View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Electronic Thesis and Dissertation Archive - Università di Pisa

UNIVERSITÀ DI PISA
Facoltà di Scienze Matematiche Fisiche e Naturali

On Regularity for Constrained Extremum Problems

Andreea Moldovan

Advisor
Prof. Franco Giannessi

Direttore del Dottorato
Prof. Fabrizio Broglia

Acknowledgements

This thesis would have not been possible without the continuous and careful assistance of Prof. Franco Giannessi, my supervisor. I am particularly grateful to him for his important advices, constant encouragements and helpful suggestions. Special thanks are due to Prof. Letizia Pellegrini and to Dr. Giandomenico Mastroeni not only for their stimulating discussions, valuable suggestions and constructive advice, but also for their friendliness. I would like to thank Prof. Massimo Pappalardo, Dr. Barbara Panicucci, Dr. Mauro Passacantando and all the Optimisation Group for their warm welcome and support. I also want to thank Prof. Gert Wanka and Dr. Radu Ioan Boţ from Chemnitz University of Technology, Germany, for the fruitful collaboration. Many thanks are due to my colleagues and friends Luca, Demdah, Ana, Jasmin, Khaldia for having made my years at University of Pisa so enjoyable. Finally, I would like to thank Gianandrea and my family for love, patience and understanding.

Contents

Chapter 1

Introduction

The fundamental necessary optimality criterion of nonlinear mathematical programming is the Kuhn-Tucker stationary point optimality criterion and other generalisations of it. Both for the Kuhn-Tucker optimality criterion and for the saddle point sufficient optimality criterion there is no guarantee that the multiplier associated to the objective function is strictly positive. If the multiplier is equal to zero, then the respective condition has been achieved without the contribution of the objective function. Even if this degenerate case is mathematically correct, this fact means perhaps that there is a gap between the mathematical model and the real problem represented by it. Thus the question arises if the respective Kuhn-Tucker or saddle point criterion could be fulfilled with a strictly positive multiplier associated to the objective function. If the condition ensuring that the multiplier is strictly positive involves only the constraints it is called *constraint qualification* or *regularity condition* if it involves also the objective function. The existence of Lagrange multipliers is strictly connected to the fulfilment of strong duality for a pair of optimisation problems and therefore constraint qualifications and regularity condition play a crucial role in the field of duality.

In the literature of mathematical programming regularity conditions and constraint qualifications date back to the first half of the 20th century due to due to W. Karush [40], F. John [37], H.W. Kuhn and A. W. Tucker [41]. For constrained extremum problems with finite dimensional image, for different classes of problems such as differentiable, convex, locally Lipschitz, weak constraint qualifications necessary and/or sufficient for the existence of Lagrange multipliers have been found since the 1970s and implications between most of them have also been shown. We will mention the most used constraint qualifications from the literature. For differentiable problems the most used are the Mangasarian Fromowitz CQ [50], Kuhn-Tucker CQ [41], Arrow-Hurwicz- Uzawa CQ [3], Linear independence CQ, Abadie CQ [2], Zangwill CQ [75], Guignard CQ [31], [29]. For convex problems classic CQs are Slater CQ [68], Karlin CQ [39], basic CQ [44]. Calmness [14] and metric regularity [33], [34] are regularity conditions introduced a bit later and with other scopes, which are also used as regularity conditions.

Extensions of constraint qualification to nonsmooth optimization problems can be found in the literature under different assumptions on the directional differentiability of the functions involved. In [54] it is used the directional differentiability in the Hadamard sense; in [71] and in [42] utilise quasi-differentiability, i.e. Dini directional differentiability where the directional derivative can be written as the difference of two sublinear functions [17]; in [38] the Clarke directional differentiability it is employed; in [43] they applied the B-differentiable functions, i.e. Dini directionally differentiable functions, where the directional derivative is locally a first order approximation to the function; Dini-Hadamard derivatives have been used in [57]. Many authors are still concerned with the variety of constraint qualifications since the weakest from the theoretical point of view are not the easiest to verify and under certain circumstances some CQs are more easily verified than the others. Also the proof of the Kuhn-Tucker type conditions might be facilitated by the choice of the CQ or the regularity condition used. ref The literature on regularity conditions in infinite dimensional is also ever growing, mainly motivated by a certain "interiority" needed in order to prove the existence of Lagrange multipliers. From the vast literature for problems with or without differentiability, with or without convexity we will mention foe differentiable (and generalised differentiability) problems [76] , [60] , [7], for convex problems [66] [74], [73], [46], [30], [10].

The approach of the present work will be the one proposed in [21], namely the Image Space Analysis (ISA). This approach consists in introducing the space of the images of the functions involved in the optimization problem, the space called Image Space (IS). Then, a new problem is defined in the IS, which is equivalent to the given one. The optimality can be expressed as the impossibility of a system, which, using the IS notations, becomes that the intersection of a convex cone with a set is empty. Since the direct proof of the empty intersection is, in general, impracticable, a separation approach has been introduced [21] which consists in finding a functional such that the cone and the set lie in opposite level sets. The fact that, in the Kuhn-Tucker optimality condition, the multiplier associated to the objective function is strictly positive can be expressed in the IS as the necessity that the separation be regular, i.e. a face of the cone should not be included in the separation hyperplane. By exploiting the geometric and analytic insights given by the ISA, we will be able to give a general regularity condition which turns out to be necessary and sufficient for optimality and for the existence of a saddle point and/or of Lagrange multipliers, depending of the set chosen in the IS.

We will divide the work into two main parts, namely the regularity condition for problems having finite and infinite dimensional image. In the first part, in Section 2, we will start with a general condition for separation in the Euclidean space between a cone and a generic set. Further, another condition will be given in order to obtain that the separation is also regular. These conditions will be applied to two sets in the IS to get the existence os the saddle point and of the Lagrange multipliers expressed as separation between two sets in the IS. That is, we get that the John saddle point is equivalent to the separation between the image set and a cone, while the classic saddle point is equivalent to their regular separation. Using the separation between the homogenization of the image set and a cone, we obtain that the existence of the multipliers is equivalent to the separation, while the existence of the Lagrange multipliers is equivalent to their regular separation. For sufficient regularity condition we performed comparisons among our regularity condition, calmness and metric regularity. Concerning the necessary conditions, we analyse relationships among our regularity condition and calmness, metric regularity, and a constraint qualification for $\mathcal{C}$-differentiable problems, qualification that slightly generalises the classic CQs such as Guignard CQ, basic CQ and Abadie CQ. Both for sufficient and necessary conditions all the implications obtained and known relationships between the previously mentioned conditions are sustained by a long list of examples and graphical representations of the implications verified for differentiable, convex and locally Lipschitz classes of problems. Most interesting are, of course, the counterexamples showing that some implications do not hold.

The second part of this work analyses regularity condition for problems having infinite dimensional image. In Section 3.1 we start the analysis of problems having infinite dimensional image by introducing the selection approach [24], [25] which actually postpones the infinite dimensionality to the introduction of the IS. Then tools of the finite dimensional IS can be applied to the selected problem. We apply the regularity condition obtained in Section 2 and we show that this regularity condition equivalent to the regular separation of two sets in the IS is equivalent to the classic Euler equation in the C^1 case. We analyse also the sufficient saddle point conditions associated to the finite dimensional selected problem.

After showing how one can analyse an infinite dimensional image problem with tools of the finite dimensional image analysis, we continue by analysing a general infinite dimensional image problem that cannot benefit from the selection approach. In Section 3 we will show that the regularity condition necessary and sufficient for regular separation, and thus for Lagrange multipliers, obtained in Section 2 for the finite dimensional case, proves to be equivalent to the existence of Lagrange multipliers also in the case of constrained extremum problems with infinite dimensional image. Section 3.3 is devoted to an analysis of the Slater type CQs using the *quasi relative interior*(qri) of a convex set. We propose a Slater type CQ using the qri that improves the other ones given until now in the literature. Since the qri is among the most general notion of generalised interior then our condition is one of the weakest Slater type CQ implied by many other found in the literature.

1.1 Preliminaries

We will define some notions that will be used within the present work. As usual, $\mathbb{R}^n$ denotes the *n dimensional real space* for any positive integer n and with $\overline{\mathbb{R}} \cup \{\pm\infty\}$ we denote the set of *extended reals*. For any $x \in \mathbb{R}^n$, $x \geq 0$ means $x_i \geq 0$, $\forall i = 1, ..., n$, while $\mathbb{R}^n_+$ denotes $\{x \in \mathbb{R}^n : x \geq 0\}$. With O_n we denote the n-tuple, whose entries are zero. When there is no fear of confusion the subfix is omitted; for $n = 1$, the 1-tuple is identified with its element, namely, we set $O_1 = 0$.

Consider X a real normed space and X^* its continuous dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$.

$A \pm B$ stands for *vector sum/difference* between sets A and B.

For a set $M \subseteq X$ we will use dim M, which denotes the *dimension* of M, aff M for the *affine hull* of M; lin M for the linear span; cl M denotes the *closure* of M; conv M denotes the *convex hull* of M; int M denotes the *interior* of M; ri M denotes the *relative interior* of the convex set M; rbd M denotes the *relative boundary* of the convex set M.
With $d(x; M) :=\inf\{\|x - y\| : y \in M\}$ we indicate the *distance* of the point x from the set M.

Because literature contains several different definitions for it, we mention that we will call a set $C \subseteq X$ a *cone with apex* at $\bar{x} \in \text{cl}\ C$ if

$$\text{for all } x \in C \text{ and } \alpha \in]0, +\infty[\text{ it holds } \bar{x} + \alpha(x - \bar{x}) \in C.$$

When the apex $\bar{x}$ coincides with the origin then we will call it simply a *cone*. If $\bar{x} = O_X$ then the apex $\bar{x}$ is omitted from the notation of the cones. A cone C is called *convex* if $C + C \subseteq C$; it is called *pointed* if $C \cap -C = \varnothing$.

If $\bar{x} \in X$ and $M \neq \{\bar{x}\}$, the *cone generated* by M from $\bar{x}$ is the set

$$\text{cone}\ (\bar{x}; M) := \{x \in X : x = \bar{x} + \alpha(y - \bar{x}), y \in M, \alpha > 0\}.$$

If M is convex, one can prove that cone conv $(M \cup \{O_X\}) =$ cone M.
As a *tangent cone* to a set $M \subseteq X$, $M \neq \varnothing$ with apex at $\bar{x} \in \text{cl}\ M$ we will refer to the Bouligand tangent cone defined as

$$TC(\bar{x}; M) := \{\bar{x} + x \in X : \ \exists\{x^n\}_{n\geq 1} \subseteq \text{cl}\ M \text{ s.t. } \lim_{n\to+\infty} x^n = \bar{x} \text{ and}$$
$$\exists\{\alpha_n\}_{n\geq 1} \subset \mathbb{R}_+ \setminus \{0\} \text{ s.t. } \lim_{n\to+\infty} \alpha_n(x^n - \bar{x}) = x\}.$$

We stipulate that $TC(\bar{x}; \varnothing) = \varnothing$. This cone is always closed and enjoys important properties [6]. We will recall just that it is isotone, i.e. $TC(\bar{x}; M_1) \subseteq TC(\bar{x}; M_2)$ whenever $M_1 \subseteq M_2$, it preserves convexity, i.e. it is convex if the original set is convex and furthermore $M - \bar{x} \subseteq TC(\bar{x}; M)$ whenever M is convex. However, it may happen that $TC(\bar{x}; M)$ is a nonconvex cone.

In general, we have the following inclusion:

$$TC(\bar{x}; M) \subseteq \mathrm{cl}\,\mathrm{cone}(\bar{x}; M).$$

If the set M is convex, then $M \subseteq TC(\bar{x}; M)$ and $TC(\bar{x}; M) = \mathrm{cl}\,\mathrm{cone}(\bar{x}; M)$.
We introduce also the *normal cone* to a set M at $\bar{x}$ as

$$NC(\bar{x}; M) := \{\bar{x} + x \in X : \langle x, y - \bar{x}\rangle \leq 0,\ \forall y \in M\}.$$

For a cone C with apex at $\bar{x}$, the (positive) *dual (polar) cone* associated to C is

$$C^* = \{x \in X : \langle x, y - \bar{x}\rangle \geq 0, \forall y \in C\}.$$

To any hyperplane $H_0 = \{x \in X : \langle a, x\rangle = b\}$, $a \neq O_X$, we associate the positive and the negative halfspaces $H_0^+ := \{x \in X : \langle a, x\rangle \geq b\}$ and $H_0^- := \{x \in X : \langle a, x\rangle \leq b\}$, respectively.

Take the function $f : X \to \overline{\mathbb{R}}$. It is called *convex* if, for any $x, y \in X$ and any $\alpha \in [0, 1]$ one has

$$f(x\alpha + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

whenever the sum in the left-hand side is defined. When S is a nonempty convex subset of X, then the function $f : S \to \mathbb{R}$ is called *convex on* S if $\forall x, y \in S$ and $\forall \alpha \in [0, 1]$ it holds

$$f(x\alpha + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

We call $f : X \to \overline{\mathbb{R}}$, respectively $f : S \to \overline{\mathbb{R}}$ *concave*, respectively *concave on* S if $-f$ is convex, respectively convex on S. The convex function $f : S \to \overline{\mathbb{R}}$, with S convex and nonempty subset of X such that $\dim S \geq 1$ is directionally derivable at any $\bar{x} \in S$ and its *directional derivative* at $\bar{x}$ in the direction $d \in S - \bar{x}$ exists and is given by

$$f'(\bar{x}; d) = \lim_{t \searrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}.$$

We recall also the *subdifferential* of the convex function f at $\bar{x} \in \mathrm{dom}\, f$ as the set

$$\partial f(\bar{x}) := \{\sigma \in X^* : f(x) - f(\bar{x}) - \langle \sigma, x - \bar{x}\rangle \geq 0, \forall x \in X\}$$

and the *subgradient* σ as being the elements of the before mentioned set.

If $S \subseteq X$ is a convex set and $H \subseteq Y$, with Y is another normed space, is a closed and convex cone with apex at the origin, then the function $f : S \to Y$ is called $H-$*convexlike* when

$$\forall x, y \in S, \quad \forall \alpha \in [0, 1], \quad \exists \hat{x} \in X \text{ s.t. } (1 - \alpha)f(x) + \alpha f(y) - f(\hat{x}) \in H.$$

Of course, f is called $H-$*concavelike* if $-f$ is $H-$convexlike. We have that f is convex-like on X if and only if the set $f(X)+H$ is convex.

Due to its importance, the concept of differentiability has been the recipient of many generalisations. Most of them have been conceived independently of each other and for special objectives, often different form those of the theory of the extrema. Here we will use a kind of generalisation called $\mathcal{C}$-differentiability which was introduced in [21], [26] and it is suitable for achieving necessary optimality conditions. We consider $X \subseteq \mathbb{R}^n$ to be a convex set, with card $X > 1$. We denote by $\mathcal{C}$ (respectively $-\mathcal{C}$) the set of all functions $\mathcal{D}_{\mathcal{C}} f : X \times \text{cone } (X-\bar{x}) \to \mathbb{R}$ (respectively, $\mathcal{D}_{-\mathcal{C}} f : X \times \text{cone } (X-\bar{x}) \to \mathbb{R}$) which are positively homogeneous of degree one and convex (respectively, concave) with respect to the second argument. We will call a function $h : X \to \mathbb{R}$ to be $\mathcal{C}$-*differentiable* at $\bar{x}$ if there exists $\mathcal{D}_{\mathcal{C}} h(\bar{x};\cdot) \in \mathcal{C}$ such that, $\forall d \in \text{cone } (X-\bar{x})$, we have:

$$\lim_{d\to 0} \frac{1}{\|d\|}[\varepsilon(\bar{x};d) := h(\bar{x}+d) - h(\bar{x}) - \mathcal{D}_{\mathcal{C}} h(\bar{x};d)] = 0.$$

The function h is $(-\mathcal{C})$-differentiable at $\bar{x}$ when $-h$ is $\mathcal{C}$-differentiable at $\bar{x}$. We have the $\mathcal{C}$-*subdifferential* of a $\mathcal{C}$-differentiable function $h : X \to \mathbb{R}$ at $\bar{x} \in X$ given by

$$\underline{\partial}_{\mathcal{C}} h(\bar{x}) := \{\sigma \in \mathbb{R}^n : \mathcal{D}_{\mathcal{C}} h(\bar{x};d) \geq \langle \sigma, d\rangle, \forall d \in \text{cone } (X-\bar{x})\};$$

$\sigma \in \underline{\partial}_{\mathcal{C}} h(\bar{x})$ is called $\mathcal{C}$-*subgradient* of h at $\bar{x}$. If the function h is $(-\mathcal{C})$-differentiable, then

$$\overline{\partial}_{-\mathcal{C}} h(\bar{x}) := \{\sigma \in \mathbb{R}^n : \mathcal{D}_{-\mathcal{C}} h(\bar{x};d) \leq \langle \sigma, d\rangle, \forall d \in \text{cone } (X-\bar{x})\}$$

is called $(-\mathcal{C})$-*superdifferential* of h at $\bar{x}$ and $\sigma \in \overline{\partial}_{-\mathcal{C}} h(\bar{x})$ is called $(-\mathcal{C})$-*supergradient* of h at $\bar{x}$. It is worth mentioning that if f is $\mathcal{C}$-differentiable at $\bar{x} \in X$ then f is continuous at $\bar{x}$; if X is nonempty, open and convex then a convex function $f : X \to \mathbb{R}$ is $\mathcal{C}$-differentiable at any $\bar{x} \in X$ and its unique $\mathcal{C}$-derivative coincides with the directional derivative of f at $\bar{x}$.

Additionally, we will need the well known *indicator function* of M, namely $\delta_M : \mathbb{R}^n \to \overline{\mathbb{R}}$ which is defined by

$$\delta_M(x) = \begin{cases} 0 & \text{if} \quad x \in M \\ +\infty & \text{if} \quad x \notin M \end{cases}$$

and the *support function* of M with respect to x,

$$\delta^*(x;M) : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad \delta^*(x;M) := \sup_{y\in M}\langle x, y\rangle.$$

The *effective domain* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the set

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

1.2 Image Space Analysis

The study of the properties of the image of a real-valued function is an old one and it has been extended recently to multifunctions and to vector valued functions. However, in most cases, the properties of the image have not been the purpose of the study and their investigations have occurred as an auxiliary step towards other achievements.

Traces of the idea of studying the image of functions involved in a constrained extremum problem go back to the work of Carathéodory. In the 1950s, for the first time in the field of Optimisation, R. Bellman proposed, with his celebrated maximum principle, to replace the given unknown by a new one which runs in the image. However, also here the image is not the main purpose. Only in the 1960s and 1970s some authors, among which we mention M.R. Hestenes and F. Giannessi, independently from each other, brought explicitly such a study into the field of Optimisation.

The approach we refer to in this work was introduced by Giannessi [21] and then developed in other papers such as [18], [19], [23], [24], [25], [26], [27], [28], [52], [53], [57] and others.

The approach consists in introducing the Image Space(for short IS) where the images of the functions of the given optimisation problem (or variational inequality or, more general, a generalised system) run. Then, a new problem is defined in the IS, which is equivalent to the given one.

The analysis in the IS must be viewed as a preliminary and auxiliary step for studying an extremum problem. When a statement has been achieved in the IS, then certainly, we have to write the corresponding (equivalent) statement in terms of the given space. If this aspect is understood, then the IS analysis may be highly fruitful. In fact, in the IS we may have a sort of "regularisation": the conic extension (which is defined below) of the image may be convex or continuous or smooth when the given extremum problem and its image do not enjoy that property, so that convex or continuous or smooth analysis can be developed in the IS but not in the given space. If the image of a problem is finite dimensional then sometimes, see Section 3.1 it can be analysed, in IS, by means of the same mathematical concepts which are used for the finite dimensional case, even if the given space is not finite dimensional. If the image is infinite dimensional, then it is possible to postpone such an infinite dimensionality to the introduction of IS, which, therefore, can be held finite dimensional.

Assume we are given the integers m, n and p with $0 \leq p \leq m$, the nonempty set $X \subseteq B$, where B is a Banach space, and the functions $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$. Let us consider the following problem

$$\min f(x) \tag{1.2.1}$$

subject to

$$g_i(x) = 0,\ \forall i \in \mathcal{I}^0 := \{1, ..., p\}, \tag{1.2.1}$$

$$g_i(x) \geq 0,\ \forall i \in \mathcal{I}^+ := \{p+1, ..., m\}, \tag{1.2.1}$$

$$x \in X \subseteq B \tag{1.2.1}$$

where $p = 0 \Rightarrow \mathcal{I}^0 = \varnothing$, $p = m \Rightarrow \mathcal{I}^+ = \varnothing$, $m = 0 \Rightarrow \mathcal{I} := \mathcal{I}^0 \cup \mathcal{I}^+ = \varnothing$. Unless differently stated, we will assume that card $X > 1$. The *feasible region* of (1.2.1) is

$$R := \{x \in X : g(x) \in D\},$$

where $g(x) = (g_1(x), ..., g_m(x))$, $D := O_p \times \mathbb{R}_+^{m-p}$ with $O_p := (0, ..., 0) \in \mathbb{R}^p$; we stipulate that $D = \mathbb{R}_+^m$ when $p = 0$ and $D = O_m := (0, ..., 0) \in \mathbb{R}^m$ when $p = m$; $m = 0$ does not require to define D.

Definition 1.2.1. An element $\bar{x} \in R$ is said to be a *global minimum point* of the problem (1.2.1) if and only if $f(x) \geq f(\bar{x})$, $\forall x \in R$. Iff this inequality is strictly verified for $x \neq \bar{x}$, then a minimum point is said to be *strict*. Iff there exists a neighbourhood $N_\varepsilon(\bar{x})$ of $\bar{x}$, such that the above inequality is (strictly) satisfied $\forall x \in R \cap N_\varepsilon(\bar{x})$, then $\bar{x}$ is said to be a *local (strict) minimum point.*

Let be $\bar{x} \in R$ and let us introduce the sets

$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0, v \in D\},$$

$$\mathcal{H}_u := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0, v = 0\},$$

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u = f_{\bar{x}}(x) = f(\bar{x}) - f(x), v = g(x), x \in X\}$$

which is called the *image* of the problem (1.2.1),

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) := \mathcal{K}_{\bar{x}} - \text{cl}\ \mathcal{H},$$

called the *conic extension* of the image of the problem (1.2.1). The set $\mathcal{K}_{\bar{x}}$ is and the space $\mathbb{R}^{1+m}$, where both $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$ lay, is called *image space.*

We have that $\bar{x} \in R$ is a global minimum point of (1.2.1) if and only if the system(in the unknown x)

$$f_{\bar{x}}(x) > 0, \qquad g(x) \in D, \qquad x \in X \tag{1.2.1}$$

is impossible.

Proposition 1.2.2. *([26]) It holds that $\bar{x} \in R$ is a global minimum point of (1.2.1) if and only if*

$$\mathcal{H} \cap \mathcal{K}_{\bar{x}} = \varnothing. \tag{1.2.1}$$

The theory of constrained extremum problems is full of proposals for changing the data of (1.2.1) without losing the minimum and minimum points and with the scope of adding a desired property to (1.2.1). Such proposals have been made with reference to the given space, while here we will extend the image of the problem and we will often analyse the conic extension $\mathcal{E}(\mathcal{K}_{\bar{x}})$ of the image of the problem rather then the image itself.

Proposition 1.2.3. *[26] (1.2.2) holds if and only if*

$$\mathcal{H} \cap \mathcal{E}(\mathcal{K}_{\bar{x}}) = \varnothing. \tag{1.2.1}$$

Proof If. This implication follows obviously from the inclusion $\mathcal{K}_{\bar{x}} \subseteq \mathcal{E}(\mathcal{K}_{\bar{x}})$. **Only if.** Let us assume that, ab absurdo, there exists $z^1 \in \mathcal{K}_{\bar{x}}$ and $z^2 \in \text{cl}\,\mathcal{H}$, such that $z^1 - z^2 \in \mathcal{H}$. Then $z^1 = (z^1 - z^2) + z^2 \in \mathcal{H} + \text{cl}\,\mathcal{H} = \mathcal{H}$ and hence (1.2.2) is contradicted. □

Besides extension, approximation is an important tool. Often the feasible region or its intersection with a level set of the objective function are very difficult to be analysed. Therefore, for special scopes, such as necessary optimality conditions, the above set is replaced with a cone which can be considered an approximation of it in at least a neighbourhood of a given point.

Definition 1.2.4. In problem (1.2.1), suppose that f is $\mathcal{C}$-differentiable and that g is-$\mathcal{C}$-differentiable on X. Consider any $\bar{x} \in X$ and set $d := x - \bar{x}$. The set

$$\mathcal{K}^h_{\bar{x}} := \{(u,v) \in \mathbb{R}^{1+m} : u = -\mathcal{D}_{\mathcal{C}} f(\bar{x}; d);\ v_i = g_i(\bar{x}) + \mathcal{D}_{(-\mathcal{C})} g_i(\bar{x}; d),\ i \in \mathcal{I};$$
$$d \in \text{cone}\,(X - \bar{x})\}$$

is called *homogenization* of the image set $\mathcal{K}_{\bar{x}}$. When the function f and g are differentiable it is called *linearization.*

We replace in this way problem (1.2.1) with its homogenized form

$$\min\,[f(\bar{x}) + \mathcal{D}_{\mathcal{C}} f(\bar{x}; d)] \tag{1.2.2}$$

$$\text{s.t.} \quad g_i(\bar{x}) + \mathcal{D}_{(-\mathcal{C})} g_i(\bar{x}; d) = 0, \quad i \in \mathcal{I}^0, \tag{1.2.2}$$

$$g_i(\bar{x}) + \mathcal{D}_{(-\mathcal{C})} g_i(\bar{x}; d) \geq 0, \quad i \in \mathcal{I}^+, \tag{1.2.2}$$

$$d \in X - \bar{x} \tag{1.2.2}$$

or, when f and g are differentiable, with its linearized form

$$\min\,[f(\bar{x}) + \langle f'(\bar{x}), d\rangle] \tag{1.2.3}$$

$$\text{s.t.} \quad g_i(\bar{x}) + \langle g_i'(\bar{x}), d\rangle = 0, \quad i \in \mathcal{I}^0, \tag{1.2.3}$$

$$g_i(\bar{x}) + \langle g_i'(\bar{x}), d\rangle \geq 0, \quad i \in \mathcal{I}^+, \tag{1.2.3}$$

$$d \in X - \bar{x}. \tag{1.2.3}$$

When (1.2.1) has only unilateral constraints ($p = 0,\ m \geq 1$), then problems (1.2.2) and (1.2.3) gain some special importance. To show this in the next proposition, let us associate (1.2.2) with the following system

$$\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) < 0, \quad g_i(\bar{x}) + \mathcal{D}_{(-\mathcal{C})} g_i(\bar{x}; d) \begin{cases} > 0, & \text{if } i \in \mathcal{I}_N^+, \\ \geq 0 & \text{if } i \in \mathcal{I}_L^+, \end{cases} \quad d \in X - \bar{x}, \tag{1.2.3}$$

where $\mathcal{I}_N^+ := \{i \in \mathcal{I}^+ : g_i(\bar{x}) = 0, \varepsilon_i(\bar{x}; d) \neq 0\}$, $\mathcal{I}_L^+ := \mathcal{I}^+ \setminus \mathcal{I}_N^+$. When f and g are differentiable then (1.2) becomes:

$$\langle f'(\bar{x}), d \rangle < 0, \quad g_i(\bar{x}) + \langle g_i'(\bar{x}), d \rangle > 0, i \in \mathcal{I}_N^+, g_i(\bar{x}) + \langle g_i'(\bar{x}), d \rangle \geq 0, i \in \mathcal{I}_L^+, d \in X - \bar{x}. \tag{1.2.3}$$

Proposition 1.2.5. *([26])*

(i) We have

$$\mathcal{K}_{\bar{x}}^h \subseteq TC(\bar{z}; \mathcal{K}_{\bar{x}}), \tag{1.2.3}$$

and in a neighbourhood of $\bar{z}$ $\mathcal{K}_{\bar{x}}^h$ is a truncated cone with apex at $\bar{z}$. If, furthermore, X is a cone with apex at $\bar{x}$ (in particular $X = \mathbb{R}^n$), then $\mathcal{K}_{\bar{x}}^h$ is a cone with apex at $\bar{z}$.

(ii) The conic extension $\mathcal{E}(\mathcal{K}_{\bar{x}}^h)$ is convex.

(iii) (Homogenization Lemma) Let f, $-g_i$, $i \in \mathcal{I}^+$, be $\mathcal{C}$-differentiable at $\bar{x} \in X$. If $\bar{x}$ is a minimum point of (1.2.1), then the system (1.2) is impossible.

Proposition 1.2.6. *([26])*

(i) Let X be convex. The conic extension $\mathcal{E}(\mathcal{K}_{\bar{x}}^h)$ is convex if and only if the map $(f(x), -g(x))$ is cl $\mathcal{H}$-convexlike.

(ii) If f and $-g_i$, $i \in \mathcal{I}$ are $\mathcal{C}$-differentiable at $\bar{x}$, then $\mathcal{E}(\mathcal{K}_{\bar{x}}^h)$ and $\mathcal{E}(\mathcal{K}_{\bar{x}}^h - \bar{z})$ are convex.

Problem (1.2.2) and its image set $\mathcal{K}_{\bar{x}}^h$ play a crucial role at least for achieving necessary optimality conditions. To this end, the extremely important aspect would be to claim that if (1.2) holds, then also

$$\mathcal{H} \cap \mathcal{K}_{\bar{x}}^h = \varnothing \tag{1.2.3}$$

holds. Unfortunately in the general case, such a claim is false.

Proposition 1.2.7. *(1.2) holds if and only if*

$$\mathcal{H} \cap \mathcal{E}(\mathcal{K}_{\bar{x}}^h) = \varnothing.$$

The equivalence between the existence of the Lagrange multipliers and the linear separation between the cone $\mathcal{H}$ and the set $\mathcal{K}_{\bar{x}}$ has been shown in [18].

Let us analyse the regularity in the case of a linear separation function. We can introduce the following definition.

Definition 1.2.8. The sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ admit a *linear separation*, if and only if there exist $\bar{\theta} \geq 0$ and $\bar{\lambda} \in D^*$ with $(\bar{\theta}, \bar{\lambda}) \neq 0$, such that:

$$\langle \bar{\theta}, u \rangle + \langle \bar{\lambda}, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}_{\bar{x}}. \tag{1.2.3}$$

If in (1.2.8) $\bar{\theta} \neq 0$, then the separation is said to be *regular*.

Next result shows that a linear functional separates $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, if and only if it separates $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\mathcal{H}$.

Proposition 1.2.9. *([18]) Let $(\bar{\theta}, \bar{\lambda}) \in \mathcal{H}^* \backslash \{O\}$. Then the following conditions are equivalent :*

(i) $\bar{\theta} u + \langle \bar{\lambda}, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}_{\bar{x}}.$

(ii) $\bar{\theta} u + \langle \bar{\lambda}, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{E}(\mathcal{K}_{\bar{x}}).$

Proof. (i) $\Rightarrow$ (ii). Let $(h_1, h_2) \in \mathcal{H}$. Since $\bar{\theta}(-h_1) + \langle \bar{\lambda}, -h_2 \rangle \leq 0$, then

$$\bar{\theta}(u - h_1) + \langle \bar{\lambda}, v - h_2 \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}_{\bar{x}},$$

and (ii) holds.
(ii) $\Rightarrow$ (i) is obvious, since $\mathcal{K}_{\bar{x}} \subseteq \mathcal{E}(\mathcal{K}_{\bar{x}})$.
□

Lemma 1.2.10. *Let $K \subseteq \mathbb{R}^{1+m}$ be a convex set and $S \subseteq \mathbb{R}^{1+m}$. Then,*

$$conv(S + K) = conv(S) + K. \tag{1.2.3}$$

Proof. ($\subseteq$) Let $\mu_i \geq 0$ with $\sum_{i=1}^{r} = 1$. If $s_i \in S$, $k_i \in K$, $i = 1, , ..., r$, then

$$\sum_{i=1}^{r} \mu_i (s_i + k_i) = \sum_{i=1}^{r} \mu_i s_i + \sum_{i=1}^{r} \mu_i k_i \in \operatorname{conv} S + K,$$

since K is convex.
($\supseteq$) If $s_i \in S, i = 1, ..., r$ and $k \in K$, then

$$\sum_{i=1}^{r} \mu_i s_i + k = \sum_{i=1}^{r} \mu_i (s_i + k) \in \operatorname{conv}(S + K),$$

and the inclusion is proved. □

The previous result allows us to consider the following characterisation of the linear separation.

Theorem 1.2.11. *([18]) It holds that* $\mathcal{E}(\mathcal{K}_{\bar{x}})$ *and* $\mathcal{H}$ *are properly linearly separable if and only if*

$$0 \notin ri\ \ conv\,\mathcal{E}(\mathcal{K}_{\bar{x}}). \tag{1.2.3}$$

Proof. It holds that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are properly linearly separable iff

$$\text{ri conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \text{ri}\,\mathcal{H} = \varnothing. \tag{1.2.3}$$

Therefore (1.2) is equivalent to

$$O \notin [\text{ri conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}})] - \text{ri}\,\mathcal{H} = [\text{ri conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}})] - \text{ri cl}\,\mathcal{H} = \text{ri conv}\,[\mathcal{E}(\mathcal{K}_{\bar{x}}) - \text{cl}\,\mathcal{H}].$$

By Lemma 1.2.10 we have:

$$[\text{conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}})] - \text{cl}\,\mathcal{H} = \text{conv}\,[\mathcal{E}(\mathcal{K}_{\bar{x}}) - \text{cl}\,\mathcal{H}] = \text{conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}})$$

which completes the proof. □

Remark 1.2.12. We observe that $O \notin \text{ri conv}\,\mathcal{E}(\mathcal{K}_{\bar{x}})$ if and only if O and $\mathcal{E}(\mathcal{K}_{\bar{x}})$ are properly linearly separable, which means that $\mathcal{E}(\mathcal{K}_{\bar{x}})$ admits a supporting hyperplane at the origin that does not contain $\mathcal{E}(\mathcal{K}_{\bar{x}})$.

1.2.1 Conic Separation

In order to deepen the analysis of the regularity conditions for (1.2.1) it is of interest to consider a first important example of non linear separation in the IS: the conic separation.

Definition 1.2.13. *([19])*

(i) $A \subset \mathbb{R}^n$ is said to be *cone separated* from $B \subset \mathbb{R}^n$ if and only if there exists a convex cone $K \subset \mathbb{R}^n$ such that $B \subset clK$ and $A \subset cl(K^c)$; if, in addition, $B \subset \text{int}\,K$, then A is said to be *regularly cone separated* from B.

(ii) If the cone K is polyhedral (i.e. K is the intersection of a finite number of halfspaces), then we will say that A and B are *piece-wise linearly* separated.

Remark 1.2.14. *([19])* Observe that if the cone K is a halfspace, then A and B are linearly separated. Moreover, A and B are piece-wise linearly separated if and only if there exist a finite number of vectors $k_1^*, ..., k_N^*$ such that

$$\min_{i=1,..,N} \langle k_i^*, a\rangle \leq \min_{i=1,..,N} \langle k_i^*, b\rangle, \quad \forall a \in A, \forall b \in B \tag{1.2.3}$$

If B is a subset of a polyhedral cone C, then for any A such that $A \cap C = \emptyset$, A and B are piece-wise linearly separated and vectors k_i^* satisfying (1.2.14) can be found in C^*.

Next result shows a characterisation of regular conic separation between $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\mathcal{H}$.

Lemma 1.2.15. *([19]) The set $\mathcal{E}(\mathcal{K}_{\bar{x}})$ is regularly cone separated from $\mathcal{H}$ if and only if*

$$cl\ cone\ \mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H}_u = \emptyset. \tag{1.2.3}$$

Proof. Assume that $\mathcal{E}(\mathcal{K}_{\bar{x}})$ is regularly cone separated from $\mathcal{H}$, then there exists a convex cone Q such that

$$\mathcal{H} \subset \text{int } (Q) \text{ and } \mathcal{E}(\mathcal{K}_{\bar{x}}) \subset \text{cl } (Q^c). \tag{1.2.3}$$

The first inclusion in (1.2.1) implies that cl cone $\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H} = \emptyset$, while the second in (1.2.1) implies that cl cone $\mathcal{E}(\mathcal{K}_{\bar{x}}) \subseteq \text{cl } (Q)^c$, from which (1.2.15) follows, recalling that $\mathcal{H}_u \subseteq \mathcal{H}$.

Conversely, assume that (1.2.15) holds.

We preliminarily observe that, if Q is a closed convex cone in $\mathbb{R}^{1+m}$ and $\xi \in \mathbb{R}^{1+m}$ is a vector such that $\xi \notin Q$, then there exists an open convex cone M_ξ such that

$$\xi \in M_\xi \qquad \text{and} \quad M_\xi \cap Q = \emptyset.$$

By (1.2.15), we can find an open convex cone C_u such that

$$\mathcal{H}_u \subset C_u \quad \text{and} \quad \text{cl cone}\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap C_u = \emptyset.$$

Set $Q := \mathcal{H} + C_u$. Since $\mathcal{H} + \mathcal{H}_u = \mathcal{H}$, then $\mathcal{H} \subset Q$. By (1.2.15) we obtain

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap C_u = \emptyset. \tag{1.2.3}$$

We will show that (1.2.1) implies $\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap Q = \emptyset$.

Ab absurdo, if there exists $z \in \mathcal{E}(\mathcal{K}_{\bar{x}}) \cap Q$, then $z \in \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $z = h_0 + c_0$, with $h_0 \in \mathcal{H}$, $c_0 \in C_u$. Since

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) - \mathcal{H} = \mathcal{K}_{\bar{x}} - (\text{cl}\mathcal{H} + \mathcal{H}) = \mathcal{K}_{\S\bar{x}} - \mathcal{H} \subset \mathcal{E}(\mathcal{K}_{\bar{x}}),$$

then $z - h_0 \in \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $z - h_0 = c_0 \in C_u$, which contradicts (1.2.1).

Therefore $\mathcal{E}(\mathcal{K}_{\bar{x}}) \subset Q^c = \text{cl}(Q^c)$ and the proof is complete. □

Remark 1.2.16. If in the above lemma $\mathcal{H}$ is a polyhedral cone then Q is polyhedral (it is enough to note that C_u can be chosen polyhedral) and there exists a regular piece-wise linear separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$.

Next theorem shows that conic separation between $\mathcal{E}(\mathcal{K}_{\overline{x}})$ and $\mathcal{H}$ is a necessary optimality condition for (1.2.1).

Theorem 1.2.17. *([19])*

(i) If $\bar{x}$ is a solution of (1.2.1), then $\mathcal{E}(\mathcal{K}_{\overline{x}})$ and $\mathcal{H}$ are cone separated.

(ii) If $\mathcal{E}(\mathcal{K}_{\overline{x}})$ and $\mathcal{H}$ are regularly cone separated, then $\bar{x}$ is a solution of (1.2.1).

Proof. (i) It is enough to note that $\mathcal{H}$ is a convex cone such that $\mathcal{E}(\mathcal{K}_{\overline{x}}) \cap \mathcal{H} = \varnothing$.
(ii) It follows from the fact that if $\mathcal{E}(\mathcal{K}_{\overline{x}})$ and $\mathcal{H}$ are regularly cone separated then $\mathcal{E}(\mathcal{K}_{\overline{x}}) \cap \mathcal{H} = \emptyset$.
□

Remark 1.2.18. One can notice that in Proposition 1.2.9, Theorem 1.2.11 and Lemma 1.2.15, $\mathcal{K}_{\bar{x}}$ can be replaced by a generic set $K \subseteq \mathbb{R}^{1+m}$.

In the analysis of local minima of (1.2.1) we have to consider the behaviour of the image of the problem restricted to a suitable neighbourhood of a solution $\overline{x}$, which is defined by

$$\mathcal{K}_{\overline{x}}^{\epsilon} := \{(u, v) \in \mathbb{R}^{1+m} : u = f(\overline{x}) - f(x), v = g(x), x \in X \cap N_{\epsilon}(\overline{x})\}. \qquad (1.2.3)$$

Definition 1.2.19. *([19])* We will say the the local image regularity condition (for short, LIRC) holds for (1.2.1) at $\overline{x} \in X$, if and only if there exists $\epsilon > 0$ such that

$$\text{cl cone } \mathcal{E}(\mathcal{K}_{\overline{x}}^{\epsilon}) \cap \mathcal{H}_u = \emptyset. \qquad (1.2.3)$$

Theorem 1.2.20. *([19]) The relation (1.2.19) holds if and only if there exists $\alpha > 0$ such that, for all $x \in X \cap N_{\epsilon}(\overline{x})$*

$$f(\overline{x}) - f(x) \leq \alpha \ d(g(x); D), \qquad (1.2.3)$$

where we recall that $d(g(x); D)$ denotes the distance from $g(x)$ to D.

Proof. We preliminarily observe that (1.2.19) is equivalent to

$$e_u \notin \text{cl cone } \mathcal{E}(\mathcal{K}_{\overline{x}}^{\epsilon}), \qquad (1.2.3)$$

where $e_u := (1, 0_m) \in \mathbb{R}^{1+m}$. Suppose that (1.2.1) does not hold, i.e.,

$$e_u \in \text{cl cone } \mathcal{E}(\mathcal{K}_{\overline{x}}^{\epsilon}). \qquad (1.2.3)$$

Therefore there exist sequences $\{\alpha_n\} \subset \mathbb{R}_+$, $\{x_n\} \subset X \cap N_{\epsilon}(\overline{x})$, $\{h_n\} := \{(h_n^1, h_n^2)\} \subset$ cl $\mathcal{H}$ s.t.

$$\lim_{n\to\infty} \alpha_n(f(\overline{x}) - f(x_n) - h_1^n) = 1, \quad \lim_{n\to\infty} \alpha_n(g(x_n) - h_n^2) = 0.$$

It follows that

$$\lim_{n\to\infty} \frac{\|g(x_n) - h_n^2\|}{f(\overline{x}) - f(x_n) - h_1^n} = 0,$$

which implies

$$\lim_{n\to\infty} \frac{d(g(x_n); D)}{f(\overline{x}) - f(x_n) - h_1^n} = 0. \tag{1.2.3}$$

Therefore, for $n > \bar{n}$, we have that

$$d(g(x_n); D) < (1/\alpha)(f(\overline{x}) - f(x_n) - h_n^1),$$

which contradicts (1.2.20), being $h_1^n \geq 0$.

Conversely, suppose that (1.2.20) does not hold. Then for any $n > 0$, $\exists x_n \in X \cap N_\epsilon(\overline{x})$ such that

$$n\ d(g(x_n); D) < f(\overline{x}) - f(x_n). \tag{1.2.3}$$

Let $\alpha_n := \frac{1}{f(\overline{x}) - f(x_n)}$ and let $h_n \geq 0$ be such that

$$d(g(x_n); D) = g(x_n) - h_n).$$

Then by (1.2.1)

$$(1, 0) = \lim_{n\to\infty} \alpha_n(f(\overline{x}) - f(x_n), (g(x_n) - h_n) \in \text{ cl cone } \mathcal{E}(\mathcal{K}_{\overline{x}}^{\epsilon})$$

which completes the proof. □

The following definition has been introduced in [14], [65]with the purpose of giving a regularity condition.

Definition 1.2.21. Problem (1.2.1) is said to be *locally calm at* $\overline{x} \in R$, if and only if $\exists\, \rho > 0$, $\exists\, \varepsilon > 0$, s.t.

$$\forall \xi \in N_\varepsilon(O) \text{ and } \forall x \in R_\varepsilon(\xi) := \{x \in X \cap N_\varepsilon(\overline{x}) : g(x) + \xi \in D\} \neq \varnothing, \tag{1.2.3}$$

we have:

$$f(x) - f(\overline{x}) + \rho||\xi|| \geq 0. \tag{1.2.3}$$

If $R_\varepsilon(\xi)$ is replaced by $R(\xi) := \{x \in X : g(x) + \xi \in D\}$, then the problem is said to be (*globally*) *calm* at $\overline{x}$.

Theorem 1.2.22. *([19]) Let $\overline{x}$ be a local optimal solution of (1.2.1). Then (1.2.1) is calm at $\overline{x}$ if and only if the local image regularity condition (1.2.19) is fulfilled at $\bar{x}$.*

Proof. Assume that (1.2.1) is not calm at $\overline{x}$. Then, there exists sequences $\{\xi_n\} \to 0$, $\{x_n\} \subseteq R_\varepsilon(\xi_n)$, with $\|x_n - \overline{x}\| \leq \frac{1}{n}$ such that

$$n\|\xi_n\| < f(\overline{x}) - f(x_n). \tag{1.2.3}$$

Since $g(x_n) + \xi_n \in D$, then $\exists h_n \in D$ such that $(f(\overline{x}) - f(x_n), g(x_n) - h_n) \in \mathcal{E}(\mathcal{K}^\epsilon_{\overline{x}})$, for n sufficiently large, with $g(x_n) + \xi_n = h_n$. Taking $\alpha_n = \frac{1}{f(\overline{x}) - f(x_n)}$, from (1.2.1) we obtain

$$\lim_{n\to\infty} \alpha_n\,(f(\overline{x}) - f(x_n), g(x_n) - h_n) = \lim_{n\to\infty} \alpha_n(f(\overline{x}) - f(x_n), -\xi_n) = (1, 0)$$

which contradicts (1.2.1).

Conversely, assume that (1.2.19) does not hold at $\overline{x}$. By Theorem 1.2.20, it follows that, for every neighbourhood V of $\overline{x}$, and $\forall \rho > 0$, there exists $x_n \in X \cap V$, such that

$$f(\overline{x}) - f(x_n) > \rho\; d(g(x_n); D). \tag{1.2.3}$$

Setting $\xi_n := -g(x_n) + d_n$, with $\|\xi_n\| = d(g(x_n); D)$, we have that (1.2.1) becomes

$$(f(\overline{x}) - f(x_n) > \rho\|\xi_n\|$$

which contradicts (1.2.21). $\square$

Chapter 2

Regularity for Problems with Finite Dimensional Image

Given a solution of a constrained extremum problem, the existence of Lagrange multipliers consists in finding a vector of multipliers, associated to the constraints, in such a way that the pair solution-vector of multipliers is a stationary point for the Lagrangian function. This is equivalent to claim that a positive multiplier can be associated to the objective function. Classic results in this sense date back to the first half of 20th century and are due to W. Karush [40], F. John [37], H.W. Kuhn and A. W. Tucker [41].

In the literature, a condition which guarantees that the multiplier associated to the objective function is positive, is called regularity condition or constraint qualification, according to whether the condition does or does not involve the objective function, respectively.

Here, a regularity condition for problems with finite dimensional image will be established by means of the ISA [26], which has been shown to be a fundamental tool for studying many topics of the Optimisation Theory. More precisely, since the optimality of a feasible point $\bar{x}$ can be proved by means of the linear separation between two suitable subsets of the IS of a problem with finite dimensional image, we begin the study, in Section 2.1, by giving a condition equivalent to the linear separation between a convex cone C and a generic set S in the Euclidean space $\mathbb{R}^n$. This condition can be called of "Helly-type" because, if each subset of S of finite cardinality enjoys a separability property, then S itself enjoys a separability property. In Section 2.2, we propose a regularity condition for the linear separation between C and S which is given in terms of the tangent cone to a suitable approximation of the set, which allows us to include also the nonconvex case. Successively, given a constrained extremum problem, we consider in the IS a convex cone $\mathcal{H}$, which depends on the kind of constraints (equalities or inequalities), and a set, which is the image, through the maps of the given functions, of their domain. The results of Section 2.2 are specified to the convex cone $\mathcal{H}$ and the image set and allows us to achieve, in Section 2.3, the existence of regular saddle points; in this case the regularity condition plays the role of a sufficient optimality condition and it is compared with calmness [14] and metric regularity [33], [34]. In Section 2.4, by replacing the image set with its homogenization, it is proved that the regularity condition of Section 2.2 is equivalent to the existence of Lagrange multipliers with a positive multiplier associated to the objective function; hence we compare it with classic constraint qualifications and regularity conditions existing in the literature such as Clarke calmness [14], Ioffe metric regularity [33], [34], Basic constraint qualification [44] and Guignard constraint qualification [31], [29]. Even if separation arguments are developed in the finite dimensional IS, the regularity condition which we obtain holds also for the infinite dimensional extremum problems having finite dimensional image, like for instance problems of isoperimetric type, as it will be seen in Section 3.1. Examples and graphical representations are given in Sections 2.5 and 2.6 with the aim of showing the relationships among the previous conditions.

2.1 A Helly-Type Condition for Linear Separation

In this section, we will give a condition necessary and sufficient for the linear separation between two sets of $\mathbb{R}^n$ and, in a particular case, sufficient for their proper separation. We suppose that one of the two sets is a nonempty convex cone C with apex at $O \notin C$, and the other is any nonempty subset S of $\mathbb{R}^n$; set $s = \dim S$. Let $z \in \mathbb{R}^n$; denote by proj z its projection on the orthogonal complement of C, that is $C^{\perp} := \{x \in \mathbb{R}^n : \langle x, k\rangle = 0, \forall k \in C\}$. Let $p = \dim C^{\perp}$ so that $\dim C = n - p$.

In the following statement, if $p = 0$ we stipulate that (2.1.1) $-$ (2.1.1) shrinks to (2.1.1). When $p > 0$ and affinely independent $z^1, ..., z^{s+1} \in S$, such that (2.1.1) is fulfilled, do not exist, then, of course, condition (2.1.1) $-$ (2.1.1) is meant to be satisfied. We stipulate that a singleton coincides with its relative interior.

Theorem 2.1.1. *C and S are (linearly) separable if and only if for every set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S such that*

$$\left.\begin{array}{c} dim\ conv\ \{proj\ z^1, ..., proj\ z^{s+1}\} = p \quad and \\ O \in ri\ conv\ \{proj\ z^1, ..., proj\ z^{s+1}\} \end{array}\right\} \tag{2.1.0}$$

we have

$$(ri\ C) \cap ri\ conv\ \{z^1, ..., z^{s+1}\} = \varnothing. \tag{2.1.0}$$

The separation is proper if $0 \leq p \leq s$.

Proof. If. The proof will be split up into four parts.
(A) $p = 0$. C is a convex body and thus, obviously, (2.1.1) implies linear separation (even proper) between C and S.
(B) $0 \leq s \leq p-1$. Let B_C and B_S be bases for aff C and aff S, respectively; dim $B_C = n-p$, dim $B_S = s$ and dim $(B_C \cup B_S) \leq n-p+s \leq n-1$. This shows that there exists a hyperplane of $\mathbb{R}^n$ which contains C and is parallel to aff S, so that separation holds.
(C) $1 \leq p \leq s$ and (2.1.1) does not hold, in the sense that no set of affinely independent vectors of S verifies (2.1.1). Denote by proj $S \subset \mathbb{R}^n$ the projection of S on $C^{\perp}$. Since for every set of $s+1$ affinely independent vectors of S, relation (2.1.1) does not hold, then

$$O \notin \text{ri conv proj } S. \tag{2.1.0}$$

In fact, if ab absurdo $O \in$ ri conv proj S, then $\exists \alpha_1, ..., \alpha_{p+1} > 0$ with $\sum_{i=1}^{p+1} \alpha_i = 1$ and $\exists x^1, ..., x^{p+1} \in$ proj S affinely independent, such that $O = \sum_{i=1}^{p+1} \alpha_i x^i$. Thus, we would have $p+1$ affinely independent vectors of S such that $x^i =$ proj z^i, $z^i \in S$, $i = 1, ..., p+1$ and $O = \sum_{i=1}^{p+1} \alpha_i \text{proj } z^i$. Since dim $S = s$, then the set $\{z^1, ..., z^{p+1}\}$ could be augmented (if $p < s$) to form a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S which would satisfy (2.1.1), this contradicts the initial assumption.
First of all, let us consider the case where $C^{\perp}$ is a coordinated subspace. Then (2.1) becomes:

$$O_p \notin \text{int conv proj } S.$$

Applying the Hahn-Banach Theorem, we get the existence of a hyperplane of $\mathbb{R}^p$ through O_p with equation $\sum_{i=1}^{p} a_i x_i = 0$ and such that $\sum_{i=1}^{p} a_i w_i \leq 0$,
$\forall\, (w_1, ..., w_p) \in$ conv proj S. Setting $a_i = 0$, $i = p+1, .., n$, it follows that $\sum_{i=1}^{n} a_i w_i \leq 0$, $\forall\, (w_1, ..., w_n) \in$ conv S because conv and proj are permutable. The hyperplane $H_0 = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = 0\}$ contains the cone C and therefore separates C and S. Moreover, the separation is proper since S cannot be included in the hyperplane H_0, otherwise (2.1) would be contradicted. Now, if $C^{\perp}$ is not a coordinated subspace, by its definition we have that there exists a suitable rotation ρ which transforms C into a cone C^{ρ} such that $(C^{\rho})^{\perp}$ is a coordinated subspace; then the above reasoning can be repeated after having applied the rotation ρ.
(D) $1 \leq p \leq s$ and (2.1.1) holds, in the sense that there exists a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S which verifies (2.1.1). We prove that (2.1.1) implies

$$\text{ri } C \cap \text{ri conv } S = \varnothing. \tag{2.1.0}$$

Suppose that (2.1) does not hold, i.e. there exists $\bar{z} \in$ ri $C \cap$ ri conv S. Because of a well known Carathéodory Theorem, $\bar{z}$ can be expressed as a convex combination of $s+1$ affinely independent vectors of S, say $w^1, ..., w^{s+1}$, that is $\bar{z} = \sum_{i=1}^{s+1} \alpha^i w^i$, with $\alpha_i > 0$, $\forall i = 1, ..., s+1$ and $\sum_{i=1}^{s+1} \alpha_i = 1$. If these vectors verify (2.1.1), then (2.1.1) is contradicted. Therefore we have:

$$O \notin \text{ri conv } \{\text{proj } w^1, ..., \text{proj } w^{s+1}\}.$$

Firstly, let us consider the case where $C^\perp$ is a coordinated subspace. In this case, the previous relation becomes:

$$O_p \notin \text{int conv}\{\text{proj } w^1, ..., \text{proj } w^{s+1}\}$$

and thus there exists $(a_1, ..., a_p) \neq O_p$ with $\sum_{i=1}^{p} a_i (w^j)_i \leq 0$, $\forall j = 1, ..., s+1$. If we set $a_i = 0$, $\forall i = p+1, ..., n$ we get $\sum_{i=1}^{n} a_i (w^j)_i \leq 0$, and therefore also $\sum_{i=1}^{n} a_i \alpha_i (w^j)_i \leq 0$, $\forall j = 1, ..., s+1$. On the other hand, $\bar{z} \in \text{ri } C$ and thus $\langle a, z \rangle = 0$. Since the coefficients α_i are all positive, it follows that $\sum_{i=1}^{n} a_i w_i^j = 0$, $j = 1, ..., s+1$. This implies that $\sum_{i=1}^{p} a_i (w^j)_i = 0$, for all $j = 1, ..., s+1$, which contradicts $O_p \notin \text{int conv}\{\text{proj } w^1, ..., \text{proj } w^{s+1}\}$.
Therefore (2.1) is satisfied and this implies proper separation between C and S. As in the case (C), if $C^\perp$ is not a coordinated subspace, then we have that there exists a suitable rotation ρ which transforms C into a cone C^ρ such that $(C^\rho)^\perp$ is a coordinated subspace and after having applied the rotation ρ we can repeat the above proof to obtain proper separation between C and S.
Only if. By assumption, $\exists a \in \mathbb{R}^n \setminus \{O\}$ and $b \in \mathbb{R}$, such that

$$\langle a, x \rangle \geq b, \ \forall x \in C \quad \text{and} \quad \langle a, y \rangle \leq b, \ \forall y \in S.$$

Since $O \in \text{cl } C$, we can put $b = 0$. Set $H_0 := \{x \in \mathbb{R}^n : \langle a, x \rangle = 0\}$. If no set of $s+1$ affinely independent vectors of S exists, such that (2.1.1) is satisfied, then the thesis is trivial. Let us assume that there exists a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S such that (2.1.1) holds while (2.1.1) is not valid, i.e.

$$O \in \text{ri conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\}, \tag{2.1.0}$$

and

$$(\text{ri } C) \cap \text{ri conv } \{z^1, ..., z^{s+1}\} \neq \varnothing. \tag{2.1.0}$$

Let $\bar{z}$ belong to the left-hand side of (2.1); thus there exists $\alpha_i > 0$, $i = 1, ..., s+1$

with $\sum_{i=1}^{s+1} \alpha_i = 1$ such that $\bar{z} = \sum_{i=1}^{s+1} \alpha_i z^i \in$ ri C. From $\bar{z} \in$ri C we have proj $\bar{z} = O$ and from (2.1) we have that there exists $J \subseteq \{1, ..., s+1\}$ with card $J = p+1$ such that proj $z^i \neq O$ for $i \in J$. Therefore, it results proj $\bar{z} =$ proj $(\sum_{i=1}^{s+1} \alpha_i z^i) = \sum_{i=1}^{s+1} \alpha_i \text{proj } z^i = \sum_{i \in J} \alpha_i \text{proj } z^i = O$. Since $z^1, ..., z^{s+1} \in S$, then $\langle a, z^i \rangle \leq 0$, $i = 1, ..., s+1$ and hence $\langle a, \sum_{i=1}^{s+1} \alpha_i z^i \rangle \leq 0$. On the other hand, $\bar{z} \in$ ri C and thus $\langle a, \sum_{i=1}^{s+1} \alpha_i z^i \rangle \geq 0$. It follows $\bar{z} \in H_0$. From $\bar{z} \in$ ri C and C convex, we have that $\exists \beta_i > 0$, $i = 1, ..., n-p+1$ with $\sum_{i=1}^{n-p+1} \beta_i = 1$ and $\exists k^i \in C$, $i = 1, ..., n-p+1$ affinely independent, such that $\bar{z} = \sum_{i=1}^{n-p+1} \beta_i k^i$. Since $\bar{z} \in H_0$, then $\sum_{i=1}^{n-p+1} \beta_i \langle a, k^i \rangle = 0$, which implies $\langle a, k^i \rangle = 0$, $i = 1, ..., n-p+1$. Thus, conv $\{k^1, ..., k^{n-p+1}\} \subseteq H_0$ and, consequently, $C \subseteq H_0$; it follows that $a \in C^\perp$. Moreover, from $S \subseteq H_0^-$ we have proj $S \subseteq H_0^-$. Using $O =$ proj $\bar{z}$, we obtain

$$\langle a, O \rangle = \langle a, \text{proj } \bar{z} \rangle = \langle a, \sum_{i=1}^{s+1} \alpha_i \text{proj } z^i \rangle = \sum_{i=1}^{s+1} \alpha_i \langle a, \text{proj } z^i \rangle.$$

Since $\alpha_i > 0$, $i = 1, ..., s+1$, we get $\langle a, \text{proj } z^i \rangle = 0$, $i = 1, ..., s+1$; hence we have also $\{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H_0$ and, obviously, conv $\{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H_0$. Let us denote by $N_\varepsilon(O_n)$ an open ball of centre O_n and radius $\varepsilon > 0$ in $\mathbb{R}^n$ such that dim $N_\varepsilon(O_n) = p$. From (2.1) we have that $\exists\ \bar{\varepsilon} > 0$ such that

$$N_\varepsilon(O_n) \subseteq \text{conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H_0,$$

i.e. $\langle a, y \rangle = 0$, $\forall y \in N_\varepsilon(O_n)$. By assumption $a \neq O$; hence, for $\gamma := \frac{\bar{\varepsilon}}{\|a\|} > 0$, it turns out $\bar{y} := \frac{1}{2}\gamma a \in N_\varepsilon(O_n)$. Consequently, we have

$$0 = \langle a, \bar{y} \rangle = \frac{\gamma}{2} \langle a, a \rangle = \frac{\gamma}{2} \|a\|^2,$$

which contradicts the assumption $a \neq O$. □

A classic result about separation and proper separation between convex sets is given by the following theorem (see Theorem 2.39 of [63]).

Theorem 2.1.2. *Two nonempty, convex sets C_1 and C_2 in $\mathbb{R}^n$ are linearly separable if and only if $O \notin$ int $(C_1 - C_2)$. The separation must be proper if also int$(C_1 - C_2) \neq \varnothing$.*

Obviously $O \notin$ int (conv $S - C$) is equivalent to the condition (2.1.1)−(2.1.1) because both are necessary and sufficient for the linear separation between the set S and the convex cone C. It is interesting to discuss such two conditions: $O \notin \text{int}\,(\text{conv}\, S - C)$ is more compact and convenient than (2.1.1) – (2.1.1) and hence it seems preferable to the other one. However, in view of applying separation results to optimisation problem, we have to distinguish between equality and inequality constraints: it will be seen that in the separation approach via image problem illustrated in Sections 2.3 and 2.4, p will be the number of bilateral constraints in a constrained extremum problem; therefore it is useful to establish a condition which considers this number even if in this way the condition may appear more complicated than the one in Theorem 2.1.2.

Both in Theorem 2.1.1 and 2.1.2 there is a sufficient condition for proper separation. We can see that the sufficient condition int $(\text{conv}\, S - C) \neq \varnothing$ implies $0 \leq p \leq s$; in fact, if this double inequality does not hold, then $p \geq s + 1 > 0$ so that int $(\text{conv}\, S - C) = \varnothing$. The viceversa does not hold as shown in the following example, where the proper separation is implied by Theorem 2.1.1 but not by Theorem 2.1.2.

Example 2.1.3. Let be $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 \geq 0, x_3 = 0\}$ and $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -x_2 \geq 0, x_3 = 0\}$. It results int $(S - C) = \varnothing$, while the condition $1 \leq p \leq s$ is satisfied and thus Theorem 2.1.1 guarantees proper separation.

We mention another classic result which is a necessary and sufficient condition for proper separation of two nonempty sets.

Theorem 2.1.4. *[64] Two nonempty, convex sets S and C are properly separable if and only if*

$$ri\, S \cap ri\, C = \varnothing. \tag{2.1.0}$$

This condition seems to be more manageable than the condition form Theorem 2.1.1. The reason for which we will apply however Theorem 2.1.1 for application to the constrained extremum, in the following sections, is that we will need separation, not necessarily proper, between two sets in the IS. Nevertheless, Theorem 2.1.4 will be used in sufficient conditions for the existence of the Lagrange multipliers.

2.2 Regular Separation Between a Set and a Face of a Cone

In [26] Giannessi states a special separation theorem, namely a disjunctive separation between a face F of a convex cone C and a set S by means of a hyperplane which does not contain the face. Such a separation will be called regular (with respect to the face F).

Let us consider Theorem 2.2.7 of [26].

Theorem 2.2.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty and convex cone, with apex at $O \notin C$ such that $C + cl\, C = C$, and F be any face of C. Let $S \subseteq \mathbb{R}^n$ be nonempty with $O \in cl\, S$ and such that $S - cl\, C$ is convex. F is contained in every hyperplane which separates C and S, if any, if and only if*

$$F \subseteq TC(S - cl\, C),$$

where $TC(S - cl\, C)$ is the tangent cone to $S - cl\, C$ at O.

Theorem 2.2.1 assumes the convexity of $S - \text{cl}\, C$. The following example shows that if we remove such an assumption, then the necessity in the theorem does not hold.

Example 2.2.2. Let C be the following convex cone in $\mathbb{R}^3$:

$$C = \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = 0, x_3 = 0\}$$

and

$$S = \{x \in \mathbb{R}^3 : x_1 = x_2 \geq 0, x_3 = -x_1^2 - x_2^2\} \cup$$
$$\{x \in \mathbb{R}^3 : x_1 = -x_2 \geq 0, x_3 = -x_1^2 - x_2^2\}.$$

Choose $F = C$. Obviously S and $S - \text{cl}\, C$ are not convex. The plane $ccH_0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ is the unique plane which separates C and S and it contains the face F, nevertheless F is not contained in $TC(S - \text{cl}\, C)$.

In order to extend Theorem 2.2.1 to nonconvex case, we have to consider $TC(\text{conv}\ (S - \text{cl}\, C))$ in place of $TC(S - \text{cl}\, C)$. First we will state some preliminary properties by means of the following lemma.

Lemma 2.2.3. *Let $C \subseteq \mathbb{R}^n$ be a nonempty and convex cone with apex at O and S be a nonempty subset of $\mathbb{R}^n$ with $O \in cl\ (S - cl\, C)$. The following statements, where H_0 denotes a generic hyperplane of $\mathbb{R}^n$, are equivalent:*

(i) H_0 *separates* C *and* S*;*

(ii) H_0 *separates* C *and* $S - cl\, C$*;*

(iii) H_0 *separates* C *and* $conv\ (S - cl\, C)$*;*

(iv) H_0 *separates* C *and* $TC(conv\ (S - cl\, C))$*.*

Proof. $(i) \Rightarrow (ii)$ Let be H_0 a hyperplane which separates C and S. Assume that $C \subseteq H_0^+$ and $S \subseteq H_0^-$. Ab absurdo, suppose that $\exists \widehat{x} \in S - \text{cl}\, C$ such that $\langle a, \widehat{x} \rangle > b$. From $\widehat{x} \in S - \text{cl}\, C$ we get the existence of $x^1 \in S$ and $x^2 \in \text{cl}\, C$ such that $\widehat{x} = x^1 - x^2$. Therefore $\langle a, \widehat{x} \rangle > b$ implies the contradiction $b \geq \langle a, x^1 \rangle > \langle a, x^2 \rangle \geq b$, where the first inequality is implied by $x^1 \in S \subseteq H_0^-$ and the third by $x^2 \in \text{cl}\, C \subseteq H_0^+$.

Moreover, observe that, since $O \in \text{cl } C$ and $O \in \text{cl } (S - \text{cl } C)$, it follows $b = 0$.
(ii) $\Rightarrow$ *(iii)* Suppose that the hyperplane H_0, whose equation is $\langle a, x \rangle = 0$, separates C and $S - \text{cl } C$, e.g. $C \subseteq H_0^+$ and $S - \text{cl } C \subseteq H_0^-$. Let z be any element of conv $(S - \text{cl } C)$. From Carathéodory's Theorem we have the existence of $z^1, ..., z^{n+1} \in S - \text{cl } C$ and $\alpha_i \in [0, 1]$, $i = 1, ..., n+1$ with $\sum_{i=1}^{n+1} \alpha_i = 1$, such that $z = \sum_{i=1}^{n+1} \alpha_i z^i$. From $z^1, ..., z^{n+1} \in S - \text{cl } C$ we have $\langle a, z^i \rangle \leq 0$, $\forall i = 1, ..., n+1$, and hence $\langle a, \alpha_i z^i \rangle \leq 0$, $\forall i = 1, ..., n+1$. Therefore it follows $\langle a, \sum_{i=1}^{n+1} \alpha_i z^i \rangle \leq 0$ or $\langle a, z \rangle \leq 0$.
(iii) $\Rightarrow$ *(iv)* Suppose that the hyperplane H_0 separates C and conv $(S - \text{cl } C)$, i.e. $C \subseteq H_0^+$ and conv $(S - \text{cl } C) \subseteq H_0^-$. Now we will prove that conv $(S - \text{cl } C) \subseteq H_0^-$ implies $TC(\text{conv } (S - \text{cl } C)) \subseteq H_0^-$. Let $t \in TC(\text{conv } (S - \text{cl } C))$; then there exist a sequence $\{x^n\}_{n \geq 1} \subseteq \text{conv } (S - \text{cl } C)$ with $\lim_{n \to +\infty} x^n = 0$ and a sequence $\{\alpha_n\}_{n \geq 1} \subset \mathbb{R}_+ \setminus \{0\}$ such that $\lim_{n \to +\infty} \alpha_n x^n = t$. Since $x^n \in \text{conv } (S - \text{cl } C)$, $\forall n \geq 0$, then $\langle a, x^n \rangle \leq 0$, and hence $\langle a, \alpha_n x^n \rangle \leq 0$, $\forall n \geq 0$. Letting $n \to +\infty$ we obtain $\langle a, t \rangle \leq 0$ and thus $TC(\text{conv } (S - \text{cl } C)) \subseteq H_0^-$.
(iv) $\Rightarrow$ *(i)* This is an obvious consequence of the inclusions $S \subseteq S - \text{cl } C \subseteq \text{conv } (S - \text{cl } C) \subseteq TC(\text{conv } (S - \text{cl } C))$. □

Now, we give the generalisation of Theorem 2.2.1 to nonconvex case:

Theorem 2.2.4. *Let $C \subseteq \mathbb{R}^n$ be a nonempty and convex cone with apex at O and S be a nonempty subset of $\mathbb{R}^n$ with $O \in$ cl $(S -$ cl $C)$. Let F be any face of C. The following statements are equivalent:*

- *(i) There exists at least one hyperplane which separates S and C and which does not contain F;*
- *(ii) $F \nsubseteq TC($conv $(S -$ cl $C))$.*

Proof. (i)$\Rightarrow$(ii) The hypotheses imply the existence of a hyperplane of equation $H_0 := \{x \in \mathbb{R}^n : \langle a, x \rangle = 0\}$, $a \neq O$, such that $\langle a, x \rangle \leq 0$, $\forall x \in S$ and $\langle a, x \rangle \geq 0$, $\forall x \in C$ and that there exists $\bar{f} \in F$ with $\langle a, \bar{f} \rangle > 0$.
Ab absurdo, suppose $F \subseteq TC(\text{conv } (S - \text{cl } C))$. From Lemma 2.2.3 we have that H_0 separates also $TC(\text{conv } (S - \text{cl } C))$ and C, i.e. $\langle a, x \rangle \leq 0$, $\forall x \in TC(\text{conv } (S - \text{cl } C))$. Thus also $\langle a, f \rangle \leq 0$, $\forall f \in F$, which contradicts the hypothesis.
(ii) $\Rightarrow$ *(i)* From $F \nsubseteq TC(\text{conv } (S - \text{cl } C))$ it follows that $\exists f^0 \in F \setminus TC(\text{conv } (S - \text{cl } C))$. Since $TC(\text{conv } (S - \text{cl } C))$ is closed and convex, then there exists a hyperplane H_0 of equation $\langle a, x \rangle = b$ with $a \in \mathbb{R}^n \setminus \{O\}$ such that $\langle a, x \rangle \leq b < \langle a, f^0 \rangle$, $\forall x \in TC(\text{conv } (S - \text{cl } C))$. Because of $O \in TC(\text{conv } (S - \text{cl } C))$, we can set $b = 0$ and thus we have

$$\langle a, x \rangle \leq 0 < \langle a, f^0 \rangle, \ \forall x \in TC(\text{conv } (S - \text{cl } C)). \tag{2.2.0}$$

The inclusion $S - \text{cl } C \subseteq TC(\text{conv } (S - \text{cl } C))$ implies that $\langle a, x \rangle \leq 0,\ \forall x \in S - \text{cl } C$. Now we prove that $\langle a, x \rangle \geq 0,\ \forall x \in C$. Ab absurdo, suppose that $\exists k \in C$ such that $\langle a, k \rangle < 0$ and let $s \in S$. Then we have $s - \alpha k \in S - \text{cl } C,\ \forall \alpha \in \mathbb{R}_+$ so that $\lim_{\alpha \to +\infty} \langle a, s - \alpha k \rangle = +\infty$, which contradicts $\langle a, x \rangle \leq 0,\ \forall x \in S - \text{cl } C$. Therefore H_0 separates C and $S - \text{cl } C$. Because of Lemma 2.2.3, H_0 separates also C and S and from (2.2) it does not contain F. □

We call the separation between S and C *regular with respect to the face* F if F is not contained in at least one separating hyperplane.

Notice that in Theorem 2.2.4 the tangent cone $TC(\text{conv } (S - \text{cl } C))$ can be replaced by $\text{cl cone conv } (S - \text{cl } C)$; in fact, if A is a convex set, then $TC(A) = \text{cl cone } A$. Moreover, observe that in Theorem 2.2.4 it is not possible to replace $TC(\text{conv } (S - \text{cl } C))$ by $\text{conv } TC(S - \text{cl } C))$; in such a case, without the convexity assumption, it may exist a hyperplane which separates C and $TC(S - \text{cl } C)$ but does not separate C and $S - \text{cl } C$. This situation is illustrated by the following example.

Example 2.2.5. Let C be the following convex cone in $\mathbb{R}^3$:

$$C = \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = 0, x_3 = 0\} \text{ and}$$

$$S = \{x \in \mathbb{R}^3 : x_1 = x_2 \geq 0, x_3 \leq 0, x_3 = (x_1 - 1)^2 + (x_2 - 1)^2 - 2\} \cup$$
$$\{x \in \mathbb{R}^3 : x_1 = -x_2 \geq 0, x_3 \leq 0, x_3 = (x_1 - 1)^2 + (x_2 + 1)^2 - 2\}.$$

Choose $F = C$. Obviously S and $S - \text{cl } C$ are not convex. The plane $H_0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ is the unique plane which separates C and S and it contains the face F. It results:

$$TC(S - \text{cl } C) = \{x \in \mathbb{R}^3 : x_1 = x_2, x_3 \leq 0, x_3 \leq -4x_1\} \cup$$
$$\{x \in \mathbb{R}^3 : x_1 = -x_2, x_3 \leq 0, x_3 \leq -4x_1\}.$$

$TC(S - \text{cl } C)$ is not convex and we have that $F \not\subseteq \text{conv } TC(S - \text{cl } C)$. Moreover, every plane $H_a = \{x \in \mathbb{R}^3 : ax_1 + x_3 = 0\}$, with $0 < a \leq 4$, separates C and $TC(S - \text{cl } C)$ (and hence also C and $\text{conv } TC(S - \text{cl } C)$), but does not separate C and S and does not contain the face F.

Both in Example 2.2.2 and 2.2.5 we have $\text{int } C = \varnothing$. Similar examples with $\text{int } C \neq \varnothing$ can be given by putting $C = \{x \in \mathbb{R}^3 : x_1 \geq 0, -10x_1 \leq x_2 \leq 0,\ 0 \leq x_3 \leq 10x_1\}$ and choosing $F \subset C$, $F = \{x \in \mathbb{R}^3 : -10x_1 \leq x_2 \leq 0, x_3 = 0\} \subset C$.

The above theorem deals with a generic subset S of the IS. Thus, it can be used in at least two ways, using every time as C the cone in the IS $\mathcal{H}$. When we want to achieve a necessary optimality condition of Lagrange type, then we must set $S = \mathcal{K}^h_{\bar{x}}$. In fact, such a type of necessary condition is based on the separation in the IS between the cone $\mathcal{H}$ and the homogenization $\mathcal{K}^h_{\bar{x}}$ (see Definition 1.2.4) of the image set. In Section 2.4, this will be done by exploiting Theorem 2.1.1 which ensures the existence of a separating hyperplane between $\mathcal{H}$ and $\mathcal{K}^h_{\bar{x}}$, where, in the gradient vector $(\overline{\theta}, \overline{\lambda})$ of multipliers, θ is merely non-negative; if $\theta = 0$ necessarily, then according to Definition 1.2.8, the separation is irregular; if we apply Theorem 2.2.4, then, as we will see, we achieve a necessary condition with $\overline{\theta} = 1$. Indeed, Theorem 2.2.4 guarantees that, if a separating hyperplanes exist, then at least one has a gradient $(\overline{\theta}, \overline{\lambda})$ with $\overline{\theta} = 1$. A second way of using Theorem 2.2.4 deals with sufficient optimality conditions.

In Section 2.3, some sufficient optimality conditions will be established; again, with the use of Theorem 2.1.1 we will look for a separation between $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$, where the separation hyperplane is of the type $\theta u + \langle \lambda, v \rangle = 0$, and we wonder wheter or not (θ, λ) exists with $\theta = 1$; to this end, Theorem 2.2.4 will be used with $S = \mathcal{K}_{\overline{x}}$.

2.3 Regularity and Sufficient Optimality Conditions

Let us consider the particular case of the constrained extremum problem (1.2.1).

We define the following function $\mathcal{L} : X \times \mathbb{R}_+ \times D^* \to \mathbb{R}$,

$$\mathcal{L}(x; \theta, \lambda) := \theta f(x) - \langle \lambda, g(x) \rangle.$$

At $\theta = 1$, $L(x; \lambda) = \mathcal{L}(x; 1, \lambda)$ is the classic *Lagrangian* function.

Definition 2.3.1. A point $(\bar{x}; \bar{\theta}, \bar{\lambda}) \in X \times \mathbb{R}_+ \times D^*$ with $(\bar{\theta}, \bar{\lambda}) \neq (0, O)$ such that

$$\mathcal{L}(\bar{x}; \bar{\theta}, \lambda) \leq \mathcal{L}(\bar{x}; \bar{\theta}, \bar{\lambda}) \leq \mathcal{L}(x; \bar{\theta}, \bar{\lambda}),\ \forall x \in X,\ \forall \lambda \in D^*,$$

is called a *John saddle point*. When $\bar{\theta} = 1$ then $(\bar{x}; \bar{\lambda}) \in X \times D^*$ is a *saddle point* for the Lagrangian function.

The next theorem states that the linear separation between two sets of the image space, namely $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, is equivalent to the existence of a John saddle point, which is a sufficient condition for optimality.
Let be $k = \dim \mathcal{K}_{\bar{x}}$.

Theorem 2.3.2. *The following statements are equivalent:*

(i) $\mathcal{K}_{\bar{x}}$ *and* $\mathcal{H}$ *are linearly separable;*

(ii) for every set $\{z^1, ..., z^{k+1}\} \subseteq \mathcal{K}_{\bar{x}}$ *of affinely independent vectors such that*

$$O_p \in \text{int conv}\{\textit{proj } \mathrm{z}^1, ..., \textit{proj } \mathrm{z}^{\mathrm{k+1}}\} \tag{2.3.0}$$

we have

$$(\text{ri } \mathcal{H}) \cap \text{ri conv}\{\mathrm{z}^1, ..., \mathrm{z}^{\mathrm{k+1}}\} = \varnothing; \tag{2.3.0}$$

(iii) there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_+ \times D^*$ *with* $(\bar{\theta}, \bar{\lambda}) \neq (0, O)$ *such that* $(\bar{x}; \bar{\theta}, \bar{\lambda})$ *is a John saddle point.*

Proof. $(i) \Leftrightarrow (ii)$ It follows from Theorem 2.1.1 with $C = \mathcal{H}$ and $S = \mathcal{K}_{\bar{x}}$ and noticing that in this case (2.1.1) – (2.1.1) becomes (2.3.2) – (2.3.2). $(i) \Leftrightarrow (iii)$ The separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ is equivalent to the existence of $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R} \times \mathbb{R}^m$, $(\bar{\theta}, \bar{\lambda}) \neq (0, O)$, such that $\bar{\theta}u + \langle \bar{\lambda}, v \rangle \geq 0$, $\forall (u, v) \in \mathcal{H}$ and $\bar{\theta}u + \langle \bar{\lambda}, v \rangle \leq 0$, $\forall (u, v) \in \mathcal{K}_{\bar{x}}$; or to the existence of $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_+ \times D^*$ such that

$$\bar{\theta}u + \langle \bar{\lambda}, v \rangle \leq 0, \ \forall (u, v) \in \mathcal{K}_{\bar{x}}. \tag{2.3.0}$$

Substituting $(u, v) \in \mathcal{K}_{\bar{x}}$ with $(f_{\bar{x}}(x), g(x)), x \in X$, (2.3) becomes

$$\bar{\theta} f(\bar{x}) \leq \bar{\theta} f(x) - \langle \bar{\lambda}, g(x) \rangle, \ \forall x \in X. \tag{2.3.0}$$

Setting $x = \bar{x}$ we get $\langle \bar{\lambda}, g(\bar{x}) \rangle = 0$. From this and $\langle \lambda, g(\bar{x}) \rangle \geq 0$, $\forall \lambda \in D^*$, it follows that (2.3) can be written as

$$\bar{\theta} f(\bar{x}) - \langle \lambda, g(\bar{x}) \rangle \leq \bar{\theta} f(\bar{x}) - \langle \bar{\lambda}, g(\bar{x}) \rangle \leq \bar{\theta} f(x) - \langle \bar{\lambda}, g(x) \rangle, \ \forall x \in X, \ \forall \lambda \in D^*.$$

Hence the proof is complete. □

We will call *regular separation* the separation which is regular with respect to the face $\mathcal{H}_u$.
We notice that $\mathcal{H}_u \not\subseteq TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}})))$ is actually equivalent to

$$\mathcal{H}_u \cap TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \varnothing.$$

The next theorem points out that the regular linear separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ is equivalent to the existence of a saddle point for the Lagrangian function.

Theorem 2.3.3. *The following statements are equivalent:*

(i)
$$\mathcal{H}_u \cap TC(\textit{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \varnothing; \tag{2.3.0}$$

(ii) for every set $\{z^1, ..., z^{k+1}\} \subseteq \mathcal{K}_{\bar{x}}$ *of affinely independent vectors which verify (2.3.2) it holds (2.3.2) and* $\mathcal{H}_u$ *is not contained in at least one hyperplane which separates* $\mathcal{K}_{\bar{x}}$ *and* $\mathcal{H}$*;*

(iii) there exists $\bar{\lambda} \in D^*$ such that $(\bar{x};\bar{\lambda})$ is a saddle point of the Lagrangian function $L(x;\lambda)$.

Proof. $(i) \Leftrightarrow (ii)$ It follows from the combination of Theorems 2.1.1, 2.2.4 and 2.3.2 for $C = \mathcal{H}$ and $S = \mathcal{K}_{\bar{x}}$. $(ii) \Leftrightarrow (iii)$ The condition (2.3.2) $-$ (2.3.2) is equivalent to the existence of a hyperplane $H_0 = \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : \bar{\theta}u + \langle \bar{\lambda}, v\rangle = 0\}$, $(\bar{\theta},\bar{\lambda}) \neq (0,O)$, which separates $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$. $\mathcal{H}_u$ not contained in H_0 is equivalent to $\bar{\theta}u \neq 0, \forall u > 0$, which implies $\bar{\theta} > 0$. We apply the equivalence $(ii) \Leftrightarrow (iii)$ of Theorem 2.3.2 with $\bar{\theta} = 1$ to obtain the thesis. □

2.3.1 Comparison with calmness

Analysing the definition of calmness at a point $\bar{x}$, we observe that this definition can be given for any feasible point $\bar{x}$ and condition (1.2.21) implies the local optimality of the point. Therefore, we can consider the calmness as a sufficient optimality condition.

Moreover, we can see that the notion of calmness is a local notion with respect to $\bar{x}$, not only because $\bar{x}$ is a local solution of the problem, but mostly because in the definition of $R_\varepsilon(\xi)$ it is required that x belong to the neighbourhood $N_\varepsilon(\bar{x})$. Hence, in order to compare the notion of calmness with the regularity condition (2.3.3), we have to consider the regularity condition (2.3.3) in a local form.

First of all, observe that condition (2.3.3) is equivalent to

$$e_u \notin TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}}))), \tag{2.3.0}$$

where $e_u := (1, O_m) \in \mathbb{R}^{1+m}$; a local form is

$$e_u \notin TC(\text{conv } (\mathcal{K}^{\varepsilon}_{\bar{x}} - \text{cl } \mathcal{H})), \tag{2.3.0}$$

where

$$\mathcal{K}^{\varepsilon}_{\bar{x}} := \{(u,v) \in \mathbb{R}^{1+m} : u = f_{\bar{x}}(x), v = g(x), x \in X \cap N_\varepsilon(\bar{x})\}.$$

Obviously the two conditions are not equivalent, as it can be seen by considering problem (8) with $p = 0$, $m = 1$; $X =]-\infty, 0]$; $f(x) = x$ and $g_1(x) = -\sqrt{-x}e^x$.

Theorem 2.3.4. *Let us consider problem (1.2.1); let f be continuous at a local solution $\bar{x}$. If (2.3.1) holds then problem (1.2.1) is calm at $\bar{x}$.*

Proof. Ab absurdo, suppose that (1.2.1) is not calm at $\bar{x}$. Then, if we set $\rho = n$ and $\varepsilon = \frac{1}{n}$, $\forall n \geq 1$, we obtain the existence of $\xi^n \in N_{\frac{1}{n}}(O)$ and of $x^n \in R_\varepsilon(\xi^n)$, in particular $\|x^n - \bar{x}\| < \frac{1}{n}$, such that

$$f_{\bar{x}}(x^n) = f(\bar{x}) - f(x^n) > n\|\xi^n\|. \tag{2.3.0}$$

From $g(x^n)+\xi^n \in D$ it follows the existence of $d^n \in D$ such that $g(x^n)-d^n=-\xi^n$, $n \geq 1$. Since $\|x^n-\bar{x}\|<\frac{1}{n}$ and $\|\xi^n\|<\frac{1}{n}$, $\forall n \geq 1$, we have that $\lim\limits_{n\to+\infty} x^n=\bar{x}$ and $\lim\limits_{n\to+\infty} g(x^n)-d^n=O$; hence, from the continuity of f at $\bar{x}$, it results $\lim\limits_{n\to+\infty} f_{\bar{x}}(x^n)=0$. Moreover, it is obvious that $(f_{\bar{x}}(x^n), g(x^n)-d^n) \in \mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H}$, $\forall n \geq 1$. Taking $\alpha_n := \frac{1}{f_{\bar{x}}(x^n)}$ (notice that (2.3.1) implies $f_{\bar{x}}(x^n)>0$, $\forall n \geq 1$), then we get

$$\lim_{n\to+\infty} \alpha_n(f_{\bar{x}}(x^n), g(x^n)-d^n) = \lim_{n\to+\infty} \alpha_n(f_{\bar{x}}(x^n), -\xi^n) = (1, O)$$

or, equivalently, that

$$(1, O) \in \text{cl cone } (\mathcal{K}_{\bar{x}}^{\varepsilon} - (O \times \text{cl } D)). \tag{2.3.0}$$

Since $\text{cl cone } (\mathcal{K}_{\bar{x}}^{\varepsilon}-(O\times \text{cl } D)) \subseteq \text{ cl cone conv } (\mathcal{K}_{\bar{x}}^{\varepsilon}-(O\times \text{cl } D)) = TC(\text{conv } (\mathcal{K}_{\bar{x}}^{\varepsilon}-(O\times \text{cl } D))) \subset TC(\text{conv } (\mathcal{K}_{\bar{x}}^{\varepsilon}-\text{cl } \mathcal{H}))$, from (2.3.1) we have $(1, O) \in TC(\text{conv } (\mathcal{K}_{\bar{x}}^{\varepsilon}-\text{cl } \mathcal{H}))$ which contradicts the assumption (2.3.1). □

As we have mentioned in the introduction, in [19] it has been shown that the problem (1.2.1) is calm at $\bar{x}$ if and only if

$$\text{cl cone } (\mathcal{E}(\mathcal{K}_{\bar{x}})) \cap \mathcal{H}_u = \varnothing. \tag{2.3.0}$$

Condition (2.3.1) is geometrically interpreted as local cone separation between $\mathcal{K}_{\bar{x}}^{\varepsilon}$ and $\mathcal{H}$ [19]; clearly it is implied by the global condition:

$$\text{cl cone } (\mathcal{E}(\mathcal{K}_{\bar{x}})) \cap \mathcal{H}_u = \varnothing.$$

When the problem (1.2.1) is convex, both (2.3.1) and (2.3.1) become

$$TC(\mathcal{E}(\mathcal{K}_{\bar{x}})) \cap \mathcal{H}_u = \varnothing$$

and thus (2.3.1) becomes equivalent to calmness.

The following example shows that, in general, the converse statement of Theorem 2.3.4 does not hold.

Example 2.3.5. Let us consider problem (1.2.1) with: $p=m=2$; $X=\mathbb{R}$, $D=\{O_2\}$, $f(x)=-|x|$, $g_1(x)=x$, $g_2(x)=-2x^2$. Obviously $\bar{x}=0$ is the (unique) optimal solution to problem (1.2.1). It will be shown that the problem is calm at $\bar{x}=0$. Set $\xi=(\xi_1,\xi_2)$; $g(x)+\xi \in D$ is equivalent to $x+\xi_1=0$, $-2x^2+\xi_2=0$, so that:

$$R_\varepsilon(\xi) = \left\{x \in \mathbb{R} : |x|<\varepsilon, x=-\xi_1=\pm\sqrt{\frac{\xi_2}{2}}\right\},\ \xi_2 \geq 0.$$

Condition (1.2.21) becomes $|x| \leq \rho\sqrt{\xi_1^2+\xi_2^2}$; being $x = -\xi_1$, the last inequality is either an identity (if $\xi_1 = 0$) or it is equivalent to $1 \leq \rho\sqrt{1+4\xi_1^2}$, which is verified if $\rho \geq 1$ and $\varepsilon > 0$. Hence Definition 1.2.21 is fulfilled.

However, the problem is not regular. Its image set is

$$\mathcal{K}_0^\varepsilon = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = |v_1|, v_2 = -2v_1^2, |v_1| < \varepsilon\},$$

and is formed by two parabolic arcs having the bisectors of quadrants (u, v_1) and $(u, -v_1)$ as tangents at O. We notice that in this case $\mathcal{H} = \mathcal{H}_u$ and hence

$$TC(\text{conv } (\mathcal{K}_0^\varepsilon - \text{cl } \mathcal{H})) = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 \leq 0\}.$$

The unique plane separating $\mathcal{H}$ and $\mathcal{K}_0^\varepsilon$ is $H_0 = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}$ and the regularity condition (2.3.1) is not satisfied.

2.3.2 Comparison with metric regularity

Let us recall the following definition due to Ioffe [33], [34].

Definition 2.3.6. Let us consider problem (1.2.1). Let $\bar{x} \in X$. The mapping g is said to be *metrically regular* at $\bar{x}$ with respect to R if and only if there exist two real numbers $L > 0$ and $\varepsilon > 0$ such that

$$d(x; R) \leq Ld(g(x); D),\ \forall x \in X \cap N_\varepsilon(\bar{x}). \tag{2.3.0}$$

It has been proved [59] that, under locally Lipschitz assumptions, metric regularity implies calmness. Therefore, we have to compare also the metric regularity with the regularity condition (2.3.1).

We will investigate problem (1.2.1) when it is convex, i.e. when the functions f and $-g_i$, $i \in \mathcal{I}^+$ are convex and g_i, $i \in \mathcal{I}^0$ are affine. In what follows, we shall prove that, under these assumptions, the metric regularity implies the regularity condition (2.3.1). The convexity of the problem implies the convexity of $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and therefore condition (2.3.1) becomes

$$e_u \notin TC(\mathcal{E}(\mathcal{K}_{\bar{x}}^\varepsilon)). \tag{2.3.0}$$

Theorem 2.3.7. *Let $\bar{x} \in X$ be a local solution to problem* (1.2.1)*, where f and $-g_i$, $i \in \mathcal{I}^+$ are convex, g_i, $i \in \mathcal{I}^0$ are affine. If f is locally Lipschitz at $\bar{x}$ and g is metrically regular at $\bar{x}$, then the regularity condition (2.3.2) holds.*

Proof. Since f is locally Lipschitz at $\bar{x}$, we can apply Theorem 5.1 from [19] which proves our assertion. We remark that in such a theorem it is not needed to assume g locally Lipschitz at $\bar{x}$. □

Removing the convexity assumption in Theorem 2.3.7, the metric regularity is no longer sufficient for regularity condition (2.3.1). For this, consider again Example 2.3.5.

Continuation of Example 2.3.5 Recall that in this case the sets R and D are $R = \{0\}$ and $D = \{O_2\} = \{(0,0)\}$, respectively. Thus, for a given $\varepsilon > 0$, we have $d(x; R) = |x|$, $\forall x \in N_\varepsilon(0)$. On the other hand, it turns out that

$$d(g(x); D) = \|g(x)\| = \sqrt{x^2 + 4x^4} = |x|\sqrt{1 + 4x^2}, \ \forall x \in N_\varepsilon(0).$$

Setting $L = 1$, relation (2.3.6) becomes obvious at $x = 0$, while if $x \neq 0$ we have

$$1 \leq \sqrt{1 + 4x^2}, \ \forall x \in N_\varepsilon(0).$$

This means that the metric regularity condition holds, but, as we have seen, the problem is not regular.

The following example shows that also the locally Lipschitz condition on f cannot be removed in the above theorem.

Example 2.3.8. Let problem (1.2.1) be given with: $p = m = 1$; $X = [0, +\infty)$, $D = \{0\}$; $f(x) = -\sqrt{x}$ and $g(x) = x$. We have $R = \{0\}$. The functions f and $-g$ are convex but f is not locally Lipschitz at $\bar{x} = 0$, which is the (unique) optimal solution to problem (1.2.1).
It results

$$\mathcal{K}_{\bar{x}} = \{(u, v) \in \mathbb{R}^2 : u = \sqrt{v}, v \geq 0\}$$

and

$$TC(\text{conv } \mathcal{E}(\mathcal{K}_{\bar{x}})) = \{(u, v) \in \mathbb{R}^2 : v \geq 0\}.$$

One obtains $d(x; R) = |x|$ and $d(g(x); D) = |x|$, $\forall x \in X$. Thus the metric regularity condition holds but, as it can be easily seen, the regularity condition (2.3.2) does not.

Finally, the following example shows that the converse statement of Theorem 2.3.7 does not hold.

Example 2.3.9. Let us consider problem (1.2.1) with the following positions: $p = 0$, $m = 1$; $X = \mathbb{R}$, $D = [0, +\infty)$; $f(x) = x^4$ and $g(x) = -x^2$. We have $R = \{0\}$. Obviously, f and $-g$ are convex functions and $\bar{x} = 0$ is the (unique) optimal solution to problem (1.2.1).
We find

$$\mathcal{K}_{\bar{x}} = \{(u, v) \in \mathbb{R}^2 : u = -v^2, v \leq 0\}$$

and

$$TC(\text{conv } \mathcal{E}(\mathcal{K}_{\bar{x}})) = \{(u, v) \in \mathbb{R}^2 : u \leq 0, v \leq 0\}.$$

Therefore the regularity condition (2.3.2) holds.
On the other hand, it results $d(x;R) = |x|$ and $d(g(x);D) = x^2$, $\forall x \in \mathbb{R}$. Condition (2.3.6) becomes $|x| \leq Lx^2$; $\forall L > 0$ and in every neighbourhood of $\bar{x} = 0$ this inequality is not fulfilled.

2.4 Regularity and Necessary Optimality Conditions

We will consider the constrained extremum problem (1.2.1) when the functions f and $-g_i$, $i \in \mathcal{I}$, defined on a convex subset X of $\mathbb{R}^n$ with card $X > 1$ are $\mathcal{C}$-differentiable at $\bar{x}$. In this case the *homogenization* of the image set $\mathcal{K}_{\bar{x}}$ is defined as

$$\mathcal{K}_{\bar{x}}^h := \{(u,v) \in \mathbb{R}^{1+m} : u = -\mathcal{D}_{\mathcal{C}} f(\bar{x};d), v_i = g_i(\bar{x}) + \mathcal{D}_{-\mathcal{C}} g_i(\bar{x};d), i \in \mathcal{I},$$

$$d \in \text{cone } (X - \bar{x})\};$$

its conic extension is $\mathcal{E}(\mathcal{K}_{\bar{x}}^h) := \mathcal{K}_{\bar{x}}^h - \text{cl } \mathcal{H}$ and it is a convex cone.
In [23] Giannessi extends the well-known Kuhn-Tucker necessary optimality conditions to $\mathcal{C}-$differentiable problems. The next theorem claims that the linear separation between $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}^h$ is equivalent to the generalised necessary optimality condition (2.4.1). Let be h =dim $\mathcal{K}_{\bar{x}}^h$.

Theorem 2.4.1. *The following statements are equivalent:*

(i) $\mathcal{K}_{\bar{x}}^h$ *and* $\mathcal{H}$ *are linearly separable;*

(ii) for every set $\{z^1, ..., z^{h+1}\} \subseteq \mathcal{K}_{\bar{x}}^h$ *of affinely independent vectors such that*

$$O_p \in \text{int conv}\{\textit{proj } \mathrm{z}^1, ..., \textit{proj } \mathrm{z}^{\mathrm{h}+1}\} \tag{2.4.0}$$

we have

$$(\text{ri } \mathcal{H}) \cap \text{ri conv}\{\mathrm{z}^1, ..., \mathrm{z}^{\mathrm{h}+1}\} = \varnothing; \tag{2.4.0}$$

(iii) there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_+ \times D^*$ *with* $(\bar{\theta}, \bar{\lambda}) \neq (0, O_m)$ *such that*

$$O_n \in \bar{\theta} \underline{\partial}_{\mathcal{C}} f(\bar{x}) + \left\langle \bar{\lambda}, \underline{\partial}_{(-\mathcal{C})}(-g(\bar{x})) \right\rangle + TC^*(\bar{x}; X) \tag{2.4.0}$$

and

$$\langle \bar{\lambda}, g(\bar{x}) \rangle = 0.$$

Proof. $(i) \Leftrightarrow (ii)$ It follows from Theorem 2.1.1 with $C = \mathcal{H}$ and $S = \mathcal{K}_{\bar{x}}^h$. $(i) \Leftrightarrow (iii)$ The assumption (i) is equivalent to the existence of $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R} \times \mathbb{R}^m$, $(\bar{\theta}, \bar{\lambda}) \neq (0, O)$ such that $\bar{\theta} u + \langle \bar{\lambda}, v \rangle \geq 0$, $\forall (u,v) \in \mathcal{H}$ and $\bar{\theta} u + \langle \bar{\lambda}, v \rangle \leq 0$, $\forall (u,v) \in \mathcal{K}_{\bar{x}}^h$, i.e.

$(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_+ \times D^*$ with $\bar{\theta}u + \langle \bar{\lambda}, v\rangle \leq 0, \forall (u,v) \in \mathcal{K}^h_{\bar{x}}$. By definition of $\mathcal{K}^h_{\bar{x}}$, it turns out

$$\bar{\theta}(-\mathcal{D}_{\mathcal{C}} f(\bar{x}; d)) + \langle \bar{\lambda}, g(\bar{x}) + \mathcal{D}_{-\mathcal{C}} g(\bar{x}; d)\rangle \leq 0, \ \forall d \in \text{cone } (X - \bar{x}). \tag{2.4.0}$$

Setting $x = \bar{x}$ in the previous inequality we have $\langle \bar{\lambda}, g(\bar{x})\rangle = 0$. If we denote by δ the indicator function of a set, we have that $\delta(d; \text{cone}(X - \bar{x})) = \delta'_d(\bar{x}; X)$, $\forall d \in \mathbb{R}^n$, where δ'_d marks the directional derivative of the convex function δ along the direction d. Hence, (2.4) can be rewritten as

$$\bar{\theta}\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) - \langle \bar{\lambda}, \mathcal{D}_{-\mathcal{C}} g(\bar{x}; d)\rangle + \delta'_d(\bar{x}; X) \geq 0, \ \forall d \in \mathbb{R}^n,$$

or, equivalently,

$$\bar{\theta}\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) - \langle \bar{\lambda}, \mathcal{D}_{-\mathcal{C}} g(\bar{x}; d)\rangle + \mathcal{D}_{\mathcal{C}}\delta(d; \text{cone } (X - \bar{x})) \geq 0, \ \forall d \in \mathbb{R}^n. \tag{2.4.0}$$

On the other side, the assumption (iii) is equivalent to the existence of $\sigma_f \in \underline{\partial}_{\mathcal{C}} f(\bar{x})$, $\sigma_g \in \underline{\partial}_{(-\mathcal{C})}(-g(\bar{x}))$ and $y \in TC^*(\bar{x}; X) = \partial\delta(\bar{x}; \text{cone } (X - \bar{x}))$ such that

$$0 = \bar{\theta}\sigma_f + \langle \bar{\lambda}, \sigma_g\rangle + y. \tag{2.4.0}$$

Using the $\mathcal{C}$-subdifferential definition and taking into account that

$$\mathcal{D}_{-\mathcal{C}}(-g(\bar{x}; x - \bar{x})) = -\mathcal{D}_{-\mathcal{C}} g(\bar{x}; x - \bar{x})$$

we see that (2.4) becomes (2.4) and this completes the proof. □

If in the previous theorem we impose that the separation is regular, then condition (2.4.1) holds with $\bar{\theta} = 1$ and we obtain the following result.

Theorem 2.4.2. *The following statements are equivalent:*

(i)
$$\mathcal{H}_u \cap TC(\mathcal{K}^h_{\bar{x}} - cl\ \mathcal{H}) = \varnothing; \tag{2.4.0}$$

(ii) for every set $\{z^1, ..., z^{h+1}\} \subseteq \mathcal{K}^h_{\bar{x}}$ of affinely independent vectors for which (2.4.1) holds we have that (2.4.1) is true and $\mathcal{H}_u$ is not contained in at least one hyperplane which separates $\mathcal{K}^h_{\bar{x}}$ and $\mathcal{H}$;

(iii) there exists $\bar{\lambda} \in D^*$ such that

$$O_n \in \underline{\partial}_{\mathcal{C}} f(\bar{x}) + \left\langle \bar{\lambda}, \underline{\partial}_{(-\mathcal{C})}(-g(\bar{x}))\right\rangle + TC^*(\bar{x}; X) \ \textit{and} \ \langle \bar{\lambda}, g(\bar{x})\rangle = 0.$$

Proof. $(i) \Leftrightarrow (ii)$ Combining Theorems 2.1.1, 2.2.4 and 2.4.1 for $C = \mathcal{H}$ and $S = \mathcal{K}^h_{\bar{x}}$ we obtain the equivalence. $(ii) \Leftrightarrow (iii)$ The condition (2.4.1) $-$ (2.4.1) is equivalent to the existence of a separation hyperplane $H_0 := \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : \bar{\theta}u + \langle \bar{\lambda}, v\rangle = 0\}$, $(\bar{\theta}, \bar{\lambda}) \neq (0, O)$ between $\mathcal{K}^h_{\bar{x}}$ and $\mathcal{H}$. $\mathcal{H}_u$ not contained in H_0 is equivalent to $\bar{\theta}u \neq 0, \forall u > 0$, i.e. $\bar{\theta} > 0$. To obtain the thesis it is enough to apply $(ii) \Leftrightarrow (iii)$ of Theorem 2.4.1 with $\bar{\theta} = 1$. □

2.4.1 Comparison with calmness

In the literature, for locally Lipschitz problems, the calmness is known as a sufficient condition for the existence of the Lagrange multipliers with the one corresponding to the objective function equal to 1 [14]. Hence, we perform the comparison of the necessary condition (2.4.2) with the calmness condition.
Just as in the previous performed comparison with calmness as a sufficient condition, we will give the regularity condition (2.4.2) in a local form, that is $\mathcal{H}_u \cap TC(\mathcal{K}_{\bar{x}}^{h,\varepsilon} - \text{cl } \mathcal{H}) = \varnothing$ or, equivalently,

$$e_u \notin TC(\mathcal{K}_{\bar{x}}^{h,\varepsilon} - \text{cl } \mathcal{H}), \tag{2.4.0}$$

where we recall that $e_u = (1, O_m) \in \mathbb{R}^{1+m}$ and

$$\mathcal{K}_{\bar{x}}^{h,\varepsilon} = \{(u,v) \in \mathbb{R}^{1+m} : u = -\mathcal{D}_{\mathcal{C}} f(\bar{x}; x - \bar{x}),$$
$$v_i = g_i(\bar{x}) + \mathcal{D}_{-\mathcal{C}} g_i(\bar{x}; x - \bar{x}), i \in \mathcal{I}, x \in \text{cone } X \cap N_\varepsilon(\bar{x})\}.$$

Theorem 2.4.3. *If problem (1.2.1) is calm at $\bar{x}$ then the regularity condition (2.4.1) holds.*

Proof. Ab absurdo, suppose that the problem is calm but the regularity condition (2.4.1) does not hold, i.e. $\exists\ \{x^n\}_{n\geq 1} \subset X$ and $\{p^n\}_{n \geq 1} \in D$ such that

$$\lim_{n \to +\infty} \left(-\mathcal{D}_{\mathcal{C}} f(\bar{x}; x^n - \bar{x}), g(\bar{x}) + \mathcal{D}_{(-\mathcal{C})} g(\bar{x}; x^n - \bar{x}) - p^n\right) = (1, 0)$$

or, equivalently that

$$\lim_{n \to +\infty} (f_{\bar{x}}(x^n) - \varepsilon_0(\bar{x}; x^n - \bar{x}), g(x^n) - \varepsilon(\bar{x}; x^n - \bar{x}) - p^n) = (1, O). \tag{2.4.0}$$

From $x^n - \bar{x} \in$ cone $(X - \bar{x})$, $\forall n \geq 1$, it follows that $\exists\ d^n \in X - \bar{x}$ such that $x^n - \bar{x} = d^n$, $\forall n \geq 1$. Therefore we have

$$\lim_{n \to +\infty} \varepsilon_i(\bar{x}; x^n - \bar{x}) = \lim_{n \to +\infty} \frac{\varepsilon_i(\bar{x}; d^n)}{\|d^n\|} \|d^n\| = 0,\ \forall i = 0, ..., m.$$

Now, (2.4.1) becomes

$$\lim_{n \to +\infty} (f(\bar{x}) - f(x^n), g(x^n) - p^n) = \lim_{n \to +\infty} (f(\bar{x}) - f(x^n), -\xi^n) = (1, O)$$

and this implies that $\forall \rho > 0\ \exists \bar{n} \in \mathbb{N}$ such that $f(\bar{x}) - f(x^n) \geq \rho \|\xi^n\|$, $\forall n \geq \bar{n}$ and thus the calmness assumption is contradicted. □

The viceversa in the above theorem does not hold, as it is shown by the Example 2.6.18 of Section 2.6.

2.4.2 Comparison with known constraint qualifications

The most general constraint qualifications for classes of differentiable problems, convex problems, differentiable and convex problems (that is Guignard CQ, basic CQ and respectively Abadie CQ) have already given already starting from the 1970s. An extensive analysis for differentiable problems and for convex ones can be found in [6] and [61]. In these works classic but strong constraint qualifications as the linear independence CQ, Mangasarian-Fromowitz CQ, Abadie CQ, Arrow-Hurwitz CQ for differentiable problems, or Slater CQ, Karlin CQ for the convex problems are inserted in the chains of implications between different CQs. In this thesis we will try to confront the regularity condition (2.4.2) with the weakest constraint qualifications for each class of problems mentioned above.

First, we will show that classic CQs such as Slater, Mangasarian-Fromowitz imply the regularity condition (2.4.2). Let us recall these well known constraint qualifications:

Slater constraint qualification (Slater CQ): for convex problems with only inequality constraints, i.e. when $\mathcal{J}^0 = \varnothing$, there exists $\tilde{x} \in R$ such that $g_i(\tilde{x}) < 0$, $\forall i \in \mathcal{J}^+(\bar{x})$. Equivalently, Slater CQ can be written as $\exists \tilde{x} \in R$ such that $g(\tilde{x}) \in \text{int } D$.

Mangasarian-Fromowitz constraint qualification (MF CQ): for differentiable problems there exists $\tilde{x}$ such that $\nabla g_i(\bar{x})\tilde{x} < 0$, $\forall i \in \mathcal{J}^+(\bar{x})$ and $\nabla g_i(\bar{x}), i \in \mathcal{J}^0$ are linearly independent.

We mention a classic and natural generalisation of the Slater CQ. First, we will need an additional classic result from convex analysis.

Proposition 2.4.4. *[64] Let K be a convex set in $\mathbb{R}^{1+m}$ and $x \in rbd(K)$. Then K admits a supporting hyperplane at x and its normal vector belongs to aff $(K - x)$.*

Then, by means of a generalisation of the interior of convex sets in $\mathbb{R}^n$, a generalisation of Slater CQ is that $\exists \tilde{x} \in R$ such that $g(\tilde{x}) \in$ri D.

Theorem 2.4.5. *Suppose that $\mathcal{E}(\mathcal{K}_{\overline{x}})$ and $\mathcal{H}$ admit a proper linear separation and, furthermore,*

$$0 \in ri \;\; conv[g(X) - D]. \tag{2.4.0}$$

Then there exist $\bar{\theta} > 0$ and $\bar{\lambda} \in D^*$ such that:

$$\bar{\theta}u + \langle \bar{\lambda}, v \rangle \le 0, \quad \forall (u, v) \in \mathcal{E}(\mathcal{K}_{\overline{x}}). \tag{2.4.0}$$

Proof. By Theorem 2.1.4, proper linear separation is equivalent to

$$O \notin \text{ri } \; \text{conv}\, \mathcal{E}(\mathcal{K}_{\overline{x}}).$$

Since $O \in \mathcal{E}(\mathcal{K}_{\overline{x}})$ then

$$O \in \text{rbd } \text{conv}\, \mathcal{E}(\mathcal{K}_{\overline{x}}).$$

Applying Proposition 2.4.4, we obtain that there exists $(\bar{\theta}, \bar{\lambda}) \in \text{aff } \text{conv}\, \mathcal{E}(\mathcal{K}_{\overline{x}})$ such that (2.4.5) holds. Since $O \in \mathcal{E}(\mathcal{K}_{\overline{x}})$ then

$$(\bar{\theta}, \bar{\lambda}) \in \text{lin } \text{conv}\, (\mathcal{E}(\mathcal{K}_{\overline{x}})).$$

Ab absurdo suppose that $\bar{\theta} = 0$. Then

$$\bar{\lambda} \in \text{lin } \text{conv}\, [g(X) - D], \tag{2.4.0}$$

and (2.4.5) implies

$$\langle \bar{\lambda}, v \rangle \leq 0, \quad \forall v \in \text{conv}\, [g(X) - D]. \tag{2.4.0}$$

By (2.4.5), there exists a neighbourhood N of $O \in \mathbb{R}^m$ such that

$$S := N \cap \text{ lin } \text{conv}[g(X) - D] \subseteq \text{conv}\, [g(X) - D].$$

Taking into account (2.4.2), we obtain that $\gamma\bar{\lambda} \in V$ for $|\gamma| < \epsilon$, sufficiently small. Since $S \subset \text{conv}\, [g(X) - D]$, by (2.4.2) ,

$$\gamma \langle \bar{\lambda}, \bar{\lambda} \rangle \leq 0, \quad \forall \gamma : |\gamma| < \epsilon,$$

which is impossible, for $\bar{\lambda} \neq 0$. □

Remark 2.4.6. When $\text{int}\, D \neq 0$, then (2.4.5) collapses to the classic Slater condition.

When the convex problem has also affine equality constraints, i.e. $\mathcal{I}^0 \neq \varnothing$ than the Slater condition becomes

$$\text{there exists } \tilde{x} \in R \text{such that } g_i(\tilde{x}) > 0, \ \forall i \in \mathcal{I}^+ \text{ and } g_i(\tilde{x}) = 0, \ i \in \mathcal{I}^0.$$

The Slater condition is a constraint qualification that obviously implies the regularity condition (2.3.3). The next theorem is a slight modification of a theorem in [18] since we consider the problem having also affine equality constraints.

Theorem 2.4.7. *Let us consider the convex problem 1.2.1 and let be $\bar{x} \in R$ such that ri $\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H}_u = \varnothing$. If the Slater CQ holds, then the regularity condition (2.3.3) is fulfilled.*

Proof. The condition ri $\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H}_u = \varnothing$ is equivalent to linear separation between $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\mathcal{H}$, i.e. $\exists\ (\theta, \lambda) \neq O_{m+1}$, with $\theta \geq 0$, $\lambda \in D^*$ such that

$$\theta u + \langle \lambda, v \rangle \geq 0, \forall (u, v) \in \mathcal{E}(\mathcal{K}_{\bar{x}}). \tag{2.4.0}$$

It is left to prove that $\theta > 0$ in order to have regular linear separation, equivalent to regularity condition (2.3.3). For this, assume ab absurdo that $\theta = 0$ and this implies $\lambda \neq O_m$. Since Slater says that there exists $(\tilde{u}, \tilde{v}) \in \mathcal{E}(\mathcal{K}_{\bar{x}})$ with $\tilde{v}_i > 0$, $i \in \mathcal{I}^+$ and $\tilde{v}_i = 0$, $i \in \mathcal{I}^0$. It follows the contradiction

$$0 < \sum_{i \in \mathcal{I}^+} \lambda_i \tilde{v}_i \leq 0,$$

where the second inequality follows from (2.4.2). Therefore it is necessary to have $\theta > 0$. □

Remark 2.4.8. One can easily give another generalisation (as done in [18]) of the Slater CQ by exploiting its meaning in the IS. Actually, the fact that there exists an $\tilde{x} \in R$ such that $g_i(\tilde{x}) > 0$, $i \in \mathcal{I}^+$, $g_i(\tilde{x}) = 0$, $i \in \mathcal{I}^0$, is equivalent to the fact that in the IS there exists a $\tilde{v} := g(\tilde{x})$ such that $\tilde{v}_i > 0$, $i \in \mathcal{I}^+$ and $\tilde{v}_i = 0$, $i \in \mathcal{I}^0$. If the separation hyperplane between the sets $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$ exists, then the separation is regular. Therefore, for ensuring that the separation is regular, it is enough to have a set of vectors $(u^j, v^j) \in \mathcal{E}(\mathcal{K}_{\bar{x}})$, $j = 1, ..., r$, such that the sum is a vector with strictly positive components, except the first component. This means that a generalisation of the Slater CQ is that there exist $(u^j, v^j) \in \mathcal{E}(\mathcal{K}_{\bar{x}})$, $j = 1, ..., r$ such that $\sum_{j=1}^{r} v_i^j > 0$, $\forall i \in \mathcal{I}^+(\bar{x})$, and the contradiction will come this time from

$$0 < \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{r} v_i^j \right) = \sum_{j=1}^{r} \langle \lambda, v^j \rangle \leq 0.$$

The implication MFCQ implies (2.4.2) can be proved using the same idea as in the theorem before and noticing that MFCQ is actually a Slater CQ applied to the linearized image of the problem $\mathcal{K}_{\bar{x}}^h$.

Theorem 2.4.9. *[18] Let $\bar{x}$ be a minimum point of problem of the differentiable problem (1.2.1) with X open and assume that $\mathcal{K}_{\bar{x}}^h$ and $\mathcal{H}$ are linearly separable. If MF CQ is fulfilled at $\bar{x}$ then the regularity condition (2.4.2) is fulfilled.*

Proof. Linear separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ is equivalent to the existence of $(\theta, \lambda) \neq O_{1+m}$ with $\theta \geq 0$ and $\lambda \in D^*$ such that

$$\theta \nabla f(\bar{x})(x - \bar{x}) + \sum_{i \in \mathcal{I}^+} \lambda_i (g_i(\bar{x}) + \nabla g_i(\bar{x})(x - \bar{x})) + \sum_{i \in \mathcal{I}^0} \mu_i (g_i(\bar{x}) + \nabla g_i(\bar{x})(x - \bar{x})) \leq 0, \forall x \in X$$

which implies $\langle \lambda, g(\bar{x})\rangle = 0$. If, ab absurdo, $\theta = 0$ then we would have $\nabla g(\bar{x})(x-\bar{x}) \leq 0, \forall x \in X$ or that $\sum_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} \nabla g_i(\bar{x})(x-\bar{x}) \leq 0, \forall x \in X$, which contradicts MF CQ. □

Remark 2.4.10. (i) In [20] it has been shown that MFCQ is a necessary and sufficient constraint qualification for the existence of bounded Lagrange multipliers.

(ii) As it has been done for Slater CQ, MF CQ could be slightly generalised by asking the existence of $y^j := x^j - \bar{x} \in X$, $j = 1, ..., r$ such that $\sum_{j=1}^{r} \nabla g_i(\bar{x}) y^j > 0$, $\forall i \in \mathcal{I}^+(\bar{x})$ and $\nabla g_i(\bar{x})$, $i \in \mathcal{I}^0$ are linearly independent.

We will pursue by giving a new constraint qualification involving $\mathcal{C}$- differentiable functions and which collapses to basic constraint qualification [44] for convex constraints and to Guignard constraint qualification [31], [29] for differentiable constraints; both of them are known as the weakest constraint qualifications for the respective classes of constraints. In this section we assume that for the constrained extremum problem (1.2.1) the functions f and $-g_i$, $i \in \mathcal{I}$ are $\mathcal{C}$-differentiable at $\bar{x}$. We consider the following constraint qualification for problems with $\mathcal{C}-$ differentiable constraints:

$$\textbf{CCQ} \qquad TC^*(\bar{x}; R) = \left\{ - \sum_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} \lambda_i \sigma_{g_i} : \lambda_i \geq 0, \lambda_i \in \mathcal{I}^+(\bar{x}), \lambda_i \in \mathbb{R}, i \in \mathcal{I}^0; \right.$$

$$\left. \sigma_{g_i} \in \overline{\partial}_{-\mathcal{C}} g_i(\bar{x}), i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0 \right\},$$

where $\mathcal{I}^+(\bar{x}) := \{i \in \mathcal{I}^+ : g_i(\bar{x}) = 0\}$. In the sequel we will prove that CCQ implies the regularity condition (2.4.2). To this aim we need the following lemma.

Lemma 2.4.11. *If $\bar{x}$ is a solution of (1.2.1) then $\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) \geq 0, \forall d \in TC(\bar{x}; R)$; and $O \in \underline{\partial}_{\mathcal{C}} f(\bar{x}) + TC^*(\bar{x}; R)$.*

Proof. First of all, we prove that

$$\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) = \sup_{\sigma_f \in \underline{\partial}_{\mathcal{C}} f(\bar{x})} \langle \sigma_f, d\rangle = \delta^*(d; \underline{\partial}_{\mathcal{C}} f(\bar{x})), \tag{2.4.0}$$

where $\delta^*(d; \underline{\partial}_{\mathcal{C}} f(\bar{x}))$ denotes the support function of $\underline{\partial}_{\mathcal{C}} f(\bar{x})$ with respect to $d \in$ cone $(X - \bar{x})$. Ab absurdo, suppose that (2.4.2) does not hold. Then, we would have $f(\bar{x}+d) - f(\bar{x}) - \varepsilon(\bar{x}; d) > \sup_{\sigma_f \in \underline{\partial}_{\mathcal{C}} f(\bar{x})} \langle \sigma_f, d\rangle$, i.e. there exists $k \neq 0$ such that $f(\bar{x}+d) - f(\bar{x}) - \varepsilon(\bar{x}; d) > k > \sup_{\sigma_f \in \underline{\partial}_{\mathcal{C}} f(\bar{x})} \langle \sigma_f, d\rangle$. Letting $d \to 0$ we obtain $0 > k > 0$, i.e. a contradiction.

Now, observe that $\bar{x}$ solution to problem (1.2.1) is equivalent to $f(x) - f(\bar{x}) \geq 0$, $\forall x \in R$, i.e.

$$\mathcal{D}_{\mathcal{C}} f(\bar{x}; x - \bar{x}) + \varepsilon(\bar{x}; x - \bar{x}) \geq 0, \forall x \in R. \qquad (2.4.0)$$

Consider $d \in TC(\bar{x}; R)$; then $\exists \{x^n\} \subset R$ with $\lim_{n \to +\infty} x^n = \bar{x}$ and $\exists \{\alpha_n\} \subset \mathbb{R}_+$ such that $\lim_{n \to +\infty} \alpha_n (x^n - \bar{x}) = d$. Thus, from (2.4.2), it follows that $\mathcal{D}_{\mathcal{C}} f(\bar{x}; x^n - \bar{x}) + \varepsilon(\bar{x}; x^n - \bar{x}) \geq 0$, $\forall n \geq 1$. Multiplying by α_n, taking the limit when $n \to +\infty$ and then taking into account that $\mathcal{D}_{\mathcal{C}} f(\bar{x}; x^n - \bar{x})$ is positively homogeneous with respect to the second argument, we get

$$\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) \geq 0, \forall d \in TC(\bar{x}; R).$$

This implies $\mathcal{D}_{\mathcal{C}} f(\bar{x}; d) + \delta(d; TC(\bar{x}; R)) \geq 0$, $\forall d \in$ cone $(X - \bar{x})$. It is known that $\delta(d; TC(\bar{x}; R)) = \delta^*(d; TC^*(\bar{x}; R))$, $\forall d \in$ cone $(X - \bar{x})$ and hence, rewriting the last relation by means of the support function and from (2.4.2), we have

$$0 \leq \delta^* \left(d; \underline{\partial}_{\mathcal{C}} f(\bar{x})\right) + \delta^* \left(d; TC^*(\bar{x}; R)\right) = \sup_{y \in \underline{\partial}_{\mathcal{C}} f(\bar{x})} \langle d, y \rangle + \sup_{z \in TC^*(\bar{x}; R)} \langle d, z \rangle =$$

$$= \sup_{y \in \underline{\partial}_{\mathcal{C}} f(\bar{x}), z \in TC^*(\bar{x}; R)} \langle d, y + z \rangle = \delta^* \left(d; \left(\underline{\partial}_{\mathcal{C}} f(\bar{x}) + TC^*(\bar{x}; R)\right)\right), \ \forall d \in \text{cone } (X - \bar{x}).$$

It follows $0 = \langle O, d \rangle \leq \delta^*(d; (\underline{\partial}_{\mathcal{C}} f(\bar{x}) + TC^*(\bar{x}; R)))$, $\forall d \in$ cone $(X - \bar{x})$, which is equivalent to affirm that $O \in \underline{\partial}_{\mathcal{C}} f(\bar{x}) + TC^*(\bar{x}; R)$. □

It follows now the announced result:

Theorem 2.4.12. *Let $\bar{x}$ be a solution to problem (1.2.1). If CCQ holds at $\bar{x}$ then regularity condition (2.4.2) is fulfilled.*

Proof. From Lemma 2.4.11 and from CCQ we have that $\exists \sigma_f \in \underline{\partial}_{\mathcal{C}} f(\bar{x})$, $\exists\ \lambda_i \geq 0, i \in \mathcal{I}^+(\bar{x}), \exists\ \lambda_i \in \mathbb{R}, i \in \mathcal{I}^0$ and $\exists \sigma_{g_i} \in \overline{\partial}_{-\mathcal{C}} g_i(\bar{x})$, $i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0$ such that

$$O = \sigma_f - \sum_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} \bar{\lambda}_i \sigma_{g_i}.$$

Setting $\bar{\lambda}_i = 0$, $i \in \mathcal{I}^+ \setminus \mathcal{I}^+(\bar{x})$ it follows that

$$0 = -\langle \sigma_f, x - \bar{x} \rangle + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \langle \sigma_{g_i}, x - \bar{x} \rangle + \sum_{i \in \mathcal{I}} \bar{\lambda}_i g_i(\bar{x}), \ \forall x \in X. \qquad (2.4.0)$$

From the definition of subdifferential we have that

$$-\mathcal{D}_{\mathcal{C}} f(\bar{x}; x - \bar{x}) \leq -\langle \sigma_f, x - \bar{x} \rangle, \ \forall x \in X. \qquad (2.4.0)$$

and

$$\mathcal{D}_{((-\mathcal{C}))} g_i(\bar{x}; x - \bar{x}) + g_i(\bar{x}) \leq \langle \sigma_{g_i}, x - \bar{x} \rangle + g_i(\bar{x}), \forall x \in X, \forall i \in \mathcal{I}. \qquad (2.4.0)$$

Multiplying (2.4.2) by $\bar{\lambda}_i$, $\forall i \in \mathcal{I}$ and summing up with (2.4.2) we obtain

$$-\mathcal{D}_{\mathcal{C}} f(\bar{x}; x-\bar{x}) + \sum_{i\in\mathcal{I}} (\bar{\lambda}_i \mathcal{D}_{-\mathcal{C}} g_i(\bar{x}; x-\bar{x}) + \bar{\lambda}_i g_i(\bar{x})) \leq$$

$$-\langle \sigma_f, x-\bar{x}\rangle + \sum_{i\in\mathcal{I}} \bar{\lambda}_i \langle \sigma_{g_i}, x-\bar{x}\rangle + \bar{\lambda}_i g_i(\bar{x}),\ \forall x \in X.$$

Now, taking into consideration (2.4.2) and the definition of $\mathcal{K}^h_{\bar{x}}$, we get

$$u + \langle \bar{\lambda}, v\rangle \leq 0, \quad \forall (u,v) \in \mathcal{K}^h_{\bar{x}}, \tag{2.4.0}$$

where $\bar{\lambda} := (\bar{\lambda}_1, ..., \bar{\lambda}_m)$. On the other side, $u + \langle \bar{\lambda}, v\rangle \geq 0, \forall (u,v) \in \text{cl}\, \mathcal{H}$. From this and (2.4.2) it follows that the hyperplane $H_0 = \{(u,v) \in \mathbb{R}^{1+m} : u + \langle \bar{\lambda}, v\rangle = 0\}$ separates $\mathcal{K}^h_{\bar{x}}$ and $\text{cl}\, \mathcal{H}$.
Let us suppose that $TC\mathcal{E}(\mathcal{K}^h_{\bar{x}}) \cap \mathcal{H}_u \neq \emptyset$, or, equivalently, that $\mathcal{H}_u \subseteq TC\mathcal{E}(\mathcal{K}^h_{\bar{x}})$. Therefore, Theorem 2.2.1 for $C = \mathcal{H}$, $S = \mathcal{K}^h_{\bar{x}}$ and $F = \mathcal{H}_u$ implies that $\mathcal{H}_u \subseteq H_0$, that is $u + \langle \bar{\lambda}, v\rangle = 0$, $\forall (u, \bar{v}) \in \mathcal{H}_u$, which is absurd. □

If the inequality constraints are convex functions and the equalities are affine, then CCQ collapses to the basic constraint qualification BCQ [44]; in fact R is a convex set and thus the polar cone of the tangent cone to R at $\bar{x}$ coincides with its normal cone:

$$\textbf{BCQ} \quad NC(\bar{x}; R) = \left\{ -\sum_{i\in\mathcal{I}^+(\bar{x})} \lambda_i \partial g_i(\bar{x}) - \sum_{i\in\mathcal{I}^0} \lambda_i \nabla g_i(\bar{x}) : \lambda_i \geq 0, i \in \mathcal{I}^+(\bar{x}) ;\right.$$

$$\left. \lambda_i \in \mathbb{R}, i \in \mathcal{I}^0 \right\}.$$

When the constraints are differentiable, then CCQ becomes the Guignard constraint qualification [31], [29]:

$$\textbf{GCQ}\ TC^*(\bar{x}; R) = \left\{ -\sum_{i\in\mathcal{I}^+(\bar{x})\cup\mathcal{I}^0} \lambda_i \nabla g_i(\bar{x}) : \lambda_i \geq 0, i \in \mathcal{I}^+(\bar{x}); \lambda_i \in \mathbb{R}, i \in \mathcal{I}^0 \right\}.$$

If the constraints are both convex and differentiable, then CCQ, and hence GCQ too, collapse to the Abadie constraint qualification ACQ [2].

In [6] relationships among constraint qualifications have already been investigated, both for the convex and the differentiable problems; in [6, 44, 51] it has been proved that classic constraint qualifications, like Slater [68] and Mangasarian Fromowitz [50], are weaker than BCQ and GCQ, respectively. For this reason we have just investigated the relationships among the more general conditions known in literature for each class of problems (i.e., BCQ, GCQ and calmness) and condition (2.4.2). If the Dini regularity assumptions [22], guaranteeing the existence of Lagrange multipliers with $\theta = 1$, are applied to equality constraints in problem (8), it can be immediately proved that they imply GCQ. Nevertheless, the converse implication does not hold as it can be seen if in problem (8) we consider $X = \mathbb{R}^2$, $p = m = 1$, $g_1(x_1, x_2) = x_1^2 x_2^2$ and any objective function f which attains its minimum at $(0, 0)$.

Moreover, we will analyse the connection between the metric regularity condition (2.3.6) and the previous mentioned conditions. Indeed, if $X = \mathbb{R}^n$, for the convex problems the metric regularity implies BCQ [70] and for the convex and differentiable problems the metric regularity is equivalent to ACQ [45]. For differentiable problems, when $X \subseteq \mathbb{R}^n$, we will prove that metric regularity implies GCQ. To this aim we will slightly generalise the known result due to Li [45] for convex and differentiable problems and $X = \mathbb{R}^n$.

Let be $X \subseteq \mathbb{R}^n$ and the functions f and g_i, $i \in \mathcal{I}$ be differentiable at $\bar{x} \in \operatorname{int} X$. We denote by

$$A := \{x \in X : \nabla g_i(\bar{x})(x - \bar{x}) \geq 0,\ i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0\}.$$

A is a polyhedral set and thus

$$TC^*(\bar{x}; A) = \left\{ - \sum_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} \lambda_i \nabla g_i(\bar{x}) : \lambda_i \geq 0, i \in \mathcal{I}^+(\bar{x}), \lambda_i \in \mathbb{R}, i \in \mathcal{I}^0 \right\}.$$

Therefore GCQ is equivalent to

$$TC^*(\bar{x}; A) = TC^*(\bar{x}; R). \tag{2.4.0}$$

Lemma 2.4.13. *It holds that $TC^*(\bar{x}; A) \subseteq TC^*(\bar{x}; R)$.*

Proof. Let us introduce the sets $R_i := \{x \in X : g_i(x) \geq 0\}$, $i \in \mathcal{I}^+(\bar{x})$, $R_j := \{x \in X : g_j(x) = 0\}$, $j \in \mathcal{I}^0$. Since $R \subseteq X \cap \left(\bigcap_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} R_i \right)$, it follows that $TC(\bar{x}; R) \subseteq TC(\bar{x}; X) \cap \left(\bigcap_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} TC(\bar{x}; R_i) \right)$. From $\bar{x} \in \operatorname{int} X$ we get $TC(\bar{x}; X) = \mathbb{R}^n$ and further that

$$TC(\bar{x}; R) \subseteq \bigcap_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} TC(\bar{x}; R_i).$$

Therefore

$$TC^*(\bar{x}; R) \supseteq \left(\bigcap_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} TC(\bar{x}; R_i) \right)^* \supseteq \sum_{i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0} TC^*(\bar{x}; R_i).$$

We will show now that $TC^*(\bar{x}; A_i) \subseteq TC^*(\bar{x}; R_i)$, where $A_i := \{x \in X : \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \geq 0\}$, $i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0$. Let be $x \in TC(\bar{x}; R_i)$. Then $\exists\ \{x^n\}_{n \geq 1} \subseteq R_i$ and $\exists \{\alpha_n\}_{n \geq 1} \subset$

$\mathbb{R}_+ \setminus \{0\}$ such that $\lim\limits_{n\to+\infty} x^n = \bar{x}$ and $\lim\limits_{n\to+\infty} \alpha_n(x^n - \bar{x}) = x$. Since g_i, $i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0$, are differentiable at $\bar{x}$, we can write $g_i(x^n) = \langle \nabla g_i(\bar{x}), x^n - \bar{x}\rangle + \|x^n - \bar{x}\|\varepsilon(x^n - \bar{x}) \geq 0$. Multiplying by α_n and letting $n \to +\infty$, we get

$$\langle x, \nabla g_i(\bar{x})\rangle \geq 0, i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0.$$

This means that $x \in A_i$ and thus $TC^*(\bar{x}; A_i) \subseteq TC^*(\bar{x}; R_i)$, $\forall i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0$. By definition $A = \bigcap\limits_{i\in\mathcal{I}^+(\bar{x})\cup\mathcal{I}^0} A_i$ and since A_i are polyhedral it holds that $TC^*(\bar{x}; A) = \sum\limits_{i\in\mathcal{I}^+(\bar{x})\cup\mathcal{I}^0} TC^*(\bar{x}; A_i)$. As a final result, we obtain $TC^*(\bar{x}; A) \subseteq TC^*(\bar{x}; R)$. □

Theorem 2.4.14. *Let be $X \subseteq \mathbb{R}^n$ and the functions f and g_i, $i \in \mathcal{I}$ be differentiable at $\bar{x} \in int\ X$. Then condition (2.3.6) implies GCQ.*

Proof. Since GCQ is equivalent to (2.4.2) and because of Lemma 2.4.13, it is enough to prove that metric regularity implies $TC^*(\bar{x}; R) \subseteq TC^*(\bar{x}; A)$. Ab absurdo, suppose that $\exists \bar{u} \in TC^*(\bar{x}; R) \setminus TC^*(\bar{x}; A)$, i.e.

$$\langle \bar{u}, y - \bar{x}\rangle \leq 0, \forall y \in R \text{ and } \exists \bar{z} \in A \text{ such that } \langle \bar{u}, \bar{z} - \bar{x}\rangle > 0. \tag{2.4.0}$$

Let us set $\beta := \frac{\|\bar{z}-\bar{x}\|}{\langle \bar{u}, \bar{z}-\bar{x}\rangle} > 0$ and $x(\alpha) := \alpha\bar{z} + (1-\alpha)\bar{x}$, $0 < \alpha < 1$. Then we have:

$$\|x(\alpha) - \bar{x}\| = \|\alpha(\bar{z} - \bar{x})\| = \alpha\beta\langle \bar{u}, \bar{z} - \bar{x}\rangle = \beta\langle \bar{u}, x(\alpha) - \bar{x}\rangle =$$

$$= \beta(\langle \bar{u}, x(\alpha) - y\rangle + \langle \bar{u}, y - \bar{x}\rangle) \leq \beta\langle \bar{u}, x(\alpha) - y\rangle \leq \beta\|\bar{u}\|\|x(\alpha) - y\| \quad , \forall y \in R.$$

In this way we obtain that

$$d(x(\alpha); R) \geq \frac{\|x(\alpha) - \bar{x}\|}{\beta\|\bar{u}\|} > 0.$$

Since the problem is metric regular at $\bar{x}$, it follows that there exist $L > 0$ and $\varepsilon > 0$ with $d(x; R) \leq Ld(g(x); D)$, $\forall x \in X$ and such that $\|x - \bar{x}\| \leq \varepsilon$. Since $x(\alpha) \to \bar{x}$ when $\alpha \to 0^+$, we have

$$\limsup_{\alpha\to0^+} \frac{d(g(x(\alpha)); D)}{d(x(\alpha); R)} \geq \frac{1}{L} > 0. \tag{2.4.0}$$

From $\lim\limits_{\alpha\to0^+} g_i(x(\alpha)) = g_i(\bar{x}) > 0$, $i \in \mathcal{I}^+ \setminus \mathcal{I}^+(\bar{x})$ it follows

$$\lim_{\alpha\to0^+} \frac{d(g_i(x(\alpha)); D_i)}{d(x(\alpha); R)} = 0, \ i \in \mathcal{I}^+ \setminus \mathcal{I}^+(\bar{x}), \tag{2.4.0}$$

where $D_i = [0, +\infty)$, $i \in \mathcal{I}^+$. For $i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0$ let us set $e_i(\alpha) := g_i(x(\alpha)) - g_i(\bar{x}) - \langle \nabla g_i(\bar{x}), x(\alpha) - \bar{x}\rangle$. From $\lim\limits_{\alpha\to0^+} \frac{|e_i(\alpha)|}{\|x(\alpha)-\bar{x}\|} = 0$ we get

$$\limsup_{\alpha\to0^+} \frac{|e_i(\alpha)|}{d(x(\alpha); R)} \leq \limsup_{\alpha\to0^+} \frac{|e_i(\alpha)|\beta\|\bar{u}\|}{\|x(\alpha) - \bar{x}\|} = 0, \quad i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0.$$

Then it holds $g_i(x(\alpha)) = g_i(x(\alpha)) - g_i(\bar{x}) = e_i(\alpha) + \langle \nabla g_i(\bar{x}), x(\alpha) - \bar{x} \rangle = e_i(\alpha) + \langle \nabla g_i(\bar{x}), \alpha(\bar{z} - \bar{x}) \rangle \leq e_i(\alpha)$, where the last inequality follows from (2.4.2). This implies that $d(g_i(x(\alpha)); D_i) \leq |e_i(\alpha)|$. Thus we have

$$\lim_{\alpha \to 0^+} \frac{d(g_i(x(\alpha)); D_i)}{d(x(\alpha); R)} \leq \limsup_{\alpha \to 0^+} \frac{|e_i(\alpha)|}{d(x(\alpha); R)} = 0, \quad i \in \mathcal{I}^+(\bar{x}) \cup \mathcal{I}^0. \tag{2.4.0}$$

Putting together (2.4.2) and (2.4.2) we get

$$\limsup_{\alpha \to 0^+} \frac{d(g_i(x(\alpha)); D)}{d(x(\alpha); R)} = 0, \ i \in \mathcal{I},$$

which contradicts (2.4.2). □

2.5 Examples and Graphical Representation for Sufficient Conditions

In each of the following examples we will consider problem (1.2.1) with $p = 1$ and $m = 2$, i.e. with one equality and one inequality constraint; while the set X, where the functions f and g are defined, is specified in every example. The point $\bar{x}$ is a minimum point of the problem and it defines the image set $\mathcal{K}_{\bar{x}}$. We compare, among them, the following conditions: sufficient regularity condition (2.3.3), metric regularity (2.3.6) and calmness (1.2.21) for problems that may be convex, differentiable and/or locally Lipschitz. By convex, differentiable at $\bar{x}$ and locally Lipschitz around $\bar{x}$ we mean problems where the objective function and the constraints are, respectively convex (with the equality constraints being actually affine), differentiable at $\bar{x}$ and locally Lipschitz around $\bar{x}$. In the examples the following properties are frequently exploited:
if the constraints of the problem are linear, then the metric regularity condition is fulfilled;
the problem is both regular and calm if it results that $u \leq 0, \forall (u, v) \in \mathcal{K}_{\bar{x}}$ or, equivalently, that $f(x) - f(\bar{x}) \geq 0, \forall x \in X$.
Some of the examples merely correspond to known results, others illustrate the implications established in the theorems of Section 2.3 ; the most significant are the counterexamples proving that some implications do not hold or the examples showing the consistency of several conditions (see Figure 1).

Example 2.5.1. $X = \mathbb{R}$, $f(x) = x^2$, $g_1(x) = 3x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -v_2^2, v_1 = 3v_2, v_2 \in \mathbb{R}\}$.
It is obvious that the problem is convex, differentiable and locally Lipschitz; metric regularity holds since the constraints are linear and hence the problem is calm as

the problem is locally Lipschitz; moreover, regularity condition (2.3.1) is fulfilled because it is equivalent to calmness for convex problems.

Example 2.5.2. $X = \mathbb{R}$, $f(x) = |x|$, $g_1(x) = 3x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -|v_1|, v_1 = 3v_2, v_2 \in \mathbb{R}\}$.
The problem is convex, locally Lipschitz and it is not differentiable at $\bar{x}$. Like in Example 2.5.1, metric regularity, calmness and regularity condition (2.3.1) are fulfilled.

Example 2.5.3. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 + x_2^2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u \le v_2 \le 0\}$.
The problem is convex, differentiable and locally Lipschitz. The metric regularity condition (2.3.6) is not verified in any neighbourhood of $\bar{x}$ when $x_1 = 0$. The regularity condition (2.3.1) is satisfied since $u \ge 0, \forall (u, v) \in \mathcal{K}_{\bar{x}}$ and, being the problem convex, the calmness is also satisfied.

Example 2.5.4. $X = \mathbb{R}^2$, $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -\sqrt{v_2^2 - v_1}, v_1 \le 0\}$.
The problem is convex, locally Lipschitz and it is not differentiable at $\bar{x}$; it is not metric regular (see Example 2.5.3). Calmness at $\bar{x}$ and regularity follow from $f(x) - f(\bar{x}) \ge 0$, $\forall x \in X$.

Example 2.5.5. $X = \mathbb{R}_+$, $f(x) = x$, $g_1(x) = x$ and $g_2(x) = \sqrt{x} - x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_1 = -u, v_2 = \sqrt{-u} + u, u \le 0\}$.
The problem is convex, it is not locally Lipschitz and not differentiable at $\bar{x}$. The metric regularity holds because if $L = 1$ then $d(x; R) = |x| \le Ld(g(x); D) = |x|$. The problem is calm at $\bar{x}$ and regular since $f(x) - f(\bar{x}) \ge 0$, $\forall x \in X$.

Example 2.5.6. $X = \mathbb{R}_+^2$, $f(x_1, x_2) = -\sqrt{x_1} - \sqrt{x_2}$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = \sqrt{-v_1} + \sqrt{v_2}, v_1 \le 0, v_2 \ge 0\}$.
The problem is convex, not locally Lipschitz and not differentiable at $\bar{x}$. The constraints are linear and hence the problem is metric regular. Consider the following sequences $\{(u_n = \frac{1}{n},\ v_{1n} = 0,\ v_{2n} = \frac{1}{n^2},)\}_{n \ge 1}$ belonging to $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n \ge 1}$. We have that $\lim_{n \to +\infty} (u_n, v_{1n}, v_{2n}) = (0, 0, 0)$ and $\lim_{n \to +\infty} \alpha_n (u_n, v_{1n}, v_{2n}) = (1, 0, 0) \in$ cl cone $(\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Therefore condition (2.3.1) is contradicted and the problem is neither calm nor regular.

Example 2.5.7. $X = \mathbb{R}_+^2$, $f(x_1, x_2) = x_1^2 + x_2^6$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = \sqrt{x_1} - x_2\sqrt{x_2}$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -v_1^2 - (\sqrt{v_1} - v_2)^6, v_1 \ge 0\}$.
The problem is convex, not locally Lipschitz and not differentiable at $\bar{x}$. It is not metric regular because condition (2.3.1) for $x_1 = 0$ becomes $|x_2| \le L|x_2|\sqrt{x_2}$, when x_2 runs in a neighbourhood of 0. Calmness and regularity of the problem follow from $f(x) - f(\bar{x}) \ge 0$, $\forall x \in X$.

Example 2.5.8. $X = \mathbb{R} \times \mathbb{R}_+$, $f(x_1, x_2) = -x_1 - \sqrt{x_2}$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2\sqrt{x_2}$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 + \sqrt[3]{v_2^2}, v_2 \le 0\}$. The problem is convex, not locally Lipschitz and not differentiable at $\bar{x}$. The problem is not metric regular because, like in Example 2.5.7, condition (2.3.1) for $x_1 = 0$ becomes $|x_2| \le L|x_2|\sqrt{x_2}$. Choosing $\{(u_n = \frac{1}{n} + \frac{1}{n^2},\ v_{1n} = \frac{1}{n^2},\ v_{2n} = -\frac{1}{n^3})\}_{n\ge 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\ge 1}$, we obtain that $\lim_{n\to+\infty}(u_n, v_{1n}, v_{2n}) = (0, 0, 0)$, while $\lim_{n\to+\infty} \alpha_n(u_n, v_{1n}, v_{2n}) = (1, 0, 0) \in \text{cl cone}\,(\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Hence, like in Example 2.5.6, the problem is neither calm nor regular.

Example 2.5.9. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1 - x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = \begin{cases} -x_2^2 & \text{if } x_2 \ge 0 \\ x_2 & \text{if } x_2 < 0 \end{cases}$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_1 = -(u - v_2)^2 \text{ if } u \ge v_2, v_1 = u - v_2 \text{ if } u < v_2\}$.
The problem is convex, locally Lipschitz and not differentiable at $\bar{x}$. Metric regularity condition (2.3.6) does not hold at $\bar{x}$ because for $x_1 = 0$ becomes $|x_2| \le Lx_2^2$. If we set $\{(u_n = \frac{1}{n} + \frac{1}{n^2},\ v_{1n} = -\frac{1}{n^2},\ v_{2n} = \frac{1}{n^2})\}_{n\ge 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\ge 1}$, then the calmness condition (2.3.1) and the regularity too are contradicted.

Example 2.5.10. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1 - x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 \pm \sqrt{-v_1}, v_1 \le 0\}$.
The problem is convex, locally Lipschitz and differentiable on X. The metric regularity condition does not hold in any neighbourhood of $\bar{x}$. If we choose $\{(u_n = \frac{1}{n} + \frac{1}{n^2}, v_{1n} = -\frac{1}{n^2}, v_{2n} = \frac{1}{n^2})\}_{n\ge 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\ge 1}$, it is proved that the calmness condition (2.3.1) and the regularity condition do not hold.

Example 2.5.11. $X = \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $g_1(x) = 3x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = \begin{cases} e^{-\frac{1}{v_2}} & \text{if } v_2 \neq 0 \\ 0 & \text{if } v_2 = 0 \end{cases}, v_1 = 3v_2\}$.
The problem is not convex, is differentiable and locally Lipschitz on X. The metric regularity condition is satisfied since the constraints are linear. Regularity and calmness at $\bar{x}$ follow from the non negativity of $f(x) - f(\bar{x})$ for all $x \in X$.

Example 2.5.12. $X = \mathbb{R}$, $f(x) = -|x|$, $g_1(x) = |x|$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = |v_2| = v_1\}$.
The problem is not convex, locally Lipschitz on X and not differentiable at $\bar{x}$. From the linearity of the constraints it follows that the metric regularity is fulfilled. The image set $\mathcal{K}_{\bar{x}}$ is a cone such that $\{(u, 0, 0) \in \mathbb{R}^3 : u > 0\} \not\subseteq TC(\text{conv}\,(\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \text{cl}\,(\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Thus the problem is regular and calm at $\bar{x}$.

Example 2.5.13. $X = \mathbb{R}$, $f(x) = \begin{cases} x + 1 & \text{if } x \le -1 \\ -x^2 & \text{if } -1 < x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$, $g_1(x) = x$ and $g_2(x) = 2x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that

$$u = \begin{cases} -v_1 - 1 & \text{if } v_1 \leq -1 \\ v_1^2 & \text{if } -1 < v_1 < 0 \\ -v_1^3 & \text{if } v_1 \geq 0 \end{cases} \quad \text{and } v_2 = 2v_1.$$

The problem is not convex, it is locally Lipschitz and differentiable at $\bar{x}$. The metric regularity is implied by the linearity of the constraints. We have that $O \in$ int conv $(\mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H})$ while cl cone $(\mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H}) \cap \mathcal{H}_u = \varnothing$. Thus the problem is calm but not regular.

Example 2.5.14. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1^2 - x_2^2$, $g_1(x_1, x_2) = x_1^2$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -v_1 + v_2, v_1 \leq 0, v_2 \geq 0\}$. The problem is not convex, it is locally Lipschitz and differentiable on X. The metric regularity is not satisfied in any neighbourhood of $\bar{x}$ for $x_1 = 0$. The image set $\mathcal{K}_{\bar{x}}$ is a cone such that $\mathcal{H}_u \not\subseteq TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Thus the problem is regular and calm at $\bar{x}$.

Example 2.5.15. $X = \mathbb{R}$, $f(x) = \begin{cases} -\frac{x}{4} & \text{if } x \leq 0 \\ -\frac{x}{2} & \text{if } x > 0 \end{cases}$, $g_1(x) = -x$ and $g_2(x) = -\frac{x}{4}$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that

$$u = \begin{cases} -v_2 & \text{if } v_2 \geq 0 \\ -2v_2 & \text{if } v_2 < 0 \end{cases} \quad \text{and } v_1 = 4v_2.$$

The problem is not convex, is locally Lipschitz and not differentiable at $\bar{x}$. The constraints are linear, thus the metric regularity condition is satisfied. The problem is not regular since $O \in$ int $TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}})))$. On the other side $\{(u, 0, 0) \in \mathbb{R}^3 : u > 0\} \not\subseteq$ cl cone $(\mathcal{E}(\mathcal{K}_{\bar{x}}))$ and thus the problem is calm at $\bar{x}$.

Example 2.5.16. $X = \mathbb{R}$, $f(x) = |x|$, $g_1(x) = -x^2$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = \begin{cases} -\sqrt[3]{-v_2} & \text{if } v_2 \leq 0 \\ \sqrt[3]{-v_2} & \text{if } v_2 > 0 \end{cases}$ and $v_1 = -\sqrt[3]{-v_2^2}$. The problem is not convex, is locally Lipschitz on X and is not differentiable at $\bar{x}$. The metric regularity condition is not satisfied in any neighbourhood of $\bar{x}$. Since it holds $f(x) - f(\bar{x}) \geq 0$ for all $x \in X$, we get that the problem is regular and calm at $\bar{x}$.

Example 2.5.17. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} -x_2^3 - x_1^2 & \text{if } \ x_1 \geq 0 \\ -\frac{1}{2}x_2^3 - \frac{1}{2}x_1^3 & \text{if} x_1 < 0 \end{cases}$, $g_1(x_1, x_2) = -x_1^3$ and $g_2(x_1, x_2) = \begin{cases} -x_2^2 & \text{if } \ x_2 \geq 0 \\ -x_2^3 & \text{if} x_2 < 0 \end{cases}$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = \begin{cases} -\sqrt[3]{v_1^2} - v_2\sqrt{-v_2} & \text{if } v_2 \leq 0 \\ \frac{-v_1 - v_2}{2} & \text{if } v_2 > 0 \end{cases}$. The problem is not convex, it is differentiable and locally Lipschitz on X. The metric regularity condition is not verified in any neighbourhood of $\bar{x} = 0$ for $x_1 = 0$. We have that $O \in$ int $TC(\text{conv } (\mathcal{E}(\mathcal{K}_{\bar{x}})))$ and thus the problem is not regular. On the other side , $\mathcal{H}_u \not\subseteq$ cl cone $(\mathcal{E}(\mathcal{K}_{\bar{x}}))$ and hence the problem is calm at $\bar{x}$.

Example 2.5.18. $X = \mathbb{R}^2$, $f(x) = \begin{cases} -x_1 & \text{if } x_1 \leq 0 \\ -2x_1 & \text{if } x_2 > 0 \end{cases}$, $g_1(x) = x_1$ and $g_2(x) = -x_2^3$; $\bar{x} = (0,0)$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = \begin{cases} v_1 & \text{if } v_1 \leq 0 \\ 2v_1 & \text{if } v_1 > 0 \end{cases}$.
The problem is not convex, not differentiable at $\bar{x}$ and it is locally Lipschitz on X. The problem is not metric not regular at $\bar{x}$ for $x_1 = 0$. The problem is not regular while it is calm at $\bar{x}$.

Example 2.5.19. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1 + x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = x_2^3$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -v_2 - \sqrt[3]{v_1}\}$.
The problem is not convex, is differentiable and locally Lipschitz on X. The metric regularity condition is not verified in any neighbourhood of $\bar{x}$ for $x_1 = 0$. If we set $\{(u_n = -\frac{1}{n^3} + \frac{1}{n}, v_{1n} = -\frac{1}{n^3}, v_{2n} = \frac{1}{n^3})\}_{n \geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n \geq 1}$, we see that the calmness condition (2.3.1) and the regularity condition do not hold.

Example 2.5.20. $X = \mathbb{R}$, $f(x) = -|x|$, $g_1(x) = x^4$ and $g_2(x) = -2x^2$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = \sqrt{-\frac{v_2}{2}}, v_1 = -\frac{v_2^2}{4}, v_2 \leq 0\}$.
The problem is not convex, not differentiable at $\bar{x}$ and it is locally Lipschitz on X. The metric regularity condition is not verified in any neighbourhood of $\bar{x}$. Setting $\{(u_n = \frac{1}{n}, v_{1n} = -\frac{2}{n^2}, v_{2n} = \frac{1}{n^4})\}_{n \geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n \geq 1}$, we have that the calmness condition (2.3.1) and the regularity condition do not hold.

Example 2.5.21. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \left|\sin \frac{1}{x^2}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $g_1(x) = 3x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that
$u = \begin{cases} -v_2^2 \left|\sin \frac{1}{v_2^2}\right| & \text{if } v_2 \neq 0 \\ 0 & \text{if } v_2 = 0 \end{cases}$ and $v_1 = 3v_2$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition holds since the constraints are linear. The non negativity of $f(x) - f(\bar{x})$ on all the domain X implies that the problem is regular and calm at $\bar{x}$.

Example 2.5.22. $X = \mathbb{R}$, $f(x) = \sqrt{|x|}$, $g_1(x) = 2x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -\sqrt{|v_2|}, v_1 = 2v_2\}$.
The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The metric regularity, the regularity and the calmness at $\bar{x}$ are implied by the linearity of the constraints and by the non negativity of $f(x) - f(\bar{x})$ for all $x \in X$, respectively.

Example 2.5.23. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \left|\sin \frac{1}{x^2}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $g_1(x) = -x^2$ and $g_2(x) = x^4$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that
$u = -\begin{cases} v_1 \left|\sin \frac{1}{\pm\sqrt{-v_1}}\right| & \text{if } v_1 \neq 0 \\ 0 & \text{if } v_1 = 0 \end{cases}$ and $v_2 = v_1^2$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable at $\bar{x}$.

The problem is not metric regular because the condition (2.3.6) is not verified in any neighbourhood of $\bar{x}$. Since $f(x) - f(\bar{x}) \geq 0$, $\forall x \in X$, the problem is regular and calm at $\bar{x}$.

Example 2.5.24. $X = \mathbb{R}$, $f(x) = \sqrt{|x|}$, $g_1(x) = -x^4$ and $g_2(x) = x^2$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -\sqrt{|v_2|}, v_1 = -v_2^3\}$.
The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The metric regular condition (2.3.6) is not verified in any neighbourhood of $\bar{x}$. Since $f(x) - f(\bar{x}) \geq 0$ for all $x \in X$ we have that the problem is regular and calm at $\bar{x}$.

Example 2.5.25. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$, $g_1(x) = 2x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = \begin{cases} -v_2^2 \sin \frac{1}{v_2^2} & \text{if } v_2 < 0 \\ -v_2^2 & \text{if } v_2 \geq 0 \end{cases}$ and $v_1 = 2v_2$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The constraints are linear and hence the problem is metric regular. We have that $TC(\text{conv}(\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \mathbb{R}^3$ and hence the problem is not regular. Nevertheless, $\mathcal{H}_u \not\subseteq \text{cl cone } (\mathcal{E}(\mathcal{K}_{\bar{x}}))$ and thus the problem is calm at $\bar{x}$.

Example 2.5.26. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$,
$g_1(x) = \begin{cases} -2x & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$ and $g_2(x) = \begin{cases} x & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that
$u = \begin{cases} -v_2^2 & \text{if } v_2 \geq 0 \\ -v_2 & \text{if } v_2 < 0 \end{cases}$, and $v_1 = \begin{cases} -2v_2^2 & \text{if } v_2 \geq 0 \\ -v_2 & \text{if } v_2 < 0 \end{cases}$.
The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The problem is metric regular, not regular at $\bar{x}$ since $O_3 \in \text{int } TC(\text{conv}(\mathcal{E}(\mathcal{K}_{\bar{x}})))$, but calm for $\mathcal{H}_u \not\subseteq \text{cl cone } (\mathcal{E}(\mathcal{K}_{\bar{x}}))$.

Example 2.5.27. $X = \mathbb{R}$, $f(x) = \begin{cases} x^6 \sin \frac{1}{x^6} & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$, $g_1(x) = 3x^3$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = \begin{cases} -v_2^2 \sin \frac{1}{v_2^2} & \text{if } v_2 < 0 \\ -v_2 & \text{if } v_2 \geq 0 \end{cases}$ and $v_1 = 3v_2$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regular condition (2.3.6) is not verified in any neighbourhood of $\bar{x}$. We have that $TC(\text{conv}(\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \mathbb{R}^3$ and hence the problem is not regular. Anyway, $\mathcal{H}_u \not\subseteq \text{cl cone } (\mathcal{E}(\mathcal{K}_{\bar{x}}))$ and thus the problem is calm at $\bar{x}$.

Example 2.5.28. $X = \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{3}x^3 + \frac{2}{3} & \text{if } x \geq 1 \\ x & \text{if } 0 \leq x < 1 \\ x^3 \sin \frac{1}{x^3} & \text{if } x < 0 \end{cases}$, $g_1(x) = 3x^3$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that

$$u = -\begin{cases} \frac{1}{3}v_2 + \frac{2}{3} & \text{if } v_2 \geq 1 \\ \sqrt[3]{v_2} & \text{if } 0 \leq v_2 < 1 \\ v_2 \sin\frac{1}{v_2} & \text{if } v_2 < 0 \end{cases} \quad \text{and } v_1 = 3v_2.$$

The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. As in Example 2.5.27 the problem is not metric regular at $\bar{x}$. It holds that $TC(\text{conv}(\mathcal{E}(\mathcal{K}_{\bar{x}}))) = \mathbb{R}^3$ and $\mathcal{H}_u \not\subseteq \text{cl cone}\,(\mathcal{E}(\mathcal{K}_{\bar{x}}))$ and thus the problem is not regular while it is calm at $\bar{x}$.

Example 2.5.29. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $g_1(x) = x$ and $g_2(x) = \begin{cases} x^4 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = -\frac{v_2}{v_1^2}$.

The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The problem is also metric regular. Choosing $\{(u_n = \frac{1}{n}, v_{1n} = -\frac{1}{n^2}, v_{2n} = \frac{1}{n^5})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\geq 1}$, we see that the calmness condition (2.3.1) is not fulfilled and thus the problem is not regular.

Example 2.5.30. $X = \mathbb{R}$, $f(x) = \begin{cases} \sqrt{-x}\sin\frac{1}{x} & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$, $g_1(x) = 2x$ and $g_2(x) = x$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that

$$u = -\begin{cases} \sqrt{-v_2}\sin\frac{1}{v_2} & \text{if } v_2 < 0 \\ v_2^2 & \text{if } v_2 \geq 0 \end{cases} \quad \text{and } v_1 = 2v_2.$$

The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The constraints are linear and thus the problem is metric regular at $\bar{x}$. Choosing $\{(u_n = \frac{1}{\sqrt{2n\pi+\frac{3\pi}{2}}}, v_{1n} = \frac{2}{2n\pi+\frac{3\pi}{2}}, v_{2n} = \frac{2}{2n\pi+\frac{3\pi}{2}})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = \sqrt{2n\pi + \frac{3\pi}{2}}\}_{n\geq 1}$, we see that the calmness condition (2.3.1) and the regularity condition do not hold.

Example 2.5.31. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if } x < 0 \\ x^6 & \text{if } x \geq 0 \end{cases}$, $g_1(x) = 2x^3$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that

$$u = -\begin{cases} \sqrt[3]{v_2^2}\sin\frac{1}{\sqrt[3]{v_2^2}} & \text{if } v_2 < 0 \\ \sqrt[3]{v_2^2} & \text{if } v_2 \geq 0 \end{cases} \quad \text{and } v_1 = 2v_2.$$

The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition is not satisfied in any neighbourhood of $\bar{x}$. If we set $\{(u_n = \frac{1}{2n\pi+\frac{3\pi}{2}}, v_{1n} = 2\sqrt{\frac{1}{(2n\pi+\frac{3\pi}{2})^3}}, v_{2n} = \sqrt{\frac{1}{(2n\pi+\frac{3\pi}{2})^3}})\}_{n\geq 1} \subset \mathcal{K}_{\bar{x}} - \text{cl}\,\mathcal{H}$ and $\{\alpha_n = 2n\pi + \frac{3\pi}{2}\}_{n\geq 1}$, then it turns out that the calmness condition (2.3.1) and the regularity condition do not hold.

Example 2.5.32. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} x_2 \sin\frac{1}{x_2} + x_1 & \text{if } x_2 \neq 0 \\ 0 & \text{if } x_2 = 0 \end{cases}$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = -\begin{cases} \pm\sqrt{-v_2}\sin\frac{1}{\pm\sqrt{-v_2}} + v_1 & \text{if } v_2 \neq 0 \\ 0 & \text{if } v_2 = 0 \end{cases}$.

The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The metric regularity condition is not satisfied in any neighbourhood of $\bar{x}$ for $x_1 = 0$. If we set $\{(u_n = \frac{1}{2n\pi+\frac{\pi}{2}} + \frac{1}{n^5}, v_{1n} = \frac{1}{n^5}, v_{2n} = -\frac{1}{(2n\pi+\frac{\pi}{2})^2})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\alpha_n = \{2n\pi + \frac{\pi}{2}\}_{n\geq 1}$, we see that the calmness condition (2.3.1) and the regularity condition do not hold.

Next figure summarises the results illustrated by the examples.

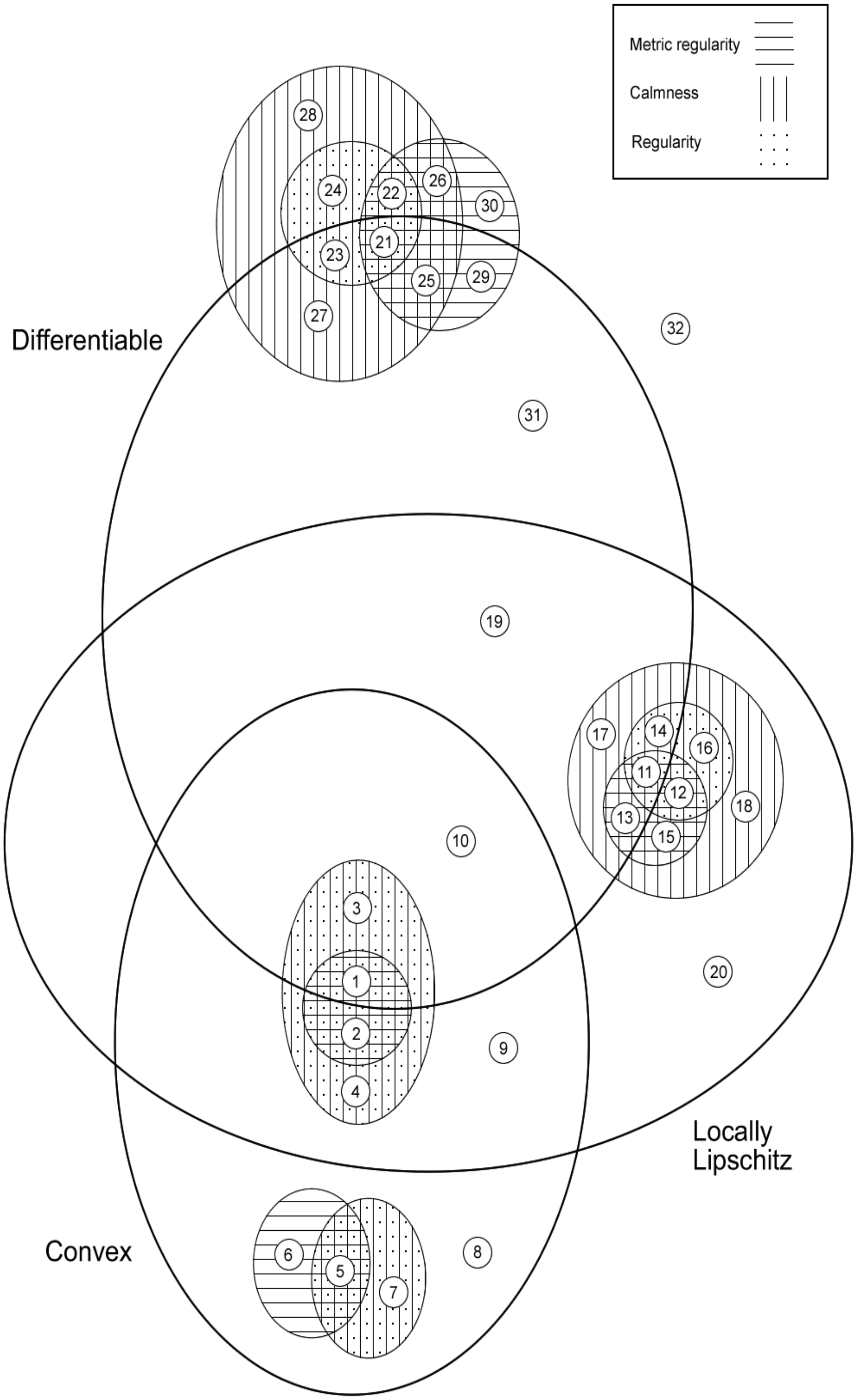
Metric regularity
Calmness
Regularity
Differentiable
Locally Lipschitz
Convex
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32

Figure 1

2.6 Examples and Graphical Representation for Necessary Conditions

As in the previous section, in the following examples we will consider problem (1.2.1) with one equality and one inequality constraint; again the set where the functions f and g are defined is specified in every example and the point $\bar{x}$ is a minimum point of the problem. We perform the comparison among the following conditions: necessary regularity condition (2.4.2), metric regularity (2.3.6), calmness (1.2.21), BCQ for convex problems, GCQ for differentiable problems. Since it is well-known that in the convex case necessary and sufficient conditions are equivalent, then the fulfilment of the regularity condition (2.4.2) (which can be applied obviously if the problem is not only convex but also $\mathcal{C}$-differentiable at $\bar{x}$) coincides, in the convex case, with the regularity condition (2.3.3). The figure 2 summarises the implications and the relationships among the above mentioned regularity conditions highlighted by the examples.

Example 2.6.1. (see Example 2.5.1). $X = \mathbb{R}$, $f(x) = x^2$, $g_1(x) = x$ and $g_2(x) = 3x$; $\bar{x} = 0$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = 0, v_1 = 3v_2\}$. The problem is convex, locally Lipschitz and differentiable on X. The constraints are linear, thus the problem is metric regular. Since $f(x) - f(\bar{x}) \geq 0$, for all $x \in X$ and the problem is convex, it follows that the problem is regular and calm at $\bar{x}$. The ACQ is satisfied.

Example 2.6.2. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 x_2^2$, $g_1(x_1, x_2) = x_1 x_2$ and $g_2(x_1, x_2) = -x_1^2 x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 = v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and differentiable on X. The metric regularity condition is not satisfied in any neighbourhood of $x_1 = x_2 < 1$. The problem is regular and GCQ is satisfied. Calmness at $\bar{x}$ follows from $f(x) - f(\bar{x}) \geq 0$, $\forall x \in X$.

Example 2.6.3. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 + x_2^2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -|x_2|$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = 0\}$.
The problem is convex, locally Lipschitz on X and not differentiable at $\bar{x}$. Metric regularity at $\bar{x}$ follows from the linearity of the constraints, while regularity is implied by $u = 0$, for all $(u, v_1, v_2) \in \mathcal{K}^h_{\bar{x}}$. Calmness at $\bar{x}$ results from the non negativity of $f(x) - f(\bar{x})$ on all X. BCQ is satisfied too.

Example 2.6.4. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 + x_2^2$, $g_1(x_1, x_2) = x_1 - x_2$ and
$$g_2(x_1, x_2) = \begin{cases} -x_1^2 - x_2^2 & \text{if } x_1 \geq 0,\ x_2 \geq 0 \\ -x_1^2 + x_2 & \text{if } x_1 \geq 0,\ x_2 < 0 \\ x_1 + x_2 & \text{if } x_1 < 0,\ x_2 < 0 \\ x_1 - x_2^2 & \text{if } x_1 < 0,\ x_2 \geq 0 \end{cases};$$
$\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = 0, v_1 = v_2\}$.

The problem is convex, locally Lipschitz on X and it is not differentiable at $\bar{x}$. The metric regularity condition is not fulfilled in any neighbourhood of $\bar{x}$ for $x_2 = 0$. Regularity follows from $u = 0$, for all $(u, v_1, v_2) \in \mathcal{K}^h_{\bar{x}}$. Calmness at $\bar{x}$ results from $f(x) - f(\bar{x}) \geq 0 \ \forall\ x \in X$. BCQ is satisfied.

Example 2.6.5. $X = \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$, $g_1(x) = x$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = 0, v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and differentiable on X. The metric regularity condition (2.3.6) is satisfied. The regularity condition is satisfied and the calmness at $\bar{x}$ is implied by $f(x) - f(\bar{x}) \geq 0$, $\forall x \in X$. GCQ is fulfilled.

Example 2.6.6. $X = \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$, $g_1(x) = x^2$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 = v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and differentiable on X. Metric regularity is not satisfied at $\bar{x}$. It is obvious that the problem is regular and it is calm at $\bar{x}$ since $f(x) - f(\bar{x}) \geq 0$, $\forall x \in X$. GCQ is not fulfilled.

Example 2.6.7. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 + x_2^2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 = 0\}$.
The problem is convex, locally Lipschitz and differentiable on X. The metric regularity condition is not satisfied in any at $\bar{x}$ for $x_1 = 0$. The problem is regular; calmness at $\bar{x}$ follows from $f(x) - f(\bar{x}) \geq 0$, $\forall\ x \in X$. ACQ is not satisfied.

Example 2.6.8. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2 + x_2^6$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -|x_1| - x_2^2$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = 0, v_1 = v_2\}$.
The problem is convex, locally Lipschitz on X and it is not differentiable at $\bar{x}$. The metric regularity condition is not verified at $\bar{x}$ for $x_1 = 0$. The problem is regular since $u = 0$, for all $(u, v_1, v_2) \in \mathcal{K}^h_{\bar{x}}$ and calm at $\bar{x}$ because $f(x) - f(\bar{x}) \geq 0$ on all X. BCQ is not satisfied.

Example 2.6.9. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1 - x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^3$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and differentiable on X. The metric regularity condition is not satisfied at $\bar{x}$ for $x_1 = 0$. The problem is not regular since $\mathcal{H}_u \subset \text{cl}\ (\mathcal{K}^h_{\bar{x}} - \text{cl}\ \mathcal{H})$. If we choose $\{(u_n = \frac{1}{n^2} + \frac{1}{n}, v_{1n} = \frac{1}{n^2}, v_{2n} = -\frac{1}{n^3})\}_{n \geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n \geq 1}$, then the calmness condition (2.3.1) does not hold. GCQ is not fulfilled.

Example 2.6.10. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1 - x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_2^2$; $\bar{x} = (0, 0)$;$\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}$.
The problem is convex, locally Lipschitz and differentiable on X. It is not metric regular since, for $x_1 = 0$, the condition (2.3.6) is not verified at $\bar{x}$ for $x_1 = 0$. As in the previous example, the problem is not regular and not calm at $\bar{x}$. While

the former is obvious, for proving the latter assertion it is enough to set $\{(u_n = \frac{1}{n^3} + \frac{1}{n}, v_{1n} = \frac{1}{n^3}, v_{2n} = -\frac{1}{n^2})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\geq 1}$. ACQ does not hold.

Example 2.6.11. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_1 + |x_2|$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = x_2^3$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and it is not differentiable at $\bar{x}$. The metric regularity is not satisfied at $\bar{x}$ for $x_2 = 0$. The problem is not regular. If we choose $\{(u_n = -\frac{1}{n^3} + \frac{1}{n}, v_{1n} = \frac{1}{n^3}, v_{2n} = -\frac{1}{n^2})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\geq 1}$ we have that the problem is not calm at $\bar{x}$.

Example 2.6.12. $X = \mathbb{R}$, $f(x) = \begin{cases} x^3 \sin\frac{1}{x^3} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$, $g_1(x) = x$ and $g_2(x) = x^3$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 = 0\}$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition is fulfilled, the problem is regular and calm at $\bar{x}$. GCQ is also satisfied.

Example 2.6.13. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} x_1^6 & \text{if} \quad x_1 \geq 0 \\ x_1^6 \sin\frac{1}{x_1^6} & \text{if} \quad x_1 < 0 \end{cases}$, $g_1(x_1, x_2) = x_1 - x_2$ and $g_2(x_1, x_2) = -x_1^6$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition does not hold at $\bar{x}$ for $x_2 = 0$. The problem is calm at $\bar{x}$ and regular, with GCQ satisfied.

Example 2.6.14. $X = \mathbb{R}$, $f(x) = \begin{cases} x^3 & \text{if} \quad x \geq 0 \\ x^6 \sin\frac{1}{x^6} & \text{if} \quad x < 0 \end{cases}$, $g_1(x) = 3x^3$ and $g_2(x) = x^3$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. As in Example 2.5.27, the problem is not metric regular and it is calm at $\bar{x}$. Regularity follows from $\mathcal{K}^h_{\bar{x}} = \{(0, 0, 0)\}$. GCQ is not satisfied.

Example 2.6.15. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if} \quad x < 0 \\ x^6 & \text{if} \quad x \geq 0 \end{cases}$, $g_1(x) = 2x^3$ and $g_2(x) = x^3$; $\bar{x} = 0$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_1 = v_2 = 0\}$.
The problem is not convex and not locally Lipschitz at $\bar{x}$, it is differentiable on X. Like in Example 2.5.31, the metric regularity condition is not satisfied in any neighbourhood of $\bar{x} = 0$ and the problem is not calm at $\bar{x}$. One can easily see that the problem is regular, while GCQ does not hold.

Example 2.6.16. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_2 + \begin{cases} x_1^2 \sin\frac{1}{x_1^2} & \text{if} \quad x_1 < 0 \\ x_1^6 & \text{if} \quad x_1 \geq 0 \end{cases}$, $g_1(x_1, x_2) = x_1^3$ and $g_2(x_1, x_2) = x_2^3$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_1 = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition (2.3.6) is not fulfilled at $\bar{x}$. Obviously, the problem is not regular; it is not calm at $\bar{x}$ since if we choose $\{(u_n = -\frac{1}{n^2} + \frac{1}{2n\pi + \frac{3\pi}{2}}, v_{1n} =$

$\frac{1}{\sqrt{(2n\pi+\frac{3\pi}{2})^3}}, v_{2n} = \frac{1}{n^6})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = 2n\pi + \frac{3\pi}{2}\}_{n\geq 1}$ the calmness condition (2.3.1) is contradicted.

Example 2.6.17. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} x_1^3 \sin \frac{1}{x_2^3} + x_2 & \text{if } x_1 \neq 0 \\ x_2 & \text{if } x_1 = 0 \end{cases}$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = \begin{cases} -x_2^2 & \text{if } x_2 \geq 0 \\ -|x_2| & \text{if } x_2 < 0 \end{cases}$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}}^h$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $v_2 \in \{0, 1\}$.
The problem is not convex, not locally Lipschitz and not differentiable at $\bar{x}$. The metric regularity condition is not satisfied at $\bar{x}$ when $x_1 = 0$. The calmness condition at $\bar{x}$ is not fulfilled if we choose $\{(u_n = -\frac{1}{n^6}\sin n^6 + \frac{1}{n}, v_{1n} = \frac{1}{n^2}, v_{2n} = -\frac{1}{n^2})\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\geq 1}$. The problem is, obviously, not regular.

Example 2.6.18. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} x_1^2 \sin \frac{1}{x_1^2} & \text{if } x_1 \neq 0 \\ 0 & \text{if } x_1 = 0 \end{cases}$, $g_1(x_1, x_2) = x_1 - x_2$ and $g_2(x_1, x_2) = -x_1^4$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}}^h = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz and not differentiable on X. The metric regularity condition (2.3.6) is not fulfilled at $\bar{x}$ for $x_2 = 0$. Obviously the problem is regular; it is not calm at $\bar{x}$ because, by choosing $\{(u_n = \frac{1}{2n\pi+\frac{3\pi}{2}}, v_{1n} = \frac{1}{n^5}, v_{2n} = \frac{1}{(2n\pi+\frac{3\pi}{2})^2}\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = 2n\pi + \frac{3\pi}{2}\}_{n\geq 1}$, the calmness condition (2.3.1) is contradicted. GCQ is fulfilled.

Example 2.6.19. $X = \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $g_1(x) = x$ and $g_2(x) = \begin{cases} x^4 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $\bar{x} = 0$; $\mathcal{K}_{\bar{x}}^h = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The problem is also metric regular. GCQ is fulfilled and the problem is regular. Let us consider the following sequences $\{(u_n = \frac{1}{n},\ v_{1n} = \frac{1}{n^2},\ v_{2n} = -\frac{1}{n^5})\}_{n\geq 1} \subseteq \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n\}_{n\geq 1}$. We have that $\lim_{n\to+\infty}(u_n, v_{1n}, v_{2n}) = (0, 0, 0)$ and $\lim_{n\to+\infty} \alpha_n(u_n, v_{1n}, v_{2n}) = (1, 0, 0) \in \text{cl cone}\,(\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Thus the problem is not calm at $\bar{x}$.

Example 2.6.20. $X = \mathbb{R}^2$, $f(x_1, x_2) = \begin{cases} x_1^2 \sin \frac{1}{x_1^2} & \text{if } x_1 \neq 0 \\ 0 & \text{if } x_1 = 0 \end{cases}$, $g_1(x_1, x_2) = \frac{x_2}{x_1^2+1}$ and $g_2(x_1, x_2) = -x_1^4$; $\bar{x} = (0, 0)$; $\mathcal{K}_{\bar{x}}^h = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = v_2 = 0\}$.
The problem is not convex, not locally Lipschitz at $\bar{x}$ and it is differentiable on X. The metric regularity condition is not verified at $\bar{x}$ for $x_2 = 0$. The problem is regular but not calm at $\bar{x}$ as it can be seen if we choose $\{(u_n = \frac{1}{2n\pi+\frac{3\pi}{2}}, v_{1n} = \frac{1}{n^5}, v_{2n} = \frac{1}{(2n\pi+\frac{3\pi}{2})^2}\}_{n\geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = 2n\pi + \frac{3\pi}{2}\}_{n\geq 1}$. GCQ is not fulfilled.

Example 2.6.21. $X = \mathbb{R}^2$, $f(x_1, x_2) = x_2^2 + x_1$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = x_2^3$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -v_1, v_2 = 0\}$.
The problem is not convex, it is locally Lipschitz and differentiable on X. The metric regularity condition is not satisfied at $\bar{x}$ for $x_1 = 0$. The problem is regular but not calm at $\bar{x}$ as it can be seen if we choose $\{(u_n = -\frac{1}{n^4} + \frac{1}{n^2}, v_{1n} = \frac{1}{n^4}, v_{2n} = \frac{1}{n^3})\}_{n \geq 1} \subset \mathcal{E}(\mathcal{K}_{\bar{x}})$ and $\{\alpha_n = n^2\}_{n \geq 1}$. GCQ is not fulfilled.

Example 2.6.22. $X = \mathbb{R}^2$, $f(x_1, x_2) = -x_1 - x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = \begin{cases} -x_2^2 & \text{if} \quad x_2 \geq 0 \\ -|x_2| & \text{if} \quad x_2 < 0 \end{cases}$; $\bar{x} = (0, 0)$; $\mathcal{K}^h_{\bar{x}}$ is the set of all $(u, v_1, v_2) \in \mathbb{R}^3$ such that $u = v_1 + v_2$ if $v_2 < 0$ and $u \geq v_1$ if $v_2 = 0$.
The problem is convex, locally Lipschitz on X and it is not differentiable at $\bar{x}$. The metric regularity condition is not satisfied at $\bar{x}$ for $x_1 = 0$. The problem is not regular and it is not calm at $\bar{x}$; as in Example 2.5.9. BCQ is not fulfilled.

Next figure summarises the results illustrated by the examples.

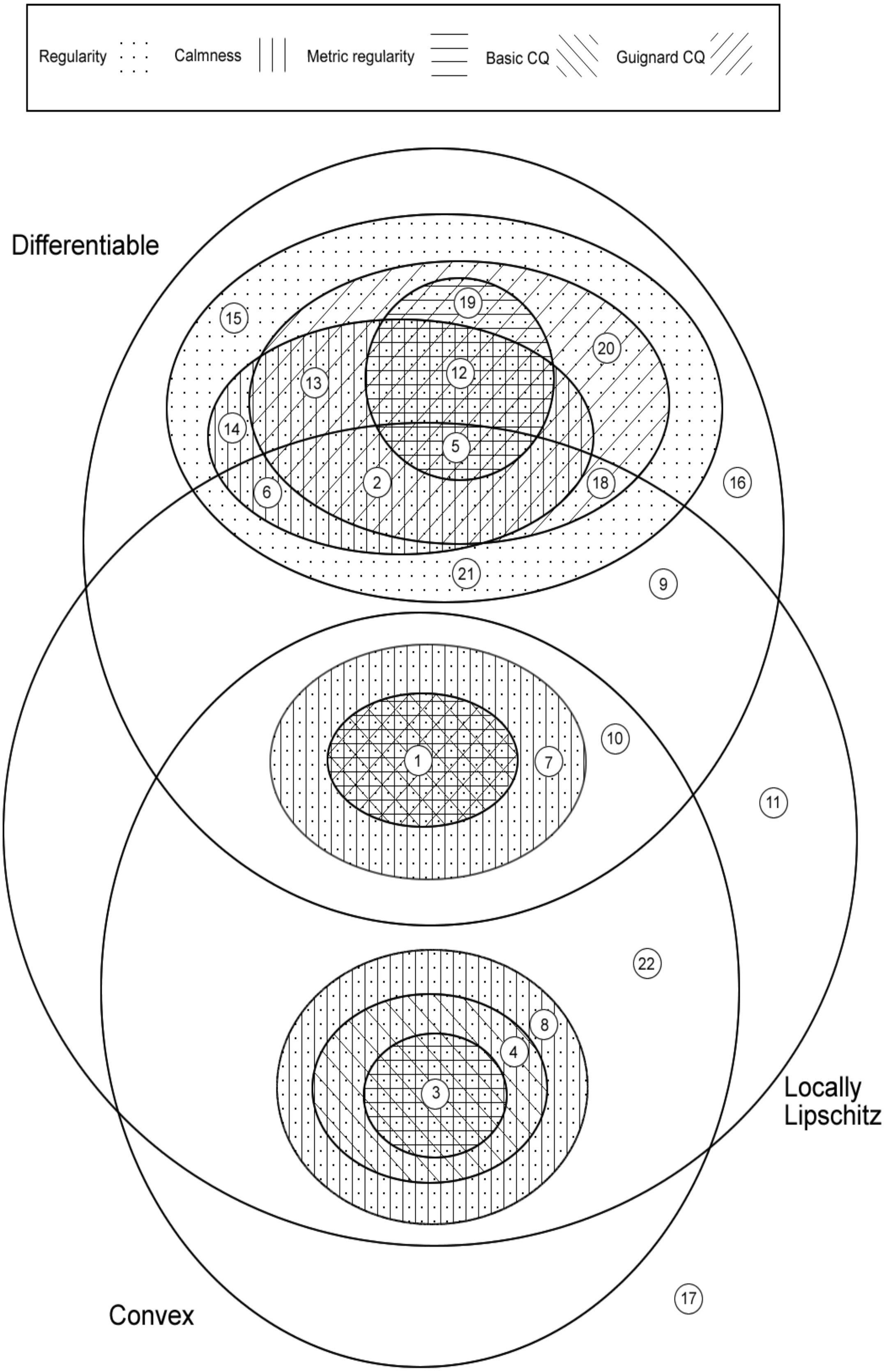
Regularity
Calmness
Metric regularity
Basic CQ
Guignard CQ
Differentiable
Locally Lipschitz
Convex
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22

Figure 2

Chapter 3

Regularity for Problems with Infinite Dimensional Image

The existence of Lagrange multipliers for problems with infinite dimensional image is a concern that dates back to the 1970s [62] and has since been developed in a a large numbers of papers. Besides the classical interior conditions, attempts have been made in order to overcome the lack of a nonempty interior. For differentiable problems we mention the works of problems [76] , [60] , [7], [31]. For convex problems with infinite dimensional image, different generalised notions of the interior of a convex set have been used, such as the *core*, the *intrinsic core* or the *strong-quasi relative interior*. In the literature regularity conditions using generalised interior have been given e.g. [66], [74], [73], [46], [30], [10], [13], [15], [16]. We mention also the class of *closedness type* conditions intensively studied recently (see, for example, [11]).

The IS analysis we performed in the previous section was valid only for problems having finite dimensional image. When the image becomes infinite dimensional we will try to extend to this case some of the results obtained for finite dimensional image. Namely, we will give the equivalent of the regularity condition (2.3.3),

$$TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}}))) \cap \mathcal{H}_u = \varnothing,$$

associated to a problem having infinite dimensional image and we will prove that it is still equivalent to the existence of Lagrange multipliers.

We will start the chapter with an approach that will still use the results of the finite dimensional IS Analysis. Following Lagrange ideas, the approach tries to postpone as long as possible the encounter with the infinite dimensionality. It consists in introducing a selection multifunction that will allow us to circumvent the infinite dimensionality and to be reduced to a problem having a finite dimensional image, called selected problem, and which is equivalent to the original one. In this way we can apply the results presented in the previous section. We will develop necessary conditions associated to the selected problem and we will show that, in the case of a selection with the selection multiplier belonging to C^1, the regularity condition analysed in the first chapter is equivalent to the classic Euler equation. Saddle point conditions and illustrative examples are also given.

In Section 3.2 it is shown that the regularity condition (2.3.3) is necessary and sufficient for the existence of Lagrange multipliers also for the case of infinite dimensional image problems. In Section 3.3 a new Slater type CQ is given using te quasi-relative interior of a convex set. We perform some analysis of other papers in the literature using the qri Slater type CQ and we conclude that our CQ improves the ones given in the literature. Our Slater CQ is a generalisation of other CQs from literature using the relative interior, the core, the intrinsic core, the strong quasi-relative interior since the quasi-relative interior generalises all these notions.

3.1 The Selection Approach

Using the selection approach for a geodesic type problem (3.1.1) we will develop the approach started in [24], [25] and developed in [27] and [28]. Starting with necessary conditions, when we have the classic Lagrange multipliers from the calculus of variations, we give and prove a linearization lemma and we will also show that the classic Euler equation can be interpreted in the IS as the regular separation between the cone $\mathcal{H}$ and the linearized image $\mathcal{K}^l_{\bar{x}}$, a separation which is equivalent to the regularity condition (2.4.2) applied to problem (3.1.1). We will go on by giving sufficient optimality condition for problem (3.1.1)

Assume we are given the integers m, n and p with $0 \leq p \leq m$, $n > 0$, the interval $T := [a, b] \subset \mathbb{R}$ with $-\infty \leq a \leq b \leq +\infty$, and the functions

$$\psi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ i = 0, 1, ..., m.$$

Let $C^0(T)^n$ denote the set of all continuous functions $x(t) = (x_1(t), ..., x_n(t))$, $t \in T$. Let $\mathbb{V}$ be the subset of $C^0(T)^n$ of the functions having continuous derivatives $x'(t) = (x'_1(t), ..., x'_n(t))$, $t \in T$, except at most for a finite number of points $\bar{t}$ at which $\lim_{t\downarrow\bar{t}} x'(t)$ and $\lim_{t\uparrow\bar{t}} x'(t)$ exist and are finite; $x'(\bar{t}) := \lim_{t\downarrow\bar{t}} x'(t)$. $\mathbb{V}$ forms a vector space on the set of real numbers, and is equipped with the norm

$$||x||_\infty := \max_{t\in T} ||x(t)||,\ x \in \mathbb{V},$$

where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^n$. X is defined as the subset of $\mathbb{V}$, whose elements satisfy the boundary condition $x(a) = x^0$ and $x(b) = x^1$, x^0 and x^1 being given vectors of $\mathbb{R}^n$.

While the results of this paper are valid for a wider class of problems, let us consider the following geodesic-type minimisation problem:

$$f^{\downarrow} := \min[f(x) := \int_T \psi_0(t, x(t), x'(t))dt], \tag{3.1.1}$$

subject to

$$\psi_i(t, x(t), x'(t)) = 0, \ \forall t \in T, \quad i \in \mathcal{I}^0 := \{1, ..., p\}, \tag{3.1.1}$$

$$\psi_i(t, x(t), x'(t)) \geq 0, \ \forall t \in T, \ i \in \mathcal{I}^+ := \{p+1, ..., m\}, \tag{3.1.1}$$

$$x \in X \subseteq C^0(T)^n \tag{3.1.1}$$

where $p = 0 \Rightarrow \mathcal{I}^0 = \varnothing$, $p = m \Rightarrow \mathcal{I}^+ = \varnothing$, $m = 0 \Rightarrow \mathcal{I} := \mathcal{I}^0 \cup \mathcal{I}^+ = \varnothing$. Unless differently stated, we will assume that card $X > 1$.

As before, we set $D := O_p \times \mathbb{R}^{m-p}_+$ with $O_p := (0, ..., 0) \in \mathbb{R}^p$. We stipulate that $D = \mathbb{R}^m_+$ when $p = 0$ and $D = O_m$ when $p = m$. Set $\psi := (\psi_1, ..., \psi_m)$. The set

$$R := \{x \in X : \psi(t, x(t), x'(t)) \in D, \quad \forall t \in T\}$$

is the feasible region of problem (3.1.1).

The analysis carried out in the following sections can also be performed locally by replacing X with $X \cap N_\rho(\bar{x})$, where $N_\rho(\bar{x})$ is the ball of center $\bar{x}$ and radius $\rho > 0$.

3.1.1 Elements of Image Space Analysis

Let us consider any $\overline{x} \in R$. Obviously, $\overline{x}$ is a global minimum point (3.1.1), if and only if the system (in the unknown x):

$$f(\overline{x}) - f(x) > 0, \quad \psi(t, x(t), x'(t)) \in D, \quad \forall t \in T, \, x \in X \tag{3.1.1}$$

is impossible.

When $x \in X$ is fixed, then ψ becomes a function of t only. The set of the functions $\tilde{\psi}(x), x \in X$ where $\tilde{\psi}(x)(t) = \psi(t, x(t), x'(t))$, is a subset of an infinite dimensional space. Therefore, unlike what happens for isoperimetric-type problems (and, of course, for problems in $\mathbb{R}^n$) the analysis of the image of (3.1.1) or (3.1.1) should be carried on in a Banach Space. Such an infinite dimensionality cannot be deleted; however, it can be postponed to the introduction of the IS. This can be done via the following approach.

The image of x through $\tilde{\psi}_i$ is again a function defined on T; the image of $\tilde{\psi}_i(x)(\cdot)$ is a subset of $\mathbb{R}$. Hence, we can introduce the multifunction which sends x into such a subset of $\mathbb{R}^{1+m}$, namely $A_{\overline{x}} : X \rightrightarrows Y \subseteq \mathbb{R}^{1+m}$, defined by:

$$A_{\overline{x}}(x) := \{(u, v) \in \mathbb{R}^{1+m} : u = f(\overline{x}) - f(x) \, v_i = \psi_i(t, x(t), x'(t)), t \in T, i \in \mathcal{I}\}.$$

$\mathcal{K}_{\bar{x}} := A_{\overline{x}}(X)$ will be the *image* of (3.1.1). By means of the above multifunction, we are able to work in a finite dimensional IS, namely $\mathbb{R}^{1+m}$; the infinite dimensionality has not been deleted, but postponed, and it will appear again later in terms of selection from $A_{\overline{x}}(X)$.

By introducing the set

$$\mathcal{H} := (\mathbb{R}_+ \backslash \{O\}) \times D,$$

it is easy to see that (3.1.1) is impossible, if and only if

$$A_{\overline{x}}(x) \nsubseteq \mathcal{H}, \ \forall x \in X. \tag{3.1.1}$$

The infinite dimensionality, which has been postponed in order to be able to introduce a finite dimensional IS, appears now with the selection, $\forall x \in X$, of an element of $A_{\overline{x}}(x)$.

Consider the functions $\omega_i : T \to \mathbb{R}, i \in \mathcal{I}$. Denote by Ω the set of vectors $\omega := (\omega_1, ..., \omega_m)$, whose elements are not all identically zero on T and such that $\omega_i \geq 0, i \in \mathcal{I}^+$; Ω represents a class of functional parameters satisfying a suitable condition, under which the integral in (3.1.1) makes sense. The selection in this case is specified to be of type:

$$\Phi(A_{\bar{x}}(x), \omega) := \left(f(\overline{x}) - f(x), \ \int_T \omega_i(t)\psi_i(t, x(t), x'(t))dt, \ i \in \mathcal{I} \right), \tag{3.1.1}$$

with $(A_{\bar{x}}(x), \omega) \subseteq 2^{\mathbb{R}^{1+m}} \times \Omega$.

Definition 3.1.1. Φ is called *generalised selection function* of $A_{\overline{x}}$ (GSF), if and only if

$$\forall x \in X, \qquad A_{\overline{x}}(x) \subseteq \mathcal{H} \Leftrightarrow \Phi(A_{\overline{x}}, \omega) \in \mathcal{H}, \ \forall \omega \in \Omega. \tag{3.1.1}$$

Observe that (3.1.1) is equivalent to:

$$\forall x \in X, \quad A_{\overline{x}}(x) \nsubseteq \mathcal{H} \Leftrightarrow \exists \omega \in \Omega \quad s.t. \quad \Phi(A_{\overline{x}}(x), \omega) \notin \mathcal{H}. \tag{3.1.1}$$

ω is called *selection quasi multiplier* (SQM).

From the so-called Fundamental Lemma of the calculus of variations [56], we immediately draw the following lemma.

Lemma 3.1.2. *Let $\alpha \in C^0[a, b]$ be such that:*

$$\int_a^b \alpha(t)\phi(t)dt \geq 0, \quad \forall \phi \in C^1_0[a, b],$$

where $C^1_0[a, b] := \{\phi \in C^1[a, b] : \phi(t) \geq 0, \forall t \in [a, b], \phi(a) = \phi(b) = 0\}$. Then

$$\alpha(t) \geq 0, \quad \forall t \in [a, b].$$

The next theorem shows that, under hypotheses of continuity of the involved functions, (3.1.1) is a GSF.

Theorem 3.1.3. *If* $C^0(T)^m \subseteq \Omega$ *and* $\psi_i,\ i \in \mathcal{I}$, *are continuous, then (3.1.1) is a GSF.*

Proof. Let $x \in X$. Suppose that $A_{\overline{x}}(x) \subseteq \mathcal{H}$, i.e. (3.1.1) holds. Then, since $\omega_i \geq 0$, $i \in \mathcal{I}^+, \forall \omega \in \Omega$, we have

$$f(\overline{x}) - f(x) > 0,\ \int_T \omega_i(t)\psi_i(t, x(t), x'(t)dt = 0,\ i \in \mathcal{I}^0,$$

$$\int_T \omega_i(t)\psi_i(t, x(t), x'(t)dt \geq 0,\ i \in \mathcal{I}^+,\ \forall t \in T. \tag{3.1.1}$$

and hence, $\Phi(A_{\overline{x}}, \omega) \in \mathcal{H},\ \forall \omega \in \Omega$. Conversely, assume that $\Phi(A_{\overline{x}}, \omega) \in \mathcal{H},\ \forall \omega \in \Omega$, that is (3.1.1) holds whatever $\omega \in \Omega$ may be. Since $x \in \mathbb{V}$, then $\psi_i(t, x(t), x'(t))$ is a bounded function continuous on T, except for a finite number of points $t_1, ..., t_k$, $i \in \mathcal{I}$. Suppose that $i \in \mathcal{I}^0$. Applying the Fundamental Lemma of calculus of variations in the interval $[a, b] := [t_j, t_{j+1}], j = 1, .., k-1$,we obtain

$$\psi_i(t, x(t), x'(t) = 0, \quad t \in T, \quad i \in \mathcal{I}^0.$$

Analogously, for $i \in \mathcal{I}^+$, applying Lemma (3.1.2) with $\alpha(t) := \psi_i(t, x(t), x'(t),\ \phi(t) := \omega(t), t \in T,\ [a, b] := [t_j, t_{j+1}], j = 1, .., k-1$, we obtain that

$$\psi_i(t, x(t), x'(t) \geq 0, \quad t \in T, \quad i \in \mathcal{I}^+. \qquad \square$$

From now on, in the rest of the chapter, we will assume that the hypotheses of Theorem (3.1.3) are fulfilled.

The previous result leads us to introduce the selected problem.

Definition 3.1.4. Let $\omega(\cdot, x) \in \Omega, x \in X$; the following problem:

$$\min f(x) := \int_T \psi_0(t, x(t), x'(t))dt, \tag{3.1.2}$$

subject to

$$g_i(x, \omega_i) := \int_T \omega_i(t, x)\psi_i(t, x(t), x'(t))dt = 0, \qquad i \in \mathcal{I}^\circ \tag{3.1.2}$$

$$g_i(x, \omega_i) := \int_T \omega_i(t, x)\psi_i(t, x(t), x'(t))dt \geq 0, \qquad i \in \mathcal{I}^+, \tag{3.1.2}$$

$$x \in X, \tag{3.1.2}$$

is called the *selected problem* of (3.1.1).

Set $g(x,\omega) := (g_1(x,\omega_1), ..., g_m(x,\omega_m))$.

Definition 3.1.5. The set:

$$\mathcal{K}_{\overline{x}}(\omega) := \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : u = f(\overline{x}) - f(x), v_i = g_i(x,\omega_i), i \in \mathcal{I}, x \in X\}$$

is called the *selected image* of problem (3.1.1).

Proposition 3.1.6. *$\overline{x} \in R$ is a (global) minimum point of problem (3.1.1), if and only if there exists a function $\overline{\omega}(t,x)$, $\overline{\omega}(\cdot,x) \in \Omega, x \in X$, such that:*

$$\mathcal{H} \cap \mathcal{K}_{\overline{x}}(\overline{\omega}) = \varnothing. \tag{3.1.2}$$

Proof. (3.1.6) is equivalent to $\Phi(x;\bar{\omega}) \notin \mathcal{H}$, $\forall x \in X$, where Φ is defined by (3.1.1). Therefore, because of (3.1.1), we have that (3.1.6) is equivalent to (3.1.1). Hence, $\overline{x}$ is a global minimum point of (3.1.1) if and only if (3.1.6) holds. □

Remark 3.1.7. Since it is known from the IS Analysis in the finite dimensional case that (3.1.6) is equivalent to

$$\mathcal{H} \cap (\mathcal{K}_{\overline{x}}(\overline{\omega}) - \text{cl } \mathcal{H}) = \varnothing, \tag{3.1.2}$$

Proposition 3.1.6 can be equivalently written using (3.1.7) instead of (3.1.6).

3.1.2 Linearization and Necessary Conditions

We will give some necessary conditions for the existence of minimum points for problem (3.1.1). To this end, we will make the following assumption.

Assumption A. Given $\bar{x} \in X$, there exists a neighbourhood $N_\rho(\bar{x})$, such that the selection multiplier ω depends only on t, namely

$$\omega(t,x) = \omega(t) \in \Omega, \quad \forall x \in X \cap N_\rho(\bar{x}).$$

In such a case ω is called selection multiplier.

In the rest of this section we will assume that $\frac{\partial}{\partial x}\psi_i(t,x(t),x'(t))$, $\frac{\partial}{\partial x'}\psi_i(t,x(t),x'(t))$, $\frac{d}{dt}\frac{\partial}{\partial x'}\psi_i(t,x(t),x'(t))$, $\forall i \in \{0\} \cup \mathcal{I}$, exist and are continuous.

Definition 3.1.8. Let $\bar{x} \in R$ and let $\delta x = (\delta x_1, ..., \delta x_n) \in X \cap N_\rho(\bar{x}) - \{\bar{x}\}$ be the vector of the variations of the elements of x, namely $\delta x_j(t) := x_j(t) - \bar{x}_j(t)$, $j = 1, ..., n$. The set

$$\mathcal{K}^l_{\bar{x}}(\omega) := \left\{ (u,v) \in \mathbb{R}^{1+m} : u = -\int_T \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\psi_0(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x'_j}\psi_0(t,\bar{x},\bar{x}')\delta x'_j \right) dt \,, \right.$$

$$v_i = \int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt + \int_T \omega_i(t) \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \right.$$

$$\left. \frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\delta x'_j \right) dt, \quad x \in X \cap N_\rho(\bar{x}),\ i \in \mathfrak{I} \Bigg\}, \quad \omega \in \Omega$$

is called *linearization* of the selected image of (3.1.1).

Indeed, in strict sense, with Definition 3.1.8 we linearise the functions ψ_i, $i \in \mathfrak{I}$; more precisely, we replace (3.1.2) (where we have put $\omega(t,x) = \omega(t)$) with its linearized form:

$$\min \left\{ \int_T \left(\psi_0(t,\bar{x},\bar{x}') + \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_0(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x'_j}\psi_0(t,\bar{x},\bar{x}')\delta x'_j \right) \right) dt \right\} \tag{3.1.3}$$

subject to

$$\int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt + \int_T \omega_i(t) \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\delta x'_j \right) dt = 0,$$

$$i \in \mathfrak{I}^0, \tag{3.1.3}$$

$$\int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt + \int_T \omega_i(t) \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\delta x'_j \right) dt \geq 0,$$

$$i \in \mathfrak{I}^+, \tag{3.1.3}$$

$$x \in X \cap N_\rho(\bar{x}). \tag{3.1.3}$$

The functions $g_i(\cdot,\omega_i)$, $i \in \mathfrak{I}$, being differentiable with respect to x at $\bar{x}$, admit the following expansion in the neighbourhood $N_\rho(\bar{x})$:

$$g_i(x,\omega_i) = \int_T \omega_i(t)\psi_i(t,\bar{x}(t),\bar{x}'(t))dt + \int_T \omega_i(t) \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \right.$$

$$\left. \frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\delta x'_j \right) dt + \int_T \omega_i(t)\varepsilon_i(t,\bar{x}(t),\bar{x}'(t);\delta x,\delta x')dt,$$

with $\lim_{x\to\bar{x}} \frac{1}{\|\delta x\|} \int_T \varepsilon_i(t,\bar{x}(t),\bar{x}'(t);\delta x,\delta x) = 0$, $i \in \mathfrak{I}$.

Let us consider, more in details, the case where (3.1.1) has only unilateral constraints ($p = 0, m \geq 1$). For this, let us associate (3.1.3) with the following system (in the unknown δx):

$$\int_T \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\psi_0(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x'_j}\psi_0(t,\bar{x},\bar{x}')\delta x'_j \right) dt < 0, \tag{3.1.4}$$

$$\int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt+\int_T \omega_i(t)\left[\sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x_j'}\psi_i(t,\bar{x},\bar{x}')\delta x_j'\right)\right] dt > 0,$$

$$i \in \mathcal{I}_N^+, \qquad (3.1.4)$$

$$\int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt+\int_T \omega_i(t)\left[\sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x_j'}\psi_i(t,\bar{x},\bar{x}')\delta x_j'\right)\right] dt \geq 0,$$

$$i \in \mathcal{I}_L^+, \qquad (3.1.4)$$

$$x \in X \cap N_\rho(\bar{x}), \quad \delta x \in X - \{\bar{x}\} \qquad (3.1.4)$$

where

$$\mathcal{I}_N^+ := \{i \in \mathcal{I}^+ : g_i(\bar{x},\omega) = 0, \int_T \omega_i(t)\varepsilon_i(t,\bar{x}(t),\bar{x}'(t);\delta x,\delta x')dt \neq 0\},$$

$$\mathcal{I}_L^+ := \mathcal{I}^+ \setminus \mathcal{I}_N^+.$$

Proposition 3.1.9. *[28] If $\bar{x}$ is a minimum point of (3.1.1), then the system (3.1.4) is impossible.*

Proof. First of all, observe that, by Proposition 3.1.6, $\bar{x}$ is a minimum point of (3.1.1) if and only if it is a minimum point of (3.1.2). For the sake of simplicity, we set

$$\psi_i'(t,\bar{x},\bar{x}';\delta x,\delta x') := \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j + \frac{\partial}{\partial x_j'}\psi_i(t,\bar{x},\bar{x}')\delta x_j'\right), \quad i \in \{0\} \cup \mathcal{I}.$$

Ab absurdo suppose that (3.1.4) is possible. Let $\delta\hat{x} = \bar{x} - \hat{x} \in X - \hat{x}$ be a solution of (3.1.4). Then $\alpha\delta\hat{x}$ is also a solution of (3.1.4), $\forall \alpha \in]0,1]$ and there exists $\alpha_1 \in]0,1]$, such that the following inequalities are satisfied $\forall \alpha \in]0,\alpha_1]$:

$$\frac{\int_T \varepsilon_0(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')dt}{\|\alpha\delta\hat{x}\|} < -\frac{\int_T \psi_0'(t,\bar{x},\bar{x}';\delta\hat{x},\delta\hat{x}')dt}{\|\delta\hat{x}\|},$$

$$\frac{\int_T \omega_i(t)\varepsilon_i(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')}{\|\alpha\delta\hat{x}\|} > -\frac{\int_T \omega_i(t)\psi_i'(t,\bar{x},\bar{x}';\delta\hat{x},\delta\hat{x}')dt}{\|\delta\hat{x}\|}, \quad i \in \mathcal{I}_N^+,$$

where ε_i is the remainder associated with the expansion of ψ_i at $\bar{x}$, $\forall i \in \{0\} \cup \mathcal{I}_N^+$. Thus, $\forall \alpha \in]0,\alpha_1]$, the following inequalities hold:

$$\int_T \left(\psi_0'(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')dt + \varepsilon_0(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')\right) dt < 0, \qquad (3.1.4)$$

$$g_i(\bar{x},\omega_i) + \int\limits_T \omega_i(t)\left(\psi_i'(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}') + \varepsilon_i(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')\right)dt > 0, i \in \mathfrak{I}_N^+. \tag{3.1.4}$$

On the other hand we have that, $\forall i \in \mathfrak{I}_L^+$, either

$$g_i(\bar{x},\omega_i) = 0 \ \text{ and } \ \int\limits_T \omega_i(t)\varepsilon_i(t,\bar{x}(t),\bar{x}'(t),\delta x(t),\delta x'(t))dt = 0,$$

or $g_i(\bar{x},\omega_i) > 0$. In the former case, we can write:

$$g_i(\bar{x},\omega_i) + \int\limits_T \omega_i(t)\left(\psi_i'(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}') + \varepsilon_i(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')\right)dt \geq 0. \tag{3.1.4}$$

In the latter case, $\exists\ \alpha_2 \in]0,1]$, such that $\forall\alpha \in]0,\alpha_2]$ we have:

$$g_i(\bar{x},\omega_i) + \int\limits_T \omega_i(t)\psi_i'(t,\bar{x},\bar{x};\alpha\delta\hat{x},\alpha\delta\hat{x}')dt > 0;$$

thus $\exists\ \alpha_3 \in]0,\alpha_2]$, such that $\forall\alpha \in]0,\alpha_3]$ it holds

$$\frac{\int\limits_T \omega_i(t)\varepsilon_i(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')dt}{\|\alpha\delta\hat{x}\|} \geq -\frac{g_i(\bar{x},\omega_i) + \int\limits_T \omega_i(t)\psi_i'(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')dt}{\|\alpha\delta\hat{x}\|}.$$

It follows that $\forall\alpha \in]0,\alpha_3]$,

$$g_i(\bar{x},\omega_i) + \int\limits_T \omega_i(t)\left(\psi_i'(t,\bar{x},\bar{x}';\alpha\delta\hat{x},\alpha\delta\hat{x}')dt + \varepsilon_i(t,\bar{x}(t),\bar{x}'(t),\alpha\delta\hat{x},\alpha\delta\hat{x}')\right)dt \geq 0. \tag{3.1.4}$$

Taking $\alpha_4 := \min\{\alpha_1,\alpha_3\}$ and invoking (3.1.2)-(3.1.2), one can conclude that the point $\hat{x}_{\alpha_4} = \bar{x} + \alpha_4\delta\hat{x} \in R$ is a solution of the system

$$f(\bar{x}) - f(x) > 0, \quad g_i(x,\omega_i) \geq 0, \ i \in \mathfrak{I}^+$$

and this contradicts the optimality of $\bar{x}$. □

From the above linearized problem it is possible to derive necessary (optimality) conditions, in particular to recover the Euler equation when $\omega \in C^1[T]$. Problem (3.1.2) is associated with the Lagrangian function $L : X \times D^* \times \Omega$ defined by

$$L(x;\lambda,\omega) := \int_T \left(\psi_0(t,x(t),x'(t)) - \sum_{i=1}^m \lambda_i\omega_i(t)\psi_i(t,x(t),x'(t))\right)dt.$$

We will denote the integrand of the Lagrangian with J, i.e.

$$L(x;\lambda,\omega)=\int\limits_T J(t,x,x';\lambda,\omega)dt.$$

In order to recover the classic Euler equation from the calculus of variations we need to assume that there exists $\frac{d\omega}{dt}$. In such a case the linearization of the selected image of the problem (3.1.2) becomes

$$\mathcal{K}^l_{\bar{x}}(\omega):=\Bigg\{(u,v)\in\mathbb{R}^{1+m}:u=-\int\limits_T\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\psi_0(t,\bar{x},\bar{x}')-\frac{d}{dt}\frac{\partial}{\partial x'_j}\psi_0(t,\bar{x},\bar{x}')\right)\delta x_j dt\ ,$$

$$v_i=\int\limits_T\omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt+\int\limits_T\omega_i(t)\sum_{j=1}^n\left[\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}')-\frac{d}{dt}\frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\right]\delta x_j dt-$$

$$\int\limits_T\frac{d\omega_i}{dt}\sum_{j=1}^n\frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}')\delta x_j dt,\quad x\in X\cap N_\rho(\bar{x}),i\in\mathcal{I}\Bigg\},\ \omega\in\Omega.$$

Definition 3.1.10. The sets $\mathcal{K}^l_{\bar{x}}(\omega)$ and $\mathcal{H}$ are *linearly separable* if and only if $\exists(\theta,\lambda)\in\mathbb{R}_+\times D^*$ such that:

$$\theta u+\langle\lambda,v\rangle\leq 0,\quad\forall(u,v)\in\mathcal{K}^l_{\bar{x}}(\omega).\tag{3.1.4}$$

We say that the linear separation is *regular* if and only if there exists at least one $\theta>0$ such that (3.1.10) holds.

Proposition 3.1.11. *Let be $\bar{x}\in ri\ X$ and $\omega(\cdot)\in C^1[T]$. Then the sets $\mathcal{K}^l_{\bar{x}}(\omega)$ and $\mathcal{H}$ are regularly linearly separable, if and only if $\exists\lambda\in D^*$ such that:*

$$\lambda_i\int\limits_T\omega_i(t)\psi_i(t,\bar{x}(t),\bar{x}'(t))dt=0,\quad\forall i\in\mathcal{I}$$

and

$$\frac{\partial}{\partial x}J(t,\bar{x},\bar{x}';\lambda,\omega)-\frac{d}{dt}\frac{\partial}{\partial x'}J(t,\bar{x},\bar{x}';\lambda,\omega)=0.\tag{3.1.4}$$

Proof. From the assumption we have that, without any loss of generality, (3.1.10) holds with $\theta=1$. Therefore, (3.1.10) can be rewritten as

$$-\int_T\psi_0{}'(t,\bar{x},\bar{x}';\delta x,\delta x')dt+\sum_{i=1}^m\lambda_i\left(g_i(\bar{x};\omega_i)+\int_T\omega_i(t)\psi_i{}'(t,\bar{x},\bar{x}';\delta x,\delta x')dt\right)\leq 0,\ \forall x\in X.$$

Equivalently, we have

$$\sum_{i=1}^{m} \lambda_i \int_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt = 0,$$

i.e.

$$\lambda_i \int\limits_T \omega_i(t)\psi_i(t,\bar{x},\bar{x}')dt = 0, \ \forall i \in \mathcal{I},$$

and

$$\sum_{j=1}^{n} \int\limits_T \Bigg[-\frac{\partial}{\partial x_j}\psi_0(t,\bar{x},\bar{x}') + \frac{d}{dt}\frac{\partial}{\partial x'_j}\psi_0(t,\bar{x},\bar{x}') + \sum_{i\in\mathcal{I}} \lambda_i \Bigg(\omega_i(t)\frac{\partial}{\partial x_j}\psi_i(t,\bar{x},\bar{x}') -$$

$$\omega_i(t)\frac{d}{dt}\frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}') - \frac{d}{dt}\omega_i(t)\frac{\partial}{\partial x'_j}\psi_i(t,\bar{x},\bar{x}') \Bigg) \Bigg] \delta x_j dt = 0, \quad \forall x \in X.$$

The above inequality holds true for any $x \in X$, that is

$$\frac{\partial}{\partial x} J(t,\bar{x},\bar{x}';\lambda,\omega) - \frac{d}{dt}\frac{\partial}{\partial x'} J(t,\bar{x},\bar{x}';\lambda,\omega) = 0. \qquad \square$$

Remark 3.1.12. We know that regular separation between $\mathcal{H}$ and $\mathcal{K}^l_{\bar{x}}(\omega)$ is equivalent to the regularity condition (2.4.2) applied to problem (3.1.2)

$$TC(\mathcal{E}(\mathcal{K}^l_{\bar{x}}(\omega))) \cap \mathcal{H}_u = \varnothing.$$

Corollary 3.1.13. *Let be $\bar{x} \in ri\ X$. The following are equivalent:*

(i) $$TC(\mathcal{E}(\mathcal{K}^l_{\bar{x}}(\omega))) \cap \mathcal{H}_u = \varnothing;$$

(ii) there exists $\exists \lambda \in D^*$ such that: $\lambda_i \int\limits_T \omega_i(t)\psi_i(t,\bar{x}(t),\bar{x}'(t))dt = 0, \quad i \in \mathcal{I}$ and (3.1.11) is satisfied.

We give now some simple examples for which the Euler equation does not hold. That shows the need of enlarging the set of classic Lagrange multipliers $\lambda(t)$ used in the calculus of variations.

3.1.3 Examples

Example 3.1.14. In (3.1.1) set $T = [0,1]$, $p = 0$, $m = 1$, $\psi_0(t,x,x') = x$, $\psi_1(t,x,x') = x^3(t)$, $\overline{x} \equiv 0$, $\forall t \in T$, $X = \{x \in \mathbb{V} : x(0) = x(1) = 0\}$. We have $\mathcal{H} = (\mathbb{R}_+ \backslash \{O\}) \times \mathbb{R}_+$. First, let us analyse the image set $\mathcal{K}_{\bar{x}} = \bigcup_{x \in X} A_{\bar{x}}(x)$ of the given problem. If $u > 0$, then $A_{\bar{x}}(x)$ is a segment (never a point, nor a line, nor a halfspace), which intersects the u-axis, and cannot be included in $\mathcal{H}$; in fact, in the contrary case, we should have $x(t) \geq 0$, $\forall t \in T$, while $u > 0$ implies $x(t) < 0$ for at least an element of T (more precisely, for at least a subinterval of T of positive length, due to the continuity of x). If $u = 0$, then $A_{\bar{x}}(x)$ can be as before and, in addition, can be a point (for $x \equiv \bar{x}$); again it is never included in $\mathcal{H}$. If $u < 0$, all the above cases are possible for $A_{\bar{x}}(x)$ and, of course, $A_{\bar{x}}(x) \nsubseteq \mathcal{H}$, since $A_{\bar{x}}(x) \subseteq \mathcal{H}$ implies $u > 0$. This shows the optimality of $\bar{x}$. For instance, set

$$x(t) = \hat{x}(t) = t \sin 2\pi t,$$

so that $\hat{x} \in X$. We find

$$f(x) = \left[\frac{1}{4\pi^2} \sin 2\pi t - \frac{t}{2\pi} \cos 2\pi t\right]_0^1 = -\frac{1}{2\pi},$$

$\tilde{\psi}_1(t) = t^3 \sin^3 2\pi t$ (with $\tilde{\psi}_1$ denotes the image of X through the function $\psi_1(t, x(t), x'(t))$ where x is fixed). Thus, we have:

$$A_{\bar{x}}(\hat{x}) = \{(u,v) \in \mathbb{R}^2 : u = \frac{1}{2\pi}, \quad \min_{t\in[0,1]} t^3 \sin^3 2\pi t \leq v \leq \max_{t\in[0,1]} t^3 \sin^3 2\pi t\} =$$

$$= \{(u,v) \in \mathbb{R}^2 : u = \frac{1}{2\pi}, -0,44 \leq v \leq 0,02\}.$$

If $\bar{x} \not\equiv 0$, then the image set changes. Taking for instance $\bar{x} \equiv \frac{3}{2}$ and $x(t) = \tilde{x}(t) = (1-t)$, we obtain:

$$\begin{aligned} A_{\frac{3}{2}}(\tilde{x}) &= \{(u,v) \in \mathbb{R}^2 : u = 1, \quad \min_{t\in[0,1]} (1-t)^3 \leq v \leq \max_{t\in[0,1]} (1-t)^3\} = \\ &= \{(u,v) \in \mathbb{R}^2 : u = 1, 0 \leq v \leq 1\} \subseteq \mathcal{H} \end{aligned}$$

and therefore $\bar{x}$ is not a minimum point of the problem.

Now, it will be shown that the Euler equation is not fulfilled. First of all, note that the given problem can be equivalently put in the form:

$$\min \int_T x_1(t)dt, \quad \text{s.t.} \quad x_1^3(t) - x_2^2(t) = 0, \quad x_1, x_2 \in X,$$

where the initial unknown $x(t)$ is now $x_1(t)$. For $\bar{x}_1 = \bar{x}_2 = 0$, the Lagrangian function becomes

$$L(x; \lambda, \omega) = \int_T \{x_1(t) - \lambda\omega(t) \left[x_1^3(t) - x_2^2(t)\right]\} dt,$$

and the Euler equation (more precisely, system) is:

$$1 - 3\lambda\omega(t)x_1^2(t) = 0, \quad 2\lambda\omega(t)x_2(t) = 0,$$

$$x_1^3(t) - x_2^2(t) = 0, \ t \in T, x_1, x_2 \in X, \lambda \in \mathbb{R}, \omega \in C^1(T)$$

and is obviously impossible at $x_1(t) = \bar{x}_1(t) \equiv 0$. Note that the second equation expresses the orthogonality (complementarity) condition, which one meets explicitly, if the given constraining inequality is not turned into an equation. It is useful to analyse the reasons of such an impossibility. Since it is conceivable to think that the theoretical foundations of problems of type (3.1.1) be the same as problems having finite dimensional image, we analyse the image set. Since Euler equation is a necessary condition, the fact that it be not satisfied, leads one to think that − as it happens for problems having a finite dimensional image − the linearized problem be irregular (in the sense introduced in [21]); of course, here, by linearized problem we must mean the linearization of the selected problem, which now becomes:

$$\min \int_0^1 1 \cdot \delta x_1 dt = \int_0^1 x_1(t) dt,$$

$$\text{s.t.} \int_0^1 \omega(t) \left[3\bar{x}_1^2(t)\delta x_1 - 2\bar{x}_2(t)\delta x_2\right] dt = 0, \quad x_1, x_2 \in X, \ \omega \in \Omega.$$

Consequently, the linearization of the selected image becomes:

$$\mathcal{K}^l_{\bar{x}}(\omega) = \{(u, v) \in \mathbb{R}^2 : u = -\int_0^1 x_1(t)dt, \ v = \int_0^1 \omega(t)(0 \cdot \delta x_1 - 0 \cdot \delta x_2)dt,$$
$$x_1, x_2 \in X\}, \ \omega \in \Omega.$$

Therefore,

$$\mathcal{K}^l_{\bar{x}}(\omega) = \{(u, v) \in \mathbb{R}^2 : v = 0\}.$$

Since now

$$\mathcal{H} = \{(u, v) \in \mathbb{R}^2 : u > 0, v = 0\},$$

the equation of a separating hyperplane of $\mathcal{H}$ and $\mathcal{K}^l_{\bar{x}}(\omega)$, is necessarily $v = 0$. Hence the present problem is irregular.

Example 3.1.15. In (3.1.1) set $T = [0, 1]$, $p = 1$, $m = 2$, $n = 2$, $\psi_0(t, x(t), x'(t)) = -x_1^2(t) + x_2(t)$, $\psi_1(t, x(t), x'(t)) = x_1(t)$, $\psi_2(t, x(t), x'(t)) = x_2^3(t)$, $\overline{x}(t) = (0, 0)$, $\forall t \in T$, $X = \{x \in \mathbb{V} : x_i(0) = x_i(1) = 0, i = 1, 2\}$. We have

$$\mathcal{H} = (\mathbb{R}_+ \backslash \{O\}) \times \{O\} \times \mathbb{R}_+.$$

It will be shown that the Euler equation is not fulfilled. The given problem can be put in the form:

$$\min \int_T \left[-x_1^2(t) + x_2(t)\right] dt,$$

$$\text{s. t. } x_1(t) = 0,\ x_2^3(t) - x_3^2(t) = 0,\quad (x_1, x_2) \in X, x_3 \in C^0(T).$$

For $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$, the classic Lagrangian function becomes:

$$L(x_1, x_2, x_3; \lambda_1, \lambda_2, \omega_1, \omega_2) = \int_T \{-x_1^2(t) + x_2(t) - \lambda_1\omega_1(t)x_1(t) - \lambda_2\omega_2(t)\left[x_2^3(t) - x_3^2(t)\right]\}dt,$$

and the Euler equation (more precisely, system) is:

$$-2x_1(t) - \lambda_1\omega_1(t) = 0,\quad 1 - 3\lambda_2\omega_2(t)x_2^2(t) = 0,\quad 2\lambda_2\omega_2(t)x_3(t) = 0,$$

$t \in T$, $(x_1, x_2) \in X$, $x_3 \in C(T)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\omega_1, \omega_2 \in C^1(T)$, and it is obviously impossible at $x_1(t) = x_2(t) = x_3(t) \equiv 0$.

The linearization of the selected problem becomes:

$$\min \int_T [-2\bar{x}_1(t)\delta x_1 + 1 \cdot \delta x_2]dt,$$

$$\text{s.t.} \int_T \omega_1(t) \cdot 1 \cdot \delta x_1 dt = 0,\quad \int_T \omega_2(t)\left[3\bar{x}_2^2(t)\delta x_2 - 2\bar{x}_3(t)\delta x_3\right] dt = 0,$$

$$(x_1, x_2) \in X,\ x_3 \in C^0(T),\ \omega_1, \omega_2 \in \Omega.$$

Consequently, the linearization of the selected image becomes :

$$\begin{aligned} \mathcal{K}^l_{\bar{x}}(\omega) &= \{(u, v_1, v_2) \in \mathbb{R}^3 : u = -\int_T \delta x_2(t)dt, v_1 = \int_T \omega_1(t)\delta x_1 dt, \\ &\quad v_2 = 0, (x_1, x_2, x_3) \in X \times C^0(T)\},\quad \omega_1, \omega_2 \in \Omega. \end{aligned}$$

Therefore,

$$\mathcal{K}^l_{\bar{x}}(\omega) = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}.$$

Since now

$$\mathcal{H} = \{(u, v) \in \mathbb{R}^3 : u > 0, v_1 = v_2 = 0\},$$

the equation of a separating hyperplane of $\mathcal{H}$ and $\mathcal{K}^l_{\bar{x}}(\omega)$ is necessarily $v_2 = 0$. Hence, the present problem is irregular.

Example 3.1.16. In (3.1.1) set $T = [0, 1]$, $p = 0$, $m = 3$, $n = 2$, $\psi_0(t, x(t), x'(t)) = -x_1(t) + x_2(t)$, $\psi_1(t, x(t), x'(t)) = (1 - x_1(t))^3 - x_2(t)$, $\psi_2(t, x(t), x'(t)) = x_1(t)$, $\psi_3(t, x(t), x'(t)) = x_2(t)$, $\overline{x}(t) = (1, 0)$, $\forall t \in T$, $X = \{x \in \mathbb{V} : x_1(0) = x_1(1) = 1, x_2(0) = x_2(1) = 0\}$. It is easy to see that $\overline{x}$ is a minimum point for (3.1.1) (actually, $-x_1(t) + x_2(t) \geq -1$, $\forall x \in R$ and $t \in T$). It will be shown that the Euler equation is not fulfilled at $\overline{x}$. The Lagrangian function associated with (3.1.1) is $L(x_1, x_2; \lambda_1, \lambda_2, \lambda_3, \omega_1, \omega_2, \omega_3) :=$

$$\int_T \{-x_1(t) + x_2(t) - \lambda_1\omega_1(t)((1 - x_1(t))^3 - x_2(t)) - \lambda_2\omega_2(t)x_1(t) - \lambda_3\omega_3 x_2(t)\}dt,$$

and the Euler equation leads to the following system:

$$-1+3\lambda_1\omega_1(t)(1-x_1(t))^2-\lambda_2\omega_2(t)=0,\quad 1+\lambda_1\omega_1(t)-\lambda_3\omega_3(t)=0,$$

$$\lambda_1\omega_1(t)[-x_2(t)+(1-x_1(t))^3]=\lambda_2\omega_2(t)x_1(t)=\lambda_3\omega_3(t)x_2(t)=0,$$

$$\lambda_i,\omega_i(t)\geq 0,\ i=1,.,3,\quad t\in T,\quad (x_1,x_2)\in X,$$

which is impossible at $x_1(t)\equiv 1, x_2(t)\equiv 0$. The linearization of the selected problem is:

$$\min\{-1+\int_T[-1\cdot\delta x_1+1\cdot\delta x_2]dt\},$$

$$\text{s.t.}\int_T\omega_1(t)(0\cdot\delta x_1-1\cdot\delta x_2)dt=0,\quad \int_T\omega_2(t)(1+\delta x_1)dt\geq 0,\int_T\omega_3(t)\delta x_2 dt\geq 0,$$

$$x\in X,\ \omega\in\Omega:=\{(\omega_1,\omega_2,\omega_3)\in[C^1(T)]^3:\omega_i\geq 0, i=1,...,3\}.$$

The linearization of the selected image becomes :

$$\begin{aligned}\mathcal{K}^l_{\bar{x}}(\omega) \;=\; \Bigg\{&(u,v_1,v_2,v_3)\in\mathbb{R}^4: u=\int_T(x_1(t)-x_2(t))dt,\ v_1=-\int_T\omega_1(t)x_2dt,\\ &v_2=\int_T\omega_2(t)x_1dt,\ v_3=\int_T\omega_3(t)x_2dt,(x_1,x_2)\in X\Bigg\},\omega_1,\omega_2\in\Omega.\end{aligned}$$

We have

$$\mathcal{H}=(\mathbb{R}_+\backslash\{O\})\times\mathbb{R}^3_+.$$

The existence of a separating hyperplane between $\mathcal{H}$ and $\mathcal{K}^l_{\bar{x}}(\omega)$, of equation

$$\theta u+\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=0,\quad \theta,\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}_+,$$

lead us to obtain the following condition

$$\int_T[(\theta+\lambda_2\omega_2(t))x_1(t)+(-\theta-\lambda_1\omega_1(t)+\lambda_3\omega_3(t))x_2(t)]dt\leq 0,\quad \forall(x_1,x_2)\in X,$$

which implies

$$-\theta-\lambda_1\omega_1(t)+\lambda_3\omega_3(t)=0,\quad \theta+\lambda_2\omega_2(t)=0,\quad \forall t\in T.$$

By the second equation, we obtain that $\theta=0$ necessarily; therefore, the present problem is irregular.

3.1.4 Saddle point and sufficient conditions

We have shown that well-known results, such as the Euler equation, can be recovered assuming that the selection multipliers do not locally depend on x (Assumption A) and that are of class C^1. The examples of the previous section show that even elementary problems may escape from the classic Lagrange multipliers theory. In order to extend the validity of such a theory, it is possible to consider a wider class of *selection quasi-multipliers*(SQM), which depend also on the unknown. This enlargement is suggested by the Image Space Analysis of Sect. 3.1.1, which has led to split the multiplier into two parts: selection from a multifunction and separation of two sets. While the latter remains unchanged, the former can be extended, namely the SQM are given by the functions $\omega_i : T \times X \to \mathbb{R}$, $i \in \mathcal{I}$, such that $\omega_i(\cdot, x) \in \Omega, \forall x \in X$ (see Definition 3.1.1). Without any fear of confusion, the domain of $\omega(t, x)$ is denoted again by Ω.

Consider the Lagrangian function associated with the selected problem (3.1.2):

$$L(x; \lambda, \omega(t, x(t))) := \int_T \left(\psi_0(t, x(t), x'(t)) - \sum_{i=1}^{m} \lambda_i \, \omega_i(t, x(t)) \psi_i(t, x(t), x'(t)) \right) dt, \tag{3.1.-2}$$

which differs from that of Sect. 3.1.1 only because of the dependence of ω_i on x.

Theorem 3.1.17. *If there exist* $\overline{\lambda} \in D^*, \overline{\omega}(t, x) \in \Omega$, *such that:*

$$L(\overline{x}; \lambda, \overline{\omega}(t, x)) \leq L(\overline{x}; \overline{\lambda}, \overline{\omega}(t, x)) \leq L(x; \overline{\lambda}, \overline{\omega}(t, x)), \ \forall x \in X, \ \forall (\lambda, \omega) \in D^* \times \Omega, \tag{3.1.-2}$$

then $\overline{x}$ *is a global minimum point of (3.1.1).*

Proof. The first of (3.1.17) is equivalent to

$$\sum_{i=1}^{m} \int_T \lambda_i \omega_i(t, x) \psi_i(t, \bar{x}, \bar{x}') dt \geq \sum_{i=1}^{m} \int_T \bar{\lambda}_i \omega_i(t, \bar{x}) \psi_i(t, \bar{x}, \bar{x}') dt, \quad \forall (\lambda, \omega) \in D^* \times \Omega. \tag{3.1.-2}$$

Let us first prove that $\bar{x} \in R$. Ab absurdo, suppose that $\exists r \in \mathcal{I}$ and $\tilde{t} \in T$ such that

$$\text{either } \psi_r(t, \bar{x}, \bar{x}') \neq 0 \text{ if } r \in \mathcal{I}^0 \text{ or } \psi_r(t, \bar{x}, \bar{x}') < 0 \text{ if } r \in \mathcal{I}^+.$$

Since $\psi_r(t, \bar{x}, \bar{x}')$ is continuous, then is it possible to find $\tilde{\omega}_r(\cdot, \bar{x}) \in C^0(T)$ such that $\int_T \tilde{\omega}(t, \bar{x}) \psi_r(t, \bar{x}, \bar{x}') dt$ has the same sign as $\psi_r(t, \bar{x}, \bar{x}')$. Therefore, by setting $\tilde{\omega}(t, \bar{x}) = 0$, $\lambda_i = \bar{\lambda}_i$, $i \in \mathcal{I} \setminus r$, we have that $\tilde{\omega} := (0, ..., \tilde{\omega}_r, ..., 0) \in \Omega$; letting λ_r go to either $+\infty$ or $-\infty$, according to respectively $\psi_r(t, \bar{x}, \bar{x}') < 0$ or $\psi_r(t, \bar{x}, \bar{x}') > 0$, the left-hand side of (3.1.4) goes to $-\infty$ and contradicts (3.1.4). Hence $\bar{x} \in R$.

Next, we prove that

$$\bar{\lambda}_i \int_T \bar{\omega}_i(t,\bar{x})\psi_r(t,\bar{x},\bar{x}')dt = 0, \qquad \forall i \in \mathfrak{I}. \tag{3.1.-2}$$

Since $\bar{x} \in R$ and $(\bar{\lambda}, \bar{\omega}(t,\bar{x})) \in D^* \times \Omega$, then

$$\bar{\lambda}_i \int_T \bar{\omega}_i(t,\bar{x})\psi_r(t,\bar{x},\bar{x}')dt \geq 0, \qquad \forall i \in \mathfrak{I}. \tag{3.1.-2}$$

Moreover, by setting $\lambda = 0$ in (3.1.4), we obtain that

$$\sum_{i\in\mathfrak{I}} \bar{\lambda}_i \int_T \bar{\omega}_i(t,\bar{x})\psi_r(t,\bar{x},\bar{x}')dt \leq 0, \qquad \forall i \in \mathfrak{I}. \tag{3.1.-2}$$

By (3.1.4) we have that equality holds in (3.1.4) and thus (3.1.4) follows.

Taking into account (3.1.4), the 2nd of (3.1.17) becomes

$$\int_T \psi_0(t,x,x')dt \geq \int_T \psi_0(t,\bar{x},\bar{x}')dt + \sum_{i\in\mathfrak{I}} \int_T \bar{\lambda}_i\bar{\omega}(t,\bar{x})\psi_i(t,x,x')dt, \quad \forall x \in X,$$

so that the above inequality implies $f(x) \geq f(\bar{x})$ for each $x \in R$. □

Regarding the examples analysed in Section 3.1.3, we will show that, using quasi-multipliers, we can overcome the presence of a positive duality gap.

Definition 3.1.18. The *Lagrange dual* associated with (3.1.2) is defined by

$$\sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} L(x;\lambda;\omega).$$

The *duality gap*, provided that $\bar{x}$ is a global optimal solution to (3.1.2), will be

$$v := f(\bar{x}) - \sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} L(x;\lambda;\omega).$$

Continuation of Example 3.1.14. It is not difficult to show that in the hypothesis where the SQM are independent on x (Assumption A), the duality gap is $v = +\infty$. Assume now that the Lagrangian is defined by (3.1.4) so that the Lagrange dual is

$$\sup_{\lambda\in D^*} \sup_{\omega\in\Omega} \inf_{x\in X} \int_0^1 (x(t) - \lambda\omega(t,x)x^3(t))dt,$$

where $D^* = \mathbb{R}_+$ and $\Omega = \{\omega : \omega(\cdot,x) \in C^1(T),\ \omega(t,x) \geq 0, \forall x \in X, \forall t \in T\}$.

If we consider the function

$$\bar{\omega}(t,x) = \begin{cases} -x^3 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases},$$

which belongs to Ω, then it is easy to show that:

$$\sup_{\lambda \in D^*} \inf_{x \in X} \int_0^1 (x(t) - \lambda\bar{\omega}(t,x)x^3(t))dt = 0$$

and thus the duality gap becomes 0. □

The image set of (3.1.1) has been defined through the point-to-set map $A_{\bar{x}}$ of Section 3.1.1. Such a definition is a quite natural extension of that for extremum problems with finite dimensional image. However, the definition of Section 3.1.1 takes into account ψ_0 only through an integration, so that a full information about ψ_0 is lost. Even if for certain purposes – like those of the previous sections – the definition of Section 3.1.1 is enough, it is useful to consider a more general definition. To this end, consider the vectors (of the IS)

$$a(t,x,\xi) := (u = \psi_0(t,\bar{x},\bar{x}') - \psi_0(t,x,\xi), v_i = \psi_i(t,x,\xi), i \in \mathcal{I}) \in \mathbb{R}^{1+m},$$

$$t \in T,\ x \in X,\ \xi \in \Xi, \qquad (3.1.\text{-}2)$$

where Ξ is a superset, say X', of the set of values of x'; then the map $A_{\bar{x}}$ of Section 3.1.1 can be replaced by

$$A_{\bar{x}}(x,\xi) := \{a(t,x,\xi) : t \in T\}, \quad x \in X, \quad \xi \in \Xi.$$

$$\mathcal{K}^f_{\bar{x}} := \{A_{\bar{x}}(x,x') : x \in X\} \text{ and } \mathcal{K}^e_{\bar{x}} := \{A_{\bar{x}}(x,\xi) : x \in X, \xi \in \Xi\} \qquad (3.1.\text{-}2)$$

are the *full image set* of (3.1.1) and an *embedding* of it, respectively. The study of properties of (3.1.4) is of fundamental importance; indeed, through the analysis of (3.1.4), it should be possible to extend known results. To this end, consider the vector function:

$$E(t,x,x',\xi) := a(t,\bar{x},\xi) - a(t,\bar{x},\bar{x}') - a'(t,\bar{x},\bar{x}')(\xi - \bar{x}'), \qquad (3.1.\text{-}2)$$

where a' denotes Jacobian with respect to x'. When $m = 0$ (so that (3.1.1) is the most classic fixed endpoint problem), to within obvious transformation, (3.1.4) is the Weierstrass excess function [6], which enjoys the well known property to be non-positive when $\bar{x}$ is a minimum point. Besides the study of (3.1.4) when $m > 0$, it would be interesting to analyse the properties of the sets

$$A_{\bar{x}}(x,\xi(\alpha)) - \left[(1-\alpha)A_{\bar{x}}(x,\xi^1) + \alpha A_{\bar{x}}(x,\xi^2)\right],\ \alpha \in [0,1],$$

where $\xi^1, \xi^2 \in \Xi$ and $\xi(\alpha) := (1-\alpha)\xi^1 + \alpha\xi^2$.

$A_{\bar{x}}(x,x')$ is an arc. It is useful to state the most general conditions under which it is regular in the sense of Jordan, namely it is a homeomorphism (on int T) and

$a(\cdot, x, x')$ has continuous prime derivatives, which do not vanish simultaneously. The same question holds for $A_{\bar{x}}(x, \xi)$ too.

The selected image of Definition 3.1.4 is obtained by projecting $A_{\bar{x}}(x, x')$ on the hyperplane (of $\mathbb{R}^{1+m}$), whose equation is:

$$u = \int_T [\psi_0(t, \bar{x}(t), \bar{x}'(t)) - \psi_0(t, x(t), x'(t))] dt,$$

by selecting an element from such a projection, and then by performing their union with respect to $x \in X$.

In Section 3.1.1 it has been shown that the selected image can be extended without changing the optimality of the minimum points and the minimum (if any); such changes imply changes in the data of (3.1.1), namely $X, \psi_0, ..., \psi_m$, which will be interesting to study and to see the numerical consequences. A more general framework is obtained, if the extensions are performed before selecting. Note that, for this purpose, it is not necessary to consider the full image with respect to ψ_0. Therefore, we can select with respect to ψ_0 and replace (3.1.4) with

$$a(t, x, \xi) := (u = f(\bar{x}) - f(x), v_i = \psi_i(t, x, \xi), i \in \mathcal{I}), \tag{3.1.-2}$$

and, consequently, a is replaced also in $A_{\bar{x}}$ and in (3.1.4). Now, consider any $A_{\bar{x}}(\hat{x}, \hat{x}') \subset \mathcal{H}$ and let $u = \hat{u}$ be the hyperplane which contains it; consider any arc, say γ, of $\mathbb{R}^{1+m}$ which lies in the hyperplane $u = \tilde{u} \leq \hat{u}$; if $A_{\bar{x}}(\hat{x}, \hat{x}') \not\subseteq \mathcal{H}$, then also $\gamma \not\subseteq \mathcal{H}$. If the full image set is extended to embrace γ, then the optimality of $\bar{x}$ is not modified. Now, we have to search for changes in the ψ_i's and for an $\hat{x}$, such that the image of it be γ.

The full image set should be the root for giving a general necessary optimality condition. The first step should be to consider the homogenized problem by following the scheme of Section 3.1.2 and then derive necessary and/or sufficient optimality conditions by means of separation arguments.

3.2 Necessary and Sufficient Condition for the Existence of Lagrange Multipliers

Let us now come to a general extremum problem having infinite dimensional image. Assume that X is a real linear topological space, S a nonempty subset of X, Y and Z be real normed space and C a nonempty, closed, convex and pointed cone of Y with apex at the origin and which partially orders Y. Let be $f : S \to \mathbb{R}$ and $g : S \to Z \times Y$, with $g = (g_1, g_2)$. Consider the following primal optimization problem

$$\inf f(x) \tag{3.2.1}$$

$$\text{subject to} \qquad g_1(x) = O \qquad , \qquad (3.2.1)$$

$$g_2(x) \in C. \qquad (3.2.1)$$

We make the assumptions that the feasible region

$$R := \{x \in S : g(x) \in D\}$$

is nonempty, where $D = O_Z \times C$ is a closed convex cone.

Let be $\bar{x} \in R$ and let us introduce the sets

$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times Z \times Y : u > 0, v \in D\},$$

$$\mathcal{H}_u := \{(u, v) \in \mathbb{R} \times Z \times Y : u > 0, v = 0\},$$

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R} \times Z \times Y : u = f_{\bar{x}}(x) = f(\bar{x}) - f(x), v = g(x), x \in S\}$$

which is the *image* of the problem (3.2.1). Its *conic extension* is defined as

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) := \mathcal{K}_{\bar{x}} - \text{cl } \mathcal{H}.$$

The space $\mathbb{R} \times Z \times Y$, where both $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$ lay, is the *image space* and in this case and we notice that it is infinite dimensional.

The Lagrange dual problem associated to (3.2.1) will be

$$\sup_{\lambda \in D^*} \inf_{x \in S} [f(x) - \langle \lambda, g(x) \rangle], \qquad (3.2.1)$$

where

$$D^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in D\} = Z^* \times C^*$$

is the *dual cone* of D and Z^* is the topological dual space of Z. Let us denote by $v(P^e)$ and $v(D_L^e)$ the optimal objective values of the primal and the dual problem, respectively. Weak duality always holds, that is

$$v(D_L^e) \leq v(P^e).$$

We say that strong duality holds, if

$$v(P^e) = v(D_L^e)$$

and (D_L^e) has an optimal solution.

Before stating the announced result we prove another auxiliary lemma.

Lemma 3.2.1. *For* $a \in \mathbb{R} \times Z^* \times C^* \setminus \{(0, O_{Z^* \times C^*})\}$, *let*

$$H = \{t \in \mathbb{R} \times Z \times Y : \langle a, t \rangle = 0\}$$

be a hyperplane in $\mathbb{R} \times Z \times Y$. *Let be* $K \subseteq \mathbb{R} \times Z \times Y$. *The following statements are equivalent:*

(i) H separates the sets K and $\mathcal{H}$;

(ii) H separates the sets $\mathcal{E}(K)$ and $\mathcal{H}$;

(iii) H separates the sets (conv $\mathcal{E}(K)$) and $\mathcal{H}$;

(iv) H separates TC(conv $(\mathcal{E}(K))$) and $\mathcal{H}$;

(v) H separates TC(conv $(\mathcal{E}(K))$) and $\mathcal{H}_u$.

Proof. (i)$\Rightarrow$(ii) Assume that

$$\langle a,t\rangle \geq 0,\ \forall\, t\in\mathcal{H} \text{ and } \langle a,t\rangle \leq 0,\ \forall\, t\in K.$$

This means that $\mathcal{H}\subseteq H^+$ and $K\subseteq H^-$, where by H^+ and H^- we denote the half-spaces $\{t\in\mathbb{R}\times Z\times Y : \langle a,t\rangle\geq 0\}$ and $\{t\in\mathbb{R}\times Z\times Y : \langle a,t\rangle\leq 0\}$, respectively. We prove that actually $\mathcal{E}(K)\subseteq H^-$.

To this end we suppose that $\exists\hat{t}\in\mathcal{E}(K)$ such that $\langle a,\hat{t}\rangle>0$. Since $\hat{t}\in\mathcal{E}(K)=K-\text{cl}\,\mathcal{H}$ we get the existence of $t^1\in K$ and $t^2\in\text{cl}\,\mathcal{H}$ such that $\hat{t}=t^1-t^2$. From $\langle a,\hat{t}\rangle>0$ we have

$$0\leq\langle a,t^2\rangle<\langle a,t^1\rangle\leq 0,$$

where the third inequality comes from $K\subseteq H^-$ and the first one from the fact that cl $\mathcal{H}\subseteq H^+$, which is an easy consequence of $\mathcal{H}\subseteq H^+$. We get a contradiction and this proves the first implication.

(ii)$\Rightarrow$ (iii) Let be $t\in$ conv $\mathcal{E}(K)$, i.e. there exists $t_1,...,t_{s+1}\in\mathcal{E}(K)$, $\alpha_1,...,\alpha_{s+1}\geq 0$ with $\sum\limits_{i=1}^{s+1}\alpha_i=1$, such that $t=\sum\limits_{i=1}^{s+1}\alpha_i t_i$, where $s=\dim\text{aff}\mathcal{E}(K)$. Since $\mathcal{E}(K)\subseteq H^-$ it derives that $\sum\limits_{i=1}^{s+1}\alpha_i\langle a,t_i\rangle=\langle a,t\rangle\geq 0$ and the conclusion follows immediately.

(iii)$\Rightarrow$(iv) Assuming now that $\mathcal{H}\subseteq H^+$ and conv $(\mathcal{E}(K))\subseteq H^-$, it follows that $TC(\text{conv }(\mathcal{E}(K)))=\text{cl}\,\text{cone}\,\text{conv }(\mathcal{E}(K))\subseteq\text{cl}\,\text{cone}\,H^-=H^-$ and this gives (iv).

(iv)$\Rightarrow$(v) Follows automatically using that $\mathcal{H}_u\subseteq\mathcal{H}$.

(v)$\Rightarrow$(i) We assume that

$$\langle a,t\rangle\geq 0,\ \forall t\in\mathcal{H}_u \text{ and } \langle a,t\rangle\leq 0,\ \forall t\in TC(\text{conv }(\mathcal{E}(K)))$$

and we prove the inclusion $\mathcal{H}\subseteq H^+$. If this is not the case, then there exists $\hat{t}\in\mathcal{H}$ such that $\langle a,\hat{t}\rangle<0$.

Consider an element $\bar{t}\in\mathcal{E}(K)$. Then $\forall\alpha\geq 0$, we have $\bar{t}-\alpha\hat{t}\in\mathcal{E}(K)\subseteq TC(\mathcal{E}(K))$. Further, it holds

$$\lim_{\alpha\to+\infty}\langle a,\bar{t}-\alpha\hat{t}\rangle=+\infty,$$

but this is a contradiction to $\langle a,t\rangle \le 0,\ \forall t \in TC(\mathcal{E}(K))$. This means that $\mathcal{H} \subseteq H^+$. Since $K \subseteq \mathcal{E}(K) \subseteq TC(\mathcal{E}(K)) \subseteq H^-$, the conclusion follows. □

We will show now that the regularity condition necessary and sufficient for optimality and for the existence of Lagrange multipliers, and thus having a zero duality gap between the primal problem and its Lagrange dual, that was used when we dealt with problems having finite dimensional image, will be equivalent to the optimality to the existence of the Lagrange multipliers also in the case of the general problem (3.2.1).

Theorem 3.2.2. *Suppose that $\bar{x} \in R$ is a feasible point of the problem (3.2.1). Then*

$$TC(conv\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))) \cap \mathcal{H}_u = \varnothing \tag{3.2.1}$$

holds if and only if $v(P^e) = v(D^e_L)$ and $\exists\bar{\lambda} \in D^*$ such that $\bar{\lambda}$ is an optimal solution of the dual. In this situation we have $\langle\bar{\lambda}, g(\bar{x})\rangle = 0$.

Proof. Only if. Suppose that $TC(\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))) \cap \mathcal{H}_u = \varnothing$. This implies that $\exists h \in \mathcal{H}_u \setminus TC(\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}})))$. Since $TC(\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}})))$ is a closed and convex set, by a separation theorem (see Theorem 3.4 in [67]) we get the existence of $a = (\theta, \lambda) \neq (0, O_{Z^*\times C^*})$ such that

$$\langle a,h\rangle > 0 \ge \langle a,t\rangle,\ \forall t \in TC(\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))).$$

Since $\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}})) \subseteq TC(\text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}})))$ it follows $\langle a,t\rangle \le 0,\ \forall t \in \text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))$.

Further we prove that $\langle a,t\rangle \ge 0,\ \forall t \in \mathcal{H}$. To this aim, assume that there exists $\hat{t} \in \mathcal{H}$ such that $\langle a,\hat{t}\rangle < 0$. Let be $\bar{t} \in \text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))$ fixed. Then we have that $\bar{t} - \alpha\hat{t} \in \text{conv}\ (\mathcal{E}(\mathcal{K}_{\bar{x}}))$, $\forall\alpha \ge 0$. But the fact that

$$\lim_{\alpha\to+\infty} \langle a, \bar{t} - \alpha\hat{t}\rangle = +\infty$$

leads to contradiction. Thus, by Lemma 3.2.1 ((iii)⇒(i)), we obtain that

$$\langle a,t\rangle \ge 0, \forall t \in \mathcal{H} \text{ and } \langle a,t\rangle \le 0, \forall t \in \mathcal{K}_{\bar{x}}.$$

This means that the hyperplane $H = \{t \in \mathbb{R}\times Z \times Y : \langle a,t\rangle = 0\}$ separates the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$. From the above inequalities we get

$$\theta u + \langle\lambda, v\rangle \ge 0,\ \forall(u,v) \in \mathcal{H} \tag{3.2.1}$$

and

$$\theta(f(\bar{x}) - f(x)) + \langle\lambda, g(x)\rangle \le 0, \forall x \in S. \tag{3.2.1}$$

Relation (3.2) implies $\lambda \in D^*$ and $\theta \ge 0$.

Now let us assume that $\theta = 0$. This would mean that $\langle a, f\rangle = 0$, $\forall f \in \mathcal{H}_u$, but this is a contradiction to the fact that there exists $h \in \mathcal{H}_u$ such that $\langle a, h\rangle > 0$. Therefore we have necessarily $\theta > 0$.

From (3.2) we obtain

$$f(x) - f(\bar{x}) - \langle\bar{\lambda}, g(x)\rangle \geq 0,\ \forall x \in S,$$

where $\bar{\lambda} = (1/\theta)\lambda \in D^*$. Taking $x = \bar{x}$ in the above inequality, we have $\langle\bar{\lambda}, g(\bar{x})\rangle \leq 0$ and since $g(\bar{x}) \in D$ and $\bar{\lambda} \in D^*$ we get

$$\langle\bar{\lambda}, g(\bar{x})\rangle = 0.$$

Thus

$$v(P^e) = f(\bar{x}) = \inf_{x\in S}[f(x) - \langle\bar{\lambda}, g(x)\rangle] = v(D_L^e)$$

and $\bar{\lambda}$ is an optimal solution of the dual.

If. Suppose that $\exists\bar{\lambda} \in D^*$ such that $\bar{\lambda}$ is an optimal solution of the dual (D_L^e) and $v(P^e) = v(D_L^e)$. Then $\langle\bar{\lambda}, g(\bar{x})\rangle = 0$ is an easy consequence of this fact. In this way we obtain

$$f(x) - f(\bar{x}) + \langle\bar{\lambda}, g(x)\rangle\rangle \geq 0,\ \forall x \in S$$

and so the hyperplane $H = \{(r, y) \in \mathbb{R} \times Z \times Y : r + \langle\bar{\lambda}, y\rangle = 0\}$ separates the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, namely $\mathcal{K}_{\bar{x}} \subseteq H^-$ and $\mathcal{H} \subseteq H^+$. By Lemma 3.2.1 ($(i) \Rightarrow (v)$) we get that

$$\mathcal{H}_u \subseteq H^+ \text{ and } TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}}))) \subseteq H^-.$$

On the other hand, assume that $TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}}))) \cap \mathcal{H}_u \neq \varnothing$, i.e. $\exists(\tilde{t}, O_y, O_Z) \in TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}})))$, with $\tilde{t} > 0$. The set $TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}})))$ being a cone, it follows that $\mathcal{H}_u \subseteq TC(\text{conv }(\mathcal{E}(\mathcal{K}_{\bar{x}})))$. Then we obtain $\mathcal{H}_u \subseteq H^-$ and so $\mathcal{H}_u \subseteq H$, which is a contradiction. □

3.3 Generalised Slater Constraint Qualification

In many theoretical and practical infinite-dimensional convex optimization problems, the interior conditions are useless since for instance, the interior of the set involved in the regularity condition is empty. This is the case, for example, when dealing with the positive cones l^p_+ and $L^p_+(T, \mu)$ of the spaces l^p and $L^p(T, \mu)$, respectively, where (T, μ) is a σ-finite measure space and $p \in [1, \infty)$. For these two cones even the strong quasi-relative interior (which is the the weakest generalised interior notion among other generalised interior notions, such as interior, relative interior, quasi interior) is empty. In order two overcome such a situation Borwein and Lewis introduced in [10] the notion of *quasi-relative interior* of a convex set, which is a further generalisation of the above mentioned interior notions. They also proved that the quasi-relative interiors of l^p_+ and $L^p_+(T,\mu)$ are nonempty.

The number of papers dealing with regularity conditions for convex optimization problem with cone (and equality) constraints in infinite dimensional spaces, formulated by using the quasi-relative interior, is not very large. An important contribution in this field is the paper of Jeyakumar and Wolkowicz [36], despite its drawback that the cone defining the constraints is assumed to have a nonempty interior. But lately we noticed an increasing number of papers on this topic which try to overcome this fact, like [13], [15] and [16].

In our paper we discuss and improve the duality results given in the aforementioned papers. We give a general strong duality theorem, the regularity condition of which being expressed by using the *quasi-relative interior* and/or the *quasi-interior* of the sets involved. We illustrate the theoretical considerations by some examples.

3.3.1 Quasi-relative interior of convex sets

Consider X a real normed space and X^* its continuous dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$.

Definition 3.3.1. Let C be a convex subset of X. The *quasi-interior* of C is the set

$$\operatorname{qi} C = \{x \in C : \operatorname{cl}\operatorname{cone}(C - x) = X\}.$$

We have the following characterisation of the quasi-interior of a convex set.

Proposition 3.3.2. *([16]) Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qi} C$ if and only if $NC(x;C) = \{O_{X^*}\}$.*

The following notion is a refinement of the quasi-interior and is due to Borwein and Lewis ([10]).

Definition 3.3.3. *([10])* Let C be a convex subset of X. The *quasi-relative interior* of C is the set

$$\operatorname{qri} C = \{x \in C : \operatorname{cl}\operatorname{cone}(C - x) \text{ is a linear subspace of } X\}.$$

Proposition 3.3.4. *([10]) Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qri} C$ if and only if $NC(x;C)$ is a linear subspace of X^*.*

It follows from the definitions above that $\operatorname{qi} C \subseteq \operatorname{qri} C$ and $\operatorname{qri}\{x\} = \{x\}$, $\forall x \in X$. Also, if $\operatorname{qi} C \neq \varnothing$, then $\operatorname{qi} C = \operatorname{qri} C$ (cf. [47]). If X is a finite dimensional space, then $\operatorname{qi} C = \operatorname{int} C$ (cf. [47]) and $\operatorname{qri} C = \operatorname{ri} C$ (cf. [10]), where $\operatorname{ri} C$ is the relative interior of C. In the following proposition we give some useful properties of the quasi-relative interior.

Proposition 3.3.5. *([9], [10]) Let us consider C and D two convex subsets of X, $x \in X$ and $\alpha \in \mathbb{R}$. Then:*

(i) $\operatorname{qri} C + \operatorname{qri} D \subseteq \operatorname{qri}(C + D)$;

(ii) $\operatorname{qri}(C \times D) = \operatorname{qri} C \times \operatorname{qri} D$;

(iii) $\operatorname{qri}(C - x) = \operatorname{qri} C - x$;

(iv) $\operatorname{qri}(\alpha C) = \alpha \operatorname{qri} C$;

(v) $t \operatorname{qri} C + (1-t)C \subseteq \operatorname{qri} C$, $\forall t \in (0,1]$, *hence* $\operatorname{qri} C$ *is a convex set;*

(vi) if C is an affine set then $\operatorname{qri} C = C$;

(vii) $\operatorname{qri}(\operatorname{qri} C) = \operatorname{qri} C$.

If $\operatorname{qri} C \neq \varnothing$ *then:*

(viii) $\operatorname{cl} \operatorname{qri} C = \operatorname{cl} C$;

(ix) $\operatorname{cl} \operatorname{cone} \operatorname{qri} C = \operatorname{cl} \operatorname{cone} C$.

Proof. For the proof of (i)-(viii) we refer to [9] and [10] for more details.
(ix) The inclusion $\operatorname{cl} \operatorname{cone} \operatorname{qri} C \subseteq \operatorname{cl} \operatorname{cone} C$ is always true. We prove that $\operatorname{cone} C \subseteq \operatorname{cl} \operatorname{cone} \operatorname{qri} C$. Consider $x \in \operatorname{cone} C$ arbitrary. There exist $\lambda \geq 0$ and $c \in C$ such that $x = \lambda c$. Take $x_0 \in \operatorname{qri} C$. Using the property (v), we obtain $tx_0 + (1-t)c \in \operatorname{qri} C$, $\forall t \in (0,1]$, so

$$\lambda t x_0 + (1-t)x = \lambda(tx_0 + (1-t)c) \in \operatorname{cone} \operatorname{qri} C, \ \forall t \in (0,1].$$

Passing to the limit as $t \searrow 0$ we get $x \in \operatorname{cl} \operatorname{cone} \operatorname{qri} C$ and the conclusion follows. □

We come now to a lemma which will prove to be useful in the following.

Lemma 3.3.6. *Let A and B be nonempty convex subsets of X such that $A \cap B \neq \varnothing$. If $O_X \in \operatorname{qi}(A - A)$ and $B \cap \operatorname{qri} A \neq \varnothing$, then $O_X \in \operatorname{qi}(A - B)$.*

Proof. Take $x \in B \cap \operatorname{qri} A$ and let $x^* \in NC(O_X; A - B)$ be arbitrary. We get $\langle x^*, a - b\rangle \leq 0, \forall a \in A, \forall b \in B$. Then

$$\langle x^*, a - x\rangle \leq 0, \forall a \in A \tag{3.3.0}$$

that is $x^* \in NC(x; A)$. As $x \in \operatorname{qri} A$, $NC(x; A)$ is a linear subspace of X^*, hence $-x^* \in NC(x; A)$, which is nothing else than

$$\langle x^*, x - a\rangle \leq 0, \forall a \in A. \tag{3.3.0}$$

The relations (3.3.1) and (3.3.1) give us $\langle x^*, a' - a''\rangle \leq 0, \forall a', a'' \in A$, so

$$x^* \in NC(O_X; A - A).$$

Since $O_X \in \operatorname{qi}(A - A)$ we have $NC(O_X; A - A) = \{O_{X^*}\}$ (cf. Proposition 3.3.2) and we get $x^* = O_{X^*}$. As x^* was arbitrary chosen we obtain

$$NC(A - B)(O_X) = \{O_{X^*}\}$$

and, using again Proposition 3.3.2, the conclusion follows. □

A classic separation theorem in locally convex spaces is the following theorem which has the inconvenience of asking the interior of one of the sets to be nonempty.

Theorem 3.3.7. *[5] Let be X a locally convex space and A and B are two nonempty convex subsets. If int $A \neq \varnothing$ and is disjoint from B then the sets are linearly separable.*

We will overcome the lack of a nonempty interior by giving now some separation theorems in terms of the quasi-relative interior.

Theorem 3.3.8. *([15], [16]) Let C be a convex subset of X and $x_0 \in C \setminus \operatorname{qri} C$. Then there exists $x^* \in X^*, x^* \neq O_{X^*}$ such that*

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle,\ \forall x \in C.$$

Viceversa, if there exists $x^* \in X^*$, $x^* \neq O_{X^*}$ such that

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle,\ \forall x \in C$$

and

$$\operatorname{cl}(TC(x_0; C) - TC(x_0; C)) = X,$$

then $x_0 \in C \setminus \operatorname{qri} C$.

Remark 3.3.9. The condition $\mathrm{cl}(TC(x_0;C)-TC(x_0;C)) = X$ in the above theorem can be reformulated as follows: $\mathrm{cl\,cone}(C-C) = X$ or, equivalently, $O_X \in \mathrm{qi}(C-C)$. Indeed, we have

$$\mathrm{cl}[\mathrm{cl\,cone}(C-x_0) - \mathrm{cl\,cone}(C-x_0)] = X \Leftrightarrow \mathrm{cl}[\mathrm{cone}(C-x_0) - \mathrm{cone}(C-x_0)] = X$$

$$\Leftrightarrow \mathrm{cl\,cone}(C-C) = X \Leftrightarrow O_X \in \mathrm{qi}(C-C),$$

where we used the following properties: $\mathrm{cl}(\mathrm{cl}\,E + \mathrm{cl}\,F) = \mathrm{cl}(E+F)$, for arbitrary sets E, F in X and $\mathrm{cone}\,A - \mathrm{cone}\,A = \mathrm{cone}(A-A)$, if A is a convex subset of X such that $0 \in A$.

The condition $x_0 \in C$ in Theorem 3.3.8 is essential (see [16]). However, if x_0 is an arbitrary element in X, we can give also a separation theorem using the following result due to Cammaroto and Di Bella (Theorem 2.1 in [13]).

Theorem 3.3.10. *([13]) Let S and T be nonempty convex subsets of X with $\mathrm{qri}\,S \neq \emptyset$, $\mathrm{qri}\,T \neq \emptyset$ and such that $\mathrm{cl\,cone}(\mathrm{qri}\,S - \mathrm{qri}\,T)$ is not a linear subspace of X. Then, there exists $x^* \in X^*$, $x^* \neq O_{X^*}$, such that $\langle x^*, s\rangle \leq \langle x^*, t\rangle$ for all $s \in S$, $t \in T$.*

The following result is a direct consequence of Theorem 3.3.10.

Corollary 3.3.11. *Let C be a convex subset of X such that $\mathrm{qri}\,C \neq \emptyset$, and $\mathrm{cl\,cone}(C-x_0)$ is not a linear subspace of X, where $x_0 \in X$. Then there exists $x^* \in X^*, x^* \neq O_{X^*}$ such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$, $\forall x \in C$.*

Proof. We apply Theorem 3.3.10 with $S := C$ and $T := \{x_0\}$. Then we use Proposition 3.3.5 (iii) and (ix) to obtain the conclusion. □

3.3.2 Slater type CQ for the Problem with Cone Constraints

In the following we deal with the convex optimization problem

$$\inf f(x), \tag{3.3.1}$$

$$\text{subject to} \quad g(x) \in D, \tag{3.3.1}$$

$$x \in S. \tag{3.3.1}$$

The feasible set

$$R = \{x \in S : g(x) \in D\},$$

expressed here only by means of cone constraints, is assumed to be nonempty. The spaces X and Y, the sets S and D and the functions f and g are considered like in the previous subsection. if an affine linear mapping and the function $(f,-g) : S \to \mathbb{R} \times Y$, defined by $(f,-g)(x) = (f(x), -g(x)), \forall x \in S$, is $\mathrm{cl}\,\mathcal{H}$ convex-like, that is the set $(f, -g)(S) + \text{cl}\,\mathcal{H}$ is convex. Let us notice that this property implies that the sets $f(S) + [0, \infty)$ and $g(S) - D$ are convex (the reverse implication does not always hold).

The Lagrange dual problem associated to (3.3.1) is having the following formulation

$$(D_L) \quad \sup_{\lambda \in D^*} \inf_{x \in S} [f(x) - \langle \lambda, g(x) \rangle]. \tag{3.3.1}$$

Our intention is to give more general strong duality theorems than the ones given in [16] and [13]. We mention first that under the convexity assumptions stated for (3.3.1) only assuming that $\text{qri}\, D \neq \varnothing$, $\text{cl}(D - D) = Y$ and the existence of $\tilde{x} \in S$ with $g(\tilde{xl}) \in \text{qri}\, D$ is not enough for having strong duality between (3.3.1) and (3.3.2). This follows also from the following example, which was given by Daniele and Giuffrè in [15].

Example 3.3.12. Let be $X = S = Y = l^2$, the Hilbert space consisting of all sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\sum\limits_{n=1}^{\infty} x_n^2 < \infty$ and $D = l^2_+ = \{x = (x_n)_{n \in \mathbb{N}} \in l^2 : x_n \geq 0, \forall n \in \mathbb{N}\}$, the positive cone of l^2. Take $f : l^2 \to \mathbb{R}$, $f(x) = \langle c, x \rangle$, where $c = (c_n)_{n \in \mathbb{N}}, c_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$ and $g : l^2 \to l^2, g(x) = Ax$, where $(Ax)_n = \frac{1}{2^n} x_n$, $\forall n \in \mathbb{N}$. Then $R = \{x \in l^2 : Ax \in l^2_+\} = l^2_+$. It holds $\text{cl}(l^2_+ - l^2_+) = l^2$ and $\text{qri}\, l^2_+ = \{x = (x_n)_{n \in \mathbb{N}} \in l^2 : x_n > 0,\ \forall n \in \mathbb{N}\} \neq \varnothing$ (cf. [10]) and one can easily find an $\tilde{x} \in l^2$ with $g(\tilde{x}) \in \text{qri}\, l^2_+$. We also have that

$$v(P) = \inf_{x \in T} \langle c, x \rangle = 0$$

and $x = O_{l^2}$ is an optimal solution of the primal problem. On the other hand, for $\lambda \in D^* = l^2_+$, it holds

$$\inf_{x \in S} [f(x) - \langle \lambda, g(x) \rangle] = \inf_{x \in l^2} [\langle c, x \rangle - \langle \lambda, g(x) \rangle]$$

$$\inf_{x = (x_n)_{n \in \mathbb{N}} \in l^2} \left(\sum_{n=1}^{\infty} \frac{1}{n} x_n - \sum_{n=1}^{\infty} \lambda_n \frac{1}{2^n} x_n \right) = \inf_{x = (x_n)_{n \in \mathbb{N}} \in l^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{\lambda_n}{2^n} \right) x_n$$

$$= \begin{cases} 0, & \text{if } \lambda_n = \frac{2^n}{n}, \forall n \in \mathbb{N}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Since $(\frac{2^n}{n})_{n \in \mathbb{N}}$ does not belong to l^2, we obtain $v(D_L) = -\infty$, hence the optimal objective values of the two problems do not coincide.

This means that along the Slater-type regularity conditions one needs to make some supplementary assumptions for ensuring strong duality. Since the regularity condition (3.2.2) is a necessary and sufficient condition, it could be of interest to give weaker regularity conditions expressed via the quasi-relative interior which prove to be (only) sufficient for having strong duality between (3.3.1) and its Lagrange dual.

Some properties of the set $g(S) - D$ are given below in the next lemma.

Lemma 3.3.13. *Suppose that* $\operatorname{cl}(D - D) = Y$ *and* $\exists \tilde{x} \in S$ *such that* $g(\tilde{x}) \in \operatorname{qri} D$. *Then the following are true:*

(a) $O_Y \in \operatorname{qi}(g(S) - D)$;

(b) $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) - D)]$ *is a linear subspace of* Y.

Proof. (a) We apply Lemma 3.3.6 with $A := D$ and $B := g(S) - D$. The sets A and B are convex and we have $O_Y \in A \cap B$. The condition $\operatorname{cl}(D - D) = Y$ implies

$$O_Y \in \operatorname{qi}(A - A),$$

while the Slater-type condition gives us

$$g(\tilde{x}) \in B \cap \operatorname{qri} A.$$

Hence, by Lemma 3.3.6 we obtain $O_Y \in \operatorname{qi}(A - B)$, that is

$$O_Y \in \operatorname{qi}(-g(S) + D),$$

which is nothing else than $O_Y \in \operatorname{qi}(g(S) - D)$.

(b) From (a) it follows that $O_Y \in \operatorname{qri}(g(S) - D)$. Applying Proposition 3.3.5 (vii) we get

$$O_Y \in \operatorname{qri}(\operatorname{qri}(g(S) - D)),$$

which means that $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) - D)]$ is a linear subspace of Y. □

In the following we give a strong duality theorem for (3.3.1) and its Lagrange dual (3.3.2) under a weak regularity condition expressed by using the quasi-relative interior of the sets involved. Different to the similar attempts in [13] and [16], we do not assume that the primal problem has an optimal solution. This situation will be treated in a corollary which will follow our main result.

Since in case $v(P) = -\infty$, strong duality obviously holds, for the rest of the paper we consider that $v(P) \in \mathbb{R}$.

The conic extension for (3.3.1) looks now like

$$\begin{aligned} \mathcal{E}(\mathcal{K}_{v(P)}) &= \{(v(P) - f(x) - \alpha, g(x) - y) : x \in S, \alpha \geq 0, y \in D\} \\ &= \mathcal{K}_{v(P)} - \operatorname{cl} \mathcal{H}, \end{aligned}$$

where with $\mathcal{K}_{v(P)}$ we denoted the set

$$\mathcal{K}_{v(P)} := \{(u,v) \in \mathbb{R} \times Y : u = v(P) - f(x), v = g(x), x \in S\}.$$

The conic extension $\mathcal{E}(\mathcal{K}_{v(P)})$ is also in this case a convex set fulfilling $(0, O_Y) \in \mathcal{E}(\mathcal{K}_{v(P)})$ if and only if the primal problem (P) has an optimal solution.

We prove first some preliminary results.

Lemma 3.3.14. *The following statements are true:*

(i) if $g(x_0) - y_0 \in \operatorname{qri}(g(S) - D)$ *then* $(v(P) - f(x_0) - t, g(x_0) - y_0) \in \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)})$, $\forall t > 0$*;*

(ii) if $(r_0, y_0) \in \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)})$ *then* $y_0 \in \operatorname{qri}(g(S) - D)$*;*

(iii) $\operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)}) \neq \varnothing$ *if and only if* $\operatorname{qri}(g(S) - D) \neq \varnothing$*.*

Proof. (a) Let us suppose that $g(x_0) - y_0 \in \operatorname{qri}(g(S) - D)$. Let $t > 0$ be fixed. Then obviously

$$(v(P) - f(x_0) - t, g(x_0) - y_0) \in \mathcal{E}(\mathcal{K}_{v(P)}).$$

Take (r^*, y^*) an arbitrary element in $NC((v(P) - f(x_0) - t, g(x_0) - y_0); \mathcal{E}(\mathcal{K}_{v(P)}))$. It holds

$$r^*(u - (v(P) - f(x_0) - t)) + \langle y^*, v - (g(x_0) - y_0)\rangle \leq 0, \ \forall (u,v) \in \mathcal{E}(\mathcal{K}_{v(P)}).$$

Choosing first in the previous inequality $u := v(P) - f(x_0) - t/2$, $v := g(x_0) - y_0$ and then $u := v(P) - f(x_0) - (3t)/2$, $v := g(x_0) - y_0$, we obtain $+\frac{t}{2} r^* \leq 0$ and $-\frac{t}{2} r^* \leq 0$, respectively, that is $r^* = 0$. Hence

$$\langle y^*, v - (g(x_0) - y_0)\rangle \leq 0, \forall (u,v) \in \mathcal{E}(\mathcal{K}_{v(P)},)$$

which is nothing else than $\langle y^*, v - (g(x_0) - y_0)\rangle \leq 0, \forall v \in g(S) - D$. Thus

$$y^* \in NC(g(x_0) - y_0; g(S) - D).$$

Since $NC(g(x_0) - y_0, g(S) - D)$ is a linear subspace of Y^*, we have also that $-y^* \in NC(g(x_0) - y_0; g(S) - D)$ and so

$$\langle -y^*, v - (g(x_0) - y_0)\rangle \leq 0, \forall v \in g(S) - D.$$

Hence

$$\langle (0, -y^*), (u - (v(P) - f(x_0) - t), v - (g(x_0) - y_0))\rangle \leq 0, \forall (u,v) \in \mathcal{E}(\mathcal{K}_{v(P)}).$$

Further, we get $-(r^*, y^*) = (0, -y^*) \in NC(v(P) - f(x_0) - t, g(x_0) - y_0; \mathcal{E}(\mathcal{K}_{v(P)}))$, showing that $NC(v(P) - f(x_0) - t, g(x_0) - y_0; \mathcal{E}(\mathcal{K}_{v(P)}))$ is a linear subspace of $\mathbb{R} \times Y^*$, that is

$$(v(P) - f(x_0) - t, g(x_0) - y_0) \in \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)}).$$

(b) Assume that $(r_0, y_0) \in \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)})$. Take an arbitrary element in the normal cone

$$y^* \in NC(y_0; g(S) - D) = \{y^* \in Y^* : \langle y^*, v - y_0 \rangle \leq 0, \forall v \in g(S) - D\}.$$

Then $(0, y^*) \in NC((r_0, y_0); \mathcal{E}(\mathcal{K}_{v(P)})) = \{(r^*, y^*) \in \mathbb{R} \times Y^* : r^*(u - r_0) + \langle y^*, v - y_0 \rangle \leq 0, \forall (u, v) \in \mathcal{E}(\mathcal{K}_{v(P)})\}$. As $NC((r_0, y_0); \mathcal{E}(\mathcal{K}_{v(P)})$ is a linear subspace of $\mathbb{R} \times Y^*$ we get $(0, -y^*) \in NC((r_0, y_0); \mathcal{E}(\mathcal{K}_{v(P)})$, that is

$$-\langle y^*, v - y_0 \rangle \leq 0, \forall v \in g(S) - D.$$

This is nothing else than $-y^* \in NC(y_0; g(S) - D)$. This means that the cone $NC(y_0, g(S) - D)$ is a linear subspace of Y^*, hence $y_0 \in \operatorname{qri}(g(S) - D)$.

(c) This assertion is a direct consequence of the statements (a) and (b). □

Proposition 3.3.15. *Assume that* $O_Y \in \operatorname{qi}[(g(S) - D) - (g(S) - D)]$. *Then* $NC(\mathit{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}))$ *is a linear subspace of* $\mathbb{R} \times Y^*$ *if and only if it holds that*

$$NC(\mathit{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})) = \{(0, O_{Y^*})\}.$$

Proof. If. The sufficiency is trivial.
Only if. Consider that $NC(\operatorname{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}))$ is a linear subspace of $\mathbb{R} \times Y^*$. Take $(\theta, \lambda) \in NC(\operatorname{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}))$ arbitrary. Then $\theta u + \langle \lambda, v \rangle \leq 0$, $\forall (u, v) \in \operatorname{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$, which implies

$$-\theta(f(x) + \alpha - v(P)) + \langle \lambda, g(x) - y \rangle \leq 0, \forall x \in S, \forall y \in D \text{ and } \forall \alpha \geq 0. \quad (3.3.\text{-}1)$$

Let $x' \in R$ be a feasible element. For $y := g(x')$ and $x := x'$ in the above inequality we obtain

$$-\theta(f(x') + \alpha - v(P)) \leq 0, \forall \alpha \geq 0,$$

hence $\theta \geq 0$ (otherwise, if $\theta < 0$, then when passing to the limit as $\alpha \to +\infty$ we obtain a contradiction).

Since $NC(\operatorname{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}))$ is a linear subspace of $\mathbb{R} \times Y^*$, the argument from above applies also for $(-\theta, -\lambda)$, implying $\theta \leq 0$. Finally, we get $\theta = 0$ and inequality (3.3.2) and relation $(0, -\lambda) \in NC(\operatorname{conv}\ (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}))$ imply

$$\langle \lambda, g(x) - y \rangle = 0, \forall x \in S \text{ and } \forall y \in D.$$

It follows that $\langle \lambda, y \rangle = 0, \forall y \in \operatorname{cl} \operatorname{cone}[(g(S) - D) - (g(S) - D)] = Y$, that is $\lambda = O_{Y^*}$. So $(\theta, \lambda) = (0, O_{Y^*})$ and the conclusion follows. □

Remark 3.3.16. (a) As $D - D \subseteq (g(S) - D) - (g(S) - D)$, we have the following implication

$$\operatorname{cl}(D - D) = Y \Rightarrow O_Y \in \operatorname{qi}[(g(S) - D) - (g(S) - D)].$$

(b) Since $\operatorname{cone}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}) = \operatorname{cone}\mathcal{E}(\mathcal{K}_{v(P)})$, we automatically get the following relation

$$\operatorname{cl}\operatorname{cone}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}) = \operatorname{cl}\operatorname{cone}\mathcal{E}(\mathcal{K}_{v(P)}).$$

Hence the normal cone

$$\begin{aligned} & NC(\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})) \text{ is a linear subspace of } \mathbb{R} \times Y^* \Leftrightarrow \\ \Leftrightarrow\ & (0, O_Y) \in \operatorname{qri}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}) \Leftrightarrow \\ \Leftrightarrow\ & \operatorname{cl}\operatorname{cone}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}) \text{ is a linear subspace of } \mathbb{R} \times Y \Leftrightarrow \\ \Leftrightarrow\ & \operatorname{cl}\operatorname{cone}\mathcal{E}(\mathcal{K}_{v(P)}) \text{ is a linear subspace of } \mathbb{R} \times Y. \end{aligned}$$

On the other hand, the condition $NC(\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})) = \{(0, O_{Y^*})\}$ is equivalent to $(0, O_Y) \in \operatorname{qi}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$, so in case $O_Y \in \operatorname{qi}[(g(S) - D) - (g(S) - D)]$, we have

$$\operatorname{cl}\operatorname{cone}\mathcal{E}(\mathcal{K}_{v(P)}) \text{ is a linear subspace of } \mathbb{R}\times Y \Leftrightarrow (0, O_Y) \in \operatorname{qi}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)})\cup\{(0, O_Y)\}),$$

or, equivalently

$$(0, O_Y) \in \operatorname{qri}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)})\cup\{(0, O_Y)\}) \Leftrightarrow (0, O_Y) \in \operatorname{qi}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)})\cup\{(0, O_Y)\}).$$

(c) If the primal problem has an optimal solution (which means that $(0, O_Y) \in \mathcal{E}(\mathcal{K}_{v(P)})$) and $O_Y \in \operatorname{qi}[(g(S) - D) - (g(S) - D)]$ we have

$$(0, O_Y) \in \operatorname{qri}\mathcal{E}(\mathcal{K}_{v(P)}) \Leftrightarrow (0, O_Y) \in \operatorname{qi}\mathcal{E}(\mathcal{K}_{v(P)}).$$

We are able now to give the following strong duality result.

Theorem 3.3.17. *Suppose that* $\operatorname{cl}(D - D) = Y$ *and* $\exists \tilde{x} \in S$ *such that* $g(\tilde{x}) \in \operatorname{qri} D$. *If* $(0, O_Y) \notin \operatorname{qri}$ *conv* $(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$, *then* $v(P) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Proof. Lemma 3.3.13 and Lemma 3.3.14 ensure that

$$\operatorname{qri}\mathcal{E}(\mathcal{K}_{v(P)}) \neq \varnothing,$$

while condition $(0, O_Y) \notin \operatorname{qri}\operatorname{conv}\,(\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$ means actually that

$$\operatorname{cl}\operatorname{cone}\mathcal{E}(\mathcal{K}_{v(P)}) \text{ is not a linear subspace of } \mathbb{R} \times Y.$$

Applying Corollary 3.3.11, we can separate now the sets $\mathcal{E}(\mathcal{K}_{v(P)})$ and $\{(0, O_Y)\}$. Thus there exists $(\theta, \lambda) \in \mathbb{R} \times Y^*$, $(\theta, \lambda) \neq (0, O_{Y^*})$ such that

$$\theta(f(x) + \alpha - v(P)) - \langle \lambda, g(x) - y\rangle \geq 0,\ \forall x \in S, \forall \alpha \geq 0, \forall y \in D. \qquad (3.3.\text{-}6)$$

We claim that $\lambda \in D^*$. If we suppose that $\exists y_0 \in D$ such that $\langle \lambda, y_0 \rangle < 0$, then the inequality

$$\theta(f(x) + \alpha - v(P)) - \langle \lambda, g(x) \rangle + t\langle \lambda, y_0 \rangle \geq 0$$

is true for every $t \geq 0$ (cf. (3.3.2)) and passing to the limit as $t \to +\infty$ (for a fixed $x \in S$ and $\alpha \geq 0$) we obtain a contradiction. Similar arguments as in the proof of the Proposition 3.3.15 show that $\theta \geq 0$.

Let us prove that actually $\theta > 0$. Assume that $\theta = 0$. Then (3.3.2) gives us

$$\langle \lambda, -g(x) + y \rangle \geq 0, \forall x \in S, \quad \forall y \in D.$$

By the hypotheses, $\exists \tilde{x} \in S$ such that $g(\tilde{x}) \in \operatorname{qri} D$. This together with $\lambda \in D^*$ show that

$$\langle \lambda, g(\tilde{x}) \rangle \geq 0.$$

Taking $y = 0_Y$ and $x = \tilde{x}$ in the inequality $\langle \lambda, -g(x) + y \rangle \geq 0, \forall x \in S, \forall y \in D$ we get

$$\langle \lambda, g(\tilde{x}) \rangle \leq 0$$

and hence $\langle \lambda, g(\tilde{x}) \rangle = 0$. Also from the inequality $\langle \lambda, -g(\tilde{x}) + y \rangle \geq 0, \forall y \in D$ we obtain

$$-\lambda \in NC(g(\tilde{x}); D).$$

As $NC(g(\tilde{x}); D)$ is a linear subspace of Y^* we get $\langle \lambda, g(\tilde{x}) - y \rangle = 0, \forall y \in D$, that is $\langle \lambda, y \rangle = 0, \forall y \in D$, hence $\langle \lambda, y \rangle = 0$, $\forall y \in \operatorname{cl}(D - D) = Y$, namely $\lambda = O_{Y^*}$. Thus $(\theta, \lambda) = (0, O_{Y^*})$ and this leads to a contradiction. We must have $\theta > 0$.

Taking in (3.3.2) $\alpha = 0$ and $y = O_Y$ we obtain

$$v(P) \leq f(x) - \frac{1}{\theta}\langle \lambda, g(x) \rangle, \forall x \in S.$$

With the notation $\overline{\lambda} := \frac{1}{\theta}\lambda \in D^*$ we get $v(P) \leq f(x) - \langle \overline{\lambda}, g(x) \rangle, \forall x \in S$. Taking the infimum with respect to $x \in S$ we have

$$v(P) \leq \inf_{x \in S} [f(x) - \langle \overline{\lambda}, g(x) \rangle],$$

hence $v(P) \leq v(D_L)$. As the opposite inequality always holds, we get $v(P) = v(D_L)$ and $\overline{\lambda}$ is an optimal solution of the dual problem (D_L). □

In case the primal problem (P) has an optimal solution we get the following strong duality result.

Corollary 3.3.18. *Suppose that the primal problem has an optimal solution,* $\operatorname{cl}(D - D) = Y$ *and* $\exists \tilde{x} \in S$ *such that* $g(\tilde{x}) \in \operatorname{qri} D$. *If* $(0, O_Y) \notin \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)})$, *then* $v(P) = v(D_L)$ *and the problem* (3.3.2) *has an optimal solution.*

Remark 3.3.19. (a) In the hypotheses of Theorem 3.3.17 one has that

$$(0, O_Y) \notin \operatorname{qri} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$$

is equivalent (see Remark 3.3.16) to

$$(0, O_Y) \notin \operatorname{qi} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}).$$

Similarly, in the hypotheses of Corollary 3.3.18 condition $(0, O_Y) \notin \operatorname{qri} \mathcal{E}(\mathcal{K}_{v(P)})$ is equivalent to $(0, O_Y) \notin \operatorname{qi} \mathcal{E}(\mathcal{K}_{v(P)})$.

(b) One has that

$$(0, O_Y) \in \operatorname{qi} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, 0)\}) \Rightarrow O_Y \in \operatorname{qi}(g(S) - D).$$

Indeed, if $(0, O_Y) \in \operatorname{qi} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$, then $\operatorname{cl} \operatorname{cone} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}) = \mathbb{R} \times Y$, thus $\operatorname{cl} \operatorname{cone} \mathcal{E}(\mathcal{K}_{v(P)}) = \mathbb{R} \times Y$. Since $\mathcal{E}(\mathcal{K}_{v(P)}) \subseteq \mathbb{R} \times (g(S) - D)$, we get $\operatorname{cl} \operatorname{cone}[g(S) - D] = Y$. The last relation is nothing else than $O_Y \in \operatorname{qi}(g(S) - D)$. Thus

$$O_Y \notin \operatorname{qi}(g(S) - D) \Rightarrow (0, O_Y) \notin \operatorname{qi} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\}).$$

Hence we have found a condition which guarantees the fulfilment of

$$(0, O_Y) \notin \operatorname{qri} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$$

(which, in case $\operatorname{cl}(D - D) = Y$, is equivalent to $(0, O_Y) \notin \operatorname{qi} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$).

Let us mention that one cannot substitute in the hypotheses of Theorem 3.3.17 $(0, O_Y) \notin \operatorname{qri} \operatorname{conv} (\mathcal{E}(\mathcal{K}_{v(P)}) \cup \{(0, O_Y)\})$ by $O_Y \notin \operatorname{qi}(g(S) - D)$, since this would be in contradiction with the other assumptions (see Lemma 3.3.13).

(c) Coming now back to Example 3.3.12, it is not surprising that there strong duality does not holds, since not all the hypotheses of Corollary 3.3.18 are fulfilled. This is what we show in the following, namely that $(0, O_{l^2}) \in \operatorname{qi} \mathcal{E}(\mathcal{K}_{v(P)})$. Take $(\theta, \lambda) \in NC(\mathcal{E}(\mathcal{K}_{v(P)}))$. Then we have

$$\theta(-\langle c, x\rangle - \alpha) + \langle \lambda, g(x) - y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+, \tag{3.3.-6}$$

that is

$$\theta\left(-\sum_{n=1}^{\infty} \frac{1}{n} x_n - \alpha\right) + \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{2^n} x_n - y_n\right) \leq 0,$$
$$\forall x = (x_n)_{n\in\mathbb{N}} \in l^2, \forall \alpha \geq 0, \ \forall y = (y_n)_{n\in\mathbb{N}} \in l^2_+.$$

Setting $\alpha = 0$ and $y_n = 0$, $\forall n \in \mathbb{N}$ in the relation above we get

$$\sum_{n=1}^{\infty} \left(-\theta \frac{1}{n} + \frac{1}{2^n} \lambda_n\right) x_n \leq 0, \forall x = (x_n)_{n\in\mathbb{N}} \in l^2,$$

which implies $\lambda_n = \theta \frac{2^n}{n}$, $\forall n \in \mathbb{N}$. Since $\lambda \in l^2$, we must have $\theta = 0$ and hence $\lambda = O_{l^2}$. Thus $NC(\mathcal{E}(\mathcal{K}_{v(P)})) = \{(0, O_{l^2})\}$ and so $(0, O_{l^2}) \in \operatorname{qi} \mathcal{E}(\mathcal{K}_{v(P)})$.

In the following example we introduce an optimization problem for which strong Lagrange duality holds. In this way we illustrate the applicability of Corollary 3.3.18.

Example 3.3.20. Let be $X = S = Y = l^2$ and $D = l^2_+$, the positive cone of l^2. For $f : l^2 \to \mathbb{R}$, $f(x) = \langle c, x\rangle$, where $c = (c_n)_{n\in\mathbb{N}}, c_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$ and $g : l^2 \to l^2, g(x) = x$, we get $R = l^2_+$ and the following optimization problem

$$\inf_{x\in R} \langle c, x\rangle.$$

Its optimal objective value $v(P)$ is equal to zero and $x = O_{l^2}$ is an optimal solution of (3.3.1). The conditions $\mathrm{cl}(D - D) = Y$ and $\exists \tilde{x} \in S$ such that $g(\tilde{x}) \in \mathrm{qri}\, D$ are obviously satisfied.

We prove that $(0, O_{l^2}) \notin \mathrm{qi}\, \mathcal{E}(\mathcal{K}_{v(P)})$. Indeed, by using relation (3.3.19) we have $(\theta, \lambda) \in NC(\mathcal{E}(\mathcal{K}_{v(P)}))$ if and only if $\theta(-\langle c, x\rangle - \alpha) + \langle \lambda, x - y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+$, or, equivalently $\langle -\theta c + \lambda, x\rangle - \theta\alpha - \langle \lambda, y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+$. It is obvious that $(\theta, \lambda) := (1, c) \in N(\mathcal{E}(\mathcal{K}_{v(P)}))$, which means that $NC(\mathcal{E}(\mathcal{K}_{v(P)})) \neq \{(0, O_{l^2})\}$ or, equivalently, $(0, O_{l^2}) \notin \mathrm{qi}\, \mathcal{E}(\mathcal{K}_{v(P)})$. The hypotheses of Corollary 3.3.18 being fulfilled, strong duality holds between (3.3.1) and (3.3.2). One can easily see that $\bar{\lambda} = c$ is an optimal solution for the dual.

Bibliography

[1] Abadie J.(Ed.), *Nonlinear Programming.* North-Holland, Amsterdam, 1967.

[2] Abadie J.M., *On the Kuhn-Tucker theorem*, Nonlinear Programming, J. Abadie, ed., North-Holland, Amsterdam, pp. 19–36, 1967.

[3] Abadie J.M., *Constraint Qualifications in Maximization Problems*, Naval Research Logistics Quarterly, Vol. 8, pp. 175–91, 1961.

[4] Abadie J.M., *Constraint Qualifications in Maximization Problems II*, Institute for Mathematical Studies in Social Sciences , Stanford, 1960.

[5] Barbu V., Precupanu T, *Convexitate si Optimizare in Spatii Banach*, Editura Academiei Republicii Socialiste Romania, Bucuresti, 1975 (in Romanian).

[6] Bazaara M.S. and Shetty C.M., *Foundations of Optimization*, in Lecture Notes in Economics and Mathematical Systems, Vol. 122, Springer-Verlag, Berlin, 1976.

[7] Ben-Tal A., Zowe J., *A Unified Theory of First and Second Order Conditions for Extremum Problems in Topological Vector Spaces*, Mathematical Programming Study, Vol. 19, pp. 39– 76, 1982.

[8] Bertsekas D.P., Qzdaglar A.E., *Pseudonormality and a Lagrange Multiplier Theory for Constrained Optimization*, Journ. of Optim. Theor. and Appl., Vol. 114, No. 2, pp 287–343, 2002.

[9] Borwein, J.M., Goebel, R., *Notions of relative interior in Banach spaces*, Journal of Mathematical Sciences, Vol. 115, No. 4, pp. 2542–2553, 2003.

[10] Borwein, J.M., Lewis, A.S., *Partially finite convex programming, part I: Quasi relative interiors and duality theory*, Mathematical Programming, Vol. 57, pp. 15–48, 1992.

[11] Boţ, R.I., Wanka, G., *An alternative formulation for a new closed cone constraint qualification*, Nonlinear Analysis: Theory, Methods & Applications, Vol. 64, No. 6, pp. 1367–1381, 2006.

[12] Bot R.I., Csetnek R., Moldovan A., *Revisiting Some Duality Theorems Via the Quasirelative Interior in Convex Optimization* to appear in Journ. of Optim. Theory and Appl., 2008.

[13] Cammaroto, F., Di Bella, B., *Separation theorem based on the quasirelative interior and application to duality theory*, Journ. of Optim. Theory and Appl., Vol. 125, No. 1, pp. 223–229, 2005.

[14] Clarke F.H., *Optimization and Nonsmooth Analysis*, J. Wiley, New York, 1983; reprint, SIAM Classics in Applied Mathematics, Vol.5, Philadelphia, 1994.

[15] Daniele, P., Giuffrè, S., *General infinite dimensional duality theory and applications to evolutionary network equilibrium problems*, Optimization Letters, Vol. 1, No. 3, pp. 227–243, 2007.

[16] Daniele, P., Giuffrè, S., Idone, G., Maugeri, A., *Infinite dimensional duality and applications*, Mathematische Annalen, Vol. 339, No. 1, pp 221–239, 2007.

[17] Demyanov V.F., Rubinov A.M., *Quasidifferentiable Calculus*, Optimization Software, New York, NY, 1986.

[18] Dien P.H., Mastroeni G., Pappalardo M., Quang P.H., *Regularity Conditions for Constrained Extremum Problems via Image Space: the Linear Case*, Preprint of University of Pisa, 1994.

[19] Dien P.H., Mastroeni G., Pappalardo M., Quang P.H., *Regularity Conditions for Constrained Extremum Problems via Image Space*, Journ. of Optim. Theory and Appl., Vol.80, pp.19–37, 1994.

[20] Gauvin J., *A Necessary and Sufficient Regularity Condition to Have Bounded Multiplier in Nonconvex Programming*, Mathematical Programming, Vol. 12, pp 136–138, 1977.

[21] Giannessi, F., *Theorems of the Alternative and Optimality Conditions*, Report No. 83, Dept. of Mathem., Univ. of Pisa, Sect. of Optimization, pp. 1–30, 1982. Published with the same title in Journ. of Optim. Theory and Appl, Vol.42, No.3, pp 331–365, 1984.

[22] Giannessi, F., *Metodi Matematici della Programmazione. Problemi Lineari e non Lineari (Mathematical methods of programming. Linear and nonlinear problems)*, Monograph No. 23 of Italian Mathem. Soc. (UMI), Pitagora Publisher, Bologna, 1982 (in Italian).

[23] Giannessi, F., *Semidifferentiable Functions and Necessary Optimality Conditions*, Journ. of Optim. Theory and Appl., Vol.60, pp.191–241, 1989.

[24] Giannessi, F., *Theorems of the Alternative for Multifunctions with Applications to Optimization. Necessary Conditions*, Tech. Paper No. 131, Optimiz. Series, Dept. of Mathem.,Univ. of Pisa, pp 1–127, 1986.

[25] Giannessi, F., *Theorems of the Alternative for Multifunctions with Applications to Optimization: General Results*, Journ. of Optim. Theory and Appl., Vol.55, No. 2, pp.233–256, 1987.

[26] Giannessi F., *Constrained Optimization and Image Space Analysis.* Vol.1: Separation of Sets and Optimality Conditions, Springer, 2005.

[27] Giannessi F., Mastroeni G. and Uderzo A., *A multifunction approach to extremum problems having infinite dimensional image: necessary conditions for unilateral constraints.* Cybernetics and System Analysis, 2002, No. 3, Plenum, pp. 39–51.

[28] Giannessi F., Mastroeni G., Uderzo A., *On Necessary Condition for Inifinite-Dimensional Extremum Problems*, Journ. of Glob. Optimization, Vol. 28, pp. 319–337, 2004.

[29] Gould F.J., Teboulle J.W., *A Necessary and Sufficient Qualification for Constrained Optimization*, Siam J. Appl. Math, Vol. 20, No. 2, pp 164–172, 1971.

[30] Gowda, M.S., Teboulle, M., *A comparison of constraint qualifications in infinite-dimensional convex programming*, SIAM Journal on Control and Optimization 28, 925-935, 1990.

[31] Guignard M., *Generalized Kuhn-Tucker Conditions for Mathematical Programming Problems in a Banach Space*, SIAM Journ. Control, Vol.7, pp.232–241, 1969.

[32] Hestenes M.R., *Calculus of Variations and Optimal Control Theory.* J. Wiley, New York, 1966.

[33] Ioffe A.D., *Regular Points of Lipschitz Functions*, Trans. Amer. Math. Soc., Vol.251, pp.533–548, 1994.

[34] Ioffe A.D, *Metric Regularity and Subdifferential Calculus*, Russian Mathematical Reviews, Vol. 55, No. 3, pp 501–558, 2000.

[35] Jahn, J., *Introduction to the theory of nonlinear optimization*, Springer, Berlin, 1996.

[36] Jeyakumar, V., Wolkowicz, H., *Generalizations of Slater's constraint qualification for infinite convex programs*, Mathematical Programming, Vol. 57, pp 85–101, 1992.

[37] John F., *Extremum Problems with Inequalities as Subsidiary Conditions*, in K.O. Friedrichs, O.E. Neugebauer and J.J. Stoker (eds.), "Studies and Essays: Courant Anniversary Volume", Interscience Publishers, New York, pp. 187–204, 1948.

[38] Jourani A., *Constraint Qualification and Lagrange Multipliers in Nondifferentiable Programming Problems*, Journ. of Optim. Theory and Applic, Vol. 81, pp. 533–548, 1994.

[39] Karlin, S., *Mathematical Methods and Theory in Games, Programming and Economics*, Addison-Wesley, Reading, Mass, Vol. 1, 1959 pp. 19–36, 1967.

[40] Karush W., *Minima of Functions of Several Variables with Inequalities as Side Conditions*, Master's Thesis, University of Chicago, Chicago, 1939.

[41] Kuhn H.W., Tucker A.W., *Nonlinear Programming*, in J. Neyman (ed.), "Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probabilty", University of California Press, Berkeley, California, pp. 481–492, 1951.

[42] Kuntz L., Scholtes S., *Constraint Qualification in Quasidifferentiable Optimization*, Mathematical Programming, Vol. 60, pp. 339–347, 1993.

[43] Kuntz L., Scholtes S., *A Nonsmooth Variant of the Mangasarian-Fromowitz Constraint Qualification*, Journ. of Optim. Theory and Applic, Vol. 82, pp. 59–75, 1994.

[44] Hiriart-Urruty J.-B., Lemaréchal C., *Convex Analysis and Minimization Algorithms I*, Springer-Verlag Berlin, 1993.

[45] Li W., *Abadie's Constraint Qualification, Metric regularity, and Error Bounds for Differentiable Convex Inequalities*, SIAM Journ. Optim., Vol. 7, No. 4, pp. 966– 978, 1997.

[46] Li C, Ng K.F., *On Constraint Qualification for an Infinite System of Convex Inequalities in a Banach Space*, SIAM Journ. Optim., Vol. 15, No. 2, pp. 488–512, 2005.

[47] Limber, M.A., Goodrich, R.K., *Quasi interiors, Lagrange multipliers, and L^p spectral estimation with lattice bounds*, Journ. of Optim. Theory and Appl., Vol.78, No.1, pp. 143–161, 1993.

[48] Madani K., Mastroeni G., Moldovan A., *Constrained Extremum Problems with Infinite Dimensional Image: Selection and Necessary Conditions* Journ. of Optim. Theory and Appl., Vol.135, No.1, pp 37–53, 2007.

[49] Madani K., Mastroeni G., Moldovan A., *Constrained Extremum Problems with Infinite Dimensional Image: Selection and Saddle Point* to appear in Journ. of Glob. Optimization, 2008.

[50] Mangasarian O.L., Fromovitz S., *The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints*, Journ. of Math. Analysis and Appl., Vol.17, pp. 37–47, 1967.

[51] Mangasarian O.L., *Nonlinear Programming*, McGraw-Hill, New York, 1969; reprint, SIAM Classics in Applied Mathematics, Vol.10, Philadelphia, 1994.

[52] Martein L., *Regularity Conditions for Extremum Problems*, Journ. of Optim. Theory and Appl., Vol. 47, No. 2, pp. 217–233, 1985.

[53] Mastroeni G., Rapcsák T., *On Convex Generalized Systems*, Journ. of Optim. Theory and Appl., Vol.104, pp 605–627, 2000.

[54] Merkovsky R.P., Ward D.E., *General Constraint Qualifications in Nondifferentiable Programming*, Mathematical Programming, Vol. 47, pp. 389–405, 1990.

[55] Moldovan A., Pellegrini L., *A Necessary and Sufficient Regularity Condition for Constrained Optimization*. Preprint, University of Pisa, 2007.

[56] Pars L.A., *An Introduction to the Calculus of Variations*, Heinemann, London, 1962.

[57] Pappalardo M., *On Image Regularity Condition*, Journ. of Optim. Theory and Appl., Vol. 121, No. 3, pp 673–678, 2004.

[58] Penot J. P., *A New Constraint Qualification*, Journ. of Optim. Theory and Appl., Vol. 48, pp. 459–468, 1986.

[59] Penot J. P., *About Linearization, Conization, Calmness, Openess and Regularity*, Nonlinear Analysis and Applications, Proc. Conference Arlington, Texas, July 28/ August 1, 1986 Lakshmikantham Editor, Chapter 56, pp 439–450, Marcel Dekker, New York, 1987.

[60] Penot J.P., *On Regularity Condition in Mathematical Programming*, MAthematical PRogramming Study, Vol. 19, pp. 167– 199, 1982.

[61] Peterson, D.W., *Review of Constraint Qualifications in Finite-Dimensional Spaces* , Siam Review, Vol.15, No. 3, pp.639–654, 1973.

[62] Psenichnyi B.N., *Necessary Conditions for an Extremum*, Marcel Dekker, New York, 1971.

[63] Rockafellar T.R., Wets R.J.-B., *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[64] Rockafellar R.T., *Convex Analysis*, Princeton University Press, 1970.

[65] Rockafellar R.T., *Lagrange Multipliers and Subderivatives of Optimal Value Functions in Nonlinear Programming*, Mathematical Programming Study, Vol. 17, pp 28–66, 1982.

[66] Rockafellar R.T., *Conjugate Duality and Optimization*, CBMS-NFS Regional Conference in Applied Mathematics 16, Society for Industrial and Applied Mahematics, Philadelphia, PA, 1974.

[67] Rudin, W., *Functional analysis. Second edition*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[68] Slater M., *Lagrange Multipliers Revisited: A Contribution to Nonlinear Programming*, Cowles Commission Discussion Paper, Mathematics 403, 1950.

[69] Stein O., *On Constraint Qualifications in Nonsmooth Optimization*, Journ. of Optim. Theory and Appl., Vol.121, No. 3, pp 647–671, 2004.

[70] Tiba D., Zălinescu C., *On the Necessity of some Constraint Qualifications Conditions in Convex Programming*, Journal of Convex Analysis, Vol. 11, No. 1, pp 95–110, 2004.

[71] Ward D.E., *A Constraint Qualification in Quasidifferentiable Programming*, Optimization, Vol. 22, pp. 661–668, 1991.

[72] Ye J.J., *Multiplier Rules under Mixed Assumptions of Differentiability and Lipschitz Continuity*, SIAM Journ. Control Optim., Vol. 39, pp.1441–1460, 2001.

[73] Zălinescu C., *Convex analysis in general vector spaces*, World Scientific, Singapore, 2002.

[74] Zălinescu C., *A Comparison of Constraint Qualifications in Infinite-Dimensional Convex Programming Revisited*, J. Austral. Math. Soc. Ser. B, Vol. 40, pp. 353–378, 1999.

[75] Zangwill, W.I. *Nonlinear Programming: A Unified Approach*, Prentice Hall, Inc. Englwood Cliffs, N. J., 1969.

[76] Zowe J., Kurcyusz S., *Regularity and Stability for the Mathematical Programming Problem in Banach Spaces*, Appl. Math. Optim, Vol. 5, pp. 49– 62, 1979.