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Electric-Magnetic Duality in Supergravity and Extremal Black Hole Attractors

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Introduction

Black Holes are space time singularities whose properties in classical theories have been studied for a long time. The interest on metric solutions has mainly grown with the study of the deep connections of Supergravities with different versions of String Theories.

In fact, $4-$dimensional $\mathcal{N}$-extended ungauged Supergravities have two types of geometry, the space-time geometry and the moduli space geometry; the symmetries of the latter are connected to those of the higher dimensional String theory. Taking into account at the same time space-time and moduli space geometry can help clarifying the properties of these connections.

Black holes are states of the graviton spin 2 field in Supergravity spectrum and, in the case of non thermal radiation these states are stable, as happens for electromagnetically charged (dyonic) black holes with zero temperature but finite entropy, a property called *extremality*; these solutions can be thus treated as solitonic systems.

In the case of supersymmetric extremal black holes solutions, the critical points of the central charge are connected to the critical points of the scalar fields in moduli space through the extremization of the black hole potential

$$\frac{\partial \phi^i}{\partial \tau}(\tau) = 0 \qquad \rightarrow \qquad dV_{BH} = 0$$

due to this property, even if the black hole has a scalar hair, its entropy does depend only on asymptotical degrees of freedom, namely, for static solutions, electric and magnetic charges determined by vector field strengths fluxes at spatial infinity.

An *attractor behaviour* was initially shown to occur for $\frac{1}{2}$-BPS extremal black holes in $\mathcal{N} = 2$, $d = 4$ ungauged Supergravity coupled to abelian vector multiplets, but it also holds in the case of $non-BPS$ extremal black holes and for both the solutions in $\mathcal{N} > 2-$extended Supergravities. In these generalizations, however, if the black holes has a regular horizon geometry and is a *large* black hole, that is it has a non-vanishing horizon area, flat directions in the Hessian matrix of the black hole potential may occur, even in the BPS case. This is due to the non-compactness of the stabilizer of the orbits of the scalar fields and is closely related to the decoupling of the hyperscalars of the $\mathcal{N} = 2$ theory from the dynamics of the black hole configuration, and, as a consequence, the moduli space of BPS attractors for $\mathcal{N} > 2$ is a quaternionic manifold, spanned by the hypermultiplets scalar degrees of freedom, as they appear in the supersymmetric reduction down to $\mathcal{N} = 2$. Even in the presence of vanishing eigenvalues of the Hessian matrix, the attractor equations, following from the above extremum condition, cancel the moduli dependence of the dynamical configuration at the black hole horizon.

The geometrical difference between $\mathcal{N} = 2$ and $\mathcal{N} > 2$ Supergravity is that the scalar manifold of the latter has to be a coset space of the form G/H (G is the U-duality group of the theory) while this is not necessarily required for the former. It is possible, however, to build an $\mathcal{N} = 2$ Supergravity with a symmetric scalar manifold, based on the quadratic series of the complex Grassmannian manifolds $SU(1,n)/U(1) \times SU(n)$, which represents the minimal coupling of n vector multiplets in the bosonic sector.

Thus, static spherically symmetric systems in $d = 4$ space-time dimensions are considered, for which the dynamic is one dimensional and allows the determination of an effective potential V depending on electromagnetic charges and scalar (moduli) fields, written in terms of dressed central (in case, also matter) charges, in $\mathcal{N} = 2$ quadratic, $\mathcal{N} = 3, 5$ extended Supergravity. All of these theories have a scalar manifold $\mathcal{M}_{scalar}$ which is a symmetric space and does not admit a $d = 5$ uplift.

The classical black hole entropy, as given by the Bekenstein-Hawking formula, in the case of $\mathcal{N} = 2$ quadratic and $\mathcal{N} = 3$ Supergravity both coupled to abelian vector multiplets in the fundamental representation of the U-duality group, is given by a quadratic expression of electric and magnetic charges in the form of the absolute value of the U-duality invariant of the scalar manifold

$$S_{BH} = \frac{A}{4} = \pi V_{BH}\big|_{\partial V_{BH}=0} = \pi|\mathcal{I}_2| \ . \tag{0.1}$$

The $\mathcal{N} = 5$ theory does not admit a quadratic invariant, since the vector fields are in the three-fold antisymmetric **20** representation of the U-duality group, which is a symplectic representation with a singlet $\mathbf{1_a}$ in the tensor product $\mathbf{20} \times \mathbf{20}$ [**71**]. Taking the tensor product of the $\mathbf{35} \times \mathbf{35}$ representation, coming from the $\mathbf{20} \times \mathbf{20}$, one find the singlet $\mathbf{1_s}$, so the invariant is quartic as expressed in terms of electric and magnetic charges; the entropy for this theory is given by

$$S_{BH} = \frac{A}{4} = \pi V_{BH}\big|_{\partial V_{BH}=0} = \pi\sqrt{|\mathcal{I}_4|} \ , \tag{0.2}$$

but in this case this formula reduces to a perfect square of a quadratic expression, once it is explicited as a function of the skew eigenvalues of the central charge matrix. The same result is valid also for $\mathcal{N} = 4$ pure Supergravity.

Due to these peculiarities it is possible, for these theories, to write an alternative expression for the Bekenstein-Hawking Entropy in terms of the *effective horizon radius* R_H, whose expression is a function of scalar charges and the geometrical radius of the horizon, r_H, following the procedures of the first order formalism. In fact, non-BPS attractor flows of extremal black holes in $d = 4$ can be described in terms of a *fake superpotential* $\mathcal{W}$ such that $\mathcal{W}(\phi_\infty, p, q) = \ r_H(\phi_\infty, p, q)$ that reduces the equations of motions for the scalars to fisrt order ones, and enters in the espression of the effective radius R_H; the importance of this description is that the latter turns out to be, for the above mentioned theories, a moduli independent quantity.

As a counterexample, $\mathcal{N} = 4$, $d = 4$ Supergravity coupled to 1 vector multiplet admits an uplift to $\mathcal{N} = 4$ pure Supergravity in $d = 5$ dimensions, but has a quartic invariant which cannot be written as a quadratic expression of the skew-eigenvalues of the central charge matrix, and the effective radius description does not hold.

The fermionic sector does not enter in the determination of the black hole configuration, but the supersymmetry transformation of gravitino is given in terms of the sections that one needs to build the symplectic embedding of the Supergravity theory under consideration.

In order to clarify the role of the symplectic structure at the basis of $\mathcal{N}$-extended supergravities, the first Chapter of the thesis is dedicated to the problem of the coupling of vector fields, invariant under duality rotations, to a theory of fermionic and bosonic fields, through Gaillard-Zumino construction [**62**], further specializing to the bosonic sector of extended supergravities.

Chapters 2 and 3 review the attractor behaviour, leading to the attractor equations, and the first order formalism, in order to provide the theoretical framework where the work of the chapters in the following will develop.

In Chapter 4, 5, and 6 black holes solutions for $\mathcal{N} = 2$ quadratic, $\mathcal{N} = 3$ matter coupled and $\mathcal{N} = 5$ Supergravities are explicitly studied and discussed, with the determination of the symplectic sections and the solution of the attractor equations; the black hole entropy as a function of the electric and magnetic charges is computed; black holes parameters, such as fake superpotential, scalar charges and effective radius are presented [**69**].

The same results for $\mathcal{N} = 4$ pure Supergravity, compared with the $\mathcal{N} = 5$ case, are added in Chapter 7 which concludes with a discussion on the dualities among bosonic sectors of different Supergravities, to explicitly show that bosonic interacting theories do not have a unique supersymmetric extension. This part also refers to [**69**].

CHAPTER 1

Duality invariant theories of gravity, vectors and fermion fields

1. The bosonic Supergravity Lagrangian

1.1. Duality invariance in Maxwell-Einstein theory. Maxwell theory of electromagnetism is a theory of an abelian gauge field $A_\mu(x)$. In the geometrical construction, it is the connection of a $U(1)$ gauge bundle over the 4-dimensional space-time manifold with metric $g_{\mu\nu}$; the action is

$$S_{EM} = \frac{1}{16\pi G}\int \sqrt{-g}\left\{R - \frac{1}{\beta}F_{\mu\nu}F^{\mu\nu}\right\}\ , \tag{1.1}$$

where β is a constant depending on the normalization of charges, leading to the equations of motion

$$\partial_\mu F^{\mu\nu} = 0\ ,$$
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}\ ,$$

where

$$^\star F = \frac{1}{2}\tilde{F}^{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}dx^\mu \wedge dx^\nu\ , \tag{1.2}$$

is the hodge dual field strength of the vector field, satisfying Bianchi identities

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad \partial_{[\mu}F_{\nu\rho]} = 0\ , \tag{1.3}$$

while the stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\alpha}F^\alpha_\nu - \frac{1}{4}g_{\mu\nu}F^2\right]\ . \tag{1.4}$$

This theory is manifestly duality invariant, in the sense that the set of equations (1.2) is unaffected by the following transformations on the vector field strength

$$F'^{\mu\nu} = (\cos\alpha + j\sin\alpha)F^{\mu\nu}\ , \qquad \alpha \in \mathbb{R}\ , \tag{1.5}$$

where the j "duality" operator is such that $jF = {}^*F$, corresponding to the following $U(1) \simeq SO(2)$ rotation of electromagnetic field

$$\begin{pmatrix} E' \\ H' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} . \tag{1.6}$$

The Lagrangian of the vector field is written in terms of invariant products of field strenghts and dual field strengths, or explicitly

$$T = \frac{1}{4\pi}\left(E^2 - H^2\right) , \tag{1.7}$$

which is an $SO(2)$ invariant expression; there's no need to require the metric to change under duality transformations, for the Einstein equations are not affected. We now notice that in this case the duality group is abelian; duality rotations, however, are not defined as transformations on the vector fields, and the Lagrangian, in further generalization of duality, won't still be invariant.

1.2. Duality invariance in a theory of vector fields. We now want to describe the generalization of duality invariance to the case of a theory of n interacting vector fields, in addition to other fields χ^i, both fermionic and bosonic. The Lagrangian is the functional

$$\mathcal{L} = \mathcal{L}(F^a, \chi^i, \chi^i_\mu) , \tag{1.8}$$

where F^a, $(a = 1, ..., n)$ are vector field strengths

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \tag{1.9}$$

and $\chi^i_\mu \equiv \partial_\mu \chi^i$. We define

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G^{a\rho\sigma} \equiv 2\frac{\partial L}{\partial F^{a\,\mu\nu}} , \tag{1.10}$$

so that for a Lagrangian as in (1.8) the equations of motions for the vector fields can be written as

$$\partial^\mu \tilde{G}^a_{\mu\nu} = 0 . \tag{1.11}$$

Bianchi identities still hold in the form

$$\partial_\mu F^{a\,\mu\nu} = 0 \tag{1.12}$$

The infinitesimal transformation that leave these equations and (1.11) invariants are the linear transformations

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} , \tag{1.13}$$
$$\delta\chi^i = \xi^i(\chi) ,$$
$$\delta(\partial_\mu \chi^i) = \partial_\mu \xi^i = \partial_\mu \chi^j \frac{\partial \xi^i}{\partial \chi^j} ,$$

where the quadratic blocks matrix is an arbitrary real $2n \times 2n$ matrix, and the functions $\xi^i(\chi)$ do not contain fields derivatives.

1.3. Variation of the Lagrangian functional. We define the duality group the one which acts linearly on the vectors of the field strengths and their duals, not affecting the dynamical equations of the theory; their covariance, indeed, put constraints on the possible duality transformations among the general linear ones. The generic variation of the Lagrangian is, from (1.8),

$$\delta\mathcal{L} = \left[\xi^i \frac{\partial \mathcal{L}}{\partial \chi^j} + \chi^j_\mu \frac{\partial}{\partial \chi^i_\mu} + (F^c A^{bc} + G^c B^{bc}) \frac{\partial}{\partial F^b} \right] \mathcal{L} , \tag{1.14}$$

if we require the covariance of the equations of motion under (1.13)

$$C = C^T , \qquad B = B^T , \qquad A = -D^T , \tag{1.15}$$

which restrict the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be an element of $Sp(2n, \mathbb{R})$. We also find the variation of the Lagrangian

$$\delta L = \frac{1}{4}(FC\tilde{F} + GB\tilde{G}) . \tag{1.16}$$

The energy momentum tensor of the vector fields is also invariant, though not necessarily symmetric.

1.4. Construction of the Lagrangian. Under a generic variation of its variables, we can write more simply $\delta\mathcal{L} = \frac{1}{4}\delta(F\tilde{G})$, so that we begin to write the functional as

$$L = \frac{1}{4} F\tilde{G} + \mathcal{L}_{inv} ,$$

where $\mathcal{L}_{inv}$ is written as a function of invariants of the duality group. But in the general case where this group is $Sp(2n, R)$ and the field strengths $\begin{pmatrix} F \\ G \end{pmatrix}$ transform as a vector in the fundamental representation, the only possible invariant function of F and G, apart from generic fields χ^i in the theory, is of the form

$$\mathcal{L}_{inv.}(F, G, \chi^i, \chi^i_\mu) = \frac{1}{4}(FI - GH) + \mathcal{L}_{inv.}(\chi^i, \chi^i_\mu) \ ,$$

where $L_{inv.}$ is an invariant functional of the χ^i fields only, so that it does not affect the equations of motion and I, H form a vector in the fundamental representation. By definition one has $\frac{\delta\mathcal{L}}{\delta F} = \frac{1}{2}\tilde{G}$, and this is actually a constraint on I and H

$$\tilde{G} - I = (F + \tilde{H})\frac{\partial\tilde{G}}{\partial F} \ . \tag{1.17}$$

the operator j introduced in the previous section, giving a field strength $T_{\mu\nu}$, satisfies

$$j\ T_{\mu\nu} = \tilde{T}_{\mu\nu} \ ,$$
$$(j)^2 = -1 \ .$$

We can write (1.17) as

$$j\,G - I = (F + jH)\frac{\partial\tilde{G}}{\partial F} \ .$$

whose general solution is

$$jG - I = -K(\chi)(F + jH) \ ;$$
$$\Downarrow$$
$$jG = I - K(\chi)(F + jH) \ .$$

Thus the effect of an infinitesimal duality transformation of $Sp(2n, R)$, (1.13), is determined by the tranformations on (F, G) and (H, I) the vectors of the fundamental representation. We find

$$\delta K(\chi) = -jC - jKBK + DK - KA \ , \tag{1.18}$$

which restricts the form of the Lagrangian to

$$\mathcal{L} = -\frac{1}{4}FKF + \frac{1}{2}F(I - jKH) + \frac{1}{4}jH(I - jKH) + \mathcal{L}_{inv.}(\chi) \ . \tag{1.19}$$

1.5. Compact Duality Rotations. Suppose $K(\chi) = 1$, then $\delta K = 0$. From (1.18) we find an other constraint on the coefficients of the transformation

$$B = -C = B^T$$
$$A = D = -A^T \ ,$$

so that now the duality group is restricted to $U(n) \in Sp(2n, R)$, the maximal compact subgroup, and rotations as in (1.20) now become compact duality trasformations. This appears more clearly if we use a complex basis in the fundamental representation, writing the vectors as

$$F^+ \equiv F + iG \ ,$$
$$F^- \equiv F - iG \ ,$$

which give (1.13) in the form

$$\delta \begin{pmatrix} F^+ \\ F^- \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix} \begin{pmatrix} F^+ \\ F^- \end{pmatrix} \ , \tag{1.20}$$

with

$$T = A - iB = -T^\dagger \ .$$

The two F^+ and F^- transform respectively under an n-dimensional $U(n)$ representation (which can be reducible in the case the actual duality group is a subgroup of $U(n)$) and its conjugate. In this notation we notice how the structure is similar to that of fermions fields of definite chirality: the two vector fields of opposite helicity transforms according to a representation and its complex conjugate, as two fermions of opposite chirality do).

We can easily generalize this notation to the non compact case, using the "duality" operator instead of the imaginary complex element i in the definitions

$$F^+ \equiv F + jG \ ,$$
$$F^- \equiv F - jG \ ,$$

in this analogy $i \to j$, and the T-matrix is just

$$Re\, T \pm i\, Im\, T \to Re\, T \pm j\ Im\ T \ .$$

Finally, if we write a complex analog of the field strength $F^{\pm}$ as $jH_{\pm} \equiv (H \pm jI)$, the Lagrangian (1.19) can be written as

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}FH_+ - \frac{1}{8}H_+^2 - \frac{1}{8}H_+H_- + \mathcal{L}_{inv.}(\chi) \ .$$

The field H_- has no dynamical meaning, since it does not appear in any of the couplings of F and for that reason all the terms containing H_- must be invariants, we see, for instance, that $\frac{1}{8}H_-H_+$ actually is, and can be reabsorbed in $\mathcal{L}_{inv.}(\chi)$. We are setting H_- to zero, reducing (H, I) to the vector $(H, -jH)$, that now has just one indipendent component. In the case of compact duality rotations, then, one simply has to introduce the tensor

$$H = \frac{1}{2} j\, H_+ \ , \tag{1.21}$$

in the same representation of F_+, to get the correct Lagrangian transformation.

Consider the invariant bilinear $FI - GH$. Taking now $H = jI$, from (1.18) it follows that $I = (F + jG)$ and one can write then

$$\frac{1}{2}(FI - GH) = \frac{1}{2}(F - jG)(F + jG) = \frac{1}{2}(F^2 + G^2) \ ,$$

which is manifestly invariant under the action of the unitary group $U(n)$ on the vector (F, G).

1.6. Non linear realizations. By now we are able to describe the theory of interacting bosonic and fermionic fields with invariance under a compact subgroup of $Sp(2n, \mathbb{R})$, but we need to generalize the description to non compact duality groups. The solution is to introduce in the theory scalar fields described by a nonlinear sigma model, taking values in the quotient space of group $\mathcal{G}$ with respect to its maximal compact subgroup $\mathcal{K}$, being the semisimple group $\mathcal{G}$ the duality group.

The quotient space has sense once the Lagrangian is invariant under the gauge transformations of the scalar fields

$$g(x) \rightarrow g(x)[k(x)]^{-1} \ , \tag{1.22}$$

The quotient defines a coset, a symmetric space. We suppose further that the rigid transformation of the moduli

$$g(x) \rightarrow g_0 g(x) \ , \tag{1.23}$$

con $g_0 \in \mathcal{G}$, is an invariance of the Lagrangian. We then need to find the constraints coming from these assumptions, in order to find its correct form.

We start studying the properties of the gauge group. Being Q_μ the connection, its transformation under (1.22) is

$$Q'_\mu = k\ Q_\mu k^{-1}\ , \tag{1.24}$$

while the covariant derivative $D_\mu g = \partial_\mu g - gQ_\mu$ changes as

$$D_\mu g \to (D_\mu g)[k(x)]^{-1}\ .$$

We notice that $g^{-1}D_\mu g$ is invariant under the global transformation (1.23); under the action of the gauge group, instead, it changes as

$$g^{-1}D_\mu g \to k(g^{-1}D_\mu g)k^{-1}\ . \tag{1.25}$$

If we write the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\ Tr\left(g^{-1}D_\mu g\right)^2\ ,$$

we then have an invariant expression.

Working in this framework, any other bosonic field, including the gravitational one, is left invariant by duality group, so that a theory including a non interacting gravitational field is allowed.

The equations of motion that follow from the variation of (1.26) with respect to the gauge fields Q_μ are

$$\delta\mathcal{L} = \mathrm{Tr}\delta Q_\mu(g^{-1}\partial_\mu g - Q_\mu) = 0\ ; \tag{1.26}$$

the trace on the elements of the duality group $\mathcal{G}$ is not degenerate and δQ_μ belongs to the Lie algebra of $\mathcal{K}$, so that (1.26) implies that $(g^{-1}\partial_\mu g - Q_\mu)$ is in the ortogonal complement of $\mathcal{K}$, and the element $g^{-1}\partial_\mu g$ is

$$g^{-1}\partial_\mu g = Q_\mu + P_\mu\ , \tag{1.27}$$

that is

$$P_\mu = g^{-1}D_\mu g\ . \tag{1.28}$$

We can then write the Lagrangian in (1.26) in a more specific form

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}P_\mu^2\ . \tag{1.29}$$

Varying this with respect to the scalar fields one finds

$$\delta\mathcal{L} = \mathrm{Tr}\,\delta g\, g^{-1}\partial_\mu(D_\mu g\, g^{-1}) = 0\,; \tag{1.30}$$

since δg is arbitrary, $\delta g\, g^{-1}$ runs on the whole Lie algebra $Lie(\mathcal{G})$, and the equations of motion for the scalar fields are

$$\begin{gathered} \partial_\mu(D_\mu g\, g^{-1}) = 0 \ , \\ \Downarrow \\ \partial_\mu(g\, P_\mu g\, g^{-1}) = 0 \ . \end{gathered} \tag{1.31}$$

If we define for P_μ a covariant derivative

$$D_\mu P_\mu \equiv \partial_\mu P_\mu - [P_\mu, Q_\mu] \ , \tag{1.32}$$

the (1.31) reveals to be

$$D_\mu P_\mu = 0 \ . \tag{1.33}$$

Consider now a rigid transformation as in (1.22),

$$g \to g\, k^{-1} \ , \tag{1.34}$$

which is one specific kind of gauge transformation (for this reason the associated current is identically zero). The rigid transformations

$$g \to k\, g \ , \tag{1.35}$$

where $k \in \mathcal{K}$, are equivalent to

$$g \to k\, g k^{-1} \ , \tag{1.36}$$

and the related current, in the algebra of the group $\mathcal{G}$ is

$$J_\mu = -g P_\mu g^{-1} \ ; \tag{1.37}$$

if one restricts g to the subgroup $\mathcal{K}$, the currents corresponding to the resulting transformations are just the correct currents of (1.35) and (1.36).

1.7. Coupling to spinor fields. We now want to consider the coupling to spinor fields that under the gauge transformation (1.22) vary as

$$\psi(x) \to k(x)\psi(x) \; ; \tag{1.38}$$

if we introduce a covariant derivative defined as

$$D_\mu \psi = \partial_\mu \psi + Q_\mu \psi \; , \tag{1.39}$$

we can build an invariant Lagrangian, including for instance Dirac terms as

$$-\frac{i}{2}\bar{\psi}\gamma^\mu(\overrightarrow{D}_\mu - \overleftarrow{D}_\mu)\psi \; , \tag{1.40}$$

but also non derivatives couplings, such as

$$\bar{\psi}_1 \gamma^\mu P_\mu \psi_2 \; , \tag{1.41}$$

where now P_μ and the spinor fields belong to a specific representation of $\mathcal{K}$. Since Q_μ and P_μ are functions of scalar fields and their derivatives, (1.40) and (1.41) give fermionic terms contribution to the conserved current, namely,

$$\begin{aligned} J'_\mu &= -i\bar{\psi}\gamma_\mu q \psi \; , \\ J''_\mu &= \bar{\psi}_1 \gamma_\mu p \psi_2 \; , \end{aligned} \tag{1.42}$$

where q and p belong to the Lie algebra of $\mathcal{K}$ and its orthogonal complement, respectively, and are defined by

$$\delta g = (q + p) g \; . \tag{1.43}$$

The group $\mathcal{G}$ being non-compact, in order not to introduce ghost fields in the theory, we have to require that the fields we are coupling are invariant under transformations (1.23). We notice that the connection Q_μ can be taken, in a sort of Palatini formalism, as an independent field, so that its equation of motion now receives contribution from the fermionic sector.

Finally, we recall that one can restrict the description of the scalar fields to a particular gauge. Indeed, an element $g \in \mathcal{G}$ can always be rewritten, using gauge invariance, as

$$g' = e^P = g k^{-1} \; , \tag{1.44}$$

where $P \in \mathrm{Lie}(\mathcal{K})^\perp$, and the scalar fields are the elements $P(x)$ parametrizing the coset space $\mathcal{G}/\mathcal{K}$. This turns out to be important for a kind of $N = 2$ and all $N > 2$-extended

Supergravities, since their scalar manifold is an homogeneous space. Transformations (1.23) are then

$$(g_0 g) = (g_0 e^P) = e^{P'} = g_0 e^P k^{-1} \ , \tag{1.45}$$

$P'(g_0, P)$ and $k(g_0 P)$ depending on the group structure.

What we have built, starting from the transformations

$$P \to P'(g_0, P) \ , \tag{1.46}$$

is a non linear realization of (1.23). Since we worked in a particular gauge, suitable transformations of the fields ψ and Q_μ are needed in order to remain in the gauge (1.44). Still, invariance under (1.23) is manifest, but, after the gauge fixing, we loose the a priori invariance under (1.22).

2. $d = 4$ Supergravity bosonic sector

2.1. Duality rotations and covariance under the action of symplectic group. The N-extended Supergravity theory in $d = 4$, has in the bosonic sector, apart from the metric field, vector and scalar fields, the latter being the coordinates of a scalar manifold, the manifold of duality transformations acting on the vector fields. The generic form of this bosonic part of the action is the one we built in the above sections, that we can write as

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{2} R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma,\ \mu\nu} + \frac{1}{2\sqrt{-g}} \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \right.$$
$$\left. + \frac{1}{2} g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \right) \ . \tag{1.47}$$

More precisely, scalar fields are described by a σ-model on the scalar manifold $\mathcal{M}_{scalar}$ whose real dimension is $m = 2 \cdot \#\ complex\ scalar\ fields$ and are coupled to the vector fields by the matrix

$$\mathcal{N}_{\Lambda\Gamma} = \mathcal{N}_{\Lambda\Gamma}(\Phi) \ . \tag{1.48}$$

$\mathcal{N}(\Phi)$ is a symmetric matrix $n_V \times n_V$, with n_V number of vector fields, depending on their representation of Gaillard Zumino Symplectic group. Different Supergravity theories thus correspond to different scalar manifolds and number of vector multiplets, and, since usually scalar fields belong to the same multiplets as vectors, the action of the vector isometry group $\mathcal{M}_{scalar}$ is deeply connected to their transformations. This results on the embedding of the isometry group in the duality group, whose explicit form relies on the specific Supergravity theory we are considering. Once we have this correspondence, we find the matrix $\mathcal{N}$ in its explicit form.

It is crucial, then, to study duality transformations in details, in the form of linear action on the (abelian) vector field strengths and their dual forms. As seen before, these transformations leave Bianchi Identities and equations of motions invariant, and generalize electromagnetic duality. In what follows we will see the Gaillard Zumino construction at work in the Supergravity framework.

2.2. Duality Rotations and symplectic covariance. We deal with a theory of vectors and scalar fields which is invariant under the action of a duality group, in $d = 4$. The gauge fields are n_V abelian fields A^Λ_μ, whose dynamic is described by the field strengths in the action (1.47). We can separately write the dual and anti-dual field strength

$$\begin{aligned} F^{\pm} &= \frac{1}{2}(F \pm i\,{}^\star F) \;, \\ {}^\star F^{\pm} &= \mp i F^{\pm} \;, \end{aligned} \tag{1.49}$$

and rewrite the vector part of the action as

$$\begin{aligned} \mathcal{L}_{vec} &= i\;\left[F^{-,T}\bar{\mathcal{N}}F^{-} - F^{+,T}\mathcal{N}F^{+}\right] = \\ &= -i\left(F^{+\;T}\;,\;F^{-\;T}\right)\begin{pmatrix}\mathcal{N} & 0\\ 0 & -\bar{\mathcal{N}}\end{pmatrix}\begin{pmatrix}F^{+}\\ F^{-}\end{pmatrix}\;. \end{aligned} \tag{1.50}$$

Following the Gaillard-Zumino construction we introduce the tensor $G^\Lambda_{\mu\nu}$ defined as

$${}^\star G^\Lambda_{\mu\nu} \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}} \;, \tag{1.51}$$

that, for the theory under examination, is

$${}^\star G_{\Lambda\,\mu\nu} = \mathrm{Im}\;\mathcal{N}_{\Lambda\Sigma}\;F^\Sigma_{\mu\nu} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,{}^\star F^\Sigma_{\mu\nu}\;. \tag{1.52}$$

The equations of motion and Bianchi identities are

$$
\begin{aligned}
\nabla^{\mu\star} F^{\Lambda}_{\mu\nu} &= 0 \ , \\
\nabla^{\mu\star} G_{\Lambda\,\mu\nu} &= 0 \ , \\
&\Downarrow \\
\nabla^{\mu} \mathrm{Im} F^{\pm\ \Lambda} &= 0 \ , \\
\nabla^{\mu} \mathrm{Im} G^{\pm}_{\Lambda\,\mu\nu} &= 0 \ ,
\end{aligned}
\tag{1.53}
$$

where we also write $G_{\Lambda\,\mu\nu}$ separating its self-dual and anti self-dual part

$$
\begin{aligned}
G^{\pm} &= \frac{1}{2}(G \pm i\,{}^{\star}G) \ , \\
{}^{\star}G^{\pm} &= \mp i G^{\pm} \ ,
\end{aligned}
\tag{1.54}
$$

whose relation on the field strength F is given by

$$
\begin{aligned}
G^{+} &= \mathcal{N} F^{+} \ , \\
G^{-} &= \bar{\mathcal{N}} F^{-} \ .
\end{aligned}
\tag{1.55}
$$

The vector part of the Lagrangian, if written in terms of F and G as in (1.51), takes the compact form

$$
\begin{aligned}
\mathcal{L}_{vec} &= i\left[F^{-\,T} G^{-} - F^{+\,T} G^{+}\right] = \\
&= -i \begin{pmatrix} F^{+\ T} \ , \ F^{-\ T} \end{pmatrix} \begin{pmatrix} G^{+} \\ G^{-} \end{pmatrix} \ .
\end{aligned}
\tag{1.56}
$$

Moreover, we introduce the $n_V + n_V$ components vector

$$
\mathbf{V} \equiv \begin{pmatrix} {}^{\star}F \\ {}^{\star}G \end{pmatrix} \ ,
$$

and we get equations of motion, from the variation of the vector fields, in the form

$$
\partial \mathbf{V} = 0 \ ; \tag{1.57}
$$

duality transformations are then simply described by

$$
\mathbf{V}' = \mathcal{S}\,\mathbf{V} \ , \tag{1.58}
$$

where

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ , \tag{1.59}$$

is a priori a matrix in $GL(2n_v, \mathbb{R})$, and, since we always required duality invariance during the construction, the equations of motion for the vector $\mathbf{V}'$ are still given by $\partial\, \mathbf{V}' = 0$.

2.3. Transformation of the kinetic matrix required by duality invariance. Since the vector fields in action (1.47) are coupled to scalars via the $\mathcal{N}(\Phi)$ matrix, a duality transformation acting on V would imply a corresponding transformation on the scalar fields (coming from the action of a diffeomorphism of $\mathcal{M}_{scalar}$), hence on $\mathcal{N}$. An homomorphism

$$\iota_\xi : \quad C^\infty(\mathcal{M}_{scalar}) \quad \rightarrow \quad GL(2n_V, \mathbb{R}) \ , \tag{1.60}$$

that maps a given diffeomorphism on $\mathcal{M}_{scalar}$ to a transformation in $GL(2n_V, \mathbb{R})$, allows us to define the following transformations

$$\xi \rightarrow \begin{cases} \mathbf{V} & \rightarrow \mathbf{V}' = \mathcal{S}_\xi \mathbf{V} \\ \Phi & \rightarrow \Phi' = \xi(\Phi) \\ \mathcal{N}(\Phi) & \rightarrow \mathcal{N}'(\xi(\Phi)) \end{cases} \ , \tag{1.61}$$

where $\mathcal{S}_\xi = \iota_\xi \mathbf{V}$. If one defines, then,

$$\mathcal{S}_\xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ , \tag{1.62}$$

the transformed Lagrangian for the bosonic sector in our Supergravity theory is

$$\mathcal{L}' = i \left[F^{-T}(A + B\bar{\mathcal{N}})^T \bar{\mathcal{N}}'(A + B\bar{\mathcal{N}})F^- - F^{+T}(A + B\bar{\mathcal{N}})^T \mathcal{N}'(A + B\bar{\mathcal{N}})F^+ \right] \tag{1.63}$$

The transformed dual tensor G' has to be consistent with its definition in (1.51), and this requirement gives the constraint needed to restrict the transformation of the $\mathcal{N}$ matrix to the form

$$N'(\xi(\Phi)) = (C + D\mathcal{N}(\Phi))(A + B\mathcal{N}(\phi))^{-1} \ ; \tag{1.64}$$

also knowing that $\mathcal{N}'$ is a symmetric matrix, we can finally identify the duality rotation matrix as being $\mathcal{S} \in Sp(2n_V, \mathbb{R})$, in perfect agreement with Gaillard Zumino construction.

In general a diffeomorphism in $C^\infty(\mathcal{M}_{scalar})$ implies a transformation of the scalar part in the Lagrangian (1.47). A duality rotation, then, does not correspond to an invariance of the action, unless we require $\mathcal{S}_\xi$ matrices corresponding, in the sense of (1.61), to isometries of the scalar manifold, unaffecting, by definition, the metric g_{rs} and thus the scalar part of the action. Eventually also the kinetic $\mathcal{N}$ matrix does not have to change. In this case we look for a homomorphism that maps

$$\iota_\xi \; : \; \mathrm{Iso}(M_{scalar}) \to Sp(2n_V, \mathbb{R}) \; ; \tag{1.65}$$

the relation (1.64) now becomes

$$N(\xi(\Phi)) = (C + D\mathcal{N}(\Phi))(A + B\mathcal{N}(\phi))^{-1} \; . \tag{1.66}$$

One could also study the case in which duality transformations are a symmetry of the Lagrangian. The vector kinetic term in the Lagrangian indeed transforms as

$$\begin{aligned} \mathrm{Im}(F'^{-\,\Lambda} G'^{\Lambda}) = \mathrm{Im}\, \big[& F^{-\,\Lambda} G^-_\Lambda + 2(C^T B)^\Sigma_\Lambda (F^{-\,\Lambda} G^-_\Sigma) + \\ & + (C^T A)_{\Lambda\Sigma} F^{-\Lambda} F^{-\Sigma} + (D^T B)^{\Lambda\Sigma} G^{-\,\Lambda} G^{-\,\Sigma} \big] \; , \end{aligned} \tag{1.67}$$

where the expression (1.56) and the definition (1.52) have been used. We see then that $B = C = 0$ transformations are symmetries of the Lagrangian, and we notice that its variation in the case $B \neq 0, C = 0$ is given by

$$(C^T A)_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \, {}^\star F^{\Sigma\,\mu\nu} \; , \tag{1.68}$$

which is a topological term; finally, it is worth pointing out that among all the $\mathcal{N}$ matrix transformations (1.66), we also find the mapping

$$\mathcal{N} \to -\frac{1}{\mathcal{N}} \; , \tag{1.69}$$

which provides the interchange of the strongly and weakly coupled sectors in the theory.

CHAPTER 2

The Black Hole Attractor Mechanism

1. Black Holes in Supergravity Spectrum

In the classical description of Einstein Maxwell theory, black holes can be considered solitons of general relativity; such interpretation, however, breaks down in a quantum interpretation, since a spontaneous particle creation would take place, by the gravitational and electromagnetic fields, responsible of black hole instability. Thermal radiation is not allowed in the case of zero-temperature black holes, for which only loss of angular momentum or charge can be responsible for their instability, thus, non-rotating systems in a theory whose elementary fields are not charged are stable. This is the case of Reissner-Nordstrom Black Holes in extended ungauged supergravities.

2. Gravity and non linear sigma model

We restrict the attention to dynamics and fields equations for the bosonic sector of Supergravity theories, that is to massless scalars and n vector fields coupled to gravity. The scalars describe a non linear σ-model over a manifold $\bar{G}/\bar{H}$, the vector fields transform accordingly to a certain representation of the global (ungauged) symmetry group G.

As we discussed before, we are interested in stationary solutions, that is to that systems allowing for a time-like Killing vector field; these can be dimensionally reduced to a 3 dimensional theory. The resulting scalar fields are the scalars of the non linear σ-model in the higher dimensional theory, two scalar fields from the reduction of gravity, and electric and magnetic potentials from each vector fields, a total of $(2n + 2)$ scalars. The two scalars from gravity alone would describe an $SL(2)/SO(2)$ σ-model, but we ask for a larger symmetry of the enlarged set of scalars, so that they together describe a σ model of G/H. To link the discussion to the case at hand, we require the original quotient space $\bar{G}/\bar{H}$ to be a non-compact coset space, and the coupling with vector fields to respect Gaillard-Zumino construction, that is, if we fix a particular gauge for gravitino, the twist potential scalar field does not enter in the final sigma model. Adding the vectors also has to lead to positive energy density T_{00}. The resulting enlarged σ-model refers to the same symmetry group G but now H is a non compact form of the maximal compact subgroup, say H', of G, thus G/H is a pseudo-Riemannian symmetric pace.

In $N > 2$ extended Supergravities a vector field is present as graviphoton in the gravitational multiplet, and one or more vectors can be added as vector multiplets, depending on the Supergravity theory under consideration. In these case, a non linear sigma model arises as a consequence of the duality invariance of the theory. In the $N = 2$ theory, instead, vectors and scalars are in the same multiplet.

2.1. Dimensional reduction to three dimensions. We derive the three dimensional effective metric in the case of static spherically symmetric black holes, for a non linear sigma model coupled to gravity. The equations of motion for the enlarged set of scalar fields are geodesics.

The 4-dimensional space time manifold Σ has metric $g_{\alpha\beta}$ and the original action for the non linear sigma model is

$$S_\phi = \int_\Sigma \sqrt{g} dx \left[-\frac{1}{2} R(x) + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \bar{\phi}^i \partial_\beta \bar{\phi}^j G_{ij} \right] \ , \tag{2.1}$$

from which we derive the equations of motion

$$R_{\alpha\beta} - \partial_\alpha \bar{\phi}^i \partial_\beta \phi^j = 0 \ , \tag{2.2}$$

$$D^\alpha \partial_\alpha \bar{\phi}(x) = 0 \ . \tag{2.3}$$

We recall that solutions to (2.3) are harmonic maps from the (pseudo) Riemannian manifold $(\Sigma \, , g_{ij})$ to $(G/H \, , G_{ij})$ At this point we consider a theory admitting everywhere a time-like Killing vector field, which is orthogonal to the reduced 3 dimensional space Σ_3 which allows $SO(3)$ symmetries, namely spherical symmetry. The metric thus decomposes as

$$g_{\alpha\beta} = \begin{pmatrix} e^{2U} & 0 \\ 0 & -e^{-2U} h_{ab} \end{pmatrix} \ , \tag{2.4}$$

and the metric on Σ_3, h_{ab}, can be parametrized in terms of a function $f(r)$ so that

$$\begin{aligned} ds^2 &= -e^{2U} dt^2 + e^{-2U} \left(dr^2 + f(r)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right) \ , \\ &\equiv -e^{2U} dt^2 + e^{-2U} h_{ab} dx^a dx^b \ . \end{aligned} \tag{2.5}$$

The effective Lagrangian for the reduced three dimensional system is

$$\frac{1}{2}\hat{R} - \frac{1}{2}\gamma^{mn}\partial_m\phi^a\partial_n\phi^b G^{ab} \tag{2.6}$$

where $c^2 = \frac{\kappa A}{4pi} = 2ST$, and G^{ab} is now the metric of the enlarged scalar manifold, so that actually

$$\phi^a = (U, \bar{\phi}^a, \psi^\Lambda, \chi_\Lambda) \ . \tag{2.7}$$

The metric, by gauge invariance of the theory, cannot depend on ψ^Λ and χ_Λ, and as a consequence electric and magnetic charges become constants of motion, precisely, from (2.6),

$$\begin{aligned} p^\Lambda &= \hat{G}^{\Lambda\Sigma}\frac{d\hat{\chi}_\Sigma}{d\tau} \ , \\ q_\Lambda &= \hat{G}_{\Lambda\Sigma}\frac{d\hat{\psi}^\Sigma}{d\tau} \ . \end{aligned} \tag{2.8}$$

The equations of motion in this case are

$$\begin{aligned} f^{-2}\frac{d}{dr}\left(f^2\frac{d\phi^i}{dr}\right) + \Gamma^i_{jk}(\phi)\frac{d\phi^j}{dr}\frac{d\phi^k}{dr} &= 0 \ , \\ R_{rr} = -2f^{-1}\frac{d^2 f}{dr^2} &= G_{ij}(\phi)\frac{d\phi^j}{dr}\frac{d\phi^k}{dr} \ , \\ \sin^2\theta R_{\phi\phi} = R_{\theta\theta} = f^{-2}\left(\frac{d}{dr}f\frac{df}{dr} - 1\right) &= 0 \ . \end{aligned} \tag{2.9}$$

From the last one we find

$$f(r)^2 = (r - r_0)^2 + \tilde{c} \ , \tag{2.10}$$

thus, if we define the harmonic function on (Σ_3, h)

$$\tau(r) \equiv -\int_r^\infty f^{-2}(s)ds \ , \tag{2.11}$$

then being

$$f^{-2}(r) = -\frac{d\tau}{dr} \ , \tag{2.12}$$

we find that the first in (2.9) is

$$-\left(\frac{d\tau}{dr}\right)^2\frac{d}{dr}\left(f^2\frac{dr}{d\tau}\frac{d}{d\tau}\phi^i\right) + \Gamma^i_{jk}(\phi)\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau}\left(\frac{d\tau}{dr}\right)^2 = 0 \ , \tag{2.13}$$

that is, the geodesic equation

$$\frac{d^2\phi(\tau)}{d\tau^2} + \Gamma^i_{jk}(\phi)\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau} = 0 \ . \tag{2.14}$$

The geodesic map ϕ satisfies the condition

$$G_{ij}(\phi)\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} = 2c^2 \ ; \tag{2.15}$$

comparing with the general solution for $f(r)$ in (2.10), we can set $\tilde{c} = -c^2$.

To write the metric in (2.5) using τ coordinate we compute, from the definition (2.11)

$$\begin{aligned}
(r - r_0)^2 - c^2 &= \frac{c^2}{\sinh^2(c\tau)} \\
&\Downarrow \\
(r - r_0)^2 &= c^2 \coth(c\tau) \ , \\
dr^2 &= \frac{c^4}{\sinh^4(c\tau)} d\tau^2 \ , \\
f^2(r(\tau)) &= \frac{c^2}{\sinh^2(c\tau)} \ ,
\end{aligned} \tag{2.16}$$

so that we find

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{c^4 d\tau^2}{\sinh^4(c\tau)} + \frac{c^2}{\sinh^2(c\tau)}(d\theta^2 + \sin^2\theta d\phi^2)\right] \ . \tag{2.17}$$

2.2. The vector sector. The coupling with the vector fields is given through the embedding of the symmetry group in the symplectic group and it allows one to determine the kinetic matrix for the vector term in the Lagrangian. As discussed in the previous section, the bosonic action is given by

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_{EH} + \mathcal{S}_{scalar} + \mathcal{S}_V = \\
&= \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}R + \frac{1}{2}G_{rs}(\Phi)D_\mu\Phi^r D^\mu\Phi^s - \frac{1}{4}F_{\alpha\beta}\left(\mu F^{\alpha\beta} - \nu\, {}^*F^{\alpha\beta}\right)\right) \ ,
\end{aligned} \tag{2.18}$$

with $\mu_{\Lambda\Sigma} = -\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$, $\nu_{\Lambda\Sigma} = -\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$ are real symmetric matrices. To write the contribution of S_V to Einstein equations we need to compute the energy-momentum tensor

$$T_V^{\mu\nu} = \frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}L_V)}{\partial g_{\mu\nu}} - \partial_\lambda\frac{\partial(\sqrt{-g}L_V)}{\partial(\partial_\lambda g_{\mu\nu})}\right] \ ; \tag{2.19}$$

since

$$^*F^{\Lambda\alpha\beta} = \frac{1}{2\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}F^{\Lambda}_{\gamma\delta} \ , \tag{2.20}$$

then

$$\frac{\partial}{\partial g_{\mu\nu}}(F^{\Lambda}_{\alpha\beta}\, {}^*F^{\Sigma\alpha\beta}) = F^{\Lambda}_{\alpha\beta}\frac{\partial\, {}^*F^{\Sigma\alpha\beta}}{\partial g_{\mu\nu}} \ , \tag{2.21}$$

and

$$\frac{\partial\, {}^*F^{\Lambda\alpha\beta}}{\partial g_{\mu\nu}} = \frac{1}{2}g^{\mu\nu}\, {}^*F^{\Lambda\alpha\beta} \ . \tag{2.22}$$

We then have

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}L_V + \sqrt{-g}\left[-\frac{1}{2}F^{\Lambda\sigma}_{\mu}\mu_{\Lambda\Sigma}F^{\Sigma}_{\nu\sigma} + \frac{1}{2}\cdot\frac{1}{4}g^{\mu\nu}F^{\Lambda}_{\alpha\beta}\nu_{\Lambda\Sigma}\, {}^*F^{\Sigma\alpha\beta}\right] \tag{2.23}$$

and finally

$$T^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F^{\Lambda}_{\alpha\beta}\mu_{\Lambda\Sigma}F^{\Sigma\,\alpha\beta} - F^{\Lambda}_{\mu\sigma}\mu_{\Lambda\Sigma}F^{\Sigma\,\sigma}_{\nu} \ . \tag{2.24}$$

Scalard fields equations are then modified by

$$\frac{\delta L_V}{\delta\phi^i} = -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}\left(\frac{\delta\mu}{\delta\phi^i}F^{\alpha\beta} - \frac{\delta\nu}{\delta\phi}\, {}^*F^{\alpha\beta} \ .\right) \tag{2.25}$$

Let's consider the dual field strength defined as in (1.52), that is

$$^*G_{\Lambda\,\mu\nu} = \mu_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu} + \nu_{\Lambda\Sigma}\, {}^*F^{\Sigma}_{\mu\nu} \ , \tag{2.26}$$

and the symplectic vector

$$\mathcal{F} = \begin{pmatrix} F \\ G \end{pmatrix} \ , \tag{2.27}$$

we can write (2.25) and (2.24) in a manifestly symplectic way introducing the matrix

$$\mathcal{M} = \begin{pmatrix} \mu + \nu\mu^{-1}\nu & \nu\mu^{-1} \\ \mu^{-1}\nu & \mu^{-1} \end{pmatrix} \ , \tag{2.28}$$

so that (2.25) and (2.24) become

$$T^{\mu\nu} = -\frac{1}{2}\mathcal{F}^{\Lambda}_{\mu\gamma}\mathcal{M}_{\Lambda\Sigma}\mathcal{F}^{\Sigma\,\gamma}_{\nu} \ , \tag{2.29}$$

and

$$\frac{\delta L_V}{\delta\phi^i} = -\frac{1}{8}\sqrt{-g}\mathcal{F}^{\Lambda}_{\mu\nu}\frac{\delta\mathcal{M}_{\Lambda\Sigma}}{\delta\phi^i}F^{\Sigma\mu\nu} \ , \tag{2.30}$$

For time-independent solutions that preserve spherical symmetry, the 4-dimensional system reduces to a one-dimensional effective theory, described, with the assumption of the metric as in (2.17) theory, by integrating over $\mathbb{R}_t \times S^2$ and discarding constant integration factors at (spatial) infinity. Due to the integration, only conserved electromagnetic charges defined in (2.8) appear in the Lagrangian and in the equations of motion. The integrated expression for the equations (2.29) and (2.30) can be written in terms of an effective black hole potential

$$V_{BH} = \frac{1}{2} Q^{T\,\Lambda} \mathcal{M}_{\Lambda\Sigma} Q^{\Sigma} \ , \tag{2.31}$$

$$Q^{\Lambda} = \begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix} \ , \tag{2.32}$$

is the vector of the charges

$$\begin{aligned} p^{\Lambda} &= \frac{1}{4\pi} \int_{S^2} F^{\Lambda} \ , \\ q_{\Lambda} &= \frac{1}{4\pi} \int_{S^2} G_{\Lambda} \ , \end{aligned} \tag{2.33}$$

The resulting effective action is given by integrating over the remaining radial coordinate $S = \int d\tau \mathcal{L}$ the Lagrangian [**4**]

$$\mathcal{L} = \left(\frac{dU}{d\tau} \right)^2 + G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau} + \mathrm{e}^{2U} V_{BH} - c^2 \ . \tag{2.34}$$

This holds quite general for any 4-dimensional gravity theory. The explicit form of the effective potential actually select the theory under consideration. The dynamics is though constrained, as discussed before, and in this coordinates the Hamiltonian constraint takes the form

$$\left(\frac{dU}{d\tau} \right)^2 + G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau} - \mathrm{e}^{2U} V_{BH} = c^2 \ , \tag{2.35}$$

Black holes are solutions to the equations of motion derived from the lagrangian (2.34)

$$\frac{d^2 U}{d\tau^2} = 2\mathrm{e}^{2U} V_{BH}(\phi, p, q), \tag{2.36}$$

$$\frac{D\phi^a}{D\tau^2} = \mathrm{e}^{2U} \frac{\partial V_{BH}}{\partial \phi^a}, \tag{2.37}$$

and constrained by (2.35); $c^2 = 2ST$ [**4**] where S is the entropy and T the temperature of the black hole. Extremal black holes have zero temperature and can now equivalently be characterized by $c = 0$.

2.3. Near horizon dynamics. The metric of the static spherically symmetric system can be described with coordinates

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2\right] , \tag{2.38}$$

where the horizon is located at negative infinity in terms of the coordinate τ. If it has a finite area then the term e^{-2U} has to behave as

$$e^{-2U} \to \left(\frac{A}{4\pi}\right)\tau^2 , \qquad as \;\; \tau \to -\infty . \tag{2.39}$$

The scalar term in the Lagrangian remains finite near the horizon if

$$G_{ij}\partial_m\phi^i\partial_n\phi^j\gamma^{mn} < \infty , \tag{2.40}$$

that is, in our coordinates,

$$G_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}e^{2U}\tau^4 < \infty . \tag{2.41}$$

The near horizon behaviour is then given by

$$G_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}(\frac{4\pi}{A})\tau^2 \to X^2 , \qquad \text{as} \;\; \tau \to -\infty , \tag{2.42}$$

that gives the condition, substituting in the constraint (2.35) in the extremal case $c = 0$, near the horizon,

$$A \leq 4\pi V_{BH}(p, q, \phi_H) , \tag{2.43}$$

and the metric is

$$ds^2 \approx -\frac{4\pi}{A\tau^2}dt^2 + \left(\frac{A}{4\pi}\right)\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2\right] . \tag{2.44}$$

The $AdS_2 \times S^2$ horizon geometry of the extremal black hole appears explicitly once the metric is written in terms of the coordinate

$$\omega = \log\rho , \qquad \rho = -\frac{1}{\tau} , \tag{2.45}$$

since the metric becomes

$$ds^2 \approx -\frac{4\pi}{A}e^{2\omega}dt^2 + \left(\frac{A}{4\pi}\right)d\omega^2 + \left(\frac{A}{4\pi}\right)(d\theta^2 + \sin^2\theta d\phi^2) . \tag{2.46}$$

The condition (2.42) becomes

$$G_{ij}\frac{d\phi^i}{d\omega}\frac{d\phi^j}{d\omega}(\frac{4\pi}{A}) \to X^2 \ , \qquad \text{as} \ \ \omega \to \infty \ ; \tag{2.47}$$

the only allowed value of X^2 is then $X^2 = 0$, in order for the moduli dynamic to be regular at the horizon, since a non-zero constant value of $\frac{d\phi^a}{d\omega}$

$$\frac{d\phi^a}{d\omega} = const. \qquad \text{as} \ \ \omega \to \infty \ , \tag{2.48}$$

would give a linear dependence on ω that would prevent regular moduli dynamics at the horizon. The only possibility is then

$$\frac{d\phi^a}{d\omega} = 0 \ , \tag{2.49}$$

so that the constraint (2.35) in the extremal case now strictly requires

$$\frac{A}{4\pi} = V_{BH}(p, q, \phi_H) \ . \tag{2.50}$$

In the case of constant scalar fields the black hole is double-extremal, its area is still given by V_{BH}, following immediately from (2.35), and it is equal to the area of an extremal black hole with the same electric and magnetic charges

$$A_{extr}(p, q) = A_{double-extr}(p, q) = 4\pi V_{BH}(p, q, \phi_\infty) \ . \tag{2.51}$$

The behaviour of the scalars near the horizon, taking into account that $\frac{d\phi^a}{d\omega} = 0$, follows from the equation of motion (2.37) for which

$$\frac{d^2\phi^a}{d\tau^2} \to \frac{1}{2}\frac{\partial V_{BH}}{\partial \phi^a}\left(\frac{4\pi}{A\tau^2}\right) \ , \tag{2.52}$$

whose solution, recalling that a linear dependence on τ coordinates would give a singular dilaton field at the horizon, is

$$\phi^a \approx \phi^a_H + \left(\frac{2\pi}{A}\right)\frac{\partial V_{BH}}{\partial \phi^a}\log\tau \ . \tag{2.53}$$

The regularity requirement now gives the following extremum condition on the potential

$$\left(\frac{\partial V_{BH}}{\partial \phi^a}\right)_{hor} = 0 \ . \tag{2.54}$$

In this picture the black hole is a solution corresponding to dynamical trajectories in the moduli space $\mathcal{M}_\phi$ from the asymptotic point ϕ_∞ to the critical point ϕ_h. Double extremal black holes correspond to trivial trajectories, while scalars running between two different critical points do not correspond to asymptotically flat solutions.

CHAPTER 3

First Order Formalism

The properties of black holes in Supergravity theories depend on the values ϕ_∞ of the massless scalar fields parametrizing the different vacua of the theory. The entropy of the black hole, $S = \frac{A}{4}$, however, in order to be consistent with the microstate counting interpretation in string theory, has to be independent, in the extreme case, of the particular ground state being determined only by the conserved elecric and magnetic charges (dyonic black hole).

1. Scalar charges and Black Hole asymptotic moduli dependence

The expansion of the scalar fields at spatial infinity

$$\phi^a = \phi^a_\infty + \frac{\Sigma^a}{r} + O(\frac{1}{r^2}) \ , \tag{3.1}$$

defines the scalar charges $\Sigma^a = \Sigma^a(A, q_\Lambda, p^\Lambda, \phi^a_\infty)$. In the presence of scalar fields, the first law of thermodynamics for a static dyonic black hole has to be replaced by

$$dM = TdA + \psi^\Lambda dq_\Lambda + \chi_\Lambda dp^\Lambda + (\frac{\partial M}{\partial \phi^a})d\phi^a \ , \tag{3.2}$$

where the black hole temperature is $T = \frac{\kappa}{2\pi}$, and ψ^Λ, χ_Λ are electric and magnetic scalar potentials, respectively.

The potential $V(\phi, p, q)$ defines a symmetric tensor that satisfies the convexity condition

$$V_{ab} \equiv \nabla_a \nabla_b V \geq 0 \ , \tag{3.3}$$

on the scalar manifold M_ϕ. Moreover, if V_{ab} is strictly positive and the sclar charges vanish, the scalar fields have to be frozen to $\phi^a(\tau) = \phi^a_\infty$.

The mass of the black hole, by comparison with the asymptotic Gravitational potential, is given by

$$M = \left(\frac{dU}{d\tau}\right)_{\tau=0} \tag{3.4}$$

and this substitution in the constraint (2.35) evaluated at spatial infinity ($\tau = 0$) leads to

$$M^2 + G_{ab}(\phi_\infty)\Sigma^a\Sigma^b - V(\phi_\infty, p, q) = 4S^2T^2 \ . \tag{3.5}$$

The scond term on the left is the contribution

$$\left(\frac{\partial M}{\partial \phi^a}\right) = -G_{ab}(\phi_\infty)\Sigma^b \tag{3.6}$$

in expression (3.2). The right hand side is related to the black hole configuration described by the metric (2.17) by

$$c = 2ST \ . \tag{3.7}$$

Differentiating with respect to the moduli at infinity gives the dependence of the system on the moduli space

$$M\frac{\partial M}{\partial \phi^c_\infty} + G_{ab}(\phi_\infty)\Sigma^a\nabla_c\Sigma^b - \frac{1}{2}\frac{\partial V}{\phi^c_\infty} = 2c\left(\frac{\partial ST}{\partial \phi^c_\infty}\right) \ . \tag{3.8}$$

For extremal black holes, the attractor mechanism fixes the moduli at the horizon in terms of electric and magnetic charges

$$\phi_{H,extr} = \phi_{fix}(p,q) \ , \tag{3.9}$$

and the extreme point can be found, for a given charge configuration, as

$$\left.\frac{\partial M_{extr}}{\partial \phi}\right|_{\phi=\phi_{extr}} = 0 \ . \tag{3.10}$$

From (3.6), the above condition is equivalent to

$$\Sigma^a(\phi_{fix}, p, q) = 0 \ , \tag{3.11}$$

thus defining

$$\phi_{fix} = \phi_{fix}(p,q) \ , \tag{3.12}$$

and, as stated in the previous section, this identify double extremal black holes, with constant moduli fields throughout the radial trajectory

$$\phi(r) = \phi_{H,extr} = \phi_\infty \ , \tag{3.13}$$

so that the horizon configuration is given by the asymptotic moduli ϕ_∞. A black hole with frozen moduli reduces, in this treatment, to the Reissner-Nørdstrom black hole, with both electric and magnetic charges. In particular, the entropy of the extremal black hole is independent on ϕ_∞, being

$$S = \frac{A}{4} = \pi V_{BH}(\phi_{fix}, p, q) \ . \tag{3.14}$$

The scalar charge is not conserved, the flux of the gradient vanishes at the horizon, and it reveals that it resides entirely outside the horizon. Equivalently moduli at infinity or the scalar charges have to be added to the mass M, the charges (q, p) and, in the non static case, to the angular momentum J to completely characterize the black hole solution.

2. First order *fake Supergravity* formalism for $d = 4$ Extremal Black Holes

In the context of $d = 4$ static, spherically symmetric, black holes, with asymptotically flat dyonic *extremal* $(c = 0)$ configuration, For $d = 4$ supergravities a general formula for a black hole effective potential holds,

$$V_{BH} = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I, \tag{3.15}$$

where $Z_{AB} = Z_{[AB]}$ $(A, B = 1, ..., \mathcal{N})$ is the *central charge matrix*, and Z_I $(I = 1, ..., n)$ are the *matter charges*, where $n \in \mathbb{N}$ is the number of matter multiplets coupled to the gravity multiplet in the considered Supergravity theory. Equivalently, in the *first order formalism* (see Eq. (23) of [**39**]):

$$V_{BH} = \mathcal{W}^2 + 4G^{i\bar{j}} \left(\partial_i \mathcal{W}\right) \overline{\partial}_{\bar{j}} \mathcal{W} = \mathcal{W}^2 + 4G^{i\bar{j}} \left(\nabla_i \mathcal{W}\right) \overline{\nabla}_{\bar{j}} \mathcal{W}, \tag{3.16}$$

where $\mathcal{W}$ is the moduli-dependent so-called *first order fake superpotential*, and ∇ denotes the relevant covariant differential operator.

In fact, the secon order equations of motion (2.36) and (2.37) can be derived by a first order system, by performing the Ansatz

$$\dot{U} = e^U \mathcal{W}(\phi, \tau) \ , \tag{3.17}$$

where $\dot{U} = \frac{dU}{d\tau}$. The extremal solution corresponds to

$$\partial_\tau \mathcal{W} = 0 \ , \tag{3.18}$$

in this case the fake superpotential loose the dependence on the radial coordinate.

2.1. Extremal solutions. Differentiating equation (3.17) with respect to τ gives the equation of motion for the field $U(\tau)$ and the identification of

$$V_{BH} = \mathcal{W}^2 + e^{-U}\dot{\phi}^a\partial_a\mathcal{W} \; . \tag{3.19}$$

It follows from the constraint (2.35) that

$$\ddot{U} - \dot{U}^2 = \frac{1}{2}\dot{\phi}^a\dot{\phi}^b G_{ab} = \dot{\phi}^a\partial_a\mathcal{W}e^U \; , \tag{3.20}$$

which, disregarding contributions that do not affect the entropy, is solved by

$$\dot{\phi}^a = 2e^U g^{rs}\partial_s\mathcal{W} \; , \tag{3.21}$$

where the last equation is a first order type BPS-like condition. The effective potential becomes, as stated above,

$$V_{BH} = \mathcal{W}^2 + 2G^{ab}\partial_a\mathcal{W}\partial_b\mathcal{W} \; . \tag{3.22}$$

Extremization of V_{BH} corresponds to

$$\partial_a V_{BH} = 2\partial_b\mathcal{W}(\mathcal{W}\delta_a^b + 2G^{bc}\nabla_a\partial_c\mathcal{W}) = 0 \; , \tag{3.23}$$

which means that in the first order formalism the attractor point for scalar fields at the horizon of extremal black holes is directly related to the extrema of $\mathcal{W}$. From the first order equations and the spatial asymptotic configuration, defined by the expansion

$$-\frac{1}{\tau}e^{-U} \rightarrow -\frac{1}{\tau} + M_{ADM} + \mathcal{O}(\tau) \; , \qquad \tau \rightarrow 0^- \; , \tag{3.24}$$

assuming regularity conditions on functions of moduli, so that we can perform the radial asymptotical ($\tau \rightarrow 0^-$) and near horizon ($\tau \rightarrow -\infty$) limits. The *covariant scalar charges* and the *squared ADM mass* [**58**] can be written as

$$\Sigma_i = 2\lim_{\tau\rightarrow 0^-}\nabla_i\mathcal{W} = 2\lim_{\tau\rightarrow 0^-}\partial_i\mathcal{W} \; ; \tag{3.25}$$

$$M_{ADM}^2 = r_H^2 = \lim_{\tau\rightarrow 0^-}\left[V_{BH} - 4G^{i\bar{j}}\left(\partial_i\mathcal{W}\right)\overline{\partial}_{\bar{j}}\mathcal{W}\right] = \lim_{\tau\rightarrow 0^-}\mathcal{W}^2, \tag{3.26}$$

where $\tau \equiv (r_H - r)^{-1}$. One can introduce then an *effective horizon radius* (in the extremal case we are discussing $R_{+,c=0} = R_{-,c=0} \equiv R_H$), defined as

$$\begin{aligned} R_H^2 &\equiv \lim_{\tau\rightarrow -\infty} V_{BH} = V_{BH}|_{\partial V_{BH}=0, V_{BH}\neq 0} = \lim_{\tau\rightarrow -\infty}\mathcal{W}^2 = \mathcal{W}^2\big|_{\partial\mathcal{W}=0,\mathcal{W}\neq 0} = \\ &= \frac{A_{eff}(p,q)}{4\pi} = \frac{S_{BH}(p,q)}{\pi}, \end{aligned} \tag{3.27}$$

where (p, q) denotes the set of magnetic and electric BH charges, A_{eff} (simply named A in the Introduction) is the *effective area* of the BH (*i.e.* the area of the surface pertaining to R_H), S_{BH} is the classical BH entropy, and the Bekenstein-Hawking entropy-area formula has been used.

Total derivative with respect to the radial parameter for the potential $\mathcal{W}$ gives

$$\frac{d\mathcal{W}}{d\tau} = 2G^{ab}\partial_a\mathcal{W}\partial_b\mathcal{W}e^U \geq 0 \ , \tag{3.28}$$

revealing $\mathcal{W}$ as a monotonic function.

Due to the symmetric nature of the scalar manifold in the Supergravity theories under consideration, R_H^2 can be expressed in terms of a suitable power of the invariant of the relevant representation of the U-duality group G, determining the symplectic embedding of the vector field strengths. In $d = 4$ S_{BH} is homogeneous of degree two in (p, q), and only two possibilities arise:

$$R_H^2 = \left|\mathcal{I}_2\left(p,q\right)\right| , \ \ or \ R_H^2 = \sqrt{\left|\mathcal{I}_4\left(p,q\right)\right|}, \tag{3.29}$$

where $\mathcal{I}_2$ and $\mathcal{I}_4$ respectively denote U-invariants *quadratic* and *quartic* in BH charges.

Total derivative with respect to the radial parameter for the potential $\mathcal{W}$ gives

$$\frac{d\mathcal{W}}{d\tau}(\phi, p, q) = 2G^{ab}\partial_a\mathcal{W}\partial_b\mathcal{W}e^U \geq 0 \ , \tag{3.30}$$

revealing that $\mathcal{W}$ is a monotonic function (a c-function for extremal black holes).

In the extremal case $c = 0$ the monotonicity of $\mathcal{W}$ implies following inequality

$$\begin{aligned} M_{ADM}^2\left(z_\infty, \overline{z}_\infty, p, q\right) &= \lim_{\tau\to 0^-}\left[V_{BH} - 4G^{i\overline{j}}\left(\partial_i\mathcal{W}\right)\overline{\partial}_{\overline{j}}\mathcal{W}\right] = \\ &= \lim_{\tau\to 0^-}\mathcal{W}^2 \equiv r_H^2\left(z_\infty, \overline{z}_\infty, p, q\right) \\ &\geqslant R_H^2\left(p,q\right) = \lim_{\tau\to -\infty}\mathcal{W}^2 = \lim_{\tau\to -\infty}V_{BH}, \end{aligned} \tag{3.31}$$

where the radius r_H of the BH event horizon was introduced, so that the relevant relation becomes

$$r_H^2\left(z_\infty, \overline{z}_\infty, p, q\right) \geqslant R_H^2\left(p,q\right), \ \forall \left(z_\infty, \overline{z}_\infty\right) \in \mathcal{M}_\infty, \tag{3.32}$$

holding in the whole asymptotical scalar manifold $\mathcal{M}_\infty$.

2.2. Effective radius in the $\mathcal{N}$-extended studied Supergravity theories. In the minimally matter coupled $\mathcal{N}=2$, $d=4$ Supergravity, as well as in $\mathcal{N}=3$, *pure* $\mathcal{N}=4$ and $\mathcal{N}=5$, $d=4$ supergravity, it is possible to specialize further the inequality (3.32).

The formula for the entropy of the extremal black holes in these cases is

$$S_{BH}=\frac{A}{4}=\pi\ V_{BH}|_{\partial V_{BH}=0}\equiv\pi R_H^2\left(p,q\right)=$$
$$=\pi\left[r_H^2\left(\varphi_\infty,p,q\right)-\frac{1}{2}G_{a\bar{b}}\left(\varphi_\infty\right)\Sigma^a\left(\varphi_\infty,p,q\right)\bar{\Sigma}^{\bar{b}}\left(\varphi_\infty,p,q\right)\right]=\begin{cases}\pi\sqrt{|\mathcal{I}_4|}\\ or\\ \pi\left|\mathcal{I}_2\right|\end{cases},\quad(3.33)$$

where r_H is the radius of the unique (event) horizon of the extremal BH, Σ^a denotes the set of *scalar charges* asymptotically associated to the scalar field φ^a and defined in (3.25), and G_{ab} is the covariant metric tensor of the scalar manifold. The crucial feature of the considered theories, expressed by Eq. (3.35) is thendisappearance of the dependence on the asymptotical moduli $(z_\infty,\overline{z}_\infty)$ in the combination of quantities separately depending on moduli, as

$$r_H^2-G_{i\bar{j}}\Sigma^i\overline{\Sigma}^{\bar{j}}\ . \tag{3.34}$$

as it can be seen from the second line of (3.33), which is a *moduli-independent* combination of *moduli-dependent* quantities, thus revealing the moduli independent nature of the effective radius for these theories. $\mathcal{N}=2$ quadratic, $\mathcal{N}=3$ and $\mathcal{N}=4$-pure Supergravities have complex scalar manifold, and the effective radius is

$$R_H^2\left(p,q\right)\equiv\frac{S_{BH}\left(p,q\right)}{\pi}=r_H^2\left(z_\infty,\overline{z}_\infty,p,q\right)-G_{i\bar{j}}\Sigma^i\overline{\Sigma}^{\bar{j}}=$$
$$=r_H^2\left(z_\infty,\overline{z}_\infty,p,q\right)-4\lim_{\tau\to0^-}G^{i\bar{j}}\left(\partial_i\mathcal{W}\right)\overline{\partial}_{\bar{j}}\mathcal{W}, \tag{3.35}$$

clearly yielding the inequality (3.32) by the presence of non-vanishing scalar charges and the positive definiteness of $G_{i\bar{j}}$.

Equation (3.35) is nothing but a many-moduli generalization of the formula holding (also in the *non-extremal case*) for the (axion-)dilaton BH [**54**] in the Maxwell-axion-dilaton supergravity (see *e.g.* [**54, ?**], and also [**46**]), in [**46**] Eq. (3.35) was proved to hold in the *extremal case* for the whole sequence of $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to Abelian vector multiplets [**52**], in terms of the invariant $\mathcal{I}_2$ of the U-duality group $G = SU(1,n)$, which is *quadratic* in charges:

$$R_H^2(p,q) = r_H^2(z_\infty, \overline{z}_\infty, p, q) - 4 \lim_{\tau \to 0^-} G^{i\overline{j}}(\partial_i \mathcal{W}) \overline{\partial}_{\overline{j}} \mathcal{W} = |\mathcal{I}_2(p,q)|. \tag{3.36}$$

These results will be reported in Subsections 6.1 and 6.2.

By exploiting thefirst order formalism for $d = 4$ extremal BHs, it can be proved that the same happens for $\mathcal{N} = 3$ matter coupled Supergravity, as intuitively expected by the strict similarity with the minimally coupled $\mathcal{N} = 2$ theory, $\mathcal{N} = 5$ [**51**] and pure $\mathcal{N} = 4$ Supergravities [**50**], with $|\mathcal{I}_2|$ replaced by $\sqrt{|\mathcal{I}_4|}$.

While $\mathcal{N} = 5$ theory cannot be coupled to matter, in the case $\mathcal{N} = 4$ *matter coupling* is allowed, but (3.35) holds only in $\mathcal{N} = 4$ pure supergravity.

CHAPTER 4

$\mathcal{N} = 2$ Supergravity black holes

We explicit the black hole attractor equations for Supergravity theories in which the black hole entropy is given by an invariant of the scalar manifold that can be written as a quadratic expression as function of the electric and magnetic charges or the skew-eigenvalues of the central charge function, once the scalar fields satisfy the attractor condition. The first theory we consider is the $\mathcal{N} = 2$ supergravity theory with symmetric scalar manifold given by the projective space $\mathbf{C}\mathrm{P}(\mathrm{n})$.

Since in $\mathcal{N} = 1, 2$ Supergravity scalar fields are not part of the gravity multiplet, they can be introduced coupling the theory to additional multiplets, such as chiral multiplets, in the case $\mathcal{N} = 1$, and vector multiplets or hypermultiplets in the case $N = 2$.

In $\mathcal{N} = 1$ supergravity the kinetic term for the scalars is

$$e^{-1}\mathcal{L} = G_{I\bar{J}}(Z^I, \bar{Z}^I)\partial_\mu Z^I \partial_{nu} \bar{Z}^J g_{\mu\nu} \ , \tag{4.1}$$

where $G_{I\bar{J}}$ is the metric of the scalar manifold that is necessarily a Kähler manifold, that is there exists a scalar holomorphic "Kähler potential" from which the metric is derived

$$G_{I\bar{J}} = \partial_I \partial_{\bar{J}} \mathcal{G}(Z^I, \bar{Z}^I) \ . \tag{4.2}$$

In $\mathcal{N} = 2$ Supergravity, apart from the scalar fields in the hypermultiplet, which span a quaternionic manifold, the scalar manifold of the vector multiplet has again a Kähler structure, but of the special kind. That is we can define homogeneous coordinates

$$Z^I \equiv \frac{X^I}{X^0} \ , \tag{4.3}$$

and the scalar Kähler potential

$$\begin{aligned} \mathcal{G}(\phi^a, \bar{\phi}^a) &= \log \phi^i K_{ij}(\phi^a, \bar{\phi}^a)\bar{\phi}^j \ , \\ K_{ij}(\phi^a, \bar{\phi}^a) &= \tfrac{1}{4}\partial_i \partial_j F(X^i) + h.c. \ , \end{aligned} \tag{4.4}$$

where $F(X^I)$ is a omogeneous holomorphic function of degree 2, so that the holomorphic sections are simply

$$\begin{aligned} X^I &= (1, Z^1, ..., Z^n) \ , \\ F^I &= \partial_I F(X) \ . \end{aligned} \tag{4.5}$$

The geometry in this case is restricted in the sense that, being $F(X) = (X^0)^2 f(Z)$, the manifold is determined once we choose the holomorphic function $f(Z)$, instead of having an arbitrary real function $\mathcal{G}(Z, \bar{Z})$. In fact, from (4.4), the curvature tensor satisfies

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{j}}G_{k\bar{l}} + G_{i\bar{l}}G_{k\bar{j}} - C_{ikm}\bar{C}_{\bar{j}\bar{l}\bar{m}}G^{m\bar{m}} \ , \tag{4.6}$$

where C_{ijk} is a completely symmetric covariantly holomorphic tensor

$$D_{\bar{i}}C_{jkl} = 0 \ , \tag{4.7}$$

defining the covariant derivative of the holomorphic sections V over the symplectic bundle.

We do not deal with hypermultiplets. The reason is that the fermionic gravitino, gaugino and hyperino fields respectively trasform, under supersymmetry variations with chiral and antichiral parameters ε_A and ε^A as [**68**]

$$\begin{cases} \delta\psi_{A\mu} = D_\mu \varepsilon_A + \epsilon_{AB}\varepsilon^B T^-_{\mu\nu}\gamma^\nu \ , \\ \delta\lambda^{iA} = i\varepsilon^A \gamma_\mu \partial_\mu z^i + \epsilon^{AB}\varepsilon_B \mathcal{F}^{i-}_{\mu\nu}\gamma^{\mu\nu} \ , \\ \delta\zeta_\alpha = i\epsilon_{AB}\varepsilon^A \mathcal{U}^{B\beta}_u \gamma^\mu \mathcal{C}_{\alpha\beta}\partial_\mu q^u \ . \end{cases}$$

The hyperinos, then, transform indipendently of the vector fields, whereas the gaugino's transformations depend on the vector fields. This means that the hyperscalars do not contribute to the dynamics of the other fields, in particular the flow of the scalars z^i is indipendent of them, and the attractor behaviour of the black hole horizon as well. Moreover, the hyperinos transformation does not put constraints on their asymptotic configurations.

1. N=2 Supersymmetric Black Holes with Symmetric scalar manifolds

We study $N = 2$ Supergravity coupled to n vector fields. Their kinetic term is defined by the geometry of the scalar manifold having the scalar fields as coordinate maps which is

$$\mathcal{M} = \frac{SU(1,n)}{SU(n) \times U(1)} \ . \tag{4.8}$$

This manifold is the quotient space of a non-compact group with respect to its maximal compact subgroup, it is then a symmetric space. The $n+1$ vector field strengths and their duals sit in the fundamental $\mathbf{n+1}$ representation of the U-duality group[1] $SU(1,n)$ embedded, as discussed in Chapter 1, in the symplectic group $Sp(2n+2,\mathbb{R})$

1.1. The scalar manifold is a Kähler manifold. In the case of $SU(1,n)/SU(n)\times U(1)$, the Kähler structure is defined by the prepotential

$$F(X) = -\frac{i}{2}(X^{0\,2} - X^{\,2}) \ . \tag{4.9}$$

Due to the projective geometry of the scalar manifold $\mathcal{M}_n = C\mathbb{P}^n$, we can deal with *special coordinates*

$$z^i = \frac{X^i}{X^0} \ , \tag{4.10}$$

and write the fields $X^\Lambda = (1, z^1, ..., z^n)$, and the prepotential

$$F(X) = -\frac{i}{2}\left(1 - \sum_i (z^i)^2\right) \ . \tag{4.11}$$

The holomorphic sections are

$$\begin{aligned} (X^\Lambda, F_\Lambda) &= e^{-K/2}(f^\Lambda, h_\Lambda) \ , \\ \partial_{\bar{i}}(X^\Lambda, F_\Lambda) &= 0 \ , \end{aligned} \tag{4.12}$$

and their dependence on special coordinates is given by

$$\begin{aligned} X^\Lambda &= (1, z^1, ..., z^n) \ , \\ F_\Lambda \equiv \partial_\Lambda F &= (-i, iz^1, ..., iz^n) \ . \end{aligned} \tag{4.13}$$

2. Attractor equations

The black hole potential at the attractor point is given by one of the quadratic invariants of the scalar manifold [**48**],

$$V_{BH} = I_1 = |Z|^2 + |D_i Z|^2 \ , \tag{4.14}$$

[1]Throughout the analysis the semiclassical limit of large, then continuous electric and magnetic charges is considered.

where D is the covariant derivative the condition for the horizon to be an attractor point coincides with the requirement that it is a critical point V_{BH}, namely that

$$\partial_i V_{BH}|_h = 0 \ , \tag{4.15}$$

which gives a constraint on the central charge and its covariant derivatives, since, from (4.14), we have

$$\begin{aligned} \partial_i V_{BH} &= \partial_i(|Z|^2 + |D_i Z|^2) = \\ &= \partial_i(Z\bar{Z}) + \partial_i(G^{k\bar{l}} D_k Z D_{\bar{l}} \bar{Z}) = \\ &= \bar{Z} D_i Z + G^{k\bar{l}} D_i(D_k Z D_{\bar{l}} \bar{Z}) = \\ &= \bar{Z} D_i Z + G^{k\bar{l}}(D_i D_k Z D_{\bar{l}}) + G^{k\bar{l}} D_k Z D_i D_{\bar{l}} \bar{Z} \ . \end{aligned} \tag{4.16}$$

The special geometry of the scalar manifold gives the equations, which hold for a symplectic section V,

$$\begin{aligned} D_i D_k V &= i C_{ijk} G^{k\bar{k}} D_{\bar{k}} \bar{V} \ , \\ D_i D_{\bar{k}} \bar{V} &= G_{i\bar{k}} \bar{V} \ , \\ D_i \bar{Z} &= 0 \ , \end{aligned} \tag{4.17}$$

where C_{ijk} is a completely symmetric tensor depending on the Kähler space of the theory we are studying. In particular, for the series of spaces of the form $SU(1,n)/SU(n)\times U(1)$, $C_{ijk} \equiv 0$. The central charges are linear functions of $(X^\Lambda \, , F_\Lambda)$ so that we can apply the above equations, obtaining

$$\begin{aligned} D_i D_k Z &= 0 \ , \\ D_i \bar{Z} &= 0 \ , \end{aligned} \tag{4.18}$$

giving

$$\begin{aligned} \partial_i V_{BH} &= \bar{Z} D_i Z + G^{k\bar{l}} D_k Z G_{i\bar{l}} \bar{Z} = \\ &= 2\bar{Z} D_i Z \ . \end{aligned} \tag{4.19}$$

The extremum condition is satisfied whenever at the horizon

- $D_i Z = 0 \ , \quad Z \neq 0 \ , \quad$ BPS ;
- $D_i Z \neq 0 \ , \quad Z = 0 \ , \quad$ non-BPS ;

which refer to a supersymmetric and non-supersymmetric black hole solution, respectively.

3. BPS Black hole

In this case, attractor equations allow us to write electric and magnetic charges , defined by the integral of field strengths fluxes on a sphere at infinity, as functions of Z and holomorphic sections as

$$\begin{cases} p^\Lambda = ie^{K/2}(\bar{Z}X^\Lambda - Z\bar{X}^\Lambda) \\ q_\Lambda = ie^{K/2}(\bar{Z}F_\Lambda - Z\bar{F}_\Lambda) \end{cases} .$$

Summing the two equations we get

$$X^\Lambda q_\Sigma - p^\Lambda F_\Sigma = ie^{K/2}Z(\bar{X}^\Lambda F_\Sigma - X^\Lambda \bar{F}^\Sigma) \ , \tag{4.20}$$

in which coordinates z^i and $\bar{z}^i$ and the central charge function Z take values at the horizon. This equation, following directly by the attactor equation, allow us to write the stabilization equations for the scalar fields. We can indeed write the different components explicitely and we find

$$\begin{aligned} q_0 + ip^0 &= 2\ e^{K/2}Z \ , \\ q_i - iz^i p^0 &= -\ e^{K/2}Z(z^i + \bar{z}^i) \ , \\ z^i q_0 + ip^i &= e^{K/2}Z(\bar{z}^i + z^i) \ , \\ z^i(q_i - ip^i) &= -2e^{K/2}Z|z^i|^2 \ , \end{aligned} \tag{4.21}$$

so that the fields at the horizon are given, as functions of electric and magnetic charges, by

$$z^i = -\frac{q_i + ip^i}{q_0 - ip^0} \ . \tag{4.22}$$

Also, the central charge is

$$Z = \frac{1}{2}e^{-K/2}(q_0 + ip^0) \ , \tag{4.23}$$

which is consistent with the definition [**59**]

$$Z = e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \ . \tag{4.24}$$

The Kähler potential is defined as

$$\begin{aligned} e^{-K} &= i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) = \\ &= 2(1-\sum_i |z^i|^2) \ , \end{aligned} \tag{4.25}$$

yielding the metric constraint $1-|z|^2>0$, and if we substitute this expression in (4.23), we find

$$Z = (q_0+ip^0)\left(\frac{(q_0^2+p^{0\,2})-(q_i^2+p^{i\,2})}{2(q_0^2+p^{0\ 2})}\right)^{1/2} \ . \tag{4.26}$$

One can also write

$$Z = (q_0+ip^0)\left(\frac{Q^2+P^2}{2(q_0^2+p^{0\ 2})}\right)^{1/2} \ . \tag{4.27}$$

where we defined

$$\begin{aligned} Q^2 &= q_\Lambda \eta^{\Lambda\Sigma} q_\Sigma \ , \\ P^2 &= p^\Lambda \eta_{\Lambda\Sigma} p^\Sigma \ , \end{aligned} \tag{4.28}$$

and $\eta_{\Lambda\Sigma}$ is the metric of $SO(1,n)$, $\eta_{\Lambda\Sigma} = diag(1,-1...,-1)$. To explicit the symmetry of the scalar manifold we define complex charges as

$$z^\Lambda \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \ , \tag{4.29}$$

so that the central charge becomes

$$Z = (q_0+ip^0)\left(\frac{z^0\bar{z}^0 - z^i\bar{z}^i}{2(q_0^2+p^{0\ 2})}\right)^{1/2} \ . \tag{4.30}$$

Black hole entropy at the attractor points is given by the modulus of the second quadratic invariant of the symmetric space,

$$S = |I_2| = \big||Z|^2 - |D_iZ|^2\big| \ , \tag{4.31}$$

where $|D_iZ|^2 = G^{i\bar{\imath}} D_iZ\, D_{\bar{\imath}}\bar{Z}$; in the BPS case, $D_iZ=0$, and we have then

$$\begin{aligned} S_{N=2-Symm-BPS} = |Z|^2 &= \frac{1}{2}(q_0^2+p^{0\,2}-(q_i^2+p^{i\,2})) = \\ &= \frac{1}{2}(z^0\bar{z}^0 - z^i\bar{z}^i) \ . \end{aligned} \tag{4.32}$$

We can show that the attractor point corresponds, in the BPS case, to a minimum of the potential. In fact, from (4.19), the Hessian metric for the black hole potential is

$$\begin{aligned} V_{BH\,\bar{j}i} &= 2D_{\bar{j}}\bar{Z}D_iZ + 2\bar{Z}D_{\bar{j}}D_iZ \Big|_{Z_i=0} = \\ &= 2\bar{Z}D_{\bar{j}}D_iZ \Big|_{Z_i=0} \ , \end{aligned} \tag{4.33}$$

and, from (4.18), we get

$$V_{BH\,\bar{j}i} = 2G_{\bar{j}i}|Z|^2 \ ; \tag{4.34}$$

since the metric is positive defined, this matrix has no null-eigenvalues, which means that there are no "flat directions" for the scalar fields, and the residual moduli space in the BPS solution to the quadratic series of $N = 2$ Supergravity is empty. We notice that, from the definition of the black hole potential in (4.14), this result depends only on the special geometry equations, that is, on the Kähler nature of the scalar manifold.

4. Non-BPS solution

Non supersymmetric solution is given by $D_iZ \neq 0$, together with the condition

$$\begin{gathered} Z = 0 = \frac{q_0 + ip^0 + \sum_i (q_i - ip^i)z^i}{\sqrt{2(1 - \sum_i z^i \bar{z}^i)}} \\ \Downarrow \\ \sum_i (q_i - ip^i)z^i = -(q_0 + ip^0) \ , \end{gathered} \tag{4.35}$$

we therefore have one condition leaving $n - 1$ undetermined moduli at the horizon. By its definition the central charge is

$$Z = e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \ , \tag{4.36}$$

so that

$$\begin{aligned}
D_i Z &= e^{K/2}(D_i X^\Lambda\ , q_\Lambda - D_i F_\Lambda\ , p^\Lambda) = \\
&= e^{K/2}(q_i - ip^i) + \partial_i K\, Z = \\
&= \frac{1}{\sqrt{2(1-\sum_i z^i \bar{z}^i)}}\left[q^i - ip^i + \partial_i K(q_0 + ip^0 + (q_l - ip^l)z^l)\right] = \\
&= \frac{1}{\sqrt{1-|z|^2}}\left[q_i - ip^i + \frac{\bar{z}^i}{1-|z|^2}(q_0 + ip^0 + z^l(q_l - ip^l))\right] = \\
&= \frac{1}{\sqrt{2}\,(1-\sum_i z^i\bar{z}^i)^{3/2}}\left[(q_i - ip^i)(1-\sum_i z^i\bar{z}^i) + 2(q_0+ip^0)\bar{z}^i + 2(q_l - ip^l)z^l\bar{z}^i\right]
\end{aligned} \tag{4.37}$$

In the non-BPS case the central charge is null at the horizon, and we find for $D_i Z|_{hor}$ the expression

$$D_i Z|_{hor} = \frac{q_i - ip^i}{\sqrt{2(1-\sum_i |z^i|^2)}}\ , \tag{4.38}$$

but this time the $z^i|_{hor}$ are not stabilized The black hole entropy in this case is

$$\begin{aligned}
S_{N=2,non-BPS} &= |I_2| = \left||Z|^2 - |D_i Z|^2\right| = \\
&= -D_i Z D_{\bar{j}} G^{i\bar{j}}\ ,
\end{aligned}$$

where $G^{i\bar{j}}$ is the inverse metric given in (A.11)

$$G^{i\bar{j}} = (1-|z|^2)\left(\delta^{i\bar{j}} - \bar{z}^i z^{\bar{j}}\right)\ , \tag{4.39}$$

and we find

$$\begin{aligned}
S_{N=2,non-BPS} &= -\frac{1}{2}(q_i - ip^i)(q_{\bar{j}} + ip^{\bar{j}})(\delta^{i\bar{j}} - z^i\bar{z}^j) = \\
&= -\frac{1}{2}(q_i^2 + p^{i\,2} - z^i(q_i - ip^i) - \bar{z}^{\bar{j}}(q_{\bar{j}} + ip^{\bar{j}}))\ ,
\end{aligned} \tag{4.40}$$

and, by the attractor condition $Z=0$ and (4.35), it can be written as

$$\begin{aligned}
S_{N=2,non-BPS} &= \frac{1}{2}(q_0^2 + p^{0\,2} - (q_i^2 + p^{i\,2})) = \\
&= \frac{1}{2}(z^0\bar{z}^0 - z^i\bar{z}^i)\ ,
\end{aligned} \tag{4.41}$$

where complex charges are defined by (4.29).

We have checked the important result that the entropy of the black hole does not depend on the nature (BPS or non-BPS) of the solution, but only on the asymptotic (initial) configuration of electric and magnetic charges. We notice that in the BPS case $I_2 > 0$, while here $I_2 < 0$. Electric and magnetic charges thus select the nature of the solution, depending on the sign of their combination

$$|Z|^2 - |D_i Z|^2 = \frac{1}{2}\left(q_0^2 + p^{0\,2} - (q_i^2 + p^{i\,2})\right) \ , \tag{4.42}$$

which is invariant in the moduli space.

5. Invariant expressions

Black hole entropy, as well as black hole potential, are invariant expressions of the charges, and can be written as

$$\begin{aligned} V_{BH} &= -\frac{1}{2}Q^t \mathcal{M}(\mathcal{N}) Q \ , \\ S_{BH} &= \frac{1}{2}Q^t \mathcal{M}(\mathcal{F}) Q \ , \end{aligned} \tag{4.43}$$

where $\mathcal{N}$ is the matrix in the vector fields kinetic term, and $\mathcal{F} \equiv \ F_{\Lambda\Sigma} \ = \ \partial_\Lambda \partial_\Sigma F(X)$. We find, if the prepotential is (A.2), that

$$M(\mathcal{F}) = \begin{pmatrix} Im\mathcal{F} & 0 \\ 0 & (Im\mathcal{F})^{-1} \end{pmatrix} \ , \tag{4.44}$$

where $Im\mathcal{F} = Id_2 \otimes \eta_{\Lambda\Sigma}$, $\eta_{\Lambda\Sigma} = diag_n(-1, 1, ..., 1)$ and we compute the multiplications to find

$$\begin{aligned} S_{BH} &= \frac{1}{2}(p^\Lambda \eta_{\Lambda\Sigma} p^\Sigma + q_\Lambda \eta^{\Lambda\Sigma} q_\Sigma) = \\ &= \frac{1}{2}\left(q_0^2 + p^{0\,2} - \sum_i (q_i^2 + p^{i\,2})\right) \ , \end{aligned} \tag{4.45}$$

which is determined only by the charges configuration at infinity.

6. Black Hole Parameters in $\mathcal{N}=2$ minimally coupled Supergravity

6.1. Black Hole Parameters for $\frac{1}{2}$-BPS Flow. The first order fake superpotentials for $\frac{1}{2}-BPS$ and non-BPS ($Z=0$) attractor flows, are [**39**]

$$\begin{aligned}\mathcal{W}^2_{\left(\frac{1}{2}-\right)BPS} &= |Z|^2 = \alpha_1^2 = \\ &= \frac{\left[q_0 + ip^0 + (q_i - ip^i)\, z^i\right]\left[q_0 - ip^0 + (q_j + ip^j)\, \overline{z}^{\overline{j}}\right]}{2\,(1-|z|^2)}; \end{aligned} \tag{4.46}$$

$$\begin{aligned}\mathcal{W}^2_{non-BPS(,Z=0)} &= G^{i\overline{j}}\,(D_i Z)\,\overline{D}_{\overline{j}}\overline{Z} = \alpha_2^2 = \\ &= \frac{1}{2\left(1-|z|^2\right)^2}\left(\delta^{i\overline{j}} - z^i \overline{z}^{\overline{j}}\right)\cdot \\ &\quad\cdot\left[(q_i - ip^i)(1-|z|^2) + (q_0 + ip^0)\overline{z}^{\overline{i}} + (q_r - ip^r) z^r \overline{z}^{\overline{i}}\right]\cdot \\ &\quad\cdot\left[(q_j + ip^j)(1-|z|^2) + (q_0 - ip^0) z^j + (q_n + ip^n)\overline{z}^{\overline{n}} z^j\right]. \end{aligned} \tag{4.47}$$

Thus, by using the explicit expressions of $\mathcal{W}^2_{BPS}$ and the differential relations of special Kähler geometry of $\mathcal{M}_{\mathcal{N}=2}$ [**59**], exploiting the first order (fake supergravity) formalism the expressions of the ADM mass, covariant scalar charges and effective horizon radius for the $\frac{1}{2}$-BPS attractor flow can be explicitly written as[2] [**46**]:

$$r^2_{H,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=M^2_{ADM,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=\mathcal{W}^2_{BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=$$

$$=\lim_{\tau\to 0^-}|Z|^2\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=$$

$$=\frac{\left[q_0+ip^0+\left(q_i-ip^i\right)z^i_\infty\right]\left[q_0-ip^0+\left(q_j+ip^j\right)\overline{z}^{\overline{j}}_\infty\right]}{2\left(1-|z_\infty|^2\right)}; \quad (4.48)$$

$$\Sigma_{i,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=2\lim_{\tau\to 0^-}\left(\partial_i\mathcal{W}_{BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=$$

$$=\frac{1}{M_{ADM,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)}\lim_{\tau\to 0^-}\left(\overline{Z}D_iZ\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=$$

$$=\frac{1}{\sqrt{2}\left(1-|z_\infty|^2\right)^{3/2}}\sqrt{\frac{q_0-ip^0+\left(q_j+ip^j\right)\overline{z}^{\overline{j}}_\infty}{q_0+ip^0+\left(q_k-ip^k\right)z^k_\infty}}\cdot$$

$$\cdot\left[(q_i-ip^i)(1-|z_\infty|^2)+(q_0+ip^0)\overline{z}^{\overline{i}}_\infty+(q_r-ip^r)z^r_\infty\overline{z}^{\overline{i}}_\infty\right]; \quad (4.49)$$

$$R^2_{H,BPS}=\lim_{\tau\to 0^-}\left[\begin{array}{l}\mathcal{W}^2_{BPS}\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)+\\ \\ -4G^{i\overline{j}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\left(\partial_i\mathcal{W}_{BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\cdot\\ \\ \cdot\left(\overline{\partial}_{\overline{j}}\mathcal{W}_{BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\end{array}\right]=$$

$$=\mathcal{I}_2\left(p,q\right)=V_{BH,BPS}=\frac{S_{BH,BPS}\left(p,q\right)}{\pi}. \quad (4.50)$$

[2] All the considered functions $f\left(z,\overline{z},p,q\right)$ admit the limit

$$\left(f\left(z,\overline{z},p,q\right)\right)_\infty\equiv\lim_{\tau\to 0^-}f\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=f\left(z_\infty,\overline{z}_\infty,p,q\right),$$

and are assumed $f\left(z,\overline{z},p,q\right)$ to be smooth enough to split the asymptotical limit of a product into the product of the asymptotical limits of the factors.

Eq. (4.50) proves Eq. (3.36) for the $\frac{1}{2}$-BPS attractor flow of the $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to $n \equiv n_V$ Abelian vector multiplets.

Notice that in the extremality regime ($c=0$) the *effective horizon radius* R_H, and thus A_H and the Bekenstein-Hawking entropy S_{BH} are *independent* on the particular vacuum or ground state of the considered theory, *i.e.* on $\left(z^i_\infty, \overline{z}^{\overline{i}}_\infty\right)$, but rather they depend *only* on the electric and magnetic charges q_Λ and p^Λ, which are *conserved* due to the overall $(U(1))^{n+1}$ gauge-invariance. The independence on $\left(z^i_\infty, \overline{z}^{\overline{i}}_\infty\right)$ is of crucial importance for the consistency of the *microscopic state counting interpretation* of S_{BH}, as well as for the overall consistency of the macroscopic thermodynamic picture of the BH. However, it is worth recalling that the ADM mass M_{ADM} generally does depend on $\left(z^i_\infty, \overline{z}^{\overline{i}}_\infty\right)$ *also in the extremal case*, as yielded by Eq. (4.48) for the considered $\frac{1}{2}$-BPS attractor flow.

6.2. Black Hole Parameters for Non-BPS ($Z=0$) Flow. Once again from the explicit expressions of $\mathcal{W}^2_{non-BPS}$ in (4.47), using the differential relations of special Kähler geometry of $\mathcal{M}_{\mathcal{N}=2,mc,n}$ and exploiting the first order formalism the expressions of the ADM mass, covariant scalar charges and effective horizon radius for the non-BPS $Z=0$ attractor flow [**46**]:

$$
\begin{aligned}
r^2_{H,non-BPS}\left(z_\infty, \overline{z}_\infty, p, q\right) &= M^2_{ADM,non-BPS}\left(z_\infty, \overline{z}_\infty, p, q\right) = \mathcal{W}^2_{non-BPS}\left(z_\infty, \overline{z}_\infty, p, q\right) = \\
&= \lim_{\tau\to 0^-}\left[G^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{Z}\right]\left(z\left(\tau\right), \overline{z}\left(\tau\right), p, q\right) =
\end{aligned}
$$

that explicitly becomes

$$
\begin{aligned}
r^2_{H,non-BPS}\left(z_\infty, \overline{z}_\infty, p, q\right) = &\frac{1}{2\left(1-|z_\infty|^2\right)^2}\left(\delta^{i\overline{j}} - z^i_\infty \overline{z}^{\overline{j}}_\infty\right)\cdot \\
&\cdot\left[(q_i - ip^i)(1-|z_\infty|^2) + (q_0 + ip^0)\overline{z}^{\overline{i}}_\infty + (q_r - ip^r)z^r_\infty \overline{z}^{\overline{i}}_\infty\right]\cdot \\
&\cdot\left[(q_j + ip^j)(1-|z_\infty|^2) + (q_0 - ip^0)z^j_\infty + (q_n + ip^n)\overline{z}^{\overline{n}}_\infty z^j_\infty\right].
\end{aligned}
\tag{4.51}
$$

The scalar charges are

$$\begin{aligned}\Sigma_{i,non-BPS}\left(z_{\infty},\overline{z}_{\infty},p,q\right) &= 2\lim_{\tau\rightarrow 0^{-}}\left(\partial_{i}\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)= \\ &= \frac{1}{M_{ADM,non-BPS}\left(z_{\infty},\overline{z}_{\infty},p,q\right)}\lim_{\tau\rightarrow 0^{-}}\left(\overline{Z}D_{i}Z\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)= \\ &= \frac{1}{\sqrt{2}}\frac{1}{\left(1-|z_{\infty}|^{2}\right)}\cdot \\ &\quad\cdot\left[q_{0}-ip^{0}+\left(q_{j}+ip^{j}\right)\overline{z}_{\infty}^{\overline{j}}\right]\cdot \\ &\quad\cdot\left[(q_{i}-ip^{i})(1-|z_{\infty}|^{2})+(q_{0}+ip^{0})\overline{z}_{\infty}^{\overline{i}}+(q_{m}-ip^{m})z_{\infty}^{m}\overline{z}_{\infty}^{\overline{i}}\right]\cdot \\ &\quad\cdot\left[\begin{array}{l}(\delta^{n\overline{p}}-z_{\infty}^{n}\overline{z}_{\infty}^{\overline{p}})\cdot \\ \\ \cdot\left[(q_{n}-ip^{n})(1-|z_{\infty}|^{2})+(q_{0}+ip^{0})\overline{z}_{\infty}^{\overline{n}}+(q_{s}-ip^{s})z_{\infty}^{s}\overline{z}_{\infty}^{\overline{n}}\right]\cdot \\ \\ \cdot\left[(q_{p}+ip^{p})(1-|z_{\infty}|^{2})+(q_{0}-ip^{0})z_{\infty}^{p}+(q_{w}+ip^{w})\overline{z}_{\infty}^{\overline{w}}z_{\infty}^{p}\right]\end{array}\right]^{-1/2}\end{aligned} \tag{4.52}$$

$$R_{H,non-BPS}^{2}=\lim_{\tau\rightarrow 0^{-}}\left[\begin{array}{l}\mathcal{W}_{non-BPS}^{2}\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)+ \\ \\ -4G^{i\overline{j}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\left(\partial_{i}\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\cdot \\ \\ \cdot\left(\overline{\partial}_{\overline{j}}\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\end{array}\right]=$$

$$=-\mathcal{I}_{2}\left(p,q\right)=V_{BH,non-BPS}=\frac{S_{BH,non-BPS}\left(p,q\right)}{\pi}. \tag{4.53}$$

Eq. (4.53) proves Eq. (3.36) for the non-BPS $Z=0$ attractor flow of the the $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to $n\equiv n_{V}$ Abelian vector multiplets. The considerations made at the end of Subsect. 6.1 hold also for the considered attractor flow.

It is worth noticing out that Eqs. (4.50) and (4.53) are consistent, because, as pointed out above, the ($\frac{1}{2}$-)BPS- and non-BPS ($Z=0$)- supporting BH charge configurations in the considered theory are respectively defined by the *quadratic* constraints $\mathcal{I}_{2}\left(p,q\right)>0$ and $\mathcal{I}_{2}\left(p,q\right)<0$.

As yielded by Eqs. (4.49) and (4.52) for both *non-degenerate* attractor flows of the considered theory it holds the following relation among *scalar charges* and *ADM mass*:

$$\Sigma_i = \frac{1}{M_{ADM}} \lim_{\tau \to 0^-} D_i \left(|Z|^2 \right). \tag{4.54}$$

CHAPTER 5

$\mathcal{N} = 3$ Supergravity black holes

1. Embedding of the noncompact group in the symplectic group

The scalar manifold for the non linear σ-model of the scalar fields in $N = 3$ Supergravity is [**53**]

$$\frac{SU(3,n)}{SU(3) \times SU(n) \times U(1)} \; , \tag{5.1}$$

whose coordinates are the $3n$ complex scalar fields. The isometry group $SU(3,n)$ is the duality group of the $(n+3)$ vector fields and is a subgroup of the symplectic group $Sp(2(3+\ n), \mathbb{R})$ [**62**]. Given

$$H = \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array} \right) \; , \tag{5.2}$$

the generic element parametrizing the coset space is

$$L = \exp(H) = \begin{pmatrix} \sqrt{1 + XX^\dagger} & X \\ X^\dagger & \sqrt{1 + X^\dagger X} \end{pmatrix} \; ; \tag{5.3}$$

let us consider the embedding of the isometry group $SU(3,n)$ into the symplectic group

$$\begin{aligned} SU(3,n) &\rightarrow Sp(2(3+n), \mathbb{R}) \; , \\ g \equiv L(z) &\rightarrow S(g) \equiv S(L(z)) \; , \end{aligned} \tag{5.4}$$

the matrix S is given by the block matrix

$$S(g) = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} \; , \tag{5.5}$$

which is an element $S \in SU(3,n) \subset Sp(2(3+n), \mathbb{R})$, so that the sub-blocks ϕ_0 and ϕ_1 satisfy the relations

$$\begin{aligned} \phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 &= 1 \; , \\ \phi_0^\dagger \phi_1^* - \phi_1^\dagger \phi_0^* &= 0 \; . \end{aligned} \tag{5.6}$$

In the Gaillard Zumino construction the vector fields kinetic Lagrangian is

$$\mathcal{L}^{kin}_{vec} = F^{+}_{\Lambda\mu\nu}F^{+}_{\Sigma\mu\nu}\mathcal{N}^{\Lambda\Sigma} + F^{-}_{\Lambda\mu\nu}F^{-}_{\Sigma\mu\nu}\bar{\mathcal{N}}^{\Lambda\Sigma} \ , \tag{5.7}$$

where the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ is symmetric and is defined as

$$\mathcal{N}_{\Lambda\Sigma} = (\phi_0^\dagger + \phi_1^\dagger)^{-1}(\phi_0^\dagger - \phi_1^\dagger) \ . \tag{5.8}$$

The embedding is defined once we write the matrix S as a functions of X, we have

$$S = \left(\begin{array}{cc|cc} \sqrt{1+XX^\dagger} & 0 & 0 & X \\ 0 & \sqrt{1+X^TX^*} & X^T & 0 \\ \hline 0 & X^* & \sqrt{1+X^*X^T} & 0 \\ X^T & 0 & 0 & \sqrt{1+X^\dagger X} \end{array}\right) \ , \tag{5.9}$$

that is

$$\phi_1 = \begin{pmatrix} 0 & X^* \\ X^\dagger & 0 \end{pmatrix} \ ,$$

$$\phi_0 = (1+\phi_1^*\phi_1)^{1/2} = \begin{pmatrix} \sqrt{1+XX^\dagger} & 0 \\ 0 & \sqrt{1+X^TX^*} \end{pmatrix} \ . \tag{5.10}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}$ can be written in terms of symplectic sections as

$$\mathcal{N}_{\Lambda\Sigma} = (h\, f^{-1})_{\Lambda\Sigma} \ , \tag{5.11}$$

where the explicit dependence of $\mathbf{f}$ and $\mathbf{h}$ on the sublocks of $S(X)$ is given by

$$\mathbf{f} = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1) \ ,$$

$$\mathbf{h} = \frac{-i}{\sqrt{2}}(\phi_0 - \phi_1) \ , \tag{5.12}$$

and in terms of the coordinates of the coset space

$$\mathbf{f} = f^\Lambda_\Sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1+XX^\dagger} & X^* \\ X^\dagger & \sqrt{1+X^TX^*} \end{pmatrix} = \tag{5.13}$$

$$= (f^\Lambda_{AB}, \bar{f}^\Lambda_{\bar{I}}) \ ,$$

$$\mathbf{h} = h_{\Lambda\Sigma} = \frac{-i}{\sqrt{2}}\begin{pmatrix} \sqrt{1+XX^\dagger} & -X^* \\ -X^\dagger & \sqrt{1+X^TX^*} \end{pmatrix} = \tag{5.14}$$

$$= (h_{\Lambda AB}\bar{h}_{\Lambda\bar{I}}) \ .$$

If we write the equations (5.6) in terms of symplectic sections we find

$$\begin{aligned} i(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) &= 1 \ , \\ \mathbf{h}^T \mathbf{f} - \mathbf{h}\mathbf{f}^T &= 0 \ . \end{aligned} \tag{5.15}$$

1.1. Central charges and matter charges. Charges are defined as the integral over a sphere at infinity of the dressed graviphoton and matter field strengths, so that we have

$$\begin{aligned} Z_{AB} &= -\int_{S^2} T_{AB} = -\int_{S_2} T^-_{AB} = \\ &= f^\Lambda_{AB} q_\Lambda - h_{\Lambda AB} p^\Lambda \ , \end{aligned} \tag{5.16}$$

$$\begin{aligned} Z_I &= -\int_{S^2} T_I = -\int_{S_2} T^-_I = \ , \\ &= f^\Lambda_I q_\Lambda - h_{\Lambda I} p^\Lambda \ . \end{aligned} \tag{5.17}$$

Using the explicit expression for the symplectic sections given in (5.13) and (5.15), we find for the charges $Z = (Z_{AB}\ ,\ Z_I)$ ($C = 1, .., 3$ and $I, i = 1, ..., n$)

$$Z_{AB} = \frac{1}{\sqrt{2}} \left[\sqrt{1 + XX^\dagger}^{C}_{(AB)} (q_C + ip^C) + (X^*)^i_{(AB)} (q_i - ip^i) \right] \ , \tag{5.18}$$

$$Z_I = \frac{1}{\sqrt{2}} \left[(X^\dagger)^C_I (q_C - ip^C) + \sqrt{1 + X^T X^*}^{i}_{I} (q_i + ip^i) \right] \ . \tag{5.19}$$

1.2. Attractor equations and V_{BH} critical points. We impose to the black hole potential

$$V_{BH} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^{\bar{I}} \ , \tag{5.20}$$

an extremum condition, in order to get a regular dynamic for the scalar fields at the horizon, therefore

$$dV_{BH}|_{hor} = \frac{1}{2} (DZ_{AB}) \bar{Z}^{AB} + (DZ_I) \bar{Z}^I + \frac{1}{2} Z_{AB} D\bar{Z}^{AB} + Z_I D\bar{Z}^{\bar{I}} = 0 \ . \tag{5.21}$$

Depending on the geometry of the scalar manifold, one can write the expression of the covariant derivatives in terms of the embedded vielbien $\mathcal{P}$, defined in a suitable central/-matter indices decomposition. For $N = 3$ its only non-zero components are P^I_{AB}, so that the above equation becomes

$$dV_{BH}|_{hor} = P^I_{AB}\bar{Z}^{AB}Z_I + c.c. = 0\ . \tag{5.22}$$

It admits the two solutions

- $Z_{AB} \neq 0$, $Z_I = 0$,
  $V_{BH}|_{hor} = \frac{1}{2}|Z_{AB}|^2$,
  BPS solution
- $Z_{AB} = 0$, $Z_I \neq 0$,
  $V_{BH}|_{hor} = |Z_I|^2$,
  non-BPS solution.

1.3. Fake superpotentials. Once the charges have been written in (5.18) and (5.19), one finds that the superpotentials take the following form

$$\begin{aligned}
\mathcal{W}^2_{(\frac{1}{3}-)BPS} &= \frac{1}{2}Z_{AB}\overline{Z}^{AB} = \mathcal{Z}_1^2 = \\
&= \frac{1}{2}\left[(q_C - ip^C)\left(\sqrt{1+XX^\dagger}\right)^{\ C} + (q_i + ip^i)\left(X^T\right)^i\right]\cdot \\
&\quad \cdot\left[\left(\sqrt{1+XX^\dagger}\right)^{\ D}(q_D + ip^D) + \overline{X}^j(q_j - ip^j)\right] = \\
&= \frac{1}{2}\left[(1+XX^\dagger)^{AB}(q_A - ip^A)(q_B + ip^B) + \right. \\
&\quad +(\sqrt{1+XX^\dagger}\,X)^{A\,i}(q_i + ip^i)(q_A + ip^A) + \\
&\quad +(X^\dagger\sqrt{1+XX^\dagger})^{j\,B}(q_A - ip^B)(q_j - ip^j) + \\
&\quad \left. +(X^\dagger X)^{kl}(q_l + ip^l)(q_k - ip^k)\right]\ ;
\end{aligned} \tag{5.23}$$

$$
\begin{aligned}
\mathcal{W}^2_{non-BPS(,Z_{AB}=0)} &= Z_I \overline{Z}^I = \rho^2 = \\
&= \frac{1}{2}\left[(q_D + ip^D)X^D + (q_l - ip^l)\left(\sqrt{1+X^T\overline{X}}\right)^{\ l}\right] \cdot \\
&\quad \cdot \left[(X^\dagger)^C(q_C - ip^C) + \left(\sqrt{1+X^T\overline{X}}\right)^{\ i}(q_i + ip^i)\right] = \\
&= \frac{1}{2}\left[(XX^\dagger)^{CD}(q_C + ip^C)(q_D - ip^D) + \right. \\
&\quad +(\sqrt{1+X^\dagger X}\, X^\dagger)^{l\,C}(q_l - ip^l)(q_C - ip^C) + \\
&\quad +(X\sqrt{1+X^\dagger X})^{D\,i}(q_D + ip^D)(q_i + ip^i) + \\
&\quad \left. +(1+X^\dagger X)^{l\,i}(q_l - ip^l)(q_i + ip^i)\right] \ .
\end{aligned}
\tag{5.24}
$$

Notice that, since all the contractions of $SU(3)$ and $SU(n)$ indices of electric and magnetic BH charges are uniquely defined with respect to the row or columns of the matrix X, every transposition index has been suppressed in Eqs. (5.23) and (5.24).

By introducing the *complexified graviphoton* and *matter* BH charges respectively as follows:

$$Q_C \equiv q_C + ip^C; \tag{5.25}$$

$$Q_i \equiv q_i + ip^i, \tag{5.26}$$

Eqs. (5.23) and (5.24) can be rewritten as follows:

$$
\begin{aligned}
\mathcal{W}^2_{BPS} = \frac{1}{2}&\left[(1+XX^\dagger)^{AB}\overline{Q}_A Q_B + (\sqrt{1+XX^\dagger}\, X)^{\ Ai} Q_i Q_A + \right. \\
&\left. +(X^\dagger\sqrt{1+XX^\dagger})^{j\,B}\overline{Q}_B\overline{Q}_j + (X^\dagger X)^{k\,l}\overline{Q}_k Q_l\right] \ ;
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
\mathcal{W}^2_{non-BPS} = \frac{1}{2}&\left[(XX^\dagger)^{AB} Q_A \overline{Q}_B + (\sqrt{1+X^\dagger X}\, X^\dagger)^{i\,A}\overline{Q}_i\overline{Q}_A + \right. \\
&\left. +(X\sqrt{1+X^\dagger X})^{B\,j} Q_B Q_j + (1+X^\dagger X)^{k\,l}\overline{Q}_k Q_l\right] \ .
\end{aligned}
\tag{5.28}
$$

2. Black Hole entropy

The isometry group $SU(3,n)$ only has the quadratic invariant

$$I_2 = \frac{1}{2}(Z_{AB}\bar{Z}^{AB}) - (Z_I\bar{Z}^{\bar{I}})^2 \tag{5.29}$$

and in this case, as for $N=2$, the entropy at the attractor point is the modulus of I_2,

$$S_{BH} = |\frac{1}{2}Z_{AB}\bar{Z}^{AB} - Z_I\bar{Z}^I| \ . \tag{5.30}$$

Using (5.27) and (5.28), the entropy is then given by

$$\begin{aligned} S_{BH} &= \frac{1}{2}\left|\sum_C (q_C^2 + p^{C\,2}) - \sum_l (q_l^2 + p^{l\,2})\right| = \\ &= \frac{1}{2}\left|Q_A\bar{Q}_A - Q_i\bar{Q}_i\right| \ , \end{aligned} \tag{5.31}$$

for, since the matrix $XX^\dagger$ is hermitian,

$$\begin{aligned} (XX^\dagger)^{CD}Q_C\bar{Q}_D &= \langle Q_C, Q_C\rangle_{XX^\dagger} = \\ &= \langle \bar{Q}_C, \bar{Q}_C\rangle_{XX^\dagger} = (XX^\dagger)^{CD}\bar{Q}_C Q_D \ , \end{aligned} \tag{5.32}$$

and we have used the identity, which holds for any matrix A,

$$A^\dagger\sqrt{1+AA^\dagger} = \sqrt{1+A^\dagger A}\ A^\dagger \ , \tag{5.33}$$

whose hermitian conjugate is

$$\sqrt{1+AA^\dagger}\ A = A\sqrt{1+A^\dagger A} \ . \tag{5.34}$$

We notice that the entropy in (5.31) generalizes as expected the entropy of the black hole we found for $N=2$ supergravity, where the scalar manifold was $SU(1,n)/SU(1)\times SU(n)$, but now residual flat directions for the scalar fields appear both in the BPS and in the non-BPS solution.

3. BPS $N=3$ solution

In the BPS case $Z_I = 0$ so that the black hole potential is

$$V_{BH} = \frac{1}{2}|Z_{AB}|^2 = S_{BH} \tag{5.35}$$

Imposing $Z_I = 0$, we have n equations that will allow us to stabilize $\frac{1}{3}$ of the complex scalar fields. From equation (5.19), we have

$$(X^\dagger)^C\bar{z}_C = -\sqrt{1+X^\dagger X}^{\ i} z_i \ . \tag{5.36}$$

Inserting this relation and its hermitian conjugate in (5.27), using also (5.33) and (5.34) we find

$$
\begin{aligned}
Z_{AB}\bar{Z}^{AB} &= (1+XX^\dagger)^{CD}\bar{Q}_C Q_D + (X\sqrt{1+X^\dagger X})^{Di} Q_i Q_D + \\
&\quad +(\sqrt{1+X^\dagger X}X^\dagger)^{lC}\bar{Q}_C\bar{Q}_l + (X^\dagger X)^{li}\bar{Q}_l Q_i = \\
&= \bar{Q}_C Q^C + (XX^\dagger)^{CD}\bar{Q}_C Q_D - (XX^\dagger)^{DC} Q_D\bar{Q}_C + \\
&\quad -(1+X^\dagger X)^{li}\bar{Q}_l Q_i + (X^\dagger X)^{li}\bar{Q}_l Q_i = \\
&= \sum_C Q_C\bar{Q}_C - \sum_i Q_i\bar{Q}_i \ .
\end{aligned} \tag{5.37}
$$

The entropy in the BPS case is

$$
\begin{aligned}
S_{BH} &= \frac{1}{2} Z_{AB}\bar{Z}^{AB} = \\
&= \frac{1}{2}\left[\sum_C (q_C^2 + p^{C\,2}) - \sum_l (q_l^2 + p^{l\,2})\right] = \\
&= \frac{1}{2}\left[\sum_C Q_C\bar{Q}_C - \sum_i Q_i\bar{Q}_i\right] \ .
\end{aligned} \tag{5.38}
$$

The condition $Z_I = 0$ is a set of n complex equations that does not fix all the $3n$ complex scalar fields, the residual $2n$ flat directions defining the moduli space

$$
\frac{SU(2,n)}{SU(2)\times SU(n)\times U(1)} \ . \tag{5.39}
$$

4. Non-BPS Solution

We get three constraints on the scalar fields from the extremum condition $Z_{AB} = 0$ potential in the non BPS case. Explicitly we have

$$
\sqrt{1+XX^\dagger}^{\,C}(q_C + ip^C) = -(X^\dagger)^i(q_i - ip^i) \ , \tag{5.40}
$$

or

$$
\sqrt{1+XX^\dagger}^{\,C} Q_C = -(X^\dagger)^i \bar{Q}_i \ . \tag{5.41}
$$

The black hole potential at the horizon is

$$
V_{BH} = Z_I\bar{Z}^I = S_{BH} \ . \tag{5.42}
$$

From the expression (5.28), analogously to what done in the BPS case, we compute

$$\begin{aligned} 2|Z_I|^2 &= (XX^\dagger)^{CD} Q_C \bar{Q}_D + (X^\dagger \sqrt{1+XX^\dagger})^{lC} \bar{Q}_l \bar{Q}_C + \\ &\quad + (\sqrt{1+XX^\dagger} X)^{D\,i} Q_D Q_i + (1+X^\dagger X)^{l\,i} \bar{Q}_l Q_i = \\ &= (XX^\dagger)^{CD} Q_C \bar{Q}_D - (1+XX^\dagger)^{CD} Q_C \bar{Q}_D + \\ &\quad - (X^\dagger X)^{li} \bar{Q}_l Q_i + (1+X^\dagger X)^{l\,i} \bar{Q}_l Q_i = \\ &= \sum_i Q_i \bar{Q}_i - \sum_C Q_C \bar{Q}_C \ . \end{aligned} \tag{5.43}$$

The entropy is given again, as expected, by the formula

$$\begin{aligned} S_{BH} &= Z_I \bar{Z}^I = \\ &= -\frac{1}{2} \left[\sum_C Q_C \bar{Q}_C - \sum_i Q_i \bar{Q}_i \right] \ . \end{aligned} \tag{5.44}$$

In this case we have 3 equations that stabilize only $\frac{1}{n}$ the scalar fields, so that the moduli space for the non-BPS solution is

$$\frac{SU(3,n-1)}{SU(3) \times SU(n-1) \times U(1)} \ , \tag{5.45}$$

which has $dim_{\mathbb{C}} = 3(n-1)$.

5. Black Hole Parameters for $\frac{1}{3}$-BPS Flow

By using the *Maurer-Cartan Eqs.* of $\mathcal{N}=3$, $d=4$ supergravity (see *e.g.* [**65, 63, 64**]), one finds [**39**]

$$\partial_i \mathcal{Z}_1 = \partial_i \mathcal{W}_{BPS} = \frac{1}{2\sqrt{2}} \frac{P_{IAB,i} \overline{Z}^I \overline{Z}^{AB}}{\sqrt{Z_{CD} \overline{Z}^{CD}}} = \frac{1}{4\mathcal{Z}_1} P_{IAB,i} \overline{Z}^I \overline{Z}^{AB}, \tag{5.46}$$

where $P_{IAB} \equiv P_{IAB,i} dz^i$ is the holomorphic Vielbein of $\mathcal{M}_{\mathcal{N}=3,n}$. Here, ∇ denotes the $U(1)$-Kähler and $H_{\mathcal{N}=3,n}$-covariant differential operator. In the first order formalism the relevant parameters ADM mass,covariant scalar charges and effective horizon radius for the $\frac{1}{3}$-BPS attractor flow are computed to be

$$r^2_{H,BPS}(z_\infty,\overline{z}_\infty,p,q) = M^2_{ADM,BPS}(z_\infty,\overline{z}_\infty,p,q) = \mathcal{W}^2_{BPS}(z_\infty,\overline{z}_\infty,p,q) =$$

$$= \frac{1}{2}\lim_{\tau\to 0^-}\left(Z_{AB}\overline{Z}^{AB}\right)(z(\tau),\overline{z}(\tau),p,q) =$$

$$= \frac{1}{2}\left[\begin{array}{l}(1+X_\infty X_\infty^\dagger)^{AB}\overline{Q}_A Q_B + (\sqrt{1+X_\infty X_\infty^\dagger}\,X_\infty)^{\ Ai} Q_i Q_A + \\ \\ +(X_\infty^\dagger\sqrt{1+X_\infty X_\infty^\dagger})^{j\,B}\overline{Q}_B\overline{Q}_j + (X_\infty^\dagger X_\infty)^{k\,l}\overline{Q}_k Q_l\end{array}\right]; \tag{5.47}$$

$$\Sigma_{i,BPS}(z_\infty,\overline{z}_\infty,p,q) = 2\lim_{\tau\to 0^-}(\partial_i\mathcal{W}_{BPS})(z(\tau),\overline{z}(\tau),p,q) =$$

$$= \frac{1}{\sqrt{2}}\left[\frac{P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}}{\sqrt{Z_{CD}\overline{Z}^{CD}}}\right]_\infty = \frac{1}{2}\left[\frac{1}{\mathcal{Z}_1}P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}\right]_\infty =$$

$$= \frac{1}{2M_{ADM,BPS}(z_\infty,\overline{z}_\infty,p,q)}\left(P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}\right)_\infty; \tag{5.48}$$

$$\begin{aligned} R^2_{H,BPS} = \lim_{\tau\to 0^-}\Big[&\mathcal{W}^2_{BPS}(z(\tau),\overline{z}(\tau),p,q) + \\ &- 4G^{i\overline{j}}(z(\tau),\overline{z}(\tau))(\partial_i\mathcal{W}_{BPS})(z(\tau),\overline{z}(\tau),p,q)\cdot \\ &\cdot\left(\overline{\partial}_{\overline{j}}\mathcal{W}_{BPS}\right)(z(\tau),\overline{z}(\tau),p,q)\Big] = \\ = \mathcal{I}_2(p,q) = V_{BH,BPS} = \frac{S_{BH,BPS}(p,q)}{\pi}&, \end{aligned} \tag{5.49}$$

where

$$X_\infty \equiv \lim_{\tau\to 0^-} X(\tau). \tag{5.50}$$

The subscript "∞" indicates the point at radial infinity z^i_∞.

Eq. (5.49) proves Eq. (3.36) for the $\frac{1}{3}$-BPS attractor flow of the considered $\mathcal{N}=3$, $d=4$ supergravity. Such a result was obtained by using Eq. (5.46) and computing that

$$\begin{aligned} 4G^{i\overline{j}}\left(\partial_i\mathcal{W}_{BPS}\right)\overline{\partial}_{\overline{j}}\mathcal{W}_{BPS} &= 4G^{i\overline{j}}\left(\partial_i\mathcal{Z}_1\right)\overline{\partial}_{\overline{j}}\mathcal{Z}_1 = \\ &= \frac{G^{i\overline{j}}P_{IAB,i}\overline{P}_{JEF,\overline{j}}\overline{Z}^I Z^J\overline{Z}^{AB}Z^{EF}}{2Z_{CD}\overline{Z}^{CD}} = Z_I\overline{Z}^I = \rho^2, \end{aligned} \tag{5.51}$$

where the relation

$$G^{i\overline{j}}P_{IAB,i}\overline{P}_{JEF,\overline{j}} = \delta_{IJ}\left(\delta_{AE}\delta_{BF}-\delta_{AF}\delta_{BE}\right) \tag{5.52}$$

was exploited.

The considerations made at the end of Subsect. 6.1 hold also for the considered attractor flow.

As pointed out above, the same also holds for ($\frac{1}{2}$-BPS attractor flow of) $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to Abelian vector multiplets (see Eq. (150) of [**46**]), in which the (*unique*) invariant of the U-duality group $SU(1,n)$ is *quadratic* in BH electric and magnetic charges. Such a similarity is ultimately due to the fact that $SU(m,n)$ is endowed with a pseudo-Hermitian *quadratic* form built out of the *fundamental* $\mathbf{m}+\mathbf{n}$ and *antifundamental* $\overline{\mathbf{m}+\mathbf{n}}$ representations.

6. Black Hole Parameters for Non-BPS ($Z_{AB}=0$) Flow

By using the *Maurer-Cartan Eqs.* of $\mathcal{N}=3$, $d=4$ supergravity (see *e.g.* [**65, 63, 64**]), one gets [**39**]

$$\partial_i\rho = \partial_i\mathcal{W}_{non-BPS} = \frac{1}{4}\frac{P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}}{\sqrt{Z_J\overline{Z}^J}} = \frac{P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}}{4\rho}. \tag{5.53}$$

The relevant non-BPS flow parameters are

$$r^2_{H,non-BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=M^2_{ADM,non-BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=\mathcal{W}^2_{non-BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=$$

$$=\lim_{\tau\to 0^-}\left(Z_I\overline{Z}^I\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=$$

$$=\frac{1}{2}\left[\begin{array}{l}(X_\infty X^\dagger_\infty)^{AB}Q_A\overline{Q}_B+(\sqrt{1+X^\dagger_\infty X_\infty}\,X^\dagger_\infty)^{i\,A}\overline{Q}_i\overline{Q}_A+\\ \\ +(X_\infty\sqrt{1+X^\dagger_\infty X_\infty})^{B\,j}Q_BQ_j+(1+X^\dagger_\infty X_\infty)^{k\,l}\overline{Q}_kQ_l\end{array}\right];\tag{5.54}$$

$$\Sigma_{i,non-BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=2\lim_{\tau\to 0^-}\left(\partial_i\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)=$$

$$=\frac{1}{2}\left[\frac{P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}}{\sqrt{Z_J\overline{Z}^J}}\right]_\infty=\frac{1}{2}\left[\frac{P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}}{\rho}\right]_\infty=$$

$$=\frac{1}{2M_{ADM,non-BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)}\left(P_{IAB,i}\overline{Z}^I\overline{Z}^{AB}\right)_\infty;\tag{5.55}$$

$$R^2_{H,non-BPS}=\lim_{\tau\to 0^-}\left[\begin{array}{l}\mathcal{W}^2_{non-BPS}\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)+\\ \\ -4G^{i\overline{j}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\left(\partial_i\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\cdot\\ \\ \cdot\left(\overline{\partial}_{\overline{j}}\mathcal{W}_{non-BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)\end{array}\right]=$$

$$=-\mathcal{I}_2\left(p,q\right)=V_{BH,non-BPS}=\frac{S_{BH,non-BPS}\left(p,q\right)}{\pi}.\tag{5.56}$$

Eq. (5.49) proves Eq. (3.36) for the non-BPS ($Z_{AB}=0$) attractor flow of the considered $\mathcal{N}=3$, $d=4$ supergravity. Such a result was obtained by using Eq. (5.53) and computing that

$$
\begin{aligned}
4G^{i\bar{j}}\left(\partial_i\mathcal{W}_{non-BPS}\right)\overline{\partial}_{\bar{j}}\mathcal{W}_{non-BPS} &= 4G^{i\bar{j}}\left(\partial_i\rho\right)\overline{\partial}_{\bar{j}}\rho = \\
&= \frac{1}{4}\delta_{IK}\left(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}\right)\frac{\overline{Z}^{AB}Z^{CD}\overline{Z}^{I}Z^{K}}{Z_J\overline{Z}^{J}} = \\
&= \frac{1}{2}Z_{AB}\overline{Z}^{AB} = \mathcal{Z}_1^2. \qquad (5.57)
\end{aligned}
$$

The considerations made at the end of Subsect. 6.1 hold also for the considered attractor flow.

It is worth noticing out that Eqs. (5.49) and (5.56) are consistent, because, as pointed out above, the ($\frac{1}{3}$-BPS)- and non-BPS ($Z_{AB}=0$)- supporting BH charge configurations in the considered theory are respectively defined by the *quadratic* constraints $\mathcal{I}_2\left(p,q\right)>0$ and $\mathcal{I}_2\left(p,q\right)<0$.

As yielded by Eqs. (5.48) and (5.55) for both *non-degenerate* attractor flows of the considered theory it holds the following relation among *scalar charges* and *ADM mass*:

$$
\Sigma_i = \frac{1}{2M_{ADM}}\lim_{\tau\to 0^-} P_{IAB,i}\overline{Z}^{I}\overline{Z}^{AB}. \qquad (5.58)
$$

7. Black Hole Entropy in Minimally Coupled $\mathcal{N}=2$ and $\mathcal{N}=3$ Supergravity

It is here worth remarking that the classical Bekenstein-Hawking [**49**] $d=4$ BH entropy S_{BH} for *minimally coupled* $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity is given by the following $SU\left(m,n\right)$-invariant expression:

$$
\frac{S_{BH}}{\pi} = \frac{1}{2}\left|q^2+p^2\right|, \qquad (5.59)
$$

where $q^2\equiv\eta^{\Lambda\Sigma}q_\Lambda q_\Sigma$ and $p^2\equiv\eta_{\Lambda\Sigma}p^\Lambda p^\Sigma$, $\eta^{\Lambda\Sigma}=\eta_{\Lambda\Sigma}$ being the Lorentzian metric with signature $\left(m,n\right)$. As said above, $\mathcal{N}=2$ is obtained by putting $m=1$, whereas $\mathcal{N}=3$ is given by $m=3$. Thus, in Eq. (5.59) the positive signature pertains to the *graviphoton charges*, while the negative signature corresponds to the charges given by the fluxes of the vector field strengths from the matter multiplets.

The supersymmetry-preserving features of the attractor solution depend on the sign of q^2+p^2. The limit case $q^2+p^2=0$ corresponds to the so-called *small* BHs (which however, in the case $\mathcal{N}=3$, do *not* enjoy an enhancement of supersymmetry, contrarily to what usually happens in $\mathcal{N}\geqslant 4$, $d=4$ supergravities; see *e.g.* the treatment in [**48**]).

By setting $n=0$ in $\mathcal{N}=3$, $d=4$ supergravity (with resulting U-duality $U(3)$ which, due to the absence of scalars, coincides with the $\mathcal{N}=3$ $\mathcal{R}$-symmetry $U(3)$ [**66**]), one gets

$$\frac{S_{BH}}{\pi}=\frac{1}{2}\left[q_1^2+q_2^2+q_3^2+\left(p^1\right)^2+\left(p^2\right)^2+\left(p^3\right)^2\right], \tag{5.60}$$

which is nothing but the sum of the entropies of three *extremal* (and thus BPS; see *e.g.* the discussion in [**46**]) Reissner-Nördstrom BHs, *without any interference terms*. Such a result can be simply understood by recalling that the generalization of the Maxwell electric-magnetic duality $U(1)$ to the case of n Abelian gauge fields is $U(n)$ [**62**], and that the expression in the right-hand side of Eq. (5.60) is the unique possible $U(3)$-invariant combination of charges.

Moreover, it is here worth noticing that $\mathcal{N}=3$, $d=4$ supergravity is the only $\mathcal{N}>2$ supergravity in which the gravity multiplet does *not* contain any scalar field at all, analogously to what happens in the case $\mathcal{N}=2$. Thus, in *minimally coupled* $\mathcal{N}=2$[1] and $\mathcal{N}=3$, $d=4$ supergravity the *pure* supergravity theory, obtained by setting $n=0$, is *scalarless*, with the U-duality coinciding with the $\mathcal{R}$-symmetry [**66**].

This does *not* happen for all other $\mathcal{N}>2$ theories. For instance, the $\mathcal{N}=4$, $d=4$ gravity multiplet does contain one complex scalar field (usually named *axion-dilaton*) and six Abelian vectors; thus, the *pure* $\mathcal{N}=4$ theory, obtained by setting $n=0$, is not scalarless. By further truncating four vectors out (*i.e.* by performing a $(U(1))^6\rightarrow(U(1))^2$ gauge truncation) and analyzing the bosonic field content, one gets the bosonic sector of $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to one vector multiplet, the so-called *axion-dilaton supergravity*.

[1]Let us notice also that $\mathcal{N}=2$ *minimally coupled* theory is the only (symmetric) $\mathcal{N}=2$, $d=4$ supergravity which yields the *pure* $\mathcal{N}=2$ supergravity simply by setting $n=0$.

CHAPTER 6

N=5 Supergravity black holes

The 10 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the three-fold antisymmetric irreducible representation **20** of the U-duality group $G = SU(1,5)$ (or equivalently of the compact form $SU(6)_{\mathbb{C}}$), andnot in itsfundamental representation **6**.

$Z_{AB} = Z_{[AB]}$, $A, B = 1,2,3,4,5 = \mathcal{N}$ is the central charge matrix. By means of a suitable transformation of the $\mathcal{R}$-symmetry $H_{\mathcal{N}=5} = U(5)$, Z_{AB} can be skew-diagonalized writing the matrix in its normal form

$$Z_{AB} = \begin{pmatrix} \mathcal{Z}_1 \epsilon & & \\ & \mathcal{Z}_2 \epsilon & \\ & & 0 \end{pmatrix}, \tag{6.1}$$

where $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{R}_0^+$ are the $\mathcal{N} = 5$ skew-eigenvalues, which can be ordered as $\mathcal{Z}_1 \geqslant \mathcal{Z}_2$ without any loss of generality, and can be expressed as

$$\begin{cases} \mathcal{Z}_1 = \frac{1}{\sqrt{2}}\sqrt{I_1 + \sqrt{2I_2 - I_1^2}}; \\ \\ \mathcal{Z}_2 = \frac{1}{\sqrt{2}}\sqrt{I_1 - \sqrt{2I_2 - I_1^2}}; \end{cases} \Longleftrightarrow \begin{cases} I_1 = \mathcal{Z}_1^2 + \mathcal{Z}_2^2; \\ \\ I_2 = \mathcal{Z}_1^4 + \mathcal{Z}_2^4, \end{cases} \tag{6.2}$$

where

$$I_1 \equiv \frac{1}{2} Z_{AB} \overline{Z}^{AB}, \tag{6.3}$$

$$I_2 \equiv \frac{1}{2} Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA}, \tag{6.4}$$

are the two unique (moduli-dependent) $H_{\mathcal{N}=5}$ invariants.

1. Symplectic sections

In $N = 5$ Supergravity no matter coupling is allowed, so we only deal with central charges. The scalar manifold of the theory is

$$\frac{SU(1,5)}{U(5)} \,, \tag{6.5}$$

From the Lagrangian of $N = 5$ supergravity [**51**] we read

$$\mathcal{L}^{vec}_{kin} = -\frac{1}{8}V(2S^{ij,kl} - \delta^{ik}\delta^{jl})F^{+}_{\mu\nu\,ij}F^{+\,\mu\nu}{}_{kl} + \text{h.c.}\ , \tag{6.6}$$

and, accordingly to the Gaillard Zumino construction, (read eq. (36) from [**48**])

$$\begin{aligned}\mathcal{L}^{vec}_{kin} &= -i\,\mathcal{N}_{ij,kl}F^{+\,ij}F^{+\,kl} + h.c. = \\ &= \sqrt{-g}\,\frac{i}{4}\mathcal{N}_{ij,kl}F^{+\,ij}_{\mu\nu}F^{+\,kl,\mu\nu} + h.c.\ ,\end{aligned} \tag{6.7}$$

so we can identify the kinetic matrix $N^{ij,kl}$ with

$$N^{ij,kl} = i(S^{ij,kl} - \frac{1}{2}\delta^{ik}\delta^{jl})\ . \tag{6.8}$$

The matrix S satisfies the relation

$$(\delta^{ij}_{kl} - \bar{S}^{ij,kl})S^{kl,mn} = \delta^{ij}_{mn}\ , \tag{6.9}$$

where, for a suitable choice of the scalar fields,

$$\bar{S}^{ij,kl} = -\frac{1}{2}\epsilon^{ijkla}\phi_a\ . \tag{6.10}$$

We then find

$$S^{ij,kl} = \frac{1}{1-(\phi_i)^2}(\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijkla}\phi_a - 2\delta_{[i\,[k}\phi_{l]}\phi_{j]})\ , \tag{6.11}$$

where the last term is normalized as

$$\delta^{[i}_{[k}\phi^{j]}\phi_{l]} = \frac{1}{4}(\delta^i_k\phi^j\phi_l \pm perm...) \tag{6.12}$$

so that we can write the kinetic matrix as

$$\mathcal{N}_{ij,kl} = \frac{\alpha}{1-(\phi_i)^2}\left(\frac{1}{2}(1+(\phi_i)^2)\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijkla}\phi_a - 2\delta_{[i\,[k}\phi_{l]}\phi_{j]}\right)\ , \tag{6.13}$$

where α is a factor to be determined by the relations satisfied by $\mathbf{f}$ and $\mathbf{h}$ as symplectic sections.

Since we can write the supersymmetry transformation for the vector field as [**67**]

$$\delta A^{ij}_\mu = 2f^{ij\,AB}\bar{\psi}_{A\mu} + 2f^{ij}_{AB}\bar{\psi}^A_\mu \epsilon^B \ , \tag{6.14}$$

we compare this formula with the one from [**51**]

$$\delta A^{ij}_\mu = (\bar{S}^{ij,kl} - \delta^{ij,kl})(C^{-1})_{kl}{}^{AB}(\bar{\epsilon}^C\gamma_\mu\chi_{ABC} + 2\sqrt{2}\bar{\epsilon}_A\psi_B) \ , \tag{6.15}$$

where

$$\begin{aligned} C_{ij}{}^{kl} &= \frac{1}{e_1}\delta^{ij}_{kl} - 2\frac{e_2}{e_1}\delta^{[i}_{[k}\phi^{j]}\phi_{l]} \ , \\ (C^{-1})_{ij}{}^{kl} &= \left(e_1\delta^{kl}_{ij} + 2e_2\delta^{[i}_{[k}\phi^{j]}\phi_{l]}\right) \ , \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} e_1^2 &= \frac{1}{1-|\phi|^2} \ , \\ e_2 &= \frac{1}{|\phi|^2}(1-e_1) \ . \end{aligned} \tag{6.17}$$

We simply get, then

$$f^{ij}{}_{AB} = \left(e_1\delta^{ij}_{AB} + \frac{e_1}{2}\epsilon^{ijABm}\phi_m + 2e_2\delta^{[A}_{[i}\phi^{B]}\phi_{j]}\right) \ , \tag{6.18}$$

The symplectic section $\mathbf{h}$ is

$$(\mathbf{h})_{ij,AB} = \mathcal{N}_{ij,mn}(\mathbf{f})^{mn}{}_{AB} \ , \tag{6.19}$$

and explicitely

$$\begin{aligned} h_{ij,AB} = {} & \frac{\alpha}{1-(\phi_i)^2}\left(\frac{1}{2}(1+(\phi_i)^2)\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijkla}\phi_a - 2\delta_{[i\,[k}\phi_{l]}\phi_{j]}\right)\cdot \\ & \cdot\left(e_1\delta^{kl}_{AB} + \frac{e_1}{2}\epsilon^{klABm}\phi_m + 2e_2\delta^{[A}_{[k}\phi^{B]}\phi_{l]}\right) = \\ = {} & \alpha\left(\frac{e_1}{2}\delta^{ij}_{AB} - \frac{e_1}{4}\epsilon^{ijABm}\phi_m + e_2\delta^{[A}_{[i}\phi^{B]}\phi_{j]}\right) \ . \end{aligned} \tag{6.20}$$

We now check our results and fix the numerical factor in front of $\mathcal{N}$, by writing explicitely the identities

$$(\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f}) = -i1 \ , \tag{6.21}$$

$$\mathbf{f}^T\mathbf{h} - \mathbf{h}^T\mathbf{f} = 0 \ . \tag{6.22}$$

We begin with the second which is easly computed, once we recall that

$$\begin{aligned}
&\frac{1}{4}\epsilon^{ijABa}\epsilon^{ijCDb}\phi_a\phi_b = (\phi_i)^2\delta^{AB}_{CD} - 2\delta_{[A[C}\phi_{D]}\phi_{B]} \ , \\
&\delta^{[A}_{[i}\phi^{B]}\phi_{j]}\delta^{[C}_{[i}\phi^{D]}\phi_{j]} = \frac{1}{2}(\phi_i)^2\delta^{[A[C}\phi^{D]}\phi^{B]} \ ,
\end{aligned} \tag{6.23}$$

since

$$\begin{aligned}
(\mathbf{f}^T\mathbf{h})^{AB}_{CD} &= (\mathbf{h}^T\mathbf{f})^{AB}_{CD} = \\
&= \alpha\left(\frac{e_1}{2}\delta^{ij}_{AB} - \frac{e_1}{4}\epsilon^{ijABm}\phi_m + e_2\delta^{[A}_{[i}\phi^{B]}\phi_{j]}\right)\cdot \\
&\quad \cdot\left(e_1\delta^{ij}_{CD} + \frac{e_1}{2}\epsilon^{ijCDm}\phi_m + 2e_2\delta^{[C}_{[i}\phi^{D]}\phi_{j]}\right) = \\
&= \alpha\left(\frac{e_1^2}{2}\delta^{AB}_{CD}(1-\phi_i^2) + e_1^2\delta_{[A[C}\phi_{D]}\phi_{B]} + e_2^2(\phi_i)^2\delta^{[A[C}\phi^{D]}\phi^{B]} + \right. \\
&\qquad \left. +2e_1e_2\, Re\left\{\delta^{[C}_{[A}\phi^{D]}\phi_{B]}\right\}\right) \ ,
\end{aligned} \tag{6.24}$$

We write now

$$\begin{aligned}
\mathbf{f}^\dagger\mathbf{h} &= f_{ij}{}^{AB}h_{ij\,CD} = \\
&= \alpha\left(e_1\delta^{AB}_{ij} + \frac{e_1}{2}\epsilon_{ijABm}\phi^m + 2e_2\delta^{[i}_{[A}\phi^{j]}\phi_{B]}\right)\cdot \\
&\quad \cdot\left(\frac{e_1}{2}\delta^{ij}_{CD} - \frac{e_1}{4}\epsilon^{ijCDa}\phi_a + e_2\delta^{[C}_{[i}\phi^{D]}\phi_{j]}\right) = \\
&= \alpha\left[\frac{e_1^2}{2}\delta^{AB}_{CD} + \frac{e_1^2}{4}\epsilon_{ABCDm}\phi^m + e_1e_2\delta^{[C}_{[A}\phi^{D]}\phi_{B]} + \right. \\
&\quad -\frac{e_1^2}{4}\epsilon^{ABCDa}\phi_a - \frac{e_1^2}{2}\delta^{AB}_{CD} + e_1^2\delta^{[C}_{[A}\phi^{D]}\phi_{B]} + \\
&\quad \left. + e_1e_2\delta^{[C}_{[A}\phi^{D]}\phi_{B]} + e_2^2|\phi|^2\delta^{[C}_{[A}\phi^{D]}\phi_{B]}\right] \ ,
\end{aligned} \tag{6.25}$$

and since $|\phi|^2 e_2 = 1 - e_1$, and $e_1^2 + e_1e_2 + e_2 = 0$,

$$f_{ij}{}^{AB}\ h_{ij\,CD} = \alpha\left[\frac{1}{2}\delta^{AB}_{CD} + \frac{i}{2}e_1^2 Im(\epsilon_{ABCDm}\phi^m)\right] \ . \tag{6.26}$$

By an analogous calculation we see that

$$\begin{aligned}
\mathbf{h}^\dagger\mathbf{f} &= (h_{ij\,CD})^\dagger f_{ij}{}^{AB} = \\
&= \alpha^*\left[\frac{1}{2}\delta^{AB}_{CD} + \frac{i}{2}e_1^2 Im(\epsilon^{ABCDa}\phi_a)\right] \ ,
\end{aligned} \tag{6.27}$$

so that, if we take $\alpha = -i$, identity (6.21) is satisfied.

2. Central charges

[1] We build central charges by their definition in terms of electric and magnetic ones, in the same representation of Z_{AB}, as

$$Z_{AB} = f^{ij}_{AB} q_{ij} - h_{ij,AB} p^{ij} \ , \tag{6.29}$$

and from (6.18) and (6.20), using complex electromagnetic charges, we find

$$Z_{AB} = \left(e_1 q^{AB} + \frac{e_1}{2} \epsilon^{ABijm} \bar{q}^{ij} \phi_m - 2 e_2 \phi^{[A} q^{B]N} \phi_N \right) \ . \tag{6.30}$$

The U-duality invariant for $N = 5$ theory is

$$I_{N=5} = 4Tr(A^2) - (Tr\, A)^2 \ , \tag{6.31}$$

where

$$Tr\, A = Z_{AB} \bar{Z}^{BA} \ , \tag{6.32}$$

$$Tr(A)^2 = Z_{AB} \bar{Z}^{BC} Z_{CD} \bar{Z}^{DA} \ , \tag{6.33}$$

and we notice that $TrA = -2V_{AB}$. The first order fake superpotential, corresponding to the non degenerate attractor flow, the $\frac{1}{5}$-BPS one, is

$$\begin{aligned} \mathcal{W}^2_{(\frac{1}{5}-)BPS} &= \frac{1}{2} \left[\frac{1}{2} Z_{AB} \overline{Z}^{AB} + \sqrt{Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} - \frac{1}{4} \left(Z_{AB} \overline{Z}^{AB} \right)^2} \right] = \\ &= \frac{1}{2} \left[I_1 + \sqrt{2 I_2 - I_1^2} \right] = \mathcal{Z}_1^2. \end{aligned} \tag{6.34}$$

3. Attractor Equations

The black hole potential, in absence of matter charges, is

$$V_{BH} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} \ , \tag{6.35}$$

---

[1] We rescale the section $\mathbf{f}$ by a factor $\frac{1}{2}$, and the $\mathcal{N}$ matrix at the same time by a factor of 2. The definition of $\mathbf{h}$ section and the identity (6.21) are left unchanged, but we avoid with this redefinition an unsuitable rescaling of magnetic charges, once we have to deal with complex charges, such as

$$q^{AB} \equiv \frac{1}{2} (q_{AB} + i p^{AB}) \ . \tag{6.28}$$

and the minimum condition becomes

$$0 = dV_{BH} = \frac{1}{2} DZ_{AB} \bar{Z}^{AB} + \frac{1}{2} Z_{AB} D\bar{Z}^{AB} \ . \tag{6.36}$$

The Maurer-Cartan equations for the symplectic sections define the covariant derivative of the charges as

$$\begin{aligned} DZ_{AB} &= \frac{1}{2} \bar{Z}^{AB} P_{ABCD} = \\ &= \frac{1}{2} \bar{Z}^{AB} \epsilon_{ABCDE} P^E \ , \end{aligned} \tag{6.37}$$

so that (6.36) becomes

$$dV_{BH} = 0 = \epsilon_{ABCDE} \bar{Z}^{AB} \bar{Z}^{CD} P^E + \epsilon^{ABCDE} Z_{AB} Z_{CD} P_E \ , \tag{6.38}$$

and we find the two equations

$$\begin{aligned} \epsilon^{ABCDE} Z_{AB} Z_{CD} &= 0 \ , \\ \epsilon_{ABCDE} \bar{Z}^{AB} \bar{Z}^{CD} &= 0 \ . \end{aligned} \tag{6.39}$$

We explicite the first one, using the expression for the central charge written in (6.30), as

$$\begin{aligned} 0 &= \epsilon^{ABCDE} Z_{AB} Z_{CD} = \\ &= e_1^2 \epsilon^{ABCDE} q^{AB} q^{CD} + e_1^2 (\epsilon^{ABCDn} \bar{q}^{AB} \bar{q}^{CD} \phi_n) \phi_E + 8 e_1^2 (q\bar{q}) \phi_E + \\ &\quad + 16 e_1^2 \phi_n q^{nA} \bar{q}^{AE} - 4 e_1 e_2 \epsilon^{ABCDE} q^{AB} \phi^C q^{Di} \phi_i + \\ &\quad - 8 e_1 e_2 (\phi_l q^{Dl} \bar{q}^{CD} \phi^C) \phi_E - 16 e_1 e_2 |\phi|^2 \phi_l q^{il} \bar{q}^{iE} \ . \end{aligned} \tag{6.40}$$

The criticality conditions (6.39) and (6.40) are satisfied for a unique class of critical points, identifying the ($\frac{1}{5}$-)BPS solution

$$\mathcal{Z}_2 = 0, \ \ \mathcal{Z}_1 > 0. \tag{6.41}$$

It is worth counting here the degrees of freedom related to eqs. (6.39) and (6.39), or equivalently to the unique $\frac{1}{5}$-BPS solution given by (6.41). Equations (6.39) and (6.39) are 10 *real* equations, but actually only 6 *real* among them are independent. Thus a moduli space of $\frac{1}{5}$-BPS attractors, spanned by the 2 complex scalars unstabilized by (6.41) does [**45**] exist. This counting of flat directions of V_{BH} at its $\frac{1}{5}$-BPS critical points are given in terms of $\mathcal{N} = 2$ hyperscalars' degrees of freedom in the $\mathcal{N} = 5 \rightarrow \mathcal{N} = 2$ supersymmetry reduction.

4. Black Hole Parameters for $\frac{1}{5}$-BPS Flow

By using the *Maurer-Cartan Eqs.* of $\mathcal{N}=5$, $d=4$ supergravity (see *e.g.* [**65, 63, 64**]), one gets [**39**]

$$\begin{aligned}\partial_i\mathcal{Z}_1 &= \partial_i\mathcal{W}_{BPS} = \\ &= \frac{P_{,i}}{\sqrt{2}}\sqrt{\frac{1}{2}Z_{AB}\overline{Z}^{AB}-\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}} = P_{,i}\mathcal{Z}_2,\end{aligned} \tag{6.42}$$

where $P\equiv P_{1234}$, $P_{ABCD}\equiv P_{ABCD,i}dz^i=\epsilon_{ABCDE}P^E$ being the holomorphic Vielbein of $\mathcal{M}_{\mathcal{N}=5}$. Here, ∇ denotes the $U(1)$-Kähler and $H_{\mathcal{N}=5}$-covariant differential operator.

Thus, by using the explicit expressions of $\mathcal{W}^2_{BPS}$ given by Eq. (6.34), using the *Maurer-Cartan Eqs.* of $\mathcal{N}=5$, $d=4$ supergravity (see *e.g.* [**65, 63, 64**]), following the treatment of the first order formalism, one respectively obtains the following expressions of the *(square) ADM mass*, *covariant scalar charges* and *(square) effective horizon radius* for the $\frac{1}{5}$-BPS attractor flow:

$$\begin{aligned}r^2_{H,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right) &= M^2_{ADM,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)=\mathcal{W}^2_{BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)= \\ &= \frac{1}{2}\lim_{\tau\to0^-}\left[\frac{1}{2}Z_{AB}\overline{Z}^{AB}+\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}\right]= \\ &= \left.\mathcal{Z}_1^2\right|_\infty;\end{aligned} \tag{6.43}$$

$$\begin{aligned}\Sigma_{i,BPS}\left(z_\infty,\overline{z}_\infty,p,q\right) &\equiv 2\lim_{\tau\to0^-}\left(\partial_i\mathcal{W}_{BPS}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right),p,q\right)= \\ &= \sqrt{2}\lim_{\tau\to0^-}\left[P_{,i}\sqrt{\frac{1}{2}Z_{AB}\overline{Z}^{AB}-\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}}\right]= \\ &= 2\left(P_{,i}\mathcal{Z}_2\right)_\infty;\end{aligned} \tag{6.44}$$

$$
\begin{aligned}
&R^2_{H,BPS} = \mathcal{W}^2_{BPS}\left(z_\infty,\overline{z}_\infty,p,q\right)+\\
&-4G^{i\overline{j}}\left(z_\infty,\overline{z}_\infty\right)\left(\partial_i\mathcal{W}_{BPS}\right)\left(z_\infty,\overline{z}_\infty,p,q\right)\left(\overline{\partial}_{\overline{j}}\mathcal{W}_{BPS}\right)\left(z_\infty,\overline{z}_\infty,p,q\right)=\\
&=\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}=\\
&=\sqrt{2\mathcal{Z}_1^4+2\mathcal{Z}_2^4-\left(\mathcal{Z}_1^2+\mathcal{Z}_2^2\right)^2}=\\
&=\mathcal{Z}_1^2-\mathcal{Z}_2^2=\sqrt{\mathcal{I}_4\left(p,q\right)}>0.
\end{aligned}
\tag{6.45}
$$

Eq. (6.45) proves Eq. (3.36) for the $\frac{1}{5}$-BPS attractor flow of the considered $\mathcal{N}=5$, $d=4$ supergravity. Such a result was obtained by using Eq. (6.42) and computing that

$$
\begin{aligned}
&4G^{i\overline{j}}\left(\partial_i\mathcal{W}_{BPS}\right)\overline{\partial}_{\overline{j}}\mathcal{W}_{BPS}=4G^{i\overline{j}}\left(\partial_i\mathcal{Z}_1\right)\overline{\partial}_{\overline{j}}\mathcal{Z}_1=\\
&=2G^{i\overline{j}}P_{,i}\overline{P}_{,\overline{j}}\left[\frac{1}{2}Z_{AB}\overline{Z}^{AB}-\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}\right]=\\
&=\frac{1}{2}\left[\frac{1}{2}Z_{AB}\overline{Z}^{AB}-\sqrt{Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2}\right]=\mathcal{Z}_2^2,
\end{aligned}
\tag{6.46}
$$

where the relation

$$4G^{i\overline{j}}P_{,i}\overline{P}_{,\overline{j}}=1 \tag{6.47}$$

was used.

It is worth noticing out that Eq. (6.45) is consistent, because, as pointed out above, the $\frac{1}{5}$-BPS-supporting BH charge configurations in the considered theory is defined by the *quartic* constraints $\mathcal{I}_4\left(p,q\right)>0$.

CHAPTER 7

The case of $\mathcal{N}=4$ Supergravity and *Dualities*

1. N=4 Pure Supergravity

The special Kahler scalar manifold is

$$\mathcal{M}_{\mathcal{N}=4,pure}=\frac{G_{\mathcal{N}=4,pure}}{H_{\mathcal{N}=4,pure}}=\frac{SU(1,1)\times SU(4)}{U(1)\times SU(4)}=\frac{SU(1,1)}{U(1)},\ dim_{\mathbb{R}}=2, \tag{7.1}$$

spanned by the complex scalar

$$s\equiv a+ie^{-2\varphi},\ a,\varphi\in\mathbb{R}, \tag{7.2}$$

where a and φ are usually named *axion* and *dilaton*, respectively. The invariant of the scalar manifold is a quartic expression in terms of electric and magnetic charges

$$\mathcal{I}_4=4\left[p^2q^2-(p\cdot q)^2\right]\ . \tag{7.3}$$

It is worth noticing that when the symplectic index are $\Lambda,\Sigma=1,2$, the theory corresponds to the truncation $(U(1))^6\rightarrow(U(1))^2$ of the gauge group, and $\mathcal{I}_4$ is a *perfect square* that, if expressed as a function of the skew-eigenvalues of the central charge function of the truncated Supergravity, it reduces to

$$\mathcal{I}_4=\left[(Z_1)^2-(Z_2)^2\right]^2\ , \tag{7.4}$$

thus reproducing the *quadratic* invariant $\mathcal{I}_2$ of the minimally coupled $\mathcal{N}=2$, $d=4$ sequence. Also in this case it is possible to apply the first order formalism.

From the symplectic structure [**63, 64, 48**] the symplectic sections are

$$f^{\Lambda}_{AB}=e^{\varphi}\delta^{\Lambda}_{AB},\quad h_{\Lambda|AB}=se^{\varphi}\delta_{\Lambda|AB}=\left(ae^{\varphi}+ie^{-\varphi}\right)\delta_{\Lambda|AB}, \tag{7.5}$$

and the kinetic vector matrix is

$$\mathcal{N}_{\Lambda\Sigma}=\left(\mathbf{hf}^{-1}\right)_{\Lambda\Sigma}=s\delta_{\Lambda\Sigma}. \tag{7.6}$$

We can then write the central charge matrix

$$Z_{AB}=f^{\Lambda}_{AB}q_{\Lambda}-h_{\Lambda|AB}p^{\Lambda}\ =e^{\varphi}\delta^{\Lambda}_{AB}q_{\Lambda}-se^{\varphi}\delta_{\Lambda|AB}p^{\Lambda}=-e^{\varphi}\left(sp_{AB}-q_{AB}\right), \tag{7.7}$$

and, from (6.2)-(6.4), the invariants are

$$I_1 \equiv \frac{1}{2} Z_{AB}\overline{Z}^{AB} = \mathcal{Z}_1^2 + \mathcal{Z}_2^2 = \\ = \left(e^{2\varphi}a^2 + e^{-2\varphi}\right)p^2 + e^{2\varphi}q^2 - 2ae^{2\varphi}p\cdot q; \tag{7.8}$$

$$I_2 \equiv \frac{1}{2} Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA} = \mathcal{Z}_1^4 + \mathcal{Z}_2^4 = \\ = \frac{1}{2}e^{4\varphi}\left(sp_{AB} - q_{AB}\right)\left(\overline{s}p^{BC} - q^{BC}\right)\left(sp_{CD} - q_{CD}\right)\left(\overline{s}p^{DA} - q^{DA}\right), \tag{7.9}$$

where $p^2 \equiv (p^1)^2 + ... + (p^6)^2$, $q^2 \equiv q_1^2 + ... + q_6^2$, and $p \cdot q \equiv p^\Lambda q_\Lambda$.

Only $\frac{1}{4}$-BPS attractor flow is non-degenerate (*i.e.* corresponding to *large* black holes [**48**]), and the corresponding *first order fake superpotential* is identical to the one of the $\frac{1}{5}$-BPS attractor flow in $\mathcal{N} = 5$, $d = 4$ supergravity [**39**], given by (6.34), which in the considered framework can be further elaborated as follows:

$$\mathcal{W}^2_{\left(\frac{1}{4}-\right)BPS} = \frac{1}{2}\left[I_1 + \sqrt{2I_2 - I_1^2}\right] = \mathcal{Z}_1^2 = \\ = \frac{e^{2\varphi}}{4}\left[\begin{array}{l} \left(sp_{AB} - q_{AB}\right)\left(\overline{s}p^{AB} - q^{AB}\right) + \\ + \sqrt{\begin{array}{l} 4\left(sp_{AB} - q_{AB}\right)\left(\overline{s}p^{BC} - q^{BC}\right)\left(sp_{CD} - q_{CD}\right)\left(\overline{s}p^{DA} - q^{DA}\right) + \\ - \left[\left(sp_{AB} - q_{AB}\right)\left(\overline{s}p^{AB} - q^{AB}\right)\right]^2 \end{array}} \end{array}\right]. \tag{7.10}$$

In the case of matter coupled $\mathcal{N} = 4$ Supergravity the scalar manifold is real

$$\mathcal{M}_{\mathcal{N}=4} = \frac{G_{\mathcal{N}=4}}{H_{\mathcal{N}=4}} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6)\times SO(n)},\ dim_{\mathbb{R}} = 6n + 2. \tag{7.11}$$

The *quartic* $G_{\mathcal{N}=4}$-invariant $\mathcal{I}_4$ of $\mathcal{N} = 4$, $d = 4$ supergravity is the following unique (moduli-independent) $G_{\mathcal{N}=4}$-invariant combination of $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ [**65**]:

$$\mathcal{I}_4 \equiv \mathbb{I}_1^2 - \mathbb{I}_2\mathbb{I}_3 = \mathbb{I}_1^2 - |\mathbb{I}_2|^2 = \\ = \left(\mathcal{Z}_1^2 - \mathcal{Z}_2^2\right)^2 + \left(\rho_1^2 + \rho_2^2\right)^2 - 2\left(\mathcal{Z}_1^2 + \mathcal{Z}_2^2\right)\left(\rho_1^2 + \rho_2^2\right) + \\ +4\mathcal{Z}_1\mathcal{Z}_2\left[\rho_1^2 + \rho_2^2 cos\left(2\theta\right)\right] - \left[\rho_1^4 + \rho_1^4 + 2\rho_1^2\rho_2^2 cos\left(2\theta\right)\right]. \tag{7.12}$$

$\mathcal{I}_4$ This last expression is a non-trivial *perfect square* of a function of degree 2 of $\mathcal{Z}_1$, $\mathcal{Z}_2$, ρ_1, ρ_2 and θ *only* in the *pure* supergravity theory (obtained by setting $n=0$), *i.e.* only in the case $\rho_1=\rho_2=0$. In such a limit, equation (7.12) consistently reduces to (7.14).

This can be explicitly read, as an exemple, computing the invariant in the case $n=1$ (which uplifts to *pure* $\mathcal{N}=4$, $d=5$ supergravity) which acquires the following form:

$$\mathcal{I}_4=(\mathcal{Z}_1-\mathcal{Z}_2)^2\left(\mathcal{Z}_1+\mathcal{Z}_2+\sqrt{2}\rho_1\right)\left(\mathcal{Z}_1+\mathcal{Z}_2-\sqrt{2}\rho_1\right), \tag{7.13}$$

which is not to a perfect square of $\mathcal{Z}_1$, $\mathcal{Z}_2$ and ρ_1.

2. Peculiarity of *Pure* $\mathcal{N}=4$ and $\mathcal{N}=5$ Supergravity

The expression of R_H^2 in the extremal case $c=0$ given by (3.35) has been shown to hold in $d=4$ for

- *minimally coupled* $\mathcal{N}=2$ theory;
- $\mathcal{N}=3$;
- $\mathcal{N}=5$;
- $\mathcal{N}=4$ *pure.*

The crucial difference among this theories is that, whereas the U-invariant of minimally coupled $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity is quadratic, the U-invariant of $\mathcal{N}=5$ and pure $\mathcal{N}=4$ theories is quartic in the black hole charges.

Moreover, the form of their Attractor Equations are structurally identical to the ones of the minimally coupled $\mathcal{N}=2$ and $\mathcal{N}=3$, and actually also to the very structure of $\mathcal{W}_{BPS}^2$. As already pointed out, the invariant $\mathcal{I}_4(p,q)$ of $G_{\mathcal{N}=5}$ and $G_{\mathcal{N}=4,pure}$ is a perfect square of a quadratic expression when written in terms of the moduli-dependent skew-eigenvalues $\mathcal{Z}_1$ and $\mathcal{Z}_2$

$$\mathcal{I}_4(p,q)\equiv Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2=Tr\left(A^2\right)-\frac{1}{4}\left(Tr\left(A\right)\right)^2=\left(\mathcal{Z}_1^2-\mathcal{Z}_2^2\right)^2 1, \tag{7.14}$$

but such a result does not generally hold for all other $\mathcal{N}>2$, $d=4$ supergravities with *quartic* U-invariant (*i.e.* for $\mathcal{N}=4$ *matter coupled* and $\mathcal{N}=6,8$ theories, as well as for $\mathcal{N}=2$ supergravity whose scalar manifold does *not* belong to the aforementioned sequence of complex Grassmannians).

This allows one to state that the relation (in the *extremal case* $c=0$) between the square effective horizon radius R_H^2 and the square BH event horizon radius r_H^2 for the non-degenerate attractor flows of such supergravities, if any, is structurally different from the one given by Eq. (3.35). Of course, in such theories one can still compute the quantity $r_H^2(z_\infty,\overline{z}_\infty,p,q)-G_{i\overline{j}}\Sigma^i\overline{\Sigma}^{\overline{j}}$ (in case, within a real parametrization of the scalar fields), but, also in the extremal case, it will be moduli-dependent, thus not determining $R_H^2(p,q)$.

In the non-extremal case (*i.e.* $c\neq 0$) the expression generalizing Eq. (3.35) is

$$R_+^2(z_\infty,\overline{z}_\infty,p,q)\equiv\frac{S_{BH,c\neq 0}(z_\infty,\overline{z}_\infty,p,q)}{\pi}\equiv R_+^2(z_\infty,\overline{z}_\infty,p,q)= \\ =r_+^2(z_\infty,\overline{z}_\infty,p,q)-G_{i\overline{j}}\Sigma^i\overline{\Sigma}^{\overline{j}} \tag{7.15}$$

can be only guessed, but at present cannot be rigorously proved. Indeed, for non extremal black holes a first order formalism is currently unavailable, so there is no way to compute the scalar charges beside the direct integration of the equations of motion of the scalars.

3. $\mathcal{N}\geqslant 2$ Supergravities with the same Bosonic Sector and "Dualities"

Consider the following relations among 4 dimensional Supergravities.

I)

- $\mathcal{N}=2$ (*matter coupled*) *magic* supergravity based on the degree 3 complex Jordan algebra $J_3^{\mathbb{H}}$;
- $\mathcal{N}=6$ supergravity.

The scalar manifold of both such theories (which can be uplifted to $d=5$) is $\frac{SO^*(12)}{SU(6)\times U(1)}$. It is a rank-3 homogeneous symmetric special Kähler space. In both theories the 16 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the left-handed spinor representation **32** of the U-duality group $SO^*(12)$, which is symplectic and contains the symmetric singlet $\mathbf{1}_a$ in the tensor product $\mathbf{32}\times\mathbf{32}$. The vector fields representation is thus irreducible with respect to both $SO^*(12)$ and $Sp(32,\mathbb{R})$.

II)

- $\mathcal{N}=2$ supergravity minimally coupled to $n=n_V=3$ Abelian vector multiplets;
- $\mathcal{N}=3$ supergravity coupled to $m=1$ matter multiplet.

These two theories are matter coupled and have a quadratic U-invariant, but are not upliftable $d=5$ dimensions. They share the same scalar manifold, $\frac{SU(1,3)}{SU(3)\times U(1)}$, a rank-1 symmetric special Kähler space. The 4 vector field strengths and their duals, as well as their asymptotical fluxes, sit in thefundamental **4** representation of the U-duality group

Orbit	$\mathcal{N}=2$ *minimally coupled*, $n_V=3$	$\mathcal{N}=3,\ m=1$
$\frac{SU(1,3)}{SU(3)}$	$\mathcal{O}_{\frac{1}{2}-BPS}$, *no mod. space*, $\mathcal{I}_{2,\mathcal{N}=2}>0$	$\mathcal{O}_{non-BPS,Z_{AB}=0}$, *no mod. space*, $\mathcal{I}_{2,\mathcal{N}=3}<0$
$\frac{SU(1,3)}{SU(1,2)}$	$\mathcal{O}_{non-BPS,Z=0}$, *mod. space* $=\frac{SU(1,2)}{SU(2)\times U(1)}$, $\mathcal{I}_{2,\mathcal{N}=2}<0$	$\mathcal{O}_{\frac{1}{3}-BPS}$, *mod. space* $=\frac{SU(1,2)}{SU(2)\times U(1)}$, $\mathcal{I}_{2,\mathcal{N}=3}>0$

TABLE 1. **$\mathcal{N}$-dependent BPS-interpretations of the classes of *non-degenerate* orbits of the symmetric special Kähler manifold** $\frac{SU(1,3)}{SU(3)\times U(1)}$

$SU(3,1)$ which is reducible with respect to $SU(3,1)$, but irreducible with respect to $Sp(8,\mathbb{R})$.

The fermionic sector contains 8 fields (because of the supersymmetry invariance of the theory, it is the same number of bosonic fields) for both these theories, but the spin/field content is different, explicitly

$$\begin{aligned}&\mathcal{N}=2\ \textit{minimally coupled},\ n_V=3: && \left[1\left(2\right),2\left(\tfrac{3}{2}\right),1\left(1\right)\right],3\left[1\left(1\right),2\left(\tfrac{1}{2}\right),1_{\mathcal{C}}\left(0\right)\right];\\ &\mathcal{N}=3,\ m=1: && \left[1\left(2\right),3\left(\tfrac{3}{2}\right),3\left(1\right),1\left(\tfrac{1}{2}\right)\right],1\left[1\left(1\right),4\left(\tfrac{1}{2}\right),3_{\mathcal{C}}\left(0\right)\right].\end{aligned} \tag{7.16}$$

It then follows that one can switch between the two theories by transforming 1 gravitino in 1 gaugino. The relation among the various classes of non-degenerate extremal BH attractors is given in Table 1.

When switching between $\mathcal{N}=2$ and $\mathcal{N}=3$, the flip in sign of the quadratic U-invariant $\mathcal{I}_2=q^2+p^2$ can be understood by recalling that $q^2\equiv\eta^{\Lambda\Sigma}q_\Lambda q_\Sigma$ and $p^2\equiv\eta_{\Lambda\Sigma}p^\Lambda p^\Sigma$, with $\eta^{\Lambda\Sigma}=\eta_{\Lambda\Sigma}=diag\left(1,-1,-1,-1\right)$ in the case $\mathcal{N}=2$, and $\eta^{\Lambda\Sigma}=\eta_{\Lambda\Sigma}=diag\left(1,1,1,-1\right)$ in the case $\mathcal{N}=3$ (recall Eq. (5.59)). The positive signature pertains to the *graviphoton charges*, while the negative signature corresponds to the charges given by the asymptotical fluxes of the vector field strengths from the matter multiplets. As

Orbit	$\mathcal{N}=2,\ n_V=7$	$\mathcal{N}=4,\ n=2$
$\frac{SU(1,1)\times SO(2,6)}{SO(2)\times SO(6)}$	$\mathcal{O}_{\frac{1}{2}-BPS}$, *no mod. space*, $\mathcal{I}_{4,\mathcal{N}=2}>0$	$\mathcal{O}_{non-BPS,Z_{AB}=0}$, *no mod. space*, $\mathcal{I}_{4,\mathcal{N}=4}>0$
$\frac{SU(1,1)\times SO(2,6)}{SO(2)\times SO(2,4)}$	$\mathcal{O}_{non-BPS,Z=0}$, *mod. space* $=\frac{SO(2,4)}{SO(2)\times SO(4)}$ $\mathcal{I}_{4,\mathcal{N}=2}>0$	$\mathcal{O}_{\frac{1}{4}-BPS}$, *mod. space* $=\frac{SO(2,4)}{SO(2)\times SO(4)}$ $\mathcal{I}_{4,\mathcal{N}=4}>0$
$\frac{SU(1,1)\times SO(2,6)}{SO(1,1)\times SO(1,5)}$	$\mathcal{O}_{non-BPS,Z\neq 0}$, *mod. space* $=SO(1,1)\times\frac{SO(1,5)}{SO(5)}$ $\mathcal{I}_{4,\mathcal{N}=2}<0$	$\mathcal{O}_{non-BPS,Z_{AB}\neq 0}$, *mod. space* $=SO(1,1)\times\frac{SO(1,5)}{SO(5)}$ $\mathcal{I}_{4,\mathcal{N}=4}<0$

TABLE 2. **$\mathcal{N}$-dependent BPS-interpretations of the classes of *non-degenerate* orbits of the reducible symmetric special Kähler manifold $\frac{SU(1,1)}{U(1)}\times\frac{SO(2,6)}{SO(2)\times SO(6)}$.** The structure of the *"duality"* is analogous to the one pertaining to the manifold $\frac{SO^*(12)}{SU(6)\times U(1)}$ (see point **I** above, as well as Table 9 of [**28**])

yielded by Table 1, the supersymmetry-preserving features of the attractor solutions depend on the sign of $\mathcal{I}_2$.

III)

- $\mathcal{N}=2$ supergravity coupled to $n_V=n+1=7$ Abelian vector multiplets, with scalar manifold $\frac{SU(1,1)}{U(1)}\times\frac{SO(2,6)}{SO(2)\times SO(6)}$;
- $\mathcal{N}=4$ supergravity coupled to $n_m=2$ Abelian vector multiplets (matter multiplets).

The scalar manifold is the same for the two theories

$$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,6)}{SO(2)\times SO(6)}\ , \tag{7.17}$$

and is an homogeneous symmetric reducible special Kähler space, with rank 3. In both theories the 8 vector field strengths and their duals, as well as their asymptotical fluxes, are in the (spinor/doublet)-vector representation $(\mathbf{2},\mathbf{8})$ of the U-duality group $SU(1,1)\times SO(2,6)$, which is symplectic and contains the antisymmetric singlet $\mathbf{1}_a$ in the tensor product $(\mathbf{2},\mathbf{8})\times(\mathbf{2},\mathbf{8})$. It is thus irreducible with respect to both $SU(1,1)\times SO(2,6)$ and $Sp(16,\mathbb{R})$.

Due to the isomorphism $\mathfrak{so}(6,2)\sim\mathfrak{so}^*(8)$, the "dual" supersymmetric interpretation of the scalar manifold $\frac{SU(1,1)}{U(1)}\times\frac{SO(2,6)}{SO(2)\times SO(6)}$ can be considered, disregarding the axion-dilaton sector $\frac{SU(1,1)}{U(1)}$, as a "subduality" of the "duality" discussed in **I**.

Once again we notice that, even if the number of fermion fields is the same, 16 bosons and 16 fermions, the theories have different relevant spin/field contents:

$$\begin{aligned}&\mathcal{N}=2\ \text{“cubic”},\ n_V=7: && \left[1\,(2)\,,2\left(\tfrac{3}{2}\right),1\,(1)\right],7\left[1\,(1)\,,2\left(\tfrac{1}{2}\right),1_{\mathcal{C}}\,(0)\right];\\ &\mathcal{N}=4, n_m=2: && \left[1\,(2)\,,4\left(\tfrac{3}{2}\right),6\,(1)\,,4\left(\tfrac{1}{2}\right),1_{\mathcal{C}}\,(0)\right],2\left[1\,(1)\,,4\left(\tfrac{1}{2}\right),3_{\mathcal{C}}\,(0)\right].\end{aligned}\tag{7.18}$$

From this it follows that one can switch between such two theories by transforming 2 gravitinos in 2 gauginos. The correspondences among the various classes of non-degenerate extremal BH attractors of these two theories is given in Table 2.

Such a "duality" is pretty similar to the one considered above at point **I**; that the sign of the *quartic* U-invariant is unchanged by the *"duality"* relation, and, in this sense, it differs from the *"duality"* between $\mathcal{N}=2$ *minimally coupled*, $n_V=3$ and $\mathcal{N}=3$, $m=1$ considered at point **II**.

All these cases present evidences that interacting bosonic field theories have a unique supersymmetric extension. The sharing of the same bosonic backgrounds with different supersymmetric completions implies the "dual" interpretation with respect to the supersymmetry-preserving properties of non-degenerate extremal BH attractor solutions (see Table 1 and Table 2).

4. Conclusions

Black holes in Supergravity have an extremely rich structure and give an interplay between space-time singularities of Einstein solutions and the solitonic, particle-like structure of these configuration, such as mass, spin and charge.

The bosonic action of Supergravity has been studied and its specific form determined by the embedding of the U-duality group in the symplectic group. Extremal black holes in these theories satisfy an attractor condition, fixing the solution in terms of asymptotical electric and magnetic charges, and erasing any dependence on the scalar hair. The general property of extremization of the central charge in the moduli space has been explicited in the case of $\mathcal{N} = 2$ quadratic, $\mathcal{N} = 3$ matter coupled and $\mathcal{N} = 5$ ungauged Supergravities; all these theories cannot be extended to $d = 5$ space-time dimensions. For the same theories, and for $\mathcal{N} = 4$ Supergravity, the extremal black hole parameters of the non degenerate attractor flows have been formulated in terms of the first order (fake Supergravity) formalism and the resulting effective radius has revealed to be, in these cases, moduli independent.

The Supergravity theories considered are the only ones admitting a quadratic invariant or a quartic invariant that reduces to a perfect square of a quadratic expression, if written in terms of the skew eigenvalues of the central charge matrix, so that this property has revealed to be crucial for the definition of an effective radius, whose dependence on the scalar fields would eventually cancel.

APPENDIX A

$\mathcal{N} = 2$ explicit calculations

1. Some formulas following from definitions, with the explicit dependence on scalar fields (not at the attractor point)

In the case of

$$SU(1,n)/SU(n) \times U(1) \ , \tag{A.1}$$

the Kähler structure of the scalar manifold is entirely defined by the prepotential

$$F(X) = -\frac{i}{2}(X^{0\,2} - X^i X_i) \ ; \tag{A.2}$$

the holomorphic sections are then

$$\begin{aligned} (X^\Lambda, F_\Lambda) &= e^{-K/2}(f^\Lambda, h_\Lambda) \ , \\ \partial_{\bar{i}}(X^\Lambda, F_\Lambda) &= 0 \ , \end{aligned} \tag{A.3}$$

and their explicit dependence in terms of scalar fields is

$$\begin{aligned} X^\Lambda &= (1, z^1, ..., z^n) \ , \\ F_\Lambda \equiv \partial_\Lambda F &= (-i, iz^1, ..., iz^n) \ . \end{aligned} \tag{A.4}$$

Kähler potential is defined as

$$\begin{aligned} e^{-K} &= i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) = \\ &= 2(1 - \sum_i |z^i|^2) \ , \end{aligned} \tag{A.5}$$

while the central charge is

$$\begin{aligned} Z &= e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \\ Z &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 - \sum_i |z^i|^2}} \left[q^0 - ip^0 + z^l(q^l - ip^l) \right] \ . \end{aligned} \tag{A.6}$$

We have

$$2e^K = \frac{1}{1-|z|^2} \ , \tag{A.7}$$

$$\begin{aligned} \partial_i K e^{-K} &= 2\bar{z}^i \ , \\ \partial_i K &= \frac{\bar{z}^i}{1-|z|^2} \ , \\ \partial_{\bar{\jmath}}\partial_i K &= 2\delta_{ij}e^K + 2\bar{z}^i\partial_{\bar{\jmath}}e^K = \\ &= \frac{\delta_{i\bar{\jmath}}}{1-|z|^2} + \frac{\bar{z}^i z^{\bar{\jmath}}}{(1-|z|^2)^2} \ . \end{aligned} \tag{A.8}$$

The metric for the Kähler space is given by

$$G_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}K = \frac{\delta_{i\bar{\jmath}}}{1-|z|^2} + \frac{\bar{z}^i z^{\bar{\jmath}}}{(1-|z|^2)^2} \ . \tag{A.9}$$

To find the inverse matrix we write the general form

$$G^{i\bar{\jmath}} = A\delta^{i\bar{\jmath}} + Bz^i\bar{z}^{\bar{\jmath}} \ , \tag{A.10}$$

and we find

$$G^{i\bar{\jmath}} = (1-|z|^2)\left(\delta^{i\bar{\jmath}} - z^i\bar{z}^{\bar{\jmath}}\right) \ . \tag{A.11}$$

2. Explicit check of the quadratic invariant $|Z|^2 - |D_iZ|^2$ in N=2 quadratic series supergravity

We want to compute

$$\begin{aligned} I_2 &= |Z|^2 - |D_iZ|^2 = \\ &= |Z|^2 - D_iZD_{\bar{\jmath}}\bar{Z}G^{i\bar{\jmath}} \ , \end{aligned} \tag{A.12}$$

to this aim we define

$$\begin{aligned} &\alpha \equiv (q_0 + ip^0 + z^l(q_l - ip^l)) \equiv (q^0 + ip^0 + \beta) \equiv (a+\beta) \ , \\ &c = a\bar{a} \equiv p^{0\,2} + q_0^2 \ , \end{aligned} \tag{A.13}$$

so that we can write the identities

$$\begin{aligned} \alpha\bar{\alpha} \equiv (q_0^2 + p^{0\,2} + \gamma) &\equiv (c+\gamma) \ , \\ \gamma &= a\bar{\beta} + \bar{a}\beta + \beta\bar{\beta} \ , \\ \alpha\bar{\beta} + \bar{\alpha}\beta &= \gamma + \beta\bar{\beta} \ . \end{aligned} \tag{A.14}$$

We then have

$$Z\bar{Z}=\frac{\alpha\bar{\alpha}}{2(1-|z|^2)}\ , \tag{A.15}$$

and

$$\begin{aligned} D_iZD_{\bar{j}}\bar{Z}G^{i\bar{j}}=\frac{1}{2}&\left[q_i-ip^i+\frac{\bar{z}^i}{1-|z|^2}(q_0+ip^0+z^l(q_l-ip^l))\right]\cdot\\ &\cdot\left[q_{\bar{j}}+ip^{\bar{j}}+\frac{z^{\bar{j}}}{1-|z|^2}(q_0-ip^0+z^l(q_l+ip^l))\right]\cdot\\ &\cdot\left[\delta^{\bar{i}j}-\bar{z}^{\bar{i}}z^j\right]\ ; \end{aligned} \tag{A.16}$$

with the notations above

$$\begin{aligned} D_iZD_{\bar{j}}\bar{Z}G^{i\bar{j}}&=\frac{1}{2}\left[\sum_l(q_l^2+p^{l\,2})-\beta\bar{\beta}+\frac{\alpha\bar{\beta}}{1-|z|^2}-\frac{|z|^2\alpha\bar{\beta}}{1-|z|^2}+\right.\\ &\quad+\frac{\bar{\alpha}\beta}{1-|z|^2}-\frac{|z|^2\bar{\alpha}\beta}{1-|z|^2}+\\ &\quad\left.\frac{\alpha\bar{\alpha}}{(1-|z|^2)^2}-\frac{|z|^2\alpha\bar{\alpha}}{(1-|z|^2)^2}\right]=\\ &=\frac{1}{2}\left[\sum_l(q_l^2+p^{l\,2})-\beta\bar{\beta}+\alpha\bar{\beta}+\bar{\alpha}\beta+\alpha\bar{\alpha}\frac{|z|^2}{1-|z|^2}\right]=\\ &=\frac{1}{2}\left[\sum_l(q_l^2+p^{l\,2})+\gamma-\alpha\bar{\alpha}\frac{|z|^2}{1-|z|^2}\right]\ . \end{aligned} \tag{A.17}$$

The invariant, from (A.15) and (A.16), using (A.13) and (A.14), becomes

$$\begin{aligned} I_2&=\frac{1}{2}\left[\alpha\bar{\alpha}-\gamma-\sum_l(q_l^2+p^{l\,2})\right]=\\ &=\frac{1}{2}\left[(q_0^2+p^{0\,2})-\sum_l(q_l^2+p^{l\,2})\right]\ , \end{aligned} \tag{A.18}$$

where, as expected, the moduli dependence has vanished.

APPENDIX B

$SU(5,1)$ invariant.

We wrote in (6.31) the U-duality invariant, which is actually the invariant of the compact group $SU(6)$, of which $SU(5,1)$ is a non-compact form, in terms of the central charge matrices. Since it is indipendent on the moduli fields, and since in the parametrization we are considering the origin of coordinates, $\phi = 0$, is the invariant point under the action of the whole isotropy group $SU(5)$ (see [**51**], eq. (2.17)), we can find its explicit dependence on electromagnetic charges simply computing $Z_{AB}(\phi^i = 0)$. From (6.30) we have

$$Z_{AB}(\phi^i = 0) = q^{AB} \ , \tag{B.1}$$

so that, from (6.31), (6.32) and (6.33), with $q_{AB} \equiv (q^{AB})^*$, we find

$$I_4 = 4\, q_{AB} q^{BC} q_{CD} q^{DA} - (q_{AB}\, q^{AB})^2 \ . \tag{B.2}$$

We recall that electromagnetic charges are in the same $SU(5)$ 3-fold antisymmetric representation as the vector fields, for which the embedding in $SU(5,1)$ is given by

$$t^{abc} = \frac{1}{3!} \epsilon^{abcde6} t_{de6} \ , \tag{B.3}$$

so that the invariant (B.5) can be rewritten in terms of q_{ABC} in the $SU(5,1)$ representation. We are going to show that it is

$$I_4 = \frac{1}{4!} \epsilon^{ABCA'B'C'''} \epsilon^{A''B''C''A'''B'''C'} q_{ABC} q_{A'B'C'} q_{A''B''C''} q_{A'''B'''C'''} \ . \tag{B.4}$$

To this aim, we explicite the entries of "6" among other indices and we count for each term, writing then

$$\begin{aligned} I_4 = \frac{1}{4!} \Big[& 9\, \epsilon^{6BCA'B'C'''} \epsilon^{6B''C''A'''B'''C'} q_{6BC} q_{A'B'C'} q_{6B''C''} q_{A'''B'''C'''} \cdot \\ & 6 \cdot 2\, \epsilon^{ABC6B'C'''} \epsilon^{6B''C''A'''B'''C'} q_{ABC} q_{6B'C'} q_{6B''C''} q_{A'''B'''C'''} \cdot \\ & 3 \cdot 2\, \epsilon^{ABCA'B'6} \epsilon^{6B''C''A'''B'''C'} q_{ABC} q_{6B'C'} q_{A''B''C''} q_{A'''B'''6} \cdot \\ & 4\, \epsilon^{ABC6B'C'''} \epsilon^{A''B''C''6B'''C'} q_{ABC} q_{6B'C'} q_{A''B''C''} q_{6B'''C'''} \cdot \\ & \epsilon^{ABCA'B'6} \epsilon^{A''B''C''A'''B'''6} q_{ABC} q_{A'B'6} q_{A''B''C''} q_{A'''B'''6} \Big] \ , \end{aligned} \tag{B.5}$$

thus, recalling that $q_{6AB} \equiv q_{AB}$ and $q_{ABC} = \frac{1}{3!}\epsilon_{ABCDE}q^{DE}$, we can write (B.5) in terms of two-indices charges and, performing the sum on the antisimmetric tensors contracted indices we find

$$\begin{aligned} I_4 = & \frac{1}{4!}\left[4\left(q_{AB}q^{BC}q_{CD}q^{DA} + (q_{AB}q^{AB})^2\right) + \right. \\ & +16\left(2\,q_{AB}q^{BC}q_{CD}q^{DA} - (q_{AB}q^{AB})^2\right) + \\ & +16\left(2\,q_{AB}q^{BC}q_{CD}q^{DA} - (q_{AB}q^{AB})^2\right) + \\ & +16\,q_{AB}q^{BC}q_{CD}q^{DA} + \\ & \left. +4\,(q_{AB}q^{AB})^2\right] = \\ = & \, 4\,q_{AB}q^{BC}q_{CD}q^{DA} - (q_{AB}q^{AB})^2 \ . \end{aligned} \tag{B.6}$$

1. Black Hole entropy

The only solution to equations (6.38) for a black hole with non-zero area is the BPS one which corresponds, in terms of the central charge matrix skew-eigenvalues, to $Z_2 = 0$, $Z_1 > 0$, and the black hole potential at the attractor point is

$$V_{BH} = |Z_1|^2 = \sqrt{|I_4|} \ , \tag{B.7}$$

so that the black hole entropy is again indipendent on the moduli fields and is

$$S_{BH} \propto \sqrt{|4\,q_{AB}q^{BC}q_{CD}q^{DA} - (q_{AB}q^{AB})^2|} \ . \tag{B.8}$$

We notice that the invariant turns out to be the square of a function of $Z_1,\ Z_2$, namely

$$I_4 = \left[(Z_1)^2 - (Z_2)^2\right]^2 \ . \tag{B.9}$$

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