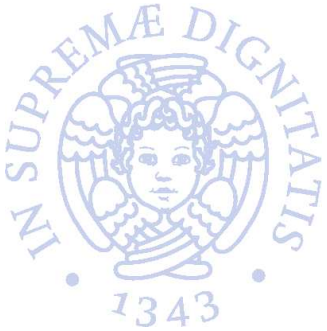

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Graduate Course in Physics

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Brunel University West London

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Ph.D. thesis:

"Effective theories of finite volume QCD"

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Abstract

Finite volume QCD close to the chiral limit cannot be described by chiral Perturbation Theory using the usual p-expansion when the correlation length of pions becomes larger than the size of the box. An alternative approach to this problem was proposed by Gasser and Leutwyler in 1987, it is referred to as ϵ-expansion. In 1993 Shuryak and Verbaarschot conjectured that the spectral properties of the leading order of this alternative expansion were shared with a simpler theory called chiral Random Matrix Theory. In the following years this equivalence was widely used.

In the first part of this work we prove this equivalence for any value of masses and for both zero and non-zero chemical potential. In particular the equivalence of all the low energy spectral properties imply the equivalence of the individual eigenvalue distributions, which are particularly useful to determine low energy constants from Lattice QCD with chiral fermions.

In the second part, working in ϵ-expansion with an accuracy up to the next to the leading order, we determine the volume and mass dependence of scalar and pseudoscalar two-point functions in N_f-flavour QCD, in the presence of an isospin chemical potential. Thanks to the non-vanishing chemical potential these correlation functions show a dependence on both chiral condensate and pion decay constant already at leading order.

The present thesis contains results published
by the author in [71, 88, 114].

Contents

Chapter 1

Introduction

According to Quantum Chromodynamics (QCD) the interactions between quarks and gluons are highly non-perturbative at energies below the breaking scale of chiral symmetry, and, as a consequence, the description of the low-energy hadronic world in terms of partonic degrees of freedom seems to be, after more than 30 years, still an unfeasible task. On the other hand we know from experiments that the spectrum of the theory is rather simple at low energies, containing only the octet of light pseudoscalar mesons, and that, at very low energies, these pseudoscalar mesons interact weakly, both with each other and with nucleons.

This is the framework where *Chiral Perturbation Theory* (χPT) lives, an effective theory where the fundamental degrees of freedom are pseudoscalar mesons and perturbative computations can be performed.

The basic principle of χPT, as of any effective field theory, is that, in a given energy range, only few degrees of freedom are relevant and need to be described by dynamical fields. The remaining degrees of freedom of the more general theory can be integrated out, leading to effects that are encoded in the coefficients of appropriate local operators.

Fortunately (or unfortunately, it depends) despite this theory being much simpler than fundamental QCD, it has not yet been studied and solved in all its ingredients.

In the present work we will focus on χPT defined in a finite volume box, or to be even more accurate to a particular regime of finite volume χPT where the correlation length of the fundamental degrees of freedom (mesons) is the same size as (or even bigger than) the dimensions of the box. This regime was first introduced by Gasser and Leutwyler [1, 2] and is usually called ϵ-*regime*. The importance of this theory lies in the fact that since it explicitely considers finite-size effects it allows a comparison with lattice calculations even when finite size effects may not be disregarded, as an example when chiral transition is approached. The values of quantities like the low energy constants can be extrapolated

before chiral transition is reached, with a much smaller computational effort.

The main feature of this ε-regime is that, since the correlation length is at least the same size as the box, the zero modes of the mesons cannot be treated perturbatively, and consequently they have to be considered separately from non-zero modes (which can be still treated perturbatively).

The collective zero modes are described by group elements of the broken flavour symmetry (in standard χPT with N_f flavours this symmetry group is the unitary group $U(N_f)$) and the functional integral has to be performed integrating with respect to the Haar measure.

An old result by Banks and Casher [3] was already suggesting the low energy spectral properties of finite volume QCD may not depend on the dynamic of pions but only on the collective modes (the Banks-Casher relation relates the chiral condensate to the spectral density of the Dirac operator in zero: $\Sigma \equiv |< \overline{\psi}\psi >| = \lim_{V\to\infty} \frac{\pi\rho(0)}{V}$ and there is no explicit dependence on the pion decay constant F_π), but the big impulse in the study of the zero-modes was given by a paper by Leutwyler and Smilga [4] where it was shown that, for small enough quark masses ($m \ll \frac{F_\pi^2}{2L^2\Sigma}$), the low energy spectral properties (for energies $\lambda \ll \frac{F_\pi^2}{2L^2\Sigma}$) of the finite volume χPT were only depending on the collective modes. Using this property they first computed sum rules for the sums of inverse powers of the Dirac operator eigenvalues. These sum rules are usually referred to as *Leutwyler-Smilga sum-rules*.

This is the origin of the interest in the spectral properties of the ε-regime of χPT.

In [5] Shuryak and Verbaarschot argued that since χPT, as an effective theory, is solely based on the symmetries of QCD, and hence the low energy spectral properties depend on the symmetries of the Dirac operator as well, if one starts out with a theory with the same global symmetries as QCD but different dynamical input (or even no dynamical input) one should arrive at exactly the same spectral properties. And this was exactly what they did: they introduced a Random Matrix Theory (that is a static theory) miming the chiral structure of the QCD Dirac operator (in the following we will call this theory *Chiral Random Matrix Theory*, χRMT). The proof they gave that the partition functions (and hence the Leutwyler-Smilga sum rules) were the same was a strong argument in favour of their equivalence conjecture.

This conjecture was soon extended to QCD-like theories with real (QCD with quarks in the adjoint representation of the colour group) or pseudo-real (2 colour QCD) Dirac operator [6].

This conjecture was of great help in the studies of the spectral properties of the Dirac operator, in fact computations in the χPT framework (that need integrations with respect to the Haar measure over classical group manifolds or even over super-group[1] manifolds) are usually much more involved than the corresponding ones in χRMT. By matching the

[1]Super-groups, sometimes called graded-groups, are manifolds involving commuting and anticommuting degrees of freedom, see appendix A.

spectral properties of the Dirac operator (spectral density, spectral correlation functions and individual eigenvalue distribution) measured on lattice simulations with the prediction provided by RMT and χPT, one can extract the low energy constants Σ and F_π. This technique still needs some improvements (so far a proper way to deal with finite volume effects is lacking), nevertheless it is considered interesting due to its remarkable speed; see [7] for an overview of recent results.

The study of RMT and of χPT in the ϵ regime has been generalised in many directions: theories with baryon chemical potential or isospin chemical potential were studied [8], or even QCD with bosonic quarks or with both bosonic and fermionic quarks, and the conjectured equivalence has always been considered true. In many cases computations were performed in both frameworks, and every time results were in perfect agreement.

In this work we prove this 15-year old conjecture developing a systematic way to link χRMT to χPT. This link holds for any number of bosonic and fermionic quarks and for any kind of chemical potential; thanks to it we can write χPT in a case where its form was previously unknown (the theory with a real chemical potential and both bosonic and fermionic quarks) and we learn how to compute some integrals occurring in computing the corrections in the ε-expansion. Another result that we achieve is the proof of the equivalence, in the infinite volume limit, of two different χRMT describing QCD with real or imaginary chemical potential (introduced respectively by Stephanov [8] and Osborn [9] for real chemical potential).

The work is organised as follows: in the first chapter we provide all the necessary informations, what is chiral Perturbation Theory and what does it mean to consider its ε-expansion (sect. 1.1), how to study a non-zero chemical potential through χPT, what is Chiral Random Matrix Theory and how it can be used to describe the $\mathcal{O}(\varepsilon^0)$ limit of QCD (sect. 1.3), and how to generate the spectral properties through the resolvent method and partially-quenched QCD (sect. 1.4). In chapter 2 we give the most important result in this paper, that is the proof of the equivalence of the spectral properties in the two effective theories. In chapter 3 the fundamental superbosonisation theory used in chapter 2 is introduced and an original proof is provided. In chapter 4 we show, as a corollary to the result in chapter 2, the equivalence of two different matrix models describing the same limit of QCD with chemical potential. In chapter 5 we go beyond the χRMT $\mathcal{O}(\varepsilon^0)$ approximation computing the finite volume expectation values of current-current correlations (for both scalar and pseudoscalar, neutral and charged currents) in the presence of a chemical potential up to including $\mathcal{O}(\varepsilon^2)$ terms.

In appendix A we briefly show the superanalysis concepts used in this work, in appendix B we focus on the well-definiteness of the δ functions over supermanifolds used in chapter 3, and in appendices C and D some mathematical details on computations in the work are provided.

1.1 Chiral Perturbation Theory

1.1.1 The chiral Lagrangian

Chiral Perturbation Theory (χPT) is an effective quantum field theory describing the low energy sector of QCD. In principle the reduction to a low energy effective theory should be done integrating out all the higher energy modes in the fundamental theory, but nowadays nobody knows how to perform this integration. χPT, as we know it now, is written down starting from first principles and from working hypotheses compatible with experimental data [10]. Essentially the starting points are that:

- the global chiral symmetry is spontaneously broken;
- the Goldstone bosons generated by this breakdown are the only massless particles contained in the spectrum of asymptotic states and low energy dynamics is dominated by poles due to the exchange of these particles;
- the vertices admit a Taylor series expansion in powers of the momenta;
- the mass term of the light quarks (which explicitely breaks the chiral symmetry) can be treated as small perturbations around the chiral limit.

The conditions above can be considered true for energies below the scale Λ_χ of the mass of the lightest "non-Goldstone" particles (physically ρ-mesons and nucleons, that is $\Lambda_\chi \simeq 1$ GeV):

$$E \ll \Lambda_\chi. \tag{1.1}$$

The problem of writing χPT starting from the hypotheses above and not from the fundamental theory is that the most general theory compatible with the symmetry transformation of the Goldstone fields necessarily contains an infinite number of terms that cannot be derived from QCD but need to be measured. Nevertheless only a finite number of operators contribute at any given order in E/Λ_χ expansion [11]. The values of the coefficients (whose number is infinite) have to be measured from experiments or from lattice data.

Formally the partition function in Euclidean space-time is:

$$\int [d\mu(U(x))]\, e^{-\int_V dx\, \mathcal{L}(U,\partial U,\partial^2 U,\ldots,M)} \tag{1.2}$$

where $\int [d\mu(U(x))]$ stays for a quantum field integral of the field $U(x)$ that belongs to the broken symmetry group (usually $U(N_f)$) and lives in the space-time volume, $d\mu(\cdot)$ is the Haar measure over the broken group[2]. Without getting too much into details but referring to existing reviews (see [11]) we say that additional external sources may be inserted: vector, axial, scalar (like the masses) and pseudo-scalar.

[2] A way to write explicitely the Haar measure in a given parametrisation framework can be found in [4].

In the following we will consider only the lowest non-vanishing order in the energy expansion (E/Λ_χ) of the Lagrangian:

$$\begin{aligned}\mathcal{L}(U,\partial U,\partial^2 U,\ldots,M) &\rightarrow \\ &\rightarrow \mathcal{L}^{(2)}(U,\partial U,M) = \frac{F^2}{4}Tr\left[\partial_\mu U\partial_\mu U^\dagger\right] - \frac{1}{2}\Sigma Tr\left[MU + MU^\dagger\right]\end{aligned} \tag{1.3}$$

where M is the light quark mass matrix ($M_{ii} \ll \Lambda_\chi$), and at this order in the momentum expansion (*p-expansion*), F is exactly the pion decay constant and Σ is the chiral condensate $|<\overline{\psi}\psi>|$. The last two quantities are external input in this theory.

If considering all the vector (v_μ), axial (a_μ), scalar (s) and pseudo-scalar (s) external sources the Lagrangian is [1]:

$$\mathcal{L}^{(2)}(U,\partial U,v_\mu,a_\mu,s,p) = \frac{1}{4}F^2 Tr\left[\nabla_\mu U\nabla_\mu U^\dagger\right] - \frac{1}{2}\Sigma Tr\left[s(U+U^\dagger) - ip(U-U^\dagger)\right] \tag{1.4}$$

where

$$\begin{aligned}\nabla_\mu U &\equiv \partial_\mu U - i(v_\mu + a_\mu)U + iU\,(v_\mu - a_\mu) \\ \nabla_\mu U^\dagger &\equiv \partial_\mu U^\dagger - i(v_\mu - a_\mu)U^\dagger + iU^\dagger(v_\mu + a_\mu).\end{aligned} \tag{1.5}$$

The name of the external sources come after their transformation properties under chiral rotations, they are the same as the external sources coupling to the quark currents in QCD, and, since χPT is a field theory over a representation of the chiral group, we conclude that these quantities provide a representation in χPT of the corresponding QCD quantities. Eq. (1.4) is a realisation of:

$$\mathcal{L}^{(0)}_{QCD} + \overline{\psi}\gamma_\mu(v_\mu + a_\mu\gamma_5)\psi - \overline{\psi}(s - ip\gamma_5)\psi. \tag{1.6}$$

As in QCD, the aim of this insertion is that once computed the partition function with these sources one can obtain the (physical observable) quark-current correlation functions through simple differentiations.

1.1.2 Finite volume χPT and the ϵ regime

The problem of a χPT on a discretized torus (that is a lattice with periodic boundary conditions) was first considered in [2]. The authors saw that in the chiral limit $M \rightarrow 0$ the usual exponential representation of the meson fields

$$U(x) \equiv e^{i\frac{\sqrt{2}}{F}\phi(x)}, \tag{1.7}$$

where $\phi(x)$ belongs to the Lie algebra generated by the broken generators, becomes meaningless as far as the chiral limit is approached. Expanding the action according to eq. (1.7)

we have:

$$\begin{aligned}\mathcal{L}^{(2)}(e^{i\frac{\sqrt{2}}{F}\phi(x)}, \partial e^{i\frac{\sqrt{2}}{F}\phi(x)}, M) \; &= \\ = \; -\Sigma\, Tr\,[M] + \frac{1}{2} Tr\left[\partial_\nu\phi(x)\partial_\nu\phi(x) + \frac{2\Sigma M}{F^2}\phi^2(x)\right] &+ \mathcal{O}(\phi^4)\end{aligned} \tag{1.8}$$

where we are considering the quartic and higher order terms in ϕ as (perturbative) interactions of the free fields. We can see that when considering discretized momenta $p = \frac{2\pi}{L}(n_1\hat{x}_1 + n_2\hat{x}_2 + n_3\hat{x}_3 + n_4\hat{x}_4)$ the zero modes enter into the action only through the mass term $\frac{2\Sigma M}{F^2}\phi_0^2(x)$, and hence if this one vanishes too it completely disappears from the action invalidating the standard perturbative method based on gaussian integral of the quadratic free fields. This failure of the standard chiral expansion can be seen too by considering the propagator [2, 12],

$$G(x) = \frac{F^2}{2\,\Sigma\, M\, V} + \frac{1}{V}\sum_{\mathbf{n}\neq 0} \frac{1}{\left(\frac{2\pi}{L}\right)^2 |\mathbf{n}|^2 + \frac{2\,\Sigma\, M}{F^2}} e^{i\frac{2\pi}{L}\mathbf{n}\cdot x} \tag{1.9}$$

whose zero mode part explodes in the chiral limit. In order to have a representation valid near the chiral limit we have to resum all the graphs involving an arbitrary number of zero modes propagator. The standard p-expansion is no longer valid.

This breakdown has a deep physical reason: in the broken phase the mesonic correlation length diverges and all around the space-time volume the fields have to be considered as fluctuations over a non trivial vacuum alignment (the zero mode). It happens any time the pion Compton wave length overcomes the typical length of the box:

$$\frac{1}{\Lambda_\chi} \ll L \ll \frac{1}{m_\pi}, \tag{1.10}$$

where m_π is the pion mass given by $\frac{\sqrt{2\Sigma M}}{F}$. The way to avoid this problem is to consider the zero mode of the pion field separately from the other modes, in a non perturbative way:

$$U(x) = U_0 \cdot \tilde{U}(x) \tag{1.11}$$

In this scheme the partition function, considering only the quadratic order in the chiral expansion E/Λ_χ, is:

$$\begin{aligned}&\int \left[d\mu(\tilde{U}(x))\right]' \,\mathrm{Exp}\left[-\int_V dx\, \frac{F^2}{4} Tr\left[\partial_\mu\tilde{U}(x)\partial_\mu\tilde{U}^\dagger(x)\right]\right] \\ &\times \int d\mu(U_0)\,\mathrm{Exp}\left[\int_V dx \frac{1}{2}\Sigma Tr\left[MU_0\tilde{U}(x) + M\tilde{U}^\dagger(x)U_0^\dagger\right]\right]\end{aligned} \tag{1.12}$$

where in $\left[d\mu(\tilde{U}(x))\right]'$ the prime means that the integration is over the non-zero modes. As the Haar measure is invariant under multiplication the change of variables in eq. (1.11) does not generate any Jacobian.

In order to perform functional integrals like the one above through a perturbative computation, Gasser and Leutwyler have introduced a new power counting: instead of the usual counting rules where the expansion was performed in order of the energies or momenta p/Λ_χ (p-expansion)

$$\frac{m_\pi}{\Lambda_\chi} = \frac{\sqrt{2\Sigma M}}{F\,\Lambda_\chi} \sim \frac{p}{\Lambda_\chi} \sim \frac{1}{L\,F} \tag{1.13}$$

a different one, usually called ε-*expansion*, was chosen:

$$\frac{m_\pi}{\Lambda_\chi} \sim \frac{p^2}{\Lambda_\chi^2} \sim \frac{1}{L^2F^2} \sim \varepsilon^2 \tag{1.14}$$

In this expansion the terms involving only collective modes are dominant (using $p \sim 1/L$ one can check they are of order $M\Sigma V \sim m_\pi^2 F^2 V \sim \varepsilon^0$) and all the other terms, including the quartic or higher order terms in the fields $\mathcal{L}^{(4)}$, are at least ε^2-suppressed (in the observables all the term $\mathcal{O}(\varepsilon)$ are identically zero due to $\int dx\,\xi(x) = 0$).

This systematic expansion allows to perform perturbative calculations, computing expectation values of the observable in the two separate ensembles of the zero modes and propagating modes.

Most of the work employing this expansion has been devoted to computing quantities only in the leading order of the ε-expansion $\mathcal{O}(\varepsilon^0)$ (usually called ε-regime) or to the computation of finite volume corrections to the low energy constant of χPT [2] at $\mathcal{O}(\varepsilon^2)$, but in the last few years computation of dynamic correction to static modes has attracted the attention of many groups (see chapter 5): the possibility of investigating QCD near chiral limit by simulations on a small lattice[3] fulfilling the ε-regime conditions is appealing.

1.1.3 Chiral Perturbation Theory at $\mathcal{O}(\varepsilon^0)$

When considering the leading order in the ε-expansion of eq. (1.12), static and dynamical contribution completely decouple, since the only couplings with the external sources are in the static part the dynamical gaussian free field integration can be factorised and disregarded in that equation:

$$Z(M) = \int_{SU(N_f)} d\mu(U_0)\ \mathrm{Exp}\left[\frac{1}{2}V\Sigma\ Tr\left[MU_0 + MU_0^\dagger\right]\right] \tag{1.15}$$

where the integral is performed over the manifold of the broken symmetries, that is the quotient group G/H where G is the whole symmetry group of the action and H is the unbroken symmetry group. In QCD this manifold is

$$(SU_R(N_f) \times SU_L(N_f) \times U(1))/(SU_V(N_f) \times U(1)) = SU_A(N_f)$$

[3]Close to chiral limit the compton length of pions diverges, and p-regime approach requires using lattices with length bigger than this compton length. This requirement is absent in ε-regime.

This expression is equivalent to the fundamental theory

$$\begin{aligned} Z(M,\theta) =& \int \left[dG\, d\psi\, d\psi^\dagger\right] \mathrm{Exp}\Big[- \int dx\ \frac{1}{4g^2} G_{\mu\nu} G_{\alpha\beta} - i\theta \frac{1}{32} G_{\alpha\beta} \overline{G}_{\mu\nu} \\ & + \sum_f^{N_f} \overline{\psi}_f (-i\mathcal{D} + m_f) \psi_f \Big] \\ =& \int [dG]\ e^{-S_{YM}[G] + i\theta\nu} \prod_f^{N_f} \mathcal{D}et\left[-i\mathcal{D} + m_f\right] \\ =& \sum_{\nu\in\mathbb{Z}} Z_\nu(M)\, e^{i\nu\theta} \end{aligned} \tag{1.16}$$

where $\mathcal{D}$ is the (Hermitian) Dirac operator in the gauge field G and ν is the integer winding number. According to the Atiyah-Singer index theorem ν is equal to the difference between the number the right-handed eigenvalues and the number of left-handed ones, that is equal[4] to the number of exact zero eigenvalues. Thanks to the axial symmetry the non-zero eigenvalues come in opposite pairs $\pm\lambda_n$. The determinant in the equation above can be written as:

$$\prod_f^{N_f} \mathcal{D}et\left[-i\mathcal{D} + m_f\right] = \left(\mathcal{D}et_f\left[M\right]\right)^\nu \prod_{\lambda_n > 0} \mathcal{D}et_f\left[\lambda_n + M\right] \tag{1.17}$$

where M is the (real diagonal) quark mass matrix, the subscript f means that the determinant is in the quark flavour space, the product is performed only over positive eigenvalues.

$$\begin{aligned} Z(M,\theta) &= \int [dG]\ e^{-S_{YM}[G] + i\theta\nu}\ \left(\mathcal{D}et_f\left[M\right]\right)^\nu \prod_{\lambda_n>0} \mathcal{D}et_f\left[\lambda_n^2 + M\cdot M^\dagger\right] \\ &= \int [dG]\ e^{-S_{YM}[G]}\ \left(\mathcal{D}et_f\left[M\, e^{i\frac{1}{N_f}\theta}\right]\right)^\nu \prod_{\lambda_n>0} \mathcal{D}et_f\left[\lambda_n^2 + M\cdot M^\dagger\right]. \end{aligned}$$

From the equation above we can see that the partition function depends on the mass matrix and the vacuum angle only through the product $M\, e^{i\theta/N_f}$, a change in the phase of the mass matrix is equivalent to a change in θ [4].

Applying this result to eq. (1.15) we can make explicit the θ-dependence in the chiral Lagrangian and hence obtain the partition function for a given topological charge performing the fourier transform; the result is:

$$Z_\nu(M) = \int_{U(N_f)} d\mu(U_0)\ \mathcal{D}et\left[U_0\right]^\nu \mathrm{Exp}\left[\frac{1}{2} V\Sigma\ Tr\left[MU_0 + MU_0^\dagger\right]\right] \tag{1.18}$$

The integral above is the archetype of the integrals of χPT in the ε-regime. In [4] it was computed for degenerate masses and taking its derivatives it was used to obtain a constraint

[4]This sentence is true only if disregarding those configurations with left and right-handed eigenstates whose eigenvalues are accidentally zero. These configurations may be disregarded since they give no contributions to the integrals.

for the low energy part of the Dirac spectra of QCD in a box fulfilling the ε-regime conditions. E.g. if we consider degenerate masses and differentiate the partition function at fixed winding number with respect to this mass we have:

$$\begin{aligned}\partial_m Z_\nu(m\mathbf{1}_{N_f}) &= \partial_m\left(m^{N_f\,\nu}\int [dG]_\nu\; e^{-S_{YM}[G]}\prod_{\lambda_n>0}\left(\lambda_n^2+m^2\right)^{N_f}\right)\\ &= N_f\,\nu\frac{Z_\nu}{m}\;+\;N_f m^{N_f\,\nu}\int [dG]_\nu\; e^{-S_{YM}[G]}\\ &\quad\times\prod_{\lambda_n>0}\left(\lambda_n^2+m^2\right)^{N_f}\sum_{\lambda_n>0}\frac{2m}{\lambda_n^2+m^2}\end{aligned}\tag{1.19}$$

The last term in the equation above gives, in the chiral limit an information about the sum of the inverse square of the eigenvalues of the Dirac Operator:

$$m^{N_f\,\nu}\int [dG]_\nu\; e^{-S_{YM}[G]}\prod_{\lambda_n>0}\left(\lambda_n^2+m^2\right)^{N_f}\sum_{\lambda_n>0}\frac{1}{\lambda_n^2+m^2}\to\left\langle\sum_{\lambda_n>0}\frac{1}{\lambda_n^2}\right\rangle_\nu .\tag{1.20}$$

As a consequence from the knowledge of the explicit dependence of the partition function from the quark masses this constraint follows on the eigenvalues distribution. If differentiating twice or n-times one obtains constraints on the sum of the eigenvalues power minus 4 or minus $2n$. These equations take the name of *Leutwyler-Smilga sum rules*. In [4] the group integral was performed for $\nu=0$ obtaining for the first time one of these sum rules:

$$\left\langle\frac{1}{V^2}\sum_{\lambda_n>0}\frac{1}{\lambda_n^2}\right\rangle_{\nu=0}=\frac{\Sigma^2}{4N_f}.\tag{1.21}$$

One could point out that asymptotically for large λ the density of eigenvalues grows like $V\lambda^3$ (see fig. 1.1), and hence some of these sums are diverging. There is an implicit cut-off in this sum: both the description of QCD through χPT and the zero-order ϵ-expansion approximation give constraints for the domain of validity of the sum-rules. The spectrum of QCD and that one of χPT are supposed to be the same for energies smaller then the scale of lightest non-goldstone particle Λ_χ. The description provided by the leading order in the epsilon-expansion of χPT is valid only at energies that are not influenced by the pions' dynamic, or equivalently, at energies whose pion compton wavelength fits in the box. In formulas [13]:

$$E\ll\frac{1}{\frac{\Sigma}{F^2}L^2}.\tag{1.22}$$

This quantity take the name of Thouless energy after its equivalent in mesoscopic systems (it is indicated with a m_c in the schematic spectrum in fig. 1.1). The same value of Thouless energy was found in [14] starting from partially quenched χPT (see sect. 1.4.2) and in [15] considering a diffusion process in a stochastic QCD-like theory.

We can conclude that the Leutwyler-Smilga sum rules have to be considered summing only up to the smaller of these cut-offs.

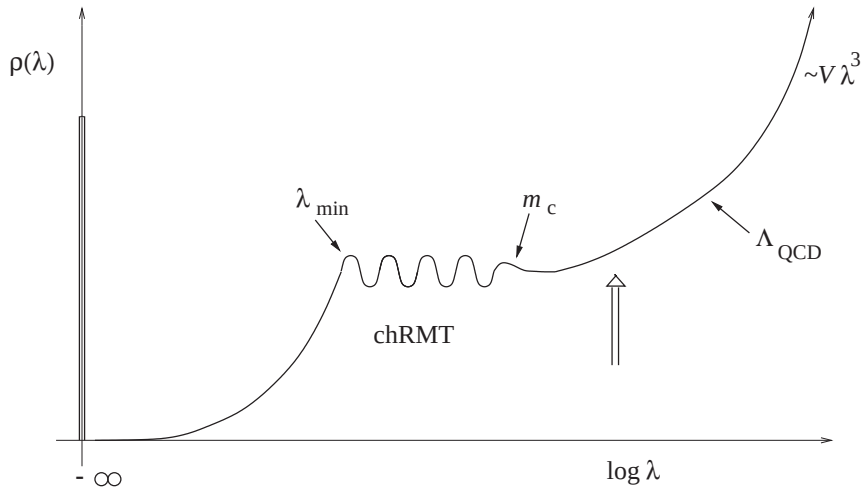

Figure 1.1: Schematic picture of the QCD Dirac spectrum. The quantity m_c, called Thouless energy, bounds the region described by the $\mathcal{O}(\varepsilon^0)$ in the ε-regime. Picture taken from [16].

Integrals like eq. (1.18) are known in literature for non-degenerate masses both for zero [17] and non-zero winding number [18]. The same approach was used on 2-colour QCD and in QCD with quarks in the adjoint representation of the colour group [19].

It is important to note that only through the differentiation with respect to the masses of equations like eq. (1.18) one cannot obtain the shape of the eigenvalues spectrum.

1.1.4 Chiral Perturbation Theory at $\mathcal{O}(\varepsilon^2)$

In the previous section we have seen what we can learn investigating only the leading order in the ε-expansion (1.14) of eq. (1.12): such equations describe a theory where the only fields are the mesons but in a regime where energies are too small to consider interactions and distances are too small to allow propagations of particles. Under these hypotheses we loose any information on the proper dynamic of QCD-χPT and what remain describes how the vacuum depends on the external parameters (masses).

The introduction in the action of higher ε order terms allows to investigate dynamical properties of the theory, like quark current correlation functions. The problem of dynamical ($\mathcal{O}(\varepsilon^2)$) correction to the static ($\mathcal{O}(\varepsilon^0)$) quantities is considered through a systematic approach in chapter 5. We briefly show in this introductory section some well known results.

The key point is that the perturbative expansion of the propagating pion field is performed around the non-perturbative zero-mode [2, 12, 20]:

$$U(x) = U_0 \cdot \tilde{U}(x) \equiv U_0 \cdot e^{i\frac{\sqrt{2}}{F}\xi(x)} \simeq U_0 \cdot \left(\mathbf{1} + i\frac{\sqrt{2}}{F}\xi(x) - \frac{1}{F^2}\xi^2(x) + \dots\right) \tag{1.23}$$

where the field ξ has a vanishing zero-momenta $\int dx\, \xi(x) = 0$. The basic assumption in this

expansion is that the field $\xi(x)$ is a quantity of order[5] $\varepsilon \cdot F$ [2].

$$\frac{|\,\xi\,|}{F} = \mathcal{O}(\varepsilon) \tag{1.24}$$

Considering the terms up to order $\mathcal{O}(\varepsilon^2)$ in the Lagrangian, eq. (1.3) reduces to:

$$\begin{aligned}\mathcal{L}^{(2)}(U_0,\xi,\partial\xi;M) \;=& Tr\left[\frac{1}{2}\partial_\mu\xi(x)\partial_\mu\xi(x) - \frac{1}{2}\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right] \\ &+Tr\left[\frac{\Sigma}{2F^2}\left(M\,U_0\xi^2(x) + \xi^2(x)\,M\,U_0^\dagger\right)\right] + \ldots\end{aligned} \tag{1.25}$$

In the equation above we have omitted an irrelevant 4-term interaction in the ξ fields; for details we refer to chapter 5.

In analogy with the standard p-expansion we consider masses as small quantities and perform perturbative calculations. The partition function up to the second order in ε is[6]:

$$\begin{aligned}Z(M) \;&=\; \int d\mu(U_0)\; e^{Tr\left[\frac{1}{2}V\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right]} \int [d\xi(x)]_{U_0}\, e^{-\int dx\, Tr\left[\frac{1}{2}\partial_\mu\xi(x)\partial_\mu\xi(x)\right]} \\ &\quad\times \mathrm{Exp}\left[-\int dx\, Tr\left[\frac{\Sigma}{2F^2}\left(M\,U_0\xi^2(x) + \xi^2(x)\,M\,U_0^\dagger\right)\right]\right] \\ &\simeq\; \int d\mu(U_0)\; e^{Tr\left[\frac{1}{2}V\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right]} \int [d\xi(x)]_{U_0}\, e^{-\int dx\, Tr\left[\frac{1}{2}\partial_\mu\xi(x)\partial_\mu\xi(x)\right]} \\ &\quad\times \left(1 - \frac{\Sigma}{2F^2} Tr\left[\left(M\,U_0 + M\,U_0^\dagger\right)\int dx\,\xi^2(x)\right]\right)\end{aligned} \tag{1.26}$$

where $[d\xi(x)]_{U_0}$ indicates a modified path integral over the field ξ. It differs from the standard path integral measure referred to as $[d\xi(x)]$ in an additional Jacobian factor coming from the change of variables $U \to U_0 \cdot \mathrm{Exp}\left[i\frac{\sqrt{2}}{F}\xi\right]$. The first non-vanishing correction is in order ξ^2 and is given by [20, 21]:

$$[d\xi(x)]_{U_0} = [d\xi(x)]\left(1 - \frac{N_f}{3F^2}\frac{1}{V}\int dx\, Tr\left[\xi^2(x)\right]\right). \tag{1.27}$$

This Jacobian generates a mass term that gives an ε^2 correction to the ξ propagator that is irrelevant to the quantities computed here.

It is important to note that the vacuum expectation value is taken respect to the ensembles defined by the zero order in ε-expansion. At $\mathcal{O}(\varepsilon^0)$ order the integration over the non-zero modes is not considered at all since it completely decouples from the zero-modes and has no explicit dependences on the masses or other external sources.

Integration over the propagating fields ξ must be performed in a perturbative way, projecting ξ on the generator of the $SU(N_f)$ group

$$\xi(x) = \sum_a \xi^a(x)\, T^a_{ij} \tag{1.28}$$

[5]This assumption is consistent with the observation that whenever two fields $\xi(x)$ are contracted in the computation of an expectation value a propagator $\overline{\Delta}(x)$ is obtained, and $|\overline{\Delta}(x)/F^2| < |\overline{\Delta}(0)/F^2|$ that is an ε^2 quantity.

[6]For simplicity of notations we consider only the case with zero topology.

that propagates according to a free propagator:

$$\overline{\Delta}(x)^{(ab)} \equiv 2\,\overline{\Delta}(x)\delta^{ab} = 2\,\delta^{ab}\frac{1}{V}\sum_{p\neq 0}\frac{e^{ipx}}{p^2}. \tag{1.29}$$

The quantity in eq. (1.26) can be computed using standard field theory methods:

$$\begin{aligned}
&= \int d\mu(U_0)\; e^{Tr\left[\frac{1}{2}V\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right]} \int [d\xi(x)]\, e^{-\int dx\, Tr\left[\frac{1}{2}\partial_\mu\xi(x)\partial_\mu\xi(x)\right]} \\
&\quad \times \left(1 - \frac{\Sigma}{2F^2} Tr\left[\left(M\,U_0 + M\,U_0^\dagger\right)\int dx\,\xi^a(x)\,T^a\xi^b(x)\,T^b\right]\right) \\
&= \int d\mu(U_0)\, e^{Tr\left[\frac{1}{2}V\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right]} \left(1 - \frac{N_f^2-1}{2N}\frac{\Sigma V}{F^2}\overline{\Delta}(0)\,Tr\left[M\,U_0 + M\,U_0^\dagger\right]\right) \\
&\simeq \int d\mu(U_0)\, e^{Tr\left[\frac{1}{2}V\Sigma\left(M\,U_0 + M\,U_0^\dagger\right)\right]} \mathrm{Exp}\left[-\frac{N_f^2-1}{2N}\frac{\Sigma V}{F^2}\overline{\Delta}(0)\,Tr\left[M\,U_0 + M\,U_0^\dagger\right]\right]
\end{aligned} \tag{1.30}$$

where the generators are normalised according to:

$$\begin{aligned}
Tr\left[T^a T^b\right] &= \frac{1}{2}\delta^{ab}\ , \\
\left[T^a, T^b\right] &= i\, f_{abc} T^c\ .
\end{aligned} \tag{1.31}$$

This normalisation gives raise to the following useful relations:

$$\begin{aligned}
&\sum_{a,b} f_{abc} f_{abd} = N\,\delta_{c,d}\ , \\
&\sum_{a\neq 0} T^a T^a = \frac{N^2-1}{2N}\mathbf{1}_N\ , \\
&\sum_{a\neq 0} Tr\left[T^a\,A\,T^a\,B\right] = -\frac{1}{2N}Tr\left[A\,B\right] + \frac{1}{2}Tr\left[A\right]Tr\left[B\right]\ , \\
&\sum_{a\neq 0} Tr\left[T^a\,A\right]Tr\left[T^a\,B\right] = -\frac{1}{2N}Tr\left[A\right]Tr\left[B\right] + \frac{1}{2}Tr\left[A\,B\right]\ .
\end{aligned} \tag{1.32}$$

The result in the last of eqs. (1.30) may be absorbed in the definition of the Σ:

$$\Sigma \to \Sigma_{\text{eff}} = \Sigma - \Sigma\frac{N_f^2-1}{N}\frac{1}{F^2}\overline{\Delta}(0)\ . \tag{1.33}$$

The value of the propagator $\overline{\Delta}(0)$ can be computed in a dimensional regularisation obtaining the result:

$$\overline{\Delta}(0) = -\frac{\beta_1}{\sqrt{V}} + \mathcal{O}(1/V) \tag{1.34}$$

and β_1 is the shape coefficient [22]. This equation gives the finite volume correction to the chiral condensate [2] (Σ is the value of $\langle\overline{\psi}\psi\rangle$ for an infinite volume and Σ_{eff} is the one for a finite volume). The consistency of the approximation is shown by the fact that according

to (1.14) $\overline{\Delta}(0)/F^2$ is a $\mathcal{O}(\varepsilon^2)$ term. This useful result allows to obtain the chiral condensate from a lattice computation before reaching the infinite volume limit.

In order to compute current correlation functions external sources may be added like in eq. (1.4). We show here this procedure only in the simplest case, that is neutral scalar-scalar currents. This insertion is equivalent to replace the quark-masses with

$$M \to M + s(x) \tag{1.35}$$

in the partition function. The scalar-scalar correlation function is given by:

$$\begin{aligned}\langle S(x)S(0)\rangle \; =& \frac{\delta}{\delta s(x)}\frac{\delta}{\delta s(0)} Z_{\nu,\text{eff}}(M, s(x))_{|s(x)\equiv 0} \qquad (1.36)\\ =& \Big\langle \Big\langle Tr\left[U_0 \cdot e^{i\frac{\sqrt{2}}{F}\xi(x)} + e^{-i\frac{\sqrt{2}}{F}\xi(x)} \cdot U_0^\dagger\right]\\ & \times Tr\left[U_0 \cdot e^{i\frac{\sqrt{2}}{F}\xi(0)} + e^{-i\frac{\sqrt{2}}{F}\xi(0)} \cdot U_0^\dagger\right] \Big\rangle_{[\xi]} \Big\rangle_{U_0,\text{eff}}\\ =& \Big\langle \Big\langle (Tr\left[U_0 + U_0^\dagger\right])^2 - \frac{2}{F^2} Tr\left[\xi(x)(U_0 - U_0^\dagger)\right] Tr\left[\xi(0)(U_0 - U_0^\dagger)\right]\\ & - \frac{1}{F^2} Tr\left[(U_0 + U_0^\dagger)(\xi^2(x) + \xi(0))\right] \Big\rangle_{[\xi]} \Big\rangle_{U_0,\text{eff}} \qquad (1.37)\end{aligned}$$

The integration has to be performed like the one in the previous case, the result of the field theory integration is:

$$\begin{aligned}\langle S(x)S(0)\rangle \; =& \; \Big\langle (Tr\left[U_0 + U_0^\dagger\right])^2 - 4\frac{\overline{\Delta}(x)}{F^2}\Big(Tr\left[(U_0 - U_0^\dagger)^2\right] \qquad (1.38)\\ & - \frac{1}{N_f} Tr\left[U_0 - U_0^\dagger\right]^2\Big) - \; 4\frac{N_f^2 - 1}{N_f}\frac{V\overline{\Delta}(0)}{F^2} Tr\left[(U_0 + U_0^\dagger)\right] \Big\rangle_{U_0,\text{eff}}\end{aligned}$$

The last step is to perform the group integrals. This may be an involved task, specially if considering complicated extensions of this theory. The results are usually obtained or through group integral identities [23] or explicit integration formulas using character expansion [24].

1.2 Non-zero chemical potential

The standard QCD approach fails when applied in high density systems like neutron stars, supernova explosions or heavy ion collision, chemical potential term $\mu\overline{\psi}\gamma_0\psi$ has to be inserted in the Lagrangian. The study of QCD with non-zero chemical potential is a tremendous challenge: the insertion of the chemical potential breaks the hermiticity of the Dirac operator invalidating the fundamental tools provided by lattice Monte Carlo simulations.

Despite non-zero chemical potential QCD being so involved, its low energy sector may be studied in a way which is not conceptually different from vacuum QCD: whenever chiral symmetry is still spontaneously broken in the vacuum and the conditions expressed in sect. 1.1 are fulfilled one can write a low energy effective theory. The only additional problem is how to include the chemical potential term.

The insertion of the chemical potential in the effective Lagrangian can be seen as an insertion of an interaction with an external (imaginary) vector current, see eq. (1.6):

$$\overline{\psi}_f i\mathcal{D}\psi_f \to \overline{\psi}_f i\mathcal{D}\psi_f + \mu_f \overline{\psi}_f \gamma_0 \psi_f \equiv \overline{\psi}_f i\mathcal{D}\psi_f + \overline{\psi}_f B_f^{(\eta)} \gamma_\eta \psi_f \tag{1.39}$$

by taking $B_f^{(\eta)} = \delta_{0,\eta}\mu_f$ a matrix in the flavour space, where $\{\mu_f\}$ is a set of (complex) numbers, we are considering the possibility of studying both baryon chemical potential, isospin chemical potential or even more general cases. In order to write the coupling with the pion fields the idea [25] is to promote the global flavour symmetry to a local one, considering the coupling with B as a gauge coupling leading to the invariant quantity $i\mathcal{D} + B_\eta\gamma_\eta$. The way to couple a gauge coupling for a matter field in the adjoint representation of the gauge group is through a commutator:

$$\partial_\eta \Upsilon \to \nabla_\eta \Upsilon \equiv \partial_\eta \Upsilon + \left[\Upsilon, B^{(\eta)}\right] \ . \tag{1.40}$$

It is important to point out that the symmetry that we are gauging is the (global, broken) flavour symmetry and that quarks fall in the fundamental representation of this group, but the mesons, that are the particles described by χPT, lie in the adjoint one.

The matrix B is a diagonal matrix whose entries are given by the chemical potential values for any flavour, $B_{f,g} = \delta_{f,g}\mu_f$. For a baryon chemical potential it is proportional to the identity, for an isospin chemical it will be by a series of plus or minus μ_I on the diagonal. The result is that the chiral Lagrangian is [1, 11, 26]:

$$\mathcal{L}^{(2)} = \frac{1}{4}F^2 Tr\left[\nabla_\eta U \nabla_\eta U^\dagger\right] - \frac{1}{2}\Sigma Tr\left[MU + MU^\dagger\right]. \tag{1.41}$$

Not surprisingly this equation is equivalent to eq. (1.4) computed for an external source vector current $v_\eta = -i\delta_{0,\eta}B$, the i term is due to the fact that for real chemical potential ($\mu \in \mathbb{R}$) the Lagrangian is no longer Hermitian.

As for vacuum QCD, we can study finite volume QCD with the systematic approach provided by the ϵ-regime power counting. A scaling law for μ has to be considered together with those in eq. (1.14) and the prescription is that the leading order (the static mode) gives a contribution to the partition function of the same order as the static mass term:

$$\mu^2 F^2 V \sim \varepsilon^0 \quad \to \quad \frac{\mu}{\Lambda_\chi} \sim \varepsilon^2. \tag{1.42}$$

The $\mathcal{O}(\varepsilon^0)$ part of the partition function given by eq. (1.41) is:

$$Z_\nu = \int_{U(N_f)} d\mu(U)\ \mathcal{D}et\,[U]^\nu\ \mathrm{Exp}\left[\frac{1}{2}\Sigma Tr\left[MU + MU^\dagger\right] - \frac{1}{2}VF^2 Tr\left[B\,U\,B\,U^\dagger\right]\right] \tag{1.43}$$

This partition function is based only on the symmetries of the microscopical Dirac operator and is independent of the B matrix: with $B = \mu \mathbf{1}_{N_f}$ we can describe real chemical potential,

$B = i\mu \mathbf{1}_{N_f}$ describes imaginary chemical potential and $B = \mu\sigma_3 \times \mathbf{1}_{N_f/2}$ and $B = i\mu\sigma_3 \times \mathbf{1}_{N_f/2}$ are, respectively, for real and imaginary [27] isospin chemical potential.

It is worthwhile to spend here a few words explaining why one could be interested in studying theories with a chemical potential matrix different from the real baryon chemical potential. First of all, all these theories do not suffer from the sign problem and, as a consequence, can be simulated using standard Montecarlo method [28] providing useful checks. Obviously this reason is not sufficient to justify an interest, in fact, beyond it, there are arguments saying that it is possible to obtain information on the behaviour of QCD at real chemical potential from these chemical potential-like theories. The simplest one is to perform an analytic continuation in the μ plane [29, 30]; this method had been tested on 2 colour QCD, that allows both imaginary and real chemical potential simulations, and a good agreement between the two systems was shown [31].

The interest in the real isospin potential lies in the fact that though the fermion determinant remains real and positive (and thus amenable to numerical simulations), the eigenvalues of the single quarks acquire a non-vanishing imaginary part breaking the hermiticity of the Dirac operator; this feature resembles the real chemical potential theory [32, 33, 34]. On the contrary, it could seems strange but what makes the imaginary chemical potential QCD an interesting theory is that it is not like the real chemical potential theory: imaginary isospin does not alter the hermiticity of the Dirac operator itself, and, as a consequence, it can be used as a parameter deforming the real Dirac operator spectra. As an example we show picture 1.2 where the introduction of an imaginary isospin chemical potential alters the spectral 2-points correlation function spreading the δ function contribution arising at equal points; this approach has been recently used to give a direct measurement of the pion decay constant F_π from the spectrum [27, 35, 36, 37, 38, 39].

1.3 Random Matrix Theory

1.3.1 A brief introduction to RMT

Random Matrix Theory (RMT) is nothing but a powerful tool used to describe some specific properties of complex or chaotic systems.

RMT describes ensembles of matrices with random numbers as matrix elements, in particular distributions and correlations of the eigenvalues of these matrices are usually computed when their dimensions approach infinity. The set of the possible matrix ensembles is rather small[7] and the choice of the proper one is done simply checking the symmetries of the system to be described. For many simple problems the description is possible through the use of a single random matrix [44, 45], in some cases it may be useful to consider RMT

[7] They have been classified in [40, 41] in the Hermitian case and in [42, 43] for the non-Hermitian case.

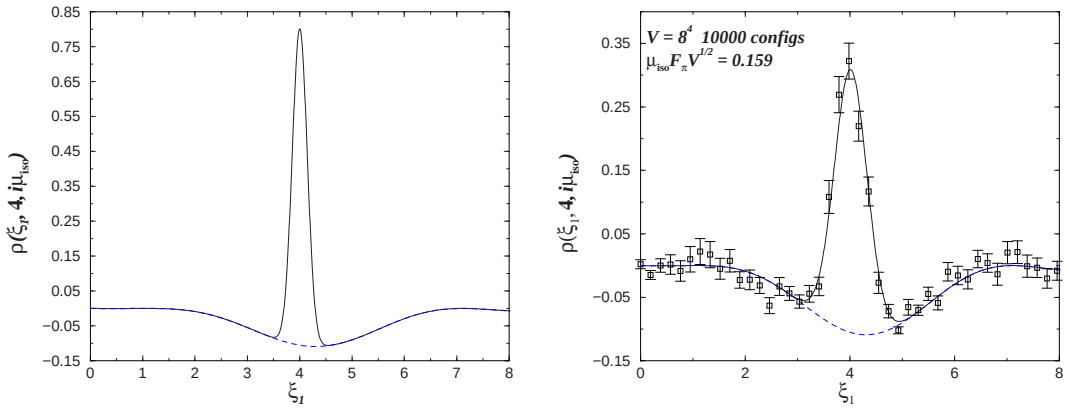

Figure 1.2: On the left hand side we show the two point correlation function $\rho(\xi_1, \xi_2)$ with one eigenvalue fixed at $\xi_2 = 4$, for for zero (dashed) and non zero isospin potential (full). The δ-function peak at $\xi_1 = \xi_2$ for μ_{iso} has not been shown. On the other side we show the two point correlation function at fixed $\xi_2 = 4$ measured on a 8^4 lattice. Pictures taken from [27].

with two or more random matrices [9, 46, 47]. The main value of these theories is that in many cases they are integrable systems and in most of them the computation are relatively easy, and, more importantly, have already been done analytically.

As said above RMT is a tool, and every tool has limitations: the strongest one (that from some point of view is a value too) is that RMTs depend on really few external (physical) parameters, in most cases it is just one (like in χRMT where the only input is the chiral condensate Σ): they can be used only to describe either simple systems or systems in particular regimes where all the relevant dependences are on few (one or two) physical parameters.

Despite these strong limitations, RMT has been applied in many different fields: excited states of heavy nuclei[8], complex molecules, transport properties of mesoscopic systems, two dimensional gravity, conformal field theory, growth problem, non-trivial zeros of the Riemann ζ function and, obviously, QCD. An important conjecture[9] providing criteria of applicability of RMT was provided by Bohigas Giannoni and Schmidt [49][10]: their claim is that the spectra of systems whose classical analogues are ergodic chaotic systems show the same fluctuation properties as the proper RMT. A good review on the history and on the applications of RMT can be found in [45].

We will not get into the details of the classification of RMTs here, we will just cite the

[8] Originally, RMT was designed by Wigner to deal with the statistics of eigenvalues of complex manybody quantum systems, having in mind the particular case of the scattering of neutrons with heavy nuclei. This application of RMT may be seen as a formal implementation of Bohrs compound nucleus hypothesis.

[9] Altough a formal proof has not yet been provided remarkable progress in that direction has been made, see [48].

[10] An earlier version of this conjecture was stated in [50].

most important theories introduced by Wigner and Dyson: a) the first one introduced is the ensemble of Hermitian matrices with real, complex and quaternion numbers, respectively known as orthogonal ensemble (OE or, according to the Dyson classification, $\beta = 1$), unitary ensemble (UE or $\beta = 2$) and symplectic ensemble (SE or $\beta = 4$); b) the second set introduced is the one of unitary matrices with real, complex and quaternion numbers, known as circular orthogonal ensemble (COE) and so on. In the present work we will not deal with any of these classical ensemble, we will focus on an implementation of the unitary ensemble taking into account the chiral structure (called chiral unitary ensemble, χUE) introduced by Shuryak and Verbaarschot [5] to describe the properties of the Dirac operator of QCD. We mention that χRMT may be defined for real and quaternion numbers [6, 19] giving rise to effective theories describing 2-colour QCD (χOE) and QCD with quarks in the adjoint representation (χSE).

Concerning the classification of RMTs we have focused the attention only on the matrix sets without taking in account the weight function. The reason for this choice lies in the fact that there is strong evidence, and in some cases there are proofs too (see [45]), of the independence from the infinite matrix dimension limit on the particular choice of the weight function (under some broad hypotheses). This property is usually referred to as RMT universality and justifies the usual simple choice of the gaussian weight function.

The problem of RMT universality is currently being studied in mathematical literature, see [51].

1.3.2 Chiral Random Matrix Theory

Chiral Random Matrix Theory (χRMT) was first introduced in [5] for QCD with 3 or more colours and quarks in the fundamental representation (this model is usually referred to as χUE). The theory was derived starting only from the symmetries of the Dirac operator and from its topological structure. We will briefly summarise this procedure.

The starting point is the Dirac operator in the gauge field A_ν:

$$i\mathcal{D} \equiv i\gamma_\nu \mathcal{D}_\nu = i\gamma_\nu \partial_\nu + \gamma_\nu A_\nu. \tag{1.44}$$

Its chiral symmetry $\{i\mathcal{D}, \gamma_5\} = 0$ implies that the eigenvalues will occur in pairs $\pm\lambda$ apart from zero eigenvalues. The last relation forces the eigenfunctions of the Dirac operator ϕ_λ to be related by the property $\phi_{-\lambda} = \gamma_5 \phi_\lambda$. The number of zero eigenvalues is fixed by the Atiyah-Singer index theorem and is equal to the winding number ν. Let us consider a finite volume discretized theory, one can turn to a chiral basis $\psi_{L,k}, \psi_{R,k}$ with $\gamma_5 \psi_{R,k} = \psi_{R,k}$ and $\gamma_5 \psi_{L,k} = -\psi_{L,k}$. Zero eigenvalues may be either right handed or left handed. In this chiral basis expansion the massless action becomes:

$$\int d^4x \; \overline{\psi} \; i\mathcal{D}\psi = \sum_{k,l} \begin{pmatrix} \chi^*_{R,k} \\ \chi^*_{L,k} \end{pmatrix} \begin{pmatrix} 0 & iD_{LR} \\ iD_{RL} & 0 \end{pmatrix} \begin{pmatrix} \chi_{R,l} \\ \chi_{L,l} \end{pmatrix} \tag{1.45}$$

where $\chi_{S,k}$ are the coefficients of the expansion of the field ψ in the chiral basis $\psi = \sum_k \chi_{R,k}\psi_{R,k} + \chi_{L,k}\psi_{L,k}$ and the matrix elements are given by

$$D_{LR,kl} = \int d^4x \; \psi^*_{R,k} \, \mathcal{D} \, \psi_{L,l} \tag{1.46}$$

and from the anti-hermiticity[11] of the euclidean Dirac operator $D_{RL} = D^\dagger_{LR}$. The functions ϕ are the eigenfunctions of of the Dirac operator for a given configuration A, and only for that configuration the matrix D_{LR} is diagonal; for all the other configurations it will be a rectangular matrix with a dimension exceeding the other of ν (due to the presence of ν zero eigenvalues of the Dirac operator). Let us say we have N_- left-handed eigenfunctions and $N_+ = N_- + \nu$ right-handed ones.

The ones above are just algebraic manipulations, and the key observations have still to be been done. We know from other approaches (χPT) that there are properties of QCD only resulting from the chiral structure and from overall symmetries (like the Leutwyler-Smilga sum rules, see sect. 1.1.3), disregarding the particular dynamics of the theory; starting from this wisdom the main idea is that whenever we are only interested in these properties we can substitute the Yang-Mills action, that is a weight-function for the matrix D_{LR}, with a simpler weight function with the same structure satisfying the same symmetries but disregarding the microscopical QCD dynamic:

$$\left(D_{LR,kl}[A_\mu], e^{S_{YM}[A_\mu]}\right) \to (T_{kl}, w(T)) \tag{1.47}$$

where T is a $N_+ \times N_-$rectangular complex matrix[12]. The simplest choice possible for the weight function is the one of a gaussian wight-function

$$w(T) = \mathrm{Exp}\left[-\sigma N \; Tr\left[T^\dagger T\right]\right] \tag{1.48}$$

where the quantity σ is a dimensionless free parameter (the only parameter in the theory) and the factor $N = N_+ + N_-$ has been introduced due to a useful convention. The simple gaussian choice for the wight function is supported by results assuring that, according to reasonable hypotheses, the $N \to \infty$ limit is independent of the particular choice of the weight function [52, 53, 54].

The substitution above is expected to be valid only for investigating the energy spectrum in given conditions, conditions which, we already know from different ways, ensure that the spectrum can be studied by means of "universal" effective theories [5]: in our case it is valid

[11] The possible presence of anti-unitary symmetry may imply that the matrix D_{LR} is a real or quaternion matrix. If such a symmetry is absent the matrix is a complex one.

[12] If one considers the elements of T real number or quaternions one obtains chOE and chSE respectively. These theories may be used to describe properties of QCD-like theories with additional anti-unitary symmetries.

only when considering the ε-regime. The resulting RMT is:

$$\int dT\ e^{-\sigma N T r[TT^\dagger]} \prod_{f=1}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT \\ iT^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}. \tag{1.49}$$

This model take the name of chGUE.

Naively speaking the argument above can be summarised saying: there are quantities, like the Leutwyler-Smilga sum rules, that are functions of the lower part of the spectra, which does not depend on the particular QCD dynamic but only on the symmetries *hence* they are universal; since they are universal they can be described by any theory with the same symmetries. This one may seem to the reader a rash conclusion, a conjecture rather than an argument, and in fact it is a conjecture. When it was formulated in [5] it was introduced together with an explicit computations showing that it was possible to obtain the very same Leutwyler-Smilga sum rules as χPT starting from χRMT. This was a strong argument in favour of the conjectured equivalence but was not at all explaining why QCD should show some universality feature[13].

A subtle argument in favour of the universality was proposed in [55] where it is shown that under well accepted hypotheses (like the pion-pole dominance, the Gell-Mann-Oaks-Renner relation or semiclassical arguments), if one considers the eigenvalues of the 4-dimensional Dirac operator like the eigenvalues of a quantum Hamiltonian in a 4+1 dimensional theory an ergodic dynamic in this additional (Schwinger) time it follows from the ε-regime range energy of 4-dimensional QCD[14]. This result, together with the Bohigas-Giannoni-Schmidt conjecture [49], gives an explanation too of the equivalence between χPT and χRMT: spectra of classically ergodic chaotic systems may be described through "proper" random matrices. The "proper" one for this particular case has to be chosen in order to verify the same symmetries and topological structure as QCD and, hence, it is the one in eq. 1.49.

Here we will not show details that can be found in [55], its easier and impressive to show some numerical results where comparison between the spectrum obtained by lattice QCD simulations and the one predicted by RMT is made.

The picture in fig. 1.3, taken from [56], "provides direct evidence for the conjecture" above. Simulations were in 2-colour quenched QCD using the staggered Dirac operator and the comparison was done with χGSE[15] predictions in [57].

Nowadays the fact that RMT may be used to describe the low energy spectra of QCD in the ε-regime is widely accepted, no matter that a real proof showing how RMT descends directly from QCD is still lacking. In this work we will not fill this gap (deriving low energy

[13] In the same paper the sum rules very verified for an instanton liquid model with the same symmetries.

[14] The same Schwinger time approach was used in [15] to derive pq-χPT starting from stochastic QCD-like theory.

[15] Two colour QCD has an anti-unitary symmetry, its universality class may be χGSE or χGOE according to the type of fermions are used [6].

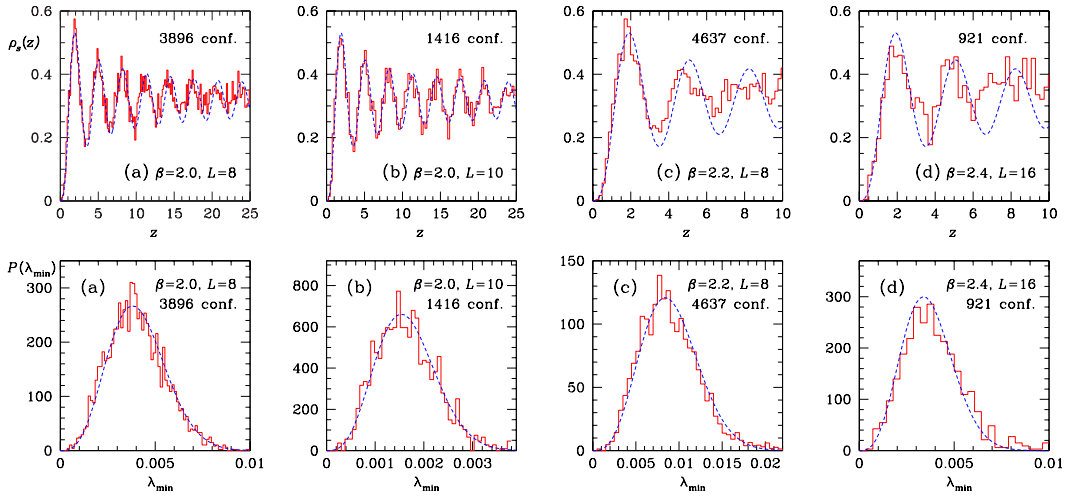

Figure 1.3: QCD spectrum (upper line) and first eigenvalue distribution (lower) from lattice simulations and RMT predictions at different gauge couplings β and lattice volume L^4. Picture taken from [56].

properties from microscopical QCD is really an ambitious task!), but we will prove the existence of a mathematical link with the well accepted χPT.

1.3.3 An example: chGUE

In order to understand how RMT works it may be instructive to show an explicit computation as an example; we compute here the spectral correlation function for the chGUE with N_f dynamical fermions like in eq. (1.49). For simplicity we show the computation for zero topological charge as it was shown for the first time in [58]. The starting point is that any complex matrix T can be diagonalised according to $T = V \cdot \Lambda \cdot W$, with $V \in U(N)$, $W \in U(N)/U(1)^N$ and Λ is a positive definite diagonal matrix with entries $\lambda_1, \ldots, \lambda_N$. The measure transforms according to

$$dT = d\mu(U)\, d\mu(W) \prod_i d\lambda_i \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 \prod_k \lambda_k. \tag{1.50}$$

The measures over the unitary matrices $d\mu(U)$, $d\mu(W)$ are Haar measures and the integration gives a trivial contribution since these angular degrees of freedom decouple from the rest of integral.

The N-eigenvalue distribution, also called *joint probability density function* (jpdf) is given by:

$$\rho_N(\lambda_1, \ldots, \lambda_N) = \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 \prod_{k=1}^{N} \left(\lambda_k \prod_f^{N_f} (m_f^2 + \lambda_k^2) \right) \mathrm{Exp}\left[-2\sigma N \sum_{k=1}^{N} \lambda_k^2 \right]. \tag{1.51}$$

The spectral density function (or any k-point correlation function) can be obtained integrating the jpdf over the remaining $N-1$ $(N-k)$ eigenvalues. The technical tool used to solve this integration is provided by the orthogonal polynomials [44]: one defines a scalar product in a function space, in this case:

$$< f, g >= \int d\lambda^2 \prod_f^{N_f} (\lambda^2 + m_f^2) e^{-2\sigma N \lambda^2} f(\lambda)\, g(\lambda). \tag{1.52}$$

Orthogonal polynomials are obtained performing Gram-Schmidt orthogonalisation to the monomials $\lambda^{2j},\ j \in \mathbb{N}$. The term $\prod_{k<l}(\lambda_k^2 - \lambda_l^2)$ is written as a Vandermonde determinant and this determinant is expanded using the Cramer's rule, but instead of writing λ_k^{2j} one can consider any monic $P_j(\lambda_k^2)$. All the terms involving integrated variables can be integrated using the orthogonalisation relation $< P_k, P_s >= r_k\, \delta_{k,s}$. The result is a sum of $P_k \cdot P_k$. This sum can be performed through the Christoffel-Darboux formula. For $m_f = 0$ the polynomials are known[16], they can be written in terms of Laguerre polynomials L_a^b and the result for the spectral density function is:

$$\begin{aligned} \rho_1(\lambda) \;\; = \;\; & \frac{2\sigma N\ N!}{(N+N_f-1)!} (2\sigma N\lambda^2)^{N_f+1/2} \mathrm{Exp}\left[-2\sigma N\lambda^2\right] \\ & \times \left(L_{N-1}^{N_f}(2\sigma N\lambda^2) L_{N-1}^{N_f+1}(2\sigma N\lambda^2) - L_N^{N_f}(2\sigma N\lambda^2) L_{N-2}^{N_f+1}(2\sigma N\lambda^2) \right) \end{aligned} \tag{1.53}$$

Rather than in the quantity above, we are interested in its *microscopic limit* ρ_s, that is $N \to \infty$ limit keeping $x = 2N\lambda$ constant. The reason for this interest lies in the fact that this limit is not sensitive to the particular choice of the weight function (restricted to a broad class of function) [52, 53, 54] and, obviously, is the one that is believed to describe QCD spectrum. The quantity to be kept fixed, $\sigma\, N\, \lambda$, strongly reminds the quantity $\Sigma\, V\, \lambda$ in the Leutwyler-Smilga sum rule (1.21), where σ plays the role of the chiral condensate Σ and the number of eigenvalues N is related to the volume V. The microscopic limit of (1.53) can be obtained using the limit:

$$\lim_{N\to\infty} \frac{1}{N^\alpha} L_N^\alpha \left(\frac{y}{N}\right) = y^{-\alpha/2} J_\alpha(2\sqrt{y}) \tag{1.54}$$

where J is a Bessel function of the first kind. The result is

$$\rho_s(x) = \sigma x \left(J_{N_f}^2\left(\sqrt{2\sigma}x\right) - J_{N_f+1}\left(\sqrt{2\sigma}x\right) J_{N_f+1}\left(\sqrt{2\sigma}x\right)\right). \tag{1.55}$$

The same result was obtained later in [60] starting from χPT and was confirmed through lattice simulations in [61]. The explicit result for the k-point correlation function can be found in [52].

The $m_f \neq 0$ explicit solution can be read off from [54, 59]. Looking at eq. (1.51), and considering that $z^2 + m_f^2 = (z + i m_f)(z - i m_f)$, one can see that any flavour can be treated

[16] They are also known for $m_f \neq 0$, see [54, 59].

as an additional imaginary eigenvalue (modulo some easy to calculate mass-dependent prefactor). After this remark we see that the microscopic limit has to be considered scaling the quark masses keeping $m \cdot N$ fixed. The analogous of this condition in χPT is that we are considering only quantities depending on the masses scaling like ε^0, see sect. 1.1.3.

1.3.4 Non Hermitian chiral RMT

RMTs may be used to describe non Hermitian theories too. We consider here only the case relevant to the purposes of this work, that is the chiral case used to describe low energy properties of QCD with a chemical potential (see [62] for a review). The latter is a theory whose Dirac operator is no more Hermitian, and it can be approached with the very same idea used in the Hermitian cases; an equivalent of the "heuristic" substitution of eq. (1.47) may be provided:

$$\left(\begin{pmatrix} 0 & iD_{LR}[A]+\mu \\ iD_{RL}[A]+\mu & 0 \end{pmatrix}, e^{S_{YM}[A]}\right) \rightarrow (D_{RMT}, w(D_{RMT})) \tag{1.56}$$

where this time D_{RMT} is a non-Hermitian matrix.

Two different matrix structures have been introduced for D_{RMT}, the first by Stephanov [8] which consists of adding a Hermitian constant term (miming the chemical potential) to the standard anti-Hermitian random matrix part (Dirac operator), in formulas:

$$\begin{pmatrix} 0 & iD_{LR}[A]+\mu \\ iD_{RL}[A]+\mu & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & iT+\mu \\ iT^{\dagger}+\mu & 0 \end{pmatrix}. \tag{1.57}$$

The other model was introduced by Osborn [9] and consists of two independent random matrix parts, one Hermitian and one anti-Hermitian:

$$\begin{pmatrix} 0 & iD_{LR}[A]+\mu \\ iD_{RL}[A]+\mu & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & iT+\mu W \\ iT^{\dagger}+\mu W^{\dagger} & 0 \end{pmatrix}. \tag{1.58}$$

In both models the quantity μ is a dimensionless parameter (< 1) playing the role of the chemical potential. The usual choice for the random matrix weight function is the gaussian one.

Despite the fact that Osborn's model is a two matrix model and that doubling the number of variables may seem to be increasing the complexity of calculations this is not always the case, there are quantities, like the spectral density function [9], whose computation in this framework is much easier than in the other one [63]. On the other side Stephanov's model is more efficient in other computations (like the study of QCD phase diagrams, e.g. see [64]). Every time computations (or simulations) have been performed in the two models they were in agreement in the thermodynamic limit ([65, 9, 63, 66]). These two models are completely equivalent and in chapter 4 we will give a mathematical proof of it.

The partition functions for the two models are ($N_+ \geq N_-$)

$$Z = \int dT\, e^{-\sigma N Tr[TT^\dagger]} \prod_{f=1}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT + \mu \mathbf{1}_{N_-} \\ iT^\dagger + \mu \mathbf{1}_{N_-} & m_f \mathbf{1}_{N_-} \end{bmatrix} \tag{1.59}$$

(the off diagonal parts are $N_+ \times N_-$ matrices, and the identity matrix $\mathbf{1}_{N_-}$ has to be intended like the biggest identity matrix fitting in this rectangular) for Stephanov's model and

$$Z = \int dT\, dW\, e^{-\sigma N Tr[TT^\dagger + WW^\dagger]} \prod_{f=1}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT + \mu W \\ iT^\dagger + \mu W^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix} \tag{1.60}$$

for Osborn's one.

We briefly mention the way used by Osborn to solve his model, it is not different conceptually from the one sketched in sect. 1.3.3 for the Hermitian model. The starting point is to note that the blocks of the Dirac operator can be simultaneously "triangularised"[17]

$$\begin{aligned} iT + \mu W &= U(X + R)V \\ iT^\dagger + \mu W^\dagger &= V^\dagger(X + R)U^\dagger \end{aligned} \tag{1.61}$$

and that the upper triangular parts V and R are irrelevant both to the Jacobian of this triangularisation and to the argument of the integration. As a consequence it is possible to express the integrals in terms of the (diagonal) elements of X and Y, or even better, just of the complex (diagonal) elements of $X \cdot Y$. Once obtained for the N-eigenvalues distribution an analogous of eq. (1.51) in order to obtain the k-point function one integrates the remaining $N-k$ complex variables through the complex-orthogonal polynomials method [67, 68, 69]. Conceptually it is not different from the real one already encountered in the previous section, the only difference lies in the fact that the integration defining the scalar product is over the whole complex plane. An explicit solution can be written in terms of Laguerre polynomials for finite N and its N-infinite limit in terms of Bessel functions. Showing a picture of the typical density function is more clarifying than writing the explicit expression (for that we refer to [9, 62]), see fig. 1.4.

In the previous section we have pointed that the $N \to \infty$ limit can be written only when the mass changes with N like N^{-1}. An analogous of this scaling property exists for the "chemical potential" too: $N \cdot \mu^2$ has to be kept fixed. This is usually called weak non-hermiticity limit, in order to distinguish it from the strong one where μ^2 stays finite (see [70] for an overview on the topic). The weak non-hermiticity limit is the RMT equivalent of the power counting (1.42) in χPT.

The quenched spectrum predicted by this model[18] has been checked with the one obtained from lattice simulations [72, 73] showing a good agreement between the data (see fig.

[17] The block are not square matrices due to the topological charge.

[18] Strictly speaking the model was not this one, it was an older eigenvalues-model introduced by Akemann [68] having the same $m \cdot N \to 0$ as Osborn's one. This model was not introduced starting from a matrix model.

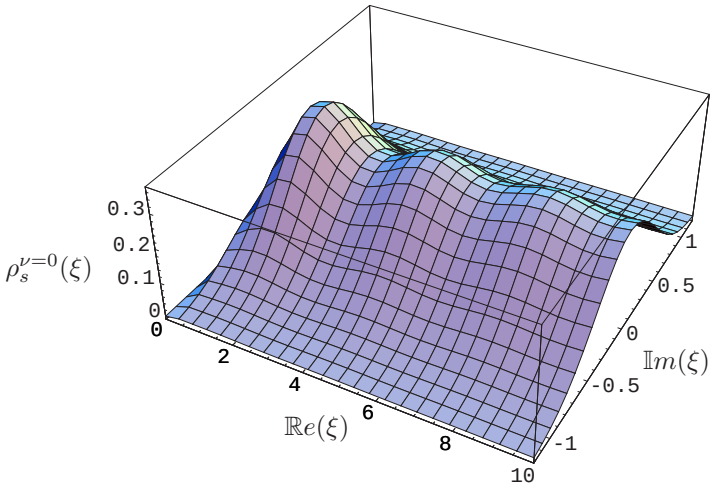

Figure 1.4: The quenched spectral density for non-zero chemical potential for zero topological charge. Picture taken from [62].

1.5).

What we have shown up to this point is the non-Hermitian chUE, but exactly like UE and chUE, non-Hermitian chiral RMT (sometimes referred to as χR2MT) has a orthogonal and symplectic partner too, whether one considers random matrices with real or quaternionic entries. These two models are particularly interesting since they describe QCD-like theories that can be simulated on the lattice. In [74] the predictions for the symplectic ensemble [69, 71] have been successfully compared with the lattice data coming from 2-colour QCD with staggered fermions at non-zero chemical potential.

1.3.5 Isospin chemical potential RMT

As in QCD the chemical potential matrix may be chosen not proportional to the identity. In general we can define the Dirac operator for a quark with mass m_f and chemical potential μ_f, it is given by [75]:

$$i\mathcal{D}_f + m_f \equiv \begin{pmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{pmatrix} , \tag{1.62}$$

the partition function is:

$$\begin{aligned} Z_{pq} =& \left\langle \prod_f^{n_f} \mathcal{D}et \left[i\mathcal{D}_f + m_f \right] \right\rangle \\ =& \int dA\, dB\, e^{-\sigma N Tr[AA^\dagger + BB^\dagger]} \prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix} \end{aligned} \tag{1.63}$$

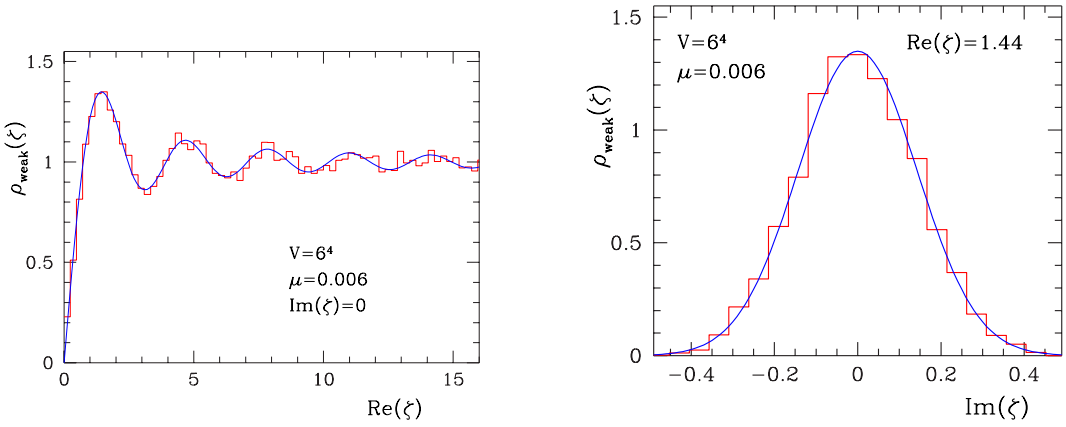

Figure 1.5: Density of small Dirac eigenvalues of a 6^4 lattice at non-zero chemical potential, cut along the real axis (left) and parallel to the imaginary axis at the first maximum (right). The histogram represents lattice data, and the solid curve is theoretical prediction. Picture taken from [72].

where A and B are complex $N_+ \times N_-$ random matrices with Gaussian weights, and, as in eq. (1.49) the measures dA and dB are the flat measure in the independent entries of the matrices.

This model was introduced and solved for isospin chemical potential[19] for real case in [66] (it follows as a particular case of [9]) and imaginary[20] one in [75]. These models are defined following Osborn's prescription [9]: the chemical potential term is a coupling between two random matrices; this is just an apparent complication indeed this introduction makes the model simpler: for two different chemical potentials one can go to an eigenvalue basis and use bi-orthogonal polynomials.

The properties making these two isospin theories interesting have been already summarised in section 1.2, an example of the results obtained is the F_π-depending deformation of the two point correlation function in fig 1.2.

The issue of universality is more subtle here because the matrices A and B will couple after changing variables. We refer to [75] for a more detailed discussion.

1.4 Partially quenched QCD and superanalysis

We will introduce here an important instrument for deriving the properties of the spectrum of an operator: the resolvent method. This is a general method that can be applied both to QCD and to RMT, though in the latter case it is inconvenient compared to the simpler orthogonal polynomial method. It leads to the introduction of a QCD-like theory with

[19] The chemical potential values are coupled in $+\mu$ and $-\mu$.

[20] In this case the solution given holds for N_1 quark with chemical potential $i\mu_1$ and N_2 with $i\mu_2$.

fermionic and bosonic quarks called *partially quenched QCD* (pq-QCD). In order to deal with complex (bosonic) and Grassmannian (fermionic) variables we introduce super-analysis (see [76, 77] or appendix A).

1.4.1 The resolvent method - Hermitian case

We start with a simple but clarifying example to explain the idea of resolvent method. Consider a real number λ_0 and a compact surface Ω in the complex plane whose contour is denoted by $\partial\Omega$. Whenever $\lambda_0 \notin \partial\Omega$ we have that:

$$\oint_{\partial\Omega} dz \, \frac{1}{z-\lambda_0} = \chi_\Omega[\lambda] \equiv \begin{cases} 1 \text{ if } \lambda_0 \in \Omega \\ 0 \text{ if } \lambda_0 \notin \Omega \end{cases} \tag{1.64}$$

Similarly given a set of $\lambda_i \in \mathbb{R}$:

$$\oint_{\partial\Omega} dz \, \sum_i \frac{1}{z-\lambda_i} = \sum_i \chi_\Omega[\lambda_i]. \tag{1.65}$$

The poles of the integrand on the l.h.s. lie along the real line, the path of integration may be deformed in two pieces, one above and one below the real line.

$$\lim_{\varepsilon\to 0} \int_{\Omega\cap\mathbb{R}} dz \, \sum_i \left(\frac{1}{z-i\varepsilon-\lambda_i} - \frac{1}{z+i\varepsilon-\lambda_i} \right) = \sum_i \chi_\Omega[\lambda_i]. \tag{1.66}$$

The equation above explains the main idea of the resolvent method: on the l.h.s. we have a line integration, on the r.h.s. we have a function that counts the eigenvalues inside Ω. Unfortunately this naive equation cannot be implemented giving rise the spectral density function: in order to do that one should perform the average of functional in eq. (1.64) according to the statistics of ensemble, but that functional is not defined over all the dominium of the integration of the eigenvalues[21].

A proper approach to the resolvent method can be found in [78]; we define the spectral density function of an operator whose eigenvalues λ_i are distributed according to a given ensemble like

$$\rho_1(z) \equiv \langle \sum_i \delta(z-\lambda_i) \rangle \tag{1.67}$$

and the (1-point) resolvent:

$$G_1(z) \equiv \langle \sum_j \frac{1}{z-\lambda_j} \rangle = \int_{\mathbb{R}} d\lambda \, \rho_1(\lambda) \frac{1}{z-\lambda} \tag{1.68}$$

the latter is defined only outside the support of spectral density ρ_1. The equation above can be inverted [78]:

$$\rho_1(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0} \left[G_1(\lambda - i\varepsilon) - G_1(\lambda + i\varepsilon) \right]. \tag{1.69}$$

[21] Eq. (1.64) is valid only for $\lambda_0 \notin \partial\Omega$.

The same formula can be used to invert the integrals for the k-point correlation functions: once defined the k-point density correlation ρ_k and the k-point resolvent G_k:

$$\rho_k(z_1,\ldots,z_k) \equiv \langle \prod_k \left(\sum_i \delta(z_k - \lambda_i) \right) \rangle$$
$$G_k(z_1,\ldots,z_k) \equiv \langle \prod_{j=1}^{k} \sum_{\lambda \in \text{e.v.}} \frac{1}{z_j - \lambda} \rangle = \int \prod_{j=1}^{k} d\lambda_j \frac{1}{z_j - \lambda_j} \rho_k(\lambda_1,\ldots,\lambda_k) \tag{1.70}$$

we obtain the k-point density correlations by computing the discontinuities with respect to all arguments (see e.g. [79]):

$$\rho_k(\eta_1,\ldots,\eta_k) = \frac{1}{(2\pi i)^k} \lim_{\varepsilon \to 0^+} \sum_{\{\sigma\},\sigma_j=\pm 1} \left(\prod_{j=1}^{k} \sigma_j \right) G_k\left(\eta_1 - i\sigma_1\varepsilon,\ \ldots,\ \eta_k - i\sigma_k\varepsilon\right). \tag{1.71}$$

The k-point correlation function can be obtained from the k-point density function by removing the singularity arising whenever two or more arguments are coinciding [45].

We have seen how to generate the spectral properties from the resolvents, but how to compute the resolvents? The idea is that since the λs are eigenvalues of an operator D we can write:

$$\sum_j \frac{1}{z - \lambda_j} = \frac{\partial}{\partial z'} \frac{\prod_j (z' - \lambda_j)}{\prod_i (z - \lambda_i)} \Big|_{z'=z} = \frac{\partial}{\partial z'} \frac{\mathcal{D}et\,[z' - D]}{\mathcal{D}et\,[z - D]} \Big|_{z'=z} \tag{1.72}$$

and substitute this equation in the definitions (1.68) and (1.70). The problem has now turned to computing the expectation value of a ratio of determinants; if for RMT this is a technical problem, for χPT (that is for QCD) this is equivalent to considering a different theory with both fermionic and bosonic quarks.

1.4.2 Partially quenched χPT

The starting point is a gaussian integral: given a complex number α with $\mathrm{Re}[\alpha] > 0$ it holds that:

$$\int_{\mathbb{C}} d_{\mathbb{C}} z\, e^{-\alpha\, z z^*} = \frac{\pi}{\sqrt{\alpha}}. \tag{1.73}$$

It is well known that it can be generalised to an integral over $\mathbb{C}^{n_b}$ of

$$\int_{\mathbb{C}^{n_b}} d_{\mathbb{C}}^{n_b} z\ \mathrm{Exp}\left[-z^\dagger \cdot A \cdot z\right] = \frac{\pi^{n_b}}{\sqrt{\mathcal{D}et\,[A]}} \tag{1.74}$$

where A is a matrix whose Hermitian part is positively defined.

Let us consider the Grassmannian counterpart of the gaussian integral (for conventions and notations on Grassmann variables and superanalysis see app. A):

$$\int d\theta^*\, d\theta\ e^{-\alpha\, \theta^* \theta} = \int d\theta^*\, d\theta\ \ (1 - \alpha\, \theta^* \theta) = \frac{\alpha}{\pi} \tag{1.75}$$

it can be generalised to Grassmann vectors too

$$\int d^{n_f}\overline{\theta}\, d^{n_f}\theta \ \mathrm{Exp}\left[-\theta^\dagger \cdot A \cdot \theta\right] = \frac{1}{\pi^{n_b}} \mathcal{D}et\left[A\right] \tag{1.76}$$

Complex integration is used to describe bosonic fields and Grassmann integration is used for fermionic fields, eq. (1.16) comes from the integration of the fermionic fields through eq. (1.76).

If we want to compute the k-point spectral function of the anti-Hermitian Dirac operator $i\mathcal{D}$ in QCD through the resolvent method, according to eqs. (1.71,1.72) we have to compute the ratio:

$$Z_{pq} \equiv \left\langle \prod_i^k \frac{\mathcal{D}et\left[z_i' - i\mathcal{D}\right]}{\mathcal{D}et\left[z_i - i\mathcal{D}\right]} \right\rangle_{QCD} = \left\langle \prod_i^k \frac{\mathcal{D}et\left[z_i' - i\mathcal{D}\right]}{\mathcal{D}et\left[z_i - i\mathcal{D}\right]} \prod_j^{N_f} \mathcal{D}et\left[m_i - i\mathcal{D}\right] \right\rangle_{YM} \tag{1.77}$$

where the first expectation value is in QCD (gauge + fermions) and the second one is in the Yang Mills theory (only gauge). From this equation we can see that the quantity that we have to compute is the partition function of a QCD-like theory with $N_f + k \equiv n_f$ fermions and $k \equiv n_b$ bosons; this theory is usually called *partially quenched QCD* (pq-QCD). Its name comes after the fact that it was used for the first time in lattice QCD [80] to simulate a theory with both dynamical and valence quarks. The k additional couples of bosonic and fermionic quarks are sometime referred to as *valence quark* (if $z_i = z_i'$ the loop diagrams involving these quark cancel due to supersymmetry) and the physical ones are called *sea quarks*.

The next step is to find a χPT-like theory able to describe the low energy sector of pq-QCD. The right answer was given in [60], and it claims that the partition function, at fixed topology ν, is given by:

$$Z_\nu[\mathcal{M}] = \int_{\hat{G}l(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\left[U\right]^\nu \mathrm{Exp}\left[-\frac{1}{2} V\Sigma\ Str\left[\mathcal{M}U + \mathcal{M}U^{-1}\right]\right]. \tag{1.78}$$

In the formula above U and $\mathcal{M}$ are supermatrices (see app. A) written according the boson-fermion convention, $\mathcal{S}det\left[\cdot\right]$ and $Str\left[\cdot\right]$ are the superdeterminant and the supertraces (sometime also referred to as graded-determinant and graded-trace). Matrix $\mathcal{M}$ is a diagonal matrix whose entries are given by $m_1, \ldots, m_{N_f}, z_1', \ldots, z_k'$ in the fermion-fermion sector and $z_1, \ldots, z_k$ in the boson-boson one. Matrix U belongs to the maximal Riemannian submanifold for the symmetric superspace $Gl(n_b|n_f)$, in formulas $\hat{Gl}(n_b|n_f)$, and $Gl(n_b|n_f)$ is the the manifold of $(n_b|n_f) \times (n_b|n_f)$ matrices with a non-vanishing superdeterminant. The measure $d\mu_{\mathcal{S}}(U)$ is the Haar measure (invariant with respect to the matrix product) over this supermanifold.

A proper mathematical definition of $\hat{Gl}(n_b|n_f)$ can be found in [81].

$\hat{Gl}(n_b|n_f)$ is a supermanifold (that is a "manifold" whose degrees of freedom are described by commuting and anticommuting numbers) whose base manifold (that is the manifold

described by the complex number part of the commuting entries in the matrix representation) is $Gl(N_b)/U(n_b) \oplus U(n_f)$. As an example, a matrix representation of the elements of this group can be provided by:

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \cdot \mathrm{Exp} \begin{bmatrix} 0 & \Omega \\ \Xi & 0 \end{bmatrix} \tag{1.79}$$

where $U \in Gl(n_b)/U(n_b)$, $V \in U(n_f)$ and Ω and Ξ are $n_b \times n_f$ matrix whose entries are Grassmann variables, or, using the same notation, another representation may be:

$$\begin{pmatrix} U & \Omega \\ \Xi & V \end{pmatrix}. \tag{1.80}$$

Before [60] Lie superunitary group $U(n_b|n_f)$ was commonly used as the chiral group in pq-QCD instead of $\hat{Gl}(n_b|n_f)$, it is worth to explain why this choice has been dropped.

To begin with the Haar measure integrated over the whole group vanishes [76]:

$$\int_{U(n_b|n_f)} d\mu_{\mathcal{S}}(U) = 0 \tag{1.81}$$

and it implies that we could obtain a vanishing partition function; its geometry is non-Riemannian (it has a metric tensor with positive and negative eigenvalues) and this implies that a possible kinetic term $Str\left[\partial_\alpha \Phi \partial_\alpha \Phi\right]$ would be not bounded from below.

There is a third reason to choose $\hat{Gl}(n_b|n_f)$ instead of $U(n_b|n_f)$ which relies on the microscopical theory we want to describe. When considering the $(n_b|n_f) = (n_b|0)$ case we should recover the purely bosonic χPT. The integration manifold consistent with $U(n_b|0)$ is $U(n_b)$, but it is not consistent with the symmetries of bosonic QCD: let us investigate its integration manifold studying the simple $n_b = 1$ case [82]. We write the inverse power of the Dirac operator determinant through a complex number gaussian integration:

$$\mathcal{D}et \begin{bmatrix} m & i\mathcal{D} \\ i\mathcal{D}^\dagger & m \end{bmatrix} = \frac{1}{\pi^2} \int d_{\mathbb{C}}\phi_1 \, d_{\mathbb{C}}\phi_2 \, \mathrm{Exp} \left[\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} m & i\mathcal{D} \\ i\mathcal{D}^\dagger & m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right] \tag{1.82}$$

The equation above (and hence the theory with bosonic quarks itself) is valid only when the integral on the r.h.s. is convergent: $\begin{pmatrix} 0 & i\mathcal{D} \\ i\mathcal{D}^\dagger & 0 \end{pmatrix}$ is anti-Hermitian and hence has only imaginary eigenvalues, as a result the convergence of the integral (that is the positivity of the matrix) exclusively relies on m that, consequently, is required to have positive real part.

The most general flavour symmetry group acting on this action is $Gl(2)$. We can parametrise this manifold with:

$$U = e^H \cdot V \tag{1.83}$$

with $H = H^\dagger$ Hermitian and V unitary. The invariance of the action with $m = 0$ implies

$$H = \begin{pmatrix} s & \\ & -s \end{pmatrix}, \text{ with } s \in \mathbb{R}, \text{ and } V = \mathbf{1} \cdot e^{i\theta}. \tag{1.84}$$

H may be seen as a representation of $Gl(1)/U(1) \sim \mathbb{R}$. The e^H symmetry is explicitely broken by the mass $m \neq 0$ and the V symmetry is not; hence we can consider $Gl(1)/U(1)$ as the broken axial symmetry and $U(1)$ as the conserved vectorial current. This argument can be generalised to n_b bosonic quarks, leading to $Gl(n_b)/U(n_b)$ and $U(n_b)$ as the broken and the conserved symmetry.

This result is in contrast with the purely bosonic case of $U(n_b|0)$, and hence the choice $U(n_b|n_f)$ is not admissible.

1.4.3 The resolvent method - non Hermitian case

In section 1.4.1 we have explained how to generate the spectral properties from resolvents provided that the eigenvalues lie on a line[22]. If the spectrum is spread on $\mathbb{C}$ (or on a non-trivial 2-dimensional subset of it) we have to use a different method to generate the δ functions:

$$\frac{1}{\pi}\,\partial_{z^*}\frac{1}{z} = \delta_{\mathbb{C}}(z). \tag{1.85}$$

For the sake of simplicity let us consider the spectral density. It can be written:

$$\rho_1(z) = \left\langle \sum_j \delta_{\mathbb{C}}(z-\lambda_j) \right\rangle = \frac{1}{\pi}\left\langle \sum_j \partial_{z^*}\frac{1}{z-\lambda_j} \right\rangle = \frac{1}{\pi}\,\partial_{z^*}\,G_1(z) \tag{1.86}$$

where we have defined the complex resolvent exactly as in the Hermitian case in eq. (1.68).

Generalisation to k-point spectral density is straightforward:

$$\rho_k(z_1,\ldots,z_k) = \frac{1}{\pi^k}\,\partial_{z_1^*}\cdots\partial_{z_k^*}\,G_k(z_1,\ldots,z_k). \tag{1.87}$$

The idea to generate resolvents starting from ratios of determinants is the same as in the Hermitian case (eq. (1.72)), however a regularisation is needed due to convergence requirement.

1.4.4 Partially quenched χPT for non-Hermitian QCD

Let us consider the partially quenched partition function in eq. (1.77) for a non-Hermitian theory, $i\mathcal{D} \to i\mathcal{D} + \mu\gamma_0$:

$$\left\langle \frac{\prod_k^{n_f} \mathcal{D}et\left[m_k - i\mathcal{D} - \mu\gamma_0\right]}{\prod_j^{n_b} \mathcal{D}et\left[m_j - i\mathcal{D} - \mu\gamma_0\right]} \right\rangle_{YM} \tag{1.88}$$

In contrast with the Hermitian case this quantity cannot be directly derived from a theory involving fermionic and bosonic quarks. According to what is said in appendix A writing an inverse determinant in terms of a bosonic integral is possible only if the matrix has a

[22]This is an essential requirement to transform a contour integration in an integration over $\mathbb{R}$. The integration over $\mathbb{R}$ is subsequently "inverted" giving the spectral properties, or on a formal level we have that the resolvent can be defined only outside the spectrum.

positive definite Hermitian part. This requirement has been fulfilled in sect. 1.4.2 imposing $\mathbb{R}e(m_b) > 0$. For non-Hermitian operators this requirement is fulfilled through an Hermitisation procedure [83, 84] and a new particle content of the theory comes out [85, 63, 86].

Let us consider the simplest example, $n_b = 1$, $n_f = 0$ [86]. The same sign problem that invalidates the importance sampling method in lattice simulations adds here a term depending on the chemical potential to the Hermitian part of the Dirac operator, it is:

$$\mathcal{D}_H = \begin{pmatrix} m & \mu \\ \mu & m \end{pmatrix} \tag{1.89}$$

which has a negative eigenvalue whenever $m < \mu$. The problem may be circumvented by an Hermitisation procedure [83, 84]; for any bosonic quark with parameters (m, μ) we have to add an additional couple of bosonic and fermionic conjugated quarks with parameters $(-m^*, -\mu)$:

$$\begin{aligned} \frac{1}{\mathcal{D}et\left[m - i\mathcal{D} - \mu\gamma_0\right]} &= \frac{\mathcal{D}et\left[m_j^* + i\mathcal{D} - \mu\gamma_0\right]}{\mathcal{D}et\left[m_j^* + i\mathcal{D} - \mu\gamma_0\right]\mathcal{D}et\left[m - i\mathcal{D} - \mu\gamma_0\right]} \\ &= \lim_{\varepsilon\to 0^+} \frac{\mathcal{D}et\left[m_j^* + i\mathcal{D} - \mu\gamma_0\right]}{\mathcal{D}et\begin{bmatrix} \varepsilon\mathbf{1} & m - i\mathcal{D} - \mu\gamma_0 \\ -m_j^* - i\mathcal{D} + \mu\gamma_0 & \varepsilon\mathbf{1} \end{bmatrix}}. \end{aligned} \tag{1.90}$$

Apart from the ε term the matrix in the denominator in the last equation is anti-Hermitian. The introduction of this additional ε term ensures the positivity of the matrix and hence it can be written in terms of a converging bosonic integral. The generalisation to any number n_b of bosonic uncoupled quarks is straightforward: $(N_b|0) \to (N_b + N_b^*|N_b^*)$.

We can conclude that a partially quenched partition function can be derived from QCD with a real chemical potential only when bosonic quarks appear in conjugate couples.

This Hermitisation may seem just a mathematical trick, but this is not the case, the need of writing inverse ratios of determinants in terms of converging integrals is a feature of the underlying theory, and to write the underlying theory it is necessary to derive χPT Lagrangian.

It is remarkable that whereas Hermitisation is needed for the computations in QCD this is not the case in RMT where there is no need of writing inverse determinants in terms of convergent integrals and their expectation values are properly defined for complex spectra too [87, 66, 86].

The partition function for QCD with a couple of conjugated bosonic quarks was derived in [63] both starting from the symmetries of the theory and starting from a supposed equivalent

RMT:

$$Z^{(n_b=1+1^*)} = \lim_{\varepsilon\to 0}\int_{Q\in GL(2),\, Q=Q^\dagger} \frac{dQ}{\mathcal{D}et\,[Q^2]}\theta(Q) \qquad (1.91)$$
$$\mathrm{Exp}\left[-\frac{V\Sigma}{2}Tr\left[M^T(Q+I\cdot Q^{-1}\cdot I)\right]-\frac{VF^2}{2}Tr\left[B\cdot Q\cdot B\cdot Q^{-1}\right]\right]$$

where

$$M=\begin{pmatrix}\varepsilon & -iz\\ -iz^* & \varepsilon\end{pmatrix},\quad I=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad B=\begin{pmatrix}\mu & \\ & -\mu\end{pmatrix} \qquad (1.92)$$

and the symbol $\theta(Q)$ is a step function that is 1 whenever Q is positive definite and 0 otherwise.

One of the original results of the present work is the form of the partition function involving an unequal number of fermions and bosons (already published in [88]), it will be presented in sect. 2.3.

An additional remark may be made on the well definiteness of the limit $\varepsilon \to 0$. This limit may diverge [63]. Despite we have not solved this problem, in section 2.3.1 we will consider this problem a bit more in detail, explaining why this divergence arises in some cases (an additional invariance under a non-compact subgroup when computing the integral directly in the limit $\varepsilon = 0$) and why we think it should not be present in other cases.

Chapter 2

The RMT-χPT equivalence

THE $N \to \infty$ limit of χRMT and $\mathcal{O}(\varepsilon^0)$ order of χPT are two effective theories describing the same properties as QCD, and, as already hinted in sect. 1.3.2, though a proof is still lacking, nowadays their equivalence is a widely accepted fact. In this chapter we will show that all the spectral properties of χRMT and χPT are indeed equivalent.

The proof is based on the fact that spectral properties may be derived through resolvents starting from the partially quenched partition functions, see sect. 1.4, and hence the equivalence of the spectral properties follows from the equivalence of these partially quenched partition functions. The way to show that pq-χPT and pq-χRMT have the same partition functions recalls the method provided in [5] for the standard (purely fermionic) χPT and χRMT. This work is not just a blind reproduction for a mixed bosonic-fermionic theory of the calculation done for the fermionic case: the introduction of bosons, and hence the need of superanalysis formalism, invalidates the standard approach to Hubbard-Stratonovich[1] transformation that was used in the old proof. We have substituted this transformation (together with a subsequent saddle point approximation) with a new superbosonisation theorem. The proof of the latter is provided in chapter 3.

The proof can be schematically summarised in the following scheme:

$$\begin{array}{rcl} Z_{pq} \text{ in } \chi \text{ RMT} & = & Z_{pq} \text{ in } \varepsilon\chi \text{ PT} \\ & \Downarrow & \\ \rho_k(\cdots) \text{ in } \chi\text{RMT} & = & \rho_k(\cdots) \text{ in } \varepsilon\chi \text{ PT} \end{array}$$

where the equivalence between the partially quenched partition functions follows from *superanalysis computation* and it is heredited by the spectral correlation functions thanks to the *resolvent method*.

[1] As an example see [90].

As already said the equivalence of these theories is already widely accepted: they both descend, through symmetry arguments, from QCD, and, more importantly, any time a quantity has been computed in both frameworks the results were the same. We recall here (up to our knowledge) these partial results. The partition functions with only fermions were shown to agree for zero [5] and non-zero μ [63, 65, 89, 62]. The equivalence for partition functions with only bosons at $\mu = 0$ follows from [90]. The generating functional of the $(N_f + 1|1)$ supergroup integral leading to the spectral density was computed in [14, 60] including N_f massless fermions and checked with the already known result from RMT. This work was extended to include the quenched two-point density in [91], all for $\mu = 0$. Furthermore, the quenched density at real $\mu \neq 0$ was computed from both $\epsilon\chi$PT [92] and χRMT [9] and found to be in agreement. For imaginary isospin $\mu \neq 0$ the equivalence was established up to the two-point function in [27, 35, 36, 75]. It was pointed out [93] in principle how to compute the distributions of the k-th individual eigenvalue from $\epsilon\chi$PT, using all k-point density correlation functions, in order to reproduce previous χRMT results [94, 95]. The same strategy can be applied for non-zero μ [96, 97].

It is important to note that proving the equivalence of all the k-point correlation function is not just a mathematical exercise secondary to the equivalence of the spectral density: there are quantities, like the Dirac operator individual eigenvalue distribution, whose measurement on lattice became particularly popular during the last years and whose knowledge cannot be derived either from the spectral density alone or from any finite set of k-point correlation functions.

2.1 Zero chemical potential

We will start from the simplest case: the Hermitian χRMT corresponding to QCD with zero chemical potential. In principle this theory could be seen as a particular case of the imaginary μ case studied in sect. 2.2 by setting $\mu = 0$. Though computations are not too much involved not even in the imaginary chemical potential case, formulas may seem too odd, we have hence preferred to present the easiest computation as a separate case.

We will prove that, up to an irrelevant constant, the integrals describing the partially quenched $(k|N_f + k)$ partition functions in χRMT and χPT coincide. According to what is said in sect. 1.4.1 from this partition function we can generate the resolvents for all spectral correlation functions, and hence all spectral correlation functions themselves. So by proving the equivalence of the partially quenched partition function, without computing the spectra themselves by taking the discontinuities, it follows that all the k-point correlation functions are equal, and, hence all spectral properties agree; it holds in particular for the individual eigenvalues distribution functions. In order to actually compute any given correlation function we may thus choose either theory, and within that theory we may even use any other

equivalent method that is simpler.

According to what was explained in sect. 1.4 if we want to generate the k-point spectral correlations of the Dirac operator we have to consider a partially quenched theory with $n_f = N_f + k$ fermions and $n_b = k$ bosons. We start from the partially quenched version of the chGUE in eq. (1.49):

$$Z_{pq} = \left\langle \frac{\prod_f^{n_f} \mathcal{D}et\left[\mathcal{D}_f + m_f\right]}{\prod_b^{n_b} \mathcal{D}et\left[\mathcal{D}_b + m_b\right]} \right\rangle = \int dA\, e^{-\sigma N Tr\left[A^\dagger A\right]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}} \tag{2.1}$$

where A is a complex $N_+ \times N_-$ random matrix. It describes a fixed topological sector of QCD with a topological charge $\nu = N_+ - N_-$, where ν will be kept fixed while considering the $N_+ + N_- \equiv N \to \infty$ limit. We have indicated the flat measure in the independent entries of complex matrix A as dA.

Since we are interested in finding the spectrum and the k-point correlation functions using the resolvent method [98, 14, 60, 79], according to what was said in sect. 1.4, we consider the more general theory with n_f fermionic quarks and n_b bosonic ones and compute its partition function.

As already said in sect. 1.4.2 for convergence reasons we have to require $\mathbb{R}e(m_b) > 0 \,\forall\, b$ and as a consequence we can generate the resolvents only in the upper half-plane. The result for $\mathbb{R}e(m_b) < 0$ can be easily recovered from eq. (2.1)[2].

At this point we follow the common procedure [45] (see appendix A) of writing the determinants in the numerator in terms of Gaussian fermionic integrals, and the ones in the denominator in terms of bosonic integrals; we introduce two sets of N_+ and N_- complex-supervectors[3] in $(n_b|n_f)$. We remember that we are using the boson-fermion convention for ordering elements in supervectors. We write these vectors in a matrix form $\psi_{g,\alpha}, \phi_{g,\beta}, \psi^*_{g,\alpha}$ and $\phi^*_{g,\beta}$ where Latin indices run over the $(n_b|n_f)$ superflavours and Greek indices run over the N_+ (or N_-) eigenvalues,

$$\begin{aligned} Z_{pq} = & \int dA\, d(\psi, \psi^*, \phi, \phi^*)\, e^{-\sigma N Tr\left[A^\dagger A\right]} \\ & \times \mathrm{Exp}\left[- \sum_{g \in (n_b|n_f)} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & iA_{\alpha,\beta'} \\ iA^\dagger_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix} \right]. \end{aligned} \tag{2.2}$$

Here $d(\psi, \psi^*, \phi, \phi^*)$ is a shorthand notation for the product of the flat measures of the independent entries of the supervectors[4]. The integral above depends on the random matrices

[2] In the literature a diagonal matrix S with elements $s_i = \pm 1$ is often introduced to be able to work in the whole complex plane at once. For the sake of simplicity we have omitted such a notation.

[3] The anticommuting number part of supervectors and its conjugate are independent Grassmann variables.

[4] That is a $d_{\mathbb{C}}\psi_{g\alpha}$ term for any complex bosonic degree and $d\psi_{g_\alpha}\, d\psi^*_{g_\alpha}$ for any couple of fermionic ones.

only in a Gaussian way:

$$\text{Exp}\left[-\sigma N\ A_{\alpha,\beta}A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - iA^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right]\ . \tag{2.3}$$

We can thus perform the Gaussian matrix integration by completing the squares:

$$\begin{aligned} Z_{pq} \propto \int & d(\psi,\psi^*,\phi,\phi^*)\ \text{Exp}\left[-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha} + m_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]\right] \\ & \times\ \text{Exp}\left[-\frac{1}{\sigma N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right]\ . \end{aligned} \tag{2.4}$$

In the following we introduce the matrix notation $M_{gh} = N\cdot\delta_{gh}m_g$, anticipating the correct scaling later in the large-N limit to obtain N-independent quantities in the thermodynamic limit [4]. This expression is a Gaussian integral in any of the two sets of supervectors. We can easily perform one of the two supervector Gaussian integrations $(\phi,\phi^\dagger)$ obtaining a superdeterminant as a result (app. A):

$$\begin{aligned} Z_{pq} \propto \int & d(\psi,\psi^*)\ \text{Exp}\left[-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right] \\ & \times\ \mathcal{S}det\left[\frac{1}{N}M_{gh} + \frac{1}{\sigma N}\left(\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha}\right)\right]^{-N_-}\ . \end{aligned} \tag{2.5}$$

The equation above depends on the supervectors only through the sum of outer products. We can now apply the superbosonisation theorem to be proven in chapter 3:

$$\frac{1}{N_+}\sum_{\alpha=1}^{N_+}\psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha} \to U_{gh}\ \in\ \hat{Gl}(n_b|n_f) \tag{2.6}$$

where $\hat{Gl}(n_b|n_f)$ is the maximal Riemannian submanifold of the linear group in the $(n_b|n_f)$ superspace [81]; we will use here the matrix representation given in 1.80; in this representation the Haar measure, indicated as $d\mu_{\mathcal{S}}(a)$, can be expressed in terms of usual integrations through (see eq. (3.19)):

$$\begin{aligned} \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}\begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2\end{pmatrix} \ =\ & \int_{H_1=H_1^\dagger} dH_1\ \theta(H_1)\int_{U(n_f)} d\mu(H_2)\ \mathcal{D}et\left[H_2\right]^{n_f} \\ & \times\ \int d\Theta\, d\Theta^\dagger\, \mathcal{S}det\begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2\end{bmatrix}^{n_f-n_b}\ . \end{aligned} \tag{2.7}$$

Here, dH_1 is the flat measure on the Hermitian matrices H_1, $\theta(H_1)$ is the product of the step function in the eigenvalues ($\theta(H_1)>0 \iff H_1$ is positive definite), $d\mu$ is the Haar

measure on the unitary matrices H_2 and $d\Theta d\Theta^\dagger$ is the flat Grassmannian integration in the independent entries of the boson-fermion block[5]. Other parametrisations of $\hat{Gl}(n_b|n_f)$ have been provided for some specific values of $(n_b|n_f)$ [60, 91, 99].

The result is ($N_+ = \nu + N_-$):

$$\begin{aligned} Z_{pq} \propto & \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{N_+}\, e^{-Str\left[\frac{N_+}{N}MU\right]} \mathcal{S}det\left[\sigma\frac{M}{N_+}+U\right]^{-N_-} \\ = & \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{\nu}\, e^{-Str\left[\frac{N_+}{N}MU\right]} \mathcal{S}det\left[1+\sigma\frac{1}{N_+}M\cdot U^{-1}\right]^{-N_-} . \end{aligned} \tag{2.8}$$

All the equations above hold for finite N. As a last step we can perform the $N \to \infty$ limit. Here we send the masses m_h to zero while keeping M_{gh} fixed. Therefore the $N \to \infty$ merely leads to an expansion of the superdeterminant (app. A), without the need to perform any further approximation:

$$\begin{aligned} \lim_{N\to\infty} Z_{pq} &= \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{\nu}\, e^{Str\left[-\frac{1}{2}MU-\sigma M\cdot U^{-1}\right]} \\ &\propto \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{\nu}\, e^{-\sqrt{\frac{\sigma}{2}}Str\left[M\cdot U+M\cdot U^{-1}\right]} . \end{aligned} \tag{2.9}$$

This equation is equivalent to the partially-quenched partition function in eq. (1.78) or, e.g., in [60], after matching parameters

$$m_i^{(RMT)} N\sqrt{\frac{\sigma}{2}} = m_i^{(\chi PT)}\frac{\Sigma V}{2} . \tag{2.10}$$

We denote by V the volume in χPT and by Σ the chiral condensate[6], the first low energy constant in χPT. Applying the resolvent method to both eq. (2.1) in the limit $N \to \infty$, and eq. (2.9) while keeping eq. (2.10) finite, we obtain the claimed $\epsilon\chi$PT-χRMT equivalence at $\mu = 0$ of all k-point correlation functions in the microscopic limit.

2.2 Imaginary chemical potential

The simplest generalisation of the arguments above is obtained by adding imaginary chemical potentials. As already pointed in sect. 1.3.5 this is a Hermitian theory too and, hence, the resolvent method described in sect. 1.4.1 may be applied as well. The only difference with the former section lies in a different random matrix Dirac operator, eq. (1.62), containing one more random matrix.

Our starting point is the partially quenched version of the RMT partition function in eq. 1.63:

[5]The Grassmann variables $\Theta_{\alpha,b}$ and $\Theta^\dagger_{b,\alpha}$ are independent real Grassmann variables.
[6]The chiral condensate Σ has to be considered in the limit $V \to \infty$.

$$Z_{pq} = \left\langle \frac{\prod_f^{n_f} \mathcal{D}et\left[\mathcal{D}_f + m_f\right]}{\prod_b^{n_b} \mathcal{D}et\left[\mathcal{D}_b + m_b\right]} \right\rangle \tag{2.11}$$
$$= \int dA\, dB\, e^{-\sigma N Tr\left[AA^\dagger + BB^\dagger\right]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + i\mu_f B \\ iA^\dagger + i\mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA + i\mu_b B \\ iA^\dagger + i\mu_f B^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}} .$$

where m_f and μ_f are generic masses and chemical potentials of fermionic quarks, and m_b are μ_f are those of bosonic ones. For convergence reasons we take $m_b > 0 \,\forall\, b$. Despite solutions are known only for isospin chemical potential, that is having only two different chemical potentials, in the present work we do not need to explicit integrate the matrices, and hence the equivalence is proved for any kind of imaginary chemical potential set.

Here A and B are complex $N_+ \times N_-$ random matrices with Gaussian weights. Through the resolvent method this partition function generates the n_b-point correlation function for the theory with $N_f = n_f - n_b$ dynamical quarks.

In principle one could also use as a starting point Stephanov's partition function [8] with only one random matrix and the (imaginary) μ-term proportional to unity. We will show in chapter 4 that they are equivalent, and hence we can use the one here allowing less involved computations.

From this point on, most of the steps are equivalent to the ones performed in the previous section: we introduce two sets of N_+ and N_- complex-supervectors in $(n_b|n_f)$ to write the ratio of determinants as a Gaussian integral

$$Z_{pq} = \int dA\, dB\; d(\psi, \psi^*, \phi, \phi^*)\; \mathrm{Exp}\left[-\sigma N \left(Tr\left[A^\dagger A\right] + Tr\left[B^\dagger B\right]\right)\right] \tag{2.12}$$
$$\times\, \mathrm{Exp}\left[-\sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & (iA + i\mu_g B)_{\alpha,\beta'} \\ \left(iA^\dagger + i\mu_g B^\dagger\right)_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix} \right] .$$

We reduce all the dependence on random matrices in terms of Gaussian functions:

$$\mathrm{Exp}\left[-\sigma N\; A_{\alpha,\beta} A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - iA^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right] \tag{2.13}$$
$$\times\, \mathrm{Exp}\left[-\sigma N\; B_{\alpha,\beta} B^*_{\alpha,\beta} - i\mu_g B_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - i\mu_g B^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right] .$$

Next we can perform the Gaussian integration completing the squares:

$$Z_{pq} \propto \int d(\psi,\psi^*,\phi,\phi^*)\ \mathrm{Exp}\left[-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}+m_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]\right]$$
$$\times\ \mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right]$$
$$\times\ \mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\mu_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\mu_h\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right]. \tag{2.14}$$

Here we introduce again the matrix notation $M_{gh}\equiv N\,\delta_{gh}\,m_g$, as well as $B^{(\mu)}_{gh}=\sqrt{\frac{N}{2}}\,\delta_{gh}\,\mu_g$, anticipating also the proper scaling of μ with N below. This expression is a Gaussian integral in any of the two sets of supervectors. As before we explicitly integrate one set of supervectors and express the remaining outer product in terms of an integration over $\hat{Gl}(n_b|n_f)$. The result is:

$$Z_{pq} \propto \int d(\psi,\psi^*)\mathrm{Exp}\left[-Str\left[\frac{1}{N}M\cdot\sum_\alpha \psi_\alpha\otimes\psi^\dagger_\alpha\right]\right] \tag{2.15}$$
$$\times\ \mathcal{S}det\left[\frac{1}{N}M+\frac{1}{\sigma N}\sum_\alpha \psi_\alpha\otimes\psi^\dagger_\alpha+\frac{2}{\sigma N^2}B^{(\mu)}\cdot\sum_\alpha \psi_\alpha\otimes\psi^\dagger_\alpha\cdot B^{(\mu)}\right]^{-N_-}.$$

We can use again the theorem of chapter 3 obtaining:

$$Z_{pq} \propto \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^\nu\, e^{-Str\left[\frac{N_+}{N}MU\right]} \tag{2.16}$$
$$\times\ \mathcal{S}det\left[1+\sigma\frac{1}{N_+}M\cdot U^{-1}+\frac{2}{N}B^{(\mu)}UB^{(\mu)}U^{-1}\right]^{-N_-}.$$

This result is again exact for any finite N. If we now take the large-N limit while keeping M and $B^{(\mu)}$ fixed[7] we obtain finally:

$$\lim_{N\to\infty} Z_{pq} = \int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^\nu\, e^{-\sqrt{\frac{\sigma}{2}}Str\left[M\cdot U+M\cdot U^{-1}\right]-Str\left[B^{(\mu)}UB^{(\mu)}U^{-1}\right]} \tag{2.17}$$

where the only approximation we have used is the expansion of the superdeterminant.

This expression is equivalent to the $\epsilon\chi$PT effective partition [27, 35, 36], where in order to match we use the following relations [89, 62]

$$\begin{aligned} m_i^{(RMT)}N\sqrt{\frac{\sigma}{2}} &= m_i^{(QCD)}\frac{\Sigma V}{2}\ , \\ \mu_i^{(RMT)}\sqrt{N} &= \mu_{D_i}^{(\chi PT)}F\sqrt{V}\ . \end{aligned} \tag{2.18}$$

[7] In the theory with real chemical potential in the next section this limit is called weak non-hermiticity limit [70, 100]. While we inherit the same scaling here our operators are always Hermitian.

Here μ_D is a dimensional constant, instead of the μ used above that is dimensionless. The additional parameter F is the pion decay constant, the second low energy constant in the leading order chiral Lagrangian. Thus we have established the $\epsilon\chi$PT-χRMT equivalence for all k-point correlation functions with imaginary chemical potential.

2.3 Real chemical potential

As before the starting point is the partially quenched version of the partition function of χRMT, for this case it is the one in eq. (1.60).

As explained in sect. 1.4.4 the presence of bosonic quarks requires the Hermitisation of the boson-boson part of Dirac operator [85]. Despite it is not a necessary step to solve the RMT [87, 86], it will be a necessary regularisation for the computation below.

$$\begin{aligned} Z_{pq} &= \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right]}{\prod_i^{n_v} \mathcal{D}et\left[D_i + \tilde{z}_i\right]} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle \\ &= \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right] \mathcal{D}et\left[D_i + \tilde{z}_i\right]^*}{\prod_i^{n_v} \mathcal{D}et\left[D_i + \tilde{z}_i\right] \mathcal{D}et\left[D_i + \tilde{z}_i\right]^*} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle \\ &= \lim_{\varepsilon \to 0} \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right] \mathcal{D}et\left[D_i + \tilde{z}_i\right]^*}{\prod_i^{n_v} \mathcal{D}et\left[(D_i + \tilde{z}_i)(D_i^\dagger + \tilde{z}_i^*) + \varepsilon^2\right]} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle . \end{aligned} \tag{2.19}$$

The first line is the proper definition of the resolvent, the second, that is obtained from the first by a trivial step, is the Hermitised version, the last line is the regularised integral.

In the second line the additional valence quarks are in conjugate pairs $(n_b + n_b^* | n_v + n_v^* + N_f)$ (the replacement $\tilde{z}_i \to z_i$ in the numerator is irrelevant). It will be sufficient to consider the conjugate fermionic quarks as independent quarks with mass $-m^*$ and chemical potential $-\mu$; concerning the bosonic conjugate quarks, as already seen in sect. 1.4.4, there is a deep reason that forces us to treat them simultaneously.

We mention in passing that a purely bosonic theory with bosons coming in conjugated pairs $(n_b + n_b^*, 0)$ occurs when applying the replica trick or Toda lattice equation [8, 66, 63, 86].

In the following we will prove the equivalence of the partition functions of $\epsilon\chi$PT and χRMT at non zero chemical potential with n_f fermionic quarks with given masses and baryon chemical potential (m_f, μ_f), and n_b couples of conjugated bosonic quarks, with parameters (m_b, μ_b) and $(-m_b^*, -\mu_b)$. The partially quenched theory with N_f physical quarks, n_v couples of conjugated fermionic quarks and n_v couples of bosonic quarks will result as a special case. The equivalence of the spectra follows applying the resolvent method in both theories.

We will again use the the two-matrix model as was introduced by Osborn [9], but the same calculation can be done also for the Stephanov model, see chapter 4.

The equivalent of the QCD Dirac operator for a quark with mass m_f and chemical potential μ_f is given by:

$$\mathcal{D}_f + m_f \equiv \begin{pmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{pmatrix} . \tag{2.20}$$

where A and B are complex $N_+ \times N_-$ random matrices.

The partition function equivalent to partially quenched QCD with quarks occurring in conjugated pairs is:

$$\begin{aligned} Z_{pq} \;=\; & \int dA\; dB\; w(A)\; w(B)\; \prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix} \\ & \times \left(\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix} \mathcal{D}et \begin{bmatrix} -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} \end{bmatrix} \right)^{-1} , \end{aligned} \tag{2.21}$$

with a Gaussian weight function

$$w(X) = \mathrm{Exp}\left[-\sigma\; N\; Tr\left[X^\dagger X \right] \right] . \tag{2.22}$$

In order to write the inverse determinants as bosonic Gaussian integrals we perform the same regularisation as in eq. (1.90), obtaining an anti-Hermitian matrix apart from an ε times the identity. The regularised denominator is:

$$\begin{aligned} & \mathcal{D}et \left[\begin{pmatrix} m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \end{pmatrix} \cdot \begin{pmatrix} -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} \end{pmatrix} - \varepsilon^2 \mathbf{1}_{N_+ + N_-} \right] = \\ & = (-)^{N_+ - N_-} \mathcal{D}et \left[\begin{pmatrix} \varepsilon \mathbf{1}_{N_+} & 0 & m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ 0 & \varepsilon \mathbf{1}_{N_-} & iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \\ -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B & \varepsilon \mathbf{1}_{N_+} & 0 \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} & 0 & \varepsilon \mathbf{1}_{N_-} \end{pmatrix} \right] \\ & = (-)^{N_+ - N_-} \mathcal{D}et \left[\begin{pmatrix} \epsilon + \frac{m - m^*}{2} & iA & \frac{m + m^*}{2} & \mu B \\ iA^\dagger & \epsilon + \frac{m - m^*}{2} & \mu B^\dagger & \frac{m + m^*}{2} \\ -\frac{m + m^*}{2} & -\mu B & \epsilon - \frac{m - m^*}{2} & -iA \\ -\mu B^\dagger & -\frac{m + m^*}{2} & -iA^\dagger & \epsilon - \frac{m - m^*}{2} \end{pmatrix} \right] \end{aligned} \tag{2.23}$$

We introduce two sets of N_+ and N_- complex supervectors in $(2n_b|n_f)$ to write the ratio

of determinants as a Gaussian integral

$$
\begin{aligned}
Z_{pq} = \int dA\; dB\; d(\psi,\psi^*,\phi,\phi^*) \mathrm{Exp}\left[-\sigma N\left(Tr\left[A^\dagger A\right] + Tr\left[B^\dagger B\right]\right)\right] \qquad (2.24)\\
\times \mathrm{Exp}\left[-\sum_f^{n_f} \begin{pmatrix}\psi^*_{f,\alpha} \\ \phi^*_{f,\beta}\end{pmatrix} \begin{pmatrix} m_f \mathbf{1}_{\alpha,\alpha'} & (iA+\mu_f B)_{\alpha,\beta'} \\ \left(iA^\dagger + \mu_f B^\dagger\right)_{\beta,\alpha'} & m_f \mathbf{1}_{\beta,\beta'}\end{pmatrix} \begin{pmatrix}\psi_{f,\alpha'} \\ \phi_{f,\beta'}\end{pmatrix}\right]\\
\times \mathrm{Exp}\left[-\sum_b^{n_b} \begin{pmatrix}\psi^*_{2b-1,\alpha} \\ \phi^*_{2b-1,\beta} \\ \psi^*_{2b,\alpha} \\ \phi^*_{2b,\beta}\end{pmatrix} \begin{pmatrix} \epsilon+\frac{m_b-m_b^*}{2} & iA_{\alpha\beta'} & \frac{m_b+m_b^*}{2} & \mu_b B_{\alpha\beta'} \\ iA^\dagger_{\beta\alpha'} & \epsilon+\frac{m_b-m_b^*}{2} & \mu_b B^\dagger_{\beta\alpha'} & \frac{m_b+m_b^*}{2} \\ -\frac{m_b+m_b^*}{2} & -\mu_b B_{\alpha\beta'} & \epsilon-\frac{m_b-m_b^*}{2} & -iA_{\alpha\beta'} \\ -\mu_b^* B^\dagger_{\beta\alpha'} & -\frac{m_b+m_b^*}{2} & -iA^\dagger_{\beta\alpha'} & \epsilon-\frac{m_b-m_b^*}{2}\end{pmatrix} \begin{pmatrix}\psi_{2b-1,\alpha'} \\ \phi_{2b-1,\beta'} \\ \psi_{2b,\alpha'} \\ \phi_{2b,\beta'}\end{pmatrix}\right]
\end{aligned}
$$

As before we have reduced all the dependence on random matrices in terms of Gaussian functions:

$$
\begin{aligned}
&\mathrm{Exp}\left[-\sigma N\, A_{\alpha,\beta}A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\sum_f \psi^*_{f,\alpha}\phi_{f,\beta} + \sum_b\left(\psi^*_{2b-1,\alpha}\phi_{2b-1,\beta} - \psi^*_{2b,\alpha}\phi_{2b,\beta}\right)\right)\right.\\
&\qquad \left. -iA^*_{\alpha,\beta}\left(\sum_f \phi^*_{f,\beta}\psi_{f,\alpha} + \sum_b\left(\phi^*_{2b-1,\beta}\psi_{2b-1,\alpha} - \phi^*_{2b,\beta}\psi_{2b,\alpha}\right)\right)\right] \qquad (2.25)\\
&\times \mathrm{Exp}\left[-\sigma N\, B_{\alpha,\beta}B^*_{\alpha,\beta} - B_{\alpha,\beta}\left(\sum_f \mu_f\psi^*_{f,\alpha}\phi_{f,\beta} + \sum_b \mu_b\left(\psi^*_{2b-1,\alpha}\phi_{2b,\beta} - \psi^*_{2b,\alpha}\phi_{2b-1,\beta}\right)\right)\right.\\
&\qquad \left. -B^*_{\alpha,\beta}\left(\sum_f \mu_f\phi^*_{f,\beta}\psi_{f,\alpha} + \sum_b \mu_b\left(-\phi^*_{2b,\beta}\psi_{2b-1,\alpha} + \phi^*_{2b-1,\beta}\psi_{2b,\alpha}\right)\right)\right] ,
\end{aligned}
$$

and we can perform the Gaussian integration completing the squares:

$$
\begin{aligned}
&\mathrm{Exp}\left[-\frac{1}{\sigma N}\sum_{\alpha,\beta}\psi^*_{g,\alpha}\Gamma^A_{gl}\phi_{l,\beta}\phi^*_{m,\beta}\Gamma^A_{m,n}\psi_{n,\alpha} + \frac{2}{\sigma N^2}\sum_{\alpha,\beta}\psi^*_{g,\alpha}\Gamma^B_{gl}\phi_{l,\beta}\phi^*_{m,\beta}\Gamma^B_{m,n}\psi_{n,\alpha}\right]\\
&= \mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\Gamma^A\sum_\alpha\left(\phi_\alpha\otimes\phi^\dagger_\alpha\right)\Gamma^A\sum_\beta\left(\psi_\beta\otimes\psi^\dagger_\beta\right) - \right.\right. \qquad (2.26)\\
&\qquad \left.\left. -\frac{2}{N}\Gamma^B\sum_\alpha\left(\phi_\alpha\otimes\phi^\dagger_\alpha\right)\Gamma^B\sum_\beta\left(\psi_\beta\otimes\psi^\dagger_\beta\right)\right]\right] .
\end{aligned}
$$

Here we have introduced the $(2n_b|n_f)\times(2n_b|n_f)$ supermatrices

$$
\Gamma^A = \begin{pmatrix} \left.\begin{matrix}1 & 0 \\ 0 & -1\end{matrix}\right\}\times n_b & 0 \\ 0 & \mathbf{1}_{n_f}\end{pmatrix} , \quad \Gamma^B = \sqrt{\frac{N}{2}}\begin{pmatrix} \left.\begin{matrix}0 & \mu_b \\ -\mu_b & 0\end{matrix}\right\}\times n_b & 0 \\ 0 & \left.\mu_f\right\}\times n_f\end{pmatrix} , \qquad (2.27)
$$

and the mass matrix:

$$M = N \begin{pmatrix} \left.\begin{matrix} \varepsilon + \frac{m_b - m_b^*}{2} & \frac{m_b + m_b^*}{2} \\ -\frac{m_b + m_b^*}{2} & \varepsilon - \frac{m_b - m_b^*}{2} \end{matrix}\right\} \times n_b & 0 \\ 0 & \left. m_f \right\} \times n_f \end{pmatrix}, \tag{2.28}$$

anticipating their N-dependence below. We can rewrite the partition function (2.24):

$$Z_{pq} = \int d(\psi, \psi^*, \phi, \phi^*) \, \mathrm{Exp}\left[-Str\left[\frac{1}{N} M \cdot \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \right] \right] \mathrm{Exp}\left[-Str\left[\sum_\alpha \phi_\alpha \otimes \phi_\alpha^\dagger \cdot \right.\right.$$
$$\left.\left. \cdot \left(\frac{1}{N} M + \frac{1}{\sigma N} \Gamma^A \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \Gamma^A - \frac{2}{\sigma N^2} \Gamma^B \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \Gamma^B \right) \right] \right]. \tag{2.29}$$

From this point on the procedure is the same as before: we integrate explicitly the sets of supervectors ϕ, ϕ^* and use the superbosonisation theorem

$$\begin{aligned} Z_{pq} &= \int d(\psi, \psi^*) \, \mathrm{Exp}\left[-Str\left[\frac{1}{N} M \cdot \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \right] \right] \\ &\quad \times \mathcal{S}det \left[\frac{1}{N} M + \frac{1}{\sigma N} \Gamma^A \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \Gamma^A - \frac{2}{\sigma N^2} \Gamma^B \sum_\beta \psi_\beta \otimes \psi_\beta^\dagger \Gamma^B \right]^{-N_-} \\ &\propto \int_{\hat{G}l(2n_b|n_f)} d\mu_{\mathcal{S}}(U) \, \mathcal{S}det\,[U]^{N_+} \, \mathrm{Exp}\left[-Str\left[\frac{N_+}{N} M \cdot U \right] \right] \\ &\quad \times \mathcal{S}det \left[\frac{1}{N} M + \frac{N_+}{\sigma N} \Gamma^A \cdot U \cdot \Gamma^A - \frac{2N_+}{\sigma N^2} \Gamma^B \cdot U \cdot \Gamma^B \right]^{-N_-}. \end{aligned} \tag{2.30}$$

This result is valid for finite-N. Once performing the $N \to \infty$ weak non-hermiticity limit [70, 100], keeping M fixed as well as Γ^B, the following result is obtained

$$\lim_{N\to\infty} Z_{pq} = \int_{\hat{G}l(2n_b|n_f)} d\mu_{\mathcal{S}}(U) \, \mathcal{S}det\,[U]^{\nu} \, \mathrm{Exp}\left[Str\left[-\sqrt{\frac{\sigma}{2}} M \left(U + \Gamma^A \cdot U^{-1} \cdot \Gamma^A \right) + \right.\right.$$
$$\left.\left. + \Gamma^A \Gamma^B U \Gamma^B \Gamma^A U^{-1} \right] \right]. \tag{2.31}$$

Rotating all the matrices under the superunitary transformation $X \to T^\dagger \cdot X \cdot T$ with

$$T = \begin{pmatrix} \left.\begin{matrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{matrix}\right\} \times n_b & 0 \\ 0 & \mathbf{1}_{n_f} \end{pmatrix}, \tag{2.32}$$

we obtain the new result for Hermitised $\epsilon\chi$PT with real $\mu \neq 0$, generalising previous results in the literature [63, 86]:

$$Z_{pq} = \int_{\hat{G}l(2n_b|n_f)} d\mu_{\mathcal{S}}(U) \, \mathcal{S}det\,[U]^{\nu} \, \mathrm{Exp}\left[Str\left[-\sqrt{\frac{\sigma}{2}} \hat{M} \left(U + I \cdot U^{-1} \cdot I \right) + B_+^{(\mu)} U B_-^{(\mu)} U^{-1} \right] \right], \tag{2.33}$$

where

$$\hat{M} \equiv T^\dagger \cdot M \cdot T = \begin{pmatrix} \left.\begin{matrix} \varepsilon N & -im_b N \\ -im_b^* N & \varepsilon N \end{matrix}\right\} \times n_b & & 0 \\ & & \\ 0 & & m_f N\} \times n_f \end{pmatrix},$$

$$B_+^{(\mu)} \equiv \sqrt{\frac{N}{2}} T^\dagger \cdot \Gamma^A \cdot \Gamma^B \cdot T = \begin{pmatrix} \left.\begin{matrix} -\mu_b & 0 \\ 0 & \mu_b \end{matrix}\right\} \times n_b & & 0 \\ & & \\ 0 & & \mu_f\} \times n_f \end{pmatrix},$$

$$B_-^{(\mu)} \equiv \sqrt{\frac{N}{2}} T^\dagger \cdot \Gamma^B \cdot \Gamma^A \cdot T = \begin{pmatrix} \left.\begin{matrix} \mu_b & 0 \\ 0 & -\mu_b \end{matrix}\right\} \times n_b & & 0 \\ & & \\ 0 & & \mu_f\} \times n_f \end{pmatrix},$$

$$I \equiv T^\dagger \cdot \Gamma^A \cdot T = \begin{pmatrix} \left.\begin{matrix} 0 & -i \\ i & 0 \end{matrix}\right\} \times n_b & & 0 \\ & & \\ 0 & & \mathbf{1}_{n_f} \end{pmatrix}. \tag{2.34}$$

It is expressed in term of Σ, F and chemical potential using eq. (2.18). For details on the Haar measure $d\mu_{\mathcal{S}}(U)$ we refer to the next chapter.

The expression above is a generalisation of $\epsilon\chi$PT with one pair of bosons [63, 86], and of $\epsilon\chi$PT with N_f fermions [82]. The signature of the boson-boson block in the metric I slightly differs from [63] because of a different (anti-)Hermitisation used here. It is important to note that the two mass terms have different signs if considering bosonic or fermionic quarks, see eqs. (102) and (131) of [63], respectively.

The particular cases of eq. (2.33) already known in literature [63, 86, 82] have been derived starting from the symmetries of the microscopic theory under vector and axial transformations too, and we suppose that the same arguments can be applied to the general case. The existence of two different matrices $B_\pm^{(\mu)}$ is due to the fact that the covariant derivative has a different behaviour on bosonic and fermionic quarks in $\epsilon\chi$PT as explained in [86].

2.3.1 * Divergences in ε in the one boson non-Hermitian theory

We have marked this section by a ***** since in this section we are not going to show any conclusive result but just introducing a still open question (and our proposal to solve it) related to the previous section. The analogous problem of the definition of supersymmetric partition function in non-Hermitian Gaussian Random Matrix model (introduced in [100]) was recently considered in detail in [101].

χPT partition function (2.33) in the $(1 + 1^*|0)$ case, that is a couple of conjugated bosonic quarks, was explicitely computed in [63] and it was seen the its $\varepsilon \to 0$ limit is a

diverging quantity. This is not really a problem since physical quantities are always related to logarithmic derivatives of the partition function, and hence this multiplying divergent quantity cancels out. Problems seem to arise when considering the $(1+1^*|1^*)$ case: on one side we have that since $Gl(2)/U(2) \subset \hat{Gl}(2|1)$ we expect to have the same divergence, but on the other side χRMT computation may be performed explicitely through the Cauchy transform of the orthogonal polynomial [102, 87] and the result of the $\varepsilon \to 0,\ N \to \infty$ is a finite quantity [86]. We show here a possible way to approach the problem: in the $(1+1^*|0)$ case it works properly, but we have not yet been able to implement it to the $(1+1^*|1^*)$ case.

The partition function for the two coupled conjugated bosons in presence of chemical potential is:

$$\int_{Gl(2)/U(2)} d\mu(U)\, \mathcal{D}et\,[U]^{\nu}\, \mathrm{Exp}\left[Tr\left[-\sqrt{\frac{\sigma}{2}} M_{\varepsilon}\cdot\left(U + I\cdot U^{-1}\cdot I\right) + B_+\cdot U\cdot B_-\cdot U^{-1}\right]\right] \tag{2.35}$$

where (for simplicity we are dropping some useless indices and the dependence on N):

$$M_{\varepsilon} = \begin{pmatrix} \varepsilon & -im \\ -im^* & \varepsilon \end{pmatrix} \equiv \varepsilon\, \mathbf{1}_2 + M\ , \tag{2.36}$$

$$I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ ,$$

$$B_+ = -B_- = B \equiv \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}\ .$$

We parametrise the elements of $Gl(2)/U(2)$ through

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = X\cdot\tilde{U}\cdot X \equiv X\cdot\begin{pmatrix} u & u_{12} \\ u_{21} & u \end{pmatrix}\cdot X = \begin{pmatrix} u\,x^2 & u_{12} \\ u_{21} & u\,x^{-2} \end{pmatrix} \tag{2.37}$$

where

$$X = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \quad x \in \mathbb{R}^+\ , \tag{2.38}$$

and the measure is

$$d\mu(U) = dU \frac{1}{\mathcal{D}et\,[U]^2} = 4\, dx \frac{1}{x}\ du\, du_{12}\, du_{21} \frac{u}{\mathcal{D}et\left[\tilde{U}\right]^2} \tag{2.39}$$

The Lagrangian, apart from the term involving ε, is invariant under the symmetry $U \to X^{-1}\cdot U\cdot X^{-1}$. We can hence factorise the integration:

$$Z^{(1+1^*)} = \int du\, du_{12}\, du_{21} \frac{u}{\mathcal{D}et\left[\tilde{U}\right]^2}\ e^{-\mathcal{L}_{inv}(\tilde{U})} \times \int_{\mathbb{R}^+} dx\, \frac{1}{x}\, \mathrm{Exp}\left[-\varepsilon\sqrt{\frac{\sigma}{2}} Tr\left[X^2\cdot(\tilde{U} + I\cdot\tilde{U}^{-1}\cdot I)\right]\right]\ . \tag{2.40}$$

The second term can be integrated explicitely

$$\int_{\mathbb{R}^+} dx\, \frac{1}{x}\, \mathrm{Exp}\left[-\varepsilon\sqrt{\frac{\sigma}{2}}u\left(1+\frac{1}{\mathcal{D}et\left[\tilde{U}\right]}\right)\left(x^2+\frac{1}{x^2}\right)\right]$$
$$= K_0\left(\varepsilon\sqrt{2\sigma}u\left(1+\frac{1}{\mathcal{D}et\left[\tilde{U}\right]}\right)\right)\ , \tag{2.41}$$

its leading order in $\varepsilon \to 0$ limit is $-\ln\varepsilon$ and is independent of the value of $\tilde{U}$. Our idea is that instead of considering the ε-regularised theory integrating over $Gl(2)/U(2)$ and sending $\varepsilon \to 0$ at the end recovering the additional $\mathbb{R}^+$ invariance in the result, we could directly put $\varepsilon = 0$ and integrate using $Gl(2)/(U(2)\otimes\mathbb{R}^+)$ as integration manifold.

Let us turn now to the $(1+1^*|1^*)$ case. The factorisation

$$U = \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \tilde{U} \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \tag{2.42}$$

is possible for the $\hat{Gl}(2|1)$ ensemble too, and as before the only ε-dependent term is an overall integration over x:

$$\int_{\mathbb{R}^+} dx\, \frac{1}{x}\, \mathrm{Exp}\left[-\varepsilon\sqrt{\frac{\sigma}{2}}Tr\left[\begin{pmatrix} x^2 & \\ & x^{-2} \end{pmatrix}\cdot(\tilde{U}_{bb} + I\cdot(\tilde{U}^{-1})_{bb}\cdot I)\right]\right]$$

where the subscript bb means that we are just taking the boson-boson part of $\tilde{U}$ and of $\tilde{U}^{-1}$. For the last term we have

$$(\tilde{U}^{-1})_{bb} = (\tilde{U}_{bb})^{-1} + \text{ nilpotents} \tag{2.43}$$

Without showing all the computations we just say that the result of the integrations is

$$K_0(\varepsilon\,\Upsilon) + \varepsilon\, n_1\, K_1(\varepsilon\Upsilon) + \varepsilon^2\, n_2\, K_2(\varepsilon\Upsilon) \tag{2.44}$$

where for simplicity we have introduced $\Upsilon = \varepsilon\sqrt{2\sigma}u\left(1+\frac{1}{\mathcal{D}et\left[\tilde{U}_{bb}\right]}\right)$, and n_1, n_2 are known nilpotent terms. When considering the $\varepsilon \to 0$ limit only the first term has a diverging limit:

$$\to -(\ln\varepsilon + \ln\Upsilon + \gamma_{EM}) + \frac{n_1}{\Upsilon} + \frac{n_2}{\Upsilon^2}\ . \tag{2.45}$$

The first part does not depend on any of the grassmann variables. Supposing there exists a parametrisation of the grassmann variables such that the Lagrangian has no explicit dependence on one integration variables, that is there is an additional invariance of the Lagrangian under a grassmann integration variable, this together with the property of grassmann integration $\int d\theta = 0$, should be sufficient to conclude that the divergent part disappears from

the result. A good candidate for such an invariance under a grassmann integration may be:

$$\tilde{U} \to I \cdot A \cdot I \cdot \tilde{U} \cdot A^{-1} \tag{2.46}$$

where A is the supermatrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\alpha\beta}{2} & \alpha \\ 0 & \beta & 1 + \frac{\alpha\beta}{2} \end{pmatrix}. \tag{2.47}$$

It can be easily seen that, apart from the ε-dependent part, the lagrangian is invariant under the transformation (2.46).

Unfortunately we have not yet been able achieve the desired result: to factorise this $U(1|1)/U(1)^2$ group from the Haar measure (2.7) on $\hat{Gl}(2|1)/\mathbb{R}^+$ seems to be a highly non-trivial task, the Z grading of the manifold changes and consequently an Efetov-Wegner term (see sect. A.2) arises.

Chapter 3

Superbosonisation theorem

THE aim of this section is to find a way to express integrals of functions of outer products of supervectors in terms of integrals over a smaller space. In formulas we are going to study

$$\int d(\psi,\psi^*)\ f\left(\sum_k \psi_k\otimes\psi_k^\dagger\right), \tag{3.1}$$

where $\psi_k,\psi_k^\dagger$ are complex vectors of n_b bosons and n_f fermions, $d(\psi,\psi^*)$ is a shorthand notation for $\prod_k^N d\psi_k\ d\psi_k^*$ and f is a function defined over $(n_b|n_f)$ supermatrices.

The main idea is to embed $\sum_k \psi_k\otimes\psi_k^\dagger$ in some manifold where we can define a δ-function and where computations are feasible. At a merely symbolic level we have:

$$\int d(\psi,\psi^*)\ f\left(\sum_k \psi_k\otimes\psi_k^\dagger\right)=\int_{\mathcal{M}} da\, f(a)\int d(\psi,\psi^*)\ \delta_{\mathcal{M}}\left(a-\sum_k \psi_k\otimes\psi_k^\dagger\right)\ . \tag{3.2}$$

In order to give a meaning to the equation above we have to specify which manifold $\mathcal{M}$ we use and which measure we use on it (the δ-function depends on it). The use of a δ-function requires f to be continuous with respect to the metric chosen.

A good choice is to take as $\mathcal{M}$ in eq. (3.2) the manifold of super-Hermitian matrices. Doubt was been expressed in [103] about the possibility of defining a δ-function (or more precisely δ-distribution) to be used with outer products of supervectors. In appendix B we show that, using the definition of δ-distributions over commuting and anticommuting numbers known in literature [77], such a δ-distribution is defined in a meaningfully mathematical way. The δ-function can be written using an extension of the usual Fourier-transform representation of the δ-function on Hermitian matrices (see appendix B).

A crucial ingredient in this proof will be the possibility to flip the order of integration. Changing the order between commuting and anti-commuting variables is not a problem, but problems can arise when considering the case of two commuting variables. In order to see

where these problems come from, and how to avoid them, let us consider a simple example, the one of a single commuting number vector of length 1:

$$\begin{aligned}\int_{\mathbb{C}} d^2z\, f(z\cdot z^*) &= \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} dx\, f(x)\,\delta(x-zz^*) \\ &= \int_{\mathbb{R}} dx\, f(x) \int_{\mathbb{C}} d^2z\,\delta(x-zz^*) \\ &= \int_{\mathbb{R}} dx\, f(x) \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} dy\, e^{iy(x-zz^*)}\ .\end{aligned} \tag{3.3}$$

The first change of variable is always allowed when considering converging integrals. At this point we need to flip the order of integration of y and z, z^*, and this is an illicit step since the integrals are not converging. This problem may be avoided considering a real quantity $\eta > 0$

$$\begin{aligned}\int_{\mathbb{C}} d^2z\, f(zz^*) &= \int_{\mathbb{C}} d^2z\, f(zz^*) e^{(\eta-\eta)zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iy(x-zz^*)} e^{-\eta zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iyx} \int_{\mathbb{C}} d^2z e^{-(\eta+iy)zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iyx} \frac{(-i)\pi}{y-i\eta} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x}\, e^{-\eta x}\pi\,\theta(x) = \pi \int_0^\infty dx\, f(x)\ .\end{aligned} \tag{3.4}$$

and all the steps are mathematically rigorous any time we can apply a δ-distribution as a functional to the function $f(x)e^{\eta x}$. The symbol $\theta(x)$ indicates the step function.

This easy example is conceptually not too different from the proof of the following theorem.

Theorem. (Superbosonisation) *Let f be a function defined on the $(n_b|n_f)$ supermatrices, then the following identity holds*

$$\int \prod_k^N d\psi_k d\psi_k^*\ f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) \propto \int_{\hat{G}l(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^N\ \ f(U)\ , \tag{3.5}$$

whenever the integral on the l.h.s. is well defined. The first integration is performed over N complex supervectors in $(n_b|n_f)$ and $d\mu_{\mathcal{S}}(U)$ denotes the Haar measure over $\hat{G}l(n_b|n_f)$.

The idea of expressing integrals of a function of an outer product $\sum_k \psi_k \otimes \psi_k^\dagger$ in terms of an integral of the same function over a simpler space has been widely used in physics: it was derived for the first time in [104] for outer products of Grassmannian vectors appearing in exponential function (the term "bosonisation" comes from this kind of application),

subsequently in [90] for commuting number vectors and in [105, 106, 107, 108, 109] for supervectors. This (super)bosonisation may be seen as an application of the Riesz Representation Theorem[1]. However, up to our knowledge, a graded version of the last theorem is lacking.

By coincidence the superbosonisation theorem has been independently developed at the very same time[2] of an analogous theorem on superbosonisation by Littelmann, Sommers and Zirnbauer[3] [111]. Anyway the proof they give is different from ours, theirs is an algebraic proof based on invariant theory over supervectors, ours uses analysis instruments. The main difference is that we base our proof on the existence of a δ-function fulfilling eq. (3.2), and that collaboration develops a powerful apparatus in order not to use such an equation [111]. For this reason we have chosen to show in appendix B all the details concerning the mathematical rigorousness of our definition.

Proof: As already hinted above we write the l.h.s. of eq. (3.5) introducing an additional integration over super-Hermitian matrices:

$$\begin{aligned}
&\int d(\psi,\psi^*)\, f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) = \\
&= \int_{H=H^\dagger} dH\ f(H)\ e^{\eta Str[H]} \int d(\psi,\psi^*)\, \delta\left(\sum_k \psi_k \otimes \psi_k^\dagger - H\right) e^{-\eta \sum_k \psi_k^\dagger \cdot \psi_k} \\
&\propto \int_{H=H^\dagger} dH\ f(H)\ e^{\eta Str[H]} \int d(\psi,\psi^*)\, e^{-\eta \sum_k \psi_k^\dagger \cdot \psi_k} \int_{F=F^\dagger} dF\ e^{iF\left(H - \sum_k \psi_k \otimes \psi_k^\dagger\right)} \\
&= \int_{H=H^\dagger} dH\ f(H)\ e^{\eta Str[H]} \int_{F=F^\dagger} dF\ e^{iFH}\ \mathcal{S}det\left[\eta + iF\right]^{-N}
\end{aligned} \tag{3.6}$$

where $d(\psi,\psi^*) \equiv \prod_k^N d\psi_k d\psi_k^*$ and $d\psi_k d\psi_k^* = \prod_b^{n_b} d^2_{\mathbb{C}}\psi_{k,b} \prod_f^{n_f} d\psi_{k,f} d\psi_{k,f}^*$. We denote by $\psi^\dagger = \psi^{*\,T}$, and $\psi \otimes \psi^\dagger$ is the outer product and $\psi^\dagger \cdot \psi$ the scalar product in $(n_b|n_f)$. The measure dH is the flat measure in the independent entries of the super-Hermitian matrix.

From now on let us focus on the second integral. This quantity reminds of a quantity already computed in [109], unfortunately the explicit expression in terms of eigenvalues provided in that paper seems to be not fitting in the purposes of the present theorem: when diagonalising the super-Hermitian integration variable H Efetov-Wegner terms arise in the measure, and their value in the general case has never been computed explicitely in literature. In the following we will never change the Z-gradings of the integration manifold, consequently no boundary terms will arise (see app. A or [112]).

Writing F and H in terms of blocks, $dF = dF_1\, dF_2\, d\Phi\, d\Phi^\dagger$

$$F = \begin{pmatrix} F_1 & \Phi^\dagger \\ \Phi & F_2 \end{pmatrix}\ , \quad H = \begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{pmatrix}\ , \tag{3.7}$$

[1]This theorem ensures that any bounded linear functional of functions over a locally compact Hausdorff space may be computed as an integral of that function over that space using a proper measure [110].

[2]This result was already presented by the author in "QCD in extreme conditions", Frascati 6-8 August 2007.

[3]Part of these results are explained in [103] too.

we obtain:

$$\begin{aligned}\mathcal{I} &\equiv \int_{F=F^\dagger} dF\ e^{iFH}\ \mathcal{S}det\left[F-i\eta\right]^{-N} \\ &= \int dF_1\, dF_2\, d\Phi\, d\Phi^\dagger\ \left(\frac{\mathcal{D}et\left[F_2-i\eta-\Phi(F_1-i\eta)^{-1}\Phi^\dagger\right]}{\mathcal{D}et\left[F_1-i\eta\right]}\right)^N \\ &\quad\times \mathrm{Exp}\left[iTr\left[F_1H_1+\Phi^\dagger\Theta\right]-iTr\left[H_2F_2+\Phi\Theta^\dagger\right]\right].\end{aligned} \tag{3.8}$$

As a first step we note that if we diagonalise the matrix F_1 and make an analytic continuation in the eigenvalues we see that the only poles in the expression above are the ones where at least one eigenvalue of F_1 is $i\eta$. Considering the integration over the eigenvalues as a complex contour integral and closing the integral in the upper semicircle (or lower one, depending on the signature of the matrix H_1[4]) we have non vanishing contributions only if every path of the eigenvalues winds the pole in $i\eta$. We can consider this integral as a contour integral around $i\eta$ whenever $\theta(H_1)>0$, otherwise it is zero. Since there are no poles apart from the ones in $i\eta$ we can fix the contour integral as we prefer. We choose the modulus of the eigenvalues as equal to 1, hence we have that the matrix $F_1-i\eta$ may be analytically continued obtaining a unitary matrix.

$$\begin{aligned}\mathcal{I} &= \theta(H_1)e^{-\eta Tr[H_1]}\oint_{U(n_b)} dF_1 \int dF_2\, d\Phi\, d\Phi^\dagger\ \left(\frac{\mathcal{D}et\left[F_2-i\eta-\Phi F_1^{-1}\Phi^\dagger\right]}{\mathcal{D}et\left[F_1\right]}\right)^N \\ &\quad\times \mathrm{Exp}\left[i\left[F_1H_1+\Phi^\dagger\Theta\right]-i\left[H_2F_2+\Phi\Theta^\dagger\right]\right]\ .\end{aligned} \tag{3.9}$$

The notation $\oint_{U(n_b)} dF_1$ stands for an integral over the manifold $U(n_b)$ considering as a measure the analytic continuation of the flat measure of Hermitian matrices. It is related to the Haar measure $d\mu$ integration by:

$$\oint_{U(n_b)} dF_1\ \equiv \int_{U(n_b)} d\mu(F_1)\ \mathcal{D}et\left[F_1\right]^{n_b}\ . \tag{3.10}$$

The relation between these two integrals is the same as between the two integrals below describing a circuitation around zero, with $z=e^{i\theta}$:

$$\int_0^{2\pi} d\theta = \oint_{|z|=1} dz\,\frac{1}{z} \neq \oint_{|z|=1} dz. \tag{3.11}$$

[4]The matrices H_1 and H_2 are complex number Hermitian matrices.

Eq. (3.10) can be derived diagonalising the matrix $F_1 = U \cdot \mathsf{f} \cdot U^\dagger$ where $\mathsf{f}_j = e^{i\theta_j}$:

$$\begin{aligned}
\oint_{U(n_b)} dF_1 &\equiv \prod_i \oint d\mathsf{f}_i \Delta^2(\{\mathsf{f}\}) \int d\mu(U) \\
&= \prod_i \oint \frac{d\mathsf{f}_i}{\mathsf{f}_i} \mathcal{D}et\left[\mathsf{f}\right] \ \Delta(\{\mathsf{f}\}) \Delta\left(\{\mathsf{f}^{-1}\}\right) \mathcal{D}et\left[\mathsf{f}\right]^{n_b-1} \int d\mu(U) \\
&= \prod_i \int_0^{2\pi} d\theta_i \left|\Delta(\{e^{i\theta}\})\right|^2 \mathcal{D}et\left[\mathsf{f}\right]^{n_b} \int d\mu(U) \\
&\equiv \int_{U(n_b)} d\mu(F_1) \ \mathcal{D}et\left[F_1\right]^{n_b} \ . \qquad (3.12)
\end{aligned}$$

We consider now that

$$\begin{aligned}
\int_{A=A^\dagger} dA \ \mathcal{D}et\left[A\right]^N e^{-iTr[AH_2]} &= \int_{A=A^\dagger} dA \ \mathcal{D}et\left[A-i\eta\right]^N e^{-iTr[(A-i\eta)H_2]} \\
&= \int_{A=A^\dagger} dA \ \mathcal{D}et\left[A-i\eta-\Phi^\dagger F_1^{-1}\Phi\right]^N e^{-iTr\left[(A-i\eta-\Phi F_1^{-1}\Phi^\dagger)H_2\right]} \ . \qquad (3.13)
\end{aligned}$$

The first equality comes from the analyticity of the integrand in the diagonal entries of the matrix and the second comes from the analogue of the contour invariance in superanalysis [77], applied to the real and imaginary parts of the Hermitian matrix entries. Applying this equivalence we obtain for eq. (3.9)

$$\begin{aligned}
\mathcal{I} = \theta(H_1) e^{\eta Tr[H_2]-\eta Tr[H_1]} \oint_{U(n_b)} dF_1 \int dF_2 \, d\Phi \, d\Phi^\dagger \ \left(\frac{\mathcal{D}et\left[F_2\right]}{\mathcal{D}et\left[F_1\right]}\right)^N \qquad (3.14) \\
\times \ \mathrm{Exp}\left[iTr\left[F_1H_1 + \Phi^\dagger\Theta\right] - iTr\left[H_2F_2 + \Phi\Theta^\dagger\right] - iTr\left[H_2\Phi F_1^{-1}\Phi^\dagger\right]\right]
\end{aligned}$$

The subsequent step is to perform the $d\Phi \, d\Phi^\dagger$ integration, using

$$\begin{aligned}
&Tr\left[\Phi^\dagger\Theta\right] - Tr\left[\Phi\Theta^\dagger + H_2\Phi F_1^{-1}\Phi^\dagger\right] = \qquad (3.15) \\
&= -Tr\left[\Theta\Phi^\dagger + \Phi\Theta^\dagger + H_2\Phi F_1^{-1}\Phi^\dagger\right] \\
&= -Tr\left[\Theta F_1 F_1^{-1}\Phi^\dagger + H_2\Phi\Theta^\dagger H_2^{-1} + H_2\Phi F_1^{-1}\Phi^\dagger\right] \\
&= -Tr\left[(\Theta F_1 + H_2\Phi)\cdot\left(F_1^{-1}\Phi^\dagger + \Theta^\dagger H_2^{-1}\right) - \Theta F_1\Theta^\dagger H_2^{-1}\right] .
\end{aligned}$$

We can transform the fermionic variables of integration $\Phi \to \tilde{\Phi} \equiv H_2\Phi + \Theta F_1$ and $\Phi^\dagger \to \tilde{\Phi}^\dagger \equiv F_1^{-1}\Phi^\dagger + \Theta^\dagger H_2^{-1}$ and thus perform the Gaussian integration,

$$\begin{aligned}
&\int d\Phi \, d\Phi^\dagger \ \mathrm{Exp}\left[-iTr\left[H_2\Phi F_1^{-1}\Phi^\dagger\right] + iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]\right] \\
&\qquad = \frac{\mathcal{D}et\left[H_2\right]^{n_b}}{\mathcal{D}et\left[F_1\right]^{n_f}} \ e^{iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]} \int d\tilde{\Phi} \, d\tilde{\Phi}^\dagger e^{-iTr\left[\tilde{\Phi}\tilde{\Phi}^\dagger\right]}, \qquad (3.16)
\end{aligned}$$

where the last integration is just a constant. Getting back to eq. (3.14) we have:

$$\begin{aligned}\mathcal{I} \;&\propto\; \theta(H_1)\,\mathcal{D}et\,[H_2]^{n_b}\; e^{-\eta Str[H]} \oint_{U(n_b)} dF_1 \int dF_2\, \frac{\mathcal{D}et\,[F_2]^N}{\mathcal{D}et\,[F_1]^{N+n_f}} \\ &\qquad \times e^{iTr[F_1H_1]-iTr[H_2F_2]+iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]} \\ &=\; \theta(H_1)\,\mathcal{D}et\,[H_2]^{n_b}\; e^{-\eta Str[H]} \int_{F_2=F_2^\dagger} dF_2\; \mathcal{D}et\,[F_2]^N\, e^{-iTr[H_2F_2]} \\ &\qquad \times \oint_{U(n_b)} dF_1\; \frac{1}{\mathcal{D}et\,[F_1]^{N+n_f}}\; e^{i\left[F_1\left(H_1-\Theta^\dagger H_2^{-1}\Theta\right)\right]}.\end{aligned} \tag{3.17}$$

Let us consider again the whole eq. (3.6). We can now perform the integration in F_1 and F_2, using the integrals in appendix C:

$$\begin{aligned}&\int d(\psi,\psi^*)\, f\left(\sum_k \psi_k\otimes\psi_k^\dagger\right) \propto \\ &\propto \int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \int_{H_2=H_2^\dagger} dH_2\; \mathcal{D}et\,[H_2]^{n_b} \int d\Theta\, d\Theta^\dagger\; f(H) \\ &\quad\times \int_{F_2=F_2^\dagger} dF_2\; \mathcal{D}et\,[F_2]^N\, e^{-iTr[H_2F_2]} \oint_{U(n_b)} dF_1 \frac{e^{i\left[F_1\left(H_1-\Theta^\dagger H_2^{-1}\Theta\right)\right]}}{\mathcal{D}et\,[F_1]^{N+n_f}} \\ &\propto \int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \int_{H_2=H_2^\dagger} dH_2\; \mathcal{D}et\,[H_2]^{n_b} \int d\Theta\, d\Theta^\dagger\; f(H) \\ &\quad\times \int_{F_2=F_2^\dagger} dF_2\; \mathcal{D}et\,[F_2]^N\, e^{-iTr[H_2F_2]}\; \mathcal{D}et\left[H_1-\Theta^\dagger H_2^{-1}\Theta\right]^{N+n_f-n_b} \\ &\propto \int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\; \frac{\mathcal{D}et\left[H_1-\Theta^\dagger H_2^{-1}\Theta\right]^{N+n_f-n_b}}{\mathcal{D}et\,[H_2]^{N+n_f-n_b}} f(H) \\ &= \int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\; \mathcal{S}det \begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{N+n_f-n_b} f(H)\;.\end{aligned} \tag{3.18}$$

The last step missing is to find a relation between the measure used in the equation above and the Haar measure. This relation is:

$$\begin{aligned}&\int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\; \mathcal{S}det \begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{n_f-n_b} \\ &= \int_{H_1=H_1^\dagger} dH_1\; \theta(H_1) \int_{U(n_f)} d\mu(H_2)\; \mathcal{D}et\,[H_2]^{n_f} \int d\Theta\, d\Theta^\dagger\, \mathcal{S}det \begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{n_f-n_b} \\ &= \int_{\hat{G}l(n_b|n_f)} d\mu_{\mathcal{S}} \begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{pmatrix}.\end{aligned} \tag{3.19}$$

It is proved in the following. The manifolds we are integrating are the same, in fact the $H_1 = H_1^\dagger > 0$ is equivalent to the boson-boson [113] base manifold of $\hat{G}l(n_b|n_f)$ that is $Gl(n_b)/U(n_b)$ [81, 60]. The analytic continuation of the eigenvalues of $H_2 = H_2^\dagger$ making a circuitation around zero is equivalent to the fermion-fermion part $U(n_f)$ [81].

We have now to consider the integration measures in eq. (3.19). The point is that the measure on the l.h.s. is the Haar measure of super-Hermitian matrices (a group closed under addition) computed on the analytically continued manifold. It is induced by the flat metric:

$$Str\left[dU \cdot dU\right] \ , \tag{3.20}$$

but the Haar measure on the r.h.s. is the analytical continuation of the Haar measure on super-Unitary matrices [81, 113, 60] (a group closed under multiplication), induced by the metric [60]:

$$Str\left[dU \cdot dU^{-1}\right] = Str\left[U^{-1}dU \cdot U^{-1}dU\right]. \tag{3.21}$$

In order to find a relation between these two matrices one has to compute the Berezinean of the transformation [77, 81]:

$$\mathfrak{B} = \mathcal{S}det_{ij,mn}\left[\frac{(U^{-1}\,dU)_{ij}}{dU_{mn}}\right]. \tag{3.22}$$

Since this is not a change of variables (the differentials dU are the same) and hence no Z-gradings are changed, no boundary (Efetov-Wegner) terms arise in the superintegration. We write this Berezinean matrix in the block structure (Latin indices stay for bosons and Greek for fermions):

$$\begin{aligned}\frac{U^{-1}\,dU}{dU} &= \begin{pmatrix} bb/bb & bb/ff & bb/fb & bb/bf \\ ff/bb & ff/ff & ff/fb & ff/bf \\ fb/bb & fb/ff & fb/fb & fb/bf \\ bf/bb & bf/ff & bf/fb & bf/bf \end{pmatrix} \\ &= \begin{pmatrix} U^{-1}_{ab}\otimes \mathbf{1}_{n_b} & 0 & U^{-1}_{a\beta}\otimes \mathbf{1}_{n_b} & 0 \\ 0 & U^{-1}_{\alpha\beta}\otimes \mathbf{1}_{n_f} & 0 & U^{-1}_{\alpha b}\otimes \mathbf{1}_{n_f} \\ U^{-1}_{\alpha b}\otimes \mathbf{1}_{n_b} & 0 & U^{-1}_{\alpha\beta}\otimes \mathbf{1}_{n_b} & 0 \\ 0 & U^{-1}_{a\beta}\otimes \mathbf{1}_{n_f} & 0 & U^{-1}_{ab}\otimes \mathbf{1}_{n_f} \end{pmatrix}.\end{aligned} \tag{3.23}$$

Computing the superdeterminant of the matrix above we obtain

$$\mathfrak{B} = \mathcal{S}det_{ij,mn}\left[\frac{(U^{-1}\,dU)_{ij}}{dU_{mn}}\right] = \mathcal{S}det\left[U\right]^{n_f-n_b}. \tag{3.24}$$

As a consequence we have:

$$d\mu_{\mathcal{S}}\begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{pmatrix} = dH_1\,dH_2\,d\Theta\,d\Theta^\dagger\,\mathcal{S}det\begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{n_f-n_b}, \tag{3.25}$$

that is eq. (3.19). Together with eq. (3.18) we obtain the proof of our theorem eq. (3.5). ■

Chapter 4

The Stephanov-Osborn RMT equivalence

As already introduced in sect. 1.3.4 there are two different RMTs describing low energy QCD in presence of a chemical potential, the one matrix model introduced by Stephanov [8] adding a constant matrix to the off-diagonal part of the the Dirac operator, see eq. (1.59):

$$Z = \int dT\, e^{-\sigma N Tr[TT^\dagger]} \prod_{f=1}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT + \mu \mathbf{1}_{N_-} \\ iT^\dagger + \mu \mathbf{1}_{N_-} & m_f \mathbf{1}_{N_-} \end{bmatrix} , \tag{4.1}$$

and the two matrix model introduced by Osborn [9] adding a random matrix in the same class as the Dirac operator but with opposite hermiticity property[1], see eq. (1.60):

$$Z = \int dT\, dW\, e^{-\sigma N Tr[TT^\dagger + WW^\dagger]} \prod_{f=1}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT + \mu W \\ iT^\dagger + \mu W^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix} . \tag{4.2}$$

Both models have been used for computation, see sect. 1.3.4, but there was only proof for the fermionic partition function and the spectral density function linking them. As already anticipated, thanks to the instruments developed above, in this short chapter we are able to provide this proof as a corollary to the one in chap. 2.

Both eq. (4.2) and eq. (4.1) may be used for imaginary chemical potential as well, see [75] for the solution of the second one. The proof provided here holds for both real and imaginary chemical potential.

The idea of the proof is to perform the same computation as in sections 2.2 and 2.3 but starting from Stephanov's RMT in eq. 1.59. We will find that this theory too can be

[1] In general one can even consider the chemical potential μ_f depending on the the flavour index f. In particular this allows to study isospin chemical potential theories.

mapped, in the $N \to \infty$ limit, to χPT, the equivalence of all the spectral properties of the two models, in that limit, follows as a consequence.

We start from the partially quenched RMT partition function in the case of imaginary chemical potential. The Stephanov's version reads

$$\begin{aligned} Z =& \int dT\, e^{-\sigma N Tr[TT^\dagger]} \frac{\prod_{f=1}^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iT + i\mu_f \mathbf{1}_{N_-} \\ iT^\dagger + i\mu_f \mathbf{1}_{N_-} & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_{b=1}^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iT + i\mu_b \mathbf{1}_{N_-} \\ iT^\dagger + i\mu_b \mathbf{1}_{N_-} & m_b \mathbf{1}_{N_-} \end{bmatrix}} \\ =& \int dT\, d(\psi, \psi^*, \phi, \phi^*)\ \mathrm{Exp}\left[-\sigma N Tr\left[TT^\dagger\right]\right] \\ & \times \mathrm{Exp}\left[-\sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & (iT + i\mu_g \mathbf{1})_{\alpha,\beta'} \\ (iT^\dagger + i\mu_g \mathbf{1})_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix} \right] \\ =& \int d(\psi, \psi^*, \phi, \phi^*)\, \mathrm{Exp}\left[-\frac{1}{\sigma N} Str \left[\sum_\beta \phi_{g,\beta} \otimes \phi^\dagger_{h,\beta} \cdot \sum_\alpha \psi_{h,\alpha} \otimes \psi^\dagger_{g,\alpha} \right] \right] \\ & \times \mathrm{Exp}\left[-i\mu_g \psi^\dagger_{g,\alpha} \delta_{\alpha,\beta} \phi_{g,\beta} - i\mu_g \phi^\dagger_{g,\beta} \delta_{\beta,\alpha} \psi_{g,\alpha} - m_g \psi^\dagger_g \cdot \psi_g - m_g \phi^\dagger_g \cdot \phi_g \right] \end{aligned} \tag{4.3}$$

As before we have written the ratio of determinants in term of a gaussian superintegration and performed explicitely the integration of the random matrix.

The difference with the computations of the previous chapter is the presence of the terms[2] $\phi^\dagger_{g,\beta} \delta_{\beta,\alpha} \psi_{g,\alpha}$ and their conjugated. We define the non-Hermitian positive definite and invertible supermatrix

$$C_{hg} = \frac{1}{\sigma N} \sum_\alpha \psi_{h,\alpha} \otimes \psi^\dagger_{g,\alpha} + m_g \delta_{h,g} \tag{4.4}$$

and rewrite

$$\begin{aligned} & \int d(\psi, \psi^*, \phi, \phi^*)\ \mathrm{Exp}\left[-m_g \psi^\dagger_g \cdot \psi_g\right] \\ & \quad \times \mathrm{Exp}\left[-\sum_\beta \left(\phi^\dagger_{h,\beta} C_{hg} \phi_{g,\beta} + i\mu_g \psi^\dagger_{g,\beta} \phi_{g,\beta} + i\mu_g \phi^\dagger_{g,\beta} \psi_{g,\beta} \right) \right] \\ =& \int d(\psi, \psi^*, \phi, \phi^*)\, \mathrm{Exp}\left[-\sum_\beta \psi^\dagger_{j,\beta} \mu_j C^{-1}_{jk} \mu_k \psi_{k,\beta} - m_g \psi^\dagger_g \cdot \psi_g \right] \\ & \quad \times \mathrm{Exp}\left[-\left(\phi^\dagger_{h,\beta} + i\psi_{j,\beta} \mu_j C^{-1}_{jh} \right) C_{hg} \left(\phi_{g,\beta} + i C^{-1}_{gk} \mu_k \psi_{k\beta} \right) \right] \\ =& \int d(\psi, \psi^*)\ \mathcal{S}det\left[C\right]^{-N_-} \mathrm{Exp}\left[-\sum_\beta \psi^\dagger_{j,\beta} \mu_j C^{-1}_{jk} \mu_k \psi_{k,\beta} - m_g \psi^\dagger_g \cdot \psi_g \right] \end{aligned} \tag{4.5}$$

[2]The domain of the indices are : $\alpha = 1, \ldots, N_+$ and $\beta = 1, \ldots, N_-$.

where we have used that whenever the bosonic integral is convergent it holds:

$$\int d(\xi,\xi^\dagger)\, e^{-(\xi^\dagger+\chi_1^\dagger)M(\xi+\chi_2)} = \mathcal{S}det\,[M]^{-1}\,. \tag{4.6}$$

Thanks to this property we get rid of the term linear in both ψ and ϕ. At this point we can follow the already used approach, using the superbosonisation theorem $\frac{1}{N_+}\sum_\alpha^{N_+}\psi_\alpha\otimes\psi_\alpha^\dagger \to U \in \hat{Gl}(n_b|n_f)$ and performing the $N\to\infty$ limit. We define as before a mass matrix $M_{gh}=N\,\delta_{gh}\,m_g$ and a chemical potential matrix $B^{(\mu)}_{gh}=\sqrt{\frac{N}{2}}\mu_g\delta_{gh}$.

$$\begin{aligned}
&\int d(\psi,\psi^*)\ \mathcal{S}det\,[C]^{-N_-}\ \mathrm{Exp}\left[-Str\left[\left(\frac{2}{N}B^{(\mu)}C^{-1}B^{(\mu)}+\frac{1}{N}M\right)\cdot\sum_\beta\psi_\beta\otimes\psi_\beta^\dagger\right]\right]\\
&=\int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{N_+}\ \mathcal{S}det\left[\frac{1}{2\sigma}U+\frac{1}{N}M\right]^{-N_-}\\
&\qquad\times\mathrm{Exp}\left[-Str\left[B^{(\mu)}\left(\frac{1}{2\sigma}U+\frac{1}{N}M\right)^{-1}B^{(\mu)}U+\frac{1}{2}M\cdot U\right]\right]\\
&\to\int_{\hat{Gl}(n_b|n_f)} d\mu_{\mathcal{S}}(U)\ \mathcal{S}det\,[U]^{\nu}\ \mathrm{Exp}\left[-Str\left[2\sigma\,B^{(\mu)}U^{-1}B^{(\mu)}U+\frac{1}{2}M\cdot U+\sigma\,M\cdot U^{-1}\right]\right].
\end{aligned}\tag{4.7}$$

Modulo an irrelevant redefinition of the matrix $B^{(\mu)}\to\frac{1}{\sqrt{2}}B^{(\mu)}$ and a rescaling of the integration variable U, the expression above is exactly the same as the one in eq. (2.17) obtained starting from Osborn's model. Since that is true for any set of masses and imaginary chemical potential the spectral properties of the two theories are coinciding.

The proof for the real chemical potential case is similar to the one above, the only difference lies in the hermitisation process.

$$
\begin{aligned}
Z \;=&\int dT\,\mathrm{Exp}\left[-\sigma N Tr\left[TT^\dagger\right]\right]\prod_{f=1}^{n_f}\mathcal{D}et\begin{bmatrix} m_f\mathbf{1}_{N_+} & iT+\mu_f\mathbf{1}_{N_-}\\ iT^\dagger+\mu_f\mathbf{1}_{N_-} & m_f\mathbf{1}_{N_-}\end{bmatrix}\\
&\times\left(\prod_{b=1}^{n_b}\mathcal{D}et\begin{bmatrix} m_b\mathbf{1}_{N_+} & iT+\mu_b\mathbf{1}_{N_-}\\ iT^\dagger+\mu_b\mathbf{1}_{N_-} & m_b\mathbf{1}_{N_-}\end{bmatrix}\mathcal{D}et\begin{bmatrix} -m_b^*\mathbf{1}_{N_+} & iT-\mu_b\mathbf{1}_{N_-}\\ iT^\dagger-\mu_b\mathbf{1}_{N_-} & -m_b^*\mathbf{1}_{N_-}\end{bmatrix}\right)^{-1}\\
=&\lim_{\varepsilon\to 0}\int dT\,\mathrm{Exp}\left[-\sigma N Tr\left[TT^\dagger\right]\right]\prod_{f=1}^{n_f}\mathcal{D}et\begin{bmatrix} m_f\mathbf{1}_{N_+} & iT+\mu_f\mathbf{1}_{N_-}\\ iT^\dagger+\mu_f\mathbf{1}_{N_-} & m_f\mathbf{1}_{N_-}\end{bmatrix}\\
&\times\left(\prod_{b=1}^{n_b}\mathcal{D}et\begin{bmatrix} \varepsilon+\frac{m_b-m_b^*}{2} & iT & \frac{m_b+m_b^*}{2} & \mu_b\\ iT^\dagger & \varepsilon+\frac{m_b-m_b^*}{2} & \mu_b & \frac{m_b+m_b^*}{2}\\ -\frac{m_b+m_b^*}{2} & -\mu_b & \varepsilon-\frac{m_b-m_b^*}{2} & -iT\\ -\mu_b & -\frac{m_b+m_b^*}{2} & -iT^\dagger & \varepsilon-\frac{m_b-m_b^*}{2}\end{bmatrix}\right)^{-1}\\
=&\lim_{\varepsilon\to 0}\int dT\,d(\psi,\psi^*,\phi,\phi^*)\,e^{-Str\left[\frac{1}{N}M\cdot\left(\sum_\alpha\psi_\alpha\otimes\psi_\alpha^\dagger\sum_\beta\phi_\beta\otimes\phi_\beta^\dagger\right)\right]}\;e^{-\sigma N Tr\left[TT^\dagger\right]}\\
&\times\mathrm{Exp}\left[-iT_{\alpha\beta}\sum_{g_1,g_2}\Gamma^A_{g_1,g_2}\psi^*_{g_1\alpha}\phi_{g_2,\beta}-iT^*_{\alpha\beta}\sum_{g_1,g_2}\Gamma^A_{g_1,g_2}\phi^*_{g_2,\beta}\psi_{g_1\alpha}\right]\\
&\times\mathrm{Exp}\left[-i\sqrt{\frac{2}{N}}\delta_{\alpha\beta}\sum_{g_1,g_2}\Gamma^B_{g_1,g_2}\psi^*_{g_1\alpha}\phi_{g_2,\beta}-i\sqrt{\frac{2}{N}}\delta_{\alpha\beta}\sum_{g_1,g_2}\Gamma^B_{g_1,g_2}\phi^*_{g_2,\beta}\psi_{g_1\alpha}\right]
\end{aligned}
\tag{4.8}
$$

where the $(2n_b|n_f)\times(2n_b|n_f)$ supermatrices Γ^A and Γ^B are defined in eq. (2.27). We integrate explicitely the random matrix T and we integrate the ϕ set of supervectors completing the squares as before but defining the C matrix in a slightly different way:

$$
C_{hg}=\frac{1}{\sigma N}\Gamma^A\cdot\sum_\alpha\psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\cdot\Gamma^A+m_g\delta_{h,g} \tag{4.9}
$$

This matrix too fulfils the positivity requirements as before.

At this point we can use the superbosonisation to get rid of the remaining supervector integration performing the same limit as before and using the same matrix manipulation as in sect. 2.3 we obtain:

$$
Z_{pq}=\int_{\hat{G}l(2n_b|n_f)}d\mu_{\mathcal{S}}(U)\;\mathcal{S}det\left[U\right]^\nu\mathrm{Exp}\left[Str\left[-\sqrt{\frac{\sigma}{2}}\hat{M}\left(U+I\cdot U^{-1}\cdot I\right)+2\,B_+^{(\mu)}UB_-^{(\mu)}U^{-1}\right]\right], \tag{4.10}
$$

that, modulo a factor 2, is the same equation obtained starting from Osborn's model in eq. (2.33). Since the equivalence in the $N\to\infty$ limit has been proved for any set of masses and chemical potential the equivalence of the spectral properties follows as a consequence.

Chapter 5

Finite volume correlation of currents

In the present chapter we will compute some current-current correlation in (small) finite volume QCD in the presence of chemical potential. With the term "small" volume we mean that the system may be considered in the ε-regime.

If on one side this chapter may be considered as a self standing work (the problem may be posed and solved in some cases without using any RMT result) on the other side it can be seen as a application of χRMT description of the zero-mode of χPT. This practical application may help to consider the chapters above in a broader context.

The results in this chapter can be also found in [114].

As shown in the introduction the main interest in computing the zero-mode integrals in χPT lies in the possibility of computing observables in a finite volume, theory taking into account the finite volume effect. It was used to compute the finite volume correction to the chiral condensate [2], current-current correlators[1] [20], charged correlators and the topological susceptibility [12]. Ten years after the first works on finite volume corrections a new interest in the subject arose, the computations of finite volume corrections in quenched theories increased the possibility of testing the predictions with lattice simulations. Corrections to the chiral condensate [115], scalar and pseudo-scalar current correlator [23, 116, 117] (in the last one computations are performed in partially quenched theories), vector and axial correlator [118] were computed. These relations have been used to extract low energy constant from small lattice without reaching the chiral limit [116, 119, 120, 121].

In [122] the effective pion decay constant F appearing in finite volume partition function in partially quenched theory has been computed using a theory with an imaginary chemical

[1] In this paper these quantities are computed summing over all the topologies.

potential; however their result lacks a term that we first give here for the unquenched case in eq. (5.16).

The original results for the meson correlators computed in this chapter show an interesting dependence on F already at the leading order $\mathcal{O}(\varepsilon^0)$: it can be used as an alternative way to measure this LEC on lattice.

5.1 The $\mathcal{O}(\varepsilon^2)$ improved χPT

When computing expectation values at a given order j in the ε-expansion, two integrals have to be performed: the exact zero-mode integration and the field-theoretical dynamic-mode integration. In general these two integrations do not factorise, but a simplification in this sense may be obtained introducing a $\mathcal{O}(\varepsilon^j)$-improved effective theory at given order j.

We consider the action $\mathcal{S}[s, v_\alpha; U_0, \xi, \partial\xi]$ given by the Lagrangian in eq. (1.4) in presence of non-vanishing scalar (mass) and vector (imaginary chemical potential) external sources. We split this action in three different terms, the first one depending only on the zero modes U_0, the second one only on the dynamical modes $\xi(x)$ and the third part containing all the interactions between U_0 and $\xi(x)$:

$$\mathcal{S}[s, v_\alpha; U_0, \xi, \partial\xi] = \mathcal{S}_0[s, v_\alpha; U_0] + \mathcal{S}'[\xi, \partial\xi] + \mathcal{S}'_0[s, v_\alpha; U_0, \xi, \partial\xi] \ . \tag{5.1}$$

Since the term $\mathcal{S}'[\xi, \partial\xi]$ has to be invariant under $U(N)$ global rotations[2] it cannot have any coupling to the external sources.

In this section there is no need to explicitate these terms (we refer to the next one), we need only to point out some general properties: the external sources give rise to terms suppressed in the ε-power counting (apart from the terms in S_0, see eq. (1.43)) and the theory with $s = 0$, $v_\alpha = 0$ is invariant under global $U(N)$ transformation, that is the U_0 integration is trivial.

In general we have to compute the expectation value of an observable $\mathcal{Q}$:

$$\begin{aligned}
&\langle \mathcal{Q}[U_0, \xi] \rangle_{QCD} = \\
&= \frac{\int d\mu(U_0) \int [d\xi(x)]' \, e^{-\mathcal{S}_0[s, v_\alpha; U_0] - \mathcal{S}'[\xi, \partial\xi] - \mathcal{S}'_0[s, v_\alpha; U_0, \xi, \partial\xi]} \, J[\xi] \, \mathcal{Q}[U_0, \xi,]}{\int d\mu(U_0) \int [d\xi(x)]' \, e^{-\mathcal{S}_0[s, v_\alpha; U_0] - \mathcal{S}'[\xi, \partial\xi] - \mathcal{S}'_0[s, v_\alpha; U_0, \xi, \partial\xi]} J[\xi]} \\
&\equiv \frac{\int d\mu(U_0) \, e^{-\mathcal{S}_0[s, v_\alpha; U_0]} \, Z_\xi[s, v_\alpha; U_0] \times \langle \, \mathcal{Q}[U_0, \xi] \rangle_\xi}{\int d\mu(U_0) e^{-\mathcal{S}_0[s, v_\alpha; U_0]} \, Z_\xi[s, v_\alpha; U_0]} \\
&\equiv \frac{\langle Z_\xi[s, v_\alpha; U_0] \times \langle \, \mathcal{Q}[U_0, \xi] \rangle_\xi \rangle_{U_0}}{\langle Z_\xi[s, v_\alpha; U_0] \rangle_{U_0}}
\end{aligned} \tag{5.2}$$

where the factor $J[\xi]$ is the Jacobian (1.27) coming from the exponential map of the non-zero modes [20, 21] and the measure $[d\xi(x)]'$ is a path integration over the non-zero modes. In

[2] $U(N)$ global rotations interfere only with U_0.

order to perform the field theoretical integration, we have introduced the partition function Z_ξ and the expectation value over the dynamical degrees of freedom in the presence of a vacuum alignment U_0:

$$Z_\xi[s, v_\alpha; U_0] \equiv \int [d\xi(x)]' \, e^{-\mathcal{S}'[\xi,\partial\xi]-\mathcal{S}'_0[s,v_\alpha;U_0,\xi,\partial\xi]} J[\xi] \ , \tag{5.3}$$

$$\langle \mathcal{Q}[\xi] \rangle_{\xi,s,v_\alpha} \equiv \frac{1}{Z_\xi[s, v_\alpha; U_0]} \int [d\xi(x)]' \, e^{-\mathcal{S}'[\xi,\partial\xi]-\mathcal{S}'_0[s,v_\alpha;U_0,\xi,\partial\xi]} J[\xi] \ \mathcal{Q}[\xi] \ ,$$

and we have analogously defined the expectation value over the U_0 integration with respect to the action $\mathcal{S}_0[s, v_\alpha; U_0]$:

$$\langle \mathcal{P}[U_0] \rangle_{U_0} = \frac{\int d\mu(U_0) \, e^{-\mathcal{S}_0[s,v_\alpha;U_0]} \, \mathcal{P}[U_0]}{\int d\mu(U_0) \, e^{-\mathcal{S}_0[s,v_\alpha;U_0]}} \ . \tag{5.4}$$

At this point we have to find a way to compute the expectation values in the last line of eq. (5.2); in order to proceed we have to write the dependence of $Z_\xi[s, v_\alpha; U_0]$ on the zero-mode variable U_0 explicitely.

Since U_0 appears in the action $\mathcal{S}'_0$ only in terms involving the external sources and since those quantities are small quantities in the ε-bookkeeping, the partition function can be computed through a perturbative expansion in the external sources[3] up to the given order ε^j. Scalar sources (masses) are of order ε^2 and vectors sources (imaginary chemical potential) of order ε. These derivatives can be computed through field theoretical methods:

$$\begin{aligned} &\frac{1}{Z_\xi[0,0;U_0]} \partial_s^j \partial_{v_\alpha}^k Z_\xi[s, v_\alpha; U_0] = \\ &\quad = \frac{1}{Z_\xi[0,0;U_0]} \int [d\xi(x)]' \, J[\xi] \, e^{-S'[\xi,\partial\xi]} \partial_s^j \partial_{v_\alpha}^k e^{-S'_0[s,v_\alpha;U_0,\xi,\partial\xi]} \\ &\quad = \left\langle \partial_s^j \partial_{v_\alpha}^k e^{-S'_0[s,v_\alpha;U_0,\xi,\partial\xi]} \right\rangle_{\xi,s=0,v_\alpha=0} \end{aligned} \tag{5.5}$$

The expression above is an expectation value of an observable in the ensemble of the fields ξ described by the action[4] $S'[\xi, \partial\xi]$. The derivatives have to be considered decomposing the sources in a $U(N)$ basis.

We will see in the next section that the result of this computation at the second order in ε can be re-exponentiated and absorbed in the action leading to the definition of an ε^2 improved effective action, with new volume dependent effective low energy constants.

5.2 The effective Lagrangian in the presence of imaginary chemical potential

Considering what said in the previous section, we can now compute the quantity $Z_\xi[s, v_\alpha; U_0]$ in a theory with quark masses (equivalent to scalar sources) and imaginary chemical potential

[3] For vanishing external sources the integration is trivial and $Z_\xi[0, 0; U_0]$ is just a constant.

[4] Plus the Jacobian $J[\xi]$, if necessary.

(real vector source). In this section we will work with general mass matrix, the real chemical potential case may be recovered by analytic continuation.

We read the Lagrangian from eq. (1.4):

$$\mathcal{L}(s, v_\alpha; U, \partial U) = \frac{1}{4}F^2 Tr\left[\nabla_\mu U \nabla_\mu U^\dagger\right] - \frac{1}{2}\Sigma Tr\left[s(U + U^\dagger)\right] \tag{5.6}$$

where $\nabla_\mu \Upsilon \equiv \partial_\mu \Upsilon - i\,[v_\mu, \Upsilon]$. We substitute the ε-expansion parametrisation $U(x) = U_0 \cdot \mathrm{Exp}\left[i\frac{\sqrt{2}}{F}\xi(x)\right]$ and expand the Lagrangian up to the second order in ε; splitting the result in three pieces we have $\int \mathcal{L}[U_0, \xi, \partial\xi] = \mathcal{S}_0[U_0] + \mathcal{S}'[\xi, \partial\xi] + \mathcal{S}'_0[U_0, \xi, \partial\xi]$, where the first term contains only dependences on U_0, the second depends only on ξ and the last one has all the interactions between the two ensemble. At this order the Jacobian $J[\xi]$ can be exponentiated giving a $\mathcal{O}(\varepsilon^2)$ mass term to the dynamic field ξ:

$$\begin{aligned}
\mathcal{S}_0[U_0] =& -\int d^4x \frac{F^2}{4} Tr\left[[v_\alpha(x), U_0^\dagger][v_\alpha(x), U_0]\right] - \int d^4x \frac{\Sigma}{2} Tr\left[s(x)(U_0 + U_0^\dagger)\right] \\
=& -\frac{F^2}{4} V Tr\left[[B, U_0^\dagger][B, U_0]\right] - \frac{\Sigma V}{2} Tr\left[\mathcal{M}(U_0 + U_0^\dagger)\right] \qquad (5.7) \\
\mathcal{S}'[\xi, \partial\xi] =& +\frac{1}{2}\int d^4x\, Tr\left[\partial_a \xi(x) \partial_a \xi(x)\right] + \frac{N_f}{3F^2}\frac{1}{V}\int dx\, Tr\left[\xi^2(x)\right] \\
&+\frac{1}{12F^2}\int d^4x\, Tr\left[[\partial_\alpha \xi(x), \xi(x)][\partial_\alpha \xi(x), \xi(x)]\right] \qquad (5.8) \\
\mathcal{S}'_0[U_0, \xi, \partial\xi] =& \frac{i}{2}\int d^4x\, Tr\left[(v_\alpha + U_0^\dagger v_\alpha U_0)[\partial_\alpha \xi(x), \xi(x)]\right] \\
&-\frac{1}{\sqrt{2}F}\int d^4x\, Tr\left[\xi(x)\partial_\alpha \xi(x)\, \xi(x)(U_0^\dagger v_\alpha U_0 - v_\alpha)\right] \\
&-\frac{1}{2}\int d^4x\, Tr\left[U_0^\dagger v_\alpha U_0\, [\xi(x), [v_\alpha, \xi(x)]]\right] \\
&+\frac{\Sigma}{2F^2}\int d^4x\, Tr\left[s\,(U_0 \xi(x)^2 + \xi(x)^2 U_0^\dagger)\right] \qquad (5.9)
\end{aligned}$$

According to the power counting in eqs. (1.14,1.42), the $\mathcal{S}_0$ contains only $\mathcal{O}(\varepsilon^0)$ terms by construction, $\mathcal{S}'$ is a massless propagator term plus ε^2 suppressed mass correction and 4-term interaction (suppressed with respect to the propagator). The first term of $\mathcal{S}'_0$ is $\mathcal{O}(\varepsilon)$ and the three others are $\mathcal{O}(\varepsilon^2)$.

As explained above we compute the ε^2 expansion of the partition function in eq. (5.3). The $\mathcal{O}(\varepsilon)$ term is given by the insertion of a single field ξ: this kind of integral vanishes identically due to $\int dx\, \xi(x) = 0$ and, consequently, no $\mathcal{O}(\varepsilon)$ contribution may arise in the partition function. Substituting the mass matrix $\mathcal{M}$ and the chemical potential matrix B

in the external sources $s(x) = \mathcal{M}$, $v_\alpha(x) = B\,\delta_{\alpha,0}$ we have:

$$
\begin{aligned}
&Z_\xi[\mathcal{M}, B; U_0] = \\
&= Z_\xi[0,0;U_0] + \\
&\quad + \int dx\, \mathcal{M} \frac{\delta}{\delta s(x)} Z_\xi[\mathcal{M},0;U_0] + \frac{1}{2}\int dx\, dy\, B \frac{\delta}{\delta v_0(x)} B \frac{\delta}{\delta v_0(y)} Z_\xi[0, v_\alpha; U_0] \\
&= Z_\xi[0,0;U_0] + Z_\xi[0,0;U_0] \times \\
&\quad \times \langle \int dx\, \mathcal{M} \frac{\delta}{\delta s(x)} \left(-\frac{\Sigma}{2F^2} \int d^4z Tr\left[s(z)\,(U_0 \xi(z)^2 + \xi(z)^2 U_0^\dagger) \right] \right) \rangle_{\xi,0,0} \\
&\quad + \frac{1}{2} Z_\xi[0,0;U_0] \, \langle \int dx\, dy\, B \frac{\delta}{\delta v_0(x)} B \frac{\delta}{\delta v_0(y)} \times \\
&\qquad\qquad \times \left(\frac{1}{2} \int d^4z Tr\left[U_0^\dagger v_\alpha(z) U_0 \left[\xi(z), [v_\alpha(z), \xi(z)] \right] \right] \right) \rangle_{\xi,0,0} \\
&\quad + \frac{1}{2} Z_\xi[0,0;U_0] \, \langle \int dx\, dy\, B \frac{\delta}{\delta v_0(x)} B \frac{\delta}{\delta v_0(y)} \times \\
&\qquad\qquad \times \frac{1}{2} \left(-\frac{i}{2} \int d^4x Tr\left[(v_\alpha + U_0^\dagger v_\alpha U_0)[\partial_\alpha \xi(x), \xi(x)] \right] \right)^2 \rangle_{\xi,0,0}
\end{aligned}
\tag{5.10}
$$

the first and the second term are tadpole diagrams already considered in [2, 20, 115, 117] and [122]. The new term in the last line is the correction to the vector-current propagator coming from the $\mathcal{O}(\varepsilon)$ vector-pion-pion interaction in the first line of eq. (5.9). This term is of the same order as the others and cannot be disregarded.

The derivatives have to be intended expanding the sources in a complete basis of the $N \times N$ matrices; as an example we can choose the generators of the $U(N)$ Lie algebra: $\mathcal{M} \frac{\delta}{\delta s(x)} f(s(x)) = \sum_a \mathcal{M}_a \frac{\delta}{\delta s_a(x)} f\left(\sum_a s_a(x)\, T_a\right)$.

We can now perform the integration using the propagator in eq. (1.29). The $\mathcal{O}(\varepsilon^2)$ mass term and the 4-pions interaction can be disregarded at this order in ε. For the first term of eq. (5.10) we have:

$$
\begin{aligned}
&\left\langle \int dx\, \mathcal{M} \frac{\delta}{\delta s(x)} \left(-\frac{\Sigma}{2F^2} \int d^4z Tr\left[s(z)\,(U_0 \xi(z)^2 + \xi(z)^2 U_0^\dagger) \right] \right) \right\rangle_{\xi,0,0} \\
&= -\frac{\Sigma}{2F^2} \left\langle \int dx \left(Tr\left[\mathcal{M}\,(U_0\, \xi^2(x) + \xi^2(x) U_0^\dagger) \right] \right) \right\rangle_{\xi,0,0} \\
&= -\frac{\Sigma}{2F^2}\, 2\overline{\Delta}(0)\, V \sum_{a\neq 0} Tr\left[\mathcal{M}\,(U_0\, T_a\, T_a + T_a\, T_a\, U_0^\dagger) \right] \\
&= -\frac{\Sigma V}{2} \frac{1}{F^2}\, \overline{\Delta}(0) \frac{N^2-1}{N} Tr\left[\mathcal{M}\,(U_0 + U_0^\dagger) \right]
\end{aligned}
\tag{5.11}
$$

Analogously we have for the second term (using the completeness relations in eq. (1.32)):

$$\begin{aligned}
& \frac{1}{2}\left\langle \int dx\, 2Tr\left[U_0^\dagger B U_0\,[\xi(x),[B,\xi(x)]]\right]\right\rangle_{\xi,0,0} \\
= \;& \frac{1}{2}\,2\overline{\Delta}(0)V\sum_{a\neq 0} Tr\left[U_0^\dagger B U_0\,[T_a,[B,T_a]]\right] \\
= \;& -N\,\overline{\Delta}(0)V\,Tr\left[U_0^\dagger B U_0\,B\right] + \overline{\Delta}(0)V\,Tr\,[B]^2
\end{aligned} \tag{5.12}$$

The $Tr\,[B]^2$ term, as being independent of U_0, gives an irrelevant contribution to eq. (5.2) and can be disregarded. The last term of (5.10) gives:

$$\begin{aligned}
& -\frac{1}{8}\langle \int dx\,dy\; Tr\left[(B+U_0^\dagger B U_0)\,[\partial_0\xi(x),\xi(x)]\right] \\
& \qquad \times Tr\left[(B+U_0^\dagger B U_0)\,[\partial_0\xi(y),\xi(y)]\right]\rangle_{\xi,0,0} \\
& = \frac{1}{2}V\sum_{a,b,c,d} Tr\left[(B+U_0^\dagger B U_0)[T^a,T^b]\right] Tr\left[(B+U_0^\dagger B U_0)[T^c,T^d]\right] \\
& \qquad \times \int dx\;\left(\delta_{ac}\delta_{bd}\overline{\Delta}(x)\,\partial_0^2\overline{\Delta}(x) + \delta_{ad}\delta_{bc}\partial_0\overline{\Delta}(x)\,\partial_0\overline{\Delta}(x)\right) \\
& = V\sum_{a,b} Tr\left[(B+U_0^\dagger B U_0)[T^a,T^b]\right] Tr\left[(B+U_0^\dagger B U_0)[T^a,T^b]\right] \\
& \qquad \times \int dx\,\overline{\Delta}(x)\,\partial_0^2\overline{\Delta}(x) \\
& = -NV\sum_a Tr\left[(B+U_0^\dagger B U_0)T^a\right] Tr\left[(B+U_0^\dagger B U_0)T^a\right]\int dx\,\overline{\Delta}(x)\,\partial_0^2\overline{\Delta}(x) \\
& = -NV\left(Tr\left[B^2+BU_0^\dagger B U_0\right] - \frac{2}{N}Tr\,[B]^2\right)\int dx\,\overline{\Delta}(x)\,\partial_0^2\overline{\Delta}(x)
\end{aligned} \tag{5.13}$$

As before the terms proportional to $Tr\left[B^2\right]$ and to $Tr\,[B]^2$ can be disregarded. At this point we can collect all the results leading to:

$$\begin{aligned}
\frac{Z_\xi[\mathcal{M},B,U_0]}{Z_\xi[0,0,U_0]} \;=\; & 1-\frac{\Sigma V}{2}\frac{1}{F^2}\,\overline{\Delta}(0)\frac{N^2-1}{N}Tr\left[\mathcal{M}\,(U_0+U_0^\dagger)\right] \\
& -N\,V\,Tr\left[U_0^\dagger B U_0\,B\right]\left(\overline{\Delta}(0)+\int dx\,\overline{\Delta}(x)\,\partial_0^2\overline{\Delta}(x)\right)
\end{aligned} \tag{5.14}$$

and introducing the quantities F_{eff} and Σ_{eff}

$$\Sigma_{eff} \;\equiv\; \Sigma\left(1-\frac{1}{F^2}\frac{N^2-1}{N}\overline{\Delta}(0)\right) \tag{5.15}$$

$$F_{eff} \;\equiv\; F\left(1-\frac{N}{F^2}\left(\overline{\Delta}(0)+\int dx\,\overline{\Delta}(x)\partial_{x_0}^2\overline{\Delta}(x)\right)\right) \tag{5.16}$$

we can exponentiate the result obtaining:

$$\frac{Z_\xi[\mathcal{M},B;U_0]}{Z_\xi[0,0;U_0]} = \mathrm{Exp}\left[V\frac{\Sigma_{eff}-\Sigma}{2}Tr\left[\mathcal{M}(U_0+U_0^\dagger)\right] + V\frac{F_{eff}^2-F^2}{2}Tr\left[U_0^\dagger B U_0 B\right]\right]. \tag{5.17}$$

Multiplying this result times the weight function $e^{-S_0[U_0]}$ we obtain an ε^2 improved effective static action, given by the same action as eq. (5.7) computed with the effective constants (5.15) and (5.16):

$$\mathrm{Exp}\left[-\mathcal{S}_{0,eff}[\mathcal{M},B;U_0]\right] \equiv \mathrm{Exp}\left[-\mathcal{S}_0[\mathcal{M},B;U_0]\right] \frac{Z_\xi[\mathcal{M},B;U_0]}{Z_\xi[0,0;U_0]}. \tag{5.18}$$

The expectation value in U_0 with respect to this weight function $e^{-\mathcal{S}_{0,eff}}$ will be denoted by $\langle\cdot\rangle_e$. Once introduced the effective static partition function we can conclude that, at the second order in ε, the expression in eq. (5.2) can be written as:

$$\langle\mathcal{Q}[U_0,\xi]\rangle_{QCD} = \left\langle \langle\mathcal{Q}[U_0,\xi]\rangle_{\xi,s,v_\alpha} \right\rangle_e = \left\langle \langle\mathcal{Q}[U_0,\xi]\rangle_{\xi,0,0} \right\rangle_e \tag{5.19}$$

and in the equation above we have used

$$\langle\mathcal{P}[\xi]\rangle_{\xi,s,v_\alpha} = \langle\mathcal{P}[\xi,\partial\xi]\rangle_{\xi,0,0} + \mathcal{O}(\varepsilon^3)\ . \tag{5.20}$$

The value of $\overline{\Delta}(0)$ appearing in eqs. (5.15) and (5.16) has been already given in eq. (1.34), the one of $\int dx\,\overline{\Delta}(x)\partial^2_{x_0}\overline{\Delta}(x)$ may be computed in an analogous way [20]:

$$\int dx\,\overline{\Delta}(x)\partial^2_{x_0}\overline{\Delta}(x) = -\frac{1}{2\sqrt{V}}\left(\beta_1 - \frac{T^2}{\sqrt{V}}k_{00}\right) \tag{5.21}$$

where T is the time extent of the box in which the system is enclosed and k_{00} is a numerical constant depending on the geometry of the box.

It is worthful to remark that in general the term $\int dx\,\overline{\Delta}(x)\partial^2_{x_0}\overline{\Delta}(x)$ may be of the same order as $\overline{\Delta}(0)$. We can explicitate its value in the case of a hypercubic lattice:

$$\begin{aligned}\int dx\,\overline{\Delta}(x)\partial^2_{x_0}\overline{\Delta}(x) &= \frac{1}{4}\int dx\,\overline{\Delta}(x)\sum_\alpha^4 \partial^2_{x_\alpha}\overline{\Delta}(x) \\ &= \frac{1}{4}\int dx\,\overline{\Delta}(x)\left(\frac{1}{V}-\delta(x)\right) = -\frac{1}{4}\overline{\Delta}(0)\end{aligned} \tag{5.22}$$

where we have used the properties of the propagator $\overline{\Delta}(x)$:

$$\sum_\alpha^4 \partial^2_\alpha\overline{\Delta}(x) = -\delta(x) + \frac{1}{V} \qquad \int_V dx\,\overline{\Delta}(x) = 0. \tag{5.23}$$

5.3 Current-current expectation values

At this point we compute the expectation value of observables within a $\mathcal{O}(\varepsilon^2)$ accuracy. The observables in which we are interested are current-current correlators for scalar and pseudo-scalar (neutral and charged) currents; these are given, according to the eq. (1.4), by the operators:

$$
\begin{aligned}
S(x) &= \frac{\Sigma}{2}\left(U(x)+U^\dagger(x)\right) \\
S_a(x) &= \frac{\Sigma}{2}T_a\left(U(x)+U^\dagger(x)\right) \\
P(x) &= i\frac{\Sigma}{2}\left(U(x)-U^\dagger(x)\right) \\
P_a(x) &= i\frac{\Sigma}{2}T_a\left(U(x)-U^\dagger(x)\right)
\end{aligned}
\tag{5.24}
$$

The way to perform these integrations is through the formula (5.19): to substitute the expansion $U(x) = U_0 \cdot \left(1 - i\frac{\sqrt{2}}{F}\xi(x) - \frac{1}{F^2}\xi^2(x)\right)$ as the argument of the expectation value (as an example $\langle S(x)S(0)\rangle_{QCD}$) and to perform the expectation value of the dynamical fields according to the action $\mathcal{S}'$ (at this order only direct propagators and tadpoles[5] give contributions). The result is an x-dependent linear combination of expectation values of U_0-dependent observables to be computed in the U_0 ensemble with respect to the ε^2-improved $\mathcal{S}_{0,eff}$ action.

The result of the ξ integration for the neutral correlators is:

$$
\begin{aligned}
\langle S_0(x)S_0(0)\rangle_{QCD} &= \frac{\Sigma_{eff}^2}{4}\langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e \\
&\quad -\frac{\Sigma^2}{2F^2}\left(\langle Tr\left[(U_0-U_0^\dagger)^2\right]\rangle_e - \frac{1}{N_f}\langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e\right)\overline{\Delta}(x).
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
\langle P_0(x)P_0(0)\rangle_{QCD} &= -\frac{\Sigma_{eff}^2}{4}\langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e \\
&\quad +\frac{\Sigma^2}{2F^2}\left(\langle Tr\left[(U_0+U_0^\dagger)^2\right]\rangle_e - \frac{1}{N_f}\langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e\right)\overline{\Delta}(x).
\end{aligned}
\tag{5.26}
$$

In [23] the $U(N)$ invariance of the action with degenerate masses was ensuring $\langle S_a(x)S_b(0)\rangle_{QCD} \propto \delta_{ab}$, here the presence of the chemical potential explicitely break this invariance invalidating the argument. In this case we are able, however, to compute the quantity $\sum_a \langle S_a(x)S_a(0)\rangle_{QCD}$,

[5] It is worthful to note that tadpole diagrams have to be considered since the the $\mathcal{S}'$ action has not been improved.

the results are:

$$\begin{aligned}\sum_a \langle S_a(x)S_a(0)\rangle_{QCD} &= \frac{\Sigma_{eff}^2}{8}\left(\langle Tr\left[(U_0+U_0^\dagger)^2\right]\rangle_e - \frac{1}{N_f}\langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e\right)\\ &+ \overline{\Delta}(x)\frac{\Sigma^2}{4F^2}\left(-\frac{1}{2}\langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e - \frac{1}{2N_f^2}(N_f^2+2)\langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e\right.\\ &\left.+2N_f^2+\frac{2}{N_f}\langle Tr\left[(U_0-U_0^\dagger)^2\right]\rangle_{U_0}\right)\ . \qquad (5.27)\\ \sum_a \langle P_a(x)P_a(0)\rangle_{QCD} &= -\frac{\Sigma_{eff}^2}{8}\left(\langle Tr\left[(U_0-U_0^\dagger)^2\right]\rangle_e - \frac{1}{N_f}\langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e\right)\\ &- \overline{\Delta}(x)\frac{\Sigma^2}{4F^2}\left(-\frac{1}{2}\langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e - \frac{1}{2N_f^2}(N_f^2+2)\langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e\right.\\ &\left.-2N_f^2+\frac{2}{N_f}\langle Tr\left[(U_0+U_0^\dagger)^2\right]\rangle_e\right) \qquad (5.28)\end{aligned}$$

These formulas are the same as in [23], with the difference that the integrations over U_0 are performed with respect to a different (F-dependent) weight function. Not all the quantities appearing in the formulas above are independent; using a set of identities derived in section 5.3.1 we can reduce their number:

$$\begin{aligned}\langle Tr\left[U_0-U_0^\dagger\right]\rangle_e &= -\frac{2\,N\,\nu}{m\,\Sigma V} \qquad (5.29)\\ \langle Tr\left[U_0-U_0^\dagger\right]^2\rangle_e &= -\frac{2}{m\,\Sigma V}\langle Tr\left[U_0+U_0^\dagger\right]\rangle_e + \frac{4N^2\nu^2}{m^2\Sigma^2V^2}\\ \langle Tr\left[(U_0-U_0^\dagger)^2\right]\rangle_e &= -\frac{2\,N}{m\,\Sigma V}\langle Tr\left[U_0+U_0^\dagger\right]\rangle_e + \frac{4\,N\,VF^2}{(m\,\Sigma V)^2}\langle Tr\left[BU_0BU_0^\dagger\right]\rangle_e\\ &+\frac{2(VF^2)^2}{(m\,\Sigma V)^2}\left(\langle Tr\left[(BU_0BU_0^\dagger)^2\right]\rangle_e - \langle Tr\left[B^2U_0B^2U_0^\dagger\right]\rangle_e\right)\\ &+\frac{4\,N\,\nu^2}{(m\,\Sigma V)^2} - \frac{4\,VF^2}{(m\,\Sigma V)^2}Tr\,[B]^2\end{aligned}$$

Up to this point we have worked with a general chemical potential matrix B, however we have found an explicit solution to the equations above only for $B^2 \propto \mathbf{1}$. This condition is equivalent to:

$$B = \ \mu\ T_B \equiv\ \mu\ \begin{pmatrix}\mathbf{1}_{N_+} & \\ & -\mathbf{1}_{N_-}\end{pmatrix}\ . \qquad (5.30)$$

Being B interacting with the fields only through commutators $[\,\cdot\,, B]$, all the equations are invariant under $B \to B + \tilde{\mu}\mathbf{1}$: any case with N_+ quarks with chemical potential μ_+ and N_- with μ_- can be reconducted to eq. (5.30).

The remaining four quantities can be obtained by differentiating a generating function

$$\begin{aligned}\mathcal{Z}_{gen}(\Gamma_1,\gamma_2,\gamma_3) &= \int_{U(N)} d_\mu(U)\ \mathrm{Exp}\Big[\ \frac{1}{2}\,Tr\left[\Gamma_1(U+U^\dagger)\right]+ \\ &\quad +\ \gamma_2\,Tr\left[T_B U T_B U^\dagger\right]+\ \gamma_3\,Tr\left[(T_B U T_B U^\dagger)^2\right]\Big]\end{aligned} \tag{5.31}$$

where Γ_1 is a diagonal matrix. This generalisation of χPT partition function was computed in [82] for the case $N_+ = N_-$ and $\Gamma_1 = m\cdot \mathbf{1}_{N_++N_-}$. In appendix D we provide the result of the generalisation of this computation for $T_B = \begin{pmatrix}\mathbf{1}_{N_+} & \\ & -\mathbf{1}_{N_-}\end{pmatrix}$ and general diagonal mass matrix.

By differentiating we obtain:

$$\begin{aligned}\langle Tr\left[U_0+U_0^\dagger\right]\rangle_e &= 2\frac{1}{\mathcal{Z}_{gen}}\frac{\partial}{\partial\,\gamma_1}\mathcal{Z}_{gen}(\gamma_1\,\mathbf{1},\gamma_2,\gamma_3)\equiv 2\,N\frac{\Sigma_\nu(\gamma_1,\sqrt{2\gamma_2})}{\Sigma}\\ \langle Tr\left[U_0+U_0^\dagger\right]^2\rangle_e &= 4\frac{1}{\mathcal{Z}_{gen}}\frac{\partial^2}{\partial\,\gamma_1^2}\mathcal{Z}_{gen}(\gamma_1\,\mathbf{1},\gamma_2,\gamma_3)\\ &= 4N\frac{\partial}{\partial\gamma_1}\frac{\Sigma_\nu(\gamma_1,\sqrt{2\gamma_2})}{\Sigma}+4N^2\frac{\Sigma_\nu(\gamma_1,\sqrt{2\gamma_2})^2}{\Sigma^2}\\ \langle Tr\left[B\,U_0\,B\,U_0^\dagger\right]\rangle_e &= \mu^2\frac{1}{\mathcal{Z}_{gen}}\frac{\partial}{\partial\,\gamma_2}\mathcal{Z}_{gen}(\gamma_1\,\mathbf{1},\gamma_2,\gamma_3)\equiv\mu^2\mathcal{Y}_\nu\\ \langle Tr\left[(B\,U_0\,B\,U_0^\dagger)^2\right]\rangle_e &= \mu^4\frac{1}{\mathcal{Z}_{gen}}\frac{\partial}{\partial\,\gamma_3}\mathcal{Z}_{gen}(\gamma_1\,\mathbf{1},\gamma_2,\gamma_3)\equiv\mu^4\mathcal{X}_\nu\end{aligned}$$

All these quantities have to be computed differentiating at the values

$$\gamma_1 = m\frac{\Sigma_{eff}V}{2},\quad \gamma_2=\mu^2\frac{VF_{eff}^2}{2},\quad \gamma_3=0. \tag{5.32}$$

We have introduced the quantity Σ_ν since this is the standard notation for the mass dependent chiral condensate at fixed topology. The explicit value of $\mathcal{Z}_{gen}$ can be found in appendix D. Recollecting all the previous relations we can write explicitely the correlation of neutral currents up to $\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned}\langle S_0(x)S_0(0)\rangle_{QCD} =&\frac{\Sigma_{eff}^2N_f}{\Sigma}\frac{\partial\Sigma_\nu(\eta,\alpha)}{\partial\eta}+\Sigma_{eff}^2N_f^2\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2}\\ &+\overline{\Delta}(x)\left[\frac{2\Sigma}{\eta F^2}(N_f^2-1)\Sigma_\nu(\eta,\alpha)\right.\\ &\left.-\frac{2\alpha^2\Sigma^2}{\eta^2F^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-Tr\left[T_B\right]^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\right]\\ \langle P_0(x)P_0(0)\rangle_{QCD} =&\frac{N_f\Sigma_{eff}^2}{\eta\Sigma}\Sigma_\nu(\eta,\alpha)-\frac{\nu^2N_f^2}{\eta^2}\Sigma_{eff}^2+\overline{\Delta}(x)\left[-\frac{2\Sigma}{F^2}\frac{\partial}{\partial\eta}\Sigma_\nu(\eta,\alpha)\right.\\ &-\frac{2N_f}{F^2}\Sigma_\nu(\eta,\alpha)^2-\frac{2N_f^2\Sigma}{\eta F^2}\Sigma_\nu(\eta,\alpha)+\frac{\Sigma^2N_f^2}{F^2}+\frac{2\nu^2\Sigma^2N_f^2}{\eta^2F^2}\\ &\left.+\frac{2\alpha^2\Sigma^2}{\eta^2F^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-Tr\left[T_B\right]^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\right]\end{aligned} \tag{5.33}$$

Where we have defined

$$\eta \equiv m\,V\,\Sigma_{eff}, \qquad \alpha^2 = \mu^2\,V\,F_{eff}^2 \tag{5.34}$$

Analogous equations may also be written for the charged scalar currents

$$\begin{aligned}
&\sum_a \langle S_a(x) S_a(0)\rangle_{QCD} = \\
&\quad = \frac{\Sigma_{eff}^2}{2}\left[-\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} - N_f\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2} - \frac{N_f^2}{\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} + \frac{\nu^2 N_f}{\eta^2} + N_f\right. \\
&\qquad\left. + \frac{\alpha^2}{\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha) - Tr\,[\Gamma_B]^2 + \frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha) - N_f)\Big)\right] \\
&\quad + \overline{\Delta}(x)\frac{\Sigma^2}{2F^2}\left[-N_f\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} - N_f^2\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2} - (N_f^2-2)\frac{\nu^2}{\eta^2} + \frac{2-3N_f^2}{N_f\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}\right. \\
&\qquad\left. + N_f^2 + \frac{4\alpha^2}{N_f\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha) - Tr\,[\Gamma_B]^2 + \frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha) - N_f)\Big)\right]\,,
\end{aligned} \tag{5.35}$$

and for the pseudo scalars

$$\begin{aligned}
&\sum_a \langle P_a(x) P_a(0)\rangle_{QCD} = \\
&\quad = \frac{\Sigma_{eff}^2}{2}\left[\frac{N_f^2-1}{\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} - \frac{\alpha^2}{\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha) - Tr\,[\Gamma_B]^2 + \frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha) - N_f)\Big)\right] \\
&\quad + \overline{\Delta}(x)\,\frac{\Sigma^2}{2F^2}\left[\frac{N_f^2+2}{N_f}\Big(\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} + N_f\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2}\Big) + 3\frac{N_f}{\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}\right. \\
&\qquad + (N_f^2-4)\left(\frac{\nu^2}{\eta^2}+1\right) \\
&\qquad\left. - \frac{4\alpha^2}{N_f\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha) - Tr\,[\Gamma_B]^2 + \frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha) - N_f)\Big)\right]
\end{aligned} \tag{5.36}$$

In order to give some ready-to-use quantities for the numerical evaluation of the quantities above we provide the explicit values for the simple two flavour case $N_+ = N_- = 1$:

$$\begin{aligned}
\frac{\Sigma_\nu^{(N=2)}}{\Sigma} &= \frac{1}{Z_\nu(\eta)}\int_0^1 d\lambda\,\lambda\; e^{\frac{1}{2}\alpha^2(4\lambda^2-2)} I_\nu(\lambda\,\eta)\Big(\frac{\nu}{\eta}I_\nu(\lambda\,\eta) + \lambda I_{\nu+1}(\lambda\,\eta)\Big) \\
\mathcal{Y}_\nu^{(N=2)} &= \frac{2}{Z_\nu(\eta)}\int_0^1 d\lambda\,\lambda\frac{1}{2}(4\lambda^2-2)\, e^{\frac{1}{2}\alpha^2(4\lambda^2-2)} I_\nu(\lambda\,\eta)^2 \\
\mathcal{X}_\nu^{(N=2)} &= \frac{1}{Z_\nu(\eta)}\int_0^1 d\lambda\,\lambda\,\big(16(\lambda^4-\lambda^2)+2\big)\, e^{\frac{1}{2}\alpha^2(4\lambda^2-2)} I_\nu(\lambda\,\eta)^2\;.
\end{aligned} \tag{5.37}$$

where Z_ν is the static partition defined in eq. (1.43) computed with the effective LECs:

$$Z_\nu(\eta) = \int_0^1 d\lambda\lambda\; e^{\frac{1}{2}\alpha^2(4\lambda^2-2)} I_\nu(\lambda\,\eta)^2\;. \tag{5.38}$$

5.3.1 Integral identities in the presence of chemical potential

The way to obtain the identities in eq. (5.29) is the same as in [23]; we define a left-handed derivative on $U(N_f)$:

$$\nabla^a \equiv i\,(T^a\,U)_{ij}\,\frac{\partial}{\partial U_{ij}} \tag{5.39}$$

(where T^a is a generator of the $U(N_f)$ group) and thanks to the left invariance of the Haar measure we are able to obtain the equation:

$$\begin{aligned} 0 = \langle -i\nabla^a G(U)\rangle &= \frac{1}{Z}\int_{U(N_f)} d_\mu(U)\ (-i)\nabla^a\left(G(U)\,\mathcal{D}et\,[U]^\nu\,e^{-\mathcal{S}[U]}\right) \\ &= \left\langle -i\nabla^a G + G\,Tr\left[t^a\left(\nu + \frac{V\Sigma}{2}(UM - MU^\dagger) + \frac{VF^2}{2}[UBU^\dagger, B]\right)\right]\right\rangle . \end{aligned} \tag{5.40}$$

Since in this section we are explicitely dealing only with group integrals over $U(N)$, there is no ambiguity and we can drop the indeces 0 and e in U_0 and $\langle\cdot\rangle_e$.

In the case of degenerate masses $M = m\,\mathbf{1}_N$, using the completeness relations (1.32) and the formula above, we can derive a set of equations expressing $\left\langle Tr\left[U - U^\dagger\right]\right\rangle$, $\left\langle Tr\left[U - U^\dagger\right]^2\right\rangle$ and $\left\langle Tr\left[(U \pm U^\dagger)^2\right]\right\rangle$ in terms of $\left\langle Tr\left[U + U^\dagger\right]^p\right\rangle$ and $\left\langle Tr\left[(UB\,U^\dagger B)^p\right]\right\rangle$, for $p = 1, 2$:

$$\begin{aligned} 0 &= \sqrt{2N}\left\langle \nabla^0 1\right\rangle = N\,\nu + m\frac{\Sigma V}{2} Tr\left[U_0 - U_0^\dagger\right] \\ 0 &= \sqrt{2N}\left\langle \nabla^0 Tr\left[U - U^\dagger\right]\right\rangle = Tr\left[U + U^\dagger\right] + N\,\nu\,Tr\left[U - U^\dagger\right] + m\frac{\Sigma V}{2} Tr\left[U - U^\dagger\right]^2 \\ 0 &= \sum_a \left\langle -i\nabla^a Tr\left[T^a(U - U^\dagger)\right]\right\rangle = \frac{N}{2}\left\langle Tr\left[U + U^\dagger\right]\right\rangle + \frac{\nu}{2}\left\langle Tr\left[U - U^\dagger\right]\right\rangle \\ &\quad + m\frac{\Sigma V}{4}\left\langle Tr\left[(U - U^\dagger)^2\right]\right\rangle + \frac{VF^2}{4}\left\langle Tr\left[[UB\,U^\dagger, B](U - U^\dagger)\right]\right\rangle \\ 0 &= \sum_a \left\langle -i\nabla^a Tr\left[T^a[UBU^\dagger, B]\right]\right\rangle = N\left\langle Tr\left[UBU^\dagger B\right]\right\rangle - \left\langle Tr\left[B\right]^2\right\rangle + \\ &\quad + m\frac{V\Sigma}{4}\left\langle Tr\left[[UBU^\dagger, B](U - U^\dagger)\right]\right\rangle + \frac{VF^2}{2}\left\langle Tr\left[(UBU^\dagger B)^2\right] - Tr\left[B^2 UB^2 U^\dagger\right]\right\rangle . \end{aligned} \tag{5.41}$$

The equations above are true for generic B, however we have not found any way to compute $\langle Tr\left[B^2 UB^2 U^\dagger\right]\rangle$ but requiring $B^2 \propto \mathbf{1}_{N_+ + N_-}$.

Chapter 6

Conclusions

In the present work we have obtained two new independent results:

- the equivalence of all the k-point spectral correlation functions of χRMT and χPT in the ε-regime;
- the current-current correlation function for scalar and pseudo-scalar, charged and uncharged currents in finite volume QCD in presence of a non-zero chemical potential.

The equivalence of the spectral properties of χRMT and χPT was conjectured for a long time [5] and particular quantities were computed in both frameworks but a general proof was lacking. It is worth to say that, in particular, the equivalence of the individual Dirac eigenvalue spectral densities follows from the equivalence of all the spectral correlation functions; their measurament on the lattice is a remarkably simple procedure.

The equivalence has been proved for non-degenerate masses and for both zero and non-zero general chemical potential (real and imaginary), see chap. 2. As a corollary we have proved the equivalence of two different Matrix Models that are both commonly believed to describe the low energy spectrum of QCD with non-zero chemical potential, see chap. 4. This equivalence may be considered as a sign of a possible universality of non-Hermitian χUE.

An essential ingredient for the proof of the equivalence is the *superbosonisation theorem* that we have developed independently and at the very same time of a different group [111]. Our simple and original proof is in chap. 3. We mention the explicit parametrisation of the Haar measure over the supermanifold $\hat{G}l(n_b|n_f)$ that we provide in eq. (3.19).

As already pointed out in [123] we think superbosonisation may be an important method to study universality of the spectral correlation functions in RMT, that is the independence of the result on the particular choice of the weight function. The footpath should not be

different from the one followed in [109]: to write ratios of characteristic polynomials in terms of Gaussian superintegrals, to integrate the random matrix formally in order to obtain the Fourier transformation of the weight function and thanks the key observation

$$Tr\left[\left(\sum_{\alpha}\psi_{\alpha,i}\otimes\psi^{\dagger}_{\alpha,j}\right)^{k}\right]=Str\left[\left(\sum_{i}\psi_{\alpha,i}\otimes\psi^{\dagger}_{\beta,i}\right)^{k}\right] \tag{6.1}$$

to express everything in terms of $\sum_i \psi_{\alpha,i}\otimes\psi_{\beta,i}$; at this point invoking the superbosonisation theorem it should be possible to see explicitely how the generating functions dependend on N and on the coefficients of the weight function. The procedure is quite general and may be applied to different matrix ensembles; this is a project on which we are working at the present.

A more technical open question for which we are trying to find an answer is expressed in sect. 2.3.1: we would like to show how to get rid of the divergences coming from the integration of the boson-boson sector in the bosonic non-Hermitian χPT, possibly finding a new manifold where to perform the integration without any regularisation.

The computation of scalar-scalar (pseudoscalar-pseudoscalar) correlation function in chap. 5 provides a way to measure F_π as an extension of an existing proposal. It is well known that the pion decay constant may be computed by measuring the vector-vector (or axial-axial) correlation current or that it can be measured by studying the alteration of low energy spectral properties for imaginary chemical potential, see fig. 1.2. The vector-vector correlator for vanishing chemical potential does not appear at the leading order in the ε-expansion, on the contrary the scalar-scalar correlator shows a dependence on F_π already at $\mathcal{O}(\varepsilon^0)$; this result can be used to improve the quality of the fits for F_π provided by the measurement of the spectral properties.

The computation of current-current correlators in presence chemical potential could be extended including the vector and axial current.

Appendix A

Superanalysis

In this appendix we briefly recall the main superanalysis concept used in this work. We will follow mainly [112, 77, 124]

A.1 Basic definitions

We consider N anticommuting variables ξ_i, $i = 1, \ldots, N$:

$$\xi_i \xi_j = -\xi_j \xi_i \tag{A.1}$$

A consequence of the anticommutation relations is that $\xi_i^2 = 0 \; \forall \, i$, and that for any set of coefficients a_i:

$$\left(\sum_i^N a_i \xi_i \right)^{N+1} = 0 \tag{A.2}$$

Due to this nilpotency property, we have that N anticommuting numbers generate a 2^N dimensional algebra (usually denoted Λ_N). The elements of this algebra (or of its $N \to \infty$ limit) are called *supernumber*. Every supernumber z can be expressed in the form

$$z = z_B + z_S \tag{A.3}$$

where z_B is a complex number and is called *base* and z_S is called *soul*. The latter can be written in terms of the generators of the algebra:

$$z_S = \sum_n \frac{1}{n!} c_{a_1, \ldots, a_n} \xi^{a_i} \cdots \xi^{a_n} \; . \tag{A.4}$$

For any finite N the soul is nilpotent (A.2). Supernumbers can be splitted into commuting (*c-number*) and anticommuting (*a-number*) part. The expansion of a commuting number

contains only products of even numbers of ξ_i. The anticommuting part contains only products of odd numbers of ξ_i.

A complex conjugation may be defined for super-numbers. Its behaviour on the generator of the algebras can be summarised in:

$$\begin{aligned}(\xi_i)^* &= \xi_i^* \\ (\xi_i^*)^* &= -\xi_i \\ (\xi_i\xi_j)^* &= \xi_i^*\xi_j^*\end{aligned} \tag{A.5}$$

The complex conjugation is not uniquely defined, these rules are the most commonly used and determine the so-called *second kind conjugation*. An alternative definition is considering a $+$ instead of the $-$ in the second line, and it is called *first kind conjugation*. It is important to note that ξ_i and ξ_i^* are independent quantities:

$$\xi_i\xi_i^* = -\xi_i^*\xi_i \neq 0 \tag{A.6}$$

The set of commuting numbers invariant under conjugation is called $\mathbb{R}_c$, analogously the set of real anticommuting numbers is $\mathbb{R}_a$.

Starting from analytic functions we can define *superanalytic* functions through its Taylor series:

$$f(u) = f(u_B + u_S) \equiv \sum_{n=0} f^{(n)}(u_B)\, u_S^n. \tag{A.7}$$

As a consequence we have that u^{-1} may be defined whenever $u_B \neq 0$.

From now on we will consider only commuting and anticommuting numbers separately. Supernumbers may be organised in vectors and matrices as well. A $(n_b|n_f)$ *supervector* v is defined as a set of commuting numbers z_i followed by one of anti-commuting numbers ζ_i (this ordering convention is called *boson-fermion convention*):

$$v = \begin{pmatrix} z_1 \\ \vdots \\ z_{n_b} \\ \zeta_1 \\ \vdots \\ \zeta_{n_f} \end{pmatrix}. \tag{A.8}$$

Analogously a $(n_b|n_f) \times (n_b|n_f)$ *supermatrix*

$$M = \begin{pmatrix} A & \Lambda \\ \Xi & B \end{pmatrix} \tag{A.9}$$

is defined with A and B being respectively $n_b \times n_b$ and $n_f \times n_f$ commuting number matrices, and Λ and Ξ being $n_b \times n_f$ and $n_f \times n_b$ anticommuting number matrices. The *base of a matrix* is defined by taking the complex number parts of its elements: $M_B \equiv \begin{pmatrix} A_B & 0 \\ 0 & B_B \end{pmatrix}$.

The transpose of a supermatrix is:

$$M^T = \begin{pmatrix} A^T & -\Xi^T \\ \Lambda^T & B^T \end{pmatrix} \tag{A.10}$$

and the inverse is defined as usually solving $M^{-1} \cdot M = \mathbf{1}$. A supermatrix may be inverted iff its base may be inverted.

The *supertrace* of a matrix is defined as:

$$Str\,[M] = Tr\,[A] - Tr\,[B]\,. \tag{A.11}$$

This definition follows from the requirement of the cyclic property $Str\,[M \cdot N] = Str\,[N \cdot M]$. Analogously to the standard case the *superdeterminant* may be defined integrating the equation $\delta \ln \mathcal{S}det\,[M] = Str\left[M^{-1}\delta M\right]$ with the initial condition $\mathcal{S}det \begin{bmatrix} \mathbf{1}_{n_b} & 0 \\ 0 & \mathbf{1}_{n_f} \end{bmatrix} = 1$. The result can be expressed in terms of the matrix block: whenever $\mathcal{D}et\,[B] \neq 0$ (or equivalently $\mathcal{D}et\,[A] \neq 0$)

$$\mathcal{S}det\,[M] = \frac{\mathcal{D}et\left[A - \Lambda \cdot B^{-1} \cdot \Xi\right]}{\mathcal{D}et\,[B]} = \left(\frac{\mathcal{D}et\left[B - \Xi \cdot A^{-1} \cdot \Lambda\right]}{\mathcal{D}et\,[A]}\right)^{-1}. \tag{A.12}$$

A.2 Integrations

A function $f(\xi)$ of a single anticommuting variable ξ may always be written (A.7) as $f(\xi) = a + b\,\xi$. By linearity the integral over an anticommuting variable is uniquely determined by the prescription:

$$\begin{aligned} \int d\xi &\equiv 0 \\ \int \xi\; d\xi &\equiv Z \end{aligned} \tag{A.13}$$

where Z is a conventional constant. We define $Z = 1$. A delta function for this integration may be easily written, $\delta(\xi) = \xi$

$$\int f(\xi)\,\delta(\xi)\,d\xi = \int (a + b\,\xi)\;\xi\,d\xi = a = f(0) \tag{A.14}$$

In analogy with the integration over $\mathbb{R}$, where a $\delta(x)$ function may be written using the exponential notation $\delta(x) = \frac{1}{2\pi}\int e^{i\,p\,x} dp$, we have:

$$\delta(\xi) = \xi = i\int e^{i\,\phi\,\xi} d\phi = i\int (1 + i\,\phi\,\xi)\,d\phi = \xi \tag{A.15}$$

The measure $d\xi$ transforms under a linear transformation $\xi = b \cdot \zeta$ according to $d\xi = \frac{1}{b} d\zeta$. The extension of this integration to higher numbers of variables is straightforward, for a set $\xi = (\xi_1, \ldots, \xi_{n_f})$, $d\xi \equiv \prod_{i=1}^{n_f} d\xi_i$. The multidimensional measure transforms under a linear transformation $\xi = B \cdot \zeta$ according to:

$$d\xi = \mathcal{D}et\,[B]^{-1}\, d\zeta. \tag{A.16}$$

A line integration may be defined on paths in $\mathbb{R}_c$. Let $x(x_B) = x_B + x_S(x_B)$ be a path on $\mathbb{R}$ whose soul is unambiguously determined by its body, and $x_B \in [a_B, b_B]$, and let f a superanalytic function, we have [77]:

$$\int_a^b f(x)\, dx = \sum_{n=0} \frac{1}{n!} \int_{a_B}^{b_B} f^{(n)}(x_B) x_S^n \left(1 + x_S'(x_B)\right)\, dx_B = \cdots = F(b) - F(A) \tag{A.17}$$

where $F(x_B)$ is a primitive function of $f(x_B) = \partial_{x_B} F(x_B)$. The integration of a function over a path in $\mathbb{R}$ depends only on its endpoints. The integration $\int_{\mathbb{R}_c}$ is defined as an integration over any path whose endpoints x tend to infinity, $x(x_B \to \pm\infty) = \pm\infty$.

The δ-function over $\mathbb{R}_c$ is defined through superanalytic continuation:

$$\delta(x) = \delta(x_B + x_S) = \sum_{n=0}^{\infty} \delta^{(n)}(x_B) \frac{x_S^n}{n!} \tag{A.18}$$

The transformation of the integration measure under transformations involving commuting and anti-commuting degrees of freedom at the same time is not a trivial generalisation of the Jacobian law. For the transformation

$$\begin{pmatrix} x \\ \xi \end{pmatrix} \to \begin{pmatrix} z(x,\xi) \\ \zeta(x,\xi) \end{pmatrix} \tag{A.19}$$

we define the *Berezinean*:

$$Ber\left[\frac{\partial(z,\zeta)}{\partial(x,\xi)}\right] = \mathcal{S}det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial \xi} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial \xi} \end{bmatrix}. \tag{A.20}$$

The measure transforms according to:

$$dx\, d\xi \to dz\, d\zeta = dx\, d\xi\, Ber\left[\frac{\partial(z,\zeta)}{\partial(x,\xi)}\right] + dx\, d\xi\, D \tag{A.21}$$

where the additional term D is a differential nilpotent operator. This term gives rise to an anomalous term usually referred to as *Efetov-Wegner term*. A comprehensive explanation may be found in [112]; the nilpotent property of the Grassmannian coordinates induces a Z-gradings of the set of functions: $f(x,\xi)$ has degree 0 if $\partial_{\xi_a} f = 0$ for all a, it has degree n if $\partial_{\xi_b} f$ has degree $n-1$ for all b. The anomalous term arises whenever we consider transformations changing the Z-gradings. An important property of the anomalous term is that when computed on a analytic function it gives an exact form, and hence whenever

the commuting number integration is done over a manifold with periodic boundaries (as in the integration over $U(n)$), or is performed on functions decreasing sufficiently fast at the boundaries, the Efetov-Wegner effect is vanishing. The anomalous term cannot be disregarded if computed on non-analytic functions. A way to compute this term has been provided in [112], however this procedure is very difficult and it has never be implemented but in a very few cases [112, 125, 126].

Once pointed out that the Efetov-Wegner term depends on the alterations of the Z-gradings, the fact that anomalous terms computed on superanalytic functions reduce to boundary terms may be seen as a consequence of the supercontour invariance in (A.17): the term coming from the addition of a nilpotent part to a commuting integration variable (that is changing the Z-grading) is just a boundary term.

A particularly useful superintegral is the Gaussian one. Choosing complex variables z_i, and anticommuting (independent) variables $\zeta_\alpha, \zeta^*_\alpha$, we define the vectors:

$$v = \begin{pmatrix} z \\ \zeta \end{pmatrix}, \quad v^* = \begin{pmatrix} z^* \\ \zeta^* \end{pmatrix} . \tag{A.22}$$

Let M be a supermatrix as in (A.9), we define the Gaussian superintegral:

$$I(M) = \int dv\, dv^*\ \mathrm{Exp}\left[-v^\dagger \cdot M \cdot v\right] . \tag{A.23}$$

Supposing $\mathcal{D}et\,[A] \neq 0$ we can reduce M to a block diagonal form

$$M = \begin{pmatrix} A & \Lambda \\ \Xi & B \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n_b} & 0 \\ \Xi \cdot A^{-1} & \mathbf{1}_{n_f} \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B - \Xi \cdot A^{-1} \cdot \Lambda \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_{n_b} & A^{-1} \cdot \Lambda \\ 0 & \mathbf{1}_{n_b} \end{pmatrix} \tag{A.24}$$

and defining[1]

$$\overline{v} = \begin{pmatrix} \mathbf{1}_{n_b} & A^{-1} \cdot \Lambda \\ 0 & \mathbf{1}_{n_b} \end{pmatrix} \cdot v, \quad \overline{v}^* = \begin{pmatrix} \mathbf{1}_{n_b} & 0 \\ \Xi \cdot A^{-1} & \mathbf{1}_{n_f} \end{pmatrix}^T \cdot v^* \tag{A.25}$$

we factorise the commuting and the anticommuting integrations:

$$I(M) = \int d\overline{v}\, d\overline{v}^*\ \mathrm{Exp}\left[-\overline{v}^\dagger \cdot \begin{pmatrix} A & 0 \\ 0 & B - \Xi \cdot A^{-1} \cdot \Lambda \end{pmatrix} \cdot \overline{v}\right] . \tag{A.26}$$

The integration over the anticommuting part gives (no convergence requirements have to be fulfilled):

$$\mathcal{D}et\left[B - \Xi \cdot A^{-1} \cdot \Lambda\right] . \tag{A.27}$$

[1] These two definitions seem to be not self-consistent since they give independent results for z and z^*. This is not a trouble, the definitions differs only in the souls of these two quantities, and according to (A.17) these are irrelevant to the purpose of the integration.

For the integration of the commuting variables we have to require A to be positive definite[2]. If this is the case it gives:

$$\pi^{n_b}\ \mathcal{D}et\left[A\right]^{-1}. \tag{A.28}$$

As a result we have:

$$I(M)=\pi^{n_b}\ \mathcal{D}et\left[A\right]^{-1}.\mathcal{D}et\left[B-\Xi\cdot A^{-1}\cdot\Lambda\right]=\pi^{n_b}\ \mathcal{S}det\left[M\right]^{-1} \tag{A.29}$$

A different choice in the constant Z in the definition (A.13) could lead to $I(M)=\mathcal{S}det\left[M\right]^{-1}$.

A.3 Supermanifolds

In this section we give a naive introduction to supermanifold, a good mathematical-physics one can be found in [81].

Let us consider the set of invertible supermatrices ($\mathcal{S}det\left[M\neq 0\right]$); it is an easy exercise to see that this set has a group structure (with respect to the matrix product) and it provides a representation of the *superlinear group* $Gl(n_b|n_f)$. Analogously the invertible matrices fulfilling $M^\dagger\cdot M=\mathbf{1}_{n_b+n_f}$ are a representation of the superunitary group $U(n_b|n_f)$.

We write M through an exponential map $M=\mathrm{Exp}\left[m\right]=\mathrm{Exp}\begin{bmatrix}a & \lambda\\ \xi & b\end{bmatrix}$, and consider the distance induced by the left-right-invariant form

$$\begin{aligned} Str\left[dM\cdot d(M^{-1})\right] &= -Str\left[dm\cdot dm\right] \\ &= -Tr\left[da\cdot da\right]-Tr\left[d\lambda\cdot d\xi\right]+Tr\left[d\xi\cdot d\lambda\right]+Tr\left[db\cdot db\right] \end{aligned} \tag{A.30}$$

For $M\in U(n_b|n_f)$ we have $m=-m^\dagger$ and hence we read off from the formula above that the metric form $Str\left[dM\cdot d(M^{-1})\right]$ has not a positive-definite c-number part. In this sense $U(n_b|n_f)$ is not a Riemannian superspace.

Obviously $Gl(n_b|n_f)$ is not Riemannian either; we take the maximal Riemannian submanifold of the last manifold by reducing the number of generators of the base manifold (that is da_B and db_B) in order to give a definite metric $Tr\left[da\cdot da\right]-Tr\left[db\cdot db\right]$. This result can be achieved in two ways, by taking da anti-Hermitian and db Hermitian or the opposite. In the present work we have taken the generators in the boson-boson sector to be Hermitian (generating a non-compact $\mathcal{M}_b=Gl(n_b)/U(n_b)$ manifold) and the ones in the fermion-fermion to be anti-Hermitian (generating a compact $\mathcal{M}_f=U(n_f)$ manifold). The form in eq. (A.30) forces the base of the maximal Riemannian submanifold to have a compact/non-compact structure. The generators obtained posing this (anti-) hermiticity condition on the base generators of $Gl(n_b|n_f)$ cannot be closed to give rise to a Lie algebra.

The maximal Riemannian submanifold of $Gl(n_b|n_f)$ is the pair $\hat{Gl}(n_b|n_f)\equiv(Gl(n_b|n_f),\mathcal{M}_b\times\mathcal{M}_f)$, its elements are the elements of $Gl(n_b|n_f)$ whose base manifold belongs to $M_b\times\mathcal{M}_f$. The

[2]This condition can be weakened requiring the positivity only for the Hermitian part of A.

requirement on the base manifold ensures the positivity of the measure in the complex variable integrations. As the integration of the fermionic degrees of freedom is a differentiation, we do not need additional requirements to ensure a well-definiteness of this integral. The $\hat{G}l(n_b|n_f)$ supermanifold, exactly as $U(n_b|n_f)$, is parametrised by $n_b^2 + n_f^2$ real variables and by $2\,n_b \times n_f$ grassmann variables.

Appendix B

Defining δ-functions on super-Hermitian matrices

THE question answered here is whether or not one can define a δ-distribution such that

$$f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) = \int dM\, f(M)\delta\left(M - \sum_k \psi_k \otimes \psi_k^\dagger\right). \tag{B.1}$$

In order to clarify the idea we take the same simple but nontrivial example as in [103], where the problem was raised, that is $k = 1$, $(n_b|n_f) = (0|2)$. The outer product $\psi_k \otimes \psi_k^\dagger$ is a 2×2 matrix, its terms are only nilpotent commuting numbers $\psi_i \cdot \psi_j^*$ for $i, j = 1, 2$. We can consider this matrix as belonging to a superanalytic continuation of a manifold of real dimension 4 (at least). We could take as a such manifold $U(2)$, $Gl(2)/U(2)$, Hermitian matrices or real matrices. Despite in the rest of the work we have used Hermitian matrices, for this simple example we will use the one with the simplest notation, real matrices:

$$a \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \equiv \psi \otimes \psi^\dagger = \begin{pmatrix} \psi_1\psi_1^* & \psi_1\psi_2^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* \end{pmatrix} . \tag{B.2}$$

Once considering the outer product of vectors as an element of a commuting number manifold, the well definiteness of the δ-distribution is inherited from the one of the base manifold by superanalyticity in the real parameters of the manifold. In formulas:

$$f(a) \equiv f(a_{11}, a_{12}, a_{21}, a_{22}) = \int dm\, \delta(m - a)\, f(m_{11}, m_{12}, m_{21}, m_{22}) , \tag{B.3}$$

where the integration manifold is the base manifold (real matrices) $\int dm \equiv \int_{-\infty}^{\infty} \prod_{i,j=1}^{2} dm_{ij}$, and the δ distribution is the one of the base manifold too, $\delta(m - a) \equiv \prod_{i,j=1}^{2} \delta(m_{ij} - a_{ij})$.

The last thing to be verified is that the superanalytic continuation of the δ-distribution is properly defined: its well-behaving definition can be found in literature [77]. Given a

quantity $x = x_B + x_S$, where the first part is the body of the number, and the second is the nilpotent part. Given also a function with a sufficient number of derivatives in x_B such that $f(x_B + x_S) = \sum_{n=0}^{\infty} f^{(n)}(x_B)\frac{x_S^n}{n!}$ is properly defined, we have:

$$\begin{aligned}\int_{-\infty}^{+\infty} dy\, \delta(y - x_B - x_S)\, f(y) \equiv \\ \equiv \int_{-\infty}^{+\infty} dy \sum_{n=0}^{\infty} \delta^{(n)}(y - x_B)\frac{(-x_S)^n}{n!}\, f(y) \\ = \sum_{n=0}^{\infty} \frac{(x_S)^n}{n!} \int_{-\infty}^{+\infty} dy\, \delta(y - x_B)\, f^{(n)}(x_B) = f(x_B + x_S)\ .\end{aligned} \tag{B.4}$$

Completing the discussion on the example above we can write explicitly $\delta(m - a)$:

$$\delta(m-a) = \delta(m) - \sum_{i,j=1}^{2} \psi_i\psi_j^*\partial_{m_{ij}}\delta(m) + \psi_1\psi_1^*\psi_2\psi_2^*(\partial_{m_{11}}\partial_{m_{22}} + \partial_{m_{12}}\partial_{m_{21}})\delta(m) \tag{B.5}$$

where m is a real number 2×2 matrix.

The generalisation for what is done above to eq. (B.1) is straightforward: one considers the outer product as an element belonging to the superanalytic continuation of a manifold. The manifold we choose is the one of super-Hermitian matrices: the boson-boson and the fermion-fermion block are Hermitian matrices, and the δ is defined as the superanalytic continuation of the product of δ over the real and imaginary parts of the independent elements of the matrices; boson-fermion and fermion-boson blocks are made of independent Grassmann integration variables and δ can be represented as, eq. (A.15):

$$\delta(\theta - \tilde{\theta}) \propto (\theta - \tilde{\theta}) \propto \int d\xi\, e^{i\xi(\theta - \tilde{\theta})}. \tag{B.6}$$

As an exponential representation of δ is allowed for both commuting and anticommuting variables, such a representation of δ-function on super-Hermitian matrices is allowed too:

$$\begin{aligned} f(a) &= \int_{M=M^\dagger} dM\, f(M)\, \delta(M - a) \\ &\propto \int_{M=M^\dagger} dM\, f(M) \int_{F=F^\dagger} dF\, e^{iStr[F(M-a)]} \end{aligned} \tag{B.7}$$

where dM and dF stand for the flat measure over the independent entries of super-Hermitian matrices:

$$\begin{aligned} dM \equiv & \prod_{i=1}^{n_b}\prod_{\alpha=1}^{n_f} dM_{i,\alpha}\, dM_{\alpha,i} \prod_{i=1}^{n_b} dM_{i,i} \prod_{i>j} d\mathbb{R}e\, M_{i,j}\, d\mathbb{I}m\, M_{i,j} \\ & \times \prod_{\alpha=1}^{n_f} dM_{\alpha,\alpha} \prod_{\alpha,\beta} d\mathbb{R}e\, M_{\alpha,\beta}\, d\mathbb{I}m\, M_{\alpha,\beta}\ . \end{aligned} \tag{B.8}$$

Appendix C

The integrals in eq. (3.17)

C.1 Boson-boson block

Let us start from

$$\begin{aligned}
&\oint_{U(n_b)} dF_1 \; \frac{1}{\mathcal{D}et\,[F_1]^{N+n_f}} \; \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \; = \\
&= \int_{U(n_b)} d\mu(F_1) \; \mathcal{D}et\,[F_1]^{-N-n_f+n_b} \; \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \\
&= \mathcal{D}et\left[\tilde{H}\right]^{N+n_f-n_b} \int_{U(n_b)} d\mu(F_1) \; \mathcal{D}et\left[F_1\tilde{H}\right]^{-N-n_f+n_b} \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \; ,
\end{aligned} \tag{C.1}$$

where for simplicity of notation we use $\tilde{H} = H_1 - \Theta H_2^{-1}\Theta^\dagger$. The integral in the last line can be performed using the character expansion. This is a particular case of a more general integral computed in [24]; the result is:

$$\sum_r \frac{\alpha_r^{(0)}}{d_r} \alpha_r^{(N+n_f-n_b)} \chi_r(0) \tag{C.2}$$

where the sum is over the irreducible representations of $Gl(n_b)$. The quantity $\chi_r(0)$ is zero for all representations apart from the trivial one. The result is just a constant[1].

$$\oint_{U(n_b)} dF_1 \frac{1}{\mathcal{D}et\,[F_1]^{N+n_f}} e^{iTr[F_1\tilde{H}]} \propto \mathcal{D}et\left[\tilde{H}\right]^{N+n_f-n_b}. \tag{C.3}$$

This result, together with the argument of analytic continuation of Hermitian matrices, gives an alternative way for computing the Ingham-Siegel integral as was done in [90].

[1] In order to compute the exact value one has just to substitute the relation $\alpha_r^{(\nu)} = \det_{ij}\left[\frac{1}{n_j-\nu+i-j}\right]$, where $r = (n_1, \ldots, n_b)$ are the labels of the representation.

C.2 Fermion-fermion block

The fermion-fermion block integral has already been computed by Guhr [109]. The result is:

$$\int_{F_2=F_2^\dagger} dF_2\, \mathcal{D}et\left[F_2\right]^N \mathrm{Exp}\left[-iTr\left[F_2 H_2\right]\right] \propto \prod_j \delta^{(N+n_f-1)}(h_j)\ . \tag{C.4}$$

Here we do not have any problem with possible Efetov-Wegner terms arising from these diagonalisation: both F_2 and H_2 are complex Hermitian matrices and no Z-grading will be changed during diagonalisation [112].

If we integrate this functional with a given function analytic in the matrix entries, using the Cauchy integral formula we obtain:

$$\begin{aligned}
&\int d\mu(U) \prod_j \int dh_j\, \Delta(\{h\})^2\, g(UhU^\dagger) \prod_j \delta^{(N+n_f-1)}(h_j) \\
&\propto \int d\mu(U) \oint \prod_j dh_j \frac{1}{h_j^{N+n_f}}\, \Delta(\{h\})^2\, g(UhU^\dagger) \\
&= \oint_{U(N_f)} dH_2\ \frac{1}{\mathcal{D}et\left[H_2\right]^{N+n_f}}\ g(H_2)\ .
\end{aligned} \tag{C.5}$$

Appendix D

The generalised partition function in eq. (5.31)

In this appendix we compute the following generalised partition function eq. (5.31)

$$\mathcal{Z}_{gen}(\{\eta\}) \equiv \int_{U(N_f)} d\mu(U_0)\ \det[U_0]^{\nu}\ \mathrm{Exp}\left[\frac{1}{2}Tr\left[\mathcal{M}(U_0+U_0^{\dagger})\right] + \sum_p a_p Tr\left[(U_0\Gamma_B U_0^{\dagger}\Gamma_B)^p\right]\right] , \tag{D.1}$$

where $\mathcal{M} = \mathrm{diag}(\eta_{f1=1},\cdots,\eta_{N_1},\eta_{f2=1},\cdots,\eta_{N_2})$ contains the rescaled masses which may now be different and T_B plays the role of the chemical potential matrix (5.30):

$$T_B = \begin{pmatrix} \mathbf{1}_{N_1} & 0 \\ 0 & -\mathbf{1}_{N_2} \end{pmatrix} .$$

Without loss of generality we choose $N_2 \geq N_1$ in $N_f = N_1 + N_2$. The volume V and higher order coupling constants are all absorbed into the coefficients a_p, where the sum can take finitely or infinitely many terms. We only require that the integrals converge.

The result for this generalised partition function was given in [82] for degenerate masses and $N_1 = N_2$. Our generalisation follows closely [65]. Because we differentiate the generalised partition function with respect to the couplings a_p in order to generate expectation values, we have to keep track of all constants that depend on the a_p. The unitary matrix

U_0 can be parametrised as follows [63, 65]

$$\begin{aligned} U_0 &= \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \Lambda \begin{pmatrix} v_1^\dagger & 0 \\ 0 & v_2^\dagger \end{pmatrix} , \\ \Lambda &\equiv \begin{pmatrix} \hat{\lambda} & \sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 0 \\ \sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & -\hat{\lambda} & 0 \\ 0 & 0 & -\mathbf{1}_{N_2 - N_1} \end{pmatrix} . \end{aligned} \tag{D.2}$$

Here we denote by $\hat{\lambda}$ the matrix $\hat{\lambda} \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_1})$ containing the real numbers $\lambda_k \in [0,1]$ for $k = 1, \ldots, N_1$. The unitary submatrices are $u_1, v_1 \in U(N_1)$, $u_2 \in U(N_2)$ and $v_2 \in \tilde{U}(N_2) \equiv U(N_2)/(U(1)^{N_1} \times U(N_2 - N_1))$. The matrix Λ is Hermitian and we observe that

$$Tr\left[(U_0 \Gamma_B U_0^\dagger \Gamma_B)^p\right] = Tr\left[(\Lambda \Gamma_B \Lambda \Gamma_B)^p\right] , \tag{D.3}$$

so all unitary degrees of freedom drop out. From

$$(\Lambda \Gamma_B)^2 = \begin{pmatrix} 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & -2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 0 \\ 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & 0 \\ 0 & 0 & \mathbf{1}_{N_2 - N_1} \end{pmatrix} , \tag{D.4}$$

we obtain

$$Tr\left[(U_0 \Gamma_B U_0^\dagger \Gamma_B)^p\right] = Tr\left[\begin{pmatrix} 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & -2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} \\ 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 2\hat{\lambda}^2 - \mathbf{1}_{N_1} \end{pmatrix}^p\right] - Tr\left[\mathbf{1}_{N_2 - N_1}\right] . \tag{D.5}$$

The $2N_1$ eigenvalues can be written in diagonal matrix form as

$$X_\pm \equiv 2\hat{\lambda}^2 - \mathbf{1}_{N_1} \pm i\, 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} . \tag{D.6}$$

We thus arrive at

$$\begin{aligned} Tr\left[(U_0 \Gamma_B U_0^\dagger \Gamma_B)^p\right] + (N_2 - N_1) &= Tr\left[X_+^p + X_-^p\right] \\ &= \sum_{i=1}^{N_1} 2 \sum_{q=0}^{[\frac{p}{2}]} \binom{p}{2q} \Big(2\lambda_i^2 - 1\Big)^{p-2q} \Big(-4\lambda_i^2(1-\lambda_i^2)\Big)^q = \sum_{i=1}^{N_1} 2T_{2p}(\lambda_i) . \end{aligned} \tag{D.7}$$

The real polynomial $T_{2p}(\lambda_i)$ of degree $2p$ we obtain are the Chebyshev polynomials of the first kind[1].

Coming back to the integral eq. (D.1) we can now go to an eigenvalue basis using the parametrisation eq. (D.2). The Jacobian computed in [63] gets cancelled up to a product over λ_i, after integrating out all unitary degrees of freedom factorising into $U(N_1)$ and $U(N_2)$ group integrals. Because of the decoupling of the sum over p from the unitary degrees of

[1] In [82] the polynomial was given in the form $\cos(2p \cos^{-1}(\lambda))$.

freedom in eq. (D.7) the calculation is identical to the one presented in [65] section 3., to where we refer the reader for details.

Collecting all the results we obtain the following expression for N_f flavours

$$\mathcal{Z}_{gen}(\{\eta\}) \;=\; \frac{const.}{\Delta_{N_1}(\{\eta_{f1}\})\Delta_{N_2}(\{\eta_{f2}\})} \times \tag{D.8}$$
$$\times\; \mathcal{D}et \begin{bmatrix} \mathcal{Z}_{gen}(\eta_{f1=1},\eta_{f2=1}) & \cdots & \mathcal{Z}_{gen}(\eta_{f1=1},\eta_{N_2}) \\ \cdots & \cdots & \cdots \\ \mathcal{Z}_{gen}(\eta_{N_1},\eta_{f2=1}) & \cdots & \mathcal{Z}_{gen}(\eta_{N_1},\eta_{N_2}) \\ I_\nu(\eta_{f2=1}) & \cdots & I_\nu(\eta_{N_2}) \\ \eta_{f2=1}^{N_2-N_1-1} I_\nu^{(N_2-N_1-1)}(\eta_{f2=1}) & \cdots & \eta_{N_2}^{N_2-N_1-1} I_\nu^{(N_2-N_1-1)}(\eta_{N_2}) \end{bmatrix},$$

where $\Delta_{N_1}(\{\eta_{f1}\} = \prod_{j>i}^{N_1}(\eta_j^2 - \eta_i^2)$ is the Vandermonde determinant within each flavour. The generalised $N_f = 2$ flavour partition function $\mathcal{Z}_{gen}(\eta_1,\eta_2)$ that is used inside the determinant above is given by

$$\mathcal{Z}_{gen}(\eta_1,\eta_2) \;\equiv\; \int_0^1 d\lambda\, \lambda\, \mathrm{Exp}\left[\sum_p a_p\Big(2T_{2p}(\lambda) - (N_2 - N_1)\Big)\right] I_\nu(\lambda\eta_1) I_\nu(\lambda\eta_2)\ . \tag{D.9}$$

With $I_\nu^{(k)}(\eta_{f1})$ we denote the k-th derivative of the I-Bessel function. The I-Bessel function itself in fact is a one-flavour partition function, $\mathcal{Z}_\nu = I_\nu(\eta)$ which is independent of the a_p. The constant in front in eq. (D.8) only contains numbers, from the the Taylor expansion of the unitary group integrals leading to the derivatives of the Bessel functions. Eq. (D.8) is the main new result of this appendix, the solution of the generalised partition function given by the group integral (D.1). It generalises the previous result [82] to nondegenerate masses and $Tr\,[\Gamma_B] \neq 0$.

As an example for $p = 1$ we get

$$2\ T_2(\lambda) \;=\; 4\lambda^2 - 2\ , \tag{D.10}$$

which gives back the standard partition function eq. (5.38) for $a_1 = \frac{1}{2}\alpha^2$ (and mass η). It is explicitly given in eq. (5.38), up to a constant prefactor. For $p = 2$ we get for the second polynomial

$$2\ T_4(\lambda) \;=\; 16(\lambda^4 - \lambda^2) + 2\ , \tag{D.11}$$

leading to the generalised partition function with $a_1 = \gamma_2$ and $a_2 = \gamma_3$, that is needed in eq. (5.31).

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