now our attention to the term
$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0))} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau,$$
We start by splitting the spatial domain in a suitable way:
$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0))} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau =$$
$$\int_0^t \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))dxd\tau +$$
$$\int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\left[\frac{1}{\epsilon}(R(\rho,\tau)-|x-x_0|)\frac{1}{h}\int_\tau^{\tau+h} T_m(x,\theta)ds\right]dxd\tau$$
then computing term by term the respective derivative we have:
$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0))} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau =$$
$$\int_0^t \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))dxd\tau +$$
$$\int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(\frac{\partial}{\partial\tau}R(\rho,\tau))\frac{1}{h}\int_\tau^{\tau+h} T_m(\theta(x,\tau))dsdxd\tau +$$
$$\int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(R(\rho,\tau)-|x-x_0|)\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))dxd\tau$$
Now we recall that the functions
$$j(r) = \int_0^r \psi(\theta(x,s))ds$$
is convex in $\mathbb{R}$ and increasing in $\mathbb{R}^+$, (decreasing in $\mathbb{R}^-$). Hence it is straightforward to verify , using the definition of T_m, that
$$\psi(\theta(x,\tau))(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau))) = j'(\theta(x,\tau))(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))$$
$$\leq j(T_m(\theta(x,\tau+h))) - j(T_m(\theta(x,\tau)))$$

and we can write:

$$\int_0^t \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))dxd\tau$$
$$\leq \int_0^t \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(j(T_m(\theta(x,\tau+h))) - j(T_m(\theta(x,\tau))))dxd\tau$$
$$= \int_t^{t+h} \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(j(T_m\theta(x,\tau)))dxd\tau$$
$$- \int_0^h \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}j(T_m(\theta(x,\tau)))dxd\tau$$

Passing to the limit when $h \to 0$, for almost all ρ we have

$$\int_0^t \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau))dxd\tau$$
$$\leq \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau))j(T_m(\theta(x,\tau))dx \quad - \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta_0)j(T_m u(0))dx$$

and so:

$$\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi(\theta(x,\tau))\frac{\partial}{\partial\tau}\tilde{\zeta}dxd\tau \leq \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau)j(T_m\theta(x,\tau)) \mid_{\tau=0}^{\tau=t}$$
$$+ \int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(\frac{\partial}{\partial\tau}R(\rho,\tau))T_m(\theta(x,\tau))dxd\tau$$
$$+ \lim_{h\to 0} \int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(R(\rho,\tau) - |x-x_0|)$$
$$\times\frac{1}{h}(T_m(\theta(x,\tau+h)) - T_m(\theta(x,\tau)))\ dxd\tau$$

Rewriting in spherical coordinates the term

$$\int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(R(\rho,\tau)-|x-x_0|)\frac{1}{h}(T_m(\theta(x,\tau+h))-T_m(\theta(x,\tau))dxd\tau$$

we have

$$\int_0^t \int_{R(\rho,\tau)-\epsilon}^{R(\rho,\tau)} \rho^{N-1}d\rho \int_{S^{N-1}} \psi(\theta(x,\tau))\frac{1}{\epsilon}(R(\rho,\tau)-\rho)\frac{1}{h}(T_m(\theta(\tau+h))-T_m(\theta(\tau)))d\omega d\tau$$

and so, having

$$\frac{1}{\epsilon}(R(\rho,\tau) - \rho) < 1$$

by the definition of the cut-off functions, we can conclude that the entire integral goes to zero when ϵ and h goes to 0.

Moreover from de definition of the energy domain we have

$$\frac{\partial}{\partial \tau} R(\rho,\tau) = -\|W\|_{L^\infty(Q_T)},$$

so we have

$$\int_0^t \int_{B_{R(\rho,\tau)}} \psi(\theta(x,\tau)) \frac{\partial}{\partial \tau}\tilde{\zeta} dx d\tau \leq \int_{\{|x-x_0|<R(\rho,\tau)-\epsilon\}} \psi(\theta(x,\tau)) j(T_m(\theta(x,\tau))) \mid_{\tau=0}^{\tau=t}$$
$$-\int_0^t \int_{\{R(\rho,\tau)-\epsilon<|x-x_0|<R(\rho,\tau)\}} \psi(\theta(x,\tau)) \frac{1}{\epsilon}\|W\|_{L^\infty(Q_T)} T_m(\theta(x,\tau)) dx d\tau.$$

Recall now that for the second term of (3.2.4) the following equality holds:

$$\begin{aligned} &\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \psi W \cdot \nabla \zeta dx d\tau = \\ &\int_{\partial_l P(\rho,\tau)} \psi(\theta(x,\tau)) \zeta W \cdot \mathbf{n_x} d\Gamma d\tau - \int_{P(\rho,\tau)} \psi(\theta(x,\tau)) div(W) \zeta dx d\tau, \end{aligned} \tag{3.2.8}$$

and remembering that weak solution of our problem lives in free-divergence spaces, as stated in definition 3.1.1, the second term above is zero.
Now taking the $\lim_{\epsilon\to 0}$ and returning to 3.2.7 we can write

$$\begin{aligned} &\int_0^t \int_{B_{R(\rho,\tau)}(x_0)} \nabla\varphi(\theta(x,\tau) \nabla T_m(\theta(x,\tau)) dx d\tau \\ &\leq \int_{\partial_l P(\rho,\tau)} \theta(x,\tau) \nabla\varphi(\theta(x,\tau)) \cdot n_x d\Gamma d\tau \\ &- \left[\int_{B(R(\rho,\tau)} (\psi(\theta(x,\tau)) T_m(\theta(x,\tau)) - j(T_m(\theta(x,\tau)))) dx \right]_{\tau=0}^{\tau=t} \\ &- \int_{\partial_l P(\rho,\tau)} \psi(\theta(x,\tau)) T_m(\theta(x,\tau)) (\|W\|_{L^\infty(Q_T)} + W \cdot n_x) d\Gamma d\tau, \end{aligned} \tag{3.2.9}$$

Taking now the limit, when $m \to \infty$, and taking in account the vanishing hypotesis on θ_0 and f, in theorem (3.2.1), we are allowed to state the following integral inequality:

$$\begin{aligned} &i_1 + i_2 + i_3 = \\ &\int_{B_{R(\rho,\tau)}(x_0)\cap\{\tau=t\}} G(x,\theta(x,\tau)) dx + \int_0^t \int_{B_{R(\rho,\tau)}} \nabla\varphi(\theta(x,\tau)) \nabla\theta(x,\tau) dx d\tau \\ &\leq \int_{\partial_l P} |\theta(x,\tau)| |\nabla\psi(\theta(x,\tau))| |\mathbf{n_x}| d\Gamma \\ &- \int_{\partial_l P} \psi(\theta(x,\tau)) \theta(x,\tau) (\|W\|_{L^\infty(Q_T)} + W \cdot \mathbf{n_x}) d\Gamma = j_1 + j_2. \end{aligned} \tag{3.2.10}$$

We are finally ready to start with the core of the local energy method, that is the deduction of an ordinary differential inequality in the *local energy functions* defined in (3.2.3).
We begin by estimating the term j_1. On one hand, the function

$$\rho \to \int_{\partial_l P} |\nabla \varphi(\theta)|^p d\Gamma$$

is well defined for a.e.$\rho \in [0, \rho_0]$ because of the regularity $\theta \in L^2(0,T; H^1(\Omega))$.
We need Hölder inequality to get

$$j_1 \leq M_1 \left(\int_0^t \int_{\partial B_{R(\rho,\tau)}(x_0)} |\nabla \theta(x,\tau)|^p ds d\tau \right)^{\frac{p-1}{p}} \left(\int_0^t \int_{\partial B_{R(\rho,\tau)}(x_0)} |\theta(x,\tau)|^p dx d\tau \right)^{\frac{1}{p}}. \tag{3.2.11}$$

On the other hand, by 3.2.2 we have

$$\frac{\partial E}{\partial \rho} = \int_{\partial_l P} |\nabla \theta(x,\tau)|^p d\Gamma$$

a.e. $\rho \in I = (tu + \epsilon,]$, we have

$$j_1 \leq \|\theta\|_{L^p(\partial_l P(\rho,t))} \left(\frac{\partial E}{\partial \rho}(\rho, t) \right)^{\frac{p-1}{p}}.$$

Now, taking into account that $\partial_l P(\rho,t) = \bigcup_{\tau \in (0,t)} \partial B_{\rho,\tau}$, and applying the *interpolation-trace* inequality, we obtain

$$\|\theta\|_{L^p(\partial B(\rho,\tau))} \leq C_1 \left(\|\nabla \theta\|_{L^p(B(\rho,\tau))} + R(\rho,\tau)^\delta \|\theta\|_{L^{\gamma+1}(B(\rho,\tau))} \right)^\eta \|\theta\|^{1-\eta}_{L^{\gamma+1}(B(\rho,\tau))}, \tag{3.2.12}$$

with $C_1 > 0$ an universal constant and with

$$\eta := \frac{N(p-1-\gamma)+1+\gamma}{N(p-1-\gamma)+p(\gamma+1)} \in (0,1) \text{ and } \delta := -\frac{N(p-1-\gamma)+p(\gamma+1)}{p(\gamma+1)}.$$

Notice that since $\rho \in I$

$$R(\rho,\tau)^\delta \leq \left(\min_{\tau \in (0,t)} (\rho - u\tau) \right)^\delta \leq \epsilon^\delta.$$

Defining $L := C_1 \max\{1, \epsilon^{\eta\delta}\}$ and integrating expression 3.2.12 in $(0,t)$, we obtain

$$\int_0^t \|\theta\|^p_{L^p(\partial B(\rho,\tau))} d\tau \leq L \int_0^t \left(\|\nabla \theta\|_{L^p(B(\rho,\tau))} + \|\theta\|_{L^{\gamma+1}(B(\rho,\tau))} \right)^{p\eta} \|\theta\|^{p(1-\eta)}_{L^{\gamma+1}(B(\rho,\tau))} d\tau,$$

now thanks to the inequality $(a+b)^p \leq 2^p(a^p+b^p)$ and to Holder's inequality with exponent $\frac{1}{\eta}$ we obtain

$$\int_0^t \|\theta\|^p_{L^p(\partial B(\rho,\tau))} d\tau \leq \tilde{L} \left(\int_0^t \|\nabla\theta\|^p_{L^p(B(\rho,\tau))} + \|\theta\|^p_{L^{\gamma+1}(B(\rho,\tau))} d\tau \right)^\eta \left(\int_0^t \|\theta\|^p_{L^{\gamma+1}(B(\rho,\tau))} \right)^{1-\eta}. \tag{3.2.13}$$

Noting that $\gamma < 1$, and that the energy function

$$b(\rho,t) := \text{ess sup}_{\tau\in(0,t)} \int_{B(\rho,\tau)} |\theta(x,\tau)|^{\gamma+1} dx$$

is nondecreasing with respect to ρ, and substituting the expressions of energies given in (3.2.3) we obtain

$$\begin{aligned} &\left(\int_0^t \|\theta\|^p_{L^p(\partial B(\rho,\tau))} d\tau \right)^{\frac{1}{p}} \\ &\leq Ct^{\frac{1-\eta}{p}} \left(E(\rho,t) + tb(\rho_0,t)^{\frac{p}{\gamma+1}-1} b(\rho,t) \right)^{\frac{\eta}{p}} b(\rho,t)^{\frac{(1-\eta)}{(\gamma+1)}}, \end{aligned}$$

and therefore, defining

$$K_0(t) := Ct^{\frac{1-\eta}{p}} \max\{1, Tb(\rho_0,T)^{\frac{p}{\gamma+1}-1}\}, \tag{3.2.14}$$

and observing that, due to the hypothesis of finiteness of the total energy $D(\theta)$, we can always find a finite constant K such that

$$K_0(t) \leq t^{\frac{1-\eta}{p}} CD(\theta) \leq Kt^{\frac{1-\eta}{p}},$$

we finally get

$$\left(\int_0^t \|\theta\|^p_{L^p(\partial B(\rho,\tau))} \right)^{\frac{1}{p}} \leq Kt^{\frac{1-\eta}{p}} (b(\rho,t) + E(\rho,t))^\nu$$

with

$$\nu := \frac{\eta}{p} + \frac{1-\eta}{\gamma+1} < 1 \tag{3.2.15}$$

We conclude from (3.2.13) that

$$j_1 \leq Kt^{\frac{1-\eta}{p}} \left(\frac{\partial E(\rho,t)}{\partial \rho} \right)^{\frac{p-1}{p}} (b(\rho,t) + E(\rho,t))^\nu. \tag{3.2.16}$$

So let us pass now to the estimate of the term j_2. But this is very simple because it is sufficient to observe that, just due to the appropriate choice of the energy domain where we have applied the process of localization, we have:

$$\|W\|_\infty + W \cdot \mathbf{n_x} \geq 0,$$

and so we can estimate

$$j_2 \leq 0. \tag{3.2.17}$$

Hence we have from (3.2.10) and the structural hypothesis, the following inequality:

$$\begin{aligned} &M\left(\int_{p(t)} |\theta|^{\gamma+1} dx d\tau + E(\rho,\tau)\right) \leq \\ &i_1 + i_2 + i_3 \leq Kt^{\frac{1-\eta}{p}} \left(\frac{\partial E(\rho,t)}{\partial \rho}\right)^{\frac{p-1}{p}} (b(\rho,t) + E(\rho,t))^{\nu}, \end{aligned} \tag{3.2.18}$$

with $M = \min(M_1, M_5)$. Since functions E, b and $\frac{\partial E}{\partial \rho}$ are non decreasing in time, (3.2.18) remains valid if we change $\int_{P(\rho,t)} |\theta|^{\gamma+1}$ by $b(\rho,t)$. We get

$$M(b(\rho,t) + E(\rho,t))Kt^{\frac{1-\eta}{p}} \left(\frac{\partial E(\rho,t)}{\partial \rho}\right)^{\frac{p-1}{p}} (b(\rho,t) + E(\rho,t))^{\nu},$$

and therefore raising both sides to the exponent $\frac{p}{p-1}$:

$$(b(\rho,t) + E(\rho,t))^{\kappa} \leq \bar{K} t^{\frac{1-\eta}{p-1}} \frac{\partial E(\rho,t)}{\partial \rho} \quad \text{a.e}\rho \in I, \tag{3.2.19}$$

with

$$\kappa := \frac{p(1-\nu)}{p-1}, \tag{3.2.20}$$

ν given by eq. (3.2.15).
Finally, due to the crucial assumption $\gamma < 1$ we have that $\kappa < 1$ and a direct integration of (3.2.19) in the interval (ρ, ρ_0) with $\rho \in I$ leads to

$$E^{1-\kappa}(\rho,t) \leq E^{1-\kappa}(\rho_0,t) - \tilde{M} t^{-\frac{1-\eta}{p-1}} (\rho_0 - \rho). \tag{3.2.21}$$

Now recalling the hypothesis of boundedness of the *total energy* $D(\theta) < D^*$, we have

$$E^{1-\kappa}(\rho,t) \leq D^{*1-\kappa} - \tilde{M} t^{-\frac{1-\eta}{p-1}} (\rho_0 - \rho). \tag{3.2.22}$$

We define

$$\alpha(t) := \rho_0 - \tilde{M} t^{\frac{1-\eta}{p-1}} D^{*1-\kappa}. \tag{3.2.23}$$

Observe now that

$$\alpha(0) = \rho_0$$

since $\lim_{t\to 0} D^{*1-\kappa} t^{\frac{1-\eta}{p-1}} = 0$, and that $\alpha(t)$ is continuous and decreasing with

$$\lim_{t\to\infty} \alpha(t) = -\infty.$$

Therefore $\alpha(t)$ has a unique positive zero which we denote by t_0. With this definition of $\alpha(t)$ we have that if $t \in (0, t^*)$ for some $t^* \in (0, \min\{\hat{t}, t_0\}]$ and $\rho \in (ut + \epsilon, \alpha(t))$, then 3.2.22 implies that $E(\rho, t) \equiv 0$. Notice that the interval $(ut + \epsilon, \alpha(t))$ is nonempty if

$$\rho_0 > \tilde{M} t^{\frac{1-\eta}{p-1}} D^{*1-\kappa} + ut + \epsilon$$

that we can ensure by taking t^*, ϵ small enough. Finally from the definition of $E(\rho, t)$ we deduce that

$$\theta(x, t) \equiv 0 \quad \text{if} \quad |x| \leq \rho - ut \leq \alpha(t) - ut := r(t),$$

with $r(0) = \rho_0$ and $r(t) > 0$ in $(0, t^*)$. That is, the solution possesses the finite speed of propagation property, and the proof is completed. $\square$

Appendix A

The cut-off functions

In the whole implementation of the local energy method we have made an heavily use of suitables cut-off functions, with the aim to provide the different localization in in the so called energy domain, we want now to make rigorous their employment. we shall trate here only the case of the elliptic diffusion equation, being the extension to the parabolic case very similar. We start with the definition of wake solution for the general *elliptic reaction-diffusion equation* presented in chapter two.

$$-divA(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) = f(x,t), \tag{A.0.1}$$

with the same structural hypotheses stated in chapter one.
Recall now that we have the following definition:

Definition A.0.1. *Let $f \in L^1_{loc}(\Omega)$. A locally integrable function u is said to be a weak solution of (A.0.1) if*

1. $u \in W^{1,p}_{loc}(\Omega)$;
2. $B(\cdot, u, \nabla u), A_i(\cdot, u, \nabla u) \in L^1_{loc}(\Omega), i = 1, ...N$;
3. $C(\cdot, u) \in L^1_{loc}(\Omega)$;
4. for any test function $\varphi \in C_0^\infty(\Omega)$, the equality

$$\int_\Omega \{A(x,u,\nabla u) \cdot \nabla\varphi + B(x,u,\nabla u)\varphi + C(x,u)\varphi\}dx = \int_\Omega f\varphi dx \tag{A.0.2}$$

holds.
So our task is to prove that the cutoff function like defined in (A.0.3) belongs to $W_0^{1,p}(B_\rho) \cap L^\infty(B_\rho)$, and then utilizing that the set $C_0^\infty(B_\rho)$ is dense in

$W_0^{1,p}(B_\rho) \cap L^\infty(B_\rho)$ we can conclude they are admissibile test function. The proof for the function defined in the other case is the same, so let us start to consider $T_k(u) = \min(k, |u|) sign\ u$ with $k \in \mathbb{N}$, and

$$\psi_n(r) = \left\{ \begin{array}{cc} 1 & if\ r \in [0, \rho - \frac{1}{n}], \\ n(\rho - r) & if\ r \in [\rho - \frac{1}{n}, \rho], \\ 0 & if\ r \in [\rho, \rho_0],\ n \in \mathbb{N}. \end{array} \right\}, \tag{A.0.3}$$

and so the cut-off function $\Gamma_{n,k}(u) = \psi_n(r) T_k(u)$. We will make use of a version of a lemma due to Stampacchia [5], and for it we will need a very rasonable hypothesis on the domain *Omega*,, we will suppose for Ω the *segment property*, (see [2]).

Lemma A.0.1. *Let $\Omega \subset \mathbb{R}^n$ and bounded with the* segment,property*, $u \in W^{1,p}(\Omega)$ and be $t \to G(t)$ an uniformly Lipschitz function, that is*

$$|G(t') - G(t'')| \leq L|t' - t''|$$

defined for all $t \in \mathbb{R}$, such that $G(0) = 0$, then $G(u) \in W^{1,p}(\Omega)$.

Proof. Let $\{u_n\}$ be a sequence in $C^1(\bar{\Omega})$, and $u_n \to u$ in $W^{1,p}(\Omega)$, putting

$$v_m = G(u_m)$$

we have that every v_m is continous in Ω and Lipschitz in $\bar{\Omega}$, so $\frac{\partial v_m}{\partial x_i} \in L^\infty(\Omega)$, i.e. $\sup_\Omega |\frac{\partial v_m}{\partial x_i}| \leq C_m K$; then $v_m \in W^{1,p}$ for the boundedness of Ω. It is now easy to show that

$$\|v_m - G(u)\|_{L^p(\Omega)} \leq K \|u_m - u\|_{L^p(\Omega)}$$

and

$$\limsup_{m\to\infty} \|\frac{\partial v_m}{\partial x_i}\|_{L^p(\Omega)} \leq K \limsup_{m\to\infty} \|\frac{\partial u_m}{\partial x_i}\|_{L^p(\Omega)} \leq K\|u\|_{w^{1,p}(\Omega)}.$$

Then $\frac{\partial v_m}{\partial x_i}$ belongs to a bounded subset of $L^p(\Omega)$. So we can extract a sub sequence still denoted by v_m such that v_m converges weakly in $W^{1,p}(\Omega)$ to $G(u)$. Now for the Banach-Saks theorem, the sequence w_m defined in the following manner:

$$w_m = c_1 v_1 + \cdots + c_m v_m \ \text{ with } c_j > 0,\ \sum c_j = 1,$$

converges strongly in $W^{1,p}(\Omega)$ to $G(u)$, so we conclude that $G(u) \in W^{1,p}(\Omega)$. □

Now returning to the original problem, the proof is elementary, in fact it is sufficient to observe that the functions $T_k(u)$ is uniformly Lipschitz, and so we have that

$$u \in W^{1,p}_{loc}(\Omega) \Rightarrow T_k u \in W^{1,p}_{loc}(\Omega)$$

for every $k \in \mathbb{N}$ and then by definition $T_k(u) \in W^{1,p}(B_\rho) \cap L^\infty(B_\rho)$ holds too, for some $\rho > 0$. Finally the function $\psi_n(r)T_k(u)$ belong to $W^{1,p}_0(\Omega)$, then it is an admissibile test function for our PDE equation.
To conclude we show that the sequence of cut-off functions really stisfy that

$$\|\Gamma_{n,k}(u) - u\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

It is sufficient to show that $\|T_k(u) - u\|_{W^{1,p}(\Omega)} \to 0$, but this follows directly from the definition of $T_k(u)$, in fact we have

$$\nabla T_k(u) = \begin{cases} \nabla u \ \text{ if } x \in B^k_{\rho_0} = \{x : x \in B_{\rho_0}, |u| < k\}, \\ 0 \ \text{ if} x \in B_{\rho_0} \setminus B^k_{\rho_0}, \end{cases}$$

with meas $\left\{B_{\rho_0} \setminus B^k_{\rho_0}\right\} \to 0$ as $k \to \infty$. Then we have

$$\|T_k(u) - u\|_{W^{1,p}} \leq \int_{\{|u|>k\}\cap B_{\rho_0}} |ksign\ u - u|^p dx + \int_{\{|u|>k\}\cap B_{\rho_0}} |\nabla u|^p dx$$

and the second term goes to zero as $k \to \infty$ since $u \in L^\infty(B_{\rho_0})$.

Appendix B

An ordinary differential inequality

We give here a proof of an ordinary differential inequality that plays a fundamental role in the implementation of the local energy methods.

Lemma B.0.2. *Let $E \in W^{1,1}_{loc}(0,R)$, with $E \geq 0$ and $E' \geq 0$, a.e. Assume that the inequality*

$$\Lambda E'(\rho) + F(\rho) \geq E(\rho)^{1-\mu} \quad for\ a.e\ \rho \in (R_0, R) \tag{B.0.1}$$

holds with some $R_0 \in (0,R)$, where $\mu \in (0,1)$, $\Lambda > 0$, $F(t) \geq 0$.
Assume that the integral

$$\Gamma(\rho) = \int_{R_0}^{\rho} (\tau - R_0)^{-\frac{1}{\mu}} F(\tau) d\tau$$

is convergent. Then the function $E(\rho)$ admits the estimate

$$E(R_0) \leq G(\rho) \equiv E(R) - (\rho - R_0)^{\frac{1}{\mu}} \left(\left(\frac{\mu}{\Lambda}\right)^{\frac{1}{\mu}} - \frac{\Gamma(\rho)}{\Lambda} \right) \tag{B.0.2}$$

for every $\rho \in (R_0, R)$, and $E(R_0) = 0$ if there exists $\rho^* \in (R_0, R)$ such that $G(\rho^*) = 0$.

Proof. The function

$$\bar{E}(\rho) := (\mu/\Lambda)^{\frac{1}{\mu}} (\rho - R_0)^{\frac{1}{\mu}}$$

satisfies the conditions

$$\bar{E}^{1-\mu} = \Lambda \bar{E}', \quad \bar{E}(R_0) = 0 \tag{B.0.3}$$

Subtracting the equation (B.0.3) from inequality (B.0.1) , we get the inequality

$$E^{1-\mu} - \bar{E}^{1-\mu} \leq \Lambda(E-\bar{E})' + F(\rho) \tag{B.0.4}$$

Now we observe that from elementary calculus we have:

$$\int_{x_1}^{x_2} f(s)d\tau = \int_0^1 f(\theta x_2 + (1-\theta)x_1)d\theta(x_2 - x_1)$$

So with $f(s) = s^{-\mu}$, we get

$$E^{1-\mu} - \bar{E}^{1-\mu} = (1-\mu)\left\{\int_0^1 (\theta E(\rho) + (1-\theta)\bar{E}(\rho))^{-\mu} d\theta\right\}(E(\rho) - \bar{E}(\rho)). \tag{B.0.5}$$

Now we proceed introducing the function

$$\phi(\rho) = exp\left(\alpha\int_{R_0}^{\rho}\left(\int_0^1 (\theta E(\tau) + (1-\theta)\bar{E}(\tau))^{-\mu} d\theta\right)d\tau\right) \tag{B.0.6}$$

with $\alpha = -(1-\mu)/\Lambda)$. Making use of (B.0.5),(B.0.6), we can rewrite (B.0.4) in the following equivalent form:

$$\frac{d}{d\rho}((E-\bar{E})\phi(\rho)) \geq -\frac{1}{\Lambda}\phi(\rho)F(\rho) \tag{B.0.7}$$

Integrating now (B.0.7) over the interval (R_0, ρ), we arrive at the inequality

$$E(\rho) \geq \bar{E}(\rho) + \frac{E(R_0)}{\phi(\rho)} - \frac{1}{\Lambda\phi(\rho)}\int_{R_0}^{\rho}\phi(\tau)F(\tau)d\tau. \tag{B.0.8}$$

Observe now that the function $E(\rho)$ is increasing in the interval (R_0, R) and $\phi(\rho) < 1$; then we can relax inequality (B.0.8) rewriting it in the form

$$\begin{aligned} E(R) &\geq E(R_0) + \bar{E}(\rho) - \frac{1}{\Lambda}\int_{R_0}^{\rho}\frac{\phi(\tau)}{\phi(\rho)}F(\tau)d\tau \\ &= E(R_0) + E(\bar{\rho}) - \frac{1}{\Lambda}\int_{R_0}^{\rho} F(\tau) \\ &\times exp\left(\frac{1-\mu}{\Lambda}\int_{\tau}^{\rho}\left(\int_0^1 (\theta E(\tau) + (1-\theta)\bar{E}(\tau))^{-\mu} d\theta\right)d\tau\right)d\sigma \end{aligned} \tag{B.0.9}$$

Next, the following chain of relations is true:

$$exp\left(-\frac{1-\mu}{\Lambda}\int_{\tau}^{\rho}\left(\int_0^1 (\theta E(s) + (1-\theta)\bar{E}(s))^{-\mu} d\theta\right)ds\right) \tag{B.0.10}$$

$$\leq exp\left(-\frac{1-\mu}{\Lambda}\left(\int_{\tau}^{\rho}\bar{E}(s)^{-\mu}\int_0^1 (1-\theta)^{-\mu} d\theta ds\right)\right) \tag{B.0.11}$$

$$= exp\left(\int_{\tau}^{\rho}\frac{\bar{E}(s)'}{\bar{E}(s)}ds\right) = exp\left(ln\frac{\bar{E}(\rho)}{E(\sigma)}\right) = \frac{\bar{E}(\rho)}{\bar{E}(\sigma)}, \quad \sigma < \rho.$$

Applying(B.0.3), we infer from (B.0.9),(B.0.10) that

$$0 \leq E(R_0) \leq E(R) - \bar{E}(\rho)\left(1 - \frac{1}{\Lambda}\int_{R_0}^{\rho} \frac{F(\sigma)}{\bar{E}(\sigma)} d\sigma\right) = G(\rho).$$

thus if the equation $G(\rho) = 0$ has a solution in (R_0, R), then $E(R_0) = 0$. $\square$

Appendix C

The interpolation trace inequality

Here we present a very useful interpolation inequality that arises from the theory of embedding of Sobolev Spaces. Its peculiarity is to control of the norm of the boundary trace of a function defined on an open set with some interpolation of suitable integral norms of the same function in the whole open set. It extends of the classical Sobolev embedding theorems and is an important tool in several situations such as for example, in the application of divergence theorems.

Lemma C.0.3. *Suppose G is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$ with C^1 boundary, ∂G and let $0 \leq \sigma \leq q < \infty$. Then there exists a constant C depending on σ , q, and G such that for any $v \in W^{1,q+1}(G)$ we have*

$$\|v\|_{L^{q+1}(\partial G)} \leq C(\|Dv\|_{L^{q+1}(G)} + \|v\|_{L^{\sigma+1}(G)})^{\theta}\|v\|^{1-\theta}_{L^{\sigma+1}(G)}, \tag{C.0.1}$$

where $\theta = \frac{N(q-\sigma)+\sigma+1}{N(q-\sigma)+(\sigma+1)(q+1)}$.

Proof. For the sake of simplicity we restrict ourselves to $v \in C^1(\bar{G})$ since $C^1(\bar{G})$ is dense in $W^{1,q+1}(G)$, The proof of Lemma is divided into four steps. *First step.* From a result of ([3] pp. 45-52), for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for any $v \in C^1(\bar{G})$ the following holds:

$$\|v\|_{L^{q+1}(G)} \leq \epsilon\|Dv\|_{L^{q+1}(G)} + C_\epsilon\|v\|_{L^{\sigma+1}(G)} \tag{C.0.2}$$

if we set $C_2 = \max(1+\epsilon, C_\epsilon|G|^{1-\frac{1}{\sigma+1}})$ we get

$$\|v\|_{W^{1,q+1}(G)} \leq C_2(\|Dv\|_{L^{q+1}(G)} + \|v\|_{L^{\sigma+1}(G)}). \tag{C.0.3}$$

Second step. We start from the classical trace result ([2]pp164-166, theo.5.36) : there exists $C_3 > 0$ such that for any $u \in C^1(\bar{G})$ we have

$$\|u\|_{L^1(\partial G)} \leq C_3 \|u\|_{W^{1,1}(G)} \tag{C.0.4}$$

and for $q > 0$ we apply (C.0.4) to $u = v|v|^q, \ v \in C^1(\bar{G})$, so

$$\int_{\partial G} |v|^{q+1} d\sigma \leq C_3 \left\{ (q+1) \int_G |v|^q |Dv| dx + \int_G |v|^{q+1} dx \right\}.$$

Since

$$\int_G |v|^q |Dv| dx \leq \|Dv\|_{L^{q+1}(G)} \|v\|^q_{L^{q+1}(G)}$$

we get

$$\int_{\partial G} |v|^{q+1} d\sigma \leq C_3 \{(q+1)\|Dv\|_{L^{q+1}(G)} \|v\|^q_{L^{q+1}(G)} + \|v\|^{q+1}_{L^{q+1}(G)}\},$$

which implies

$$\|v\|_{L^{q+1}(\partial G)} \leq ((q+1)C_3)^{\frac{1}{q+1}} \|v\|^{\frac{1}{q+1}}_{W^{1,q+1}(G)} \|v\|^{\frac{q}{q+1}}_{L^{q+1}(G)}. \tag{C.0.5}$$

Third step. Set $0 \leq \sigma \leq q < \infty$. We claim that there exists a constant $C_4 > 0$ such that for any $v \in C^1(\bar{G})$ we have

$$\|v\|_{L^{q+1}(G)} \leq C_4 \|v\|^{\frac{((q+1)\theta-1)}{q}}_{W^{1,q+1}(G)} \|v\|^{\frac{(q+1)(1-\theta)}{q}}_{L^{\sigma+1}(G)}. \tag{C.0.6}$$

Case 1. Assume $q+1 < N$. By classical interpolation in L^p spaces, if $\tau > q+1$ we have

$$\|v\|_{L^{q+1}(G)} \leq \|v\|^{1-\lambda}_{L^\tau(G)} \|v\|^{\lambda}_{L^{\sigma+1}(G)}, \tag{C.0.7}$$

where $\frac{1}{q+1} = \frac{\lambda}{(\sigma+1)} + \frac{(1-\lambda)}{\tau}$, that is

$$\lambda = \frac{(q+1)(\sigma+1)}{N(q-\sigma)+(q+1)(\sigma+1)}.$$

Hence with Sobolev inequality,which gives $\|v\|_{L^\tau(G)} \leq C\|v\|_{W^{1,q+1}(G)}$ with $\frac{1}{\tau} = \frac{1}{q+1} - \frac{1}{N}$, we obtain

$$\|v\|_{L^{q+1}(G)} \leq C^{1-\lambda} \|v\|^{1-\lambda}_{W^{1,q+1}(G)} \|v\|^{\lambda}_{L^{\sigma+1}(G)}, \tag{C.0.8}$$

and

$$1-\lambda = \frac{N(q-\sigma)}{N(q-\sigma)+(q+1)(\sigma+1)} = \frac{(q+1)\theta - 1}{q}, \quad \lambda = \frac{(q+1)(1-\theta)}{q},$$

wich proves (C.0.6) in case 1.

Case 2. Assume $q+1 \geq N$. We set $\alpha = \frac{(N+1)}{2}$, $\rho = \frac{2(q+1)}{(N+1)}$, $\beta = \frac{(\sigma+1)(N+1)}{2(q+1)}$ and $\alpha^* = \frac{\alpha N}{N-\alpha}$ ($\alpha^* = \infty$ if $N=1$), so that $\alpha\rho = q+1$ and $\beta\rho = \sigma+1$. By interpolation we have

$$\|u\|_{L^\alpha(G)} \leq \|u\|_{L^{\alpha^*}(G)}^{1-\lambda} \|u\|_{L^\beta(G)}^{\lambda}, \tag{C.0.9}$$

where $\frac{1}{\alpha} = \frac{(1-\lambda)}{\alpha^*} + \frac{\lambda}{\beta}$. Note that (C.0.9) is valid even if $0 < \beta < 1$ with a simple change of function. From Sobolev inequality we get

$$\|u\|_{L^\alpha(G)} \leq C_5 \|u\|_{W^{1,\alpha}(G)}^{1-\lambda} \|u\|_{L^\beta(G)}^{\lambda}. \tag{C.0.10}$$

Now we set $u = v|v|^{\rho-1}$ and we have

$$\|u\|_{L^\alpha(G)} = \|v\|_{L^{\alpha\rho}(G)}^{\rho} = \|v\|_{L^{q+1}(G)}^{\rho},$$

$$\|u\|_{L^\beta(G)} = \|v\|_{L^{\beta\rho}(G)}^{\rho} = \|v\|_{L^{\sigma+1}(G)}^{\rho},$$

$$\|u\|_{W^{1,\alpha}(G)} = \|v\|_{L^{q+1}(G)}^{\rho} + \left(\int_G (\rho|v|^{\rho-1}|Dv|)^\alpha dx \right)^{\frac{1}{\alpha}}$$

and

$$\int_G (|v|^{\rho-1}|Dv|)^\alpha dx \leq \left(\int_G |v|^{\alpha\rho} dx \right)^{1-\frac{1}{\rho}} \left(\int_G |Dv|^{\alpha\rho} dx \right)^{\frac{1}{\rho}},$$

which yields $\|u\|_{W^{1,\alpha}(G)} \leq \rho \|v\|_{L^{q+1}(G)}^{\rho-1} \|v\|_{W^{1,q+1}(G)}$. Hence (C.0.10) becomes

$$\|v\|_{L^{q+1}(G)}^{\rho} \leq C_6 \rho^{1-\lambda} \|v\|_{L^{q+1}(G)}^{(\rho-1)(1-\lambda)} \|v\|_{W^{1,q+1}(G)}^{1-\lambda} \|v\|_{L^{\sigma+1}(G)}^{\lambda\rho}. \tag{C.0.11}$$

If we compute the exponents we get

$$\frac{1-\lambda}{\lambda\rho+1-\lambda} = \frac{N(q-\sigma)}{(q+1)(\sigma+1)+N(q-\sigma)} = \frac{(q+1)\theta-1}{q}$$

and

$$\frac{\lambda\rho}{\lambda\rho+1-\lambda} = \frac{(q+1)(\sigma+1)}{N(q-\sigma)+(q+1)(\sigma+1)} = \frac{(q+1)(1-\theta)}{q},$$

which is (C.0.6).

Fourth step. End of the proof. We use (C.0.5) and (C.0.6) and get

$$\|v\|_{L^{q+1}(\partial G)} \leq C_7 \|v\|_{W^{1,q+1}(G)}^{\frac{1}{q+1}} \|v\|_{W^{1,q+1}(G)}^{\frac{(q\theta+\theta-1)}{(q+1)}} \|v\|_{L^{\sigma+1}(G)}^{1-\theta}, \tag{C.0.12}$$

where $\theta = \frac{N(q-\sigma)+\sigma+1}{N(q-\sigma)+(q+1)(\sigma+1)}$; using (C.0.3) we get finally

$$\|v\|_{L^{q+1}(\partial G)} \leq C(\|Dv\|_{L^{q+1}(G)} + \|v\|_{L^{\sigma+1}(G)})^{\theta} \|v\|_{L^{\sigma+1}(G)}^{1-\theta}. \tag{C.0.13}$$

$\square$

As a consequence of the above Lemma we have the following:

Corollary C.0.1. *If in lemma C.0.3 we suppose that $G = B_\rho(x_0)$, $\rho > 0$, then for any $u \in W^{1,q+1}(B_\rho(x_0))$ we have*

$$\|u\|_{L^{q+1}(\partial B_\rho(x_0))} \leq C(\|Du\|_{L^{q+1}(B_\rho(x_0))} + \rho^\delta \|u\|_{L^{\sigma+1}(B_\rho(x_0))})^\theta \|u\|^{1-\theta}_{L^{\sigma+1}(B_\rho(x_0))} \tag{C.0.14}$$

where $\delta = -\frac{N(q-\sigma)+(\sigma+1)(q+1)}{(q+1)(\sigma+1)}$ *and* $C = C(N, \sigma, q)$.

Proof. For the sake of simplicity we suppose $x_0 = 0$ and we perform the following change of variable: $x = \rho y$, $x \in B_\rho(0)$, $y \in B_1(0)$. If $u \in W^{1,q+1}(B_\rho(0))$, the function v defined by $v(y) = u(x)$ belongs to $W^{1,q+1}(B_1(0))$ and from (C.0.1) we have

$$\|v\|_{L^{q+1}(S_1(0))} \leq C(\|Dv\|_{L^{q+1}(B_1(0))} + \|v\|_{L^{\sigma+1}(B_1(0))})^\theta \|v\|^{1-\theta}_{L^{\sigma+1}(B_1(0))}. \tag{C.0.15}$$

But $Dv(y) = \rho Du(x)$,

$$\|v\|_{L^{\sigma+1}(B_1(0))} = \rho^{-\frac{N}{\sigma+1}} \|u\|_{L^{\sigma+1}(B_\rho(0))},$$

$$\|Dv\|_{L^{q+1}(B_1(0))} = \rho^{1-\frac{N}{q+1}} \|Du\|_{L^{q+1}(B_\rho(0))},$$

and

$$\|v\|_{L^{q+1}(\partial B_1(0))} = \rho^{-\frac{N-1}{q+1}} \|u\|_{L^{q+1}(\partial B_\rho(0))}.$$

As

$$1 - \frac{N}{q+1} + \frac{N-1}{\theta(q+1)} - \frac{1-\theta}{\theta}\frac{N}{\sigma+1} = 0$$

and

$$-\frac{N}{\sigma+1} + \frac{N-1}{\theta(q+1)} - \frac{1-\theta}{\theta}\frac{N}{\sigma+1} = -\frac{N(q-\sigma)+(\sigma+1)(q+1)}{(q+1)(\sigma+1)}$$

we get (C.0.14). □

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