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Linear and Nonlinear Perturbed Wave Equations

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Contents

Frequently used notations

Differential operators

Symbol	Description
∇	$(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ (gradient on $\mathbb{R}^n$)
Δ	$\partial_{x_1}^2 + \partial_{x_2}^2 + \ldots + \partial_{x_n}^2$ (Laplace operator on $\mathbb{R}^n$)
$\square$	$\partial_t^2 - \Delta$ (d'Alembert operator)
Δ_g	Laplace–Beltrami operator in the metric g
$\square_g$	Laplace–Beltrami operator in the metric g of index 1
$\nabla_\pm$	$\partial_t \pm \partial_r$

Function spaces

Symbol	Description
$\mathscr{C}^k$	k-times differentiable continuous functions (k nonnegative integer or $k = \infty$)
$\mathscr{C}_0^k$	$\mathscr{C}^k$ compactly supported functions
L^p	Lebesgue space ($p \in [1, \infty]$)
$\mathrm{L}^{s,p}$	Sobolev space ($s \in \mathbb{R}$)
H^s	$\mathrm{L}^{s,2}$
$\dot{\mathrm{H}}^s$	homogeneous space
$\dot{\mathrm{B}}^s_{p,q}$	Besov space

Other symbols

Symbol	Description
$\doteq$	equal by definition
$\equiv$	identically equal
$f \lesssim g$	$\exists C > 0 \,:\, f \leqslant Cg$
$f \sim g$	$f \lesssim g$ and $g \lesssim f$
$\mathbb{S}^n$	$\{x \in \mathbb{R}^{n+1} \,:\, \lvert x\rvert = 1\}$ (unitary sphere)
$\langle x \rangle$	$\sqrt{1 + \lvert x\rvert^2}$
$\blacksquare$	Q.E.D. (end of the proof)

Chapter 1

Introduction

In this chapter, the whole content of the thesis is presented, along with some preliminary information reported not only for the sake of completeness, but also to provide a frame in which to put the following results.

In particular, Section 1.1 introduces some known results on the semilinear wave equations in the Minkowski space. From this problem, certainly interesting on its own, stems the interest for most of the original results presented in this dissertation, i.e. for the problem of a semilinear wave equation in a curved background, that is, more precisely, in the Schwarzschild metric (this problem is studied in the Chapters 4 and 5). The results and the techniques illustrated in this section can be profitably compared with the ones that appear in the Schwarzschild setting.

Section 1.2 presents some known results related to the wave equation perturbed with a potential. Such a kind of perturbation is related to the problem considered in Chapter 6, where we prove dispersive estimates for the linear wave equation with an electromagnetic potential, but also to the problem of a wave equation in the Schwarzschild metric: indeed, under suitable hypotheses, we can reduce such an equation to a wave equation in the Minkowski space perturbed through a potential.

Section 1.3 contains the assertions and some comments on the original results: in particular, their mathematical and physical interest, and the main difficulties that one has to afford to prove them.

In Section 1.4 we discuss some problems related to the ones previously considered.

Finally, Section 1.5 briefly describes the structure of this dissertation.

1.1 Some known results on the semilinear wave equation

For each $n \geqslant 1$, let Δ and $\Box$ be respectively the Laplace and the d'Alembert operators, defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\,, \qquad \Box = \frac{\partial^2}{\partial t^2} - \Delta\,,$$

acting on a function $u(t,x) = u(t, x_1, \ldots, x_n)$.

Let us consider the following semilinear Cauchy problem:

$$\begin{cases} \Box u = |u|^p & \text{in } [0,\infty[\times\mathbb{R}^n\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x) & \text{in } \mathbb{R}^n\,. \end{cases} \tag{1.1}$$

The solution u represents a wave in the flat Minkowski space under the influence of a nonlinear source, that is given by $|u|^p$, and with initial data u_0 and u_1. One is interested in the large-time behavior of this solution under suitable hypotheses on the initial data and the exponent p. In particular, for physical reasons, one needs to know wether the solution is defined for every time, i.e. we have a global in time solution, or it presents a singularity, that is it blows up in finite time.

The first answer to this problem can be found in a work of Fritz John published in 1979, [32], followed by a paper of the same author published two years later, [33]. In these works, the author considers the case $n = 3$ and sufficiently regular compactly supported initial data u_0 and u_1, and shows two main results: (1) if $p \in]1, 1+\sqrt{2}]$ and u_0 and u_1 are nonnegative (and nontrivial), the solution blows up in finite time; (2) if $p > 1+\sqrt{2}$, the solution exists globally in time, provided the initial data are sufficiently small. In other words, we have a critical exponent, $p_c = 1+\sqrt{2}$, below which one has a blow-up phenomenon and above which one has global existence.

After these results, Strauss conjectured that this situation should reproduce for each $n \geqslant 2$ and a suitable critical exponent $p_c(n)$, which is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0$$

(see [48]). Note that, in particular, $p_c(3) = 1 + \sqrt{2}$. Since then, several mathematicians have contributed to the proof of this conjecture, which has been only recently completely proved. It has been shown that the solution develops a singularity also in the critical case $p = p_c(n)$. In other words, one has a blow-up phenomenon for each $p \in]1, p_c(n)]$ (and nontrivial solutions), while one has global existence for each $p \in]p_c(n), \infty[$ and small data. Remember that the data are assumed compactly supported and in suitable spaces.

Let us cite the papers containing these results. As anticipated, in his works of 1979 and 1981 ([32] and [33]), F. John considered the case $n = 3$ and showed blow-up for $1 < p < p_c(3)$ and global existence for $p > p_c(3)$. The analogous results for $n = 2$ were proved by R. Glassey in [29], a paper published in 1981. In 1984, T.C. Sideris established the blow-up result in the case $n \geqslant 4$ when $1 < p < p_c(n)$ ([47]), while the corresponding global existence result for $n \geqslant 4$ and $p > p_c(n)$ was obtained definitely later by V. Georgiev, H. Lindblad and C. Sogge in [24], a work of 1997. Finally, the blow-up in the critical case was shown by J. Schaeffer in the case $n = 2$ and $n = 3$ ([46], 1985), while the similar result in the case $n \geqslant 4$ has been reached only recently, in 2005, by B.T. Yordanov and Q.S. Zhang, in [55], completing the proof of the conjecture of Strauss.

Of course, during the years, most of the proofs have been simplified. We want to spend some words about the main techniques involved in the proof of these results, since they are useful to understand the difficulties met in the proof of the results explained in this thesis.

As far as the blow-up is concerned, the main idea is to consider the space-average function

$$F(t) = \int_{\mathbb{R}^n} u(t,x)\, dx$$

and show that F blows up in finite time. The problem is hence reduced to the proof of a blow-up for the solution to an ordinary differential equation (depending only on t) and can be solved through a lemma due to T. Kato contained in

[35] (1980). It says that if $F \in \mathscr{C}^2$ satisfies

$$F(t) \geqslant C(t+R)^a \,, \qquad F'(t) \geqslant 0 \,, \qquad F''(t) \geqslant C(t+R)^{-q} F(t)^p \,,$$

where $p > 1$, $a \geqslant 1$ and

$$0 \leqslant q < (p-1)a + 2 \,,$$

then F blows up in finite time (see also Section 5.2 for more details). Exploiting the hypothesis on the initial data and the definition of u, and choosing suitable a, p and q, one can prove that these conditions are satisfied and F blows up. This implies in particular that there exists a positive T such that

$$||u(t)||_{\mathrm{L}^2(\mathbb{R}^n)} \longrightarrow \infty$$

as $t \uparrow T$. This clarifies also the meaning of the blow-up of u. However, it is important to notice that this technique works straightforward only in the strictly subcritical case $1 < p < p_c(n)$, while in the case $p = p_c(n)$, because of the presence of badly-behaved terms in the proof of the aforementioned estimates, one needs to resort to an auxiliary weighted average function:

$$F_1(t) = \int_{\mathbb{R}^n} u(t,x)\varphi(x)\, dx \,,$$

where the weight φ has to be carefully chosen in order to avoid the badly-behaved terms. This method was introduced in [55] and, in Chapter 5, we shall develop this idea.

The global existence results follow a very different approach: they are based on a contraction argument. The contractions are based on suitable a priori estimates. For instance, in the case $n = 3$, John proves a pointwise inequality equivalent to the following one:

$$||t(t-|x|)^{\nu-2} w||_{\mathrm{L}^\infty} \leqslant C_\nu ||t^\nu (t-|x|)^{\nu(\nu-2)} F||_{\mathrm{L}^\infty}$$

where $F(t,x) = 0$ when $t - |x| \leqslant 1$, $1 < \nu \leqslant 3$ and $w = w(t,x)$ solves the linear inhomogeneous wave equation Cauchy problem

$$\begin{cases} \Box w = F & \text{in } [0, \infty[\times \mathbb{R}^n \,, \\ w(0, x) = \partial_t w(0, x) = 0 & \text{in } \mathbb{R}^n \,. \end{cases}$$

A similar estimate cannot hold in higher dimensions; however, Georgiev, Lindeblad and Sogge have shown that, for $n \geqslant 2$, the following weighted Strichartz estimate holds:

$$||(t^2 - |x|^2)^{\gamma_1} w||_{\mathrm{L}^\nu} \leqslant C_{\nu,\gamma} ||(t^2 - |x|^2)^{\gamma_2} F||_{\mathrm{L}^{\nu/(\nu-1)}} \,,$$

provided that

$$2 \leqslant \nu \leqslant \frac{2(\nu+1)}{\nu-1} \,, \qquad \gamma_1 < n\left(\frac{1}{2} - \frac{1}{\nu}\right) - \frac{1}{2} \,, \qquad \gamma_2 > \frac{1}{\nu} \,.$$

Then, one sets $v_{-1} \equiv 0$ and, for $m = 0, 1, \ldots$, and denotes by v_m the solution to

$$\begin{cases} \Box v_m = |v_{m-1}|^p \,, \\ v_m(0, x) = u_0 \,, \quad \partial_t v_m(0, x) = u_1 \,, \end{cases}$$

where the initial data are small and supported in the ball centered at the origin and of radius $R - 1$. Eventually, one obtains the estimate

$$||((t+R)^2 - |x|^2)^\gamma (u_{m+1} - u_m)||_{\mathrm{L}^{p+1}} \leqslant \frac{1}{2} ||((t+R)^2 - |x|^2)^\gamma (u_m - u_{m-1})||_{\mathrm{L}^{p+1}}$$

for each $m \geqslant 0$ and γ such that

$$\frac{1}{p(p+1)} < \gamma < n\left(\frac{1}{2} - \frac{1}{p+1}\right) - \frac{1}{2} \,,$$

from which one deduces the global existence of the solution to the Cauchy problem (1.1).

1.2 Some known results on the wave equation with potential

A first phisically relevant way to modify the wave equation consists in perturbing it through an effective potential. We consider the Cauchy problem

$$
(1.2) \qquad \begin{cases} \Box u + Wu = F & \text{in } [0,T[\times\mathbb{R}^n\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x) & \text{in } \mathbb{R}^n\,, \end{cases}
$$

where u, W, F depend on $(t,x) \in [0,\infty[\times\mathbb{R}^n$. The function W is called effective potential and it will satisfy precise conditions, above all concerning regularity, sign, decay rate and dependence on t and x. If we know that there exists a global solution to the problem above, we can investigate a priori estimates, that is estimates of the form

$$
||wu|| \leqslant C\,(||w_0u_0||_0 + ||w_1u_1||_1 + ||w_2F||_2)\,,
$$

where C is a positive constant, $w(t,x)$ and $w_j(t,x)$ are weight functions, while $||\cdot||$ and $||\cdot||_j$ are suitable norms on $[0,\infty[\times\mathbb{R}^n$ ($j=0,1,2$). In particular, we are interested in dispersive estimates, i.e. a priori L^∞_x-estimates, which means that we have $\mathrm{L}^\infty(\mathbb{R}^n)$ norms in x.

The dispersive properties of evolution equations are very important for their physical meaning and, consequently, they have been deeply studied, though the problem in its generality is still open. However, several cases have been considered. The dispersive estimate obtained in Corollary 6.1.1 provides the natural decay rate, which is the same rate that one has for the nonperturbed wave equation (see [28, 36]), i.e. a $t^{-(n-1)/2}$ decay in time, where n is the space dimension. The generalization to the case of a potential-like perturbation has been considered widely.

Let us consider the problem

$$
(1.3) \qquad \begin{cases} (\Box + V(x))u = 0 \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^n\,, \\ u(0,x) = f_0(x) \in \mathrm{L}^{p,1}(\mathbb{R}^n)\,, \quad \partial_t u(0,x) = f_1(x) \in \mathrm{L}^p(\mathbb{R}^n)\,. \end{cases}
$$

Beals and Strauss have shown in [4] that one has the decay estimate

$$||u(t)||_{\mathrm{L}^{p'}} \leqslant Ct^{-d}\left(||f_0||_{\mathrm{L}^p} + ||\nabla f_0||_{\mathrm{L}^p} + ||f_1||_{\mathrm{L}^p}\right)$$

provided that $n \geqslant 3$,

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{n+1}, \qquad \frac{1}{p'} = \frac{1}{2} - \frac{1}{n+1}, \qquad d = \frac{n-1}{n+1},$$

$\langle x \rangle^M V \in \mathrm{L}^{(n+1)/2}(\mathbb{R}^n)$ for a suitable M and that, for some $q < (n+1)/2$, the function

$$\mu(t) \doteq t^d ||\langle x \rangle^{-M} u(t)||_{\mathrm{L}^{p'}(\mathbb{R}^n)}$$

satisfies

$$||\mu||_{\mathrm{L}^q(\mathbb{R}^+)} + ||\mu||_{\mathrm{L}^\infty(\mathbb{R}^+)} \leqslant C\left(||f_0||_{\mathrm{L}^p} + ||\nabla f_0||_{\mathrm{L}^p} + ||f_1||_{\mathrm{L}^p}\right).$$

Beals alone has completed this result proving in [3] the estimate

$$||u(t)||_{\mathrm{L}^\infty} \leqslant Ct^{-(n-1)/2}||(1-\Delta)^{\lambda/2} f_1||_{\mathrm{L}^1}$$

as $t \to \infty$, $\lambda > (n+1)/2$, V rapidly decreasing either sufficiently small or non-negative, and $f_0 \equiv 0$.

Burq, Planchon, Stalker and Tahvildar-Zadeh have considered a potential of the form $V = a/|x|^2$ (inverse-square potential), where a is a real number, and have obtained the following weighted L^2-estimate:

$$||\Omega^{-1/2-2\alpha}(-\Delta + V)^{1/4-\alpha} u||_{\mathrm{L}^2} \leqslant C\left(||f_0||_{\dot{\mathrm{H}}^{1/2}} + ||f_1||_{\dot{\mathrm{H}}^{-1/2}}\right),$$

where Ω^s is the multiplication operator defined by

$$(\Omega^s \varphi)(t,x) = |x|^s \varphi(t,x)$$

and $\alpha > 0$ is bounded from above by a suitable positive quantity. They have proved a similar result also for the Schrödinger equation and have used these estimates to obtain Strichartz estimates. For instance, for the wave equation, they have deduced

$$||(-\Delta)^{\sigma/2}u||_{\mathrm{L}^p_t\mathrm{L}^q_x} \leqslant C\left(||f_0||_{\dot{\mathrm{H}}^\gamma} + ||f_1||_{\dot{\mathrm{H}}^{\gamma-1}}\right),$$

provided p, q, γ, σ satisfy some conditions (depending on n). In particular, one can take $\gamma = 1/2$. Planchon, Stalker and Tahvildar-Zadeh have also found in [41] dispersive estimates in a similar setting.

Another Strichartz estimate can be found in [16] (a sort of extension of Cuccagna's paper [15]), where Cuccagna and Schirmer consider the wave equation in $\mathbb{R}^{1+3}$ with a smooth rapidly decreasing small magnetic potential V

$$(\partial_t^2 - \Delta_V)u = f$$

with initial data (f_0, f_1), where

$$\Delta_V = \sum_{j=1}^{3}(\partial_j + iA_j(x))^2\,.$$

The achieved estimate is

$$||u||_{\mathrm{L}^{q_1}(\mathbb{R},\dot{\mathrm{B}}^\rho_{r_1,2})} \leqslant C\left(||f||_{\mathrm{L}^{q_2'}(\mathbb{R},\dot{\mathrm{B}}^\rho_{r_2',2})} + ||f_0||_{\dot{\mathrm{H}}^{2,\mu}} + ||f_1||_{\dot{\mathrm{H}}^{2,\mu-1}}\right),$$

where

$$\begin{aligned}
&\rho + 3\left(\frac{1}{2} - \frac{1}{r_j}\right) - \frac{1}{q_j} = \mu\,, \qquad 0 \leqslant \frac{2}{q_j} \leqslant \min\left\{2\left(\frac{1}{2} - \frac{1}{r_j}\right), 1\right\},\\
&\left(\frac{2}{q_j}, 2\left(\frac{1}{2} - \frac{1}{r_j}\right)\right) \neq (1,1) \qquad j = 1, 2\,,
\end{aligned}$$

and q' denote the dual exponent to q. Here we have denoted by $\dot{\mathrm{H}}^{p,s}$ the completion of $\mathscr{C}_0^\infty(\mathbb{R}^3)$ respect to the norm $||(-\Delta_V)^{s/2}f||_{\mathrm{L}^p}$, while the Besov space $\dot{\mathrm{B}}^s_{p,q}$ is the completion of the same space respect to the norm

$$\left[\sum_{j\in\mathbb{Z}}(2^{js}||\chi(2^{-j}\sqrt{-\Delta_V})f||_{\mathrm{L}^p})^q\right]^{1/q},$$

where $\chi(t) \in \mathscr{C}_0^\infty$ is an appropriate nonnegative function equal to 1 near 1 and with support in $[1/2, 2]$.

Visciglia has shown (see [51, 52]) that the following Cauchy problem for the semilinear wave equation in $\mathbb{R}^{1+3}$ perturbed through a time-dependent potential

$$\begin{cases} \Box u + a_0 \partial_t u + \displaystyle\sum_{i=1}^{3} a_i \partial_{x_i} u + V u = -u|u|^{\alpha-1} & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \,, \\ u(0,x) = f_0(x) \in \dot{\mathrm{H}}^1 \,, \quad \partial_t u(0,x) = f_1(x) \in \mathrm{L}^2 \end{cases}$$

is well-posed when the initial data are compactly supported, $1 \leqslant \alpha < 5$, and

$$a_i(t,x) \in \mathrm{L}^\infty(\mathbb{R}_t \times \mathbb{R}^3_x) \,, \qquad V(t,x) \in \mathrm{L}^\infty(\mathbb{R}_t, \mathrm{L}^3(\mathbb{R}^3_x)).$$

So, in this case, the hypotheses on the potential decay are definitely weaker than in the cases above. The proof is still based on a Strichartz estimate.

In [23], Georgiev, Heiming and Kubo have established a weighted L^∞-estimate for the solution to the linear wave equation with a smooth positive potential depending only on space variables. In particular, they prove that the inequality

$$||\, \tau_+ \, \tau_-{}^\lambda \, u||_{\mathrm{L}^\infty} \leqslant C||\, \tau_+{}^\mu \, \tau_- \, F||_{\mathrm{L}^\infty} \,,$$

where here

$$\tau_\pm = 1 + |\, t \pm |x|\,| \,, \qquad x \in \mathbb{R}^3,$$

holds provided

$$0 \leqslant \lambda < 1 \,, \qquad \mu > 2 + \lambda \,.$$

Here u is the solution to the null data Cauchy problem for the wave equation with potential, with $n = 3$. This estimate is similar to the one of F. John and allows to prove the existence of global small data solutions for the corresponding semilinear wave equation with a potential $W(x) \geqslant 0$, typically $F = |u|^p$ and $F = u|u|^{p-1}$. The result is based on the proof of weighted estimates for the resolvent of the operator $-\Delta + W$, since the representation of the solution to the perturbed wave equation can be connected with the resolvent of $-\Delta + W$.

In [27], Georgiev and Visciglia consider some estimates that hold for the nonperturbed wave equation and extend them to the case of a potential-like perturbation. They assume that the time-independent potential W satisfies

$$W(x) \leqslant \frac{C}{|x|^2(|x|^\varepsilon + |x|^{-\varepsilon})} \qquad \forall x \in \mathbb{R}^3$$

and prove, for every $\psi \in \mathscr{C}_0^\infty(]0,\infty[)$ and each $\vartheta > 0$, the decay estimate

$$||\psi(\vartheta\sqrt{-\Delta + W})u||_{\mathrm{L}^\infty(\mathbb{R}^3)} \leqslant \frac{C}{t\vartheta}||F||_{\mathrm{L}^1(\mathbb{R}^3)}\,.$$

The proof of this result relies on a suitable representation for the operators $\varphi(\sqrt{-\Delta + W})$, that is

$$\varphi(\sqrt{-\Delta + W}) = c\int_0^\infty \lambda\varphi(\lambda)[R_W(\lambda^2 + i0) - R_W(\lambda^2 - i0)]\,d\lambda\,,$$

where

$$R_W(\lambda^2 \pm i0)F = \lim_{\varepsilon\to 0} R_W(\lambda^2 \pm i\varepsilon)F$$

in a suitable L^2-weighted sense and

$$R_W(\lambda^2 \pm i\varepsilon) = [(\lambda^2 \pm i\varepsilon) + \Delta - W]^{-1}$$

for $\varepsilon \neq 0$. Moreover, they prove

$$||u||_{\mathrm{L}^4(\mathbb{R}^3)} \leqslant \frac{C}{\sqrt{t}}||F||_{\mathrm{L}^{4/3}(\mathbb{R}^3)}$$

provided $F \in \mathscr{C}_0^\infty(\mathbb{R}^3)$, and the Strichartz estimate

$$\begin{aligned} &||u||_{\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^3))} + ||u||_{\mathscr{C}^0(\mathbb{R};\dot{\mathrm{H}}^s(\mathbb{R}^3))} + ||\partial_t u||_{\mathscr{C}^0(\mathbb{R};\dot{\mathrm{H}}^{s-1}(\mathbb{R}^3))} \\ &\qquad \leqslant C\left(||u_0||_{\dot{\mathrm{H}}^s(\mathbb{R}^3)} + ||u_1||_{\dot{\mathrm{H}}^{s-1}(\mathbb{R}^3)} + ||F||_{\mathrm{L}^{\tilde{p}}(\mathbb{R};\mathrm{L}^{\tilde{q}}(\mathbb{R}^3))}\right), \end{aligned}$$

where $p, q, \tilde{p}, \tilde{q}, s \in \mathbb{R}$ satisfy

$$\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{2}\,, \qquad \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \geqslant \frac{3}{2}\,, \qquad q < \infty\,, \qquad \tilde{q} > 1\,,$$
$$\frac{1}{p} + \frac{3}{q} = \frac{1}{\tilde{p}} + \frac{3}{\tilde{q}} - 2 = \frac{3}{2} - s\,.$$

D'Ancona and Fanelli have considered in [17] the case

$$\begin{cases} (\partial_t^2 + H)u = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^3\,, \\ u(0,x) = 0\,, \quad \partial_t u(0,x) = g(x)\,, \end{cases} \tag{1.4}$$

where

$$H \doteq -(\nabla + iA(x))^2 + B(x)\,, \tag{1.5}$$

$$A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3\,, \qquad B : \mathbb{R}^3 \longrightarrow \mathbb{R}\,. \tag{1.6}$$

Under suitable conditions on A, ∇A and B, in particular

$$|A(x)| \leqslant \frac{C_0}{r\langle r\rangle(1 + |\lg r|)^\beta}\,, \qquad \sum_{j=1}^{3} |\partial_j A_j(x)| + |B(x)| \leqslant \frac{C_0}{r^2(1 + |\lg r|)^\beta}\,, \tag{1.7}$$

with $C_0 > 0$ sufficiently small, $\beta > 1$ and $r = |x|$, they have obtained the dispersive estimate

$$|u(t,x)| \leqslant \frac{C}{t} \sum_{j \geqslant 0} 2^{2j} ||\langle r\rangle w_\beta^{1/2} \varphi_j(\sqrt{H}) g||_{L^2}\,, \tag{1.8}$$

where $w_\beta \doteq r(1 + |\log r|)^\beta$ and $(\varphi_j)_{j\geqslant 0}$ is a nonhomogeneous Paley–Littlewood partition of unity on $\mathbb{R}^3$.

Another work that we want to cite is [54], a paper of Yajima that studies the existence of the Moeller wave operator for two-dimensional Schrödinger operators. Let $H_0 = -\Delta$ and let $H = -\Delta + V$ be the two-dimensional Schrödinger operator with potential V decaying at infinity according to

$$|V(x)| \leqslant \frac{C}{\langle x \rangle^{\delta}}\,, \qquad \delta > 6\,.$$

Yajima shows that the Moeller wave operator

$$W_{\pm} u = \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{itH}\, \mathrm{e}^{-itH_0}\, u$$

are bounded in $L^p(\mathbb{R}^2)$ for each $p \in]1,\infty[$ provided

$$c_0 \doteq \int V(x)\, dx \neq 0$$

and that $(1-P)G_0V(1-P)$ is invertible in $\mathrm{L}^{2,-s}(\mathbb{R}^2)$ for some $s \in]1, \delta-1[$, where

$$V_0(x) = c_0^{-1} V(x)\,, \qquad P_0 u(x) = \int u(x)\, dx\,,$$
$$G_0 u(x) = -\frac{1}{2\pi}\int (\log|x-y|) u(y)\, dy\,, \qquad P = P_0 V_0\,.$$

In other words, he gets the estimate

$$||W_{\pm}u||_{\mathrm{L}^p} \leqslant C_p ||u||_{\mathrm{L}^p}\,,$$

with C_p independent of $u \in \mathrm{L}^2(\mathbb{R}^2) \cup \mathrm{L}^p(\mathbb{R}^2)$.

1.3 Main original results

First of all, we consider a metric perturbation of the Cauchy problem explained in Section 1.1. In other words, the problem is formally the same, but we consider, instead of the d'Alembert operator $\square$, that is the Laplace operator in the Minkowski metric $(+1,-1,\dots,-1)$, its equivalent $\square_g$ in the Schwarzschild metric:

$$\square_g = \frac{1}{F}\left(\partial_t^2 - \frac{F}{r^2}\partial_r(r^2F)\partial_r - \frac{F}{r^2}\Delta_{\mathbb{S}^2}\right),$$

where

$$F(r) = 1 - \frac{2M}{r}\,, \qquad r = |x|\,, \qquad M > 0\,,$$

$\Delta_{\mathbb{S}^2}$ is the standard Laplace–Beltrami operator on the two-dimensional sphere and

$$(t, x) \in \mathbb{M} = \mathbb{R} \times \mathbf{\Omega}\,, \qquad \mathbf{\Omega} = \{(r, \omega) : r > 2M,\, \omega \in \mathbb{S}^2\} =]2M, \infty[\times \mathbb{S}^2\,.$$

The Schwarzschild metric is interesting, since it is an exact solution to the Einstein's empty space field equations

$$R_{\alpha\beta} = 0\,,$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor, and in particular it is a model for a spherically symmetric (static) black hole. This metric is studied in Chapter 3, where other physical implications are described.

We begin by considering the Cauchy problem (for the semilinear wave equation in the Schwarzschild metric)

$$\begin{cases} \Box_g u = |u|^p & \text{in } [0, \infty[\times \mathbf{\Omega}\,, \\ u(0, x) = u_0(x)\,, \quad \partial_t u(0, x) = u_1(x) & \text{in } \mathbf{\Omega}\,. \end{cases}$$

A natural question is to establish whether, also in this case, a situation similar to the one described in Section 1.1 still holds, that is to determine the existence of a critical exponent $\bar{p} \in]1, \infty[$ such that, under suitable hypotheses on the small data u_0 and u_1, the solution blows up in finite time for every $p \in]1, \bar{p}[$, while it exists globally for every $p > \bar{p}$. In this case, it is also interesting to compare $\bar{p}$ with the three-dimensional critical exponent $p_c(3) = 1 + \sqrt{2}$.

As to blow-up, we obtain two results. In both of them, we restrict ourselves to symmetrically radial solutions, so that the problem is equivalent to (see Chapter 4)

$$\begin{cases} [\partial_{tt} - \partial_{ss} + W(s)]\, v = f(s)|v|^p\,, & (t, s) \in [0, \infty[\times \mathbb{R}\,, \\ v(0, s) = v_0(s)\,, \quad \partial_t v(0, s) = v_1(s)\,, & s \in \mathbb{R} \end{cases} \tag{1.9}$$

for suitable $W(s),\ f(s) \in \mathscr{C}(\mathbb{R})$ satisfying the following estimates:

$$\begin{aligned}
&W(s) > 0\,, && f(s) > 0 && \forall s \in \mathbb{R}\,,\\
&W(s) \sim s^{-3}\,, && f(s) \sim s^{1-p} && \forall s \geqslant 1\,,\\
&W(s) \sim \mathrm{e}^{s/(2M)}\,, && f(s) \sim \mathrm{e}^{s/(2M)} && \forall s \leqslant 0\,.
\end{aligned}$$

The notation $f \lesssim g$ means that there exists a positive constant C so that $f \leqslant Cg$ and the standard notation $f \sim g$ is equivalent to $f \lesssim g$ and $g \lesssim f$.

To study the maximal time interval of existence of the solution, we choose the following initial data:

$$v_0(s) = \rho(\varepsilon)\chi_0\big(s - s_0(\varepsilon)\big)\,, \qquad v_1(s) = \rho(\varepsilon)\chi_1\big(s - s_0(\varepsilon)\big)\,, \tag{1.10}$$

where $\chi_j \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfy, for $j = 1, 2$, the conditions

$$\chi_j(s) \geqslant 0\,, \qquad s \in \mathbb{R}\,, \tag{1.11}$$

$$\chi_j(s) = 1\,, \qquad s \in [-R/2, R/2]\,, \tag{1.12}$$

$$\operatorname{supp}\chi_j \subseteq [-R, R] \tag{1.13}$$

for a positive constant R. The function $\rho(\varepsilon)$ will be chosen appropriately later on.

It is not difficult to see that

$$||v_0||_{\mathrm{H}^\sigma(\mathbb{R})} + ||v_1||_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \sim \rho(\varepsilon) \tag{1.14}$$

for each $\sigma \geqslant 1$, so the initial data in (1.10) have small $\mathrm{H}^\sigma \times \mathrm{H}^{\sigma-1}$ norms provided $\rho(\varepsilon)$ is small.

Moreover, note that a big Regge–Wheeler coordinate s corresponds to the domain where one is far away from the black hole ($\mathbb{R}^3 \setminus \mathbf{\Omega}$), i.e. the domain with almost flat metric. On the other hand, $s \to -\infty$ corresponds to the domain close to the black hole.

The first result follows closely what holds in the Minkowski space. In this case, we shall choose $\varepsilon > 0$ sufficiently small and shall set

$$s_0(\varepsilon) = \varepsilon^{-\vartheta}\,, \qquad \rho(\varepsilon) = \varepsilon\,, \tag{1.15}$$

where ϑ satisfies

$$\vartheta \geqslant \frac{b(p-1)}{-p^2+2p+1}\,, \qquad \vartheta \geqslant 1 + \frac{b(3p-5)}{-p^2+2p+1}\,, \tag{1.16}$$

and $b = p$ if $p \in [2, 1+\sqrt{2}[$, $b = p^2$ if $p \in]1,2[$.

In this case, the initial data in (1.10) have support far away from the black hole and small data solutions manifest a blow-up phenomenon in the subcritical case. We have the following theorem.

Theorem 1.3.1. *Under the above hypotheses on the initial data, for any p, $1 < p < 1+\sqrt{2}$, there exists a positive number ε_0 so that for any $\varepsilon \in]0, \varepsilon_0[$ there exists a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathscr{C}^k\big([0,T[; \mathrm{H}^{2-k}(\mathbb{R})\big)$$

of (1.9) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbb{R})} = \infty\,.$$

The above result means that, when the initial data are supported far away from the black hole, the wave equation in the Schwarzschild metric has a critical exponent similar to that one of the free wave equation. In this region, we can estimate from above the lifespan of the solution.

The situation changes completely in the second case, when one tries to approach the black hole. To have a model that simulates this phenomenon, we take initial data such that

$$s_0(\varepsilon) = -T_2(\varepsilon)\,, \tag{1.17}$$

where $T_2(\varepsilon) > 0$ grows very rapidly as $\varepsilon \to 0$. More precisely, we take $T_2(\varepsilon) \in \Sigma$, where Σ is the following class of functions:

$$\Sigma = \{T(\varepsilon) \in \mathscr{C}(]0,1]) \,:\, \forall A \geqslant 1,\, \lim_{\varepsilon \downarrow 0} \varepsilon^A T(\varepsilon) = \infty\}. \tag{1.18}$$

A typical example is $T(\varepsilon) = \mathrm{e}^{1/\varepsilon}$.

Approaching the black hole, one meets an essential difficulty to overcome the attraction force of the black hole. In this case, the coefficient $f(s)$ in the source term in (1.9) decays exponentially and this dissipative phenomenon is in competition with the blow-up properties of the source term. Because of this, the blow-up mechanism which we propose is based on a different choice of the quantity $\rho(\varepsilon)$ that measures the Sobolev norm of the initial data according to (1.14). We have to take $\rho(\varepsilon) \in \Sigma$, i.e. the initial data are large.

Then we have the following blow-up result.

Theorem 1.3.2. *For any p, $2 < p < 1 + \sqrt{2}$, there exists a positive number ε_0 so that for any $\varepsilon \in]0, \varepsilon_0[$ and any initial data satisfying the aforementioned hypotheses, there exists a function $\rho(\varepsilon) \in \Sigma$, a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathscr{C}^k\big([0, T[; \mathrm{H}^{2-k}(\mathbb{R})\big)$$

of (1.9) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbb{R})} = \infty\,.$$

The base idea for both results is to adapt the approach described in Section 1.1, but this approach meets the essential difficulty that there is no simple explicit representation of the corresponding fundamental solution to the d'Alambert operator in the Schwarzschild metric. In particular, one has to handle the sign-changing properties of the fundamental solution of the linear wave equation in Schwarzschild metric (or more generally in curved metrics). In the case of the flat $(1+3)$-Minkowski metric, the fundamental solution is nonnegative and this property is used effectively in the study of the blow-up phenomenon for the corresponding semilinear wave equation.

On the other hand, one is interested in a global existence result for a similar problem. At the moment, this problem is widely open (see Section 7.1).

Finally, we consider a different kind of perturbation of the wave equation or, more precisely, of the d'Alembert operator. Indeed, we consider a Cauchy problem for the linear wave equation in the Minkowski space with a perturbation given by an electromagnetic potential satisfying suitable hypotheses.

In particular, we investigate the dispersive properties of the linear wave equation

$$(\Box_A - B)u = F \qquad (t,x) \in [0,\infty[\times\mathbb{R}^3, \tag{1.19}$$

where

$$x = (x_1, x_2, x_3)\,, \qquad r = |x|\,, \tag{1.20}$$

$$\Box_A = \Box - A \cdot \nabla_{t,x}\,, \tag{1.21}$$

$$\Box = \partial_t^2 - \Delta = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)\,, \tag{1.22}$$

$$\nabla_{t,x} = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})\,. \tag{1.23}$$

As anticipated, we assume that the potential $A = A(t,x)$, depending on space and time, is electromagnetic, that is, A assumes imaginary values. This will play a crucial role in the development of the proof, since electromagnetic potentials are gauge invariant (see what follows). Note that also in the previous problems, where we considered the Schwarzschild metric instead of the Minkowski one, we could reduce the problem to the case of the Minkowski metric with an effective potential.

We assume further that the potential decreases sufficiently rapidly when r approaches infinity; more precisely, we suppose that

$$\sum_{j\in\mathbb{Z}} 2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A||_{L^\infty} \leqslant \delta_A \tag{1.24}$$

(that is, A is a short-range potential), where $\varepsilon_A > 0$, δ_A is a sufficiently small positive constant independent of r and the sequence $(\varphi_j)_{j\in\mathbb{Z}}$ is a Paley–Littlewood partition of unity, which means that $\varphi_j(r) = \varphi(2^j r)$ and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ is the set of all nonnegative real numbers) is a function so that

(a) $\operatorname{supp}\varphi = \{r \in \mathbb{R} \,:\, 2^{-1} \leqslant r \leqslant 2\}$;

(b) $\varphi(r) > 0$ for $2^{-1} < r < 2$;

(c) $\sum_{j\in\mathbb{Z}} \varphi(2^j r) = 1$ for each $r \in \mathbb{R}^+$.

In other words, $\sum_{j\in\mathbb{Z}} \varphi_j(r) = 1$ for all $r \in \mathbb{R}^+$ and

$$\operatorname{supp}\varphi_j = \{r \in \mathbb{R} \, : \, 2^{-j-1} \leqslant r \leqslant 2^{-j+1}\}. \tag{1.25}$$

Similarly, we assume for $B = B(t,x)$ the smallness hypothesis

$$\sum_{j\in\mathbb{Z}} (2^{-j})^2 \langle 2^{-j}\rangle^{\varepsilon_A} ||\varphi_j B||_{L^\infty} \leqslant \delta_A \, . \tag{1.26}$$

Moreover, we shall restrict ourselves to radial solutions $u = u(t,r)$, with $F = F(t,r)$, $A = (A_0, A_1, A_2, A_3)$, where

$$A_j = A_j(t,r) \in i\mathbb{R} \qquad j = 0,1,2,3\,, \tag{1.27}$$

and $B = B(t,r)$. This is another similarity with the previous problems.

Because of this assumption, setting

$$\tilde{A} = (\tilde{A}_0, \tilde{A}_1)\,, \qquad \tilde{A}_0 = A_0\,, \qquad \tilde{A}_1 = \frac{A_1x_1 + A_2x_2 + A_3x_3}{r}\,, \tag{1.28}$$

we have

$$A \cdot \nabla_{t,x} = \tilde{A} \cdot \nabla_{t,r}\,, \qquad \nabla_{t,r} = (\partial_t, \partial_r)\,. \tag{1.29}$$

It is well-known that there exists a unique global solution to the Cauchy problem

$$\begin{cases} (\Box_A - B)u = F & (t,x) \in [0,\infty[\times\mathbb{R}^3, \\ u(0,x) = \partial_t u(0,x) = 0 & x \in \mathbb{R}^3\,; \end{cases} \tag{1.30}$$

in particular, this fact holds for the smaller class of radial solutions, that is for the problem

$$\begin{cases} (\Box_A - B)u = F & (t,r) \in [0,\infty[\times\mathbb{R}^+, \\ u(0,r) = \partial_t u(0,r) = 0 & r \in \mathbb{R}^+. \end{cases} \tag{1.31}$$

Let us introduce the change of coordinates

$$\tau_\pm \doteq \frac{t \pm r}{2} \tag{1.32}$$

and the standard notation $\langle s\rangle \doteq \sqrt{1+s^2}$; our main result can be expressed as follows.

Theorem 1.3.3. *Let u be a radial solution to* (1.30)*, i.e. a solution to* (1.31)*, where $A = A(t,r)$ and $B = B(t,r)$ satisfy respectively* (1.24) *and* (1.26) *for some $\delta_A > 0$ and $\varepsilon_A > 0$. Then, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each $\delta_A \in]0,\delta]$, one has*

$$||\,\tau_+\, u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}}\,. \tag{1.33}$$

Let us introduce the differential operators

$$\nabla_\pm \doteq \partial_t \pm \partial_r\,. \tag{1.34}$$

The proof of the previous a priori estimate follows easily from the following one.

Lemma 1.3.1. *Under the same conditions of Theorem 1.3.3, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each $\delta_A \in]0,\delta]$, one has*

$$||\,\tau_+\, r\nabla_- u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}}\,. \tag{1.35}$$

An immediate consequence of Theorem 1.3.3 is the following dispersive estimate.

Corollary 1.3.1. *Under the same conditions of Theorem 1.3.3, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each*

$\delta_A \in]0, \delta]$, *one has*

$$|u(t,r)| \leqslant \frac{C}{t} || \tau_+ r^2 \langle r \rangle^\varepsilon F ||_{L^\infty_{t,r}} \tag{1.36}$$

for every $t > 0$.

To prove the lemma, one begins by thinking the potential term in (1.31) as part of the forcing term, that is, $(\Box_A - B)u = F$ can be viewed as

$$\Box u = F_1 \doteq F + \tilde{A} \cdot \nabla_{t,r} u + Bu \,. \tag{1.37}$$

Then we can rewrite this equation in terms of $\tau_\pm$ and $\nabla_\pm$, obtaining

$$\nabla_+ \nabla_- v = G \,, \tag{1.38}$$

where

$$v(t,r) \doteq r u(t,r) \qquad \text{and} \qquad G(t,r) \doteq r F_1(t,r) \,. \tag{1.39}$$

This last equation can be easily integrated to obtain a relatively simple explicit representation of $(\nabla_- v)(\tau_+, \tau_-)$ in terms of G.

Another fundamental step consists in taking advantage of the gauge invariance property of the electromagnetic potential A, which means that, set

$$A_\pm \doteq \frac{\tilde{A}_0 \pm \tilde{A}_1}{2} \,, \tag{1.40}$$

we can assume, without loss of generality, that $A_+ \equiv 0$ (see [5], p. 34). This implies that

$$\tilde{A} \cdot \nabla_{t,r} u = A_- \nabla_- u + A_+ \nabla_+ u = A_- \nabla_- u \,. \tag{1.41}$$

1.4 Related problems

When the solution blows up in finite time, it is interesting to have an estimate of the lifespan $T(\varepsilon)$ depending on the parameter ε, which provides a measure of the smallness of the initial data. For the semilinear wave equation in the $(1+n)$-Minkowski metric ($\Box_g = \Box$), Zhou has proved (see [56, 57]) that the following limit exists, provided $p < p_0$ (subcritical case):

$$T_p = \lim_{\varepsilon \to 0} \varepsilon^{k(p)} T(\varepsilon) \in]0, \infty[\,, \qquad k(p) = \frac{2p(p-1)}{2+3p-p^2}\,.$$

In the critical case $p = p_0(n)$, he has got the existence of two positive constants c and C independent of ε such that

$$\exp(c\varepsilon^{-p(p-1)}) \leqslant T(\varepsilon) \leqslant \exp(C\varepsilon^{-p(p-1)})\,.$$

All these results hold for $n = 2, 3$, while when $n = 4$ Li Ta-Tsien and Zhou Yi have shown in [38] the following estimate from below:

$$T(\varepsilon) \geqslant \exp(c\varepsilon^{-2})\,, \qquad p = p_0(4) = 2\,.$$

This problem is still open in its generality for $n \geqslant 4$. We also lack any precise result in the presence of the Schwarzschild metric, though our proof of the blow-up result suggests a rough estimate in the subcritical case (see Sections 5.3 and 5.4 for these estimates and their proofs when we have small initial data far from the black hole or large initial data next to the black hole respectively; see the end of Section 7.1 for further comments).

The large-time behavior of the solution can be investigated for other evolution equations, and in particular for the Schrödinger equation. The solution u to the Schrödinger Cauchy problem

$$\begin{cases} (i\partial_t + H_0)u = 0\,, \\ u(0) = f \end{cases}$$

is dispersive in the sense that, for each $t > 0$, one has

$$||u(t)||_{\mathrm{L}^{p'}(\mathbb{R}^n)} \lesssim t^{-n(1/p-1/2)} ||f||_{\mathrm{L}^p(\mathbb{R}^n)}\,, \tag{1.42}$$

provided $1 \leqslant p \leqslant 2$ and $1/p + 1/p' = 1$. Replacing H_0 with more general

Hamiltonians

$$H = -\Delta + V(x) ,$$

the situation becomes more complicated. Journé, Soffer and Sogge have considered in [34] potentials $V(x)$ satisfying

$$\begin{cases} \langle x \rangle^{\alpha} V(x) \,:\, \mathrm{H}^{\eta} \to \mathrm{H}^{\eta} \\ \hat{V} \in \mathrm{L}^1(\mathbb{R}^n) , \end{cases}$$

where $\alpha > n+4$, $\eta > 0$ and $\hat{V}$ is the Fourier transform of V. If $n \geqslant 3$ and P_c denotes the projection onto the continuous part of the spectrum of H, then their main result is the following: if 0 is neither an eigenvalue nor a resonance for H then, for each $t > 0$,

$$|| \, \mathrm{e}^{itH} \, P_c \psi ||_{\mathrm{L}^{p'}(\mathbb{R}^n)} \lesssim t^{-n(1/p-1/2)} ||\psi||_{\mathrm{L}^p(\mathbb{R}^n)} ,$$

with p and p' as before. Thus, if $u = \mathrm{e}^{itH} \, f$ (i.e. it is a solution to the Cauchy problem for the Schrödinger equation with Hamiltonian H and initial datum f) and f is orthogonal to the bound states of H, estimate (1.42) still holds.

Georgiev and Tarulli have studied in [26] the smoothing properties of this equation with magnetic potential $A = (A_1, \dots, A_n)$, where $A_j(t,x) \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $n \geqslant 3$. The corresponding Cauchy problem has the form

$$\begin{cases} (\partial_t - i\Delta_A) u = F , \\ u(0,x) = f(x) , \end{cases}$$

where

$$\Delta_A = \sum_{j=1}^{n} (\partial_{x_j} - iA_j)(\partial_{x_j} - iA_j) .$$

Under the essential assumption

$$\max_{1 \leqslant j \leqslant n} \sum_{k \in \mathbb{Z}} \sum_{|\beta| \leqslant 1} 2^{k(1+|\beta|)} ||D_x^{\beta} A_j(t,x)||_{\mathrm{L}_t^{\infty} \mathrm{L}^{\infty}_{|x| \sim 2^k}} \leqslant \varepsilon$$

for a suitable $\varepsilon > 0$, which avoids eigenvalues or resonances of Δ_A, the following estimate holds:

$$\int_{\mathbb{R}} \left(\sup_{k\in\mathbb{Z}} \left\| |x|_k^{-1/2} u(t) \right\|_{\dot{\mathrm{H}}_x^{1/2}} \right)^2 dt \lesssim ||f||_{\mathrm{L}_x^2}^2 + \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} \left\| |x|_k^{1/2} F(t) \right\|_{\dot{\mathrm{H}}_x^{-1/2}} \right)^2 dt\,,$$

where

$$|x|_k^{\pm 1/2} = |x|^{\pm 1/2} \varphi\left(\frac{|x|}{2^k}\right), \qquad \varphi \in \mathscr{C}_0^\infty(]1/2, 2[)\,, \qquad \varphi \geqslant 0$$

and

$$\sum_{k\in\mathbb{Z}} \varphi(|x|/2^k) = 1\,.$$

In [25], Georgiev, Stefanov and Tarulli have proved, under similar hypotheses and for suitable spaces X and X', the estimate

$$\left\| \int_{t-s>0} \mathrm{e}^{i(t-s)\Delta_A} F(s)\, d \right\|_{X'} \lesssim ||F||_X\,,$$

which implies, in particular, the smoothing Strichartz estimate

$$||u||_{X'} \lesssim ||f||_{\mathrm{L}^2} + ||F||_X\,.$$

Rodnianski and Schlag have established in [43] dispersive estimates for solutions to the linear Schrödinger equation in three dimensions

$$\frac{1}{i}\partial_t \psi - \Delta\psi + V\psi = 0\,, \qquad \psi(s) = f\,,$$

where $V(t,x)$ is a time-dependent potential that satisfies

$$\sup_t ||V(t)||_{\mathrm{L}^{3/2}(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \frac{V(\hat{\tau}, x)}{|x-y|}\, d\tau dy < c_0\,.$$

Here c_0 is a small constant and $V(\hat{\tau}, x)$ represents the Fourier transform with respect to the first variable. Under these conditions, the above problem admits solutions

$$\psi(\cdot) \in \mathrm{L}_t^\infty \mathrm{L}_x^2(\mathbb{R}^3) \cap \mathrm{L}_t^2 \mathrm{L}_x^6(\mathbb{R}^3)\,, \qquad \forall f \in \mathrm{L}^2(\mathbb{R}^3)\,,$$

satisfying the dispersive inequality

$$||\psi(t)||_{\mathrm{L}^\infty} \lesssim |t-s|^{-3/2}||f||_{\mathrm{L}^1}$$

for all times t, s. For the case of time independent potentials $V(x)$, the same estimate remains true if

$$\int_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\,dxdy < (4\pi)^2 \quad \text{and} \quad ||V||_{\mathscr{K}} \doteq \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\,dy < 4\pi\,.$$

The authors also establish the dispersive estimate with an ε-loss for large energies provided $||V||_{\mathscr{K}} + ||V||_{\mathrm{L}^2} < \infty$. Finally, they prove Strichartz estimates for the Schrödinger equations with potentials that decay like $|x|^{-2-\varepsilon}$ in dimensions $n \geqslant 3$, thus solving an open problem posed by Journé, Soffer and Sogge.

These problems can be compared with the ones that we have presented in Section 1.2, where also potentials satisfying different conditions are considered. The cited papers provide further references.

1.5 Structure of the work

The first part of this work is dedicated to the introduction of preliminary notions, both from a geometrical and a physical point of view. **Chapter 2** contains notions of Riemannian geometry, recalling rapidly but completely what needed to define the Schwarzschild metric and the Einstein's equations. We give the fundamental definitions and fix some notations. **Chapter 3** is dedicated to the description of the Schwarzschild metric, its properties and its physical interpretation. The main aim is to explain why it is interesting to consider the wave equation in the Schwarzschild metric from a physical point of view.

Then we move to the core of the work, that is the details about the original results discussed in this thesis. In **Chapter 4** we present some preliminary results for the wave equation in the Schwarzschild metric, that is the reduction of the problem to the one dimensional case of a wave equation with potential in the Minkowski metric, and a local existence theorem. We also provide some asymptotic estimates. We shall exploit either these results in Chapter 5. **Chap-**

ter 5 deals with the blow-up results concerning the semilinear wave equation in the Schwarzschild metric, i.e. Theorem 1.3.1 and Theorem 1.3.2. In **Chapter 6**, we afford the other main problem of this thesis and we prove some a priori estimates for the linear wave equation with electromagnetic potential; in particular, we show the dispersive estimate of Theorem 1.3.3, Lemma 1.3.1 and Corollary 1.3.1.

Eventually, in **Chapter 7** we present some open problems related to the main original results, i.e. concerning the semilinear wave equation in the Schwarzschild metric and the wave equation with electromagnetic potential.

Chapter 2

Elements of Pseudo-Riemannian Geometry

The aim of this chapter consists in introducing some fundamental notions —both definitions and results— of pseudo-Riemannian geometry. We shall introduce the essential stuff that we shall need in the following chapters, above all to fix notations; for a definitely more complete treatment, we refer to (13) (in particular, Chapters III and V), (1), (18) (Chapters 0–4) and (19) (Chapters 1–4). Penrose diagrams, as well as notions about the Minkowski metric, are studied in (20) and (30).

In Section 2.1, we define differentiable manifolds and local coordinates, differentiability between two manifolds, tangent vectors and differentials. We also explain and assume the Einstein convention about repeated indices, a very useful notation tool in differential geometry.

Section 2.2 is dedicated to tensors and the tensor product. We also revise the exterior product and the exterior differentiation operator.

In Section 2.3, we review metric and pseudo-Riemannian manifolds. We also introduce the Laplace–Beltrami operator respect to a metric.

Section 2.4 is dedicated to the Minkowski metric. We use several coordinate systems to study this metric and represent it through a Penrose Diagram.

We conclude with Section 2.5, where we recall linear connections and in particular Riemannian connections. These notions lead to the definition of the Ricci tensor, which is the main character of the Einstein equation, an important equation of general relativity that is the source of the topic of the following chapter and the setting of Chapters 4 and 5: the Schwarzschild metric.

2.1 Differentiable manifolds

An n-dimensional **manifold** is a Hausdorff topological space such that every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

A **chart** (U, φ) of a manifold M is an open set U of M together with a homeomorphism (a bijective bicontinuous application) $\varphi : U \to V$ onto a (necessarily) open set $V \subset \mathbb{R}^n$. The coordinates $(x^1, x^2, \dots, x^n)$ of the image $\varphi(x) \in \mathbb{R}^n$ of the point $x \in U \subset M$ are called **coordinates** of x in the chart (U, φ); this is the reason why charts are also called **local coordinate systems**.

An **atlas** of class $\mathscr{C}^k$ on a manifold M is a set $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that:

- $M = \bigcup_{\alpha\in A} U_\alpha$,
- the maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are of class $\mathscr{C}^k$.

Two $\mathscr{C}^k$-atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A'}$ are **equivalent** if and only if their union $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A\cup A'}$ is again a $\mathscr{C}^k$-atlas.

A $\mathscr{C}^k$**-manifold** is a manifold M together with a an equivalence class of $\mathscr{C}^k$-atlases. A **smooth manifold** is a $\mathscr{C}^\infty$-manifold. If we require only the differentiability of the involved maps, we have **differentiable atlases** and **differentiable manifolds**.

We use local charts to introduce the notion of differentiability on a manifold. Let us consider a real function defined on a manifold M, i.e. $f : M \to \mathbb{R}$. If (U, φ) is a chart at x, that is $x \in U$, then $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ represents f in the local chart. We say that f is **differentiable** at x if, in a chart at x, $f \circ \varphi^{-1}$ is differentiable at $\varphi(x)$. This is a suitable definition, since it does not depend on the choice of the chart. As a matter of fact, if $f \circ \varphi^{-1}$ is differentiable at $\varphi(x)$, then $f \circ \varphi'^{-1}$ is differentiable at $\varphi'(x)$ for every chart (U', φ') at x, since

$$f \circ \varphi'^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1}).$$

More generally, given two differentiable manifolds M and N, we say that a function $f : M \to N$ is **differentiable** at $x \in M$ if $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(x)$, where (U, φ) and (W, ψ) are local charts at x and at $f(x)$ respectively.

The tangent vector space at a point can be defined following several approaches. The straightest way is to say that a **tangent vector** v_x to a smooth manifold M at a point x is a linear map from the space of functions defined and differentiable on some neighborhood of $x \in M$ into $\mathbb{R}$ that satisfies the Leibniz rule, i.e.

$$\begin{aligned} v_x(af + bg) &= av_x(f) + bv_x(g) && \textit{linearity}, \\ v_x(fg) &= f(x)v_x(g) + g(x)v_x(f) && \textit{Leibniz rule} \end{aligned}$$

for all $a, b \in \mathbb{R}$ and for all real functions f, g on M differentiable at x. The space T_xM of all such vectors, endowed with addition and scalar multiplication defined by

$$(au_x + bv_x)(f) = au_x(f) + bv_x(f)\,,$$

is a vector space called **tangent vector space at the point** x.

This definition of tangent vector can be made more precise through the notion of germ. We say that two functions on M differentiable at x have the same **germ** at x if they coincide in a neighborhood of x. The equivalence class of differentiable functions at x with the same germ of a function f is called **germ** of f. Therefore, a **tangent vector** can be viewed as a derivation on the algebra of germs of differentiable functions at x.

In the chart (U, φ), the local coordinates, or **components**, of a tangent vector are the set of numbers

$$v^i = v_x(\varphi^i)\,,$$

where φ^i are the coordinates of φ in $\mathbb{R}^n$. In the local chart (U, φ), if f is a $\mathscr{C}^1$-function on a neighborhood of $x_0 \in M$, thanks to the mean value Lagrange theorem we have, with $\bar{f} = f \circ \varphi^{-1}$,

$$f(x) = \bar{f}(\varphi(x_0)) + (\varphi^i(x) - \varphi^i(x_0)) \frac{\partial \bar{f}}{\partial x^i}\bigg|_{\varphi(x_0)+s(\varphi(x)-\varphi(x_0))} \tag{2.1}$$

for some $s \in]0,1[$, where here and in the following we assume tacitly summation over repeated indices. More precisely, we adopt the **Einstein convention** about repeated indices: if the same index appears twice in the same formula, once up and once down, we suppose understood a sum over all the possible values for that index. For instance, formula (2.1) can be made explicit in the form

$$f(x) = \bar{f}(\varphi(x_0)) + \sum_{i=1}^{n} (\varphi^i(x) - \varphi^i(x_0)) \frac{\partial \bar{f}}{\partial x^i}\bigg|_{\varphi(x_0)+s(\varphi(x)-\varphi(x_0))} .$$

At x_0, the derivative of f along the vector v_x is

$$v_{x_0}(f) = v^i \frac{\partial \bar{f}}{\partial x^i}\bigg|_{\varphi(x_0)} , \qquad \text{where } v^i = v_{x_0}(\varphi^i) . \tag{2.2}$$

Each element $v_x \in T_x M$ can therefore be represented in the form

$$v_x = v^i \frac{\partial}{\partial x^i} ,$$

where the vectors $\partial/\partial x^i$ tacitly depend on x, according to (2.2). The set of these vectors, $\{\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n\}$, is a basis, called **natural basis**, for the tangent vector space, which has whence the same dimension of the manifold (note that the chart (U, φ) at x induces an isomorphism of $T_x M$ onto $\mathbb{R}^n$). The elements of the dual basis of the natural basis, called **natural cobasis**, are often denoted by dx^i, so that $dx^i(\partial/\partial x^j) = \delta^i_j$, where the Kronecker δ symbol is defined as

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j , \\ 0 & \text{if } i \neq j . \end{cases}$$

If $f : M \to N$ is a mapping differentiable at x between manifolds, we define the **differential** of f at x by

$$df_x : T_x(M) \ni v \longrightarrow w \in T_{f(x)}(N) ,$$

where for every function h differentiable at $f(x)$, we have the identity

$$w(h) = v(h \circ f)\,.$$

In the following, we shall simply write T_x instead of T_xM.

2.2 Tensor and exterior products

Let $V_1, V_2, \dots, V_n, W$ be vector spaces on $\mathbb{K}$ and let

$$M(V_1, V_2, \dots, V_n; W)$$

denote the (natural) vector space on $\mathbb{K}$ of all multilinear (i.e. linear in each component) maps from $V_1 \times V_2 \times \cdots \times V_n$ to W. We assume that $V_1, V_2, \dots, V_n$ are finite-dimensional spaces, denote by V^* the dual of a vector space V, and set

$$T = M(V_1^*, V_2^*, \dots, V_n^*; \mathbb{K})\,.$$

If $F \in M(V_1, V_2, \dots, V_n; T)$ is given by

$$F(v_1, v_2, \dots, v_n)(\varphi^1, \varphi^2, \dots, \varphi^n) = \varphi^1(v_1)\varphi^2(v_2)\cdots\varphi^n(v_n)$$

for all $v_i \in V_i$ and for all $\varphi^i \in V_i^*$, then we have the following result (see [1], page 3, Theorem 1.1.3).

Theorem 2.2.1. *With the notations above, we have that:*

(a) for each vector space W on $\mathbb{K}$ and for each $\Phi \in M(V_1, V_2, \dots, V_n; W)$, there exists a unique linear map $\tilde{\Phi} : T \to W$ such that $\Phi = \tilde{\Phi} \circ F$ (universal property of the tensorial product);

(b) if (T', F') is another couple satisfying a)*, there exists a unique isomorphism $\Psi : T \to T'$ such that $F' = \Psi \circ F$ (uniqueness of the tensorial product).*

Two couples (T_1, F_1), (T_2, F_2), where T_j are vector spaces and

$$F_j \in M(V_1, V_2, \dots, V_n; T_j)\,,$$

are called **isomorphic** if there exists an isomorphism $\Psi : T_1 \to T_2$ such that

$F_2 = \Psi \circ F_1$.

A couple (T, F) satisfying the properties of Theorem 2.2.1(a) is called the **tensor product** of the vector spaces $V_1, V_2, \ldots, V_n$ and denoted by $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Thanks to Theorem 2.2.1(b), the tensor product is well-defined modulo some isomorphisms. The elements of the form $F(v_1, v_2, \ldots, v_n)$ are denoted by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$.

In other words, if

$$v_i \in V_i\,, \qquad \varphi^i \in V_i^* \qquad i = 1, 2, \ldots, n\,,$$

we have

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n(\varphi^1, \varphi^2, \ldots, \varphi^n) = \varphi^1(v_1)\varphi^2(v_2)\cdots\varphi^n(v_n)$$

and, from the multilinearity of F,

$$\lambda(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (\lambda v_1) \otimes v_2 \otimes \cdots \otimes v_n = \cdots = v_1 \otimes v_2 \otimes \cdots \otimes (\lambda v_n)\,,$$

$$\begin{aligned} v_1 \otimes \cdots \otimes (v_i' + v_i'') \otimes \cdots \otimes v_n &= v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_n \\ &\quad + v_1 \otimes \cdots \otimes v_i'' \otimes \cdots \otimes v_n \end{aligned}$$

for all $\lambda \in \mathbb{K}$, for all $v_i', v_i'' \in V_i$.

We introduce the following spaces:

$$T_0^0(V) = T^0(V) = T_0(V) = \mathbb{K}\,, \qquad T^p(V) = T_0^p(V) = \underbrace{V \otimes \cdots \otimes V}_{\text{p times}}\,,$$
$$T_q(V) = T_q^0(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{\text{q times}}\,, \qquad T_q^p(V) = T^p(V) \otimes T_q(V)\,.$$

An element of $T_q^p(V)$ is called p**-contravariant** q**-covariant tensor**, or **tensor of type** (p, q).

We denote by $\mathfrak{S}_p$ the set of all permutations of $(1, 2, \ldots, p)$. It is known that each $\sigma \in \mathfrak{S}_p$ can be written as a composition of transpositions. This decomposition can vary but the number of transpositions is always the same. We define the **sign of a permutation** σ, composition of $r \in \mathbb{N}$ transpositions, as

$$\operatorname{sign}(\sigma) = (-1)^r .$$

Let V and W be two vector spaces over a field $\mathbb{K}$ of characteristics 0. A p-multilinear map $\varphi : V \times \cdots \times V \to W$ is said to be an **alternating map** if

$$\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(p)}) = \operatorname{sign}(\sigma)\varphi(v_1, v_2, \ldots, v_p)$$

for every $(v_1, v_2, \ldots, v_p) \in V^p$ and for every permutation $\sigma \in \mathfrak{S}_p$. We denote by $\Lambda^p(V)$ the space of all p-covariant alternating tensors —which is hence a subspace of $T^p(V)$— and, if the dimension of V is n, we set

$$\Lambda(V) = \bigotimes_{0 \leqslant p \leqslant n} \Lambda^p(V) ;$$

note that $\Lambda^0(V) = \mathbb{K}$.

In order to define a product on this space, we introduce the operator

$$A : \bigotimes_{p \geqslant 0} T^p(V) \longrightarrow \Lambda(V)$$

defined by

$$A(\alpha)(\varphi^1, \varphi^2, \ldots, \varphi^p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sign}(\sigma)\alpha(\varphi^{\sigma(1)}, \varphi^{\sigma(2)}, \ldots, \varphi^{\sigma(p)})$$

for every $\alpha \in T^p(V)$ and $\varphi^1, \varphi^2, \ldots, \varphi^p \in V^*$ (note that it is well-defined, linear and the identity on $\Lambda(V)$). Now, for each $\alpha \in \Lambda^p(V)$ and $\beta \in \Lambda^q(V)$, we set

$$\alpha \wedge \beta = \frac{(p+q)!}{p!q!} A(\alpha \otimes \beta) \in \Lambda^{p+q}(V) .$$

Extending by bilinearity, we have the **exterior product** (or **wedge product**)

$$\wedge : \Lambda(V) \times \Lambda(V) \longrightarrow \Lambda(V) .$$

The quadruple $(\Lambda(V), +, \wedge, \cdot)$ is an algebra called **exterior algebra** of V.

Now, let M be a smooth manifold and $\alpha \in \Lambda^p(M)$ —that is, $\alpha_x \in \Lambda^p(T_x)$— be of class $\mathscr{C}^k$ (in x). The **exterior differentiation operator** d maps α into $d\alpha \in \Lambda^{p+1}$ of class $\mathscr{C}^{k-1}$ and satisfies the following conditions:

(a) *linearity:* if λ is a constant,

$$d(\alpha + \beta) = d\alpha + d\beta\,, \qquad d(\lambda\alpha) = \lambda d\alpha\,;$$

(b) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$;

(c) $d^2 = 0$;

(d) if $f \in \mathscr{C}^k(M)$, then df is the usual differential of f introduced in the previous section.

These properties determine d uniquely, as shown in [13], page 200. Note that, if $f \in \mathscr{C}^k(M)$, we set

$$f\alpha = f \wedge \alpha$$

to simplify notations.

2.3 Pseudo-Riemannian manifolds

We call **tensor field** of order (p, q) a map that associates to each $x \in M$ a tensor in $T^p_q(T_x)$. For instance, a 2-covariant tensor field is a map

$$g : M \ni x \longrightarrow g_x \in T_2(T_x)\,.$$

A **pseudo-Riemannian manifold** is a couple (M, g), where M is a differentiable manifold and g is a differentiable symmetric non-degenerate 2-covariant tensor field, called **metric tensor** or also **pseudo-Riemannian metric**. In other words, a pseudo-Riemannian metric provides a map $M \ni x \mapsto g_x$, where $g_x : T_x \times T_x \to \mathbb{R}$ is a bilinear map that satisfies:

- g is differentiable, that is, if (U, φ) is a chart at x, then $g_x(\partial/\partial x^i, \partial/\partial x^j)$ is a differentiable function on $\varphi^{-1}(U) \subset \mathbb{R}^n$ into $\mathbb{R}$;

- g is symmetric, that is,

$$g_x(v,w) = g_x(w,v) \qquad \forall v,w \in T_x, \quad \forall x \in M\,;$$

- for each $x \in X$, the bilinear form g_x is non-degenerate, that is

$$g_x(v,w) = 0 \quad \forall w \in T_x \qquad \text{if and only if} \qquad v = O_x\,.$$

If we assume further that g is positive definite, i.e.

$$g_x(v,v) > 0 \qquad \forall v \in T_x \setminus O_x, \quad \forall x \in M$$

(so that each g_x is a positive definite scalar product), we call it **Riemannian metric** and say that (M,g) is a **Riemannian manifold**.

We denote by $\{e_1, e_2, \ldots, e_n\}$ a moving frame for the tangent space, that is, for each $x \in M$, we have

$$e_i\big|_x \qquad i = 1,2,\ldots,n$$

basis for T_x, while $\{\vartheta^1, \vartheta^2, \ldots, \vartheta^n\}$ will be the dual basis. Hence, if we denote the tensor g with ds^2, we have

$$g = ds^2 = g_{ij}\vartheta^i \otimes \vartheta^j = g_{ij}\vartheta^i\vartheta^j,$$

where

$$\vartheta^i\vartheta^j = \frac{1}{2}\left(\vartheta^i \otimes \vartheta^j + \vartheta^j \otimes \vartheta^i\right),$$

because g is symmetric (remember that sum over repeated indices is assumed). Moreover, we have

$$g_{ij} = g_x(e_i, e_j) \qquad e_i, e_j \in T_x$$

and an inner product on each vector space T_x defined by

$$(v|w) = g_x(v,w) \qquad \forall v,w \in T_x\,.$$

An inner product on any vector space defines a **canonical isomorphism** between the space and its dual. Indeed, for a fixed $u \in T_x$, the map

$$(u|\cdot) : T_x \ni v \longrightarrow (u|v) \in \mathbb{R}$$

is an element of T_x^*, hence the canonical isomorphism is given by

$$T_x \ni u \longrightarrow u^* \doteq (u|\cdot) \in T_x^* \,.$$

Resorting to coordinates, we have $(u|v) = g_{ij}u^i v^j$ and hence $u^* = g_{ij}u^i \vartheta^j$. With abuse of notation, one generally uses the same symbol u to indicate either u, either its image u^*. The u^i are called **contravariant components** of u, while u_i are called **covariant components** of u; these components are related by the formulae

$$u_i = g_{ij}u^i \,, \qquad u^i = g^{ij}u_j \,,$$

where (g^{ij}) is the inverse of the matrix (g_{ij}), i.e. $g^{ij}g_{jk} = \delta^i_k$. One says that indices are **raised** or **lowered** by means of the tensor g. For instance, we have the mixed components:

$$t^i{}_j = g^{ik}t_{kj} \,.$$

We have an inner product on T_x^* inducted by the canonical isomorphism, that is

$$(u^*|v^*) = (u|v) = g^{ik}u_i v_j \,,$$

and similar canonical isomorphisms, terminology and properties are used for tensors. In particular, all the following spaces are isomorphic:

$$T^p(T_x) \,, \qquad T_p(T_x) \,, \qquad T^q(T_x) \otimes T_{p-q}(T_x) \,.$$

Choosing a suitable basis $\{e'_i\}$ for T_x, we can recast the expression for the quadratic form $g_x(v,v)$ as a sum of k positive and $n-k$ negative squares:

$$g_x(v,v) = g_{ij}v^i v^j = g'_{ij}v'^i v'^j = \sum_{i=1}^{k}(v'^i)^2 - \sum_{i=k+1}^{n}(v'^i)^2 \,.$$

The number k does not depend on the choice of the basis and it is called **index** of the quadratic form, while the couple $(k, n-k)$ is called **signature**. In terms of the basis $\{\vartheta'^i\}$ dual to $\{e'^i\}$, we get

$$g_x = \sum_{i=1}^{k} (\vartheta'^i)^2 - \sum_{i=k+1}^{n} (\vartheta'^i)^2 .$$

A 4-dimensional pseudo-Riemannian manifold of index 1 is called **hyperbolic manifold**. As usual, we label with Greek indices the coordinates that take the values $0, 1, 2, 3$ and with Latin indices the ones that take the values $1, 2, 3$. A basis $\{e_\alpha\}$ on a hyperbolic manifold is called **orthonormal** if

$$(e_0|e_0) = 1 \qquad (e_i|e_i) = -1\,, \qquad (e_\alpha|e_\beta) = 0$$

for all $\alpha \neq \beta$. In terms of an orthonormal basis $\{\vartheta^\alpha\}$, one has

$$g = (\vartheta_0)^2 - \sum_{i=1}^{3} (\vartheta^i)^2 .$$

The $(1+n)$-dimensional **Minkowski space** is the space $\mathbb{R}^{1+n}$ with the metric

$$ds^2 = (dx^0)^2 - \sum_{i=1}^{n} (dx^i)^2 \tag{2.3}$$

(of index 1), called **Minkowski metric**.

Given a metric g, represented by the matrix $G = (g_{ij})$, the **Laplace–Beltrami operator** respect to g is the operator Δ_g defined by

$$\Delta_g f = \frac{1}{\sqrt{|\det G|}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{|\det G|}\, \frac{\partial}{\partial x^j} f \right) ,$$

where $f : M \to \mathbb{K}$.

In the case of the space (Riemannian manifold) $\mathbb{R}^n$ with the classical euclidean metric of signature $(n, 0)$, i.e.

$$g = \sum_{i=1}^{n} (dx^i)^2 ,$$

the Laplace–Beltrami operator reduces to the standard **Laplace operator** Δ:

$$\Delta_g f = \Delta f = \sum_{i=1}^{n} \partial_i^2 f , \qquad \partial_i = \frac{\partial}{\partial x^i} .$$

When g is a metric of index 1, we write $\Box_g$ instead of Δ_g, and we use the symbol $-\Delta_g$ only for the negative part of the metric. For instance, if we consider the $(1+n)$-dimensional Minkowski space, using the variables $(t, x_1, \ldots, x_n) \in \mathbb{R}^{1+n}$, we have

$$\Box_g = \partial_t^2 - \Delta_g = \partial_t^2 - \Delta = \partial_t^2 - \sum_{i=1}^{n} \partial_i^2 \doteq \Box ;$$

hence, in this case, $\Box_g$ reduces to the standard d'Alembert operator $\Box$.

Example 2.3.1. $\mathbb{R}^3$ with the euclidean metric $g_{ij} = \delta_{ij}$ induces on the sphere

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 \, : \, |x| = 1\}$$

the metric

$$d\omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 , \tag{2.4}$$

where we have chosen the parameters

$$(\vartheta, \varphi) \in]0, \pi[\times]0, 2\pi[$$

so that, if $x = (x^1, x^2, x^3)$, we have

$$x^1 = r \sin\vartheta \cos\varphi , \qquad x^2 = r \sin\vartheta \sin\varphi , \qquad x^3 = r \cos\vartheta . \tag{2.5}$$

The Laplace–Beltrami operator on the 2-dimensional sphere, computed respect to this metric, is

$$\Delta_{\mathbb{S}^2} = \cot\vartheta\,\partial_\vartheta + \partial_\vartheta^2 + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\,. \tag{2.6}$$

2.4 The Minkowski space-time

In the previous section, we have introduced the $(1+n)$-dimensional Minkowski space. If we consider the case $n = 3$ (three-dimensional space), we get the ordinary Minkowski space-time. From (2.3), we see that its metric, in terms of the natural coordinates $(t, x^1, x^2, x^3) \in \mathbb{R}^{1+3}$, gets the form

$$ds^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\,.$$

Using spherical polar coordinates $(t, r, \vartheta, \varphi)$, related to the natural coordinates by the relations in (2.5), we have

$$ds^2 = dt^2 - dr^2 - r^2 d\omega^2 = dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\,,$$

where $d\omega^2$ is the metric defined in (2.4) on the 2-sphere. Apparently, this metric is singular for $r = 0$ and $\sin\vartheta = 0$, but these singularities depend on the coordinate system. One can obtain regular coordinate neighborhoods restricting the ranges of the coordinates —for instance, by taking $r > 0$, $0 < \vartheta < \pi$ and $0 < \varphi < 2\pi$. It is possible to cover the whole of the Minkowski space through two such coordinate neighborhoods.

A further coordinate system is given by choosing the advanced time null coordinate $v = t + r$ and the retarded time null coordinate $w = t - r$, so that the metric becomes

$$ds^2 = dvdw - \frac{1}{4}(v - w)^2 d\omega^2\,, \qquad v, w \in \mathbb{R}\,.$$

Note that it is always $v \geqslant w$. We say that v is a **null coordinate** to indicate the fact that the surface $\{v = \text{constant}\}$ is a **null surface**, that is $v_a v_b g^{ab} = 0$ (and similarly for w).

Penrose introduced new null coordinates p and q, which are very useful in order to study the structure of infinity in the Minkowski space–time, since in these coordinates the infinities of v and w are transformed in finite values. Indeed, setting

$$\tan p = v\,, \qquad \tan q = w\,,$$

we have

$$-\frac{\pi}{2} < p\,,\, q < \frac{\pi}{2}\,, \qquad p \geqslant q\,,$$

and the metric takes the form

$$ds^2 = \sec^2 p \sec^2 q \left(dp\, dq - \frac{1}{4}(p-q) d\omega^2 \right).$$

By defining

$$t' = p + q\,, \qquad r' = p - q\,,$$

where

$$-\pi < t' + r' < \pi\,, \qquad -\pi < t' - r' < \pi\,, \qquad r \geqslant 0\,,$$

we obtain another expression for the metric of the Minkowski space, more similar to the one in polar coordinates, that is

$$ds^2 = \frac{1}{4} \sec^2 \left(\frac{1}{2}(t' + r') \right) \sec^2 \left(\frac{1}{2}(t' - r') \right) [(dt')^2 - (dr')^2 - \sin^2 r' d\omega^2]\,.$$

In terms of the null coordinates p and q, the infinity of the Minkowski space consists of the null surfaces

$$\mathscr{I}^+ = \{p = \pi/2\}\,, \qquad \mathscr{I}^- = \{q = -\pi/2\}$$

together with the (p,q)-points

$$i^+ = (\pi/2, \pi/2)\,, \qquad i^0 = (\pi/2, -\pi/2)\,, \qquad i^- = (-\pi/2, -\pi/2)\,.$$

This structure of infinity is often represented by drawing a diagram, called **Penrose diagram**, of the (t', r') plane (see Figure 2.1).

Each point of this diagram represents a sphere $\mathbb{S}^2$, except for i^+, i^0 and i^- (each of which is a single point), and points on the line $r = 0$ (where the polar

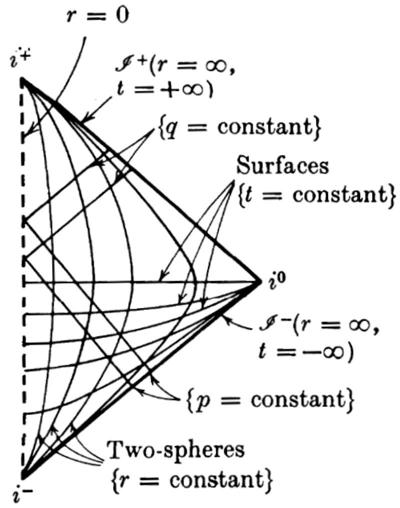

Figure 2.1 – Penrose diagram for the Minkowski metric.

coordinates are singular).

Penrose diagrams can be used to represent the structure of infinity in any spherically symmetric space-time (and we shall use them in the next chapter for the Schwarzschild metric). On such diagrams, single lines represent infinity, dotted lines the origin of polar coordinates and double lines irremovable singularities of the metric (i.e. that do not depend on the coordinate system).

2.5 Linear connections

A **vector field** is a map that associates to each point $x \in M$ a tangent vector $v_x \in T_x$. The set of all $\mathscr{C}^\infty$ vector fields on a smooth manifold M is denoted by $\mathscr{X}(M)$. If $u, v \in \mathscr{X}(M)$, we define the **Lie bracket** as

$$[u, v] = uv - vu\,;$$

the Lie bracket is still a vector field.

A **linear connection** on a smooth manifold M is a mapping $v \mapsto \nabla v$ from the germs of smooth vector fields on M into the germs of differentiable tensors of type $(1, 1)$ on M such that

(a) $\nabla(v + w) = \nabla v + \nabla w$,

(b) $\nabla(fv) = df \otimes v + f\nabla v$,

where f is a germ of a differentiable function on M. The tensor ∇v is called **covariant derivative** of v. The **connection coefficients** γ^j_{ki} are defined by the relation

$$\nabla e_i = \gamma^j_{ki}\vartheta^k \otimes e_j \,,$$

where $\{e_i\}$ and $\{\vartheta^i\}$ are dual bases.

The **covariant derivative** $\nabla_u v$ of v in the direction of u is by definition

$$\nabla_u v = (\nabla v)(u)\,.$$

To extend the covariant derivative to germs of tensors of arbitrary type, we require that the directional covariant derivative satisfies:

(a) $\nabla_v f = v(f)$ for each $f \in \mathscr{C}^1(M)$,

(b) $\nabla_v(t+s) = \nabla_v t + \nabla_v s$,

(c) $\nabla_v(t \otimes s) = \nabla_v t \otimes s + t \otimes \nabla_v s$,

(d) ∇_v commutes with the operation of contracted multiplication.

Then, if t is a tensor of type (p,q), the covariant derivative ∇t is the tensor of type $(p+1,q)$ defined by

$$(\nabla t)(v, v_1, \ldots, v_p, \omega_1, \ldots, \omega_q) = (\nabla_v t)(v_1, \ldots, v_p, \omega_1, \ldots, \omega_q)\,.$$

The **torsion operation** τ and the **curvature operation** ρ are defined by

$$\tau(u,v) = \nabla_u v - \nabla_v u - [u,v]\,, \qquad \rho(u,v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\,, \tag{2.7}$$

where $u, v \in \mathscr{X}(M)$. The **torsion tensor** $\mathbf{T}$ and the **curvature tensor** $\mathbf{R}$ are defined by

$$\mathbf{T}(\alpha, u, v) = \alpha(\tau(u,v))\,, \qquad \mathbf{R}(w, \alpha, u, v) = \alpha(\rho(u,v)w)\,, \tag{2.8}$$

where $\alpha_x \in \Lambda^1(T_x M)$. In local coordinates, the components of $\mathbf{T}$ and $\mathbf{R}$ are

(2.9) $$T^i_{\ kl} = \mathbf{T}(\vartheta^i, e_k, e_l)\,, \qquad R_i^{\ j}{}_{kl} = \mathbf{R}(e_i, \vartheta^j, e_k, e_l)\,.$$

We also introduce the **torsion forms** Θ^i and the **curvature forms** Ω^i_j by

$$\Theta^i = \frac{1}{2} T^i_{\ kl} \vartheta^k \wedge \vartheta^l\,, \qquad \Omega^i_j = \frac{1}{2} R_i^{\ j}{}_{kl} \vartheta^k \wedge \vartheta^l\,.$$

These forms can be conveniently be expressed by means of the Cartan structural equations (see [13], Chapter V, § B.1, page 306).

Theorem 2.5.1 (Cartan structural equations)**.** *If we denote by ω^j_i the connection forms $\gamma^j_{ki}\vartheta^k$, then*

$$\Theta^i = d\vartheta^i + \omega^i_l \wedge \vartheta^l\,,$$
$$\Omega^j_i = d\omega^j_i + \omega^j_m \wedge \omega^m_l\,.$$

The following results are shown, for instance, in [13], Chapter V, § B.2, page 308 and following ones.

Theorem 2.5.2. *On a pseudo-Riemannian manifold there exists a unique linear connection such that:*

(a) the torsion tensor $\mathbf{T}$ is the null tensor $(\mathbf{T} = 0)$,

(b) the covariant derivative of g vanishes $(\nabla g = 0)$.

Such a connection is called **Riemannian connection** (or **Levi–Civita connection**). If we consider such a connection, we have $\Theta^i = 0$; moreover, in an orthonormal frame, we get

$$\omega_{ij} = -\omega_{ji}\,, \qquad \Omega_{ij} = -\Omega_{ji}\,.$$

The components of the curvature tensor of a Riemannian connection, called the **Riemann tensor**, satisfy the identities

(a) $R_i^{\ j}{}_{kl} = -R_i^{\ j}{}_{lk}$,

(b) $\sum_{(ikl)} = 0$ *(first Bianchi identity),*

(c) $\sum_{(mkl)} \nabla_m R_i{}^j{}_{kl} = 0$ *(second Bianchi identity),*

(d) $R_{ijkl} = -R_{jikl}$*,*

(e) $R_{ijkl} = R_{klij}$.

Note that the symbol $\sum_{(ikl)}$ denotes the sum over cyclic permutations of three indices.

The **Ricci tensor** is a contraction of the curvature tensor and its components are, by definition,

$$R_{ik} = R_i{}^j{}_{kj}\,.$$

The Ricci tensor is symmetric: $R_{ik} = R_{ki}$. The **Riemann scalar curvature** is, by definition,

$$R = g^{ij} R_{ij}\,.$$

The tensor $\mathbf{G} \doteq R^j_k - \dfrac{1}{2}\delta^j_k R$ is called the **Einstein tensor**.

Chapter 3

The Schwarzschild Metric in Physics

In this chapter, we are going to introduce the Schwarzschild metric, which represents the setting of the problems considered in Chapters 4 and 5. This metric will be studied from both a geometrical and a physical point of view. Actually, we are interested in how it naturally arises as a solution to a physical problem and in how it is used to model some physical phenomena. Except for some relevant mathematical results, we are not going to enter into details and proof: our aim is to give an idea of why the problems that we handle are interesting and what is their physical interpretation. However, we provide precise references for all the statements of this chapter (mainly, we refer to (40), (13) and (14)).

In Section 3.1, we briefly describe the Einstein field equation, recalling its structure, its meaning and in particular its applications. Most of the assertions are not justified. Their proofs, along with a complete treatment of the Einstein equation, can be found in (40), above all in Chapter 17, where in particular are presented the main idea in the formulation of the field equation and several ways to derive it rigorously.

In Section 3.2, we find a particular class of solutions (that is solutions of a particular form) to a particular Einstein field equation, i.e. the Einstein empty space field equation. In this case, the reference is (13), page 341.

In Section 3.3, we describe very rapidly some geometrical properties of the Schwarzschild metric. We introduce the Kruskal–Szekeres and the Regge–Wheeler coordinates. Moreover, we spend some words on gravitational collapses and black holes, whose fields can be described through the Schwarzschild metric. These two topics are treated vastly in (40), Chapters 32 and 31. We also refer to (20) and (30) for further physical discussions.

3.1 The Einstein field equation

A. Einstein has shown that the Einstein tensor $\mathbf{G}$, introduced in the previous chapter (see Section 2.5), is always generated by the local distribution of matter. The Einstein tensor is a sort of average over all directions of the Riemann tensor $\mathbf{R}$. $\mathbf{G}$ is generated by a geometric object called the **stress-energy tensor** of the matter, which will be denoted by $\mathbf{S}$. No coordinates are needed to define $\mathbf{G}$ and none to define $\mathbf{S}$: like the metric tensor g, these tensors exist in the complete absence of coordinates. Moreover, in nature they are always equal, aside for a 8π factor, which is the **Einstein field equation**, i.e.

$$\mathbf{G} = 8\pi\mathbf{S}$$

or, in terms of components in an arbitrary coordinate system,

$$G_{\alpha\beta} = 8\pi S_{\alpha\beta}\,.$$

This equation shows how the stress-energy of matter generates a curvature, represented by $\mathbf{G}$, in its neighborhood. At the same time, the field equation is a propagation equation for the remaining part of the curvature, governing the external space-time curvature of a static source. It governs the generation of gravitational waves (ripples in curvature of space-time) by stress-energy in motion and their propagation through the universe. The field equation also contains in itself the equations of motion (that is, force is given by the mass multiplied by acceleration) for the matter whose stress-energy generates the curvature.

All these considerations would be sufficient to say that the Einstein field equation is very rich, despite its formal simplicity. Indeed, it has a lot of applications that make it even more valuable. For instance, the field equation governs:

- the motion of the planets in the solar system,
- the deflection of light by the sun,
- the collapse of a star to form a black hole,
- the evolution of space-time singularities at the endpoint of collapse,
- the expansion and recontraction of the universe.

We shall be interested, in particular, in the connections between the Einstein equation and black holes, since the problems considered in the Chapters 4–6 are set in a space with a particular metric, the Schwarzschild metric, which is a particular solution to the Einstein empty space field equation

$$R_{\alpha\beta} = 0\,,$$

which corresponds to the case $\mathbf{S} = 0$ (see Section 3.2).

3.2 A solution to the Einstein equation: the Schwarzschild metric

In the previous section, we have introduced the Einstein field equation, but we have not given any example of solution. In this section, we shall give such an example. Indeed, we restrict ourselves to the case of empty space, that is the stress-energy tensor $\mathbf{S}$ is null and the equation becomes simply

$$R_{\alpha\beta} = 0\,,$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor. In this case, we have already seen an exact solution to this problem, although we did not know it, that is the Minkowski metric, introduced in the previous chapter. This solution is not good in general, for instance if we want to describe the local geometry of space-time in the solar system with a good approximation. Therefore, we shall look for spherically symmetric solutions of the form

$$ds^2 = \mathrm{e}^{\mu(t,r)}\, dt^2 - \mathrm{e}^{\nu(t,r)}\, dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2 \,. \tag{3.1}$$

Note that t represents the time coordinate, $r = |x|$, with $x \in \mathbb{R}^3$ space coordinate, and (ϑ, φ) are the polar coordinates on $\mathbb{S}^2$.

In other words, we shall solve the following problem.

Problem 3.2.1. Find all metrics of the form (3.1) defined on a four-dimensional hyperbolic manifold and satisfying the Einstein empty space field equation

$$R_{\alpha\beta} = 0\,.$$

SOLUTION OF THE PROBLEM. First of all, we choose a convenient frame, that is the orthonormal frame

$$\begin{aligned}
\vartheta^0 &= \mathrm{e}^{\mu/2}\, dt\,,\\
\vartheta^1 &= \mathrm{e}^{\nu/2}\, dr\,,\\
\vartheta^2 &= r d\vartheta\,,\\
\vartheta^3 &= r\sin\vartheta d\varphi\,.
\end{aligned}$$

In regard to this frame, the metric has signature $(1,3)$. We denote by a prime the derivative with respect to r and with a point the derivate respect to t. For instance,

$$\mu' = \frac{\partial\mu}{\partial r}\,, \qquad \dot\mu = \frac{\partial\mu}{\partial t}\,.$$

Then, we have

$$\begin{aligned}
d\vartheta^0 &= \frac{1}{2}\,\mathrm{e}^{-\nu/2}\,\mu'\vartheta^1\wedge\vartheta^0\,,\\
d\vartheta^1 &= \frac{1}{2}\,\mathrm{e}^{-\mu/2}\,\dot\nu\vartheta^0\wedge\vartheta^1\,,\\
d\vartheta^2 &= \frac{1}{r}\,\mathrm{e}^{-\nu/2}\,\vartheta^1\wedge\vartheta^2\,,\\
d\vartheta^3 &= \frac{1}{r}\,\mathrm{e}^{-\nu/2}\,\vartheta^1\wedge\vartheta^2 + \frac{\cot\vartheta}{r}\vartheta^2\wedge\vartheta^3\,.
\end{aligned}$$

Note that, using the notations of Section 2.5, we have $T^i_{\ jj} = 0$, since $\tau(e_j, e_j) = 0$ (see (2.7), (2.8) and (2.7)). Thanks to the choice of an orthonormal frame, we deduce then

$$\Theta^i = \frac{1}{2} T^i_{\ kl} \vartheta^k \wedge \vartheta^l = 0$$

and the first Cartan structural equation becomes $d\vartheta^\alpha + \omega_\beta^{\ \alpha} \wedge \vartheta^\beta = 0$, as one can see from Theorem 2.5.1. Solving this linear system of equations, we can find the (Riemannian) connection forms

$$\begin{aligned}
\omega_1^{\ 0} &= \omega_{10} = -\omega_{01} = \omega_0^{\ 1} = \frac{1}{2}\,\mathrm{e}^{\nu/2}\,\mu'\vartheta^0 + \frac{1}{2}\,\mathrm{e}^{-\mu/2}\,\dot{\nu}\vartheta^1\,,\\
\omega_1^{\ 2} &= -\omega_2^{\ 1} = \frac{1}{r}\,\mathrm{e}^{-\nu/2}\,\vartheta^2\,,\\
\omega_1^{\ 3} &= -\omega_3^{\ 1} = \frac{1}{r}\,\mathrm{e}^{-\nu/2}\,\vartheta^3\,,\\
\omega_2^{\ 3} &= -\omega_3^{\ 2} = \frac{\cot\vartheta}{r}\vartheta^3\,,\\
\omega_0^{\ 2} &= \omega_2^{\ 0} = \omega_0^{\ 3} = \omega_3^{\ 0} = 0
\end{aligned}$$

(the remaining connection forms are obviously identically zero, since $\omega_{ii} = 0$, hence we have indeed a system of 24 equations in 24 unknowns). Then

$$\begin{aligned}
d\omega_1^{\ 0} &= \frac{1}{2}\left[\mathrm{e}^{-\nu}\left(\frac{1}{2}\mu'(\mu'-\nu') + \mu''\right) - \mathrm{e}^{-\mu}\left(\frac{1}{2}\dot{\nu}(\dot{\nu}-\dot{\mu}) + \ddot{\nu}\right)\right]\vartheta^1\wedge\vartheta^0\,,\\
d\omega_1^{\ 2} &= -\frac{1}{2r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}\vartheta^0\wedge\vartheta^2 - \frac{1}{2r}\,\mathrm{e}^{-\nu}\,\nu'\vartheta^1\wedge\vartheta^2\,,\\
d\omega_1^{\ 3} &= \frac{\cot\vartheta}{r^2}\,\mathrm{e}^{-\nu/2}\,\vartheta^2\wedge\vartheta^3 - \frac{1}{2r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}\vartheta^0\wedge\vartheta^3 - \frac{1}{2r}\,\mathrm{e}^{-\nu}\,\nu'\vartheta^1\wedge\vartheta^3\,,\\
d\omega_2^{\ 3} &= -\frac{1}{r^2}\vartheta^2\wedge\vartheta^3\,.
\end{aligned}$$

To compute the Riemann tensor, we can now use the second Cartan structural equation (see Theorem 2.5.1). The nonvanishing forms are therefore

$$\begin{aligned}
\Omega_0^{\ 1} &= \Omega_1^{\ 0} = d\omega_0^{\ 1}\\
&= \tfrac{1}{4}\left[\mathrm{e}^{-\nu}\left(\tfrac{1}{2}\mu'(\mu'-\nu') + \mu''\right) - \mathrm{e}^{-\mu}\left(\tfrac{1}{2}\dot{\nu}(\dot{\nu}-\dot{\mu}) + \ddot{\nu}\right)\right](\vartheta^1\wedge\vartheta^0 - \vartheta^0\wedge\vartheta^1)\,,
\end{aligned}$$

$$
\begin{aligned}
\Omega_0{}^2 &= -\omega_0{}^1 \wedge \omega_1{}^3 \\
&= -\frac{1}{4r}\,\mathrm{e}^{-\nu}\,\mu'(\vartheta^0 \wedge \vartheta^2 - \vartheta^2 \wedge \vartheta^0) - \frac{1}{4r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}(\vartheta^1 \wedge \vartheta^2 - \vartheta^2 \wedge \vartheta^1)\,, \\
\Omega_0{}^3 &= -\omega_0{}^1 \wedge \omega_1{}^2 \\
&= -\frac{1}{4r}\,\mathrm{e}^{-\nu}\,\mu'(\vartheta^0 \wedge \vartheta^3 - \vartheta^3 \wedge \vartheta^0) - \frac{1}{4r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}(\vartheta^1 \wedge \vartheta^3 - \vartheta^3 \wedge \vartheta^1)\,, \\
\Omega_1{}^2 &= -\Omega_2{}^1 = d\omega_1{}^2 - \omega_1{}^3 \wedge \omega_3{}^2 \\
&= \frac{1}{4r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}(\vartheta^2 \wedge \vartheta^0 - \vartheta^0 \wedge \vartheta^2) - \frac{1}{4r}\,\mathrm{e}^{-\nu}\,\nu'(\vartheta^1 \wedge \vartheta^2 - \vartheta^2 \wedge \vartheta^1)\,, \\
\Omega_1{}^3 &= -\Omega_3{}^1 = d\omega_1{}^3 - \omega_1{}^2 \wedge \omega_2{}^3 \\
&= \frac{1}{4r}\,\mathrm{e}^{-\nu}\,\nu'(\vartheta^3 \wedge \vartheta^1 - \vartheta^1 \wedge \vartheta^3) + \frac{1}{4r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}(\vartheta^3 \wedge \vartheta^0 - \vartheta^0 \wedge \vartheta^3)\,, \\
\Omega_2{}^3 &= -\Omega_3{}^2 = d\omega_2{}^3 - \omega_2{}^1 \wedge \omega_1{}^3 \\
&= \frac{1}{2r^2}(\mathrm{e}^{-\nu} - 1)(\vartheta^2 \wedge \vartheta^3 - \vartheta^3 \wedge \vartheta^2)\,.
\end{aligned}
$$

Since the curvature forms are defined as

$$
\Omega^i_j = \frac{1}{2} R_i{}^j{}_{kl}\vartheta^k \wedge \vartheta^l\,,
$$

we can deduce the components of the curvature tensor $\mathbf{R}$ and use them to compute the components of the Ricci tensor:

$$
\begin{aligned}
R_{00} &= R_0{}^1{}_{01} + R_0{}^2{}_{02} + R_0{}^3{}_{03} \\
&= -\tfrac{1}{2}\,\mathrm{e}^{-\nu}\left(\tfrac{1}{2}\mu'(\mu' - \nu') + \mu''\right) + \tfrac{1}{2}\,\mathrm{e}^{-\mu}\left(\tfrac{1}{2}\dot{\nu}(\dot{\nu} - \dot{\mu}) + \ddot{\nu}\right) - \frac{1}{r}\,\mathrm{e}^{-\nu}\,\mu'\,, \\
R_{01} &= R_0{}^2{}_{12} + R_0{}^3{}_{13} = -\frac{1}{r}\,\mathrm{e}^{-(\nu+\mu)/2}\,\dot{\nu}\,, \\
R_{11} &= R_1{}^0{}_{10} + R_1{}^2{}_{12} + R_1{}^3{}_{13} \\
&= \tfrac{1}{2}\,\mathrm{e}^{-\nu}\left(\tfrac{1}{2}\mu'(\mu' - \nu') + \mu''\right) - \tfrac{1}{2}\,\mathrm{e}^{-\mu}\left(\tfrac{1}{2}\dot{\nu}(\dot{\nu} - \dot{\mu}) + \ddot{\nu}\right) - \frac{1}{r}\,\mathrm{e}^{-\nu}\,\nu'\,, \\
R_{22} &= R_2{}^0{}_{20} + R_2{}^1{}_{21} + R_2{}^3{}_{23} \\
&= \frac{1}{2r}\,\mathrm{e}^{-\nu}\,\mu' + \frac{1}{r^2}(\mathrm{e}^{-\nu} - 1) - \frac{1}{2r}\,\mathrm{e}^{-\nu}\,\nu'\,,
\end{aligned}
$$

$$\begin{aligned} R_{33} &= R_3{}^0{}_{30} + R_3{}^1{}_{31} + R_3{}^2{}_{32} \\ &= \frac{1}{2r} \mathrm{e}^{-\nu} \mu' + \frac{1}{r^2}(\mathrm{e}^{-\nu} - 1) - \frac{1}{2r} \mathrm{e}^{-\nu} \nu' , \\ R_{02} &= R_{03} = R_{12} = R_{13} = R_{23} = 0 . \end{aligned}$$

From the Einstein equation, we have $R_{01} = 0$, which implies $\dot{\nu} = 0$. The remaining equations for μ and ν are

$$\begin{aligned} \tfrac{1}{2}\mu'(\mu' - \nu') + \mu'' + \frac{2}{r}\mu' = 0 , \\ \tfrac{1}{2}\mu'(\mu' - \nu') + \mu'' - \frac{2}{r}\nu' = 0 , \end{aligned}$$

$$\mathrm{e}^{-\nu}(\mu' - \nu') + \frac{2}{r}(\mathrm{e}^{-\nu} - 1) = 0 .$$

The difference between the first two equations gives $\mu' + \nu' = 0$, hence the last equation can be solved in ν obtaining

$$\mathrm{e}^{-\nu} = 1 - \frac{K}{r} ,$$

where K is a constant. It follows that

$$\mathrm{e}^{\mu} = f(t) \left(1 - \frac{K}{r}\right) ,$$

with f arbitrary function of t. We can eliminate the dependence of μ on t by making the change of coordinates

$$(t, r, \vartheta, \varphi) \mapsto (\hat{t}(t), r, \vartheta, \varphi)$$

such that

$$\frac{d\hat{t}}{dt} = \sqrt{f(t)} .$$

Therefore, we can take $F(t) = 1$ and conclude

$$ds^2 = \left(1 - \frac{K}{r}\right) dt^2 - \left(1 - \frac{K}{r}\right)^{-1} dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2\,.$$

This solves completely the problem. ■

For physical reasons, it is assumed that $K > 0$. Generally, one writes $K = 2M$, where $M > 0$ has the meaning of a mass. Setting

$$F(r) = 1 - \frac{2M}{r}\,,$$

we finally have

$$ds^2 = F(r)dt^2 - F(r)^{-1}dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2\,; \tag{3.2}$$

this metric is called the **Schwarzschild metric** and it represents the spherically symmetric empty space-time outside a spherically symmetric massive body. Note that, recalling that the standard metric on the 2-dimensional unit sphere $\mathbb{S}^2$ is

$$d\omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$$

(see Example 2.3.1), the Schwarzschild metric can be recast in the form

$$ds^2 = F(r)\, dt^2 - F(r)^{-1}\, dr^2 - r^2\, d\omega^2\,.$$

3.3 Some properties of the Schwarzschild metric

The Schwarzschild space-time geometry (3.2) seems to behave badly near $r = 2M$, where g_{tt} becomes zero and g_{rr} becomes minus infinity. However, this pathology depends on the coordinate system and not on the space-time geometry itself; in other words, one can show that the space-time geometry is not singular in the region where $r = 2M$, called **gravitational radius**. Actually, an infalling observer reaches $r = 2M$ in finite proper time but infinite coordinate time and he does not feel infinite tidal forces at the gravitational radius (for a proof, see [40], §31.2, page 820). Hence the space-time geometry is well-behaved at $r = 2M$, while the coordinate system is pathological.

On the contrary, at $r = 0$, we have infinite tidal forces, independently of the path that we choose to reach there. Indeed, in every local Lorentz frame, the curvature is infinite at $r = 0$ (the Riemann tensor has at least one infinite component as $r \to 0$), and one says that $r = 0$ is a physical singularity of spacetime.

One can exhibit several system of coordinates that avoid the pathology at the gravitational radius, like the Kruskal–Szekeres coordinates $(u, v, \vartheta, \varphi)$, where u is a dimensionless radial coordinate and v a dimensionless time coordinate related to the Schwarzschild r and t by

$$\text{when } r > 2M \qquad \begin{cases} u = (r/2M - 1)^{1/2}\,\mathrm{e}^{r/4M}\cosh(t/4M)\,, \\ v = (r/2M - 1)^{1/2}\,\mathrm{e}^{r/4M}\sinh(t/4M)\,, \end{cases}$$

$$\text{when } r < 2M \qquad \begin{cases} u = (1 - r/2M)^{1/2}\,\mathrm{e}^{r/4M}\sinh(t/4M)\,, \\ v = (1 - r/2M)^{1/2}\,\mathrm{e}^{r/4M}\cosh(t/4M)\,. \end{cases}$$

In this new coordinate system, the line element becomes

$$ds^2 = (32M^3/r)\,\mathrm{e}^{-r/2M}(dv^2 - du^2) + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,,$$

where r must be viewed as an implicite function of u and v defined by

$$(r/2M - 1)\,\mathrm{e}^{r/2M} = u^2 - v^2\,.$$

These new coordinates show interesting geometrical and physical properties (among which the possibility of the presence of warmholes connecting two asymptotically flat universes or two different regions of one universe), but they will not be used in the following and therefore, as usual, we refer to [40] for an extensive treatment (see Chapter 31).

Another interesting coordinate, which will be used massively in the following chapters, is the Regge–Wheeler coordinate. It is defined by

$$r^* = \int dr^* = \int \frac{dr}{1 - 2M/r} = r + 2M \log(r - 2M) \; . \tag{3.3}$$

Note that while the r-coordinate can freely pass through the value $r = 2M$, the r^*-coordinate can go arbitrarily far in the direction of minus infinity and still remains outside $r = 2M$. Hence there is a great difference between the description of the notion in terms of the proper time of a clock on a falling particle (described through r, which goes from a certain R to 0) and the same description in terms of the time appropriate for a faraway observer (r^* goes from R^* down to $-\infty$, so that as $r^* \to -\infty$, r is asymptotically brought down only to $2M$). Therefore the second description leaves out the whole range of values from $r = 2M$ down to $r = 0$, where we have perfectly behaved physics, which the falling particle is going to see and explore, but physics that the faraway observer can not see and never will see.

Another possibility is represented by Kruskal coordinates

$$v' = \mathrm{e}^{(t+r^*)/4M} \, , \qquad w' = -\,\mathrm{e}^{-(t-r^*)/4M} \; .$$

We can construct the Penrose diagram by defining new advanced and retarded null coordinates

$$v'' = \arctan(v'\sqrt{2M}) \, , \qquad w'' = \arctan(w'\sqrt{2M})$$

for

$$-\pi < v'' + w'' < \pi \, , \qquad -\frac{\pi}{2} < v'' \, , w'' < \frac{\pi}{2}$$

(see Figure 3.1).

One has now future, past and null infinities for each of the asymptotically flat regions I and I′. This situation may be compared with the situation represented in the Penrose diagram for the Minkowski metric portrayed in Figure 2.1, page 41.

The Schwarzschild metric is interesting and important because it illustrates clearly the highly non-Euclidean character of space-time geometry when grav-

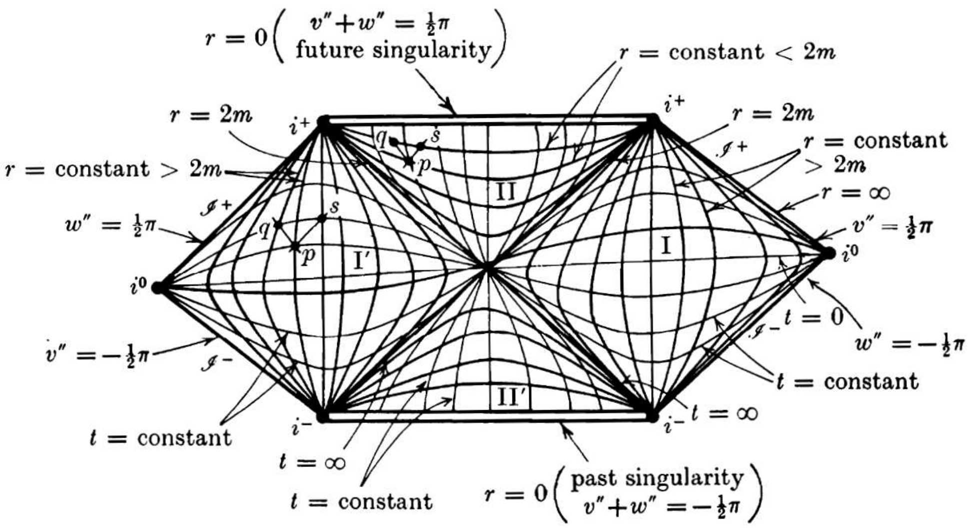

Figure 3.1 – Penrose diagram for the Schwarzschild.

ity becomes strong, but above all because, when appropriately truncated, it is the space-time geometry of a black hole and of a collapsing star. Actually, physics teaches that there is no equilibrium state at the endpoint of thermonuclear evolution for a star containing more than about twice the number of baryons of the sun. Such a star must eject the exceeding baryons — for instance by nova or supernova explosions — before settling down into its final resting state, otherwise there will be no final resting state. If a star fails to eject its exceeding baryons before the endpoint of thermonuclear evolution, since it can neither explode (it has no more thermonuclear energy to release), nor reach a static equilibrium state, the supercritical mass must collapse through its gravitational radius $r = 2M$, leaving behind a black hole.

Note that Problem 3.2.1 can be restated in the following form.

Theorem 3.3.1 (Birkhoff, 1923). *Let the geometry of a given region of space-time be spherically symmetric and a solution to the Einstein field equation in vacuum. Then that geometry is necessarily a piece of the Schwarzschild geometry.*

The collapse of an electrically neutral star endowed with spherical symmetry produces a black hole with external gravitational field described by the

Schwarzschild metric. The surface of the black hole, the horizon, is located at $r = 2M$. Only the region $r \geqslant 2M$ is relevant to external observers: events inside the horizon can never influence the exterior.

The collapse of a real star, with asymmetries and a charge, produces a black hole different from the Schwarzschild hole.

Chapter 4

Preliminary Results for the Schwarzschild Problem

In Chapter 5, we shall study two nonlinear wave equation Cauchy problems in the Schwarzschild setting. Both results are based on a preliminary reduction of the $(1+3)$-dimensional problem to the 1-dimensional case and some estimates. More precisely, we transform the original problem in a Cauchy problem for a 1-dimensional nonlinear equation with effective potential. Note that this reduction works since we restrict ourselves to radially symmetric solutions. Moreover, the proof of the local existence of the solution is given. All this stuff can be found in my joint works with Prof. V. Georgiev (9), (10) and (11), and also in my paper (7). The most extensive treatment can be found in (11).

Section 4.1 shows the reduction, while in Section 4.2 the aforementioned estimates and the local existence result are proved.

4.1 Reduction of the problem to the one-dimensional case

As widely anticipated, we consider the manifold

$$\mathbb{M} = \mathbb{R} \times \mathbf{\Omega}, \qquad \mathbf{\Omega} = \{(r,\omega) : r > 2M,\, \omega \in \mathbb{S}^2\} =]2M, \infty[\times \mathbb{S}^2,$$

equipped with the Schwarzschild metric, which has the form (see the identity (3.2) and what follows):

$$g = F(r)\, dt^2 - F(r)^{-1}\, dr^2 - r^2\, d\omega^2. \tag{4.1}$$

We recall that here

$$F(r) = 1 - \frac{2M}{r}\,, \tag{4.2}$$

the constant $M > 0$ has the interpretation of a mass and $d\omega^2$ is the standard metric on the unit sphere $\mathbb{S}^2$ (compare with Example 2.3.1).

The D'Alembert operator associated with the metric g is

$$\Box_g = \frac{1}{F}\left(\partial_t^2 - \frac{F}{r^2}\partial_r(r^2F)\partial_r - \frac{F}{r^2}\Delta_{\mathbb{S}^2}\right)$$

(see page 37), where $\Delta_{\mathbb{S}^2}$ denotes the standard Laplace–Beltrami operator on $\mathbb{S}^2$.

Now, let us consider the following wave equation in the $(1+3)$-dimensional space-time:

$$\Box_g u = \operatorname{sign}(|x| - R_0)|u|^p \qquad \text{in } [0,\infty[\times\boldsymbol{\Omega} \tag{4.3}$$

for a certain $R_0 \geqslant 2M$ and $p > 1$. Note that, for $R_0 = 2M$, we have the standard semilinear equation

$$\Box_g u = |u|^p\,.$$

An important tool — which will reduce the case of a radially symmetric wave equation in the Schwarzschild metric to the case of a 1-dimensional wave equation with suitable potential — is the use of the Regge–Wheeler coordinate

$$s(r) = r + 2M\log(r - 2M) \tag{4.4}$$

(see equation (3.3) for a brief introduction to this coordinate). We can rewrite equation (4.3) as

$$\partial_t^2 u - \partial_s^2 u - \frac{2F}{r(s)}\partial_s u - \frac{F}{r(s)^2}\Delta_{\mathbb{S}^2}u = \operatorname{sign}(r(s) - R_0)F|u|^p, \tag{4.5}$$

where

$$F = F(s) = 1 - \frac{2M}{r(s)}$$

and $r(s)$ is the inverse function to (4.4), which is well-defined, since $s(r)$ is strictly increasing.

For simplicity, we shall restrict our considerations to the case of solutions of the form $u = u(t, s)$, that is to say radially symmetric solutions. Then (4.5) is simplified to the following equation:

$$\partial_t^2 u - \partial_s^2 u - \frac{2F(s)}{r(s)}\partial_s u = \mathrm{sign}(r(s) - R_0)F(s)|u|^p. \tag{4.6}$$

Making further the substitution

$$u(t, s) = \frac{v(t, s)}{r(s)},$$

we obtain the semilinear Cauchy problem:

$$\begin{cases} [\partial_t^2 - \partial_s^2 + W(s)]\, v = \mathrm{sign}(r(s) - R_0)f(s)|v|^p, & (t, s) \in [0, \infty[\times\mathbb{R}\,, \\ v(0, s) = v_0(s), \quad \partial_t v(0, s) = v_1(s), & s \in \mathbb{R}\,, \end{cases} \tag{4.7}$$

where

$$W(s) = \frac{2MF(s)}{r(s)^3}, \qquad f(s) = F(s)r(s)^{1-p}. \tag{4.8}$$

It is easy to see that $W(s),\ f(s) \in \mathscr{C}(\mathbb{R})$ satisfy the following estimates

$$W(s) > 0, \qquad f(s) > 0 \qquad \forall s \in \mathbb{R}, \tag{4.9}$$

$$W(s) \sim s^{-3}, \qquad f(s) \sim s^{1-p} \qquad \forall s \geqslant 1, \tag{4.10}$$

$$W(s) \sim \mathrm{e}^{s/(2M)}, \qquad f(s) \sim \mathrm{e}^{s/(2M)} \qquad \forall s \leqslant 0. \tag{4.11}$$

Here and below we shall use often the notation $f \lesssim g$, which means the existence of a positive constant C so that $f \leqslant Cg$. The standard notation $f \sim g$ is equivalent to $f \lesssim g$ and $g \lesssim f$.

For this reason, the study of the large-time behavior of solutions to the wave equation in the Schwarzschild metric is reduced to the study of the semilinear 1-dimensional wave equation in (4.7). We shall study the behavior of the solutions to (4.7) without using the explicit representation (4.8), but simply assuming that

$W(s),\ f(s)$ obey the asymptotic properties (4.9), (4.10), (4.11) only.

4.2 Asymptotic estimates and the local existence theorem

The Regge–Wheeler coordinate

$$s(r) = r + 2M\log(r - 2M) \tag{4.12}$$

satisfies the relation

$$\frac{r}{r-2M}\,dr = \,ds\,.$$

If $r(s)$ denotes the function inverse to (4.12), then one can find positive constants C_1, C_2 so that we have the following asymptotic behaviors:

$$\begin{cases} C_1 s \leqslant r(s) \leqslant C_2 s & \text{if } \ s \geqslant 2\,, \\ C_1 \leqslant r(s) \leqslant C_2 & \text{if } \ |s| \leqslant 2\,, \\ C_1\,\mathrm{e}^{s/2M} \leqslant r(s) - 2M \leqslant C_2\,\mathrm{e}^{s/2M} & \text{if } \ s \leqslant -2\,; \end{cases} \tag{4.13}$$

further, the coefficient $F(r(s))$ defined in (4.2) satisfies

$$\begin{cases} |F(s) - 1| \leqslant C_2/s & \text{if } \ s \geqslant 2\,, \\ C_1 \leqslant F(s) \leqslant C_2 & \text{if } \ |s| \leqslant 2\,, \\ C_1\,\mathrm{e}^{s/2M} \leqslant F(s) \leqslant C_2\,\mathrm{e}^{s/2M} & \text{if } \ s \leqslant -2\,. \end{cases} \tag{4.14}$$

Moreover, we can use the definitions (4.8) of the potential $W(s)$ and of the coefficient $f(s)$ to conclude that (4.9), (4.10) and (4.11) are satisfied.

Now we can state and show the local existence result for the Cauchy problem (4.7).

Theorem 4.2.1. *Given any $\sigma \in [1, p+1[$ and any real number $E > 0$, one can find $T = T(E)$ so that if the initial data*

$$v_0 \in \mathrm{H}^{\sigma}(\mathbb{R}), \qquad v_1 \in \mathrm{H}^{\sigma-1}(\mathbb{R})$$

satisfy

$$\|v_0\|_{\mathrm{H}^{\sigma}(\mathbb{R})} + \|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \leqslant E\,,$$

then the Cauchy problem (4.7) *has a unique solution*

$$v(t,s) \in \mathscr{C}^0([0,T[;\mathrm{H}^\sigma(\mathbb{R})) \cap \mathscr{C}^1([0,T[;\mathrm{H}^{\sigma-1}(\mathbb{R}))\,.$$

PROOF OF THEOREM 4.2.1. It is not difficult to see that

$$G = -\partial_s^2 + W(s)$$

is a nonnegative symmetric operator in the Hilbert space $\mathrm{L}^2(\mathbb{R}, ds)$ with dense domain $\mathrm{H}^2(\mathbb{R})$. Thus the estimates (4.9), (4.10) and (4.11), combined together with the KLMN-theorem (see Theorem 10.17 in [42]), imply that G is a nonnegative self-adjoint operator.

Let us consider the Cauchy problem

$$\begin{cases} \partial_t^2 v + Gv = \Phi\,, \\ v(0,s) = v_0(s), \quad \partial_t v(0,s) = v_1(s)\,; \end{cases} \tag{4.15}$$

then the solution can be represented in the form

$$v(t) = \cos(t\sqrt{G})v_0 + \frac{\sin(t\sqrt{G})}{\sqrt{G}}v_1 + \int_0^t \frac{\sin((t-\tau)\sqrt{G})}{\sqrt{G}}\Phi(\tau)d\tau\,.$$

From this representation, we find

$$\begin{aligned} \|v(t)\|_{\mathrm{L}^2(\mathbb{R},ds)} \leqslant \|v_0\|_{\mathrm{L}^2(\mathbb{R},ds)} + t\|v_1\|_{\mathrm{L}^2(\mathbb{R},ds)} \\ + \int_0^t |t-\tau| \|\Phi(\tau)\|_{\mathrm{L}^2(\mathbb{R},ds)}\, d\tau\,, \end{aligned} \tag{4.16}$$

and for any $\sigma \geqslant 1$ we have

$$\begin{aligned} \|G^{\sigma/2}v(t)\|_{\mathrm{L}^2(\mathbb{R},ds)} \leqslant \|G^{\sigma/2}v_0\|_{\mathrm{L}^2(\mathbb{R},ds)} + \|G^{(\sigma-1)/2}v_1\|_{\mathrm{L}^2(\mathbb{R},ds)} \\ + \int_0^t \|G^{(\sigma-1)/2}\Phi(\tau)\|_{\mathrm{L}^2(\mathbb{R},ds)}\, d\tau\,; \end{aligned} \tag{4.17}$$

thus, using the equivalence

$$\|G^{\sigma/2}h\|_{\mathrm{L}^2(\mathbb{R},ds)} + \|h\|_{\mathrm{L}^2(\mathbb{R},ds)} \sim \|h\|_{\mathrm{H}^\sigma(\mathbb{R})},$$

we arrive at the energy estimate

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbb{R})} &\leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbb{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \\ &\quad + C\int_0^t (1+|t-\tau|)\|\Phi(\tau)\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})}\,d\tau \end{aligned} \tag{4.18}$$

for any $\sigma \geqslant 1$ and a suitable constant $C > 0$. Now we use the fact that in our case $\Phi = |v|^p$, and we can apply the inequality

$$\|\,|v|^p\,\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \leqslant c\|v\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})}\,\|v\|^{p-1}_{L^\infty(\mathbb{R})}$$

for $\sigma - 1 < p$ and a positive constant c (see Theorem 1 of Section 5.4.3, page 363 of [44]) to get

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbb{R})} &\leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbb{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \\ &\quad + cC\int_0^t (1+|t-\tau|)\|v\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})}\,\|v\|^{p-1}_{L^\infty(\mathbb{R})}\,d\tau\,. \end{aligned} \tag{4.19}$$

Since $\sigma \geqslant 1$, we have

$$||v||_{\mathrm{H}^{\sigma-1}} \leqslant C||v||_{\mathrm{H}^\sigma}, \qquad ||v||_{\mathrm{L}^\infty} \leqslant C||v||_{\mathrm{H}^\sigma} \tag{4.20}$$

for a suitable constant $C > 0$, hence (4.19) becomes

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbb{R})} &\leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbb{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \\ &\quad + C\int_0^t (1+|t-\tau|)\|v\|^p_{\mathrm{H}^\sigma(\mathbb{R})}\,d\tau \end{aligned} \tag{4.21}$$

(with a new positive constant C).

To conclude, it is sufficient to combine this energy estimate with the Sobolev embedding $\mathrm{H}^\sigma(\mathbb{R}) \subset \mathrm{L}^\infty(\mathbb{R})$, which holds for $\sigma > 1/2$, obtaining easily the desired local existence result for $\sigma \in [1, p+1[$ (i.e. Theorem 4.2.1). ■

Chapter 5

Blow-up in the Schwarzschild Metric

This chapter deals with the blow-up of radially symmetric solutions to the semilinear wave equation Cauchy problem under suitable assumptions. We prove two different results. The first one concerns the blow-up when the initial data are small and far from the black hole, while the exponent p of the nonlinearity satisfies $1 < p < 1 + \sqrt{2}$. In the second case, we consider large initial data close to the black hole and get a blow-up result for $2 < p < 1+\sqrt{2}$. These problems present additional difficulties if compared with the flat case and even the blow-up with large data is not trivial; consequently, in this chapter we also develop the instruments that we shall need in the proofs of the main results.
My paper (7) provides a first approach to the blow-up phenomenon in the case $p \in]1, 2]$, while the complete results can be found in my joint works with Prof. V. Georgiev: (9), (10) and above all (11).

In Section 5.1 we introduce the problem, the assumption and the results, along with the used tools. We compare the results and the difficulties with the ones that appear in the flat case.

Section 5.2 contains the statement of the Kato lemma and the statements and the proofs of its needed variants.

In Section 5.3 we prove the first blow-up result, that is for small data far from the black hole and $p \in]1, 1 + \sqrt{2}[$.

In Section 5.4 we prove the blow-up result for large data close to the black hole and $p \in]2, 1 + \sqrt{2}[$.

The following sections are a sort of appendix including technical results previously used in this chapter. In particular, Section 5.5 is dedicated to the proof of an estimate for the auxiliary function F_1 (used in the Sections 5.3 and 5.4), while Section 5.6 is devoted to the verification of some asymptotic estimates for φ_0, φ_1 and for ψ_0, all functions that we shall use in Section 5.3 and following ones, which are solutions to an associated elliptic problem.

5.1 Introduction

In this chapter we are going to study the large-time behavior of solutions to the corresponding Cauchy problem for the semilinear wave equation

$$\Box_g u = |u|^p \qquad \text{in } [0,\infty[\times\mathbf{\Omega}, \tag{5.1}$$

where $p > 1$. This problem can be considered as a natural analogue of the classical semilinear wave equation

$$\Box_{g_0} u = |u|^p \qquad \text{in } [0,\infty[\times\mathbb{R}^n, \tag{5.2}$$

where g_0 is the flat Minkowski metric

$$g_0 = dt^2 - dr^2 - r^2\, d\omega^2. \tag{5.3}$$

It is well-known (see [32], [33], [29], [46], [47], [49], [56], [58], [24], [55], [37] or the review in [22] for a more complete list of references on the subject) that, for any space dimension $n \geqslant 2$, there exists a critical value $p_0 = p_0(n) > 1$ such that the Cauchy problem for (5.2) admits a global small-data solution provided $p > p_0(n)$. For subcritical values of $p \leqslant p_0(n)$, a blow-up phenomenon in the flat background is manifested. In the case of a space dimension $n = 3$, the critical exponent is $p_0(3) = 1+\sqrt{2}$, while in the general case of a space dimension $n \geqslant 2$, the critical exponent is defined as the positive solution to

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The blow-up results in [32], [33], [29], [46], [47] require a suitable comparison principle for the free wave equation. One further remark is connected with the fact that the critical exponent $p_0(n)$ is the same for the smaller class of radially symmetric solutions.

For the case of the Schwarzschild metric, the dispersive properties of the solution to the linear problem

$$\square_g u = \Phi \qquad \text{in } [0, \infty[\times \mathbf{\Omega} \tag{5.4}$$

with zero initial data depend essentially on the distribution of resonances for the operator

$$P = \frac{F}{r^2}\partial_r(r^2 F)\partial_r + \frac{F}{r^2}\Delta_{\mathbb{S}^2}\,. \tag{5.5}$$

This problem is studied in [2], [45] (see also [12]) and in [45] it is shown that the resolvent $R(z) = (z^2 - P)^{-1}$ can be extended as a meromorphic function (as an operator from $\mathscr{C}_0^\infty(\mathbf{\Omega})$ to $\mathscr{C}^\infty(\mathbf{\Omega})$) from $\{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$ to $\mathbb{C} \setminus i\mathbb{R}$. The result in [45] shows that the resolvent can be extended further to a meromorphic function in the whole complex plane $\mathbb{C}$. The corresponding poles of the resolvent are called resonances and they are isolated and have finite rank. Moreover, there exists a strip of the type

$$\{z \in \mathbb{C} \,:\, |\operatorname{Im} z| < \varepsilon\} \tag{5.6}$$

free of resonances. However, the result contained in [45] is shown only for the De Sitter–Schwarzschild metric (see Section 7.1). This phenomenon is similar to the situation of an exterior domain of several convex obstacles, studied in [31], where a similar domain free of resonances is found. The approach in [31] leads to an exponential decay of the local energy with a derivative loss. A similar exponential energy decay with derivative losses is assumed in [39] for the case of the wave equation in the exterior of compact obstacles. In this work, some weighted space-time a priori L^2-estimates are obtained and further applications to quasilinear wave equation in the exterior of compact obstacles are done.

As shown in Chapter 4, if we restrict ourselves to symmetrically radial solutions, our problem is equivalent to

$$\text{(5.7)} \qquad \begin{cases} [\partial_{tt} - \partial_{ss} + W(s)]\, v = f(s)|v|^p, & (t,s) \in [0,\infty[\times\mathbb{R}, \\ v(0,s) = v_0(s), \quad \partial_t v(0,s) = v_1(s), & s \in \mathbb{R} \end{cases}$$

(this corresponds to the case $R_0 = 2M$), where W and f are defined and characterized in Section 4.1, and our main goal in this chapter is to show a blow-up result for $1 < p < 1+\sqrt{2}$. A natural idea here is to adapt the approach of F. John from [32], [33] to the semilinear problem (5.1) in the flat metric and to show the blow-up for some subcritical values of p. This approach meets the essential difficulty that there is no simple explicit representation of the corresponding fundamental solution to the d'Alambert operator in the Schwarzschild metric.

To study the maximal time interval of existence of solutions to (5.7), we choose the following initial data:

$$\text{(5.8)} \qquad v_0(s) = \rho(\varepsilon)\chi_0\big(s - s_0(\varepsilon)\big)\,, \qquad v_1(s) = \rho(\varepsilon)\chi_1\big(s - s_0(\varepsilon)\big)\,,$$

where $\chi_j \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfy, for $j = 1, 2$, the conditions

$$\text{(5.9)} \qquad \chi_j(s) \geqslant 0\,, \qquad s \in \mathbb{R}\,,$$

$$\text{(5.10)} \qquad \chi_j(s) = 1\,, \qquad s \in [-R/2, R/2]\,,$$

$$\text{(5.11)} \qquad \operatorname{supp}\chi_j \subseteq [-R, R]$$

for a positive constant R. The function $\rho(\varepsilon)$ will be chosen appropriately later on.

It is not difficult to see that

$$\text{(5.12)} \qquad ||v_0||_{\mathrm{H}^\sigma(\mathbb{R})} + ||v_1||_{\mathrm{H}^{\sigma-1}(\mathbb{R})} \sim \rho(\varepsilon)$$

for each $\sigma \geqslant 1$, so the initial data in (5.8) have small $\mathrm{H}^\sigma \times \mathrm{H}^{\sigma-1}$ norms provided $\rho(\varepsilon)$ is small.

Recall that a big positive Regge–Wheeler coordinate s corresponds to the domain where one is far away from the black hole ($\mathbb{R}^3 \setminus \mathbf{\Omega}$), i.e. the domain with almost flat metric. On the other hand, $s \to -\infty$ corresponds to the domain close to the black hole.

First, we shall choose $\varepsilon > 0$ sufficiently small and shall set

$$s_0(\varepsilon) = \varepsilon^{-\vartheta}\,, \qquad \rho(\varepsilon) = \varepsilon\,, \tag{5.13}$$

where ϑ satisfies

$$\vartheta \geqslant \frac{b(p-1)}{-p^2+2p+1}\,, \qquad \vartheta \geqslant 1 + \frac{b(3p-5)}{-p^2+2p+1}\,, \tag{5.14}$$

and $b = p$ if $p \in [2, 1+\sqrt{2}[$, $b = p^2$ if $p \in]1, 2[$.

In this case, the initial data in (5.8) have support far away from the black hole and our next result asserts that small data solutions manifest a blow-up phenomenon in the subcritical case.

Theorem 5.1.1. *For any p, $1 < p < 1+\sqrt{2}$, there exists a positive number ε_0 so that, for any $\varepsilon \in]0, \varepsilon_0[$ and any initial data of type* (5.8) *satisfying* (5.13)*, there exists a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathscr{C}^k\big([0, T[; \mathrm{H}^{2-k}(\mathbb{R})\big)$$

of (5.7) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbb{R})} = \infty\,.$$

The above result means that, when the initial data are supported far away from the black hole, the wave equation in the Schwarzschild metric has a critical exponent similar to that one of the free wave equation. In this region, we can estimate from above the lifespan of the solution.

The situation changes completely when one tries to approach the black hole. To have a model that simulates this phenomenon, we take initial data of the type (5.8) choosing

$$s_0(\varepsilon) = -T_2(\varepsilon)\,, \tag{5.15}$$

where $T_2(\varepsilon) > 0$ grows very rapidly as $\varepsilon \to 0$. More precisely, we take $T_2(\varepsilon) \in \Sigma$, where Σ is the following class of functions:

$$\Sigma = \{T(\varepsilon) \in \mathscr{C}(]0,1]) \, : \, \forall A \geqslant 1, \, \lim_{\varepsilon \downarrow 0} \varepsilon^A T(\varepsilon) = \infty\}. \tag{5.16}$$

A typical example is $T(\varepsilon) = \mathrm{e}^{1/\varepsilon}$.

Approaching the black hole, one meets an essential difficulty to overcome the attraction force of the black hole. In this case, the coefficient $f(s)$ in the source term in (5.7) decays exponentially and this dissipative phenomenon is in competition with the blow-up properties of the source term. Because of this, the blow-up mechanism which we propose is based on a different choice of the quantity $\rho(\varepsilon)$ that measures the Sobolev norm of the initial data according to (5.12). We have to take $\rho(\varepsilon) \in \Sigma$, i.e. the initial data are large.

Then we have the following blow-up result.

Theorem 5.1.2. *For any p, $2 < p < 1 + \sqrt{2}$, there exists a positive number ε_0 so that, for any $\varepsilon \in]0, \varepsilon_0[$ and any initial data of type* (5.8) *satisfying* (5.15)*, there exists a function $\rho(\varepsilon) \in \Sigma$, a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathscr{C}^k\big([0, T[; \mathrm{H}^{2-k}(\mathbb{R})\big)$$

of (5.7) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbb{R})} = \infty \, .$$

However, the proof we follow here suggests that the lifespan of the solution has a completely different behavior —it might be much longer than in the corresponding flat case (see Lemma 5.2.3 below).

The main difficulty to establish the blow-up of the solution is connected with the sign-changing properties of the fundamental solution of the linear wave equation in Schwarzschild metric (or more generally in curved metrics). In the case of the flat $(1+3)$-Minkowski metric, the fundamental solution is nonnegative and this property is used effectively in the study of the blow-up phenomenon for the corresponding semilinear wave equation.

Our blow-up analysis, in the case of initial data supported far away from the black hole, is based on the study of the asymptotic behavior of the following quantities:

$$U(t) = \left(\int_{\mathbb{R}} \varphi_0(s)|v(t,s)|^p f(s)\, ds\right)^{1/p}, \tag{5.17}$$

$$V(t) = \int_{\mathbb{R}} \varphi_0(s) v(t,s)\, ds\,. \tag{5.18}$$

Here φ_0 is a solution to the equation

$$\partial_{ss}\varphi_0(s) - W(s)\varphi_0(s) = 0\,. \tag{5.19}$$

Lemma 5.6.3 implies that there exists a solution φ_0 to (5.19) such that

$$\varphi_0(s) = \psi_0(s) + D \qquad \forall s \in \mathbb{R} \tag{5.20}$$

for some positive constant D and $\psi_0(s) \geqslant 0$ has the asymptotic expansion

$$\psi_0(s) \sim \begin{cases} s & s \to \infty\,, \\ \mathrm{e}^{s/(2M)} & s \to -\infty\,. \end{cases} \tag{5.21}$$

The key point in the proof of Theorem 5.1.1 is to verify the following a priori estimates for V:

$$V(0) \geqslant C_0\varepsilon^\beta\,, \qquad V'(0) \geqslant C_0\varepsilon^\beta\,, \tag{5.22}$$

$$V(t) \geqslant C_0\varepsilon^b(t+R)^a \qquad \forall t \in [T_0, T_1 + T_0[\,, \tag{5.23}$$

$$V''(t) \geqslant C_0(t+R)^{-q}V(t)^p \qquad \forall t \in [T_0, T_1 + T_0[\,, \tag{5.24}$$

and then to apply a suitable variant of the classical Kato Lemma (see Lemma 5.2.2 below).

For the case of initial data supported close to the black hole we modify U, V as follows:

$$U(t) = \left(\int_{\mathbb{R}} \psi_0(s)|v(t,s)|^p f(s)\, ds\right)^{1/p},$$

$$V(t) = \int_{\mathbb{R}} \psi_0(s) v(t,s)\, ds\,.$$

The main new phenomenon manifested in this case is the possible loss of positivity of $V(t)$. Indeed, one can show that V satisfies the differential equation (see (5.113) below)

$$V''(t) = U(t)^p + D \int vW(s)\, ds\,,$$

so the positivity of $V(t)$ cannot be obtained as a trivial consequence of the positivity of $V(0), V'(0)$ and the differential equation satisfied by $V(t)$.

We use in this case another variant of the classical Kato Lemma stated in Lemma 5.2.3 and involving two functions U and V.

5.2 Variants of the classical Kato lemma

A key point in the blow-up argument that we follow is an appropriate modification of the following lemma due to Kato (see [35]).

Lemma 5.2.1 (Classical Kato lemma). *Assume that* $a \geqslant 1$, $p > 1$ *and*

$$0 \leqslant q < (p-1)a + 2. \tag{5.25}$$

Given any positive constants R, C_0 *one can find a constant*

$$C_1 = C_1(a,p,q,R,C_0) > 0$$

so that for any $T_0 \geqslant 0$ *the condition*

a) there exists a nonnegative function $V \in \mathscr{C}^2([T_0, T_0+T[)$ *so that*

$$V(T_0) \geqslant 0\,, \qquad V'(T_0) \geqslant 0\,, \tag{5.26}$$

$$V(t) \geqslant C_0(t+R)^a \qquad \forall t \in [T_0, T_0+T[\,, \tag{5.27}$$

$$V''(t) \geqslant C_0(t+R)^{-q} V(t)^p \qquad \forall t \in [T_0, T_0+T[\tag{5.28}$$

will imply

$$T \leqslant C_1(1+T_0)\,. \tag{5.29}$$

PROOF. For $0 \leqslant T_0 \leqslant 1$ this is the classical assertion of the Kato lemma. For $T_0 \geqslant 1$ the inequality (5.29) is equivalent to $T \leqslant C_1 T_0$ and we can use a rescaling argument. More precisely, we shall perform the following transform:

$$t = T_0 \tau\,, \qquad V(t) = T_0^a \tilde{V}(\tau)\,. \tag{5.30}$$

Setting

$$\tau_1 = T/T_0\,,$$

we see that

$$\begin{cases} \tilde{V}(1) \geqslant 0\,, \qquad \tilde{V}'(1) \geqslant 0\,, \\ \tilde{V}(\tau) \geqslant C_0 \tau^a & \forall \tau \in [1, 1+\tau_1[\,, \\ \tilde{V}''(\tau) \geqslant C_0 T_0^{a(p-1)-q+2} \tau^{-q} \tilde{V}(\tau)^p & \forall \tau \in [1, 1+\tau_1[\,. \end{cases} \tag{5.31}$$

Assumption (5.25) implies that $T_0^{a(p-1)-q+2} \geqslant 1$ so, applying for (5.31) the classical Kato's argument, we get $\tau_1 \leqslant C_1$. This completes the proof. ■

The above lemma gives the following more precise information for the lifespan interval in t.

Lemma 5.2.2 (Finite lifespan for the classical Kato lemma)**.** *Assume that* $a \geqslant 1$, $p > 1$ *satisfy*

$$0 \leqslant q < (p-1)a + 2\,,$$

R, C_0 *are two positive constants and* $b \geqslant 0$*. There is a positive constant*

$$C_1 = C_1(a, p, q, R, C_0, b)\,,$$

so that for any $\varepsilon \in]0,1[$ *and for any couple of real numbers* $T_0 \geqslant 0$*,* $T_1 > 0$ *the condition*

a) there exists a nonnegative function $V \in \mathscr{C}^2([T_0, T_0 + T_1[)$ *such that* *the inequalities*

$$T_0 \geqslant C_0\varepsilon^{\alpha}, \qquad \alpha = -\frac{b(p-1)}{2+(p-1)a-q} \leqslant 0\,, \tag{5.32}$$

$$V(T_0) \geqslant C_0\varepsilon^{\beta}, \qquad V'(T_0) \geqslant C_0\varepsilon^{\beta}, \qquad \beta = -b(q-2)/(2+(p-1)a-q)\,, \tag{5.33}$$

$$V(t) \geqslant C_0\varepsilon^{b}(t+R)^{a} \qquad \forall t \in [T_0, T_0+T_1[\,, \tag{5.34}$$

$$V''(t) \geqslant C_0(t+R)^{-q}V(t)^{p} \qquad \forall t \in [T_0, T_0+T_1[\tag{5.35}$$

will imply

b) $T_1 \leqslant C_1(T_0+\varepsilon^{\alpha})\,.$

PROOF. For $b=0$ the assertion of the lemma follows directly from the classical Kato Lemma 5.2.1. Take $b>0$; making the transform

$$t = \varepsilon^{\alpha}\tau\,, \qquad V(t) = \varepsilon^{\beta}\tilde{V}(\tau) \tag{5.36}$$

and setting

$$\tau_0 = T_0\varepsilon^{-\alpha}, \qquad \tau_1 = (T_0+T_1)\varepsilon^{-\alpha},$$

we see that

$$\begin{cases} \tilde{V}(\tau_0) \geqslant C_0\,, \qquad \tilde{V}'(\tau_0) \geqslant C_0\varepsilon^{\alpha}, \\ \tilde{V}(\tau) \geqslant C_0\varepsilon^{b-\beta+\alpha a}(\tau+R\varepsilon^{-\alpha})^{a} & \forall \tau \in [\tau_0,\tau_1[\,, \\ \tilde{V}''(\tau) \geqslant C_0\varepsilon^{\beta(p-1)-\alpha(q-2)}(\tau+R\varepsilon^{-\alpha})^{-q}\tilde{V}(\tau)^{p} & \forall \tau \in [\tau_0,\tau_1[\,. \end{cases} \tag{5.37}$$

We choose α, β so that

$$b-\beta+\alpha a = \beta(p-1)-\alpha(q-2) = 0\,.$$

The solution to the above system is given by

$$\alpha = -\frac{b(p-1)}{(2+(p-1)a-q)}\,, \qquad \beta = -\frac{b(q-2)}{(2+(p-1)a-q)}\,.$$

Then (5.37) becomes

$$
\begin{cases}
\tilde{V}(\tau_0) \geqslant C_0\,, \qquad \tilde{V}'(\tau_0) \geqslant C_0\varepsilon^{\alpha}, \\
\tilde{V}(\tau) \geqslant C_0(\tau + R\varepsilon^{-\alpha})^a \geqslant C_0\tau^a \qquad \forall \tau \in [\tau_0, \tau_1[\,, \\
\tilde{V}''(\tau) \geqslant C_0(\tau + R\varepsilon^{-\alpha})^{-q}\tilde{V}(\tau)^p \qquad \forall \tau \in [\tau_0, \tau_1[\,.
\end{cases} \tag{5.38}
$$

Since $\alpha \leqslant 0$, $q \geqslant 0$ and $0 < \varepsilon < 1$, we have

$$
(\tau + R\varepsilon^{-\alpha})^{-q} \geqslant (\tau + R)^{-q}\,.
$$

Note that assumption (5.32) implies $\tau_0 = T_0\varepsilon^{-\alpha} \geqslant C_0$. We are now in the situation to apply the classical Kato Lemma 5.2.1 and get $\tau_1 \leqslant C_1(1 + \tau_0)$, which implies

b) $T_1 \leqslant C_1(T_0 + \varepsilon^{\alpha})$.

This completes the proof of the lemma. ■

We shall need another variant of the Kato lemma. For the purpose, we introduce the following class of functions:

$$
\Sigma = \{T(\varepsilon) \in C(]0,1]) \,:\, \forall A \geqslant 1,\, \lim_{\varepsilon \downarrow 0} \varepsilon^A T(\varepsilon) = \infty\}\,. \tag{5.39}
$$

Lemma 5.2.3. *Assume that* $a \geqslant 1$, $p > 1$ *satisfy*

$$
0 \leqslant q < (p-1)a + 2\,, \tag{5.40}
$$

while R, C_0, C_1 *are positive constants. Then there exist two constants*

$$
D_1 = D_1(a,p,q,R,C_0,C_1) > 0\,, \qquad \varepsilon_0 = \varepsilon_0(a,p,q,R,C_0,C_1) > 0\,,
$$

so that for any positive $T_0(\varepsilon) \in \Sigma$, *for any* $\varepsilon \in]0, \varepsilon_0[$ *and for any real number* $T_1 > 0$, *the condition*

a) there exist two functions $U \in \mathscr{C}([0, T_0(\varepsilon) + T_1[)$, $V \in \mathscr{C}^2([0, T_0(\varepsilon) +$

$T_1[$), *so that*

$$V(0) \geqslant 0\,, \qquad V'(0) \geqslant 0\,, \qquad U(t) \geqslant 0 \qquad \forall t \in [0, T_0(\varepsilon) + T_1[\,, \tag{5.41}$$

$$U(t)^p \geqslant C_0(t+R)^{-q}|V(t)|^p + C_0\varepsilon^p(t+R)^{a-2} \qquad \forall t \in [0, T_0(\varepsilon) + T_1[\,, \tag{5.42}$$

$$V''(t) \geqslant C_0 U(t)^p - C_1 U(t) \qquad \forall t \in [0, T_0(\varepsilon) + T_1[\,, \tag{5.43}$$

$$V''(t) \geqslant C_0 U(t)^p - C_1\,\mathrm{e}^{-C_0 T_0(\varepsilon)}\,U(t) \qquad \forall t \in [0, T_0(\varepsilon)[\tag{5.44}$$

implies

b) $T_1 \leqslant D_1 T_0(\varepsilon)\,.$

PROOF. Consider the function

$$K(x) = x^p - Cx\,, \tag{5.45}$$

where $C = C_1/C_0 > 0$. For

$$x > x_0 \doteq 2C^{1/(p-1)}\,, \tag{5.46}$$

we have

$$K(x) \gtrsim x^p\,. \tag{5.47}$$

In a similar way, given $T_0(\varepsilon) \in \Sigma$, we can consider the function

$$K_{T_0(\varepsilon)}(x) = x^p - C\,\mathrm{e}^{-C_0 T_0(\varepsilon)}\,x\,. \tag{5.48}$$

For

$$x > x_0(\varepsilon) \doteq 2C^{1/(p-1)}\,\mathrm{e}^{-C_0 T_0(\varepsilon)/(p-1)}\,, \tag{5.49}$$

we have

$$K_{T_0(\varepsilon)}(x) \gtrsim x^p\,. \tag{5.50}$$

Note that inequality (5.42) assures that

$$U(t) \gtrsim \varepsilon(t+R)^{(a-2)/p} \geqslant \varepsilon(T_0+R)^{(a-2)/p} \tag{5.51}$$

for $t \in [0, T_0]$ if $a \leqslant 2$, and

$$U(t) \gtrsim \varepsilon(t+R)^{(a-2)/p} \geqslant \varepsilon R^{(a-2)/p} \tag{5.52}$$

for $t \in [0, T_0]$ if $a > 2$. Now it is clear that choosing $T_0 = T_0(\varepsilon) \in \Sigma$, we can guarantee the following analogue of inequality (5.49):

$$U(t) > x_0(\varepsilon) \doteq 2C^{1/(p-1)}\,\mathrm{e}^{-C_0T_0(\varepsilon)/(p-1)} \qquad \text{for } t \in [0, T_0(\varepsilon)]\,,\ \varepsilon \in]0, \varepsilon_0[\,. \tag{5.53}$$

Indeed, the lower bound of $U(t)$ is at most polynomially decaying (in T_0) due to (5.51) and (5.52), while $x_0(\varepsilon)$ decays exponentially in $T_0(\varepsilon)$ as $\varepsilon \to 0$.

Since (5.49) implies (5.50), we conclude that

$$V''(t) \gtrsim U(t)^p \gtrsim \varepsilon^p(t+R)^{a-2} \qquad \text{for } t \in [0, T_0[\,,$$

and integrating this inequality twice, we obtain

$$V(t) \gtrsim \varepsilon^p(t+R)^a, \quad V'(t) \gtrsim \varepsilon^p(t+R)^{a-1} \quad \text{for } t \in [T_0/2, T_0[\,, \tag{5.54}$$

because of (5.41). Note that we have also the inequality

$$V''(t) \gtrsim (t+R)^{-q}V(t)^p \qquad \text{for } t \in [T_0/2, T_0]\,. \tag{5.55}$$

Now we are in the situation when estimates similar to the estimates (5.27) and (5.28) of the classical Kato lemma are fulfilled. Setting

$$a_0 = a\,, \qquad p_0 = p\,,$$

we define the recurrence sequence

$$a_{k+1} = pa_k - q + 2\,, \qquad p_{k+1} = p_kp\,. \tag{5.56}$$

It is clear that

$$a_{k+1} = a + \frac{p^{k+1}-1}{p-1}\left((p-1)a - q + 2\right), \qquad p_k = p^{k+1}, \tag{5.57}$$

so integrating twice the differential inequality (5.55) in combination with (5.54), we prove inductively in k the following estimates

$$V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1} \tag{5.58}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$. It is not difficult to see that a_k tends to infinity, due to (5.57) and (5.40).

We can choose in particular $k \geqslant 0$ so that $a_k > 2$ and fix this k. Then estimate (5.58) combined with (5.42) implies

$$\begin{cases} V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}, \\ U(t)^p \gtrsim \varepsilon^{p_{k+1}}(t+R)^{a_{k+1}-2} \end{cases} \tag{5.59}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$. Thus, we have

$$U(t) \gtrsim \varepsilon^{p_{k+1}/p}(T_0(\varepsilon)+R)^{(a_{k+1}-2)/p} > x_0 \doteq 2C^{1/(p-1)} \tag{5.60}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$, due to the property $T_0 \in \Sigma$ and the definition (5.39) of the class Σ (we assume that ε is sufficiently small). Once this inequality is satisfied, we can use (5.42), (5.60) and (5.43), and see that

$$U(t)^p \gtrsim (t+R)^{-q}|V(t)|^p + \varepsilon^p(t+R)^{a-2}, \tag{5.61}$$

$$V''(t) \gtrsim U(t)^p - CU(t) \tag{5.62}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0+T_1[$ and

$$U(t) > x_0, \quad V'(t) \geqslant 0 \quad \text{for } t \in [(1-2^{-(k+1)})T_0, T_0]. \tag{5.63}$$

Let

$$T_2 \doteq \sup\{T \in [0,T_1] \,:\, U(t) > x_0,\ t \in [(1-2^{-(k+1)})T_0, T_0+T[\}.$$

One can show that $T_2 = T_1$. Indeed, for $t \in [(1-2^{-(k+1)})T_0, T_0 + T_2[$, we have

$$V''(t) \gtrsim U(t)^p \gtrsim (t+R)^{-q}|V(t)|^p + \varepsilon^p (t+R)^{a-2}\,; \tag{5.64}$$

so we are in the situation to repeat the argument of the proof of (5.59) and we can prove inductively in m the following estimates:

$$V(t) \gtrsim \varepsilon^{p_m}(t+R)^{a_m}\,, \qquad V'(t) \gtrsim \varepsilon^{p_m}(t+R)^{a_m-1} \tag{5.65}$$

for all $t \in [(1-2^{-(k+m+1)})T_0, T_0 + T_2[$. Take $m = k$ so that $a_k > 2$. Then estimate (5.65) combined with (5.64) implies

$$\begin{cases} V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}\,, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}\,, \\ U(t)^p \gtrsim \varepsilon^{p_{k+1}}(t+R)^{a_{k+1}-2}\,, \end{cases} \tag{5.66}$$

for $t \in [(1-2^{-(2k+1)})T_0, T_0 + T_2[$. Since, provided ε small, we have

$$\varepsilon^{p_{k+1}}(T_0(\varepsilon) + R)^{a_{k+1}-2} > x_0^p$$

due to the property $T_0 \in \Sigma$, we get also $U(t) > x_0$ near $t = T_0 + T_2$. This inequality, combined with the definition of T_2, guarantees that $T_2 = T_1$.

From this conclusion we see that the functions

$$U \in \mathscr{C}([0, T_0(\varepsilon) + T_1[)\,, \qquad V \in \mathscr{C}^2([0, T_0(\varepsilon) + T_1[)\,,$$

satisfying the condition *a)*, will also obey the following stronger estimates

$$V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}\,, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}\,, \tag{5.67}$$

$$V''(t) \gtrsim (t+R)^{-q}V(t)^p \qquad \forall t \in [(1-2^{-(2k+1)})T_0(\varepsilon)\,, T_0(\varepsilon) + T_1[\,. \tag{5.68}$$

It remains to verify condition (5.33) in order to apply the lifespan estimate of Lemma 5.2.2. Indeed, (5.67) implies

$$V(T_0(\varepsilon)) \gtrsim \varepsilon^{p_k}(T_0(\varepsilon) + R)^{a_k}\,,$$

therefore the fact that $T_0 \in \Sigma$ and $a_k > 0$ imply (5.33), so applying Lemma 5.2.2 we complete the proof. ■

5.3 Blow-up for small data far from the black hole and $p \in]1, 1+\sqrt{2}[$

In this subsection, we verify the hypotheses of Lemma 5.2.2 to get Theorem 5.1.1. In particular, we take

$$p > 1\,, \qquad q = 3(p-1)\,, \qquad a = 4 - p\,, \tag{5.69}$$

and prove (5.33), (5.34) and (5.35) for

$$U(t) = \left(\int_{\mathbb{R}} \varphi_0(s) |v(t,s)|^p f(s)\, ds \right)^{1/p}, \tag{5.70}$$

$$V(t) = \int_{\mathbb{R}} \varphi_0(s) v(t,s)\, ds\,, \tag{5.71}$$

where v is a solution to the Cauchy problem (5.7), (5.8) with

$$\rho(\varepsilon) = \varepsilon\,, \qquad s_0(\varepsilon) = \varepsilon^{-\vartheta} \tag{5.72}$$

and

$$\vartheta \geqslant \frac{b(p-1)}{\delta}\,, \qquad \vartheta \geqslant 1 + \frac{b(3p-5)}{\delta}\,, \tag{5.73}$$

$$b = \begin{cases} p & \text{if } 2 \leqslant p < 1 + \sqrt{2}\,, \\ p^2 & \text{if } 1 < p < 2\,, \end{cases} \tag{5.74}$$

$$\delta = (p-1)a - q + 2 > 0\,. \tag{5.75}$$

Let us notice that (5.75) is equivalent to

$$p^2 - 2p - 1 < 0\,,$$

that is,

$$1 < p < 1 + \sqrt{2}\,;$$

in particular, we have $a > 1$. Moreover, (5.72) guarantees that

$$||v_0||_{\mathrm{H}^2(\mathbb{R})} + ||v_1||_{\mathrm{H}^1(\mathbb{R})} \sim \varepsilon\,, \tag{5.76}$$

therefore the initial data in (5.7), (5.8) have small $\mathrm{H}^2 \times \mathrm{H}^1$ norms.

First, we check (5.33). We apply Lemma 5.3.1 (see the end of this section) and get that there exists a positive constant C_0 such that

$$V(0) \geqslant C_0\varepsilon^\beta, \qquad V'(0) \geqslant C_0\varepsilon^\beta, \qquad \beta = -\frac{b(3p-5)}{\delta}$$

thanks to the second part of condition (5.73), that is $1 - \vartheta \leqslant \beta$.

Further, we have the relation

$$\begin{aligned} V''(t) &= \int_{\mathbb{R}} v_{tt}(t,s)\varphi_0(s)\,ds \\ &= \int_{\mathbb{R}} v_{ss}(t,s)\varphi_0(s)\,ds - \int_{\mathbb{R}} W(s)v(t,s)\varphi_0(s) + \int_{\mathbb{R}} f(s)|v(t,s)|^p\varphi_0(s)\,ds\,, \end{aligned}$$

so (5.19) implies

$$V''(t) = \int_{\mathbb{R}} f(s)|v(t,s)|^p\varphi_0(s)\,ds = U(t)^p \tag{5.77}$$

and we conclude that $V''(t)$ is a nonnegative function. This argument guarantees that setting

$$T_0 = T_0(\varepsilon) = 2C_1 s_0(\varepsilon)\,, \tag{5.78}$$

where $C_1 > 0$ is the constant from Lemma 5.2.2, one has

$$V(T_0) \geqslant C_0\varepsilon^\beta, \qquad V'(T_0) \geqslant C_0\varepsilon^\beta, \tag{5.79}$$

thus (5.33) is verified.

The estimates of (5.34) and (5.35) are based on a finite dependence domain argument, that is,

$$\operatorname{supp} v(t,s) \subseteq \{(t,s) \,:\, |s-s_0(\varepsilon)| \leqslant t+R\}\,. \tag{5.80}$$

This dependence domain property implies that the support of $v(t,s)$ is in the domain $s>1$ for $0\leqslant t\leqslant 2T_0$, provided $\varepsilon>0$ is sufficiently small. Hence we can use the Hölder's inequality and get, for $0\leqslant t\leqslant 2T_0$,

$$\begin{aligned}
|V(t)| &\leqslant \int_{\mathbb{R}} |v(t,s)|\varphi_0(s)\,ds \\
&= \int_{|s-s_0(\varepsilon)|\leqslant t+R} \left(f^{1/p}|v|\varphi_0^{1/p}\right)\left(f^{-1/p}\varphi_0^{(p-1)/p}\right)\,ds \\
&\leqslant \left(\int_{\mathbb{R}} f|v|^p\varphi_0\,ds\right)^{1/p}\left(\int_{|s-s_0(\varepsilon)|\leqslant t+R} f^{-1/(p-1)}\varphi_0\,ds\right)^{(p-1)/p};
\end{aligned}$$

since $f(s)^{-1/(p-1)}\varphi_0(s)\lesssim(1+|s|)^2$, we conclude

$$|V(t)|\leqslant CU(t)(t+R)^{3(p-1)/p}$$

for each $t\in[T_0,2T_0]$ (i.e. $s_0(\varepsilon)\leqslant t/2C_1$), where $C>0$ is a constant independent of ε, T_0. Thus, we have

$$U(t)^p \geqslant C(t+R)^{-q}|V(t)|^p, \qquad q=3(p-1)\,, \tag{5.81}$$

and the inequality (5.35) is verified.

It is easy to show that estimate (5.34) follows from an estimate of the type

$$U(t)^p\geqslant C\varepsilon^b(t+R)^{a-2}, \qquad t\in[T_0,2T_0]\,. \tag{5.82}$$

Indeed, using (5.77) and integrating twice (5.82) we get (5.34), since the initial data are nonnegative due to (5.79).

To verify (5.82), we use the auxiliary quantity

$$F_1(t)=\mathrm{e}^{-t/(2M)}\int_{\mathbb{R}} v(t,s)\varphi_1(s)\,ds\,, \tag{5.83}$$

where φ_1 is a solution to

$$-\partial_{ss}\varphi_1 + W(s)\varphi_1 + \frac{1}{4M^2}\varphi_1 = 0 \tag{5.84}$$

having asymptotic behavior

$$\varphi_1(s) \sim \mathrm{e}^{s/(2M)} \qquad \text{as } |s| \longrightarrow \infty\,. \tag{5.85}$$

The existence of such a function is shown in Lemma 5.6.4.

Proceeding as before, we obtain

$$\begin{aligned} F_1(t) &\leqslant \mathrm{e}^{-t/2M}\int_{\mathbb{R}} |v|\varphi_1\,ds \\ &= \mathrm{e}^{-t/2M}\int_{|s-s_0(\varepsilon)|\leqslant t+R} \left(f^{\frac{1}{p}}|v|{\varphi_0}^{\frac{1}{p}}\right)\left(f^{-\frac{1}{p}}{\varphi_0}^{-\frac{1}{p}}\varphi_1\right)\,ds \\ &\leqslant U(t)\left(\mathrm{e}^{-\frac{t}{2M}\frac{p}{p-1}}\int_{|s-s_0(\varepsilon)|\leqslant t+R} f^{-\frac{1}{p-1}}{\varphi_0}^{-\frac{1}{p-1}}{\varphi_1}^{\frac{p}{p-1}}\,ds\right)^{\frac{p-1}{p}}; \end{aligned}$$

thanks to the hypotheses on f, φ_0, φ_1 (see (4.10), (5.20), (5.21) and (5.85)), we deduce

$$F_1(t) \leqslant CU(t)\left(\int_{|s-s_0(\varepsilon)|\leqslant t+R}(1+s)^{\frac{p-2}{p-1}}\,\mathrm{e}^{\frac{s-t}{2M}\frac{p}{p-1}}\,ds\right)^{\frac{p-1}{p}}. \tag{5.86}$$

Now we distinguish two cases.

Case 1: $p \in [2, 1+\sqrt{2}[$. Let observe that, given any $\alpha \geqslant 0$, $\beta > 0$ and $R > 0$, one can find a positive constant $C = C(R,\alpha,\beta)$ so that the inequality

$$\int_{0\leqslant s\leqslant t+R+s_0(\varepsilon)} (s+R)^{\alpha}\,\mathrm{e}^{\beta(s-t)}\,ds \leqslant C\,(t+R+s_0(\varepsilon))^{\alpha}\,\mathrm{e}^{\beta s_0(\varepsilon)} \tag{5.87}$$

holds for any $t \geqslant 0$ and $\varepsilon \in]0,1]$. Applying (5.87) with

$$\alpha = \frac{p-2}{p-1} \geqslant 0\,, \qquad \beta = \frac{p}{2M(p-1)} > 0$$

in (5.86), we derive

$$F_1(t) \leqslant CU(t)(t+R+s_0(\varepsilon))^{\frac{p-2}{p}} \ \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} .$$

Now, Lemma 5.5.1 assures that

$$F_1(t) \geqslant C'\varepsilon\, \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} ;$$

so, taking $t \in [T_0, 2T_0]$ (i.e. $t \sim s_0(\varepsilon)$) and the power p on both sides, we get

(5.88) $$U(t)^p \geqslant C\varepsilon^p \, (t+R)^{2-p},$$

that is, the desired estimate (5.82) with $a = 4-p$, $b = p$.

Case 2: $p \in]1,2]$. Given any $\alpha \leqslant 0$, $\beta > 0$ and $R > 0$, one can find a positive constant $C = C(R, \alpha, \beta)$ so that the inequality

(5.89) $$\int_{0 \leqslant s \leqslant t+R+s_0(\varepsilon)} (s+R)^{\alpha}\, \mathrm{e}^{\beta(s-t)} \, ds \leqslant C \ \mathrm{e}^{\beta s_0(\varepsilon)}$$

holds for any $t \geqslant 0$ and $\varepsilon \in]0,1]$. Applying this estimate and (5.86), we derive

$$F_1(t) \leqslant CU(t)\, \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} .$$

Lemma 5.5.1 implies that

$$C'\varepsilon\, \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} \leqslant F_1(t)$$

and, hence, the inequalities

$$C'\varepsilon\, \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} \leqslant F_1(t) \leqslant CU(t)\, \mathrm{e}^{\frac{s_0(\varepsilon)}{2M}}$$

imply $U(t)^p \geqslant C\varepsilon^p$, so we get

(5.90) $$V''(t) = U(t)^p \geqslant C\varepsilon^p.$$

Integrating twice, we obtain

$$V(t) \geqslant C\varepsilon^p (t+R)^2 \tag{5.91}$$

which, substituted in (5.81), yields

$$V''(t) = U(t)^p \geqslant C\varepsilon^{p^2}(t+R)^{3-p} \tag{5.92}$$

and, consequently,

$$V(t) \geqslant C\varepsilon^{p^2}(t+R)^{5-p} \geqslant C\varepsilon^{p^2}(t+R)^a, \tag{5.93}$$

that is (5.34), with $a = 4-p$, $b = p^2$.

Now, suppose that the lifespan $T(\varepsilon)$ for the solution to (5.7) is greater than $(C_1+1)T_0(\varepsilon)$. Then we can apply Lemma 5.2.2 with $T_1 = (C_1+1)T_0(\varepsilon)$ and derive the inequality $T_1 \leqslant C_1(T_0 + \varepsilon^{-b(p-1)/\delta})$, so

$$T_0 \leqslant C_1 \varepsilon^{-b(p-1)/\delta}$$

and this is in contradiction with the choice

$$T_0 = 2C_1\varepsilon^{-\vartheta}, \qquad \vartheta \geqslant \frac{b(p-1)}{\delta}\,.$$

The contradiction shows that $V(t)$ must blow up in finite time $T(\varepsilon) \leqslant (C_1 + 1)T_0(\varepsilon)$.

To conclude, we recall that $|\varphi_0(s)| \lesssim 1+|s|$ (see (5.20) and (5.21)) and apply the Cauchy–Schwartz inequality, obtaining

$$V(t) \lesssim \int_{|s-s_0(\varepsilon)|\leqslant t+R} (1+|s|)|v(t,s)|\,ds \lesssim (t+R)^{3/2}||v(t)||_{\mathrm{L}^2(\mathbb{R})}\,. \tag{5.94}$$

Since V blows up in finite time, the same happens to v and this concludes the proof of Theorem 5.1.1.

We conclude with the statement and the proof of a result that we have used previously in this section.

Lemma 5.3.1. *There exist a positive constant* $\varepsilon_0 >$ *and a positive constant* $C_0 = C_0(R)$ *independent of* ε_0 *such that, for each* $\varepsilon \in]0, \varepsilon_0[$, *one has*

$$V(0) \geqslant C_0\varepsilon^{1-\vartheta}, \qquad V'(0) \geqslant C_0\varepsilon^{1-\vartheta}, \tag{5.95}$$

where ϑ *is the parameter introduced in* (5.72) *and* (5.73).

PROOF. First of all, let us observe that, thanks to our hypotheses on χ_j and the nonnegativity of the integrand functions, we have

$$\begin{aligned} V(0) &= \int_{\mathbb{R}} \varphi_0(s)v(0,s)\,ds = \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R} \varphi_0(s)\chi_0(s-s_0(\varepsilon))\,ds \\ &\geqslant \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\,ds \end{aligned} \tag{5.96}$$

and similarly

$$V'(0) \geqslant \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\,ds\,; \tag{5.97}$$

hence it is sufficient to prove that

$$I(\varepsilon) \doteq \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\,ds \geqslant C_0\varepsilon^{1-\vartheta} \qquad \forall \varepsilon \in]0, \varepsilon_0[\tag{5.98}$$

for suitable constants $C_0 = C_0(R) > 0$ and $\varepsilon_0 > 0$.

Since $s_0(\varepsilon) \sim \varepsilon^{-\vartheta}$ and $\varphi_0(s) \gtrsim s$ for every $s \geqslant \bar{s}$, where $\bar{s} > 0$ is sufficiently large (independent of ε_0), we can choose $\varepsilon_0 > 0$ small enough, such that

$$s_0(\varepsilon) \geqslant R\,, \qquad s_0(\varepsilon) \geqslant 2\bar{s}\,, \tag{5.99}$$

so that

$$s_0(\varepsilon) - \frac{R}{2} \geqslant \frac{s_0(\varepsilon)}{2} \geqslant \bar{s} \qquad \forall \varepsilon \in]0, \varepsilon_0[\tag{5.100}$$

and, consequently,

$$I(\varepsilon) \gtrsim \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} s\, ds = R\varepsilon s_0(\varepsilon) \sim \varepsilon^{1-\vartheta} \qquad \forall \varepsilon \in]0, \varepsilon_0[\,. \tag{5.101}$$

(note that throughout the proof, the implicit constants in "$\gtrsim$" and "$\sim$" are independent of ε and ε_0). ■

5.4 Blow-up for large data close to the black hole and $p \in]2, 1+\sqrt{2}[$

In this subsection, we verify the hypotheses of Lemma 5.2.3 to get Theorem 5.1.2. For this purpose, we set

$$U(t) = \left(\int_{\mathbb{R}} \psi_0(s) |v(t,s)|^p f(s)\, ds \right)^{1/p}, \tag{5.102}$$

$$V(t) = \int_{\mathbb{R}} \psi_0(s) v(t,s)\, ds\,, \tag{5.103}$$

where v is a solution to the Cauchy problem (5.7) with

$$T_2(\varepsilon) \in \Sigma\,, \qquad s_0(\varepsilon) = -T_2(\varepsilon)\,, \qquad \rho(\varepsilon) = \varepsilon\, \mathrm{e}^{T_2(\varepsilon)/2M}\,. \tag{5.104}$$

The function $\psi_0(s)$ can be represented as $\psi_0(s) = \varphi_0(s) - D$, where D is an appropriate constant (see Lemma 5.6.3 below), $\varphi_0(s)$ is the solution to (5.19), obeying the asymptotic properties of Lemma 5.6.3. Equation (5.19) implies further the relation

$$\partial_s^2 \psi_0 - W(s)\psi_0 = DW(s)\,. \tag{5.105}$$

We assume further

$$p \in]2, 1+\sqrt{2}[\,; \tag{5.106}$$

as to the other hypotheses, we do not make any change.

We set $T_0(\varepsilon) = T_2(\varepsilon)/2$ and suppose that $T_1 > 0$ is chosen so that

$$T_0(\varepsilon) + T_1$$

is the lifespan of the solution to (5.7), i.e. for any $T < T_1 + T_0(\varepsilon)$ there exists a solution

$$v \in \bigcap_{k=0}^{2} \mathscr{C}^k\big([0, T[; \mathrm{H}^{2-k}(\mathbb{R})\big)$$

of (5.7), (5.8).

Let notice that in this case, (5.12) and (5.104) imply

$$||v_0||_{\mathrm{H}^2(\mathbb{R})} + ||v_1||_{\mathrm{H}^1(\mathbb{R})} \sim \rho(\varepsilon) \sim \varepsilon\, \mathrm{e}^{T_2(\varepsilon)/2M}, \tag{5.107}$$

therefore the initial data in (5.8) have large $\mathrm{H}^2 \times \mathrm{H}^1$ norms.

First of all, we observe that the conditions in (5.41) are trivially satisfied. Moreover, to prove estimate (5.42), we proceed exactly as in the previous subsection, with ψ_0 instead of φ_0. In fact, we have the inequality

$$\begin{aligned} |V(t)| &\leqslant \int_{\mathbb{R}} |v(t,s)|\psi_0(s)\, ds \\ &\leqslant \left(\int_{\mathbb{R}} f|v|^p \psi_0\, ds\right)^{1/p} \left(\int_{|s-s_0(\varepsilon)|\leqslant t+R} f^{-1/(p-1)}\psi_0\, ds\right)^{(p-1)/p} \end{aligned}$$

due to the Hölder inequality. Further, from Lemma 5.6.3 and the asymptotic expansions (4.9)–(4.11), we get

$$f(s)^{-1/(p-1)}\psi_0(s) \lesssim \begin{cases} (1+s)^2 & \text{if } s \geqslant 0\,, \\ \mathrm{e}^{(p-2)s/2M(p-1)} & \text{if } s < 0\,. \end{cases} \tag{5.108}$$

Hence, we can use the assumption $p > 2$ and deduce estimate (5.81) with $q = 3(p-1)$.

To derive an estimate similar to (5.86), we use the quantity $F_1(t)$ and the estimate

$$
\begin{aligned}
F_1(t) &\leqslant \mathrm{e}^{-t/2M} \int_{\mathbb{R}} |v| \varphi_1 \, ds \\
&\leqslant U(t) \left(\mathrm{e}^{-\frac{t}{2M}\frac{p}{p-1}} \int_{|s-s_0(\varepsilon)| \leqslant t+R} f^{-\frac{1}{p-1}} {\psi_0}^{-\frac{1}{p-1}} {\varphi_1}^{\frac{p}{p-1}} \, ds \right)^{\frac{p-1}{p}};
\end{aligned} \tag{5.109}
$$

from the estimate

$$
f^{-1/(p-1)}(s) {\psi_0}^{-1/(p-1)}(s) {\varphi_1}^{\frac{p}{p-1}}(s) \lesssim \begin{cases} (1+s)^{\frac{p-2}{p-1}} \, \mathrm{e}^{\frac{ps}{2M(p-1)}} & \text{if } s \geqslant 0\,, \\ \mathrm{e}^{\frac{(p-2)s}{2M(p-1)}} & \text{if } s < 0\,, \end{cases} \tag{5.110}
$$

we get

$$
F_1(t) \leqslant C U(t) (t+R)^{\frac{p-2}{p}} \qquad \text{for } t \geqslant 0\,. \tag{5.111}
$$

To finish the proof of (5.42), it is sufficient to take $a = 4-p,\ q = 3(p-1)$ and to mention that, in this case, Lemma 5.5.1 implies

$$
F_1(t) \geqslant C' \rho(\varepsilon) \, \mathrm{e}^{s_0(\varepsilon)/(2M)} = C' \varepsilon\,.
$$

In order to prove estimate (5.43), we multiply equation (5.7) by ψ_0, integrate on $\mathbb{R}$ and then integrate by parts:

$$
\begin{aligned}
V''(t) &= \int_{\mathbb{R}} v_{tt}(t,s) \psi_0(s) \, ds \\
&= \int (v_{ss} - W(s) v) \, \psi_0 \, ds + \int f |v|^p \psi_0 \, ds \\
&= \int v \left(\psi_0'' - W(s) \psi_0 \right) \, ds + \int f |v|^p \psi_0 \, ds\,;
\end{aligned} \tag{5.112}
$$

using the relation (5.105), we deduce

$$
V''(t) = U(t)^p + D \int v W(s) \, ds\,. \tag{5.113}
$$

From the Hölder's inequality, we get

$$\begin{aligned}\int vW &\geqslant -\int |v|W \\ &= -\int \left(f^{\frac{1}{p}}|v|\psi_0^{\frac{1}{p}}\right)\left(f^{-\frac{1}{p}}W\psi_0^{-\frac{1}{p}}\right) \\ &\geqslant -U(t)\left(\int (f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.\end{aligned}$$

Using Lemma 5.6.3 and the assumptions (4.9), (4.10) and (4.11), we obtain

$$(f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}} \in \mathrm{L}^1(\mathbb{R}) \qquad \text{for } p > 2 \tag{5.114}$$

(see also (5.148) below, where this property is verified in details). Thus, we get

$$\int vW \gtrsim -U(t)\,,$$

and (5.113) implies

$$V''(t) \geqslant U(t)^p - CU(t) \tag{5.115}$$

for a suitable positive constant C.

Now, it remains to prove estimate (5.44) in $[0, T_0(\varepsilon)[$, where we recall that $T_0(\varepsilon) = T_2(\varepsilon)/2$.

To begin, let us observe that if $t \in [0, T_0(\varepsilon)[$, $\operatorname{supp} v \subset \{s < R\}$, since

$$s \leqslant s_0 + T_0(\varepsilon) + R = -T_2(\varepsilon)/2 + R = -T_0(\varepsilon) + R\,,$$

thus we can consider only (4.11) and the asymptotic behavior of ψ_0 for $s \to -\infty$, which yields

$$(f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}} \lesssim \mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}}\,. \tag{5.116}$$

Recalling the argument of the proof of (5.115), we obtain

$$
\begin{aligned}
\int vW &\geqslant -U(t)\left(\int_{|s-s_0(\varepsilon)|\leqslant t+R}(f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\
&\gtrsim -U(t)\left(\int_{|s-s_0(\varepsilon)|\leqslant t+R}\mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}}\,ds\right)^{\frac{p-1}{p}}\\
&\gtrsim -U(t)\,\mathrm{e}^{\frac{s_0+T_0+R}{2M}\frac{p-2}{p}}\gtrsim -U(t)\,\mathrm{e}^{-C_0T_0},
\end{aligned}
$$

where we have chosen

$$C_0 \leqslant \frac{p-2}{2Mp}\,;$$

from (5.113), we finally deduce estimate (5.44).

To finish the proof, we suppose that the lifespan $T(\varepsilon)$ of the solution to (5.7) is greater than $(D_1+2)T_0(\varepsilon)$, where $D_1 > 0$ is the constant from Lemma 5.2.3. Then we apply Lemma 5.2.3 with $T_1 = (D_1+1)T_0(\varepsilon)$ and get

$$(D_1+1)T_0(\varepsilon) \leqslant D_1T_0(\varepsilon)\,.$$

This is an obvious contradiction and shows that

$$T(\varepsilon) \leqslant (D_1+2)T_0(\varepsilon)\,,$$

which concludes the proof of Theorem 5.1.2

5.5 Estimate of $F_1(t)$

Lemma 5.5.1. *There exists a positive constant C' independent of ε such that*

$$F_1(t) \geqslant C'\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}$$

for all $t \geqslant 0$.

PROOF. We multiply equation (5.7) by $\psi(t,s) \doteq \mathrm{e}^{-t/2M}\,\varphi_1(s)$ and integrate over $\mathbb{R}$ in s and over $[0,\tau]$ in t:

$$\int_0^\tau \int_{\mathbb{R}} (v_{tt} - v_{ss} + W(s)v)\,\psi\,ds\,dt = \int_0^\tau \int_{\mathbb{R}} f|v|^p\psi\,ds\,dt\,.$$

Since

$$v \in \cap_{k=0}^2 \mathscr{C}^k([0,T[;\mathrm{H}^{2-k}(\mathbb{R}))\,,$$

we can apply an integration by parts argument and obtain

$$\begin{aligned}
-\int_0^\tau \int_{\mathbb{R}} v\,(\psi_{tt} - \psi_{ss} + W(s)\psi)\;ds\,dt + \int_0^\tau \int_{\mathbb{R}} f|v|^p\psi\,ds\,dt \\
= \int_{\mathbb{R}} (v_t\psi - v\psi_t)\,ds\Big|_{t=\tau} - \int_{\mathbb{R}} (v_t\psi - v\psi_t)\,ds\Big|_{t=0}\,.
\end{aligned}$$

The right-hand side of this equality can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}} (v_t\psi - v\psi_t)\,ds\Big|_{t=\tau} - \int_{\mathbb{R}} (v_t\psi - v\psi_t)\,ds\Big|_{t=0} \\
&\quad = \int_{\mathbb{R}} (v_t\psi + v\psi_t)\,ds\Big|_{t=\tau} - 2\int_{\mathbb{R}} v\psi_t\,ds\Big|_{t=\tau} \\
&\qquad - \int_{\mathbb{R}} \mathrm{e}^{-t/2M}\left(v_t + \frac{v}{2M}\right)\varphi_1\,ds\Big|_{t=0} \\
&\quad = \frac{d}{d\tau}\int_{\mathbb{R}} v\psi\,ds + \frac{1}{M}\int_{\mathbb{R}} v\psi\,ds - \int_{\mathbb{R}} \left(\frac{v_0}{2M} + v_1\right)\varphi_1\,ds
\end{aligned}$$

due to the property $\psi_t = -\psi/2M$. The relation

$$\psi_{tt} - \psi_{ss} + W(s)\psi = \mathrm{e}^{-t/2M}\left(-\partial_{ss}^2 + W(s) + \frac{1}{4M^2}\right)\varphi_1 = 0$$

implies

$$-\int_0^\tau \int_{\mathbb{R}} v\,(\psi_{tt} - \psi_{ss} + W(s)\psi)\;ds\,dt = 0\,;$$

so, being

$$F_1(t) = \mathrm{e}^{-t/2M}\int_{\mathbb{R}} v(t,s)\varphi_1(s)\,ds = \int_{\mathbb{R}} v(t,s)\psi(t,s)\,ds\,,$$

we arrive at

$$F_1'(\tau) + \frac{1}{M} F_1(\tau) = \int_0^\tau \int_{\mathbb{R}} f|v|^p \psi \, ds \, dt + \int_{\mathbb{R}} \Big(\frac{v_0}{2M} + v_1 \Big) \varphi_1 \, ds \, .$$

The right-hand side of this identity is greater than $\rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)}$ multiplied by a positive constant, since

$$\int v_j(s) \varphi_1(s) \, ds = \rho(\varepsilon) \int_{|s - s_0(\varepsilon)| \leqslant R} \chi_j\big(s - s_0(\varepsilon)\big) \varphi_1(s) \, ds \, , \tag{5.117}$$

with $j = 0, 1$. For $\varepsilon > 0$ small enough, we are in position to apply the asymptotic expansion derived in Lemma 5.6.4 and we find

$$\int v_j(s) \varphi_1(s) \, ds \geqslant C \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \int_{\mathbb{R}} \chi_j\big(s - s_0(\varepsilon)\big) \, ds \geqslant C' \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \, , \tag{5.118}$$

so we get

$$F_1'(\tau) + \frac{1}{M} F_1(\tau) \geqslant C' \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \; .$$

Now, we multiply both sides by $\mathrm{e}^{\tau/M}$ obtaining

$$\frac{d}{d\tau} \big(\mathrm{e}^{\tau/M} F_1(\tau) \big) \geqslant C' \, \mathrm{e}^{\tau/M} \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)}$$

and integrating over $[0, t]$ we deduce

$$\begin{aligned} \mathrm{e}^{t/M} F_1(t) &\gtrsim F_1(0) + M(\mathrm{e}^{t/M} - 1) \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \\ &= \int_{\mathbb{R}} v_0 \varphi_1 \, ds + M(\mathrm{e}^{t/M} - 1) \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \\ &\gtrsim [1 + M(\mathrm{e}^{t/M} - 1)] \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \gtrsim \mathrm{e}^{t/M} \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \, , \end{aligned} \tag{5.119}$$

namely

$$F_1(t) \geqslant C' \rho(\varepsilon)\, \mathrm{e}^{s_0(\varepsilon)/(2M)} \, ,$$

that is the claim. ■

5.6 Estimates for some associated elliptic linear problems

Our first step in this section is to consider the problem

$$(5.120)\qquad \begin{cases} -\varphi''(s) + W(s)\varphi(s) = 0 & s \in \mathbb{R}\,, \\ |\varphi(s) - bs| \lesssim \log(2+s) & \text{for } s \geqslant 0\,, \\ 0 < \varphi(s) \lesssim 1 & \text{for } s < 0\,, \end{cases}$$

where the potential $W(s)$ is assumed to satisfy

$$(5.121)\qquad 0 < W(s) \lesssim (1+|s|)^{-a}$$

for some $a \geqslant 3$. Note that condition (5.121) is weaker than the assumptions (4.9), (4.10) and (4.11).

Our first result is the following.

Lemma 5.6.1. *There exists a real number $b > 0$ such that problem* (5.120) *has a nonnegative solution $\varphi_0 \in \mathscr{C}^2(\mathbb{R})$ so that the limit*

$$D = \lim_{s \to -\infty} \varphi_0(s)$$

exists, $D \geqslant 0$ and the following relation

$$(5.122)\qquad 0 \leqslant \varphi_0(s) - D \lesssim |s|^{2-a} \qquad \text{for } s \to -\infty$$

holds.

PROOF. Consider the Cauchy problem

$$(5.123)\qquad \begin{cases} -y''(s) + W(s)y(s) = 0 & s \in \mathbb{R}\,, \\ y(0) = 1\,, \quad y'(0) = 0\,. \end{cases}$$

This Cauchy problem has a unique solution $y(s) \in \mathscr{C}^2(\mathbb{R})$. A qualitative analysis of the equation and the assumption $W(s) > 0$ show that the solution satisfies

$$y'(s) > 0 \qquad \text{for } s > 0\,,$$

$$y'(s) < 0 \qquad \text{for } s < 0\,,$$

and hence

$$y(s) \geqslant 1$$

for all real s.

One can rewrite problem (5.123) in the form of the following integral equation

$$y(s) = 1 + I(y)(s)\,, \tag{5.124}$$

where

$$I(y)(s) = \int_0^s \int_0^\sigma W(\vartheta)y(\vartheta)\,d\vartheta\,d\sigma = \int_0^s (s-\vartheta)W(\vartheta)y(\vartheta)\,d\vartheta\,. \tag{5.125}$$

We shall show that

$$|y(s) - d_+ s| \lesssim \log(2+s) \qquad \text{for } s \geqslant 0\,, \tag{5.126}$$

$$|y(s) - d_- s| \lesssim \log(2+|s|) \qquad \text{for } s < 0\,, \tag{5.127}$$

where

$$d_\pm = \int_0^{\pm\infty} W(\vartheta)y(\vartheta)\,d\vartheta\,.$$

Assumption (5.121) shows that the integral operator $I(y)(s)$ is well-defined (in particular $d_\pm \in \mathbb{R}$) and satisfies the estimate

$$0 \leqslant I(y)(s) \leqslant s \int_0^s W(\vartheta)y(\vartheta)\,d\vartheta\,; \tag{5.128}$$

hence (5.124) implies the inequality

$$y(s) \leqslant 1 + s \int_0^s W(\vartheta)y(\vartheta)\,d\vartheta\,. \tag{5.129}$$

Now, let us consider the case $s \geqslant 0$. The previous inequality yields

$$y(s) \leqslant 1 + s \int_0^\infty W(\vartheta)y(\vartheta)\,d\vartheta = 1 + d_+ s\,. \tag{5.130}$$

On the other hand, combining (5.124) and (5.130), we get

$$\begin{aligned} y(s) - 1 - d_+ s &= s\int_0^\infty W(\vartheta)y(\vartheta)\,d\vartheta - s\int_s^\infty W(\vartheta)y(\vartheta)\,d\vartheta \\ &\quad - \int_0^s \vartheta W(\vartheta)y(\vartheta)\,d\vartheta - d_+ s \\ &= -s\int_s^\infty W(\vartheta)y(\vartheta)\,d\vartheta - \int_0^s \vartheta W(\vartheta)y(\vartheta)\,d\vartheta \\ &\gtrsim -s\int_s^\infty \frac{d\vartheta}{(1+\vartheta)^{a-1}} - \int_0^s \frac{d\vartheta}{(1+\vartheta)^{a-2}}\,. \end{aligned}$$

But, for each $a \geqslant 3$, we have also

$$\begin{aligned} \int_s^\infty \frac{d\vartheta}{(1+\vartheta)^{a-1}} &\lesssim (1+s)^{2-a}\,, \\ \int_0^s \frac{d\vartheta}{(1+\vartheta)^{a-2}} &\lesssim \log(2+s)\,, \end{aligned}$$

and thus we deduce

$$y(s) - 1 - d_+ s \gtrsim -\log(2+s)\,.$$

From this equation and (5.130), we finally obtain the precise asymptotic estimate (5.126). Analogously, we get a similar result for $s < 0$. It is important to note that $d_+ > 0$ and

$$d_- = -\int_{-\infty}^0 W(\vartheta)y(\vartheta)\,d\vartheta < 0\,.$$

In a similar way, we can consider the Cauchy problem

$$\begin{cases} -z''(s) + W(s)z(s) = 0 & s \in \mathbb{R}\,, \\ z(0) = 0\,, \quad z'(0) = 1\,. \end{cases} \tag{5.131}$$

Obviously, this Cauchy problem has a unique solution $z(s) \in \mathscr{C}^2(\mathbb{R})$. The assumption $W(s) > 0$ guarantees that the solution satisfies

$$z'(s) > 0 \qquad \forall s \in \mathbb{R}\,,$$

so

$$z(s) > 0 \qquad \text{for } s > 0$$

and

$$z(s) < 0 \qquad \text{for } s < 0\,.$$

Equation (5.124) has to be replaced by

$$z(s) = s + I(z)(s) \tag{5.132}$$

and the argument given in the proof of estimates (5.126) and (5.127) leads to

$$|z(s) - e_+ s| \lesssim \log(2+s) \qquad \text{for } s \geqslant 0\,, \tag{5.133}$$

$$|z(s) - e_- s| \lesssim \log(2+|s|) \qquad \text{for } s < 0\,, \tag{5.134}$$

where

$$e_\pm = 1 + \int_0^{\pm\infty} W(\vartheta) z(\vartheta)\, d\vartheta \in \mathbb{R}\,.$$

Note that $e_+ > 0$ and

$$e_- = 1 - \int_{-\infty}^0 W(\vartheta) z(\vartheta)\, d\vartheta > 0\,.$$

Setting

$$\varphi(s) = e_- y(s) - d_- z(s)\,, \qquad b = e_- d_+ - d_- e_+ > 0\,,$$

we take advantage of (5.127) and (5.134), and conclude that

$$|\varphi(s)| \lesssim \log(2+|s|) \qquad \text{for } s < 0\,, \tag{5.135}$$

while from (5.126) and (5.133), we deduce

$$|\varphi(s) - bs| \lesssim \log(2+s) \qquad \text{for } s > 0\,. \tag{5.136}$$

To improve estimate (5.135), we note that $\varphi(s)$ satisfies the integral equation

$$\varphi(s) = \varphi(0) + \varphi'(0)s + I(\varphi)(s)\,. \tag{5.137}$$

As before, we have (for any $s < 0$)

$$\begin{aligned}\varphi(s) - \varphi(0) - \varphi'(0)s = -s\int_{-\infty}^{0} W(\vartheta)\varphi(\vartheta)\,d\vartheta + s\int_{-\infty}^{s} W(\vartheta)\varphi(\vartheta)\,d\vartheta \\ + \int_{-\infty}^{0} \vartheta W(\vartheta)\varphi(\vartheta)\,d\vartheta - \int_{-\infty}^{s} \vartheta W(\vartheta)\varphi(\vartheta)\,d\vartheta\end{aligned} \tag{5.138}$$

and then a combination between (5.135) and assumption (5.121) implies

$$s\int_{-\infty}^{s} W(\vartheta)|\varphi(\vartheta)|\,d\vartheta \lesssim (1+|s|)^{2-a}\log(2+|s|) \lesssim 1\,, \tag{5.139}$$

$$\int_{-\infty}^{s} \vartheta W(\vartheta)|\varphi(\vartheta)|\,d\vartheta \lesssim (1+|s|)^{2-a}\log(2+|s|) \lesssim 1\,, \tag{5.140}$$

so (5.138) yields

$$\left|\varphi(s) - \left(\varphi'(0) - \int_{-\infty}^{0} W(\vartheta)\varphi(\vartheta)\,d\vartheta\right)s\right| \lesssim 1\,.$$

Comparing this estimate with (5.135), we see that

$$\varphi'(0) - \int_{-\infty}^{0} W(\vartheta)\varphi(\vartheta)\,d\vartheta = 0\,;$$

this implies

$$|\varphi(s)| \lesssim 1 \qquad \text{for } s < 0\,. \tag{5.141}$$

We set

$$D = \varphi(0) + \int_{-\infty}^{0} \vartheta W(\vartheta)\varphi(\vartheta)\,d\vartheta \tag{5.142}$$

and observe that the function $\varphi(s)$ is positive near $s = 0$. Moreover, for $s > 0$, $\varphi(s)$ increases and is positive. It is easy to show that $\varphi(s) \geqslant 0$ for all $s < 0$. Indeed, if $\varphi(s_0) < 0$ for some $s_0 < 0$, then $\varphi(s_1) < 0$, $\varphi'(s_1) > 0$ for some $s_1 < 0$, thus the equation

$$\varphi''(s) = W(s)\varphi(s)$$

would imply that

$$\varphi(s) < 0\,, \qquad \varphi'(s) > 0\,, \qquad \varphi''(s) < 0 \qquad \text{for } s < s_1\,.$$

This contradicts (5.141) and shows that $\varphi(s) \geqslant 0$ for all $s < 0$. Hence D is non-negative, because $\varphi(s) \geqslant 0$ for all $s \in \mathbb{R}$ and

$$\lim_{s \to -\infty} (\varphi(s) - D) = 0\,.$$

We have indeed $\varphi \geqslant D$, since

$$\varphi(s) - D = \int_{-\infty}^{s} (s - \vartheta) W(\vartheta)\varphi(\vartheta)\, d\vartheta \geqslant 0\,; \tag{5.143}$$

so $\varphi(s) \geqslant D \geqslant 0$.

From these relations and (5.143), we also get

$$\varphi(s) - D \lesssim \int_{-\infty}^{s} \frac{s - \vartheta}{(1 + |\vartheta|)^a}\, d\vartheta \sim |s|^{2-a} \qquad \text{for } s \to -\infty \tag{5.144}$$

and this proves (5.122). ■

If we make a stronger assumption on the potential for negative values of s, then we can prove the positiveness of D.

Lemma 5.6.2. *If $W(s)$ satisfies estimate* (5.121) *and*

$$0 < W(s) \lesssim (1 + |s|)^{-4} \qquad \textit{for } s < 0\,, \tag{5.145}$$

then there exists a real number $b > 0$ such that problem (5.120) *has a positive* *solution* $\varphi_0 \in \mathscr{C}^2(\mathbb{R})$ *so that the limit*

$$D = \lim_{s \to -\infty} \varphi_0(s)$$

is strictly positive.

PROOF. The previous lemma guarantees that $D \geqslant 0$. Let us suppose that $D = 0$. Using (5.144) together with the assumption $a \geqslant 3$, we can get the better decay estimate

$$|\varphi(s)| + |\varphi'(s)| \lesssim |s|^{-N} \qquad \text{for } s \to -\infty \tag{5.146}$$

for any integer $N \geqslant 1$. Turning back to the equation satisfied by φ, we make the change

$$s \to \sigma = 1/s\,, \qquad \varphi \to \psi = s^{-1}\varphi\,,$$

and see that ψ satisfies the equation

$$\psi''(\sigma) = \sigma^{-4} W \psi\,;$$

moreover, estimate (5.146) guarantees that

$$\psi(0) = \psi'(0) = 0\,.$$

Therefore, our assumption (5.145) that $s^4 W(s)$ is bounded as $s \to -\infty$ implies that $\sigma^{-4} W$ is bounded, so the classical Cauchy uniqueness theorem implies that ψ is identically 0 and this contradiction implies $D > 0$. ■

Lemma 5.6.3. *If W satisfies estimates* (4.9), (4.10) *and* (4.11)*, then there exists a positive function $\varphi_0 \in \mathscr{C}^2(\mathbb{R})$ such that $(-\partial^2_{ss} + W(s))\varphi_0 = 0$ in $\mathbb{R}$ and for some positive constants b and D we have*

$$\begin{cases} |\varphi_0(s) - bs| \lesssim \log(2 + |s|) & \text{for } s \geqslant 0\,, \\ \varphi_0(s) - D \sim \mathrm{e}^{s/2M} & \text{for } s \to -\infty\,. \end{cases}$$

PROOF. Let φ satisfy

$$\varphi''(s) - W(s)\varphi(s) = 0\,.$$

Using the asymptotic estimates (4.9), (4.10) and (4.11), we find for $s < 0$

$$\begin{aligned}\varphi(s) - D &= \int_{-\infty}^{s} (s-\vartheta) W(\vartheta)\varphi(\vartheta)\,d\vartheta \\ &\sim \int_{-\infty}^{s} (s-\vartheta)\,\mathrm{e}^{\vartheta/2M}\,d\vartheta = 4M^2\,\mathrm{e}^{s/(2M)}\end{aligned} \tag{5.147}$$

as $s \to -\infty$, and this leads to

$$\varphi_0(s) - D \sim \mathrm{e}^{s/2M} \qquad \text{for } s \to -\infty\,.$$

The rest of the claim follows directly from the assertion of the previous lemma. ■

Set

$$\psi_0(s) = \varphi_0(s) - D\,,$$

we shall also need the property

$$\Psi(s) = (f\psi_0)^{-\frac{1}{p-1}} W^{\frac{p}{p-1}} \in \mathrm{L}^1(\mathbb{R}) \qquad \text{for } p > 2\,. \tag{5.148}$$

Indeed, Lemma 5.6.3 implies that

$$\psi_0(s) \sim \begin{cases} s & \text{as } s \to +\infty\,, \\ \mathrm{e}^{s/2M} & \text{as } s \to -\infty\,, \end{cases} \tag{5.149}$$

hence we get

$$\Psi(s) \sim \begin{cases} s^{-2(p+1)/(p-1)} & \text{as } s \to +\infty\,, \\ \mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}} & \text{as } s \to -\infty\,, \end{cases} \tag{5.150}$$

and we conclude that (5.148) is verified.

Now we state a corollary of the Levinson theorem (see [21], page 49, Chapter 2, §5.4), which we are going to apply to get the estimate for φ_1.

Proposition 5.6.1. *Consider the equation*

$$y^{(n)} + \sum_{k=1}^{n} \alpha_k(s) y^{(n-k)} = 0\,, \qquad s \in \mathbb{R}^+, \tag{5.151}$$

where $\alpha_k(s) \in \mathscr{C}^\infty(\mathbb{R}^+)$ are complex-valued functions such that

$$\alpha_k(s) = \beta_k + \gamma_k(s)\,, \qquad \int_{\mathbb{R}^+} |\gamma_k(s)|\, ds < \infty\,,$$

and let $q_1, q_2, \dots, q_n$ be the distinct roots of the equation

$$q^n + \sum_{k=1}^{n} \beta_k q^{n-k} = 0\,.$$

Then equation (5.151) has n linearly independent solutions

$$y_j(s)\,, \qquad j = 1, 2, \dots, n\,,$$

having the asymptotic expansion

$$y_j^{(k-1)}(s) = q_j^{k-1}\, \mathrm{e}^{q_j s}[1 + o(1)] \qquad \text{as } s \to \infty\,,$$

where $j, k = 1, 2, \dots, n$.

Lemma 5.6.4. *Given any $A > 0$, the equation*

$$(-\partial^2_{ss} + W(s) + A^2)\varphi(s) = 0\,, \qquad s \in \mathbb{R} \tag{5.152}$$

admits a positive solution $\varphi_1 \in \mathscr{C}^2(\mathbb{R})$ such that $\varphi_1(s) \sim \mathrm{e}^{A s}$ as $|s|$ approaches ∞.

PROOF. Proposition 5.6.1 guarantees that there exists a solution φ_1 of (5.152) such that $\varphi_1(s) \sim \mathrm{e}^{As}$ as $s \to -\infty$. From

$$\varphi_1'' = \big(W(s) + A^2\big)\, \varphi_1\,, \qquad W(s) + A^2 > 0$$

and a qualitative study, we get

$$\varphi_1''(s) > 0 \qquad \text{and thus } \varphi_1(s) > 0\,, \quad \varphi_1'(s) > 0 \tag{5.153}$$

for each $s \in \mathbb{R}$. Now, from Proposition 5.6.1 for $s \to +\infty$, we deduce $\varphi_1(s) \sim \lambda\,\mathrm{e}^{As} + \mu\,\mathrm{e}^{-As}$ for suitable λ, $\mu \in \mathbb{R}$ and $s \to +\infty$. Property (5.153) guarantees that $\lambda > 0$, so it is necessarily $\varphi_1(s) \sim \mathrm{e}^{As}$ for $s \to +\infty$ and the proof is finished. ■

Chapter 6

A Dispersive Estimate for a Wave Equation with Potential

In this chapter, we slightly change the subject of our investigations. Actually, we consider radial solutions to the Cauchy problem for the linear wave equation with a small short-range electromagnetic potential (the "square version" of the massless Dirac equation with a potential) and zero initial data. We prove two a priori estimates that imply, in particular, a dispersive estimate.

However, this problem presents a lot of similarities with the previous ones. We continue to consider a wave equation, even if, in this case, it is linear. We don't work in the Schwarzschild setting, whose metric can be considered as a perturbation of the Minkowski flat metric and that we recast as an effective potential, but we still have a perturbation represented by a potential under suitable conditions. Moreover, we still investigate the large-time behavior of radially symmetric solutions, providing several estimates. The results that we are going to show can be found in (8).

In Section 6.1 we introduce the problem and reformulate it in terms of conformal coordinates, and we present the main results.

Section 6.2 contains the proof of the main results, along with some other estimates and technical lemmas.

6.1 Introduction and main results

In this chapter, we investigate the dispersive properties of the linear wave equation with an electromagnetic potential, that is

$$(6.1)\qquad (\Box_A - B)u = F \qquad (t,x) \in [0,\infty[\times\mathbb{R}^3,$$

where

$$(6.2)\qquad x = (x_1, x_2, x_3)\,, \qquad r = |x|\,,$$

$$(6.3)\qquad \Box_A = \Box - A\cdot\nabla_{t,x}\,,$$

$$(6.4)\qquad \Box = \partial_t^2 - \Delta = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)\,,$$

$$(6.5)\qquad \nabla_{t,x} = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})\,.$$

The fact that the potential $A = A(t,x)$, depending on space and time, is electromagnetic means that A assumes imaginary values. This will play a crucial role in the development of the proof, since electromagnetic potentials are gauge invariant (see what follows).

We assume further that the potential decreases sufficiently rapidly when r approaches infinity; more precisely, we suppose that

$$(6.6)\qquad \sum_{j\in\mathbb{Z}} 2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A||_{L^\infty} \leqslant \delta_A$$

(that is, A is a short-range potential), where $\varepsilon_A > 0$, δ_A is a sufficiently small positive constant independent of r (see Section 6.2) and the sequence $(\varphi_j)_{j\in\mathbb{Z}}$ is a Paley–Littlewood partition of unity, which means that $\varphi_j(r) = \varphi(2^j r)$ and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ is the set of all nonnegative real numbers) is a function so that

(a) $\operatorname{supp}\varphi = \{r \in \mathbb{R} \,:\, 2^{-1} \leqslant r \leqslant 2\}\,;$

(b) $\varphi(r) > 0 \quad \text{for} \quad 2^{-1} < r < 2\,;$

(c) $\sum_{j\in\mathbb{Z}} \varphi(2^j r) = 1 \quad \text{for each} \quad r \in \mathbb{R}^+.$

In other words, $\sum_{j\in\mathbb{Z}} \varphi_j(r) = 1$ for all $r \in \mathbb{R}^+$ and

$$\operatorname{supp} \varphi_j = \{r \in \mathbb{R} \,:\, 2^{-j-1} \leqslant r \leqslant 2^{-j+1}\}. \tag{6.7}$$

Similarly, we assume for $B = B(t,x)$ the smallness hypothesis

$$\sum_{j\in\mathbb{Z}} (2^{-j})^2 \langle 2^{-j}\rangle^{\varepsilon_A} ||\varphi_j B||_{L^\infty} \leqslant \delta_A\,. \tag{6.8}$$

Moreover, we shall restrict ourselves to radial solutions $u = u(t,r)$, with $F = F(t,r)$, $A = (A_0, A_1, A_2, A_3)$, where

$$A_j = A_j(t,r) \in i\mathbb{R} \qquad j = 0,1,2,3\,, \tag{6.9}$$

and $B = B(t,r)$. Because of this assumption, setting

$$\tilde{A} = (\tilde{A}_0, \tilde{A}_1)\,, \qquad \tilde{A}_0 = A_0\,, \qquad \tilde{A}_1 = \frac{A_1x_1 + A_2x_2 + A_3x_3}{r}\,, \tag{6.10}$$

we have

$$A \cdot \nabla_{t,x} = \tilde{A} \cdot \nabla_{t,r}\,, \qquad \nabla_{t,r} = (\partial_t, \partial_r)\,. \tag{6.11}$$

It is well-known that there exists a unique global solution to the Cauchy problem

$$\begin{cases} (\Box_A - B)u = F & (t,x) \in [0,\infty[\times\mathbb{R}^3, \\ u(0,x) = \partial_t u(0,x) = 0 & x \in \mathbb{R}^3\,; \end{cases} \tag{6.12}$$

in particular, this fact holds for the smaller class of radial solutions, that is for the problem

$$\begin{cases} (\Box_A - B)u = F & (t,r) \in [0,\infty[\times\mathbb{R}^+, \\ u(0,r) = \partial_t u(0,r) = 0 & r \in \mathbb{R}^+. \end{cases} \tag{6.13}$$

Let us introduce the change of coordinates

$$\tau_\pm \doteq \frac{t \pm r}{2} \tag{6.14}$$

and the standard notation $\langle s\rangle \doteq \sqrt{1+s^2}$; our main result can be expressed as follows.

Theorem 6.1.1. *Let u be a radial solution to* (6.12)*, i.e. a solution to* (6.13)*, where $A = A(t,r)$ and $B = B(t,r)$ satisfy respectively* (6.6) *and* (6.8) *for some $\delta_A > 0$ and $\varepsilon_A > 0$. Then, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each $\delta_A \in]0,\delta]$, one has*

$$||\,\tau_+ u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}}\,. \tag{6.15}$$

Let us introduce the differential operators

$$\nabla_\pm \doteq \partial_t \pm \partial_r\,. \tag{6.16}$$

The proof of the previous a priori estimate follows easily from the following one.

Lemma 6.1.1. *Under the same conditions of Theorem 6.1.1, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each $\delta_A \in]0,\delta]$, one has*

$$||\,\tau_+ r\nabla_- u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}}\,. \tag{6.17}$$

An immediate consequence of Theorem 6.1.1 is the following dispersive estimate.

Corollary 6.1.1. *Under the same conditions of Theorem 6.1.1, for every $\varepsilon > 0$, there exist two positive constants δ and C (depending on ε) such that for each $\delta_A \in]0,\delta]$, one has*

$$|u(t,r)| \leqslant \frac{C}{t}||\,\tau_+ r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}} \tag{6.18}$$

for every $t > 0$.

The idea to prove the lemma is the following. First of all, the potential term in (6.13) can be thought as part of the forcing term, that is, $(\Box_A - B)u = F$ can be viewed as

$$\Box u = F_1 \doteq F + \tilde{A} \cdot \nabla_{t,r} u + Bu \,. \tag{6.19}$$

Then we can rewrite this equation in terms of $\tau_\pm$ and $\nabla_\pm$ (see Section 6.2), obtaining

$$\nabla_+ \nabla_- v = G \,, \tag{6.20}$$

where

$$v(t,r) \doteq ru(t,r) \qquad \text{and} \qquad G(t,r) \doteq rF_1(t,r) \,. \tag{6.21}$$

This last equation can be easily integrated to obtain a relatively simple explicit representation of $(\nabla_- v)(\tau_+, \tau_-)$ in terms of G.

Another fundamental step consists in taking advantage of the gauge invariance property of the electromagnetic potential A, which means that, set

$$A_\pm \doteq \frac{\tilde{A}_0 \pm \tilde{A}_1}{2} \,, \tag{6.22}$$

we can assume, without loss of generality, that $A_+ \equiv 0$ (see [5], p. 34). This implies that

$$\tilde{A} \cdot \nabla_{t,r} u = A_- \nabla_- u + A_+ \nabla_+ u = A_- \nabla_- u \,, \tag{6.23}$$

and hence

$$F_1 = F + A_- \nabla_- u + Bu \,, \tag{6.24}$$

thus

$$G = rF + rA_- \nabla_- u + rBu = rF + A_- \nabla_- v + \frac{1}{r} A_- v + Bv \,. \tag{6.25}$$

Obviously, one still has

$$\sum_{j\in\mathbb{Z}} 2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A_-||_{L^\infty} \leqslant \delta_A\,. \tag{6.26}$$

These simplifications, combined with the technical Lemma 6.2.1 and the estimate of Lemma 6.2.2, allow us easily to obtain Lemma 6.1.1 and Theorem 6.1.1.

The dispersive properties of evolution equations are very important for their physical meaning and, consequently, they have been deeply studied, though the problem in its generality is still open. The dispersive estimate obtained in Corollary 6.1.1 provides the natural decay rate, that is the same rate one has for the non–perturbed wave equation (see [28, 36]), i.e. a $t^{-(n-1)/2}$ decay in time, where n is the space dimension (in our case, $n = 3$). The generalization to the case of a potential-like perturbation has been considered widely (see [3, 6, 15, 16, 23, 27, 41, 51, 52, 53, 54]), also for the Schrödinger equation (see [25, 26, 34, 43]). Recently, D'Ancona and Fanelli have considered in [17] the case

$$\begin{cases} \partial_t^2 u(t,x) + Hu = 0\,, & (t,x)\in\mathbb{R}\times\mathbb{R}^3, \\ u(0,x) = 0\,, \quad \partial_t u(0,x) = g(x)\,, & \end{cases} \tag{6.27}$$

where

$$H \doteq -(\nabla + iA(x))^2 + B(x)\,, \tag{6.28}$$

$$A:\mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad B:\mathbb{R}^3\longrightarrow\mathbb{R}\,. \tag{6.29}$$

Under suitable condition on A, ∇A and B, in particular

$$|A(x)| \leqslant \frac{C_0}{r\langle r\rangle(1+|\lg r|)^\beta}\,, \qquad \sum_{j=1}^{3}|\partial_j A_j(x)| + |B(x)| \leqslant \frac{C_0}{r^2(1+|\lg r|)^\beta}\,, \tag{6.30}$$

with $C_0 > 0$ sufficiently small, $\beta > 1$ and $r = |x|$, they have obtained the dispersive estimate

$$|u(t,x)| \leqslant \frac{C}{t}\sum_{j\geqslant 0} 2^{2j}||\langle r\rangle w_\beta^{1/2}\varphi_j(\sqrt{H})g||_{L^2}\,, \tag{6.31}$$

where $w_\beta \doteq r(1+|\log r|)^\beta$ and $(\varphi_j)_{j\geqslant 0}$ is a nonhomogeneous Paley–Littlewood partition of unity on $\mathbb{R}^3$.

In this chapter, restricting ourselves to radial solutions, we are able to obtain the result in Corollary 6.1.1, which is optimal from the point of view of the estimate decay rate t^{-1} and improve essentially the assumptions on the potential, assuming the weaker condition (6.6) instead of (6.30) and allowing that it could depend on time.

We recall that the equation that we have considered, i.e. the linear wave equation with an electromagnetic potential, is strictly related to the massless Dirac equation with an electric potential. Let us introduce some notations. First, the Dirac matrices are defined by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where, here and in what follows, i denotes the imaginary unit. The Dirac operator, applied to a vector function $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ (generally called spinor), is

$$D^\star = \sum_{k=0}^{3} \gamma^k \partial_k \,,$$

with

$$\partial_0 = \partial_t \,, \qquad \partial_k = \partial_{x_k} \quad \text{for } k = 1, 2, 3 \,.$$

Note that here we are not assuming the sum over repeated up-down indices.

We consider radial solutions $u : \mathbb{R}^{1+3} \to \mathbb{C}^4$ to the Cauchy problem

$$\begin{cases} (D^\star + V_0(t,r))u = F_0 \,, \\ u(0,r) = 0 \,, \end{cases} \tag{6.32}$$

that is, the Cauchy problem for the relativistic-invariant form of the massless Dirac equation with a radial potential $V_0 = V_0(t,r)$ represented by a 4×4 complex matrix. The solution u to the Cauchy problem (6.32) must be a solution to

$$\begin{cases} i\partial_t u - Du - V(t,r)u = F(t,r) \,, \\ u(0,r) = 0 \end{cases} \tag{6.33}$$

for suitable V, F, where D is another form of the Dirac operator:

$$D = \frac{1}{i} \sum_{k=1}^{3} \alpha^k \partial_k \,,$$

with the Dirac matrices α^k defined as follows:

$$\alpha^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Note that the operator D is elliptic and self-adjoint in L^2. Moreover, the massless Dirac equation can be viewed as the square root of the wave equation in the sense that

$$-(i\partial_t - D)(i\partial_t + D) = \partial_{tt} I_4 + D^2 = (\partial_{tt} - \Delta) I_4 \,,$$

where I_4 is the 4×4 identity matrix.

To prove what affirmed, it is sufficient to multiply on the left both sides of the equation in (6.32) by $i\gamma^0$ and get

$$i \sum_{k=0}^{3} \gamma^0 \gamma^k \partial_k u + i\gamma^0 V_0 u = i\gamma^0 F_0 \,.$$

Note that

$$\gamma^0\gamma^0 = I_4\,, \qquad \gamma^0\gamma^k = \alpha^k \quad \text{for } k = 1, 2, 3\,,$$

which implies

$$(i\partial_t - D - V)u = F\,,$$

having set

$$V = -i\gamma^0 V_0 \qquad \text{and} \qquad F = i\gamma^0 F_0\,.$$

If we apply to both sides the operator $i\partial_t + D + V$, we get that u satisfies also the equation

$$(-\partial_{tt} + \Delta)u - (V^2 + DV)u + \sum_{k=1}^{3} iV\alpha^k\partial_k u = (i\partial_t + D + V)F\,.$$

Note that we have used the definition of D and the property $D^2 = -\Delta I_4$.

In other words, the solution u to the Cauchy problem

$$\begin{cases} i\partial_t u - Du - Vu = F\,, \\ u(0, r) = 0\,, \end{cases} \tag{6.34}$$

is also a solution to the Cauchy problem

$$\begin{cases} (\partial_{tt} - \Delta)u - \displaystyle\sum_{k=1}^{3} A^k\partial_k u - Bu = -(i\partial_t + D + V)F\,, \\ u(0, r) = \partial_t u(0, r) = 0\,, \end{cases}$$

where

$$A^k = iV\alpha^k \quad \text{for } k = 1, 2, 3\,, \qquad B = -V^2 - DV\,.$$

Now we observe that Theorem 6.1.1 can be easily generalized to the case of a system of wave equations of the form

$$(\partial_{tt} - \Delta)u - \sum_{k=1}^{3} A^k(t, r)\partial_k u - B(t, r)u = F(t, r)\,, \tag{6.35}$$

where $u \in \mathbb{C}^N$, and A^k, B are $\mathbb{C}^{N\times N}$ matrices that satisfy the hypotheses of the theorem. In particular, this holds for $N = 4$. In other words, one has the problem to find conditions on V such that, for the problem (6.34), similar estimates to the ones provided in Theorem 6.1.1 and Corollary 6.1.1 hold.

For some further possible applications to the results contained in this chapter, see Section 7.2.

6.2 Some a priori estimates and proof of the main results

First of all, we reformulate our problem taking advantage of the radiality of the solution u to (6.13). Indeed, since $\Delta_{\mathbb{S}^2} u(t,r) = 0$ and $v = ru$, we have

$$\Box u(t,r) = (\partial_t^2 - \Delta_x)u = \left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^2}\right) u(t,r) \tag{6.36}$$

$$= \frac{1}{r}\partial_t^2 v(t,r) - \frac{1}{r}\partial_r^2 v(t,r) \tag{6.37}$$

$$= \frac{1}{r}\nabla_+\nabla_- v(t,r) = \frac{1}{r}\nabla_-\nabla_+ v(t,r)\,. \tag{6.38}$$

Recalling (6.21) and (6.24), we get that the equation in (6.13) is equivalent to

$$\nabla_+\nabla_- v = G\,. \tag{6.39}$$

Let us notice that the support of $u(t,r)$ is contained in the domain $\{(t,r) \in \mathbb{R}^2 : r > 0,\ t > r\}$, therefore we have

$$\operatorname{supp} v(\tau_+,\tau_-) \subseteq \{(\tau_+,\tau_-) \in \mathbb{R}^2 : \tau_- > 0,\ \tau_+ > \tau_-\}\,. \tag{6.40}$$

From this fact, we get

$$\nabla_- v(\tau_+,\tau_-) = \nabla_- v(\tau_-,\tau_-) + \int_{\tau_-}^{\tau_+} G(s,\tau_-)\,ds = \int_{\tau_-}^{\tau_+} G(s,\tau_-)\,ds\,. \tag{6.41}$$

Let us observe that, for each $s \in [\tau_-,\tau_+]$, we have

$$s \leqslant \tau_+\,, \qquad\qquad s - \tau_- \leqslant \tau_+ - \tau_- = r\,, \tag{6.42}$$

hence

$$\left|\int_{\tau_-}^{\tau_+} G(s,\tau_-)\,ds\right| \leqslant \int_{\tau_-}^{\tau_+} \frac{s\langle s-\tau_-\rangle^{\varepsilon}|G(s,\tau_-)|}{\langle s\rangle\langle s-\tau_-\rangle^{\varepsilon}}\,ds$$
$$\leqslant ||\,\tau_+\langle r\rangle^{\varepsilon} G||_{\mathrm{L}^\infty_{t,r}} \int_{\tau_-}^{\tau_+} \langle s\rangle^{-1}\langle s-\tau_-\rangle^{-\varepsilon}\,ds$$

for every $\varepsilon > 0$. Applying Lemma 6.2.1 (see the end of this section), we conclude

$$\tau_+\,|\nabla_- v(\tau_+,\tau_-)| \leqslant Cr||\,\tau_+\langle r\rangle^{\varepsilon} G||_{\mathrm{L}^\infty_{t,r}}\,; \tag{6.43}$$

recalling that G satisfies (6.25), we obtain

$$\begin{aligned}\tau_+\,|\nabla_- v(\tau_+,\tau_-)| \leqslant Cr\Big(&||\,\tau_+\langle r\rangle^{\varepsilon} A_-\nabla_- v||_{\mathrm{L}^\infty_{t,r}} + ||\,\tau_+\langle r\rangle^{\varepsilon} r^{-1} A_- v||_{\mathrm{L}^\infty_{t,r}}\\ &+ ||\,\tau_+\langle r\rangle^{\varepsilon} Bv||_{\mathrm{L}^\infty_{t,r}} + ||\,\tau_+\langle r\rangle^{\varepsilon} rF||_{\mathrm{L}^\infty_{t,r}}\Big).\end{aligned} \tag{6.44}$$

Now, if we choose for the moment $\varepsilon \leqslant \varepsilon_A$, we have

$$r\langle r\rangle^{\varepsilon}\varphi_j(r)|A_-(t,r)| \leqslant C2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A_-||_{\mathrm{L}^\infty_{t,r}} \tag{6.45}$$

(here and in the following, we assume that $C = C(\varepsilon) > 0$ could change time by time), thus

$$\begin{aligned} r||\,\tau_+\langle r\rangle^{\varepsilon} A_-\nabla_- v||_{\mathrm{L}^\infty_{t,r}} &\leqslant C||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}} \sum_{j\in\mathbb{Z}} 2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A_-||_{\mathrm{L}^\infty_{t,r}}\\ &\leqslant C\delta_A||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}}\,,\end{aligned} \tag{6.46}$$

where we have used the fact that $(\varphi_j)_{j\in\mathbb{Z}}$ is a Paley–Littlewood partition of unity and property (6.26).

Moreover, $v(\tau_+,\tau_+) = 0$ because of (6.40), whence

$$v(\tau_+,\tau_-) = -\int_{\tau_-}^{\tau_+} \nabla_- v(\tau_+,s)\,ds \tag{6.47}$$

and, consequently,

$$|v(\tau_+,\tau_-)| \leqslant \int_{\tau_-}^{\tau_+} |\nabla_- v(\tau_+,s)|\,ds \leqslant r||\nabla_- v||_{\mathrm{L}^\infty_{t,r}}\,. \tag{6.48}$$

Thus we have

$$\langle r\rangle^\varepsilon \varphi_j(r)|A_-(t,r)v(\tau_+,\tau_-)| \leqslant C2^{-j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j A_-||_{\mathrm{L}^\infty_{t,r}}||\nabla_- v||_{\mathrm{L}^\infty_{t,r}}\,, \tag{6.49}$$

which implies

$$r||\,\tau_+\langle r\rangle^\varepsilon r^{-1}A_- v||_{\mathrm{L}^\infty_{t,r}} \leqslant C\delta_A||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}}\,. \tag{6.50}$$

Similarly, from (6.48) and (6.8), we get

$$\begin{aligned} r||\,\tau_+\langle r\rangle^\varepsilon Bv||_{\mathrm{L}^\infty_{t,r}} &\leqslant C||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}} \sum_{j\in\mathbb{Z}} 2^{-2j}\langle 2^{-j}\rangle^{\varepsilon_A}||\varphi_j B||_{\mathrm{L}^\infty_{t,r}} \\ &\leqslant C\delta_A||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}}\,. \end{aligned} \tag{6.51}$$

Combining this estimate with (6.46) and (6.50) in (6.44), we deduce

$$||\,\tau_+\,\nabla_- v||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\,\tau_+\,r^2\langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}}\,, \tag{6.52}$$

provided δ_A is sufficiently small. For instance, one can take δ_A such that

$$4C^2\delta_A \leqslant 1\,.$$

From the definition of v, we have

$$r\nabla_- u = \nabla_- v + u \tag{6.53}$$

and, hence,

$$|\,\tau_+\,r\nabla_- u| \leqslant |\,\tau_+\,\nabla_- v| + |\,\tau_+\,u|\,. \tag{6.54}$$

Now, thanks to the inequality in Lemma 6.2.2, we have

$$
\begin{aligned}
|\,\tau_+ u| &\leqslant \tau_+ r^2 \langle r\rangle^\varepsilon |F_1| \\
&\leqslant \tau_+ r^2 \langle r\rangle^\varepsilon |A_- \nabla_- u| + \tau_+ r^2 \langle r\rangle^\varepsilon |Bu| + \tau_+ r^2 \langle r\rangle^\varepsilon |F| \\
&\leqslant \left(\sum_{j\in\mathbb{Z}} r\langle r\rangle^{\varepsilon_A} \varphi_j |A_-| \right) |\,\tau_+ r\nabla_- u| + \left(\sum_{j\in\mathbb{Z}} r^2\langle r\rangle^{\varepsilon_A} \varphi_j |B| \right) |\,\tau_+ u| \\
&\qquad + \tau_+ r^2 \langle r\rangle^\varepsilon |F| \\
&\leqslant C\delta_A (|\,\tau_+ r\nabla_- u| + |\,\tau_+ u|) + \tau_+ r^2 \langle r\rangle^\varepsilon |F| \,,
\end{aligned}
\tag{6.55}
$$

and thus

$$
|\,\tau_+ u| \leqslant C(\delta_A |\,\tau_+ r\nabla_- u| + |\,\tau_+ r^2 \langle r\rangle^\varepsilon F|) \,. \tag{6.56}
$$

Combining this result with (6.52) in (6.54), we conclude

$$
||\,\tau_+ r\nabla_- u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2 \langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}} \,, \tag{6.57}
$$

provided $\delta_A > 0$ small enough, that is, Lemma 6.1.1.

Now we use the fact that, because of (6.53), we have

$$
|\,\tau_+ u| \leqslant |\,\tau_+ r\nabla_- u| + |\,\tau_+ \nabla_- v| \,; \tag{6.58}
$$

combining this estimate with (6.17) and (6.52), we finally conclude

$$
||\,\tau_+ u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2 \langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}} \,, \tag{6.59}
$$

and also Theorem 6.1.1 is proven.

Now we are going to prove the two lemmas that we have used previously in this section.

Lemma 6.2.1. *For each $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon)$ such that*

$$
\int_{\tau_-}^{\tau_+} \langle s\rangle^{-1} \langle s - \tau_-\rangle^{-\varepsilon}\, ds \leqslant \frac{Cr}{\tau_+} \,.
$$

PROOF. We distinguish two cases.

Case 1: $\tau_+ \geqslant 2\,\tau_-$. Let us notice that, since $r = \tau_+ - \tau_- \geqslant \tau_+/2$, in this case it is sufficient to prove that

$$\int_{\tau_-}^{\tau_+} \langle s\rangle^{-1}\langle s-\tau_-\rangle^{-\varepsilon}\,ds \leqslant C_0(\varepsilon)\,. \tag{6.60}$$

We observe that $s - \tau_- \geqslant s/2$ provided $s \geqslant 2\,\tau_-$, so

$$\begin{aligned}\int_{\tau_-}^{\tau_+} \langle s\rangle^{-1}\langle s-\tau_-\rangle^{-\varepsilon}\,ds &\leqslant \int_{\tau_-}^{\tau_-+1} \langle s\rangle^{-1}\,ds + 2^{\varepsilon}\int_{\tau_-+1}^{\tau_++1} s^{-(1+\varepsilon)}\,ds\\ &\leqslant \frac{1}{\langle \tau_-\rangle} + 2^{\varepsilon}\int_1^{\infty} s^{-(1+\varepsilon)}\,ds\\ &\leqslant 1 + C_1(\varepsilon)\,.\end{aligned}$$

Case 2: $\tau_+ < 2\,\tau_-$. We use the estimates $\langle s\rangle^{-1} < 2/\tau_+$ and $\langle s-\tau_-\rangle^{-\varepsilon} \leqslant 1$ to get

$$\int_{\tau_-}^{\tau_+} \langle s\rangle^{-1}\langle s-\tau_-\rangle^{-\varepsilon}\,ds \leqslant \frac{2}{\tau_+}(\tau_+ - \tau_-) = \frac{2r}{\tau_+}\,. \tag{6.61}$$

This concludes the proof. ∎

In the case $A \equiv B \equiv 0$ (non-perturbed equation), we have the following version of the estimate in Theorem 6.1.1. It consists in a slight modification of estimate (1.8) shown in [23], p. 2269.

Lemma 6.2.2. *Let u be the solution to*

$$\begin{cases}\Box u = F & (t,r) \in [0,\infty[\times\mathbb{R}^+,\\ u(0,r) = \partial_t u(0,r) = 0 & r \in \mathbb{R}^+.\end{cases} \tag{6.62}$$

Then, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$||\,\tau_+\,u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\,\tau_+\,r^2\langle r\rangle^{\varepsilon}F||_{\mathrm{L}^\infty_{t,r}}\,. \tag{6.63}$$

PROOF. Let us notice that u is the solution to (6.13) with $A \equiv B \equiv 0$. Then, from (6.44), we have

$$\tau_+ \, |\nabla_- v(\tau_+, \tau_-)| \leqslant C|| \, \tau_+ \, r^2 \langle r \rangle^{\varepsilon} F||_{\mathrm{L}^\infty_{t,r}} \, , \tag{6.64}$$

where $v = ru$. Using (6.48), we deduce

$$\tau_+ \, |u| = \tau_+ \, |v| r^{-1} \leqslant || \, \tau_+ \, \nabla_- v||_{\mathrm{L}^\infty_{t,r}} \tag{6.65}$$

and hence the claim. ■

Chapter 7

Open Problems

In this chapter, we present some open problems related to the original results described in this thesis. Most of them are indeed analogues of the problems considered for the semilinear wave equation in the Minkowski metric presented in Section 1.1 and in Section 1.4.

Section 7.1 deals with open problems related to the semilinear wave equation in the Schwarzschild metric with exponent $p > 1$. Summarizing, the following topics are covered: case $p > 1 + \sqrt{2}$, global existence and decay estimate, case $p < 1 + \sqrt{2}$ for small initial data, blow-up and lifespan, critical exponent, nonradial solutions.

Section 7.2 illustrates some problems related to the wave equation with electromagnetic potential.

7.1 The Schwarzschild Metric

One first important open problem is the study of the Cauchy problem for the semilinear wave equation with small initial data in the Schwarzschild metric for $p > 1 + \sqrt{2}$, that is the problem

$$\begin{cases} \Box_g u = |u|^p & \text{in } [0, \infty[\times \mathbf{\Omega}\,, \\ u(0,x) = \varepsilon u_0(x)\,, \quad \partial_t u(0,x) = \varepsilon u_1(x) & \text{in } \mathbf{\Omega} \end{cases}$$

under suitable hypotheses. One possible conjecture is that a similar behavior to the one manifested for the Minkowski metric is reproduced, that is the existence of a critical exponent $\bar{p}$ such that the solution blows up in finite time if $1 < p < \bar{p}$, while it exists globally if $p > \bar{p}$. Moreover, according to what happens in the flat case, one should also expect a blow-up phenomenon for the critical case $p = \bar{p}$.

Note that our blow-up result for such a problem is compatible with the depicted situation and, in addition to this, it suggests the value $\bar{p} = 1 + \sqrt{2}$ for the critical exponent, i.e. the same value that one has in the Minkowski metric.

Actually, it is not clear whether the solution exists globally in time or it blows up when $p > 1 + \sqrt{2}$, and if this behavior depends essentially on the position of the (compactly supported) initial data, that is the distance of their support from the horizon event $r = 2M$. Indeed, this aspect seems to constitute a main difference with respect to the flat case. We recall that we got a blow-up result for small data supported far away from the black hole or large data next to the black hole, and this latter case resulted to be nontrivial, despite the largeness of the data.

One could try a similar approach to the one described in Chapter 4, that is, the reduction of the problem to a one-dimensional semilinear wave equation with effective potential, at least in the radial case, and try to reproduce the proof used in the Minkowski metric (see Section 1.1), showing that suitable a priori estimates hold; however, because of the sign-changing property of the solution and the different decay properties of the effective potential $W(s)$ as $s \to \infty$ or $s \to -\infty$, it is not easy to obtain such a priori estimates.

In particular, while $W(s)$ decays esponentially for $s \to -\infty$, it decays only as s^{-3} for $s \to \infty$. For instance, this prevents one from applying the method described in [50], where, if $W(s)$ decays esponentially, a resolvent estimate for the associated elliptic problem is proved (see Chapter IX in [50]). This result can be used to prove a local energy decay estimate.

Similarly, because of the structure of the Schwarzschild metric, and in particular because it is asymptotically flat, the method of Sá Barreto and Zworski used in [45] to prove the meromorphic continuation of the resolvent for the De Sitter–Schwarzschild metric cannot be easily adapted. Actually, the De Sitter–Schwarzschild metric has the same form as in the exact Schwarzschild case, that is

$$\tilde{g} = \tilde{F}(r)\, dt^2 - \tilde{F}(r)^{-1}\, dr^2 - r^2\, d\omega^2,$$

where $d\omega^2$ is the standard metric on $\mathbb{S}^2$, but here we have

$$\tilde{F}(r) = \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)^{\frac{1}{2}},$$

where $m > 0$ is the mass of the black hole and the cosmological constant Λ satisfies the relation $0 < 9m^2\Lambda < 1$, while the manifold has the form

$$\tilde{\boldsymbol{\Omega}} =]r_+, r_{++}[\times \mathbb{S}^2 ,$$

where r_+ and r_{++} are the two positive roots of $\tilde{F}(r) = 0$. Note that the term Λr^2 in the definition of $\tilde{F}$ is the one that makes this metric well-behaved. A result similar to the one provided in [45] could be useful to prove a resolvent estimate and hence a local energy decay estimate.

As mentioned before, for $1 < p < 1 + \sqrt{2}$ we have no result about the behavior of the solution to the problem above when the data are small and supported close to the black hole: this is another open problem. Moreover, the case $p = 1 + \sqrt{2}$ is completely uninvestigated, both for large and small data, both for data far from and next to the horizon event.

Indeed, because of the "erratic" behavior of the solution, one could have a different critical exponent with respect to the flat case, or even several critical exponents depending on the position of the (small) initial data. Eventually, also in the case of a unique critical exponent, for instance $p = 1 + \sqrt{2}$ as in the flat case, one should verify if the expected behavior is manifested, i.e. if the solution blows up in finite time. To conclude, this problem is widely open.

When the solution exists globally in time, one can study its behavior at infinity, i.e. one can find a priori estimates, especially dispersive and decay estimates. In Section 1.4, we have shown some results concerning the nonlinear wave equation with effective potential in the Minkowski metric. One can wonder whether similar results hold also in the Schwarzschild metric or not; in particular, one could establish the decay rate of the solution. This matter is largely open.

On the other hand, when the solution blows up in finite time, one is interested in a precise expression for the lifespan $T(\varepsilon)$, where ε measures the smallness of the initial data. We have presented this problem for the semilinear wave equation in the Minkowski metric in Section 1.4. Coming back to the original results presented in this thesis, we have two estimates for the semilinear wave equation in the Schwarzschild metric.

In the first case, when $1 < p < 1 + \sqrt{2}$ and the initial data are small far from the black hole, that is they are supported in $|s - s_0(\varepsilon)| \leqslant R$, with $s_0(\varepsilon) = \varepsilon^{-\vartheta}$, $\vartheta \geqslant 0$ opportunely defined, we have shown that

$$T(\varepsilon) \leqslant C\varepsilon^{-\vartheta}$$

(see Section 5.3 for the details and in particular page 83 for this estimate).

In the second case, when $2 < p < 1 + \sqrt{2}$ and the initial data are large next to the black hole, that is they are supported in $|s - s_0(\varepsilon)| \leqslant R$, with $s_0(\varepsilon) = -\varepsilon^{1/\vartheta}$ (or similar expressions), $\vartheta \geqslant 0$ opportunely defined, we have shown that

$$T(\varepsilon) \leqslant C\varepsilon^{1/\vartheta}$$

(see Section 5.4 for the details and in particular page 89 for this estimate).

These results are different from the ones known for the flat case, recalled in Section 1.4, where $T(\varepsilon) \sim \varepsilon^{-p(p-1)}$, in particular in the second case. However, we lack an estimate from below for $T(\varepsilon)$; in other words, we do not know whether our estimates are optimal.

We have no results in the other cases. Note that, in general, the estimate from below for the lifespan is related to the proof of results in the critical and in the supercritical case.

Eventually, we recall that we have considered only radial solutions. In the flat metric, this restriction seems of little importance, since the behavior of the class of all solutions is the same as the one of the smaller class of radial solutions. However, this could not be the case in the Schwarzschild metric.

7.2 The Wave Equation with Potential

In Section 6.1, we have already noticed how the Cauchy problem for a wave equation with potential of the form

$$\begin{cases} (\Box_A - B)u = F & (t,r) \in [0,\infty[\times\mathbb{R}^+, \\ u(0,r) = \partial_t u(0,r) = 0 & r \in \mathbb{R}^+, \end{cases} \tag{7.1}$$

where $\Box_A = \Box - A \cdot \nabla_{t,x}$, is related to the Cauchy problem for a massless Dirac equation

$$\begin{cases} (D^\star + V_0)u = F_0\,, \\ u(0,r) = 0\,, \end{cases} \tag{7.2}$$

where $D^\star$ is the relativistic invariant form of the Dirac operator; this last problem can be recast in the form

$$\begin{cases} i\partial_t u - Du - Vu = F\,, \\ u(0,r) = 0\,, \end{cases} \tag{7.3}$$

where D is the Dirac operator (this formulation is not relativistic invariant). For the details, see Section 6.1, page 109 and following ones. One can wonder whether a similar result to the one provided by Theorem 6.1.1 holds or not, that is, if we have the a priori estimate

$$||\, \tau_+ \, r\nabla u||_{\mathrm{L}^\infty_{t,r}} \leqslant C||\tau_+ r^2 \langle r\rangle^\varepsilon F||_{\mathrm{L}^\infty_{t,r}} \tag{7.4}$$

or a similar one for the Cauchy problem (7.3).

Another possible application of Theorem 6.1.1 is related to another open problem: the large-time behavior of a semilinear wave equation with electromagnetic potential. In the same setting of Theorem 6.1.1, we can consider the Cauchy problem for the nonlinear equation

$$(\Box_A - B)u = |u|^p, \qquad (t,x) \in [0,\infty[\times\mathbb{R}^3\,, \tag{7.5}$$

where $p > 1$.

Now, we can consider the same questions that we have for the semilinear wave equation (in the Minkowski or in the Schwarzschild metric) $\Box u = |u|^p$, questions that we have described in Section 7.1. Summarizing, under suitable hypotheses on the initial data, we are interested in knowing the values of p such that the solution blows up in finite time and the values such that the solution exists globally; in the first case we are also interested in dispersive estimates, while in the second one we would like to know the lifespan of the solution. Finally, all these problems can be considered both in the radial and in the general case, and even for the Dirac equation

$$i\partial_t u - Du - Vu = |u|^p. \tag{7.6}$$

Note that Theorem 6.1.1 could be useful in the proof of a global existence result.

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