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On Racah-Wigner calculus for classical Lie groups via Schur-Weyl duality

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Thesis

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To D. and my family

Gratitude - is not the mention
Of a Tenderness,
But it's still appreciation
Out of Plumb of Speech -

When the Sea return no Answer
By the Line and Lead
Proves it there's no Sea, or rather
A remoter Bed?

Emily Dickinson

*In 1951, I had the good fortune of listening to Professor Racah's lecture on Lie groups at Princeton. After attending these lectures, I thought, "This is really too hard. I cannot learn all this ...
All this is too damned hard and unphysical!"*

Abdus Salam

Seminars on Theoretical Physics - Trieste (Italy), 1962.

Contents

Introduction

Representation theory of classical Lie groups (unitary, orthogonal and symplectic groups) plays a fundamental role in many areas of physics and chemistry. Orthogonal and symplectic group representation theory arises, for example, in the description of symmetrized orbitals in quantum chemistry and in fermion and boson many-body theory [1], grand unification theories for elementary particles [2], supergravity [3], interacting boson and fermion dynamical symmetry models for nuclei [4, 5], nuclear symplectic models [6, 7], and so on.

In particular, the importance of *coupling* and *recoupling* coefficients (Clebsch-Gordan coefficients, $6j$-symbols, Racah coefficients, $9j$-symbols and general $3nj$-symbols) is evident in the study of angular momentum theory which is built on the well known representation theory of $SU(2)$. Furthermore, Racah-Wigner calculus of $SU(2)$ group has notable applications in the theory of orthogonal polynomials and other special functions.

The Racah coefficients and other recoupling coefficients of unitary $SU(n)$, orthogonal $SO(n)$ and symplectic $Sp(2m)$ groups of different rank are quite useful when calculating energy levels and transition rates in atomic, molecular and nuclear theory (for example, in connection with the Jahn-Teller effect and structural analysis of atomic shells, see Judd and co-workers [8, 9] and, for a description of multi-bosonic and multi-fermionic systems and applications in the microscopic nuclear theory, consider [10, 11]), and in conformal field theory [12].

There are many approaches to the Racah coefficients, but the problem is that there are no general methods for treating various kind of coupling and recoupling issues. Any given technique applies only to a particular problem and for a particular group. Not only do the tecniques for dealing with unitary, orthogonal and symplectic groups all drastically differ from each other, but the methods for finding the various Wigner coefficients, which we are interested in, also vary from one to the other. Furthermore, analytical expressions are difficult to come by for general Lie groups, mainly because there is a *multiplicity problem* in the reduction of Kronecker products of pair of irreducible representations. Some missing labels need to be added in, for which a procedure is often difficult to do systematically. Finally, although several efficient computer codes and numerical procedures exist, they often do not permit any insight in the mathematical structure of such coefficients and, however, a general and efficient closed algorithm is still needed. Therefore, citing Jin-Quan Chen in his book *Group representation theory for physicists*, "in many cases, these methods are more of an art than a science".

The aim of this thesis is to provide a systematic and comprehensive approach to deal with the structure of coupling coefficients for classical Lie groups. The most promising strategy, from this point of view, seems to be the one building on the well-known and tight connection between symmetric and unitary groups which is called in literature *Schur-Weyl duality* and which was first pointed out by Schur at the beginning of the twentieth century [13]. Schur proved that the image of each group under its representation equals the full centralizer algebra and that the two actions are omeomorphic. This observation was developed ten years later by Brauer [14] who found the full centralizer algebra for orthogonal and symplectic groups and completed the construction of the full centralizer algebras for the classical series of the Lie groups.

Weyl used these results, gave numerous theorems concerned with irreducible representations of both classical Lie group and its centralizer algebras, and also gave application to the many-body sistems of f equivalent particles. However, the duality goes further than the one originally expressed by Schur and Weyl. Many powerful equalities between various tranformation factors of the centralizer algebras and those of the corresponding Lie group can be established. This is one of the main aims of the *invariant theory*.

Kramer [15] used explicit transformations between the bases defined in terms of different symmetric group chains (so called *Gelfand-Tzetlin chains*) to define his f symbol (our subduction factor) for a symmetric group. He showed that the f symbols were equivalent to recoupling coefficients ($6j$ and $9j$ symbols) for any unitary group and further that f symbols were also equal to coupling coefficients for $U(p+q) \supset U(p) \times U(q)$. Later [16] such results were generalized to Brauer centralizer algebras and to the corresponding orthosymplectic groups, making the problem of finding coupling and recoupling coefficients for classical Lie groups *equivalent* to the *subduction problem* for centralizer algebras.

Subduction coefficients for symmetric groups were first introduced in 1953 by Elliot *et al* to describe the states of a physical system with n identical particles as composed of two subsystems with n_1 and n_2 particles respectively ($n_1 + n_2 = n$) and then they were soon generalized to Brauer algebras [17]. Since Elliot's work, many techniques have been proposed for calculating the subduction coefficients, but the investigation is until now incomplete. The main goal to give explicit and general closed algebraic formulas has not been achieved. Only some special cases have been solved for symmetric groups. There are also numerical methods which are used to approach the issue, but, again as in the case of recoupling coefficients, no insight into the structure of the trasformation coefficients can be obtained. Another outstanding problem is the resolution of multiplicity separations in a systematic manner, indicating a consistent choice of the indipendent phases and free factors.

In this thesis, we choose an algebraic approach to the subduction problem in symmetric groups $\mathfrak{S}_n \downarrow \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ and in Brauer algebras $\mathfrak{B}_f(x) \downarrow \mathfrak{B}_{f_1}(x) \times \mathfrak{B}_{f_2}(x)$ and we analyze in detail the *linear equation method* [18], an efficient tool for deriving algebraic solutions for fixed values of n_1, n_2 and f_1, f_2 respectively. Thus we give a suitable combinatorial description of the equation system arisen from the method and we provide a new algorithm to solve it. Therefore, by solving the subduction problem for centralizer algebras, we have the solution for a unified approach to the Racah-Wigner calculus for classical Lie groups.

There are at least three possible interesting developments for this thesis:

- Racah-Wigner calculus for *quantized enveloping algebras*.

  Centralizer algebras (i.e. *Birman-Wenzl* and *type A Iwahori-Hecke algebras*) for quantized enveloping algebras are well characterized both from the algebraic and combinatorial point of view and there exists an explicit construction of their irreducible representations [19]. Thus, the linear equation method can be directly applied to this issue without any particular difficulty.

- Racah-Wigner calculus for *projective representations of classical Lie groups.*

  Projective (spinor) representations of a classical Lie group G are very useful in many situations. Finding such representations is equivalent to determining the tensorial irreducible representations of the *universal enveloping group* of G (which is another classical Lie group). An alternative approach is to find the projective representations of Brauer algebras. The Gelfand-Tzetlin basis for such representations is described in terms of combinatorial objects which are known as *stable-up-down tableaux* [20] (they are *permutation lattices with null elements*, in our language). Unfortunately, the explicit action on the irreducible invariant spaces is still unknown.

- Racah-Wigner calculus for *exceptional Lie groups*

  Racah-Wigner calculus for exceptional Lie groups also has many applications both in physics and mathematical physics. The study of the subduction problem for this case would be important for a comprehensive knowledge of Racah-Wigner calculus for all Lie groups. The centralizer algebras for exceptional Lie groups are still unknown.

The layout of the thesis is the following.

In **chapter 1**, we review some basic results in representation theory of classical Lie groups which are necessary to deal with the coupling and recoupling problem. We mainly consider the construction of standard Gelfand-Tzetlin bases and the explicit action of classical Lie algebras on such bases.

The definitions of classical coupling and recoupling coefficients for the groups $SU(2)$ and $SO(3)$ are also given and then generalized to generic classical Lie groups.

In **chapter 2**, we deal with Schur-Weyl duality and its general formulation. Besides classical duality, a description of quantum deformation groups and the corresponding centralizer algebras is given.

After that, we specialize the discussion on the $\mathfrak{B}_f(\epsilon n) - SO(n)/Sp(2n)$ duality and we highlight the relation between subduction and Racah coefficients.

In **chapter 3**, we investigate the linear equation method for symmetric groups, proposed by Chen *et al.* for the determination of the subduction coefficients as solution of a linear system. We prove that such a system, which is constituted by a complicated primal structure of dependent linear equations, can be simplified by choosing a minimal set of sufficient equations related to the concept of *subduction graph.* Furthermore, the subduction graph provides a very practical way to choose such equations and it suggests that subduction coefficients may be seen as a subspace of $\mathbb{R}^{f^\lambda} \otimes \mathbb{R}^{f^{\lambda_1} f^{\lambda_2}}$ (where f^λ, f^{λ_1}, f^{λ_2} are the dimensions of the irreducible representations involved in the subduction $\mathfrak{S}_n \downarrow \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$) obtained by the intersection of only $n-2$ explicit subspaces (each one in corrispondence with an i-layer) instead of the original $(n-2) f^\lambda f^{\lambda_1} f^{\lambda_2}$ ones. Consequently, we have a more explicit insight into the structure of the transformation from the standard basis to the split basis.

Furthermore, we propose a general form for the subduction coefficients resulting from the only requirement of orthonormality and we note that the multiplicity separation can be described in terms of the Sylvester matrix of the positive defined quadratic form τ describing the scalar product in the subduction space. Thus we are able to link the freedom to fix the multiplicity separation to the freedom to choose of the Sylvester matrix. The number of phases and free factors of the general multiplicity separation can be expressed as functions of the multiplicity μ (i.e. the dimension of the subduction space). A crucial question is the possibility to fix the Sylvester matrix to obtain all the requirements of *simplicity* given in [21] for the form of each coefficient. We conjecture that such a form only depends on the form of the eigenvalues and eigenvectors of τ.

(These results are published in [22].)

In **chapter 4**, we consider transformations between split bases and standard bases of the symmetric group $\mathfrak{S}_n$. A selection rule which allows to determine the vanishing subduction coefficients and to organize the other ones in blocks (named *islands*) is given. We prove that all islands produce the same values for the subduction coefficients and thus only a much smaller number of them really needs to be evaluated. Thus, the linear equation method, described in terms of a *reduced* subduction graph, provides a systematic and optimizated tool to calculate the unknown transformation coefficients.

As a significative example, the first multiplicity-three cases, $[4,3,2,1] \downarrow [3,2,1] \otimes [3,1]$ and its conjugate $[4,3,2,1] \downarrow [3,2,1] \otimes [2,1,1]$ for $\mathfrak{S}_{10} \downarrow \mathfrak{S}_6 \times \mathfrak{S}_4$, were dealt in detail: we give the suitable orthonormalized transformation coefficients relative to each multiplicity copy descending from the Yamanouchi phase convention.

(These results are published in [23])

Finally, in **chapter 5**, we describe the subduction problem for Brauer algebras. This problem is clearly a generalization of the corresponding one for symmetric groups because the group algebra $\mathbb{C}\mathfrak{S}_f$ is strictly included in $\mathfrak{B}_f(x)$. After the suitable combinatorial description for the Gelfand-Tzetlin basis by the introduction of *Bratteli diagrams* and *permutation lattices* (which generalize the concept of *standard Young tableau*), we provide the explicit action of Brauer algebra generators on the irreducible invariant spaces. This allows us to write the explicit form of the subduction equations deriving from the linear method.

A description of the solution of such equations is given trhough the concept of a generalized *i-layer*. We find four possible configurations for the *subduction space* and we can provide a definition of subduction graph analogous to that one given in chapter 3.

Finally, as in chapter 3, the form for the orthonormalized subduction coefficients is discussed with a special emphasis on the choice of Young-Yamanouchi phase and the other free factors.

(These results are in preparation to be submitted for publication.)

I wish to thank Massimo Campostrini for introducing me into this interesting research topic and for his valuable support during the progress of this thesis.

Chapter 1

Classical Lie algebras, classical Lie groups and Racah-Wigner calculus

This chapter is devoted to a rapid review of Racah-Wigner calculus for classical Lie groups and the envolved representation theory. In section 1, we deal with Gelfand-Tzetlin bases and explicit actions of Lie groups and algebras on such bases. In section 2, the basic results of Racah-Wigner calculus for $SU(2)$ group are presented and, in section 3, they are generalized to classical Lie groups.

1.1 Bases and operators

The simple Lie algebras over the field of complex numbers were classified in the works of Cartan and Killing in the 1930's. There are four infinite series A_n, B_n, C_n, D_n which are called the *classical Lie algebras*, and five exceptional Lie algebras E_6, E_7, E_8, F_4, G_2. The structure of these Lie algebras is uniformly described in terms of certain finite sets of vectors in a Euclidean space called the *root systems*. Due to Weyl's complete reducibility theorem, the theory of finite-dimensional representations of the semisimple Lie algebras is largely reduced to the study of irreducible representations.

The irreducible representations are parametrized by their *highest weights*. The characters and dimensions are explicitly known by the *Weyl formula*. The reader is refered to, e.g., the books of Bourbaki [24], Dixmier [25], Humphreys [26] or Goodman and Wallach [27] for a detailed exposition of the theory.

However, the Weyl formula for the dimension does not use any explicit construction of the representations. Such constructions remained unknown until 1950 when Gelfand and Tzetlin[1] published two short papers [28] and [29] (in Russian) where they solved the problem for the general linear Lie algebras (type A_n) and the orthogonal Lie algebras (types B_n and D_n), respectively. Baird and Biedenharn employed the calculus of *Young patterns* to derive the Gelfand-Tzetlin formulas. Their interest to the formulas was also motivated by the connection with the fundamental *Wigner coefficients*.

A year earlier (1962) Zhelobenko published an independent work [30] where he derived the *branching rules* for all classical Lie algebras. In his approach the representations are realized in a space of polynomials satisfying the "indicator system" of differential equations. He outlined a method to construct the *lowering operators* and to derive the matrix element formulas for the case of the general linear Lie algebra $\mathfrak{gl}_n$. An explicit "infinitesimal" form for the lowering operators as elements of the enveloping algebra was found by Nagel and Moshinsky [31] (1964) and independently by Hou Pei-yu [32] (1966). The latter work relies on Zhelobenko's results [30] and also contains a derivation of the Gelfand-Tzetlin formulas alternative to that of Baird and Biedenharn. This approach was further developed in the book by Zhelobenko [33] which contains its detailed account.

The work of Nagel and Moshinsky was extended to the orthogonal Lie algebras $\mathfrak{o}_N$ by Pang and Hecht [34] and Wong [35] who produced explicit infinitesimal expressions for the lowering operators and gave a derivation of the formulas of Gelfand and Tzetlin [29].

During the half a century passed since the work of Gelfand and Tsetlin, many different approaches were developed to construct bases of the representations of the classical Lie algebras. New interpretations of the lowering operators and new proofs of the Gelfand-Tzetlin formulas were discovered by several authors. In particular, Gould [36, 37, 38, 39] employed the *characteristic identities* of Bracken and Green [40, 41] to calculate the *Wigner coefficients* and matrix elements of generators of $\mathfrak{gl}_n$ and $\mathfrak{o}_N$. The *extremal projector* discovered by Asherova, Smirnov and Tolstoy [42, 43, 44] turned out to be a powerful

[1]Some authors and translators write this name in English as *Zetlin, Tzetlin, Cetlin,* or *Tseitlin.*

instrument in the representation theory of the simple Lie algebras. It plays an essential role in the theory of *Mickelsson algebras* developed by Zhelobenko which has a wide spectrum of applications from the branching rules and reduction problems to the classification of Harish-Chandra invarian irreducible spaces; see Zhelobenko's expository paper [45] and his book [46]. Two different *quantum minor* interpretations of the lowering and raising operators were given by Nazarov and Tarasov [47] and the author [48]. These techniques are based on the theory of quantum algebras called the *Yangians* and allow an independent derivation of the matrix element formulas.

1.1.1 Gelfand-Tzetlin basis

We now discuss the main idea which leads to the construction of the Gelfand-Tzetlin bases. The first point is to regard a given classical Lie algebra not as a single object but as a part of a chain of subalgebras with natural embeddings. We illustrate this idea using representations of the symmetric groups $\mathfrak{S}_n$ as an example. Consider the chain of subgroups

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots \subset \mathfrak{S}_n, \tag{1.1}$$

where the subgroup $\mathfrak{S}_k$ of $\mathfrak{S}_{k+1}$ consists of the permutations of the set $\{1, 2, \ldots, k+1\}$ with the index $k+1$. The irreducible representations of the group $\mathfrak{S}_n$ are indexed by partitions λ of n. A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ is depicted graphically as a *Young diagram* which consists of l left-justified rows of boxes so that the top row contains λ_1 boxes, the second row λ_2 boxes, etc. Denote by $V(\lambda)$ the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition λ. One of the central results of the representation theory of the symmetric groups is the following *branching rule* which describes the restriction of $V(\lambda)$ to the subgroup $\mathfrak{S}_{n-1}$:

$$V(\lambda)|_{\mathfrak{S}_{n-1}} = \bigoplus_{\mu} V'(\mu), \tag{1.2}$$

summed over all partitions μ whose Young diagram is obtained from that of λ by removing one box. Here $V'(\mu)$ denotes the irreducible representation of $\mathfrak{S}_{n-1}$ corresponding to a partition μ. Thus, the restriction of $V(\lambda)$ to $\mathfrak{S}_{n-1}$ is *multiplicity-free*, i.e., is contains each irreducible representation of $\mathfrak{S}_{n-1}$ at most once. This makes it possible to obtain a natural parameterization of the basis vectors in $V(\lambda)$ by taking its further restrictions to the subsequent subgroups of the chain (1.1). Namely, the basis vectors will be parametrized by sequences of partitions

$$\lambda^{(1)} \to \lambda^{(2)} \to \cdots \to \lambda^{(n)} = \lambda, \tag{1.3}$$

where $\lambda^{(k)}$ is obtained from $\lambda^{(k+1)}$ by removing one box. Equivalently, each sequence of this type can be regarded as a *standard tableau of shape* λ which is obtained by writing the numbers $1, \ldots, n$ into the boxes of λ in such a way that the numbers increase along the rows and down the columns. In particular, the dimension of $V(\lambda)$ equals the number of standard tableaux of shape λ. There is only one irreducible representation of the trivial group $\mathfrak{S}_1$ therefore the procedure defines basis vectors up to a scalar factor. The corresponding basis is called the *Young basis*. The symmetric group $\mathfrak{S}_n$ is generated by the adjacent transpositions $g_i = (i, i+1)$. The construction of the representation $V(\lambda)$ can be completed by deriving explicit formulas for the action of the elements g_i in the basis which are also due to A. Young. This realization of $V(\lambda)$ is usually called *Young's orthogonal* (or *seminormal*) *form*. The details can be found, e.g., in James and Kerber [49] and Sagan [50]; see also Okounkov and Vershik [51] where an alternative construction of the Young basis is produced.

Quite a similar method can be applied to representations of the classical Lie algebras. Consider the general linear Lie algebra $\mathfrak{gl}_n$ which consists of complex $n \times n$-matrices with the usual matrix commutator. The chain (1.1) is now replaced by

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n, \tag{1.4}$$

with natural embeddings $\mathfrak{gl}_k \subset \mathfrak{gl}_{k+1}$. The orthogonal Lie algebra $\mathfrak{o}_N$ can be regarded as a subalgebra of $\mathfrak{gl}_N$ which consists of skew-symmetric matrices. Again, we have a natural chain

$$\mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \cdots \subset \mathfrak{o}_N. \tag{1.5}$$

Both restrictions $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ are *multiplicity-free* so that the application of the argument which we used for the chain (1.1) produces basis vectors in an irreducible representation of $\mathfrak{gl}_n$ or $\mathfrak{o}_N$. With an appropriate normalization, these bases are precisely those of Gelfand and Tzetlin given in [28] and [29]. Instead of the standard tableaux, the basis vectors here are parametrized by combinatorial objects called the *Gelfand-Tzetlin patterns*. However, this approach does not work for the symplectic Lie algebras $\mathfrak{sp}_{2n}$ since the restriction $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ is not multiplicity-free. The multiplicities are given by Zhelobenko's branching rule [30] which was re-discovered later by Hegerfeldt [52]. Various approaches to fix this problem were made by several authors [53, 54, 55, 56, 57].

1.1.2 Explicit operatorial construction for $\mathfrak{gl}_n$

Now, we give some classical results of representation theory with special regard to the construction of the action of the algebra generators on irreducible invariant spaces.

Let E_{ij}, $i, j = 1, \ldots, n$ denote the standard basis of the general linear Lie algebra $\mathfrak{gl}_n$ over the field of complex numbers. The subalgebra $\mathfrak{gl}_{n-1}$ is spanned by the basis elements E_{ij} with $i, j = 1, \ldots, n-1$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{gl}_n$. The elements $E_{11}, \ldots, E_{nn}$ form a basis of $\mathfrak{h}$. Finite-dimensional irreducible representations of $\mathfrak{gl}_n$ are in a one-to-one correspondence with n-tuples of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \ldots, n-1. \tag{1.6}$$

Such an n-tuple λ is called the *highest weight* of the corresponding representation which we shall denote by $L(\lambda)$. It contains a unique, up to a multiple, nonzero vector ξ (the *highest vector*) such that $E_{ii}\,\xi = \lambda_i\,\xi$ for $i = 1, \ldots, n$ and $E_{ij}\,\xi = 0$ for $1 \leq i < j \leq n$.

The following theorem is the *branching rule* for the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$.

Theorem 1.1.1. *The restriction of $L(\lambda)$ to the subalgebra $\mathfrak{gl}_{n-1}$ is isomorphic to the direct sum of pairwise inequivalent irreducible representations*

$$L(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \bigoplus_{\mu} L'(\mu), \tag{1.7}$$

summed over the highest weights μ satisfying the betweenness conditions

$$\lambda_i - \mu_i \in \mathbb{Z}_+ \quad \textit{and} \quad \mu_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \textit{for} \quad i = 1, \ldots, n-1. \tag{1.8}$$

The rule could presumably be attributed to I. Schur who was the first to discover the representation-theoretic significance of a particular class of symmetric polynomials which now bear his name. The subsequent applications of the branching rule to the subalgebras of the chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n \tag{1.9}$$

yield a parameterization of basis vectors in $L(\lambda)$ by the combinatorial objects called the *Gelfand-Tzetlin patterns*. Such a pattern Λ (associated with λ) is an array of row vectors

$$\begin{array}{ccccccc} \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\ & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & \lambda_{21} & & \lambda_{22} & & \\ & & & \lambda_{11} & & & \end{array}$$

where the upper row coincides with λ and the following conditions hold

$$\lambda_{ki} - \lambda_{k-1,i} \in \mathbb{Z}_+, \qquad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}_+, \qquad i = 1, \ldots, k-1 \tag{1.10}$$

for each $k = 2, \ldots, n$. The *Gelfand-Tzetlin basis* of $L(\lambda)$ is provided by the following theorem. Let us set $l_{ki} = \lambda_{ki} - i + 1$.

Theorem 1.1.2. *There exists a basis $\{\xi_\Lambda\}$ in $L(\lambda)$ parametrized by all patterns Λ such that the action of generators of $\mathfrak{gl}_n$ is given by the formulas*

$$E_{kk}\,\xi_\Lambda = \left(\sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) \xi_\Lambda, \tag{1.11}$$

$$E_{k,k+1}\,\xi_\Lambda = - \sum_{i=1}^{k} \frac{(l_{ki} - l_{k+1,1}) \cdots (l_{ki} - l_{k+1,k+1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})}\, \xi_{\Lambda+\delta_{ki}}, \tag{1.12}$$

$$E_{k+1,k}\,\xi_\Lambda = \sum_{i=1}^{k} \frac{(l_{ki} - l_{k-1,1}) \cdots (l_{ki} - l_{k-1,k-1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})}\, \xi_{\Lambda-\delta_{ki}}. \tag{1.13}$$

The arrays $\Lambda \pm \delta_{ki}$ are obtained from Λ by replacing λ_{ki} by $\lambda_{ki} \pm 1$. It is supposed that $\xi_\Lambda = 0$ if the array Λ is not a pattern; the symbol $\wedge$ indicates that the zero factor in the denominator is skipped.

The vector space $L(\lambda)$ is equipped with a contravariant inner product $\langle\, , \rangle$. It is uniquely determined by the conditions

$$\langle \xi, \xi \rangle = 1 \qquad \text{and} \qquad \langle E_{ij}\,\eta, \zeta \rangle = \langle \eta, E_{ji}\,\zeta \rangle \tag{1.14}$$

for any vectors $\eta, \zeta \in L(\lambda)$ and any indices i, j. In other words, for the adjoint operator for E_{ij} with respect to the inner product we have $(E_{ij})^* = E_{ji}$.

Proposition 1.1.3. *The basis $\{\xi_\Lambda\}$ is orthogonal with respect to the inner product $\langle\, ,\, \rangle$. Moreover, we have*

$$\langle\, \xi_\Lambda, \xi_\Lambda \,\rangle = \prod_{k=2}^{n} \prod_{1\le i\le j<k} \frac{(l_{ki} - l_{k-1,j})!}{(l_{k-1,i} - l_{k-1,j})!} \prod_{1\le i<j\le k} \frac{(l_{ki} - l_{kj} - 1)!}{(l_{k-1,i} - l_{kj} - 1)!}. \tag{1.15}$$

The formulas of Theorem 1.1.2 can therefore be rewritten in the orthonormal basis

$$\zeta_\Lambda = \xi_\Lambda / \|\, \xi_\Lambda \|, \qquad \|\, \xi_\Lambda \|^2 = \langle\, \xi_\Lambda, \xi_\Lambda \,\rangle. \tag{1.16}$$

1.1.3 Gelfand-Tzetlin bases for the other classical Lie algebras

Let $\mathfrak{g}_n$ denote the rank n simple complex Lie algebra of type B, C, or D. That is,

$$\mathfrak{g}_n = \mathfrak{o}_{2n+1}, \quad \mathfrak{sp}_{2n}, \quad \text{or} \quad \mathfrak{o}_{2n}, \tag{1.17}$$

respectively. Let $V(\lambda)$ denote the finite-dimensional irreducible representation of $\mathfrak{g}_n$ with the highest weight λ. The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is not multiplicity-free in general. This means that if $V'(\mu)$ is the finite-dimensional irreducible representation of $\mathfrak{g}_{n-1}$ with the highest weight μ, then the space

$$\mathrm{Hom}_{\mathfrak{g}_{n-1}}(V'(\mu), V(\lambda)) \tag{1.18}$$

need not be one-dimensional. In order to construct a basis of $V(\lambda)$ associated with the chain of subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n \tag{1.19}$$

we need to construct a basis of the space (1.18) which is isomorphic to the subspace $V(\lambda)^+_\mu$ of $\mathfrak{gl}_{n-1}$-highest vectors of weight μ in $V(\lambda)$. The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is given by

$$V(\lambda)|_{\mathfrak{g}_{n-1}} \simeq \bigoplus_{\mu} c(\mu)\, V'(\mu), \tag{1.20}$$

where $V'(\mu)$ is the irreducible finite-dimensional representation of $\mathfrak{g}_{n-1}$ with the highest weight μ. The multiplicity $c(\mu)$ coincides with the dimension of the space $V(\lambda)^+_\mu$, and its exact value is found from the Zhelobenko branching rules [30]. In the formulas below we use the notation

$$l_i = \lambda_i + \rho_i + 1/2, \qquad \gamma_i = \nu_i + \rho_i + 1/2, \tag{1.21}$$

where the ν_i are the parameters defined in the branching rules. A parameterization of basis vectors in $V(\lambda)$ is obtained by applying the branching rules to its subsequent restrictions to the subalgebras of the chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n. \tag{1.22}$$

This leads to the definition of the Gelfand-Tzetlin patterns for the B, C and D types. Then we give formulas for the basis vectors of the representation $V(\lambda)$. We use the notation

$$l_{ki} = \lambda_{ki} + \rho_i + 1/2, \qquad l'_{ki} = \lambda'_{ki} + \rho_i + 1/2, \tag{1.23}$$

where the λ_{ki} and λ'_{ki} are the entries of the patterns defined below.

B type case. The multiplicity $c(\mu)$ equals the number of n-tuples $(\nu'_1, \nu_2, \ldots, \nu_n)$ satisfying the inequalities

$$\begin{aligned} &-\lambda_1 \geq \nu'_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n, \\ &-\mu_1 \geq \nu'_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_{n-1} \geq \nu_n \end{aligned} \tag{1.24}$$

with ν'_1 and all the ν_i being simultaneously integers or half-integers together with the λ_i. Equivalently, $c(\mu)$ equals the number of $(n+1)$-tuples $\nu = (\sigma, \nu_1, \ldots, \nu_n)$, with the entries given by

$$(\sigma, \nu_1) = \begin{cases} (0, \nu'_1) & \text{if } \nu'_1 \leq 0, \\ (1, -\nu'_1) & \text{if } \nu'_1 > 0. \end{cases} \tag{1.25}$$

Lemma 1.1.4. *The vectors*

$$\xi_\nu = z_{n0}^{\sigma} \prod_{i=1}^{n-1} z_{ni}^{\nu_i - \mu_i} z_{i,-n}^{\nu_i - \lambda_i} \cdot \prod_{k=l_n}^{\gamma_n - 1} Z_{n,-n}(k)\, \xi \tag{1.26}$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the B *type pattern* Λ associated with λ as an array of the form

$$\begin{array}{ccccccccc}
\sigma_n & & \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
& \lambda'_{n1} & & \lambda'_{n2} & & \cdots & & \lambda'_{nn} & \\
\sigma_{n-1} & & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
& \lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
\cdots & & \cdots & & \cdots & & & & \\
\sigma_1 & & \lambda_{11} & & & & & & \\
& \lambda'_{11} & & & & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \ldots, \lambda_{nn})$, each σ_k is 0 or 1, the remaining entries are all non-positive integers or non-positive half-integers together with the λ_i, and the following inequalities hold

$$\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk} \tag{1.27}$$

for $k = 1, \ldots, n$, and

$$\lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \lambda_{k-1,2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk} \tag{1.28}$$

for $k = 2, \ldots, n$. In addition, in the case of the integer λ_i the condition

$$\lambda'_{k1} \leq -1 \qquad \text{if} \quad \sigma_k = 1 \tag{1.29}$$

should hold for all $k = 1, \ldots, n$.

Theorem 1.1.5. *The vectors*

$$\xi_\Lambda = \prod_{k=1,\ldots,n}^{\rightarrow} \left(z_{k0}^{\sigma_k} \cdot \prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{ki}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{kk}-1} Z_{k,-k}(j) \right) \xi \tag{1.30}$$

parametrized by the patterns Λ form a basis of the representation $V(\lambda)$.

C type case. The multiplicity $c(\mu)$ equals the number of n-tuples of integers $(\nu_1, \ldots, \nu_n)$ satisfying the inequalities

$$\begin{aligned}
&0 \geq \nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n, \\
&0 \geq \nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_{n-1} \geq \nu_n.
\end{aligned} \tag{1.31}$$

Lemma 1.1.6. *The vectors*

$$\xi_\nu = \prod_{i=1}^{n-1} z_{ni}^{\nu_i-\mu_i} z_{i,-n}^{\nu_i-\lambda_i} \cdot \prod_{k=l_n}^{\gamma_n-1} Z_{n,-n}(k)\, \xi \tag{1.32}$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the C *type pattern* Λ associated with λ as an array of the form

$$\begin{array}{cccccccc}
 & \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
\lambda'_{n1} & & \lambda'_{n2} & & \cdots & & \lambda'_{nn} & \\
 & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
\lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
 & \cdots & & \cdots & & & & \\
 & \lambda_{11} & & & & & & \\
\lambda'_{11} & & & & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, the remaining entries are all non-positive integers and the following inequalities hold

$$0 \geq \lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk} \tag{1.33}$$

for $k = 1, \dots, n$, and

$$0 \geq \lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \lambda_{k-1,2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk} \tag{1.34}$$

for $k = 2, \dots, n$.

Theorem 1.1.7. *The vectors*

$$\xi_\Lambda = \overrightarrow{\prod_{k=1,\dots,n}} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{ki}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{kk}-1} Z_{k,-k}(j) \right) \xi \tag{1.35}$$

parametrized by the patterns Λ *form a basis of the representation* $V(\lambda)$.

D type case. The multiplicity $c(\mu)$ equals the number of $(n-1)$-tuples $(\nu_1, \ldots, \nu_{n-1})$ satisfying the inequalities

$$\begin{gathered} -|\lambda_1| \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \nu_{n-1} \geq \lambda_n, \\ -|\mu_1| \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots \geq \mu_{n-1} \geq \nu_{n-1} \end{gathered} \tag{1.36}$$

with all the ν_i being simultaneously integers or half-integers together with the λ_i. Set $\nu_0 = \max\{\lambda_1, \mu_1\}$.

Lemma 1.1.8. *The vectors*

$$\xi_\nu = \prod_{i=1}^{n-1} z_{ni}^{\nu_{i-1}-\mu_i} z_{i,-n}^{\nu_{i-1}-\lambda_i} \cdot \prod_{k=l_n}^{\gamma_{n-1}-2} Z_{n,-n}(k)\, \xi \tag{1.37}$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the *D type pattern* Λ associated with λ as an array of the form

$$\begin{array}{ccccccc} \lambda_{n1} & \lambda_{n2} & & \cdots & & \lambda_{nn} \\ & \lambda'_{n-1,1} & & \cdots & \lambda'_{n-1,n-1} & \\ \lambda_{n-1,1} & \cdots & \lambda_{n-1,n-1} & & & \\ & \cdots & \cdots & & & \\ \lambda_{21} & \lambda_{22} & & & & \\ & \lambda'_{11} & & & & \\ \lambda_{11} & & & & & \end{array}$$

such that $\lambda = (\lambda_{n1}, \ldots, \lambda_{nn})$, the remaining entries are all non-positive integers or non-positive half-integers together with the λ_i, and the following inequalities hold

$$-|\lambda_{k1}| \geq \lambda'_{k-1,1} \geq \lambda_{k2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{k-1,k-1} \geq \lambda_{kk}, \tag{1.38}$$

$$-|\lambda_{k-1,1}| \geq \lambda'_{k-1,1} \geq \lambda_{k-1,2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k-1,k-1} \geq \lambda'_{k-1,k-1} \tag{1.39}$$

for $k = 2, \ldots, n$. Set $\lambda'_{k-1,0} = \max\{\lambda_{k1}, \lambda_{k-1,1}\}$.

Theorem 1.1.9. *The vectors*

$$\xi_\Lambda = \overrightarrow{\prod_{k=2,\ldots,n}} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{k-1,i-1}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{k-1,i-1}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{k-1,k-1}-2} Z_{k,-k}(j) \right) \xi \tag{1.40}$$

parametrized by the patterns Λ form a basis of the representation $V(\lambda)$.

1.2 Coupling and recoupling for the $SO(3)$ group

1.2.1 Clebsch-Gordan coefficients

Clebsch-Gordan coefficients (CGCs) usually refer to the group $SO(3)$ and are used in physics to integrate products of three spherical harmonics. They arise in applications involving the addition of angular momentum in quantum mechanics [58]. The CGCs are variously written as $C^{j}_{m_1 m_2}$, $C^{j_1 j_2 j}_{m_1 m_2 m}$, $(j_1, j_2; m_1, m_2|j_1, j_2; j, m)$, or $\langle j_1, j_2; m_1, m_2|j_1, j_2; j, m\rangle$.

Equivalently, they are used in the representation theory of $SU(2)$ and $SO(3)$ groups to perform the explicit direct sum decomposition of the tensor product of two irreducible representations into irreducible representations, in cases where the numbers and types of irreducible components are already known. The name derives from the German mathematicians Alfred Clebsch (1833-1872) and Paul Gordan (1837-1912), who encountered an equivalent problem in invariant theory.

From the previous subsection, we know that the irreducible representations for the A_1 algebra (the Lie algebra associated to the grop $SU(2)$) are labelled by a nonnegative half-integer number (associated with the Gelfand-Tzetlin pattern) that we may denote by j and their dimension is given by $2j + 1$. Because $SU(2)$ is the covering group of $SO(3)$, j labels an irreducible representation (irrep) of $SO(3)$ if j is an integer (so-called *tensorial* irreps) and a *projective* one if j is not an integer (so-called *spinor* irreps).

Given the tensor product representation $j_1 \otimes j_2$ for $SU(2)$, it is an outstanding question which irreps are contained in its decomposition in direct sum of invariant irreducible spaces. In fact, such a decomposition has the important physical meaning of the sum of two angular momenta j_1 and j_2 respectively. By definition, CGCs are the entries of the orthogonal base changing matrix which reduces $j_1 \otimes j_2$ in a block diagonal form. Because such a decomposition is *multiplicity-free*, denoted by $|j, m\rangle$ ($m \in \{-j, -j+1, \ldots, j-1, j\}$) a generic Gelfand-Tzetlin base vector for j and by $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ for j_1 and j_2 respectively, we have

$$|j_1, j_2; j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2|j_1, j_2; j, m\rangle \tag{1.41}$$

where $\langle j_1, j_2; m_1, m_2|j_1, j_2; j, m\rangle$ are the CGCs. By using the fact that such coefficients are orthonormalized and defining the suitable raising and lowering operetors as described in the previous section, we obtain the following relation for the CGCs:

$$\begin{aligned}\sqrt{(j \mp m+1)(j \pm m)}\ \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = \\ \sqrt{(j_1 \mp m_1+1)(j_1 \pm m_1)}\ \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \mp 1\rangle + \\ \sqrt{(j_2 \mp m_2+1)(j_2 \pm m_2)}\ \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \mp 1\rangle \end{aligned} \quad (1.42)$$

which is often useful for finding the last CGCs, when the other one or two coefficients in the formula are known. Note that there are sometimes only two coefficients in that equation, the third being both invalid ($j < |m|$) and multiplied by 0. Furthermore (1.42) provides a recursion relation that can be useful for finding the explicit expression of the coefficients. In figure 1.1 a well-known table with the values of CGCs for several coupling irreps is given.

CGCs are sometimes associated to the so-called *3j-symbols* (or *Wigner coefficients*), denoted by

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix}$$

where, as usual, the entries of the symbol j_1, j_2, J, m_1, m_2 and M are either integer or half-integer [59]. $3j-symbols$ satisfy the following selection rules:

1. $m_1 \in \{-|j_1|, ..., |j_1|\}$, $m_2 \in \{-|j_2|, ..., |j_2|\}$, and $M \in \{-|J|, ..., |J|\}$.

2. $m_1 + m_2 = M$.

3. *Triangular inequalities*: $|j_1 - j_2| \leq J \leq j_1 + j_2$.

4. *Integer perimeter rule*: $j_1 + j_2 + J$ is an integer.

Note that not all these rules are independent, since rule (4) is implied by the other three. If these conditions are not satisfied the $3j$-symbol vanishes.

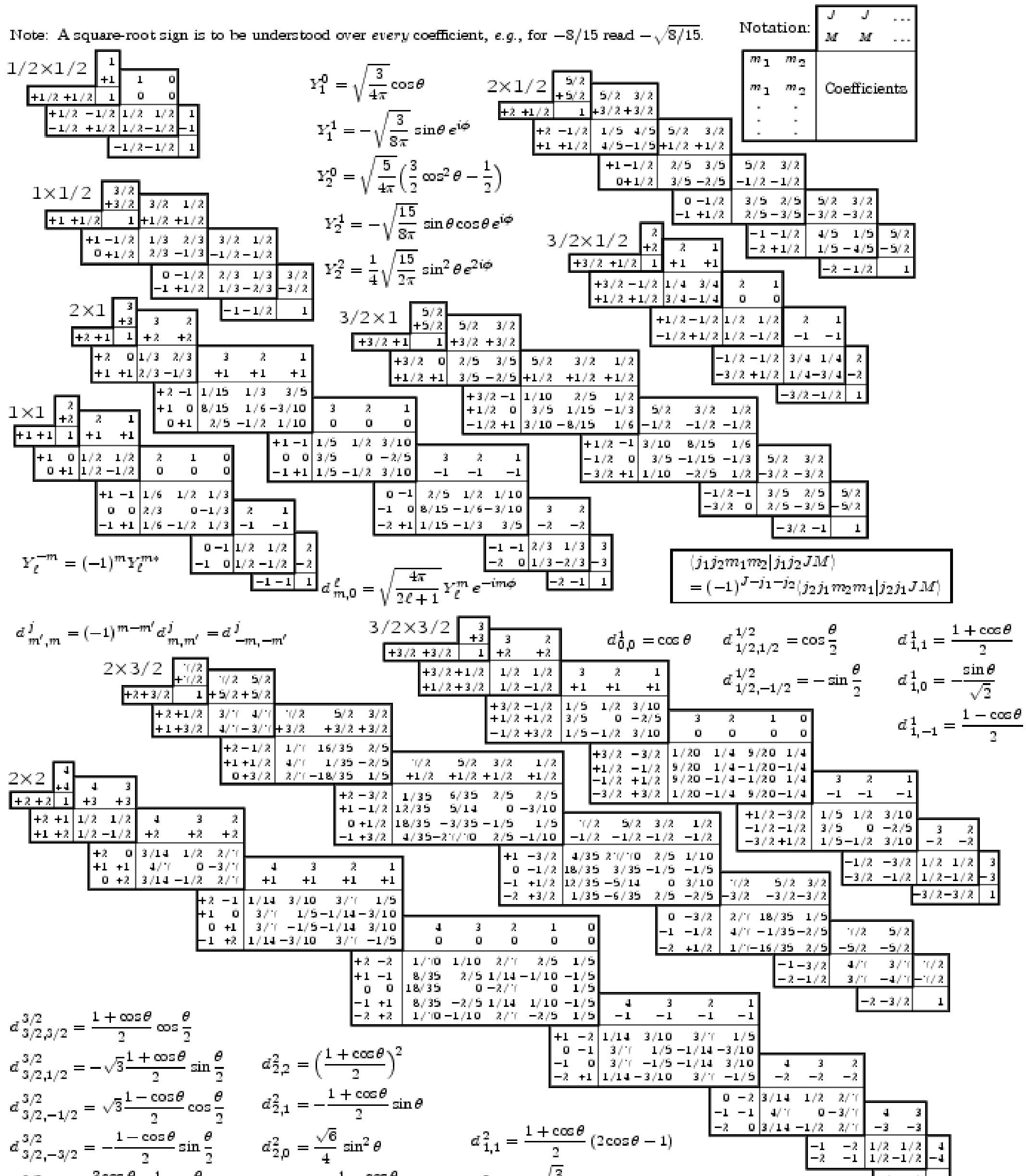


Figure 1.1: Clebsch-Gordan coefficients for several coupling (tensorial and projective) irreps of $SO(3)$. The sign convention is that of Wigner [60], also used by Condon and Shortley [61], Rose [62], and Cohen [63]. The coefficients here have been calculated using computer programs written independently by Cohen et al. at LBNL.

Furthermore, the Wigner 3j-symbols have the symmetries:

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = \begin{pmatrix} j_2 & J & j_1 \\ m_2 & M & m_1 \end{pmatrix} \tag{1.43}$$

$$= \begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m2 \end{pmatrix} \tag{1.44}$$

$$= (-1)^{j_1+j_2+J} \begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & M \end{pmatrix} \tag{1.45}$$

$$= (-1)^{j_1+j_2+J} \begin{pmatrix} j_1 & J & j_2 \\ m_1 & M & m_2 \end{pmatrix} \tag{1.46}$$

$$= (-1)^{j_1+j_2+J} \begin{pmatrix} J & j_2 & j_1 \\ M & m_2 & m_1 \end{pmatrix} \tag{1.47}$$

$$= (-1)^{j_1+j_2+J} \begin{pmatrix} j_1 & j_2 & J \\ -m_1 & -m_2 & -M \end{pmatrix} \tag{1.48}$$

and they obey the orthogonality relations

$$\sum_{j,m} (2j+1) \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ m'_1 & m'_2 & M \end{pmatrix} = \delta_{m_1,m'_1}\delta_{m_2,m'_2} \tag{1.49}$$

$$\sum_{m_1,m_2} (2j+1) \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J' \\ m'_1 & m'_2 & M' \end{pmatrix} = \delta_{J,J'}\delta_{M,M'}. \tag{1.50}$$

The connection between $3j$-symbols and Clebsch-Gordan coeficients is given by the relation:

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle = (-1)^{m+j_1-j_2}\sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}. \tag{1.51}$$

1.2.2 Racah coefficients and general recoupling symbols

Racah coefficients $U(j_1 j_2 j j_3; j_{12} j_{23})$ (sometimes called *U coefficients*) represent the elements of a unitary matrix between bases with two different coupling orders of three irreps j_1, j_2 an j_3:

$$|(j_1 j_2) j_{12}, j_3; jj'\rangle = \sum_{j_{23}} U(j_1 j_2 j j_3; j_{12} j_{23}) | j_1 (j_2 j_3) j_{23}, j_3; jj'\rangle. \tag{1.52}$$

The U coefficients satisfy the following unitay conditions

$$\sum_{j_{23}} U(j_1j_2jj_3;j_{12}j_{23})\, U(j_1j_2jj_3;\bar{j}_{12}j_{23}) = \delta_{j_{12}\bar{j}_{12}} \tag{1.53}$$

$$\sum_{j_{12}} U(j_1j_2jj_3;j_{12}j_{23})\, U(j_1j_2jj_3;j_{12}\bar{j}_{23}) = \delta_{j_{23}\bar{j}_{23}} \tag{1.54}$$

deriving from the fact that we deal with orthonormalized bases.

Sometimes one needs to use $6j-symbols$ which is defined in terms of U coefficients by [64]

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \frac{1}{\sqrt{(2j_{12}+1)(2j_{23}+1)}}\, U(j_1j_2jj_3;j_{12}j_{23}) \tag{1.55}$$

Thus, the $6j$-symbols are defined for integers and half-integers j_1, j_2, j_{12}, j_3, j, j_{23} whose triads (j_1, j_2, j_{12}), (j_1, j, j_{23}), (j_3, j_2, j_{23}), and (j_3, j, j_{12}) satisfy the following conditions

1. Each triad satisfies the triangular inequalities.

2. The sum of the elements of each triad is an integer. Therefore, the members of each triad are either all integers or contain two half-integers and one integer.

If these conditions are not satisfied, the $6j$-symbol vanishes.

Suppose we have a tetrahedron, labelled so that the three labels around each face satisfy the conditions given above, thus we have a so-called an *admissible labelling*. This tetrahedral picture is traditionally used to simply express the symmetry of the $6j$-symbol, which is naturally invariant under the full tetrahedral group S_4. In particular, they are invariant [65] under permutation of their columns, e.g.,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j & j_3 & j_{23} \end{Bmatrix} \tag{1.56}$$

and under exchange of two corresponding elements between rows, e.g.

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_3 & j & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix}. \tag{1.57}$$

The following Racah-Elliot and orthogonality relations are often useful for computing numerical values or exact expressions:

$$\sum_{j_{12}} (-1)^{2j_{12}}(2j_{12}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix} = 1 \tag{1.58}$$

$$\sum_{j_{12}} (-1)^{j_1+j_2+j_{12}} \ (2j_{12}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_2 & j_1 & j_{23} \end{Bmatrix} = \delta_{j_1 j_{23}} \sqrt{(2j_i+1)(2j_2+1)} \tag{1.59}$$

$$\sum_{j_{12}} (2j_{12}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_3 & j & j_{12} \\ j_1 & j_2 & \bar{j}_{23} \end{Bmatrix} = \frac{1}{2j_{23}+1} \delta_{j_{23}\bar{j}_{23}} \tag{1.60}$$

$$\sum_{j_{12}} (-1)^{j_{12}+j_{23}+\bar{j}_{23}} \ (2j_{12}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_3 & j & j_{12} \\ j_2 & j_1 & \bar{j}_{23} \end{Bmatrix} = \begin{Bmatrix} j_1 & j & j_{23} \\ j_2 & j_3 & \bar{j}_{23} \end{Bmatrix} \tag{1.61}$$

$$\begin{aligned} &\sum_{j_{12}} (-1)^{j_1+j_2+j_3+j+\bar{j}_3+\bar{j}+j_{23}+\bar{j}_{23}+j_{12}+\tilde{j}_{23}} \ (2j_{12}+1) \cdot \\ &\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_3 & j & j_{12} \\ \bar{j}_3 & \bar{j} & \bar{j}_{23} \end{Bmatrix} \begin{Bmatrix} \bar{j}_3 & \bar{j} & j_{12} \\ j_2 & j_1 & \tilde{j}_{23} \end{Bmatrix} = \\ &\begin{Bmatrix} \tilde{j}_{23} & \bar{j}_{23} & \tilde{j}_{23} \\ \bar{j}_3 & j_1 & j \end{Bmatrix} \begin{Bmatrix} j_{23} & \bar{j}_{23} & \tilde{j}_{23} \\ \bar{j} & j_2 & j_3 \end{Bmatrix} \end{aligned} \tag{1.62}$$

CGCs and $6j$-symbols are related by the following equation

$$\begin{aligned} &\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j_3+j}}{\sqrt{(2j_{12}+1)(2j_{23}+1)}} \cdot \\ &\sum_{m_1,m_2} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j_{12}, m_1+m_2 \rangle \ \langle j_{12}, j_3; m_1+m_2, m-m_1-m_2 | j_{12}, j_3; j, m \rangle \cdot \\ &\langle j_2, j_3; m_2, m-m_1-m_2 | j_2 j_3; j_{23}, m-m_1 \rangle \ \langle j_1, j_{23}; m_1, m-m_1 | j_1 j_{23}; j, m \rangle . \end{aligned} \tag{1.63}$$

Thus, by using (1.63) and orthonormality relations, we can also evaluate CGCs once we know $6j$-symbols.

$9j$-symbols are defined by the coupling of four irreps of $SU(2)$ and they are denoted by

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{34} & j \end{Bmatrix}.$$

They can be written in terms of $3j$-symbols:

$$\begin{pmatrix} j_{13} & j_{24} & j \\ m_{13} & m_{24} & m \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \sum_{m_1,m_2,m_3,m_4,m_{12},m_{34}} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix} \cdot$$

$$\begin{pmatrix} j_3 & j_4 & j_{34} \\ m_3 & m_4 & m_{34} \end{pmatrix} \begin{pmatrix} j_1 & j_3 & j_{13} \\ m_1 & m_3 & m_{13} \end{pmatrix} \begin{pmatrix} j_2 & j_4 & j_{24} \\ m_2 & m_4 & m_{24} \end{pmatrix} \begin{pmatrix} j_{12} & j_{34} & j \\ m_{12} & m_{34} & m \end{pmatrix} \tag{1.64}$$

and in terms of $6j$-symbols

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} =$$

$$\sum_g (-1)^{2g}(2g+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \tag{1.65}$$

A $9j$-symbol is invariant under reflection through one of the diagonals, and becomes multiplied by $(-1)^R$ upon the exchange of two rows or columns, where R is the sum of all the entries of the symbol. It also satisfies the orthogonality relationship

$$\sum_{j_{13},j_{24}} (2j_{13}+1)(2j_{24}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & \bar{j}_{12} \\ j_3 & j_4 & \bar{j}_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} =$$

$$\frac{1}{(2j_{12}+1)(2j_{34}+1)} \delta_{j_{12},\bar{j}_{12}} \delta_{j_{34},\bar{j}_{34}}. \tag{1.66}$$

So, we can see that $6j$-symbols play a leading role between the $3nj$-symbols because from those we can derive all the other ones.

1.3 Racah-Wigner calculus for general Lie groups

One can extend the definitions of CGCs, $3j$, $6j$, $9j$-symbols given in the previous section to a generic Lie group. Here, we consider the Racah coefficients and $6j$-symbols which are particularly significant in Racah-Wigner calculus, as observed in the previous section.

Given three tensorial or projective (i.e. tensorial irreps for the universal covering group) irreps $[\lambda_1]$, $[\lambda_2]$ and $[\lambda_3]$ of a (classical) Lie group G, we define the Racah coefficients as the elements of a unitary matrix between bases with two different coupling orders of $[\lambda_1]$, $[\lambda_2]$ and $[\lambda_3]$. A new major difficulty becomes now the crucial point of the question: the *mutiplicity*. In fact, the generic decomposition of tensor product of irreducible representations it is *not* multiplicity-free and we need additional labels to spot such coefficients in the multiplicity space. Thus, denoted by $\{\lambda_1\lambda_2\lambda_{12}\}$, $\{\lambda_2\lambda_3\lambda_{23}\}$, $\{\lambda_1\lambda_{23}\lambda\}$ and $\{\lambda_1\lambda_{23}\lambda\}$ the multiplicities of $[\lambda_{12}]$ in $[\lambda_1]\otimes[\lambda_2]$, $[\lambda_{23}]$ in $[\lambda_2]\otimes[\lambda_3]$, $[\lambda]$ in $[\lambda_{12}]\otimes[\lambda_3]$ and $[\lambda]$ in $[\lambda_1]\otimes[\lambda_{23}]$ respectively, and by $|(\lambda_1\lambda_2)\lambda_{12},\lambda_3;\lambda\lambda'\rangle^{t_{12}t}$, $|\lambda_1(\lambda_2\lambda_3)\lambda_{23};\lambda\lambda'\rangle^{t_{23}t'}$ the base vectors corresponding to the different orders of coupling, we have

$$|(\lambda_1\lambda_2)\lambda_{12},\lambda_3;\lambda\lambda'\rangle^{t_{12}t} = \sum_{\lambda_{23},t_{23},t'} U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{t_{12}t}_{t_{23}t'}\ |\lambda_1(\lambda_2\lambda_3)\lambda_{23};\lambda\lambda'\rangle^{t_{23}t'} \tag{1.67}$$

where $t_{12}=1,2,\ldots,\{\lambda_1\lambda_2\lambda_{12}\}$, $t_{23}=1,2,\ldots,\{\lambda_2\lambda_3\lambda_{23}\}$, $t=1,2,\ldots,\{\lambda_{12}\lambda_3\lambda\}$ and $t'=1,2,\ldots,\{\lambda_1\lambda_{23}\lambda\}$ are four multiplicity labels. The Racah coefficients satisfy the following unitary conditions:

$$\sum_{\lambda_{23},t_{23},t'} U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{t_{12}t}_{t_{23}t'}\ U(\lambda_1\lambda_2\lambda\lambda_3;\bar{\lambda}_{12}\lambda_{23})^{s_{12}s}_{t_{23}t'} = \delta_{t_{12}s_{12}}\ \delta_{ts}\ \delta_{\lambda_{12}\bar{\lambda}_{12}} \tag{1.68}$$

$$\sum_{\lambda_{23},t_{12},t} U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{t_{12}t}_{t_{23}t'}\ U(\lambda_1\lambda_2\lambda\lambda_3;\bar{\lambda}_{12}\lambda_{23})^{t_{12}t}_{s_{23}s'} = \delta_{t_{23}s_{23}}\ \delta_{t's'}\ \delta_{\lambda_{23}\bar{\lambda}_{23}}. \tag{1.69}$$

General $6j$-symbols are defined in analogy to the cases $SU(2)$ and $SO(3)$ in terms of general U coefficients, given in (1.67), by

$$\left\{\begin{matrix}\lambda_1 & \lambda_2 & \lambda_{12}\\ \lambda_3 & \lambda & \lambda_{23}\end{matrix}\right\}^{t_{12}t}_{t_{23}t'} = \frac{1}{\sqrt{D_{\lambda_{12}}D_{\lambda_{23}}}}\ U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{t_{12}t}_{t_{23}t'} \tag{1.70}$$

where D_λ represents the dimension of the (tensor or projective) irrep $[\lambda]$ of the Lie group G.

A generalized $6j$-*tetrahedron* may also be defined, with the edges labelled by combinatorial patterns describing the involved irreps. The definition of a suitable ordering relation between such patterns allows us to establish the *selection rules*, i.e. when Racah coefficients vanish, and give the $6j$-symbol symmetry properties.

Usually Racah coefficients can be obtained by using a knowledge of a few simple case to get through the extension of the Biedenharn-Elliot sum rule. This bootstrap method was developed by Bickesrstaff and Wybourne [66], Searle and Butler [67]. There are also many other methods. For example, *generating functions* can be used in some special cases [68], *isoscalar factors* can be constractively used in some cases [69], and in other situations we need to use the mathematical structure inherent the particular physical problem [11, 70, 71].

Here, we would like to emphasize the works of Kramer [15] and Chen et al [72]. They used the *Schur-Weyl duality* relation between $\mathfrak{S}_f$ and the unitary group $U(n)$, which enable them to derive $U(n)$ Racah coefficients from *subduction coefficients* of $\mathfrak{S}_f$. We develop such ideas in the next chapter with particular regard to *Brauer centralizer algebras* which allow us an unified approach to Racah-Wigner calculus for classical Lie groups.

Chapter 2

Schur-Weyl Duality

An overview of Schur-Weyl duality for classical Lie groups is presented. In section 1, we describe the centralizer algebras for unitary, orthogonal and symplectic groups an their quantum deformations. In section 2, we focus on Brauer algebras and their duality with orthogonal and symplectic groups. In section 3, we highlight the equivalence between the coupling issue for Lie groups and the subduction problem for centralizer algebras, via results of the invariant theory.

2.1 Classical Schur-Weyl duality

2.1.1 Schur's double-centralizer result

Consider the vector space $V = \mathbb{C}^n$. The simmetric group $\mathfrak{S}_r$ acts naturally on its r-fold tensor power $V^{\otimes r}$, by permuting the tensor positions. This action obviously commutes with the natural action of $GL_n = GL_n(\mathbb{C})$, acting by matrix multiplication in each tensor position. So we have a $\mathbb{C}GL_n - \mathbb{C}\mathfrak{S}_n$ bimodule structure on $V^{\otimes r}$ (here $\mathbb{C}G$ as usual denotes the *group algebra* of a group G). In 1927, Schur [13] proved that the image of each group algebra under its representation equals the full *centralizer algebra* for the other action. More precisely, if we name the representations as follows

$$\mathbb{C}GL_n \xrightarrow{\pi} End(V^{\otimes r}) \xleftarrow{\omega} \mathbb{C}\mathfrak{S}_r \tag{2.1}$$

then we have equalities

$$\pi(\mathbb{C}GL_n) = End_{\mathfrak{S}_r}(V^{\otimes r}) = \{T \in End(V^{\otimes r}) \mid Tgv = gTv,\ \forall g \in \mathfrak{S}_r,\ \forall v \in V^{\otimes r}\} \quad (2.2)$$

$$\omega(\mathbb{C}\mathfrak{S}_r) = End_{GL_n}(V^{\otimes r}) = \{T \in End(V^{\otimes r}) \mid Tgv = gTv,\ \forall g \in GL_n,\ \forall v \in V^{\otimes r}\}. \quad (2.3)$$

(Here, for a given set S operating on a vector space T through linear endomorphisms, $End_S(T)$ denotes the set of linear endomorphisms of T commuting with each endomorphism coming from S.)

Results of Carter-Lusztig [73] and J. A. Green [74] et al. show that all the above statements remain true if one replaces $\mathbb{C}$ by arbitrary infinite field $\mathbb{K}$.

2.1.2 Schur algebras

The finite-dimensional algebra in (2.2) is known as *Schur algebra*, and often denoted by $S_{\mathbb{C}}(n,r)$ or simply $S(n,r)$. The Schur algebra "sees" the part of the rational representation theory of the algebric group $GL_n(\mathbb{C})$ occurring (in some appropriate sense) in $V^{\otimes r}$. More precisely, there is an equivalence between r-homogeneous polynomial representations of $GL_n(\mathbb{C})$ and $S_{\mathbb{C}}(n,r)$-invariant irreducible spaces. Those representations (as r varies) determine all finite-dimensional rational representations.

The representation ω in (2.1) is faithful if $n \geq r$, so ω induces an isomorphism

$$\mathbb{C}\mathfrak{S}_r \simeq End_{GL_n}(V^{\otimes r}) = End_{S_{\mathbb{C}}(n,r)}(V^{\otimes r}) \qquad (n \geq r). \quad (2.4)$$

This leads to intimate connections between polynomial representations of $GL_n(\mathbb{C})$ and representations of $\mathbb{C}\mathfrak{S}_r$, a theme that has been exploited by many authors in recent years. Such connections become particularly interesting if the characteristic of the field is different from zero [75, 76]. Perhaps the most dramatic example of this is the result of Erdmann [77] (building on the previous work of Donkin and Ringel) which shows that knowing decomposition numbers for all symmetric groups in positive characteristic will determine the decomposition numbers for general linear groups in the same characteristic. Conversely, James [78] had already shown that the decomposition matrix for a symmetric group is a sub matrix of the decomposition matrix for an appropriate Schur algebra. Thus the (still open) general problem of determining the modular characters of symmmetric groups is equivalent to the similar problem for general linear groups (over infinite fields).

2.1.3 The enveloping algebra approach

Return to the basic setup, over $\mathbb{C}$. One may differentiate the action of the Lie group $GL_n(\mathbb{C})$ to obtain an action of its Lie algebra $\mathfrak{gl}_n$. Replacing the representation π in (2.1) by its derivative representation $d\pi : \mathfrak{U}(\mathfrak{gl}_n) \to End(V^{\otimes r})$ leads to the following alternative statements of Schur's result:

$$d\pi(\mathfrak{U}(\mathfrak{gl}_n)) = End_{\mathfrak{S}_r}(V^{\otimes r}) \tag{2.5}$$

$$\omega(\mathbb{C}\mathfrak{S}_r) = End_{\mathfrak{gl}_n}(V^{\otimes r}). \tag{2.6}$$

In particular, the Schur algebra (over $\mathbb{C}$) is a homomorphic image of $\mathfrak{U}(\mathfrak{gl}_n)$. All of this works over an arbitrary integral domain K if we replace $\mathfrak{U}(\mathfrak{gl}_n)$ by its "hyperalgebra" $\mathfrak{U}_K = K \otimes_{\mathbb{Z}} \mathfrak{U}_{\mathbb{Z}}$ obtained by change of ring from a suitable $\mathbb{Z}$-form of $\mathfrak{U}(\mathfrak{gl}_n)$ [79]. (One can adapt the Konstant $\mathbb{Z}$-form, originally defined for the enveloping algebra of a *semisimple* Lie algebra, to the *reductive* $\mathfrak{gl}_n$.)

2.1.4 The quantum case

Jimbo [80] extended the results of the previous subsection to the quantum case (where the quantum parameter is not a root of unity). One needs to replace $\mathfrak{S}_r$ by the Iwahory-Hecke algebra $\mathbf{H}(\mathfrak{S}_r)$ and replace $\mathfrak{U}(\mathfrak{gl}_n)$ by the quantized enveloping algebra $\mathbf{U}(\mathfrak{gl}_n)$. The analogue of the Schur algebra in this context is known as the q-Schur algebra, often denoted by $\mathbf{S}(n,r)$ or $\mathbf{S}_q(n,r)$. Dipper and James [81] have shown that q-Schur algebras are fundamental for the modular representation theory of *finite* general linear groups.

As many authors have observed, the picture in subsection 2.1.1 can also be quantized. For that one needs a suitable quantization of the coordinate algebra of the algebraic group GL_n.

There is a completely different (geometric) construction of q-Schur algebras given in [82].

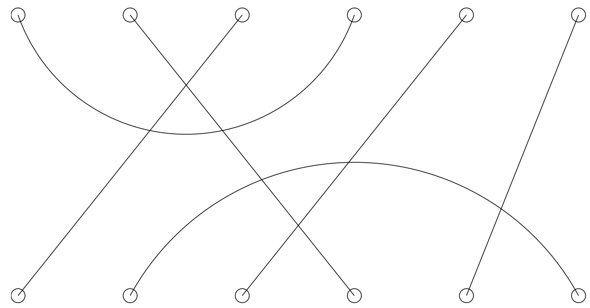


Figure 2.1: Example of a graph with $\mathbb{V}_f$ as set of vertices and $f = 6$.

2.2 $\mathfrak{B}_f(\epsilon n)$ - $O(n)$ and Sp_{2n} duality

2.2.1 Brauer centralizer algebras

Let $f \in \mathbb{N}_+$ be fixed. Denote by $\mathbb{V}_f$ the datum of $2f$ spots in a plane, arranged in two rows, one upon the other, each one of f aligned spots. Then consider the graphs with $\mathbb{V}_f$ as set of vertices and f edges, such that each vertex belongs to exactly one edge. Figure 2.1 shows an example of such a graph for $f = 6$. Such graphs are named *f-diagrams*, denoting by Δ_f the set of all of them. In general, we shall denote them by bold roman letters. These set of the f-diagrams has the same cardinality of the set of the pairings of $2f$ elements, hence $\left|\Delta_f\right| = (2f-1)!! = (2f-1)\cdot(2f-3)\cdots 5\cdot 3\cdot 1$.

We shall label the vertices in $\mathbb{V}_f$ in two ways: either we label the spots in the upper row with the numbers 1^+, 2^+, $\dots$, f^+, in their natural order from left to right, and the spots in the lower row with the numbers 1^-, 2^-, $\dots$, f^-, again from left to right, or we label them by setting i for i^+ and $f+j$ for j^- (for all $i, j \in \{1, 2, \dots, f\}$). Thus an f-diagram can also be described by specifying its set of edges: for instance, the 6-diagram in figure 2.1 is given by $\big\{\{1^+, 4^+\}, \{3^-, 5^+\}, \{2^+, 4^-\}, \{5^-, 6^+\}, \{2^-, 6^-\}, \{3^+, 1^-\}\big\}$. In general, given f-tuples $\mathbf{i} = (i_1, i_2, \dots, i_f)$ and $\mathbf{j} = (j_1, j_2, \dots, j_f)$ such that $\{i_1, \dots, i_f\} \cup \{j_1, \dots, j_f\} = \mathbb{V}_f$, we call $\mathbf{d}_{\mathbf{i},\,\mathbf{j}}$ the f-diagram obtained by joining i_k to j_k, for each $k = 1, 2, \dots, f$. So, the above diagram is $\mathbf{d}_{\mathbf{i},\mathbf{j}}$ for $\mathbf{i} = \{1^+, 2^+, 3^+, 5^+, 6^+, 2^-\}$, $\mathbf{j} = \{4^+, 4^-, 1^-, 3^-, 5^-, 6^-\}$.

When looking at the edges of an f-diagram, we shall distinguish between those which link two vertices in the same row (upper or lower), which are named *horizontal edges* or simply *bars*, and those which link two vertices in different rows, to be called *vertical edges*. Clearly, any f-diagram $\mathbf{d}$ has the same number of bars in the upper row and in the lower row: if this number is k, we shall say that $\mathbf{d}$ is a k-bar (f-)diagram. Then, letting $\Delta_{f,k} = \{\mathbf{d} \in \Delta_f \mid \mathbf{d} \textit{ is a } k\textit{-bar diagram}\}$ we have $\Delta_f = \bigcup_{k=1}^{[f/2]} \Delta_{f,k}$.

Let $\mathbf{d}$ be an f-diagram. With "bar structure of the upper row" (respectively "lower row") of $\mathbf{d}$ we shall mean the datum of the bars in the upper (respectively lower) row of $\mathbf{d}$, in their mutual positions. To put it in a nutshell, we shall use such terminology as "upper bar structure", (respectively "lower bar structure") of $\mathbf{d}$ - to be denoted with $\mathtt{ubs}(\mathbf{d})$ (respectively $\mathtt{lbs}(\mathbf{d})$ - and "bar structure of $\mathbf{d}$" - to be denoted with $\mathtt{bs}(\mathbf{d})$ - to mean the datum of both upper and lower bar structures of $\mathbf{d}$, that is $\mathtt{bs}(\mathbf{d}) = (\mathtt{ubs}(\mathbf{d}), \mathtt{lbs}(\mathbf{d}))$. Note that any upper or lower bar structure may be described by a one-row graph of vertices, arranged on a horizontal line, and some edges (the bars) joining them pairwise, so that each vertex belongs to at most one edge. Following Kerov [83], such a graph will be called a *k-bar f-junction*, or *(f,k)-junction*, where f is its number of vertices and k its number of edges. We denote the set of (f,k)-junctions by $J_{f,k}$. Then clearly $|J_{f,k}| = \binom{2f}{k}(2k-1)!!$.

Any $\mathbf{d} \in \Delta_{f,k}$ has exactly $f-2k$ vertices in its upper row, and $f-2k$ vertices in its lower row which are pairwise joined by its $f-2k$ vertical edges. Let us label with $1, 2, \ldots, f-2k$ from left to right the vertices in the upper row, and do the same in the lower row. Then there exists a unique permutation $\sigma = \sigma(\mathbf{d}) \in \mathfrak{S}_{f-2k}$ - to be called the "permutation structure", or "symmetric (group) part", of $\mathbf{d}$ - such that $\sigma(i)$ is the label of the lower row vertex of the vertical edge whose upper row vertex is labelled with i.

The outcome is that the mapping $\mathbf{d} \mapsto (\sigma(\mathbf{d}), \mathtt{bs}(\mathbf{d}))$ sets a bijection

$$\Delta_{f,k} \longrightarrow \mathfrak{S}_{f-2k} \times (J_{f,k} \times J_{f,k}) \tag{2.7}$$

and patching together these maps for all k gives a bijection $\Delta_f \longrightarrow \bigcup_{k=1}^{[f/2]} \mathfrak{S}_{f-2k} \times ({J_{f,k}}^{\times 2})$.

Let $\mathfrak{B}_f(x)$ be the $\mathbb{C}$-vector space with basis Δ_f; one can introduce a product in $\mathfrak{B}_f(x)$ by defining the product of f-diagrams and extending by linearity. So for all $\mathbf{a}, \mathbf{b} \in \Delta_f$ define the product $\mathbf{a} \cdot \mathbf{b} = \mathbf{ab}$ as follows. First, draw $\mathbf{b}$ below $\mathbf{a}$; second, connect the ith lower vertex of $\mathbf{a}$ with the ith upper vertex of $\mathbf{b}$; third, let $C(\mathbf{a}, \mathbf{b})$ be the number of cycles in the new graph obtained in the second step and let $\mathbf{a} * \mathbf{b}$ be this graph, *pruning out the*

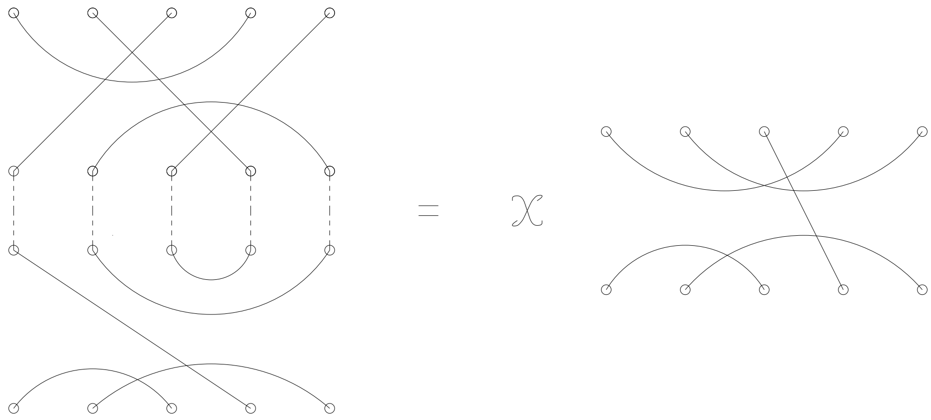


Figure 2.2: Example of product of two 5-diagrams.

cycles; then $\mathbf{a} * \mathbf{b}$ is a new f-diagram, and we set $\mathbf{ab} = x^{C(\mathbf{a},\mathbf{b})}\mathbf{a} * \mathbf{b}$. In figure 2.2 we show an example of product 5-diagrams following the definition previously given.

It is well-known that such a definition endows $\mathfrak{B}_f(x)$ with a structure of unital associative $\mathbb{C}$-algebra: this is the *Brauer algebra*, in its "abstract" form (see for instance [84]). The centralizer algebra originally considered by Brauer in the framework of invariant theory is related to this one.

Note that for $\mathbf{a}, \mathbf{b} \in \Delta_f$, the upper (respectively lower) bar structure of $\mathbf{a} * \mathbf{b}$ "contains" that of $\mathbf{a}$ (respectively $\mathbf{b}$). In particular, if $\mathbf{a} \in \Delta_{f,a}$ and $\mathbf{b} \in \Delta_{f,b}$ this gives $\mathbf{a} * \mathbf{b} \in \Delta_{f,\max(a,b)}$.

2.2.2 Brauer algebras generators and relations

Besides the construction above, we can give the Brauer algebra a presentation by generators and relations. From the previous subsection we know that $\mathfrak{B}_f(x)$ contains a copy of the symmetric group on f elements. Moreover, for any pair of distinct indices $i, j \in \{1, 2, \ldots, f\}$ we define $e_{i,j}$ to be the f-diagram with a bar joining i^+ with j^+, a bar joining i^- with j^-, and one vertical edge joining k^+ with k^- for all $k\{1, 2, \ldots, f\} \setminus \{i, j\}$. By definition, $e_{i,j} \in \Delta_{f,1}$. For instance, $e_{3,6} \in \Delta_{7,1}$ is represented in figure 2.3. The following theorem provides a presentation of Brauer algebras by generators and relations.

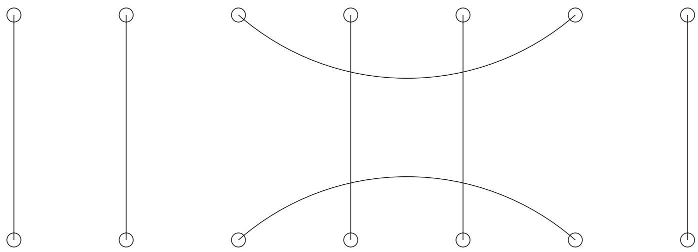


Figure 2.3: Graphical representation for $e_{3,6} \in D_{7,1}$.

Theorem 2.2.1. *$\mathfrak{B}_f(x)$ is the associative $\mathbb{C}$-algebra with generators $\mathbf{d}_\sigma$, which are in bijection with the elements of $\mathfrak{S}_f$, and elements $e_{i,j}$, for all $i, j = 1, 2, \ldots, f$ and $i \neq j$, satisfying the relations*

$$\begin{aligned} e_{i,j} &= e_{j,i} \\ \mathbf{d}_\sigma e_{i,j} \mathbf{d}_{\sigma^{-1}} &= e_{\sigma(i),\sigma(j)} \\ e_{i,j} e_{h,k} &= e_{h,k} e_{i,j} \\ e_{i,j} e_{j,k} &= e_{i,j} \mathbf{d}_{(i\,k)} \\ e_{i,j}^2 &= x\ e_{i,j} \\ e_{i,j} &= e_{i,j} \mathbf{d}_{(i,j)} \\ \mathbf{d}_\sigma\, \mathbf{d}_\tau &= \mathbf{d}_{\sigma\tau} \end{aligned} \tag{2.8}$$

for all $i, j, h, k \in \{1, 2, \ldots, f\}$ such that $\left|\{i, j, h, k\}\right| = 4$ and for all $\sigma, \tau \in \mathfrak{S}_f$.

Similar presentations are also available, which use a proper subset of the generators involved in the previously theorem [85].

Thus the theorem just given above means that $\mathfrak{B}_f(x)$ is generated by $\Delta_{f,0}$ and $\Delta_{f,1}$; even more, since $\Delta_{f,1}$ is a single $\Delta_{f,0}$-orbit (i.e. $\mathfrak{S}_f$-orbit), it is enough to take only one 1-bar f-diagram, so $\mathfrak{B}_f(x)$ is generated, for instance, by $\Delta_{f,0} \bigcup \{e_{1,2}\}$.

In particular, for any $\mathbf{d} \in \Delta_{f,k}$ there exist *unique* $\mathbf{d}_\sigma, \mathbf{d}_\rho \in \Delta_{f,0}$ such that $\mathbf{d} = \mathbf{d}_\sigma\ e_{1,2} \cdots e_{2k-1,2k}\, \mathbf{d}_\rho$; moreover, we can choose such σ and ρ so that they do not invert any of the pairs $(1, 2)$, $(3, 4)$, $\ldots$, $(2k-1, 2k)$. Then given such a factorization of $\mathbf{d}$ we may define the *sign of* $\mathbf{d}$ to be $\varepsilon(\mathbf{d}) = sgn(\sigma) \cdot (-1)^k \cdot sgn(\rho)$.

2.2.3 Schur-Weyl duality

Brauer [14] introduced the algebra $\mathfrak{B}_f(x)$ in 1936 to describe the invariants of orthogonal and symplectic groups acting on $V^{\otimes r}$. (Brauer's convenctions were slightly different; we are here following the approach of Hanlon and Wales [86], who pointed out that $\mathfrak{B}_f(-n)$ is isomorphic with the algebra defined by Brauer to deal with the symplectic case.)

Let G be $O(n)$ or $Sp(n)$, when n is even number. By restricting the action ρ considered in (2.1) we have an action of G on $V^{\otimes r}$. One can extend the action of $\mathfrak{S}_r$ to an action of $\mathfrak{B}_f(\epsilon n)$ (over $\mathbb{C}$) on $V^{\otimes r}$, where $\epsilon = 1$ if $G = O(n)$ and $\epsilon = -1$ if $G = Sp(n)$. To do this, it is enough to specify the action of the diagram $e_{i,j}$. This acts on $V^{\otimes r}$ as one of Weyl's *contraction* maps contracting in tensor positions i and j. So we have (commuting) representations

$$\mathbb{C}G \xrightarrow{\pi} End(V^{\otimes r}) \xleftarrow{\omega} \mathfrak{B}_r(\epsilon n) \tag{2.9}$$

which sotisfy Schur-Weyl duality; i.e., the image of each representation equals the full centralizer algebra of the other action

$$\pi(\mathbb{C}G) = End_{\mathfrak{B}_r(\epsilon n)}(V^{\otimes r}) \tag{2.10}$$

$$\omega(\mathfrak{B}_r(\epsilon n)) = End_G(V^{\otimes r}). \tag{2.11}$$

The algebras in equation (2.10) are the orthogonal and the symplectic Schur algebras [87, 88, 89].

If $n \geq r - 1$ the rapresentation π in (2.9) is faithful [90]; thus it induces an isomorphism $\mathfrak{B}_r(\epsilon n) \simeq End_G(V^{\otimes r})$.

2.2.4 Schur-Weyl duality in type D

In type $D_{n/2}$ (n even) the orthonormal group $O(n)$ is not connected, and contains the connected semisimple group $SO(n)$ (special orthogonal group) as subgroup of index 2. In order to handle this situation Brauer [91] defined a larger algebra $\mathfrak{D}_r(n)$ spanned by the usual r-diagrams previously defined, together with certain *partial* r-diagrams on $2r$ vertices and $r - n$ edges, in which n vertices in each of the top and bottom rows are not incident to any edge, and showed that te action of $\mathfrak{B}_r(n)$ can be extended to an action of the larger algebra $\mathfrak{D}_r(n)$ on $V^{\otimes r}$. Thus we have rapresentations

$$\mathbb{C}SO(n) \xrightarrow{\pi} End(V^{\otimes r}) \xleftarrow{\omega} \mathfrak{D}_r(\epsilon n). \tag{2.12}$$

Brauer showed that the actions of $SO(n)$ and $\mathfrak{D}_r(n)$ on $V^{\otimes r}$ satisfy Schur-Weyl duality:

$$\pi(\mathbb{C}SO(n)) = End_{\mathfrak{D}_r(\epsilon n)}(V^{\otimes r}) \tag{2.13}$$

$$\omega(\mathfrak{D}_r(\epsilon n)) = End_{SO(n)}(V^{\otimes r}). \tag{2.14}$$

The algebra (2.13) is a second Schur algebra in type D, a proper subalgebra of the algebra $End_{\mathfrak{B}_r(n)}(V^{\otimes r})$ apparing in (2.10) above.

2.2.5 The quantum case

There is a q-version of the Schur-Weyl duality considered in this section, although not as developed as in type A. One needs to replace the Brauer algebra by its q-analogue, the Birman-Murakami-Wenzl (BMW) algebra [92, 93], and replace the enveloping algebra by a suitable quantized enveloping algebra. One can think of the BMW algebra in terms of Kauffman's tangle monoid [94, 95]. (Roughly speaking, tangles are replacements for Brauer diagrams, in which one keeps track of under and over crossing, subject to certain natural relations.)

This leads to a q-analogue of the symplectic Schur algebras, in particular, which have been studied by Oehms [96]. A q-analogue of the larger algebra $\mathfrak{D}_r(n)$ (n even) considered in the previous subsection remains to be formulated.

2.3 The subduction problem and Racah coefficients

2.3.1 Reduction coefficients

Let A be an algebra and $H \subset A$ a subalgebra. Given an irreducible representation ρ of A, we have that it is, in general, a reducible representation for H. Finding which irreducible representations of H are contained in ρ is an outstanding issue in Representation Theory. It is frequently named *reduction problem* and denoted by

$$\rho \downarrow H = \bigoplus_{\nu} \{\rho;\nu\}\ \nu = \bigoplus_{\nu}\ \rho \downarrow \nu, \tag{2.15}$$

where ν is an irreducible representation of H and $\{\rho;\nu\}$ (sometimes called *Clebsch-Gordan series* for the reduction) counts the number of times that ν appears in ρ (i.e. the *multiplicity* for $\rho \downarrow \nu$).

Fixed a *standard base* for ρ, i.e. $\{|\rho;r\rangle\}$ (where r is a suitable labelling scheme for the base vectors), normally associated to a Gelfand-Tzetlin chain of G, the base which reduces ρ in a block diagonal form is called *non-standard base* and denoted by $\{|\nu;n\rangle\}$. The matrix transforming between non-standard and standard basis defined by

$$|\nu;n\rangle_\eta = \sum_r \langle\rho;r|\nu;n\rangle_\eta\ \ |\rho;r\rangle, \tag{2.16}$$

where η is a multiplicity label, is called *reduction metrix* and $\langle\rho;r|\nu;n\rangle_\eta$ are the *reduction coefficients.*

Obviously, the previous definitions can also be given for groups and subgroups.

2.3.2 Subduction coefficients

Subduction coefficients are reduction coefficients for certain algebras and subalgebras or groups and subgroups. In particular,

- reduction coefficients for the symmetric group algebras $\mathbb{C}\mathfrak{S}_n$ and the group subalgebras $\mathbb{C}(\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$, with $n_1 + n_2 = n$, are called *subduction coefficients of symmetric groups*;

- reduction coefficients for the Brauer algebras $\mathfrak{B}_f(x)$ and the subalgebras given by $\mathfrak{B}_{f_1}(x)\times\mathfrak{B}_{f_2}(x)$, with $f_1+f_2 = f$, are called *subduction coefficients of Brauer algebras*;

- reduction coefficients for the quantum deformation algebras of $\mathbb{C}\mathfrak{S}_n$, $\mathbf{H}(\mathfrak{S}_n)$, and the quantum subalgebras given by $\mathbf{H}(\mathfrak{S}_{n_1}) \times \mathbf{H}(\mathfrak{S}_{n_2})$, with $n_1 + n_2 = n$, are called *subduction coefficients of type A Iwahori-Hecke algebras*;

- reduction coefficients for the quantum deformation algebras of $B_f(x)$, $\mathbf{BW}(\mathfrak{B}_f(x))$, and the quantum subalgebras given by $\mathbf{BW}(\mathfrak{B}_{f_1}(x))\times\mathbf{BW}(\mathfrak{B}_{f_2}(x))$, with $f_1+f_2=f$, are called *subduction coefficients of Birman-Wenzl algebras.*

2.3.3 Racah coefficients

Schur-Weyl duality provides a link between the Racah coefficients of a classical Lie group G and the subduction coefficients for the centralizer algebra of G [16]. For example, if G is the orthogonal group $O(n)$ or the symplectic group $Sp(2m)$, the corresponding centralizer algebras are quotients of Brauer's $\mathfrak{B}_f(n)$ or $\mathfrak{B}_f(-2m)$, respectively. A special class of Young diagrams (usually introduced to describe the corresponding Gelfand-Tzetlin patterns) is necessary, as defined in [97]. Denoted by $P_\mu(n)$ the dimension of the irreducible representation μ of $O(n)$, a Young diagram λ is said to be *n-pemissible* if $P_\mu(n)\neq 0$ for all subdiagrams $\mu\subseteq\lambda$, where the subdiagrams μ can be obtained from λ by taking away appropriate boxes. If n is an integer, λ is n-permissible if and only if:

1. its first two columns contain at most n boxes, for n positive;
2. it contains at most m columns, for $n=-2m$ a negative even integer;
3. its first two rows contain at most $2-n$, for n odd and negative.

If these conditions are satisfied, $\mathfrak{B}_f(n)$ is isomorphic to the centralizer algebra of $O(n)$ for n positive, to the centralizer algebras of $O(2-n)$ for n negative and odd, and to the centralizer algebras of $Sp(2m)$ for $n=-2m<0$. In the following, we always assume that all irreps are n-permissible with $n\leq f-1$ for $n>0$ or $-n\leq f-1$ for negative n. The latter condition implies that $\mathfrak{B}_f(n)$ is to be considered semisimple [98].

Hence, an irrep of the centralizer algebra of $O(n)$ or $Sp(2m)$ is the same irrep of $O(n)$ or $Sp(2m)$. But the labelling schemes of the centralizer algebras of G and G are different. The former is labelled by its Brauer algebra indices, while the latter is labelled by its tensor components.

The invariants for the centralizer algebras of G defined by

$$U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{t_{12}t}_{t_{23}t'} = \sum_{r_{12}r_{23}r} \langle\lambda;r|\lambda_{12},\lambda_3;r_{12},r_3\rangle_t \ \langle\lambda_{12};r_{12}|\lambda_1,\lambda_2;r_1,r_2\rangle_{t_{12}} \cdot \langle\lambda;r|\lambda_1,\lambda_{23};r_1,r_{23}\rangle_{t'} \ \langle\lambda_{23};r_{23}|\lambda_2,\lambda_3;r_2,r_3\rangle_{t_{23}}, \quad (2.17)$$

where $\langle\lambda;r|\lambda_1,\lambda_2;r_1,r_2\rangle_\eta$ is the subduction coefficient of the Brauer algebras given by the irreps $[\lambda]$ of $\mathfrak{B}_f(x)$ and $[\lambda_1]\otimes[\lambda_2]$ of $\mathfrak{B}_{f_1}(x)\times\mathfrak{B}_{f_2}(x)$ (r, r_1, and r_2 being suitable labelling schemes and η the multiplicity label), only depend on the irreps $[\lambda_1]$, $[\lambda_2]$, $[\lambda_3]$, $[\lambda]$, $[\lambda_{12}]$, $[\lambda_{23}]$ and does not depend on the other indices. So, the summation in (2.17) can be carried out under fixed r_1, r_2, and r_3. According to the Schur-Weyl duality relation, (2.17) are also the U coefficients of the group G satisfying the unitary conditions given in the previous chapter. One can thus use (2.17) to caluculate Racah coefficients of $O(n)$ and $Sp(2m)$ from subduction coefficents of Brauer algebras $\mathfrak{B}_f(n)$.

We remark that, because $O(n)\subset U(n)$, the centralizer algebras $\mathfrak{B}_f(n)$ include the group algebras $\mathbb{C}\mathfrak{S}_n$. Therefore, the subduction coefficients of Brauer algebras are the same as those of symmetric groups when there are no trace contractions in the irreps. This implies that the Racah coefficients for $U(n)$ groups are also Racah coefficients for the corresponding irreps of $O(n)$ groups. On the other hand, by solving the general problem of finding subduction coefficients of Brauer algebras, we also solve the problem of finding Racah coefficients for all classical Lie groups.

Chapter 3

A systematic approach to the subduction problem in $\mathfrak{S}_f$ groups

In this chapter, we develop a combinatoric approach to the subduction problem in symmetric groups and we analyze in detail the *linear equation method.* In section 1 we provide some background on the subduction problem for symmetric groups and in section 2 we describe the linear equation method, giving the general structure of the resulting equation system (*subduction matrix*). In section 3 we introduce the *subduction graph* and in section 4 we relate it to the subduction matrix. The graph provides a graphic description of a minimal set of equations which are sufficient to obtain the trasformation coefficients. We find the solution space as an intersection of suitable linear subspaces of $\mathbb{R}^{f^\lambda} \otimes \mathbb{R}^{f^{\lambda_1} f^{\lambda_2}}$, where f^λ, f^{λ_1} and f^{λ_2} are the dimensions of the irreducible representations involved in the subduction $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$. Finally, in section 5, we give the general orthonormalized form for the coefficients and we discuss the choice of phases and free factors governing multiplicity separations.

3.1 The subduction problem in symmetric groups

The irreducible representations (irreps) of the symmetric group $\mathfrak{S}_n$ may be labelled by partitions $[\lambda]$ of n, i.e. sequences $[\lambda_1, \lambda_2, \ldots, \lambda_h]$ of positive integers such that $\sum_{i=1}^{h} \lambda_i =$

n and the λ_i are weakly decreasing. A partition $[\lambda]$ is usually represented by a Ferrers diagram (or Young diagram) obtained from a left-justified array with λ_j boxes on the jth row and with the kth row below the $(k-1)$th row. Standard Young tableaux are generated by filling the Ferrers diagram with the numbers $1,\ldots,n$ in such a way that each number appears exactly once and the numbers are strictly increasing along the rows and down the columns. An orthonormal basis vector of an irrep associated to the partition $[\lambda]$ may be labelled by a standard Young tableau. Such a basis corresponds to the Gelfand-Tzetlin chain $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \ldots \subset \mathfrak{S}_n$ and is usually called the *standard basis* of $[\lambda]$. We denote this basis by $\mathfrak{S}_n$*-basis* [21].

An alternative orthonormal basis for $[\lambda]$ is the *split basis*, denoted by $\mathfrak{S}_n - \mathfrak{S}_{n_1,n_2}$-*basis* [21], with $n_1 + n_2 = n$. By definition, such a basis breaks $[\lambda]$ (which is, in general, a reducible representation of the direct product subgroup $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$) in a block-diagonal form:

$$[\lambda] = \bigoplus_{\lambda_1,\lambda_2} \{\lambda;\lambda_1,\lambda_2\}\ [\lambda_1] \otimes [\lambda_2], \tag{3.1}$$

where $[\lambda_1]$ and $[\lambda_2]$ are irreps of $\mathfrak{S}_{n_1}$ and $\mathfrak{S}_{n_2}$ respectively, and $\{\lambda;\lambda_1,\lambda_2\}$, the Clebsch-Gordan series, counts the number of times (*multiplicity*) that the irrep $[\lambda_1]\otimes[\lambda_2]$ of $S_{n_1} \times S_{n_2}$ appears in the decomposition of $[\lambda]$.

The irreps of the subgroup $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ may be labelled by pairs (α, β) of Ferrers diagrams, with α corresponding to an irrep of $\mathfrak{S}_{n_1}$ and β to an irrep of $\mathfrak{S}_{n_2}$. In the same way, each element of the basis is labelled by pairs of standard Young tableaux.

Because the symmetric group $\mathfrak{S}_n$ of n elements is generated by the $n-1$ transpositions g_i each one interchanges the elements i and $i+1$, it is useful the following definition. Given a standard Young tableau m, we define the action $g_i(m)$ in the following way: if the tableau obtained from m interchanging the box with i and the box with $i+1$ (keeping the other elements fixed) is another standard Young tableau $m^{(i)}$, we set $g_i(m) = m^{(i)}$; else $g_i(m) = m$.

The g_i acts on the standard basis vectors $|\lambda; m\rangle$ of the irrep $[\lambda]$ as follows [18]:

$$g_i|\lambda;m\rangle = \begin{cases} \frac{1}{d_i(m)}|\lambda;m\rangle + \sqrt{1-\frac{1}{d_i(m)^2}}\ |\lambda;g_i(m)\rangle & \text{if } g_i(m) \neq m \\ |\lambda;m\rangle & \text{if } g_i(m) = m \end{cases}, \tag{3.2}$$

where $d_i(m)$ is the usual *axial distance* from i to $i+1$ in the standard Young tableau m [101].

The explicit action of the generators g_i ($i \neq n_1$ because g_{n_1} is not a generator of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$) on the elements of the $\mathfrak{S}_n - \mathfrak{S}_{n_1,n_2}$-basis directly follows from (3.2). In fact we have

$$g_i|\lambda_1, \lambda_2; m_1, m_2\rangle = \begin{cases} (g_i|\lambda_1; m_1\rangle) \otimes |\lambda_2; m_2\rangle & \text{if } 1 \leq i \leq n_1 - 1 \\ |\lambda_1; m_1\rangle \otimes (g_i|\lambda_2; m_2\rangle) & \text{if } n_1 + 1 \leq i \leq n - 1 \end{cases} . \tag{3.3}$$

Then, from (3.2) applied to the standard basis vectors of $[\lambda_1]$ and $[\lambda_2]$ respectively, we have the action of the generators of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ on the basis vectors $|\lambda_1; m_1\rangle \otimes |\lambda_2; m_2\rangle$.

The *subduction coefficients* (SDCs) are the entries of the matrix transforming between split and standard basis. Let $[\lambda_1] \otimes [\lambda_2]$ be a *fixed* irrep of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ in $[\lambda] \downarrow \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ and $|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ a generic vector of the split basis (where m_1 and m_2 are standard Young tableaux with Ferrers diagram λ_1 and λ_2 respectively, and η is the multiplicity label). We may expand such vectors in terms of the standard basis vectors $|\lambda; m\rangle$ of $[\lambda]$:

$$|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta = \sum_m |\lambda; m\rangle\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta. \tag{3.4}$$

Thus $\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ are the SDCs of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$ with given multiplicity label η.

Because the standard and the split basis vectors are orthogonal, the SDCs satisfy the following unitary conditions

$$\sum_m \langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta \ \langle\lambda; m|\lambda_1, \lambda_2'; m_1, m_2'\rangle_{\eta'} = \delta_{\lambda_2\lambda_2'}\delta_{m_2m_2'}\delta_{\eta\eta'} \tag{3.5}$$

$$\sum_{\lambda_2 m_2 \eta} \langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta \ \langle\lambda; m'|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta = \delta_{mm'}. \tag{3.6}$$

Notice that in (3.5) we impose orthonormality between two different copies of multiplicity. It is not necessary, but it is the most natural choice. On the other hand, it imposes a precise and explicit form for the SDCs (see section 3.5).

3.2 The linear equation method

3.2.1 Subduction matrix and subduction space

Using the linear equation method proposed by Chen and Pan [18] for Hecke algebras we may construct a matrix in such a way that the SDCs are the components of the kernel basis vectors.

From (3.3), for $l \in \{1, 2, \ldots, n_1 - 1\}$, we get

$$\langle \lambda; m|g_l|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle \lambda; m|(g_l|\lambda_1; m_1\rangle) \otimes |\lambda_2; m_2\rangle \tag{3.7}$$

and, writing $|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ and $g_l|\lambda_1; m_1\rangle$ in the $\mathfrak{S}_n$-basis and $\mathfrak{S}_{n_1}$-basis respectively, (3.7) becomes

$$\sum_p \langle \lambda; m|g_l|\lambda; p\rangle\langle \lambda; p|\lambda_1, \lambda_2; m_1, m_2\rangle = \sum_q \langle \lambda_1; q|g_l|\lambda_1; m_1\rangle\langle \lambda; m|\lambda_1, \lambda_2; q, m_2\rangle. \tag{3.8}$$

In an analogous way, for $l \in \{n_1 + 1, n_1 + 2, \ldots, n - 1\}$, we get

$$\sum_p \langle \lambda; m|g_l|\lambda; p\rangle\langle \lambda; p|\lambda_1, \lambda_2; m_1, m_2\rangle = \sum_q \langle \lambda_2; q|g_l|\lambda_2; m_2\rangle\langle \lambda; m|\lambda_1, \lambda_2; m_1, q\rangle. \tag{3.9}$$

Then, once we know the explicit action of the generators of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ on the standard basis, (3.8) and (3.9) (written for $l \in \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n - 1\}$ and all standard Young tableaux m, m_1, m_2 with Ferrers diagrams λ, λ_1 and λ_2 respectively) define a linear equation system of the form:

$$\Omega(\lambda; \lambda_1, \lambda_2)\ \chi = 0 \tag{3.10}$$

where $\Omega(\lambda; \lambda_1, \lambda_2)$ is the *subduction matrix* and χ is a vector with components given by the SDCs of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$. We call the space of the solutions of (3.10), i.e. ker $\Omega(\lambda; \lambda_1, \lambda_2)$, *subduction space*.

3.2.2 Explicit form for the subduction matrix

Denoting by f^λ, f^{λ_1} and f^{λ_2} the dimensions of the irreps $[\lambda]$, $[\lambda_1]$ and $[\lambda_2]$ respectively, (3.10) is a linear equation system with $f^\lambda f^{\lambda_1} f^{\lambda_2}$ unknowns (the SDCs) and

$(n-2)f^{\lambda}f^{\lambda_1}f^{\lambda_2}$ equations. Thus $\Omega(\lambda;\lambda_1,\lambda_2)$ is a rectangular $(n-2)f^{\lambda}f^{\lambda_1}f^{\lambda_2}\times f^{\lambda}f^{\lambda_1}f^{\lambda_2}$ matrix with real entries. Using the explicit action of g_i given by (3.2), we see that all equations of (3.10) have the form

$$\begin{aligned}
&\alpha^{(i)}_{m,m_{12}}\langle\lambda;m|\lambda_1,\lambda_2;m_1,m_2\rangle-\beta^{(i)}_m\langle\lambda;g_i(m)|\lambda_1,\lambda_2;m_1,m_2\rangle+\\
&+\beta^{(i)}_{m_{12}}\langle\lambda;m|\lambda_1,\lambda_2;g_i(m_1),m_2\rangle=0 \qquad \text{if } i\in\{1,\ldots,n_1-1\}\ , \qquad (3.11)\\
&\alpha^{(i)}_{m,m_{12}}\langle\lambda;m|\lambda_1,\lambda_2;m_1,m_2\rangle-\beta^{(i)}_m\langle\lambda;g_i(m)|\lambda_1,\lambda_2;m_1,m_2\rangle+\\
&+\beta^{(i)}_{m_{12}}\langle\lambda;m|\lambda_1,\lambda_2;m_1,g_i(m_2)\rangle=0 \qquad \text{if } i\in\{n_1+1,\ldots,n-1\} \qquad (3.12)
\end{aligned}$$

where

$$\alpha^{(i)}_{m,m_{12}}=\frac{1}{d_i(m_{12})}-\frac{1}{d_i(m)} \tag{3.13}$$

$$\beta^{(i)}_m=\sqrt{1-\frac{1}{d_i^2(m)}} \tag{3.14}$$

$$\beta^{(i)}_{m_{12}}=\sqrt{1-\frac{1}{d_i^2(m_{12})}} \tag{3.15}$$

Notice that, by definition,

$$d_i(m_{12})=\begin{cases} d_i(m_1) & \text{if } i<n_1\\ d_i(m_2) & \text{if } i>n_1\end{cases}\ . \tag{3.16}$$

3.3 Subduction graph

Given two standard Young tableaux m_1 and m_2 with the same Ferrers diagram, we say that they are *i-coupled* if $m_1=m_2$ or if $m_1=g_i(m_2)$.

If $m_{12}=(m_1,m_2)$ is a pair of standard Young tableaux with k_1 and k_2 boxes respectively, where m_1 is filled by integers from 1 to k_1 and m_2 from k_1+1 to k_1+k_2, we define

$$g_i(m_{12})=\begin{cases}(g_i(m_1),m_2) & \text{if } i<k_1\\ (m_1,g_i(m_2)) & \text{if } i>k_1\end{cases} \tag{3.17}$$

(note that the action is not defined for $i=k_1$ because g_{k_1} is not a generator of $\mathfrak{S}_{k_1}\times\mathfrak{S}_{k_2}$). Thus, denoting as m_{34} another pair (m_3,m_4), we say that m_{12} and m_{34} are *i-coupled* if $m_{12}=m_{34}$ or if $g_i(m_{12})=m_{34}$.

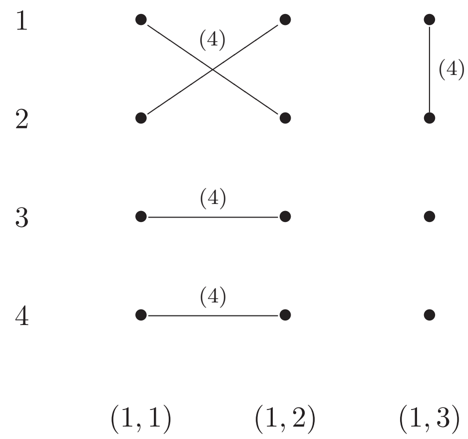


Figure 3.1: 4-layer relative to the partitions $([4,1];[1],[3,1])$. Nodes have coordinates given by the lexicografic ordering for Young tableaux with Ferrer diagram $[4,1]$ and for pairs of Young tableaux with Ferrer diagram $([1],[3,1])$. Two distinct 4-coupled nodes are joined by an edge.

Let us now consider the three partitions $(\lambda;\lambda_1,\lambda_2)$ of k, k_1 and k_2 respectively, with $k_1+k_2=k$. We call *node* each ordered sequence of three standard Young tableaux $(m;m_1,m_2)$ with Ferrers diagrams λ, λ_1 and λ_2 respectively and filled as described in the previous section. We denote it as $\langle m;m_{12}\rangle$.

The set of all nodes of $(\lambda;\lambda_1,\lambda_2)$ is called *subduction grid* (or simply *grid*). In analogy with the case of standard Young tableaux, we may define the action of g_i on a node $n=\langle m;m_{12}\rangle$ as

$$g_i(n)=\langle g_i(m);g_i(m_{12})\rangle. \tag{3.18}$$

Then we say that two nodes n_1 and n_2 are *i-coupled* if $n_1=n_2$ or if $n_1=g_i(n_2)$. Once i is fixed, it is easy to see that the i-coupling is an equivalence relation on the grid. Furthermore there are only four possible *coupling configurations* between nodes:

1. one node $n=\langle m;m_{12}\rangle$ is called *singlet* if $m=g_i(m)$ and if $m_{12}=g_i(m_{12})$;

2. two distinct i-coupled nodes $n=\langle m;m_{12}\rangle$ and $n'=\langle m';m'_{12}\rangle$ are called *vertical bridge* if $m_{12}=m'_{12}$;

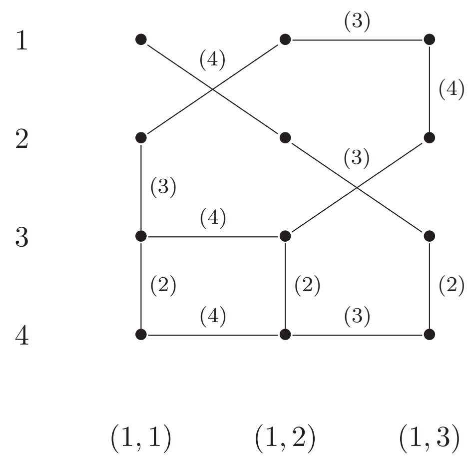


Figure 3.2: Subduction graph relative to $([4,1];[1],[3,1])$. It is obtained by the overlap of the 2-layer, 3-layer and 4-layer. Each i-layer can be distinguished by the label (i) on the edges.

3. two distinct i-coupled nodes $n = \langle m; m_{12}\rangle$ and $n' = \langle m'; m'_{12}\rangle$ are called *horizontal bridge* if $m = m'$;

4. four distinct nodes $n = \langle m; m_{12}\rangle$, $n' = \langle m'; m'_{12}\rangle$, $n'' = \langle m''; m''_{12}\rangle$ and $n''' = \langle m'''; m'''_{12}\rangle$ such that $n = g_i(n')$ and $n'' = g_i(n''')$ are called *crossing* if $m \neq m'$, $m_{12} \neq m'_{12}$, $m'' \neq m''$ and $m''_{12} \neq m'''_{12}$.

The partition of the grid related to the i-coupling relation is called *i-layer*. For each configuration it can be convenient to choose a representative node which we call *pole*. Given a pole p we denote by $\Gamma^{(i)}(p)$ the set of all nodes in its coupling configuration. For example, in figure 3.1 we show a graphic representation of the 4-layer for $([4,1];[1],[3,1])$. The nodes form a grid and their coordinates are obtained by the ordering number of the relative standard Young tableau (for example the lexicographic ordering [99]). Because each equivalence class is composed at most by two distinct nodes, we represent them as joined by an edge with a label for i. By convention, we choose the node on the top and left of the configuration as pole. We can see that $\{\langle 1;1,1\rangle, \langle 2;1,2\rangle, \langle 1;1,2\rangle, \langle 2;1,1\rangle\}$ is a crossing, $\{\langle 1;1,3\rangle, \langle 2;1,3\rangle\}$ is a vertical bridge, $\{\langle 3;1,1\rangle, \langle 3;1,2\rangle\}$ is an example of horizontal bridge and $\{\langle 2;1,3\rangle\}$ a singlet one.

We call *subduction graph* relative to $(\lambda;\lambda_1,\lambda_2)$ the *overlap* of all i-slides (by overlap between two graphs we mean the graph obtained by identification of the corresponding nodes). More simply, two distinct nodes n and n' of the grid are connected by an edge of the subduction graph if $n = g_i(n')$ for some i (notice that if n and n' are i-coupled and j-coupled, then $i = j$). In figure 3.2 the subduction graph for $([4,1];[1],[3,1])$ obtained from the overlap of the 2-layer, the 3-layer and the 4-layer is shown.

3.4 Configurations and solutions

The solution of (3.10) can be seen as an intersection of the $n-2$ subspaces of $\mathbb{R}^{f^\lambda f^{\lambda_1} f^{\lambda_2}}$ described by

$$\Omega^{(i)}(\lambda;\lambda_1,\lambda_2)\chi = 0, \tag{3.19}$$

with $i \in \{1,\dots,n_1-1,n_1+1,\dots,n-1\}$. We now construct an explicit solution of (3.19), for a fixed i, by using the concept of i-layer.

It is clear that we can associate each SDC of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$ to a node of $(\lambda;\lambda_1,\lambda_2)$ in a one-to-one correspondence. Supposed $p = \langle m; m_{12}\rangle$ as a fixed pole of a crossing configuration and $\Gamma^{(i)}(p)$ the set of all nodes of such a configuration, the solutions of the equations (3.19), written for each $n \in \Gamma^{(i)}(p)$, are the kernel vectors of the matrix

$$\Omega^{(i)}_{m;m_{12}} = \begin{pmatrix} \alpha^{(i)}_{m,m_{12}} & -\beta^{(i)}_m & \beta^{(i)}_{m_{12}} & 0 \\ -\beta^{(i)}_{g_i(m)} & \alpha^{(i)}_{g_i(m),m_{12}} & 0 & \beta^{(i)}_{m_{12}} \\ \beta^{(i)}_{g_i(m_{12})} & 0 & \alpha^{(i)}_{m,g_i(m_{12})} & -\beta^{(i)}_m \\ 0 & \beta^{(i)}_{g_i(m_{12})} & -\beta^{(i)}_{g_i(m)} & \alpha^{(i)}_{g_i(m),g_i(m_{12})} \end{pmatrix}, \tag{3.20}$$

where the following relations hold:

$$\begin{array}{cc} \alpha^{(i)}_{m,m_{12}} = -\alpha^{(i)}_{g_i(m),g_i(m_{12})}, & \alpha^{(i)}_{g_i(m),m_{12}} = -\alpha^{(i)}_{m,g_i(m_{12})}, \\ \beta^{(i)}_m = \beta^{(i)}_{g_i(m)}, & \beta^{(i)}_{m_{12}} = \beta^{(i)}_{g_i(m_{12})} \end{array} \tag{3.21}$$

(they directly discend from $d_i(m) = -d_i(g_i(m))$ and $d_i(m_{12}) = -d_i(g_i(m_{12}))$). If we put

$$\rho^{(i)}_m = \begin{pmatrix} \cos\theta^{(i)}_m & \sin\theta^{(i)}_m \\ \sin\theta^{(i)}_m & -\cos\theta^{(i)}_m \end{pmatrix}, \qquad \cos\theta^{(i)}_m = \frac{1}{d_i(m)}, \quad \sin\theta^{(i)}_m = \beta^{(i)}_m, \tag{3.22}$$

$$\rho^{(i)}_{m_{12}} = \begin{pmatrix} \cos\theta^{(i)}_{m_{12}} & \sin\theta^{(i)}_{m_{12}} \\ \sin\theta^{(i)}_{m_{12}} & -\cos\theta^{(i)}_{m_{12}} \end{pmatrix}, \qquad \cos\theta^{(i)}_{m_{12}} = \frac{1}{d_i(m_{12})}, \quad \sin\theta^{(i)}_{m_{12}} = \beta^{(i)}_{m_{12}}, \tag{3.23}$$

and we remember (3.21), then (3.20) can be written as

$$\Omega^{(i)}_{m,m_{12}} = I \otimes \rho^{(i)}_{m_{12}} - \rho^{(i)}_{m} \otimes I, \tag{3.24}$$

where I denotes the 2×2 identity matrix. It is straightforward that the kernel of $\Omega^{(i)}_{m,m_{12}}$ is generated by the vectors $e^{(i)}_m \otimes e^{(i)}_{m_{12}}$ and $\bar{e}^{(i)}_m \otimes \bar{e}^{(i)}_{m_{12}}$; here $e^{(i)}_m$ and $e^{(i)}_{m_{12}}$ are the eigenvectors of $\rho^{(i)}_m$ and $\rho^{(i)}_{m_{12}}$ respectively with eigenvalue 1, while $\bar{e}^{(i)}_m$ and $\bar{e}^{(i)}_{m_{12}}$ are the corresponding ones with eigenvalue -1; from (3.22) and (3.23) we get

$$e^{(i)}_m = \begin{pmatrix} \cos\frac{\theta^{(i)}_m}{2} \\ \sin\frac{\theta^{(i)}_m}{2} \end{pmatrix}, \qquad e^{(i)}_{m_{12}} = \begin{pmatrix} \cos\frac{\theta^{(i)}_{m_{12}}}{2} \\ \sin\frac{\theta^{(i)}_{m_{12}}}{2} \end{pmatrix}, \tag{3.25}$$

and

$$\bar{e}^{(i)}_m = \begin{pmatrix} -\sin\frac{\theta^{(i)}_m}{2} \\ \cos\frac{\theta^{(i)}_m}{2} \end{pmatrix}, \qquad \bar{e}^{(i)}_{m_{12}} = \begin{pmatrix} -\sin\frac{\theta^{(i)}_{m_{12}}}{2} \\ \cos\frac{\theta^{(i)}_{m_{12}}}{2} \end{pmatrix} \tag{3.26}$$

In the case of vertical bridge configuration, we have $\beta^{(i)}_{m_{12}} = 0$ in (3.20). Therefore we can write

$$\Omega^{(i)}_{m,m_{12}} = (d_i(m_{12})I - \rho^{(i)}_m) \oplus (d_i(m_{12})I - \rho^{(i)}_m). \tag{3.27}$$

From $m_{12} = g_i(m_{12})$ it follows that we may only consider one of the two identical copies, thus

$$\Omega^{(i)}_{m,m_{12}} = d_i(m_{12})I - \rho^{(i)}_m. \tag{3.28}$$

So, $\ker\Omega^{(i)}_{m,m_{12}}$ is generated by the eigenvector $e^{(i)}_m$ if $d_i(m_{12}) = 1$, by the eigenvector $\bar{e}^{(i)}_m$ if $d_i(m_{12}) = -1$.

In an analogous way for a horizontal bridge we have $\beta^{(i)}_m = 0$ in (3.20). By the change of basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.29}$$

and using $m = g_i(m)$, we get

$$\Omega^{(i)}_{m,m_{12}} = \rho^{(i)}_{m_{12}} - d_i(m)I. \tag{3.30}$$

Here $\ker \Omega^{(i)}_{m,m_{12}}$ is generated by the eigenvector $e^{(i)}_{m_{12}}$ if $d_i(m) = 1$, by $\bar{e}^{(i)}_{m_{12}}$ if $d_i(m) = -1$.

Finally, the case of singlet configuration is trivial because $\Omega^{(i)}_{m,m_{12}}$ is in diagonal form (both $\beta^{(i)}_m$ and $\beta^{(i)}_{m_{12}}$ are 0). We can have two possibilities:

$$\Omega^{(i)}_{m,m_{12}} = (0) \tag{3.31}$$

or

$$\Omega^{(i)}_{m,m_{12}} = (\pm 2). \tag{3.32}$$

The kernel is the one-dimensional space generated by the vector $\{1\}$ or it is the trivial space.

All these results are summarized in table 3.1, where with we deal with the various configurations, the coefficients of the linear subduction equations, their Ω *matrices* and the solution for the kernel vectors. Note that, for the crossing configuration we distinguish the case $\alpha_{m;m_{12}} \neq 0$ from the case $\alpha_{m;m_{12}} = 0$. In the latter case we draw one of the edges with a dashed line. Furthermore, in the singlet configuration, we mark the trivial kernel solution by a label 0 near the node.

3.4.1 Poles and their equivalence

We will now prove that $\Omega^{(i)}_n$, with $n \in \Gamma^{(i)}(p)$, are equivalent up to change of basis that exchanges the nodes of the configuration. In this way, only the equations relative to one node of the configuration (the pole) are needed in the subduction system.

Let us consider the crossing configuration. We first notice that

$$\rho^{(i)}_{g_i(m)} = \epsilon\rho^{(i)}_m\epsilon, \quad \rho^{(i)}_{g_i(m_{12})} = \epsilon\rho^{(i)}_{m_{12}}\epsilon, \tag{3.33}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, observing that $\epsilon^2 = I$, for the other three choices of pole we have

$$\begin{aligned} \Omega^{(i)}_{g_i(m),g_i(m_{12})} &= I \otimes \rho^{(i)}_{g_i(m_{12})} - \rho^{(i)}_{g_i(m)} \otimes I = \\ = I \otimes \epsilon\rho^{(i)}_{m_{12}}\epsilon - \epsilon\rho^{(i)}_m\epsilon \otimes I &= (\epsilon\otimes\epsilon)(I \otimes \rho^{(i)}_{m_{12}} - \rho^{(i)}_m \otimes I)(\epsilon\otimes\epsilon) = \\ &= (\epsilon\otimes\epsilon)\Omega^{(i)}_{m,m_{12}}(\epsilon\otimes\epsilon); \end{aligned} \tag{3.34}$$

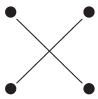
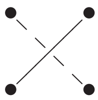
Config.	$\alpha_{m;m_{12}}$	β_m	$\beta_{m_{12}}$	$\Omega_{m;m_{12}}$	Basis
Crossing					
	$\neq 0$	$\neq 0$	$\neq 0$	$I \otimes \rho_{m_{12}} +$ $- \rho_m \otimes I$	$e_m \otimes e_{m_{12}}$ $\bar{e}_m \otimes \bar{e}_{m_{12}}$
	0	$\beta \neq 0$	$\beta \neq 0$	$I \otimes \rho +$ $- \rho \otimes I$	$e \otimes e$ $\bar{e} \otimes \bar{e}$
V. Bridge					
	$\neq 0$	$\neq 0$	0	$I - \rho_m$ $-I - \rho_m$	e_m $\bar{e}_m$
H. Bridge					
	$\neq 0$	0	$\neq 0$	$\rho_{m_{12}} - I$ $\rho_{m_{12}} + I$	$e_{m_{12}}$ $\bar{e}_{m_{12}}$
Singlet					
$\bullet$	0	0	0	(0)	1
$\bullet_0$	± 2	0	0	(± 2)	-

Table 3.1: Fundamental i-coupling configurations, Ω matrices and solution space bases.

$$\begin{gathered}\Omega^{(i)}_{m,g_i(m_{12})} = I \otimes \rho^{(i)}_{g_i(m_{12})} - \rho^{(i)}_m \otimes I = \\ = I \otimes \epsilon\rho^{(i)}_{m_{12}}\epsilon - \rho^{(i)}_m \otimes I = (I \otimes \epsilon)(I \otimes \rho^{(i)}_{m_{12}} - \rho^{(i)}_m \otimes I)(I \otimes \epsilon) = \\ = (I \otimes \epsilon)\Omega^{(i)}_{m,m_{12}}(I \otimes \epsilon);\end{gathered} \tag{3.35}$$

$$\begin{gathered}\Omega^{(i)}_{g_i(m),m_{12}} = I \otimes \rho^{(i)}_{m_{12}} - \rho^{(i)}_{g_i(m)} \otimes I = \\ = I \otimes \rho^{(i)}_{m_{12}} - \epsilon\rho^{(i)}_m\epsilon \otimes I = (\epsilon \otimes I)(I \otimes \rho^{(i)}_{m_{12}} - \rho^{(i)}_m \otimes I)(\epsilon \otimes I) = \\ = (\epsilon \otimes I)\Omega^{(i)}_{m,m_{12}}(\epsilon \otimes I).\end{gathered} \tag{3.36}$$

In any case we are able to find the suitable change of basis.

Of course, for the bridge configurations the change of pole is equivalent to a change of basis by ϵ. The singlet configuration is a trivial case.

3.4.2 Structure of the subduction space

We can now write the explicit solution space $\chi^{(i)}$ for (3.19) as a suitable subspace of $\mathbb{R}^{f^\lambda} \otimes \mathbb{R}^{f^{\lambda_1} f^{\lambda_2}}$. If we define the vectors (in components)

$$(\lambda^{(i)}_m)_k = \begin{cases} 0 & \text{if } k \text{ is not } i\text{-coupled with } m \\ (e^{(i)}_m)_k & \text{if } k \text{ is } i\text{-coupled with } m \end{cases} \tag{3.37}$$

$$(\bar{\lambda}^{(i)}_m)_k = \begin{cases} 0 & \text{if } k \text{ is not } i\text{-coupled with } m \\ (\bar{e}^{(i)}_m)_k & \text{if } k \text{ is } i\text{-coupled with } m \end{cases} \tag{3.38}$$

$$(\delta_m)_k = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases} \tag{3.39}$$

and the spaces

$$\chi^{(i)}_{m;m_{12}} = \begin{cases} \langle \alpha^{(i)}_{m;m_{12}} \delta_m \otimes \delta_{m_{12}} \rangle & \text{if } d_i(m) = \pm 1 \text{ and } d_i(m_{12}) = \pm 1 \\ \langle \lambda^{(i)}_m \otimes \delta_{m_{12}} \rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) = 1 \\ \langle \bar{\lambda}^{(i)}_m \otimes \delta_{m_{12}} \rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) = -1 \\ \langle \delta_m \otimes \lambda^{(i)}_{m_{12}} \rangle & \text{if } d_i(m) = 1 \text{ and } d_i(m_{12}) \neq \pm 1 \\ \langle \delta_m \otimes \bar{\lambda}^{(i)}_{m_{12}} \rangle & \text{if } d_i(m) = -1 \text{ and } d_i(m_{12}) \neq \pm 1 \\ \langle \lambda^{(i)}_m \otimes \lambda^{(i)}_{m_{12}}, \bar{\lambda}^{(i)}_m \otimes \bar{\lambda}^{(i)}_{m_{12}} \rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) \neq \pm 1 \end{cases}, \tag{3.40}$$

denoted by $P^{(i)}$ the set of the poles for the i-layer and observing that the set of the configurations for the i-layer is a partition of the grid, we have

$$\chi^{(i)} = \bigoplus_{\langle m;m_{12}\rangle \in P^{(i)}} \chi^{(i)}_{m;m_{12}}. \tag{3.41}$$

So the general solution of (3.10) is the intersection of $n-2$ subspaces, i.e.

$$\chi = \bigcap_{i\in K} \chi^{(i)}, \tag{3.42}$$

with $K = \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n-1\}$.

Now we can outline an algorithm (in pseudo-code) to determine the SDCs for $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$:

1. for $i \in K$:
   (a) construct the i-layer;
   (b) choose poles;
   (c) for each pole (configuration):
   construct the space $\chi^{(i)}_p$ by (3.40);
   (d) construct $\chi^{(i)}$ by (3.41);
2. determine χ as intersection of all $\chi^{(i)}$.

Step (ii) can be performed by using the subduction graph to obtain a minimal number of equations. In fact, one may associate a suitable equation deriving from (3.42) to each edge (two for the crossing) of the graph (nodes represents the unknown SDCs). Then, starting from a suitable node in the graph, we can extract such equations by applying a graph searching algorithm which is able to reach every edge [100]. In this regard it is useful to notice that equations associated to closed loops of bridge configurations are always linearly dependent.

3.5 Orthonormalization and form

The subduction space given by (3.42) has dimension μ equal to the multiplicity of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$. Then SDCs are not univocally determined. A choice of orthonormality between the different copies of multiplicity imposes a precise form for the multiplicity separations.

Let $\{\chi_1, \ldots, \chi_\mu\}$ be a basis in the subduction space. Orthonormality implies for the scalar products:

$$(\chi_\eta, \chi_{\eta'}) = f^{\lambda_1} f^{\lambda_2}\ \delta_{\eta\eta'}. \tag{3.43}$$

If we denote by χ the matrix which has the basis vectors of the subduction space as columns, we may orthonormalize it by a suitable $\mu \times \mu$ matrix σ, i.e.

$$\tilde{\chi} = \chi\sigma. \tag{3.44}$$

In (3.44) $\tilde{\chi}$ is the matrix which has the orthonormalized basis vectors of the subduction space as columns. Now we can write (3.43) as

$$\sigma^t\ \tau\ \sigma = I, \tag{3.45}$$

where I is the $\mu \times \mu$ identity matrix and τ is the $\mu \times \mu$ positive defined quadratic form given by

$$\tau = \frac{1}{f^{\lambda_1} f^{\lambda_2}}\ \chi^t \chi. \tag{3.46}$$

From (3.45) we can see σ as the Sylvester matrix of τ, i.e. the matrix for the change of basis that reduces τ in the identity form. We can express σ in terms of the orthonormal matrix O_τ that diagonalizes τ

$$\sigma = O_\tau D_\tau^{-\frac{1}{2}} O, \tag{3.47}$$

where $D_\tau^{-\frac{1}{2}}$ is the diagonal matrix with eigenvalues given by the inverse square root of the eigenvalues of τ and O a *generic* orthogonal matrix. Thus, the general form for the orthonormalized χ is

$$\tilde{\chi} = \chi O_\tau D_\tau^{-\frac{1}{2}} O. \tag{3.48}$$

(3.48) suggests some considerations on the form of the SDCs. First we notice that in case of multiplicity-free subduction, only one choice of global phase has to be made (for example Young-Yamanouchi phase convention [99]). It derives from the trivial form of the orthogonal 1×1 matrices O and O_τ.

In the general case of multiplicity $\mu > 1$, $2^{\mu-1}$ phases deriving from the O_τ matrix and 1 phase from the matrix O have to be fixed. Therefore we have $2^{\mu-1} + 1$ phases to choose. Furthermore we have other $\frac{\mu(\mu-1)}{2}$ degrees of freedom deriving from O. In sum we have a total of $(2^{\mu-1} + 1) + \frac{\mu(\mu-1)}{2}$ choices to make. We agree with [21] for the case of multiplicity 2, in which we need three phases and one extra parameter to govern the multiplicity separation.

Other aspects have to be considered if we want to find the simplest and most natural form for these symmetric group transformation coefficients. In [21] the authors expose the following suitable requirements:

1. the trasformation coefficients should be chosen to be real if possible;
2. phases and the multiplicity separation should be chosen to be indipendent from n;
3. the multiplicity separation is to be chosen so that a maximal number of zero coefficients is obtained;
4. it is desirable to have the coefficients written as a single surd of the form $a\sqrt{b}/c$, with a, b, c integers;
5. the prime numbers which occur in the surds should be as small as possible.

The first two statements are automatically verified if we assume (3.48). The last three heavily depend on the form of τ. This can be an interesting mathematical point to study (but it is not really relevant from a purely physical point of view). We think the form of eigenvalues and eigenvectors of τ are the only important factors in this regard. Non-normalized SDCs deriving from (3.42) seem always to be in a *simple* form.

Chapter 4

A reduced $\mathfrak{S}_f$ subduction graph and an example of higher multiplicity

In this chapter we first provide a *selection rule* and an *identity rule* for the subduction coefficients in symmetric groups which allow to decrease the number of unknowns and equations arising from the linear method by Pan and Chen. Thus, by using the *reduced subduction graph* approach, we may look at higher multiplicity instances. As a significant example, an orthonormalized solution for the first multiplicity-three case, which occurs in the decomposition of the irreducible representation $[4, 3, 2, 1]$ of $\mathfrak{S}_{10}$ into $[3, 2, 1] \otimes [3, 1]$ of $\mathfrak{S}_6 \times \mathfrak{S}_4$, is presented and discussed. The layout of the chapter is the following (we refer the reader to chapter 3 for definitions, notations and details on the subduction graph method). In section 1, we analyze the structure of the subduction space and we prove two theorems which are useful to reduce the number of unknowns and, consequently, the number of equations for the subduction problem. That is fundamental for an optimazed approach to very high dimension decompositions. According to McAven *et al* [21, pg 8372], we think that "the next steps in a search for a combinatorial recipe for a multiplicity separation could be to look at other multiplicity two cases and the first multiplicity three case". Therefore, in section 2, we present our determination for the significant first multiplicity-three examples in the reduction $\mathfrak{S}_{10} \downarrow \mathfrak{S}_6 \times \mathfrak{S}_4$.

4.1 Selection and identity rules

4.1.1 Crossing and bridge pairs of standard Young tableaux

Let λ be a Young diagram relative to a partition of n and (m, m') a pair of standard Young tableaux with the same diagram λ. Furthermore, we denote by $d_k(m)$ the usual *axial distance* between the numbers k and $k+1$ in the tableau m.
If $m \neq m'$, we name *cut* the minimum $i \in \{1, \ldots, n-1\}$ such that $d_i(m) \neq d_i(m')$. We give the following useful definitions:

Definition 4.1.1. *We say that (m, m') is a* crossing pair *of standard Young tableaux if there exists $i \in \{1, \ldots, n-1\}$ such that one of the following cases is verified:*

1. *$d_i(m) \neq d_i(m')$, $g_i(m) \neq m$ and $g_i(m') \neq m'$;*
2. *$d_i(m) \neq d_i(m')$, $g_i(m) = m$ and $g_i(m') = m'$.*

We call separation *for (m, m') the minimum i where one of the previous cases occurs.*

Definition 4.1.2. *We say that (m, m') is a* bridge pair *of standard Young tableaux if it is* not *a crossing pair, i.e. for all $i \in \{1, \ldots, n-1\}$ one of the following cases is verified:*

1. *$d_i(m) = d_i(m')$;*
2. *$g_i(m) = m$ and $g_i(m') \neq m'$;*
3. *$g_i(m) \neq m$ and $g_i(m') = m'$.*

Lemma 4.1.1. *Let (m, m') be a bridge pair with $m \neq m'$ and let $\bar{i}$ be the relative cut. Let us consider the application defined by*

$$g_{\bar{i}}(m, m') = (g_{\bar{i}}(m), g_{\bar{i}}(m')). \tag{4.1}$$

Then, by iteratively applying (4.1), we always obtain a crossing pair.

Proof. We first observe that, after one application of $g_{\bar{i}}$ on (m, m'), we have the following situation

$$\begin{cases} d_j(g_{\bar{i}}(m)) = d_j(g_{\bar{i}}(m')) & \text{if } j \notin \{\bar{i}-1, \bar{i}, \bar{i}+1\} \\ d_j(g_{\bar{i}}(m)) = d_j(g_{\bar{i}}(m')) + d_{j+1}(g_{\bar{i}}(m')) & \text{if } j = \bar{i}-1 \\ d_j(g_{\bar{i}}(m)) = -d_j(g_{\bar{i}}(m')) & \text{if } j = \bar{i} \\ d_j(g_{\bar{i}}(m)) = d_{j-1}(g_{\bar{i}}(m')) + d_j(g_{\bar{i}}(m')) & \text{if } j = \bar{i}+1 \end{cases} \tag{4.2}$$

thus $g_{\bar{i}}(m, m')$ has cut in $\bar{i} - 1$ because obviously $d_{\bar{i}}(g_{\bar{i}}(m')) \neq 0$.

Then, at each step of the iteration of (4.1), two cases may occur:

1. $g_{\bar{i}}(m, m')$ is a crossing pair and we have the assertion.
2. $g_{\bar{i}}(m, m')$ is a bridge pair with cut in $\bar{i} - 1$.

If case (i) never occurs, after $\bar{i} - 1$ iterations we should reach a bridge pair $(\tilde{m}, \tilde{m}')$ with cut $i = 1$. But $(\tilde{m}, \tilde{m}')$ always is a crossing pair because $g_1(\tilde{m}) = \tilde{m}$ and $g_1(\tilde{m}') = \tilde{m}'$ for each standard Young tableaux $\tilde{m}$ and $\tilde{m}'$. □

4.1.2 Islands

Let m, m_1 and m_2 be three standard Young tableaux with n, n_1 and n_2 boxes such that $n_1 + n_2 = n$ and shapes λ, λ_1 and λ_2, respectively. Denoted by $m^{(n_1)}$ the standard Young tableau obtained from m by removing the boxes with numbers $n_1 + 1, \ldots, n$, we say that m and m_1 are *compatible* if $m_1 = m^{(n_1)}$. The number of standard Young tableaux which are compatible with m_1 is equal to the number of standard skew-tableaux [101] of shape λ/λ_1 filled with the numbers $n_1 + 1, ..., n$. We denote it by f^{λ/λ_1}.

Denoted by G the grid relative to $(\lambda; \lambda_1, \lambda_2)$, we give the following

Definition 4.1.3. *Fixed the standard tableau σ with Young diagram λ_1 and varying m and m_2, with fixed Young diagrams λ and λ_2 respectively, the subset of G given by*

$$I_\sigma(G) = \{\langle m; \sigma, m_2\rangle \ \in G \mid \ m \textit{ is compatible with } \sigma\} \tag{4.3}$$

is named σ-island *of* G.

We refer to the σ-island simply saying *island* if it is not necessary to make an explicit reference to σ. Of course, the number of islands of G is given by the number of standard Young tableaux with diagram λ_1, i.e. f^{λ_1}.

Lemma 4.1.2. *Let $\langle m; m_{12}\rangle \in G$ be a node such that $(m^{(n_1)}, m_1)$ is a crossing pair. Then the corresponding SDC, $\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle$, vanishes.*

Proof. Let $i \in \{1, \dots, n_1 - 1\}$ be the separation of $(m^{(n_1)}, m_1)$. From definition 4.1.1, we need to destinguish the following situations:

- $g_i(m) \neq m$ and $g_i(m_1) \neq m_1$ (or, equivalently, $d_i(m) \neq \pm 1$ and $d_i(m_1) \neq \pm 1$).

  The action of the generator g_i on the standard base vector $|\lambda; m\rangle$ is given by [18]

$$g_i|\lambda; m\rangle = \frac{1}{d_i(m)}\ |\lambda; m\rangle\ +\ \beta^{(i)}_m\ |\lambda; g_i(m)\rangle \tag{4.4}$$

  where

$$\beta^{(i)}_m = \sqrt{1 - \frac{1}{d_i^2(m)}}. \tag{4.5}$$

  In an analogous way, the action on the split base vector $|\lambda_1, \lambda_2; m_1, m_2\rangle$ is

$$g_i|\lambda_1, \lambda_2; m_1, m_2\rangle = \frac{1}{d_i(m_1)}\ |\lambda_1, \lambda_2; m_1, m_2\rangle\ +\ \beta^{(i)}_{m_1}\ |\lambda_1, \lambda_2; g_i(m_1), m_2\rangle \tag{4.6}$$

  From (4.4) and (4.6), using $g_i^2 = 1$ and $g_i = {g_i}^\dagger$, we get

$$\left(1 - \frac{1}{d_i(m)d_i(m_1)}\right)\ \langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle\ -\ \frac{\beta^{(i)}_{m_1}}{d_i(m)}\ \langle \lambda; m|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle\ +$$

$$-\ \frac{\beta^{(i)}_m}{d_i(m_1)}\ \langle \lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle\ -\ \beta^{(i)}_m\beta^{(i)}_{m_1}\ \langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle = 0. \tag{4.7}$$

  Writing (4.7) also for the SCDs $\langle \lambda; m|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle$, $\langle \lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle$ and $\langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle$, and by using the fact that $d_i(g_i(m)) = -d_i(m)$ and $d_i(g_i(m_{12})) = -d_i(m_{12})$, we obtain the homogeneous linear system described by the matrix

$$\begin{pmatrix} 1 - \frac{1}{d_i(m)d_i(m_1)} & -\frac{\beta^{(i)}_{m_1}}{d_i(m)} & -\frac{\beta^{(i)}_m}{d_i(m_1)} & -\beta^{(i)}_m\beta^{(i)}_{m_1} \\ -\frac{\beta^{(i)}_{m_1}}{d_i(m)} & 1 + \frac{1}{d_i(m)d_i(m_1)} & -\beta^{(i)}_m\beta^{(i)}_{m_1} & \frac{\beta^{(i)}_m}{d_i(m_1)} \\ -\frac{\beta^{(i)}_m}{d_i(m_1)} & -\beta^{(i)}_m\beta^{(i)}_{m_1} & 1 + \frac{1}{d_i(m)d_i(m_1)} & \frac{\beta^{(i)}_{m_1}}{d_i(m)} \\ -\beta^{(i)}_m\beta^{(i)}_{m_1} & \frac{\beta^{(i)}_m}{d_i(m_1)} & \frac{\beta^{(i)}_{m_1}}{d_i(m)} & 1 - \frac{1}{d_i(m)d_i(m_1)} \end{pmatrix}. \tag{4.8}$$

Thereby $\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle$, $\langle \lambda; m|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle$, $\langle \lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle$ and $\langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle$ are the coordinates of a kernel vector for the matrix (4.8). It is easy to see that (4.8) has rank 3.

1. If $d_i(m) \neq -d_i(m_1)$;

   the kernel space for (4.8) is generated by the vector

$$\begin{pmatrix} 1 \\ \frac{d_i(m)d_i(m_1)(\beta_m^{(i)}+\beta_{m_1}^{(i)})}{d_i(m)+d_i(m_1)} \\ \frac{d_i(m)d_i(m_1)(\beta_m^{(i)}+\beta_{m_1}^{(i)})}{d_i(m)+d_i(m_1)} \\ -1 \end{pmatrix}, \tag{4.9}$$

   which implies

$$\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = -\langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle \tag{4.10}$$

   and

$$\langle \lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle \lambda; m|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle. \tag{4.11}$$

   Because $d_i(g_i(m)) = -d_i(m)$, $d_i(m) \neq -d_i(m_1) \Rightarrow d_i(g_i(m)) \neq d_i(m_1)$ and $d_i(m) \neq d_i(m_1) \Rightarrow d_i(g_i(m)) \neq -d_i(m_1)$. Therefore relation (4.11), written for $\langle \lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle$, yelds (remember that $g_i^2 = 1$)

$$\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle. \tag{4.12}$$

   From (4.10) and (4.12), we get

$$\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = 0. \tag{4.13}$$

2. If $d_i(m) = -d_i(m_1)$;

   the kernel space for (4.8) is generated by the vector

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.14}$$

which directly implies

$$\langle\lambda; g_i(m)|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle\lambda; m|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle \tag{4.15}$$

and

$$\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle\lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle = 0. \tag{4.16}$$

- $g_i(m) = m$ and $g_i(m_1) = m_1$ (or, equivalently, $|d_i(m)| = 1$ and $|d_i(m_1)| = 1$). The action of the generator g_i on the standard base vector $|\lambda; m\rangle$ is given by

$$g_i|\lambda; m\rangle = \pm|\lambda; m\rangle \tag{4.17}$$

and the action on the split base vector $|\lambda_1, \lambda_2; m_1, m_2\rangle$ is

$$g_i|\lambda_1, \lambda_2; m_1, m_2\rangle = \mp|\lambda_1, \lambda_2; m_1, m_2\rangle. \tag{4.18}$$

Thus (because $g_i^2 = 1$ and $g_i = {g_i}^\dagger$)

$$\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = -\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle \tag{4.19}$$

from which

$$\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = 0. \tag{4.20}$$

□

Lemma 4.1.3. *Let $\langle m; m_{12}\rangle \in G$ be a node such that $(m^{(n_1)}, m_1)$ is a bridge pair and $m^{(n_1)} \neq m_1$. Then the corresponding SDC, $\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle$, vanishes.*

Proof. Let $i \in \{1, \dots, n_1 - 1\}$ be the cut of $(m^{(n_1)}, m_1)$. For semplicity, let us suppose $g_i(m) = m$ and $g_i(m_1) \neq m_1$.
The action of the generator g_i on the standard base vector $|\lambda; m\rangle$ is given by

$$g_i|\lambda; m\rangle = \pm|\lambda; m\rangle \tag{4.21}$$

and the action on the split base vector $|\lambda_1, \lambda_2; m_1, m_2\rangle$ is

$$g_i|\lambda_1, \lambda_2; m_1, m_2\rangle = \frac{1}{d_i(m_1)}\ |\lambda_1, \lambda_2; m_1, m_2\rangle\ +\ \beta_{m_1}^{(i)}\ |\lambda_1, \lambda_2; g_i(m_1), m_2\rangle \tag{4.22}$$

Because $d_i(m_1) \neq \pm 1$, (4.21) and (4.22) imply

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = b'_{i,m_1} \langle \lambda; m | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle \tag{4.23}$$

with b'_{i,m_1} a suitable numerical factor.
In an analogous way, the case $g_i(m) \neq m$ and $g_i(m_1) = m_1$ provides

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = b''_{i,m} \langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle \tag{4.24}$$

with $b''_{i,m}$ another suitable numerical factor.
From lemma 4.1.1, by iterating the previous derivation, we may write

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = b \ \langle \lambda; \bar{m} | \lambda_1, \lambda_2; \bar{m}_1, m_2 \rangle \tag{4.25}$$

with $(\bar{m}, \bar{m}_1)$ a crossing pair and b a total numerical factor . But, from lemma 4.1.2,

$$\langle \lambda; \bar{m} | \lambda_1, \lambda_2; \bar{m}_1, m_2 \rangle = 0, \tag{4.26}$$

thus

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = 0. \tag{4.27}$$

□

It is now possible to give the following theorems:

Theorem 4.1.1 (Selection Rule). *Let $\langle m; m_{12} \rangle$ be a node of G which does not belong to any island of G. Then $\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle$ vanishes.*

Proof. Because $\langle m; m_{12} \rangle$ does not belong to any island, we have $m^{(n_1)} \neq m_1$. If $(m^{(n_i)}, m_1)$ is a crossing pair, we have the assertion by lemma 4.1.2. If $(m^{(n_i)}, m_1)$ is not a crossing pair (i.e. it is a bridge pair), we use lemma 4.1.3 and we have the proof. □

The previous theorem allows us to say that only $f^{\lambda_1} f^{\lambda_2} f^{\lambda/\lambda_1}$ SDCs may not vanishes. It provides a selection rule for the subduction coefficients which is based on the Littlewood-Richardson rule. Furthermore, we observe that, in our graph approach, it is analogous to the *block-selective rule* given in [102] and [103]. Therefore we may somehow associate our definition of island to the concept of “block” given by McAven and Butler.

Theorem 4.1.2 (Identity Rule). *All islands of G have the same corresponding SDCs, i.e.*

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = \langle \lambda; g_i(m) | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle$$

for all $i \in \{1, \ldots, n_1 - 1\}$.

Proof. Suppose $\langle m; m_{12} \rangle$ belongs to the m_1-island. Thus m is compatible with m_1 and we have $d_i(m) = d_i(m_1) = d_i$ for all $i \in \{1, \ldots, n_1 - 1\}$. We again distinguish two cases:

- $d_i \neq \pm 1$.

  It is straightforward that (4.8) is a rank 2 matrix and the kernel space is generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{\sqrt{d_i^2 - 1}} \\ 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.28}$$

  therefore we have

$$\langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle = \langle \lambda; m | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle \tag{4.29}$$

  and

$$\begin{aligned} \langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle &= \langle \lambda; g_i(m) | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle + \\ &+ \frac{2}{\sqrt{d_i^2 - 1}} \langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle. \end{aligned} \tag{4.30}$$

  Because $d_i(g_i(m)) = -d_i(m) = -d_i(m_1)$, (4.16) becomes

$$\langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle = 0 \tag{4.31}$$

  and, from (4.30),

$$\langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle = \langle \lambda; g_i(m) | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle. \tag{4.32}$$

  Thus the m_1-island and the $g_i(m_1)$-island have the same corresponding SDCs.

- $d_i = \pm 1$.

  In this case $g_i(m) = m$ and $g_i(m_{12}) = m_{12}$, thus both $\langle m; m_{12}\rangle$ and $\langle g_i(m); g_i(m_{12})\rangle$ trivially belong to the same m_1-island and, of course,

$$\langle \lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle \lambda; g_i(m)|\lambda_1, \lambda_2; g_i(m_1), m_2\rangle. \tag{4.33}$$

So the proof directly follows from the fact that the m_1-island can be transformed in to another m'_1-island by a suitable composition of g_i transformations ($i \in \{1, \ldots, n_1 - 1\}$), the same one which transforms the standard Young tableau m_1 to m'_1. □

4.1.3 Reduced subduction graph

From the previous theorems, the only SDCs we need to evaluate are the $f^{\lambda_2} f^{\lambda/\lambda_1}$ ones relative to a single island. We have a reduced linear system with $(n_2 - 1) f^{\lambda_2} f^{\lambda/\lambda_1}$ equations and $f^{\lambda_2} f^{\lambda/\lambda_1}$ unknowns instead of the $(n-2) f^{\lambda} f^{\lambda_1} f^{\lambda_2}$ and $f^{\lambda} f^{\lambda_1} f^{\lambda_2}$ primal ones.

In fact, fixed an island, the relative *reduced subduction graph* is sufficient to provide the required transformation coefficients. Such a graph is obtained by the action of the g_i trasformations, with $i \in \{n_1 + 1, \ldots, n - 1\}$, on the island nodes only, and thus it allows further on reducing the number of dependent linear equations. On the other hand, the g_i transformations with $i \in \{1, \ldots, n_1 - 1\}$ link the corresponding nodes of two different islands and thus, by the identity rule, we do not need to consider them.

In table 4.1 we deal with some subduction cases, the relative multiplicity, the number of unknowns involved in the primal linear equation system and the effective number of needed SDCs, after the application of the selection and identity rules. It is evident the drastic reduction of the number of unknowns for the subduction problem.

4.2 A higher dimension example: the first multiplicity-three case

The first multiplicity-three case for the subduction problem in symmetric groups accours in $[4, 3, 2, 1] \downarrow [3, 2, 1] \otimes [3, 1]$ of $\mathfrak{S}_{10} \downarrow \mathfrak{S}_6 \times \mathfrak{S}_4$. From the hook rule [101], the

$[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$	$\{\lambda; \lambda_1, \lambda_2\}$	$f^\lambda f^{\lambda_1} f^{\lambda_2}$	$f^{\lambda/\lambda_1} f^{\lambda_2}$
$[4,2] \downarrow [2,1] \otimes [2,1]$	1	36	6
$[3,2,1] \downarrow [2,1] \otimes [2,1]$	2	64	12
$[4,2,1] \downarrow [3,1] \otimes [2,1]$	2	210	12
$[4,3,2] \downarrow [3,2] \otimes [3,1]$	2	2520	36
$[4,3,2,1] \downarrow [3,2,1] \otimes [3,1]$	3	36864	72
$[5,4,3,2] \downarrow [4,3,2] \otimes [3,2]$	3	40360320	300
$[5,4,3,2,1] \downarrow [4,3,2,1] \otimes [4,1]$	4	899678208	480
$[6,5,4,3,2,1] \downarrow [5,4,3,2,1] \otimes [5,1]$	5	1611839486033920	3600

Table 4.1: Some examples of subduction with the relative multiplicity, the primal number of involved SDCs and the island dimension.

representation $[4,3,2,1]$ has dimension $f^\lambda = 768$, $[3,2,1]$ has dimension $f^{\lambda_1} = 16$ and $[3,1]$ dimension $f^{\lambda_2} = 3$. Thus we have $f^\lambda f^{\lambda_1} f^{\lambda_2} = 36864$ SDCs to evaluate. Many of such coefficients are zero via the selection rule provided in the previous section. Now, the number of islands is given by the dimension of $[3,2,1]$, i.e. 16. But, from theorem 2, we only need to determine the SDCs corresponding to *one* island. Because the number of standard skew-tableaux of shape $[4,3,2,1]/[3,2,1]$ is $f^{\lambda/\lambda_1} = 24$, the island is composed of $f^{\lambda/\lambda_1} f^{\lambda_2} = 72$ nodes which correspond to our unknowns.

We organize the nodes by the lexicographic ordering: first we order the tableaux and then the triplet which forms each node. We choose the usual Yamanouchi convention [99] to fix the phase freedom: we impose the *first non-zero* SDC to be *positive*.

From subduction graph and by using a suitable Mathematica program [104], we generate the homogeneous linear sistem required to obtain the SDCs. Then we find the kernel of the subduction matrix which provides a non-orthonormalized form for the coefficients. The solution space has dimension 3 (multiplicity). We orthonormalize the SDCs in such a way that the conditions (3.5) and (3.6) hold.

In table 4.2 we deal with the three copies for the SDCs (with multiplicity labels 1, 2 and 3, respectively). Such coefficients are listed in the lexicographic ordering (when they are read from left to right and up to down) and they have a fixed m along the rows and m_2 down the coloumns. Multiplicity separation can be choosen in such a way that the

1^{st} multiplicity copy			2^{nd} multiplicity copy			3^{rd} multiplicity copy		
$\frac{\sqrt{14}}{64}$	$-\frac{\sqrt{7}}{16}$	$\frac{\sqrt{21}}{32}$	$\frac{5\sqrt{2}}{64}$	$-\frac{5}{64}$	$-\frac{5\sqrt{3}}{64}$	$\frac{3\sqrt{6}}{64}$	$\frac{3\sqrt{3}}{64}$	$\frac{3}{64}$
$-\frac{5\sqrt{42}}{192}$	$-\frac{\sqrt{21}}{48}$	$\frac{3\sqrt{7}}{32}$	$-\frac{5\sqrt{6}}{192}$	$\frac{5\sqrt{3}}{192}$	$-\frac{15}{64}$	$\frac{5\sqrt{2}}{64}$	$\frac{13}{64}$	$\frac{3\sqrt{3}}{64}$
$\frac{\sqrt{42}}{64}$	$\frac{\sqrt{21}}{32}$	0	$\frac{5\sqrt{6}}{64}$	$-\frac{5\sqrt{3}}{64}$	$-\frac{15}{64}$	$\frac{9\sqrt{2}}{64}$	$\frac{3}{64}$	$\frac{5\sqrt{3}}{64}$
$\frac{\sqrt{70}}{64}$	$\frac{\sqrt{35}}{32}$	0	$-\frac{3\sqrt{10}}{64}$	$\frac{3\sqrt{5}}{64}$	$-\frac{5\sqrt{15}}{64}$	$\frac{\sqrt{30}}{192}$	$\frac{11\sqrt{15}}{192}$	$\frac{5\sqrt{5}}{64}$
$-\frac{5\sqrt{70}}{192}$	$\frac{\sqrt{35}}{24}$	$\frac{\sqrt{105}}{96}$	$-\frac{5\sqrt{10}}{192}$	$-\frac{19\sqrt{5}}{192}$	$-\frac{7\sqrt{15}}{192}$	$\frac{5\sqrt{30}}{192}$	$\frac{\sqrt{15}}{192}$	$\frac{7\sqrt{5}}{64}$
$\frac{\sqrt{210}}{64}$	0	$\frac{\sqrt{35}}{32}$	$-\frac{3\sqrt{30}}{64}$	$-\frac{5\sqrt{15}}{64}$	$-\frac{7\sqrt{5}}{64}$	$\frac{\sqrt{10}}{64}$	$\frac{5\sqrt{5}}{64}$	$\frac{7\sqrt{15}}{64}$
$\frac{\sqrt{42}}{64}$	$-\frac{\sqrt{21}}{16}$	$-\frac{\sqrt{7}}{32}$	$\frac{5\sqrt{6}}{64}$	$-\frac{5\sqrt{3}}{64}$	$\frac{5}{64}$	$\frac{9\sqrt{2}}{64}$	$\frac{9}{64}$	$-\frac{\sqrt{3}}{64}$
$-\frac{5\sqrt{14}}{64}$	$-\frac{\sqrt{7}}{16}$	$-\frac{\sqrt{21}}{32}$	$-\frac{5\sqrt{2}}{64}$	$\frac{5}{64}$	$\frac{5\sqrt{3}}{64}$	$\frac{5\sqrt{6}}{64}$	$\frac{13\sqrt{3}}{64}$	$-\frac{3}{64}$
$\frac{\sqrt{70}}{64}$	0	$-\frac{\sqrt{105}}{32}$	$\frac{5\sqrt{10}}{64}$	$\frac{3\sqrt{5}}{64}$	$-\frac{\sqrt{15}}{64}$	$\frac{3\sqrt{30}}{64}$	$-\frac{\sqrt{15}}{64}$	$\frac{3\sqrt{5}}{64}$
$-\frac{\sqrt{2}}{64}$	$\frac{1}{8}$	$-\frac{7\sqrt{3}}{32}$	$\frac{\sqrt{14}}{64}$	$\frac{7\sqrt{7}}{64}$	$-\frac{\sqrt{21}}{64}$	$-\frac{\sqrt{42}}{64}$	$\frac{3\sqrt{21}}{64}$	$\frac{3\sqrt{7}}{64}$
$-\frac{7\sqrt{10}}{64}$	$-\frac{\sqrt{5}}{32}$	$-\frac{\sqrt{15}}{16}$	$-\frac{\sqrt{70}}{64}$	$\frac{\sqrt{35}}{64}$	$\frac{\sqrt{105}}{64}$	$\frac{\sqrt{210}}{64}$	$-\frac{\sqrt{105}}{64}$	$\frac{3\sqrt{35}}{64}$
$-\frac{\sqrt{6}}{64}$	$-\frac{7\sqrt{3}}{32}$	$-\frac{5}{16}$	$\frac{\sqrt{42}}{64}$	$-\frac{\sqrt{21}}{64}$	$\frac{5\sqrt{7}}{64}$	$-\frac{3\sqrt{14}}{64}$	$\frac{3\sqrt{7}}{64}$	$\frac{5\sqrt{21}}{64}$
$\frac{\sqrt{70}}{64}$	$\frac{\sqrt{35}}{32}$	0	$\frac{5\sqrt{10}}{64}$	$-\frac{5\sqrt{5}}{64}$	$\frac{3\sqrt{15}}{64}$	$\frac{3\sqrt{30}}{64}$	$\frac{\sqrt{15}}{64}$	$-\frac{3\sqrt{5}}{64}$
$\frac{5\sqrt{42}}{192}$	$\frac{5\sqrt{21}}{96}$	0	$-\frac{5\sqrt{6}}{64}$	$\frac{5\sqrt{3}}{64}$	$\frac{15}{64}$	$\frac{5\sqrt{2}}{192}$	$\frac{55}{192}$	$-\frac{5\sqrt{3}}{64}$
$\frac{\sqrt{210}}{64}$	0	$\frac{\sqrt{35}}{32}$	$\frac{5\sqrt{30}}{64}$	$\frac{3\sqrt{15}}{64}$	$\frac{\sqrt{5}}{64}$	$\frac{9\sqrt{10}}{64}$	$-\frac{3\sqrt{5}}{64}$	$-\frac{\sqrt{15}}{64}$
$-\frac{\sqrt{6}}{64}$	$\frac{\sqrt{3}}{8}$	$\frac{7}{32}$	$\frac{\sqrt{42}}{64}$	$\frac{7\sqrt{21}}{64}$	$\frac{\sqrt{7}}{64}$	$-\frac{3\sqrt{14}}{64}$	$\frac{9\sqrt{7}}{64}$	$-\frac{\sqrt{21}}{64}$
$\frac{7\sqrt{30}}{192}$	$-\frac{\sqrt{15}}{48}$	$\frac{3\sqrt{5}}{32}$	$-\frac{\sqrt{210}}{64}$	$\frac{\sqrt{105}}{64}$	$\frac{3\sqrt{35}}{64}$	$\frac{\sqrt{70}}{192}$	$-\frac{7\sqrt{35}}{192}$	$\frac{\sqrt{105}}{64}$
$-\frac{\sqrt{10}}{64}$	$\frac{\sqrt{5}}{16}$	$\frac{3\sqrt{15}}{32}$	$\frac{\sqrt{70}}{64}$	$-\frac{\sqrt{35}}{64}$	$\frac{3\sqrt{105}}{64}$	$-\frac{\sqrt{210}}{64}$	$-\frac{\sqrt{105}}{64}$	$\frac{3\sqrt{35}}{64}$
$-\frac{35\sqrt{2}}{192}$	$\frac{7}{24}$	$-\frac{5\sqrt{3}}{96}$	$-\frac{5\sqrt{14}}{192}$	$-\frac{19\sqrt{7}}{192}$	$\frac{5\sqrt{21}}{192}$	$\frac{5\sqrt{42}}{192}$	$\frac{\sqrt{21}}{192}$	$-\frac{5\sqrt{7}}{64}$
$\frac{7\sqrt{6}}{64}$	0	$-\frac{5}{32}$	$-\frac{3\sqrt{42}}{64}$	$-\frac{5\sqrt{21}}{64}$	$\frac{5\sqrt{7}}{64}$	$\frac{\sqrt{14}}{64}$	$\frac{5\sqrt{7}}{64}$	$-\frac{5\sqrt{21}}{64}$
$-\frac{35\sqrt{6}}{192}$	$-\frac{5\sqrt{3}}{96}$	$\frac{3}{16}$	$-\frac{5\sqrt{42}}{192}$	$\frac{5\sqrt{21}}{192}$	$-\frac{3\sqrt{7}}{64}$	$\frac{5\sqrt{14}}{64}$	$-\frac{5\sqrt{7}}{64}$	$-\frac{3\sqrt{21}}{64}$
$-\frac{\sqrt{10}}{64}$	$-\frac{7\sqrt{5}}{32}$	$\frac{\sqrt{15}}{16}$	$\frac{\sqrt{70}}{64}$	$-\frac{\sqrt{35}}{64}$	$-\frac{\sqrt{105}}{64}$	$-\frac{\sqrt{210}}{64}$	$\frac{\sqrt{105}}{64}$	$-\frac{3\sqrt{35}}{64}$
$\frac{7\sqrt{10}}{64}$	$-\frac{\sqrt{5}}{16}$	$-\frac{\sqrt{15}}{32}$	$-\frac{3\sqrt{70}}{64}$	$\frac{3\sqrt{35}}{64}$	$-\frac{\sqrt{105}}{64}$	$\frac{\sqrt{210}}{192}$	$-\frac{7\sqrt{105}}{192}$	$-\frac{\sqrt{35}}{64}$
$-\frac{\sqrt{30}}{64}$	$\frac{\sqrt{15}}{16}$	$-\frac{3\sqrt{5}}{32}$	$\frac{\sqrt{210}}{64}$	$-\frac{\sqrt{105}}{64}$	$-\frac{3\sqrt{35}}{64}$	$-\frac{3\sqrt{70}}{64}$	$-\frac{3\sqrt{35}}{64}$	$-\frac{\sqrt{105}}{64}$

Table 4.2: Island subduction coefficients of $[4,3,2,1] \downarrow [3,2,1] \otimes [3,1]$ for each multiplicity copy. The coefficients are listed in the lexicographic ordering (when they are read from left to right and top to bottom) and they have the same m along the rows and m_2 down the coloumns.

coefficients are expressed as a single surd of the form $a\sqrt{b}/c$, with a, b and c integers.

By conjugation of $[4,3,2,1] \downarrow [3,2,1] \otimes [3,1]$, i.e. $[4,3,2,1] \downarrow [3,2,1] \otimes [2,1,1]$, we have another multiplicity three case for the subduction problem. The new SDCs are related to the previous ones. Denoted by $\tilde{m}$ the skew-tableau conjugate to m and by $\tilde{m}_1$ and $\tilde{m}_2$ the tableaux conjugate to m_1 and m_2 respectively, we have, for the m_1-island, the following symmetry conditions

$$\langle\tilde{\lambda};\tilde{m}|\tilde{\lambda}_1,\tilde{\lambda}_1;\tilde{m}_1,\tilde{m}_2\rangle_1 = \Lambda^{[4,3,2,1]/[3,2,1]}_{m/m_1}\ \Lambda^{[3,1]}_{m_2}\ \langle\lambda;m|\lambda_1,\lambda_2;m_1,m_2\rangle_3 \tag{4.34}$$

$$\langle\tilde{\lambda};\tilde{m}|\tilde{\lambda}_1,\tilde{\lambda}_2;\tilde{m}_1,\tilde{m}_2\rangle_2 = \Lambda^{[4,3,2,1]/[3,2,1]}_{m/m_1}\ \Lambda^{[3,1]}_{m_2}\ \langle\lambda;m|\lambda_1,\lambda_2;m_1,m_2\rangle_2 \tag{4.35}$$

$$\langle\tilde{\lambda};\tilde{m}|\tilde{\lambda}_1,\tilde{\lambda}_2;\tilde{m}_1,\tilde{m}_2\rangle_3 = \Lambda^{[4,3,2,1]/[3,2,1]}_{m/m_1}\ \langle\lambda;m|\lambda_1,\lambda_2;m_1,m_2\rangle_1, \tag{4.36}$$

where Λ^{λ}_{m} are the phase factors of the Yamanouchi basis [105] for the irrep $[\lambda]$ and

$$\Lambda^{\lambda/\lambda_1}_{m/m_1} = \Lambda^{\lambda}_{m}\Lambda^{\lambda_1}_{m_1} \qquad \text{m compatible with } m_1$$

(observe that, if m and m_1 are compatible, m/m_1 represents the skew-tableau of shape λ/λ_1 obtained by removing the first n_1 boxes from m). Denoted by m' and m'_2 the ordering number (by lexicografic ordering) for m/m_1 and m_2 respectively, our Mathematica computation provides

$$\begin{aligned} \Lambda^{[4,3,2,1]/[3,2,1]}_{m'} &= -(-1)^{\frac{m'(m'-1)}{2}} \quad && \text{with } m' \in \{1,2,3,\dots,24\} \\ \Lambda^{[3,1]}_{m'_2} &= -(-1)^{m'_2} && \text{with } m'_2 \in \{1,2,3\}. \end{aligned} \tag{4.37}$$

Chapter 5

The subduction problem for Brauer algebras

In this chapter, we follow the structure of the chapter 3 to get a suitable combinatorial and algebraic description of the *linear equation method* for Brauer algebras. In section 1, we give the irreducible representation of Brauer algebras and, by introducing the concept of generalized *permutation lattice*, we present the explicit action of the generators on the irreducible representations. In section 2, we provide the explicit form for the subduction equations and, in section 3, we link such equations to the concept of a generalized subduction graph. By using the subduction graph approach, in section 4 we are able to describe the structure of the solution space for the subduction problem. We recognize that the subduction space can be built on four tipical configurations in the i-layer: the *crossing*, the *horizontal* and *vertical bridges* and the *singlet*. Finally, in section 5, as in the case of the subduction problem in symmetric groups, we discuss the general orthonormalized form for the subduction coefficients in Brauer algebras and we define a suitable ordering relation on permutation lattices and on the *grid* (and thus on the set of the subduction coefficients) which is necessary to fix the choice of phases (i.e. the *Young-Yamanouchi phase convention*) and free factors governing the multiplicity separations.

5.1 Irreducible representations of Brauer algebras

In chapter 2, we have introduced the abstract Brauer algebras by the definition of *f-diagram* and the relative composition operation. Theorem 2.2.1 provides a construction of the Brauer algebra $\mathfrak{B}_f(x)$ by the *contraction operators* $e_{i,j}$ operators and the elements of the symmetric group $\mathfrak{S}_f$. A minimal set of generators of $\mathfrak{B}_f(x)$ is given by $\{g_1,\ g_2,\ldots,g_{f-2},\ g_{f-1},\ e_1,\ e_2\ldots,e_{f-2},\ e_{f-1}\}$, where g_i represents the elementary transposition of the symmetric group which interchanges the elements i and $i+1$ and $e_i = e_{i,i+1}$. It is clear that $\{g_1,\ g_2,\ldots,g_{f-2},\ g_{f-1}\}$ generates $\mathfrak{S}_f \subset \mathfrak{B}_f(x)$.

As we have pointed out in chapter 2, it is known that $\mathfrak{B}_f(x)$ is semisimple, i.e. it is a direct sum of full matrix algebras over $\mathbb{C}$, when x is not an integer or is an integer with $x \geq f-1$, otherwise $\mathfrak{B}_f(x)$ is not semisimple. Whenever $\mathfrak{B}(x)$ is semisimple, its irreducible representation can be labelled by a Young diagram with $f,\ f-2,\ f-4,\ \ldots,$ 1or 0 boxes. It can be seen that by removing the generators e_{f-1} and g_{f-1}, $\{g_1,\ g_2,\ldots,g_{f-2},\ e_1,\ e_2\ldots,e_{f-2}\}$ generate $\mathfrak{B}_{f-1}(x)$. By doing so repeatedly, one can establish the standard Gelfand-Tzetlin chain $\mathfrak{B}_f(x) \subset \mathfrak{B}_{f-1}(x) \subset \ldots \subset \mathfrak{B}_2(x)$. It defines the *standard basis* of $\mathfrak{B}_f(x)$. Let Υ_f be the set of all Young diagrams with $k \leq f$ boxes such that $k \geq 0$ and $f-k$ is even. If $\mathfrak{B}_f(x)$ is semisimple, it decomposes into a direct sum of full matrix algebras $\mathfrak{B}_{f,\lambda}(x)$, where $\lambda \in \Upsilon_f$. If $[f,\lambda]$ is a simple $\mathfrak{B}_{f,\lambda}(x)$ irreducible representation, it decomposes as a $\mathfrak{B}_{f-1,\lambda}(x)$ in to a direct sum

$$[f,\lambda] = \bigoplus_{\mu\leftrightarrow\lambda} [f-1,\mu] \tag{5.1}$$

where $[f-1,\mu]$ is a $\mathfrak{B}_{f-1,\mu}(x)$ irrep and μ runs through all diagrams obtained by removing or (if λ contains less than f boses) adding a box to λ.

In what follows, we always assume that $\mathfrak{B}_f(x)$ is semisimple.

5.1.1 Generalized tableaux

The branching rule given in (5.1) allow us to label the elements of the standard basis for an irrep $[f,\lambda]$ of the Brauer algebra $\mathfrak{B}_f(x)$ by defining a *generalized* Young tableau which is associated to the concept of *Bratteli diagram* [19].

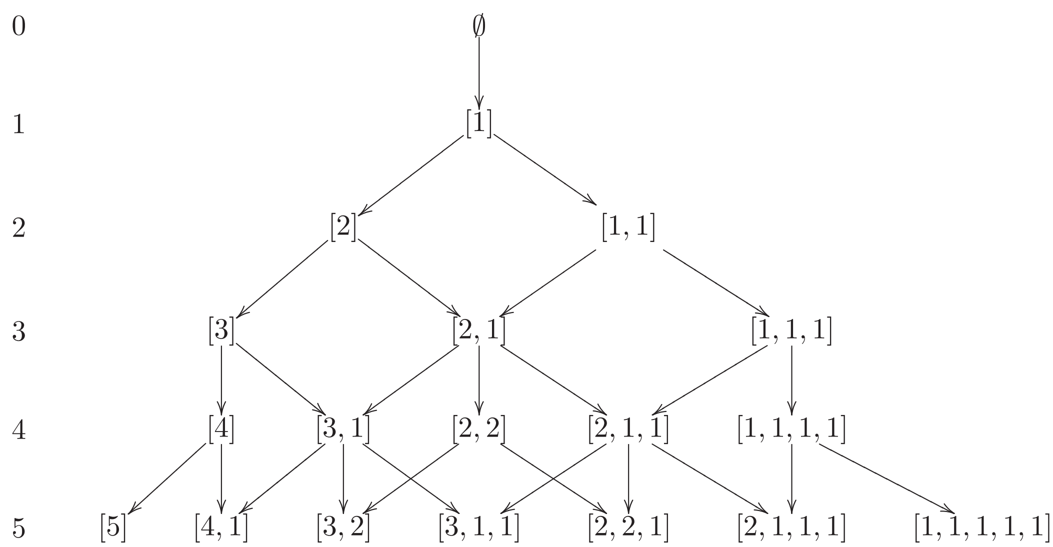


Figure 5.1: First five levels of the Bratteli diagram describing the branching rule of $\mathbb{C}\mathfrak{S}_f$ centralizer algebras. $\check{A}_k$ is the set of the partitions of k. So, the shapes are Young diagrams and $\lambda \in \check{A}_k$ is connected to $\mu \in \check{A}_{k+1}$ by an edge if μ can be obtained from λ by adding one box.

A Bratteli diagram A is a graph with vertices from a collection of sets $\check{A}_k,\ k \geq 0$, and edges that connect vertices in $\check{A}_k$ to vertices in $\check{A}_{k+1}$. One assumes that the set $\check{A}_0$ contains a unique vertex denoted by $\emptyset$. It is possible that there are multiple edges connecting any two vertices. We shall call the vertices *shapes*. The set $\check{A}_k$ is the set of shapes on *level* k. If $\lambda \in \check{A}_k$ is connected by an edge to a shape $\mu \in \check{A}_{k+1}$ we usually write $\lambda \leq \mu$.

A *multiplicity free Bratteli diagram* is a Bratteli diagram such that there is at most one edge connecting any two vertices. Here, we *assume* that all Bratteli diagrams are multiplicity free. In fact, the Bratteli diagrams, which are more interesting for our purposes, are multiplicity free and arise naturally in the representation theory of centralizer algebras. In figure 5.1 we show the Bratteli diagram describing the branching rule for the Gelfand-Tzetlin chain of $\mathbb{C}\mathfrak{S}_f$ centralizer algebras.

Let A be a multiplicity free Bratteli diagram and let $\lambda \in \check{A}_k$ and $\mu \in \check{A}_l$ where $k < l$. A *path* from λ to μ is e sequence of shapes $\lambda^{(i)}$, $k \leq i \leq l$, $P = (\lambda^{(k)}, \lambda^{(k+1)}, \ldots, \lambda^{(l)})$ such that $\lambda = \lambda^{(k)} \leq \lambda^{(k+1)}, \ldots, \lambda^{(l)} = \mu$ and $\lambda^{(i)} \in \check{A}_i$.

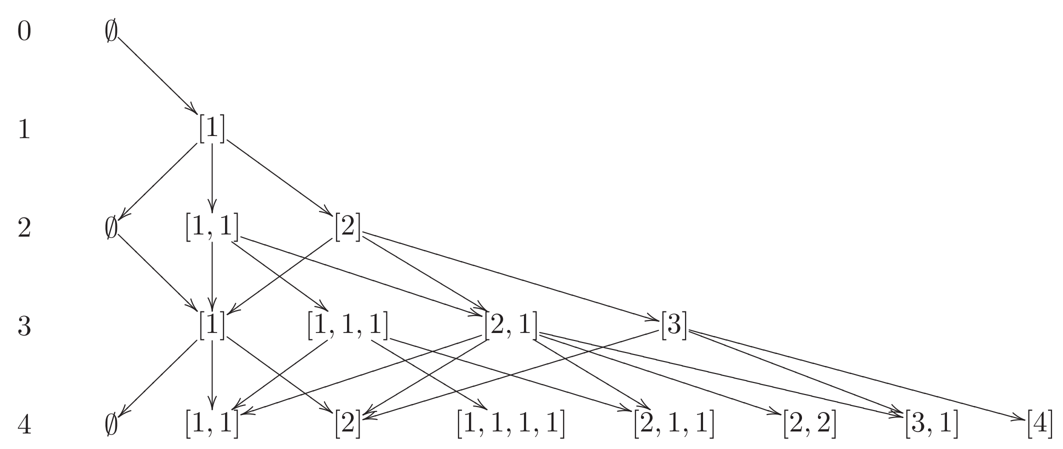


Figure 5.2: First four levels of the Bratteli diagram describing the branching rule of $\mathfrak{B}_f(x)$ centralizer algebras. Here, the shapes are Young diagrams such that $\lambda \in \check{A}_k$ is connected to $\mu \in \check{A}_{k+1}$ by an edge if μ can be obtained from λ by adding or deleting one box.

Definition 5.1.1. *A generalized* tableau τ *of shape (diagram)* λ *is a path from* $\emptyset$ *to* λ, $\sigma = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)})$*, such that* $\emptyset = \lambda^{(0)} \leq \lambda^{(1)}, \ldots, \lambda^{(k-1)} \leq \lambda^{(k)} = \lambda$ *and* $\lambda^{(i)} \in \check{A}_i$ *for each* $1 \leq i \leq l$.

The branching rule for Brauer algebras given in the previous subsection can also be described by a suitable multiplicity free Bratteli diagram, as shown in figure 5.2.

5.1.2 Permutation lattices

Let $\mathcal{W}$ be the set of all finite *words* composed of elements of $\mathbb{Z} \setminus \{0\}$. We define a *counting* function on $\mathcal{W}$ as follows:

$$\hat{\#}_w(k) = \#_w(k) - \#_w(-k), \tag{5.2}$$

where $\#_w(k)$ represents the number of times that $k \in \mathbb{Z} \setminus \{0\}$ appears in the word w.

Observe that, if w is the empty word $\emptyset$, $\#_w(k)$ vanishes by definition for all $k \in \mathbb{Z} \setminus \{0\}$. Denoting by $w^{(i)}$ the word obtained from w only considering the first i elements and neglecting the other ones, we give the following definition:

Definition 5.1.2. *A* permutation lattice *of order f is a word w composed of f elements such that*

$$\hat{\#}_{w^{(i)}}(1) \geq \hat{\#}_{w^{(i)}}(2) \geq \hat{\#}_{w^{(i)}}(3) \geq \ldots \geq 0 \tag{5.3}$$

for all $1 \leq i \leq f$ (note that $w^{(f)} = w$). The tuple $\lambda = [\hat{\#}_w(1), \hat{\#}_w(2), \hat{\#}_w(3), \ldots, \hat{\#}_w(l)]$, where $\hat{\#}_w(l)$ is the last element different from zero in the sequence (5.3), is called shape *or* diagram *of w.*

For istance, the word $w = (1, 1, 2, -1, 1, -2, 2)$ is a permutation lattice of order 7 with diagram $[2, 1]$, but $v = (1, 2, 1, -1, 2, 1, 3)$ is not a permutation lattice because $\hat{\#}_{v^{(5)}}(1) \ngeq \hat{\#}_{v^{(5)}}(2)$.

We observe that if w has only positive elements, the previous definition of permutation lattice becomes the usual one given, for example, in [101].

5.1.3 Labelling for the Gelfand-Tzetlin base

For each tableau $\tau = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(f-1)}, \lambda^{(f)})$, where $\lambda^{(0)} = \emptyset$ and $\lambda^{(f)} = \lambda$ being the shape, we can associate the f-tuple (or word) $w(\tau) = (w_1, w_2, \ldots, w_f)$ as follows:

$$w_k = \begin{cases} h & \text{if the Young diagram } \lambda^{(k+1)} \text{ is obtained from the Young diagram } \lambda^{(k)} \\ & \text{by adding one box to the } h\text{th row (from the top of the diagram);} \\ -h & \text{if the Young diagram } \lambda^{(k+1)} \text{ is obtained from the Young diagram } \lambda^{(k)} \\ & \text{by deleting one box from the } h\text{th row (from the top of the diagarm).} \end{cases} \tag{5.4}$$

For istance, the word $w(\tau)$ associated to the tableau $\tau = (\emptyset, [1], [2], [2, 1], [1, 1], [1, 1, 1], [1, 1])$, with shape (diagram) $[1, 1]$, is $w(\tau) = (1, 1, 2, -1, 3, -3)$.

Building on the previous definitions, the following proposition is straightforward:

Proposition 5.1.1. *τ is a tableau of the Bratteli diagram for the Brauer algebra $\mathfrak{B}_f(x)$ (see figure 5.2) if and only if $w(\tau)$ is a permuation lattice of order f. Furthemore, the diagram of τ coincides with the diagram of $w(\tau)$.*

Therefore, permutation lattices provide a labelling scheme for the irreducible representations of Brauer algebras. In fact, given the irrep $[f, \lambda]$ of $\mathfrak{B}_f(x)$, the relative Gelfand-

Tzetlin base vectors can be labelled by all permutation lattices of order f and diagram λ (denoted by λ_f if we need to specify the level f in the Bratteli diagram).

The dimensions of irreps of $\mathfrak{B}_f(x)$, $[f, \lambda]$, can be computed by using Bratteli diagrams inductively. One can prove that the dimension formula can be espressed [106] as

$$\dim(\mathfrak{B}_f(x); [f, \lambda]) = \frac{f!}{(f-2k)!(2k)!!} \dim(\mathfrak{S}_{f-2k}; [\lambda]) \tag{5.5}$$

where $f - 2k$ is the number of boxes which compose the diagram λ and $\dim(\mathfrak{S}_{f-2k}; [\lambda])$ is the dimension for the irrep $[\lambda]$ of $\mathfrak{S}_{f-2k}$ which can be further be espressed, for example, by Littlewood-Robinson formula for irreps of symmetric groups.

It should be noted that (5.5) provides the number of permutation lattices of order f and diagram λ once we know the number of Standard Young tableaux with diagram λ. Furthermore, the labelling scheme and the decomposition for $\mathfrak{B}_f(x)$ are the same as those for Birman-Wenzl algebras if the quantum deformation parameters q and r are not roots of unity. Thus (5.5) also applies to Birman-Wenzl algebras when q and r are not roots of unity.

5.1.4 Transpose permutation lattice

It is easily seen that the following proposition holds for any permutation lattice $w = (w_1, w_2, \ldots, w_f)$ of order f.

Proposition 5.1.2. *The word $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_f)$ defined by*

$$\bar{w}_i = \hat{\#}_{w^{(i-1)}}(w_i) + \theta(w_i), \tag{5.6}$$

where

$$\theta(w_i) = \begin{cases} 1 & \text{if } w_i > 0 \\ 0 & \text{if } w_i < 0 \end{cases} \tag{5.7}$$

is a permutation lattice of order f (note that $w^{(0)}$ is the empty word $\emptyset$ and $\hat{\#}_{w^{(0)}}(w_i) = 0$ for all $0 \leq i \leq f$).

We call $\bar{w}$ the *transpose permutation lattice* of w and denote it by w^t. One may show the following desired involution property

$$(w^t)^t = w. \tag{5.8}$$

Relation (5.8) generalizes the corresponding one for a standard Young tableau written as permutation lattice.

5.1.5 Some combinatoric functions for permutation lattices

Following [19] and rewriting in our permutation lattice language, we define

$$\bigtriangledown_i(w) = (w_i^t - w_i - x) + x\theta(w_i) \tag{5.9}$$

where $w = (w_1, w_2, \ldots, w_f)$ is, as usual, a permutation lattice of order f and $1 \leq i \leq f$. Here $x \in \mathbb{C}$ is a parameter (the same defining $\mathfrak{B}_f(x)$).

Given two permutation lattices of order f, u and v, with the same diagram λ, we can construct the "diamond" function as follows:

$$\diamondsuit_i(u, v) = \bigtriangledown_{i+1}(u) - \bigtriangledown_i(v). \tag{5.10}$$

We note that, if $u_h = v_h$ for all $h \neq i$ and $h \neq i+1$, the following simmetry property holds:

$$\diamondsuit_i(u, v) = \diamondsuit_i(v, u). \tag{5.11}$$

Further, the diamond function is related to the usual axial distance for standard Young tableaux. Precisely, given a tableau σ and the associated permutation lattice w, we have that

$$d_i(w) = \diamondsuit_i(w, w), \tag{5.12}$$

where $d_i(w)$ denotes the axial distance between the boxes i and $i+1$ in the Young diagram of σ. So, the diamond function provides a way to extend the definition of axial distance to permutation lattices. In fact, the axial distance between i and j in the permutation lattice w can be defined by

$$d_{ij}(w) = \begin{cases} \sum_{h=i}^{j-1} \diamondsuit_i(w, w) & \text{if } i < j \\ 0 & \text{if } i = j \\ -\sum_{h=j}^{i-1} \diamondsuit_i(w, w) & \text{if } i > j \end{cases} \tag{5.13}$$

Finally, following [107], for each Young diagram λ, one can define the polinomials

$$P_\lambda(x) = \prod_{(i,j)\in\lambda} \frac{x - 1 + d(i,j)}{h(i,j)}, \tag{5.14}$$

where $h(i,j)$ is the the "hook" function evaluated for the box in the ith row and jth column of λ:

$$h(i,j) = \lambda_i + \lambda'_j - i - j + 1 \tag{5.15}$$

and $d(i,j)$ is given by

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \leq j \\ -\lambda'_i - \lambda'_j + i + j - 1 & \text{if } i > j \end{cases} \tag{5.16}$$

with λ_i denoting the length of the ith row and λ'_j the length of the jth column in λ.

We remark that the polinomial function (5.14) has the property that $P_\lambda(2n+1)$ is the dimension of each irreducible representation V^λ of the orthogonal group $SO(2n+1)$.

5.1.6 Explicit actions

Now we can give the explicit action [19] for the generators of Brauer algebras $\mathfrak{B}_f(x)$ on the Gelfand-Tzetlin basis parameterized by permutation, but first we need the following definitions:

Definition 5.1.3. *Let $u = (u_1, u_2, \ldots, u_f)$ and $v = (v_1, v_2, \ldots, v_f)$ be two permutation lattices of order f which have the same diagram λ. We say that u is i-*coupled *to v (or that u and v are i-*coupled*) if*

$$u_h = v_h \tag{5.17}$$

for all $h \in \{1, \ldots, i-1, i+2, \ldots, f\}$ and we denote such a relation by $u \overset{i}{\leftrightarrow} v$.

Definition 5.1.4. *Let $u = (u_1, u_2, \ldots, u_f)$ and $v = (v_1, v_2, \ldots, v_f)$ be two i-coupled permutation lattices. We say that u is $\bar{i}$-*coupled *to v (or that u and v are $\bar{i}$-*coupled*) if*

$$u_i = -u_{i+1}, \; v_i = -v_{i+1} \tag{5.18}$$

and we denote such a relation by $u \overset{\bar{i}}{\leftrightarrow} v$.

Finally, it will be useful to introduce the $(i \setminus \bar{i})$-*coupling* as follow: given two pemutation lattices u and v, we say that u is $(i \setminus \bar{i})$-coupled to v (or that u and w are $(i \setminus \bar{i})$-*coupled*) if it results $u \overset{i}{\leftrightarrow} v$ but *not* $u \overset{\bar{i}}{\leftrightarrow} v$. We denote such a relation by $u \overset{i\setminus\bar{i}}{\leftrightarrow} v$.

Of course the previous definitions can also be given for tableaux. We simply say that two tableau σ and τ are i-coupled or $\bar{i}$-coupled if the corresponding permutation lattices $w(\sigma)$ and $w(\tau)$ are i-coupled or $\bar{i}$-coupled, respectively. Note that the i-coupling relation just given coincides with the classical i-coupling relation given in chapter three if σ and τ are standard Young tableaux.

Let $[f, \lambda]$ be an irrep for the Brauer algebras $\mathfrak{B}_f(x)$. The standard Gelfand-Tzetlin base for such an irrep can be parameterized by all permutation lattices w of order f and diagram λ: $\{|f; \lambda; w\rangle\}$. The explicit action of the $\mathfrak{B}_f(x)$ generators g_i and e_i on such vectors is described by the following theorem:

Theorem 5.1.1. *Let u and v two permutation lattices of order f and diagram λ and $|f; \lambda; u\rangle$, $|f; \lambda; v\rangle$ two standard base vectors for the irrep $[f, \lambda]$ of $\mathfrak{B}_f(x)$.*

- *If u and v are* not *i-coupled, then*

$$\langle f; \lambda; u|g_i|f; \lambda; v\rangle = \langle f; \lambda; u|e_i|f; \lambda; v\rangle = 0. \tag{5.19}$$

- *If u and v are i-coupled but* not *$\bar{i}$-coupled, then*

$$\langle f; \lambda; u|g_i|f; \lambda; v\rangle = \begin{cases} \frac{1}{d_i(u)} & \text{if } u = v \\ \sqrt{1 - \frac{1}{d_i^2(u)}} & \text{if } u \neq v \end{cases} \tag{5.20}$$

and

$$\langle f; \lambda; u|e_i|f; \lambda; v\rangle = 0, \tag{5.21}$$

where $d_i(u) = \diamondsuit_i(u, u)$ (as in (5.12)).

- *If u and v are $\bar{i}$-coupled, then*

$$\langle f; \lambda; u|g_i|f; \lambda; v\rangle = \begin{cases} \frac{1}{\diamondsuit_i(u,u)}\left(1 - \frac{P_{Y(u^{(i)})}(x)}{P_{Y(u^{(i-1)})}(x)}\right) & \text{if } u = v \\ -\frac{1}{\diamondsuit_i(u,v)} \frac{\sqrt{P_{Y(u^{(i)})}(x) P_{Y(v^{(i)})}(x)}}{P_{Y(u^{(i-1)})}(x)} & \text{if } u \neq v \end{cases} \tag{5.22}$$

and

$$\langle f;\lambda;u|e_i|f;\lambda;v\rangle = \frac{\sqrt{P_{Y(u^{(i)})}(x)P_{Y(v^{(i)})}(x)}}{P_{Y(u^{(i-1)})}(x)}, \tag{5.23}$$

where $Y(w)$ denotes the diagram of the permutation lattice w.

We observe that the previous theorem provides the same action for g_i given in chapter three if u and v are not $\bar{i}$-coupled (as the case of standard Young tableaux). Furthermore, we can easily verify that both g_i and e_i are *hermitian* operator on the invariant irreducible spaces of Brauer algebras.

5.2 The subduction problem

Subdction coefficients for the reduction $[f,\lambda] \downarrow \mathfrak{B}_{f_1}(x) \times \mathfrak{B}_{f_2}(x)$ $(f_1 + f_2 = f)$ define the base changing matrix which makes explicit the decomposition in block-diagonal form:

$$[f,\lambda] = \bigoplus_{\lambda_1,\lambda_2} \{f_1, f_2; \lambda; \lambda_1, \lambda_2\}\ [f_1,\lambda_1] \otimes [f_2,\lambda_2]. \tag{5.24}$$

Therefore, each non-standard base vector for $[f,\lambda]$ is given by the tensor product of two standard base vectors for the irreps $[f_1,\lambda_1]$ and $[f_2,\lambda_2]$. $\{f_1, f_2; \lambda; \lambda_1, \lambda_2\}$ denotes the Clebsch-Gordan series which provide the multiplicity of $[f_1,\lambda_1] \otimes [f_2,\lambda_2]$ in $[f,\lambda]$.

The irreps of $\mathfrak{B}_{f_1}(x) \times \mathfrak{B}_{f_2}(x)$ may be labelled by $(f_1, f_2; \lambda_1, \lambda_2)$ with λ_1 and λ_2 suitable partitions (shapes). In the same way, each element of the basis is labelled by pairs of permutation lattices.

As in the case of subduction problem for $\mathfrak{S}_f$ (described in chapter three), we write the non-standard base vectors $|f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle$ of $[f_1,\lambda_1] \otimes [f_2,\lambda_2]$ in terms of the standard base vectors $|f;\lambda;w\rangle$ of $[f,\lambda]$ $(f_1 + f_2 = f)$:

$$|f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle_\eta = \sum_{w\in\Xi_f^\lambda} |f;\lambda;w\rangle\langle f;\lambda;w|f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle_\eta \tag{5.25}$$

where Ξ_f^λ represents the set of all permutation lattices of order f and diagram λ. Thus $\langle f;\lambda;w|f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle_\eta$ are the SDCs (subduction coefficients) of $[f,\lambda] \downarrow [f_1,\lambda_1] \otimes [f_2,\lambda_2]$ with given multiplicity label η.

Again, the SDCs satisfy the following unitary conditions:

$$\sum_{w} \langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle_\eta \ \langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2';w_1,w_2'\rangle_{\eta'} = \delta_{\lambda_2\lambda_2'}\delta_{w_2w_2'}\delta_{\eta\eta'} \quad (5.26)$$

$$\sum_{\lambda_2 w_2 \eta} \langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle_\eta \ \langle f;\lambda;w'|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle_\eta = \delta_{ww'}. \quad (5.27)$$

5.2.1 Subduction system

Following the guidelines given for the subduction problem in symmetric groups, we now construct a matrix in such a way that the SDCs are the components of the kernel basis vectors. The dimension of such a kernel space is equal to the multiplicity for the subduction issue we are considering.

The action of g_i and e_i on the non-standard base vectors is given by

$$g_i|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle = \begin{cases} (g_i|f_1;\lambda_1;w_1\rangle)\otimes|f_2;\lambda_2;w_2\rangle & \text{if } 1\le i\le f_1-1 \\ |f_1;\lambda_1;w_1\rangle\otimes(g_i|f_2;\lambda_2;m_2\rangle) & \text{if } f_1+1\le i\le f-1 \end{cases} \quad (5.28)$$

and

$$e_i|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle = \begin{cases} (e_i|f_1;\lambda_1;w_1\rangle)\otimes|f_2;\lambda_2;w_2\rangle & \text{if } 1\le i\le f_1-1 \\ |f_1;\lambda_1;w_1\rangle\otimes(e_i|f_2;\lambda_2;m_2\rangle) & \text{if } f_1+1\le i\le f-1 \end{cases}. \quad (5.29)$$

From (5.28) and (5.29), for $1\le l\le f_1-1$, we get

$$\langle f;\lambda;w|g_l|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle = \langle f;\lambda;w|(g_l|f_1;\lambda_1;w_1\rangle)\otimes|f_2;\lambda_2;w_2\rangle \quad (5.30)$$

and

$$\langle f;\lambda;w|e_l|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle = \langle f;\lambda;w|(e_l|f_1;\lambda_1;w_1\rangle)\otimes|f_2;\lambda_2;w_2\rangle. \quad (5.31)$$

Writing $|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle$ and $g_l|f_1;\lambda_1;w_1\rangle$ in the standard basis of $[f,\lambda]$ and $[f_1,\lambda_1]$ respectively, (5.30) and (5.31) become

$$\sum_{u\in\Theta_i(w)} \langle f;\lambda;w|g_l|f;\lambda;u\rangle\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle =$$

$$\sum_{v\in\Theta_i(w_1)} \langle f_1;\lambda_1;v|g_l|f_1;\lambda_1;w_1\rangle\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;v,w_2\rangle \quad (5.32)$$

$$\sum_{u\in\bar{\Theta}_i(w)} \langle f;\lambda;w|e_l|f;\lambda;u\rangle\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle =$$

$$\sum_{v\in\bar{\Theta}_i(w_1)} \langle f_1;\lambda_1;v|e_l|f_1;\lambda_1;w_1\rangle\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;v,w_2\rangle \quad (5.33)$$

where $\Theta_i(w)$ and $\bar{\Theta}_i(w)$ denote the sets of all permutation lattices which are i-coupled and $\bar{i}$-coupled with w respectively.

In an analogous way, for $f_1+1\leq l\leq f-1$, we get

$$\sum_{u\in\Theta_i(w)} \langle f;\lambda;w|g_l|f;\lambda;u\rangle\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle =$$

$$\sum_{v\in\Theta_i(w_2)} \langle f_2;\lambda_2;v|g_l|f_2;\lambda_2;w_2\rangle\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_1,v\rangle \quad (5.34)$$

$$\sum_{u\in\bar{\Theta}_i(w)} \langle f;\lambda;w|e_l|f;\lambda;u\rangle\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_1,w_2\rangle =$$

$$\sum_{v\in\bar{\Theta}_i(w_2)} \langle f_2;\lambda_2;v|e_l|f_2;\lambda_2;w_2\rangle\langle f_1,f_2;\lambda;w|f_1;\lambda_1,\lambda_2;w_1,v\rangle. \quad (5.35)$$

Then, once we know the explicit action of the generators of $\mathfrak{B}_{f_1}(x)\times\mathfrak{B}_{f_2}(x)$ on the standard basis, (5.32), (5.33), (5.34) and (5.35) (written for all $l\in\{1,\ldots,f_1-1,f_1+1,\ldots,f-1\}$ and all permutation lattices w, w_1, w_2 of order f and diagrams λ, λ_1 and λ_2 respectively) define a linear equation system of the form:

$$\Omega(\lambda;f_1,f_2;\lambda_1,\lambda_2)\ \chi = 0 \quad (5.36)$$

where $\Omega(\lambda;f_1,f_2;\lambda_1,\lambda_2)$ is the *subduction matrix* and χ is a vector with components given by the SDCs of $[f,\lambda]\downarrow[f_1,\lambda_1]\otimes[f_2,\lambda_2]$. (5.36) is a linear equation system with $\dim(\mathfrak{B}_f(x);[f,\lambda])\cdot\dim(\mathfrak{B}_{f_1}(x);[f_1,\lambda_1])\cdot\dim(\mathfrak{B}_{f_2}(x);[f_2,\lambda_2])$ unknowns (the SDCs) and $2(f-2)\cdot\dim(\mathfrak{B}_f(x);[f,\lambda])\cdot\dim(\mathfrak{B}_{f_1}(x);[f_1,\lambda_1])\cdot\dim(\mathfrak{B}_{f_2}(x);[f_2,\lambda_2])$ equations.

5.2.2 Explicit form of the subduction system

It will be useful to give the following definitions of i-coupling and $\bar{i}$-coupling on pairs of permutation lattices:

Definition 5.2.1. *Given two pairs $w_{12} = (w_1, w_2)$ and $w'_{12} = (w'_1, w'_2)$, each one composed of two permutation lattices of order f_1 and f_2 respectively, we say that w_{12} is i-*coupled *to w'_{12} (or that w_{12} and w'_{12} are i-*coupled*) when*

$$\begin{cases} \begin{array}{l} w_1 \overset{i}{\leftrightarrow} w'_1 \\ w_2 = w'_2 \end{array} & \text{if } 1 \le i \le f_1 - 1 \\ \\ \begin{array}{l} w_1 = w'_1 \\ w_2 \overset{i-f_1}{\leftrightarrow} w'_2 \end{array} & \text{if } f_1 + 1 \le i \le f_1 + f_2 - 1 \end{cases}$$

and we denote such a relation by $w_{12} \overset{i}{\leftrightarrow} w'_{12}$.

Definition 5.2.2. *Given two pairs $w_{12} = (w_1, w_2)$ and $w'_{12} = (w'_1, w'_2)$, each one composed of two permutation lattices of order f_1 and f_2 respectively, we say that w_{12} is $\bar{i}$-*coupled *to w'_{12} (or that w_{12} and w'_{12} are $\bar{i}$-*coupled*) when*

$$\begin{cases} \begin{array}{l} w_1 \overset{\bar{i}}{\leftrightarrow} w'_1 \\ w_2 = w'_2 \end{array} & \text{if } 1 \le i \le f_1 - 1 \\ \\ \begin{array}{l} w_1 = w'_1 \\ w_2 \overset{\overline{i-f_1}}{\leftrightarrow} w'_2 \end{array} & \text{if } f_1 + 1 \le i \le f_1 + f_2 - 1 \end{cases}$$

and we denote such a relation by $w_{12} \overset{\bar{i}}{\leftrightarrow} w'_{12}$.

Of course, the $(i \setminus \bar{i})$-coupling relation on pairs of permutation lattices is defined by

$$w_{12} \overset{i\setminus\bar{i}}{\leftrightarrow} w'_{12} \iff \begin{cases} \begin{array}{l} w_1 \overset{i\setminus\bar{i}}{\leftrightarrow} w'_1 \\ w_2 = w'_2 \end{array} & \text{if } 1 \le i \le f_1 - 1 \\ \\ \begin{array}{l} w_1 = w'_1 \\ w_2 \xleftrightarrow{i\setminus\overline{i-f_1}} w'_2 \end{array} & \text{if } f_1 + 1 \le i \le f_1 + f_2 - 1 \end{cases}$$

Denoted by $\Theta_i(w_{12})$ the set of all pairs of permutation lattices which are i-coupled to the pair $w_{12} = (w_1, w_2)$ and by $\bar{\Theta}_i(w_{12})$ the set of all pairs of permutation lattices which are $\bar{i}$-coupled to the pair $w_{12} = (w_1, w_2)$, equations (5.32), (5.33), (5.34) and (5.35) can be written as

$$
\begin{aligned}
&(\langle f_1, f_2; \lambda_1, \lambda_2; w_1, w_2 | g_i | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle - \langle f; \lambda; w | g_i | f; \lambda; w\rangle) - \\
&\qquad \sum_{u \in \Theta_i'(w)} \langle f; \lambda; w | g_i | f; \lambda; u\rangle \langle f; \lambda; u | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle + \\
&\sum_{(u_1,u_2) \in \Theta_i'(w_{12})} \langle f_1, f_2; \lambda_1, \lambda_2; w_1, w_2 | g_i | f_1, f_2; \lambda_1, \lambda_2; u_1, u_2\rangle \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; u_1, u_2\rangle = 0
\end{aligned}
\tag{5.37}
$$

and

$$
\begin{aligned}
&(\langle f_1, f_2; \lambda_1, \lambda_2; w_1, w_2 | e_i | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle - \langle f; \lambda; w | e_i | f; \lambda; w\rangle) - \\
&\qquad \sum_{u \in \bar{\Theta}_i'(w)} \langle f; \lambda; w | e_i | f; \lambda; u\rangle \langle f; \lambda; u | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle + \\
&\sum_{(u_1,u_2) \in \bar{\Theta}_i'(w_{12})} \langle f_1, f_2; \lambda_1, \lambda_2; w_1, w_2 | e_i | f_1, f_2; \lambda_1, \lambda_2; u_1, u_2\rangle \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; u_1, u_2\rangle = 0,
\end{aligned}
\tag{5.38}
$$

where $\Theta_i'(w)$ and $\Theta_{\bar{i}}'(w)$ represent the sets $\Theta_i(w) \setminus \{w\}$ and $\Theta_{\bar{i}}(w) \setminus \{w\}$, respectively (and analogously for $\Theta_i'(w_{12})$ and $\Theta_{\bar{i}}'(w_{12})$).

By remembering the statement of theorem 5.1.1, we can distinguish four possible cases for the structure of the equations (5.37) and (5.38).

1. *Crossing*: $w \overset{i\backslash\bar{i}}{\leftrightarrow} w$ and $w_{12} \overset{i\backslash\bar{i}}{\leftrightarrow} w_{12}$.

   The subduction equations become of the form given in (3.12):

$$
\begin{aligned}
&\alpha^{(i\backslash\bar{i})}_{w,w_{12}} \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle - \beta^{(i\backslash\bar{i})}_{w} \langle f; \lambda; g_i(w) | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle + \\
&+\beta^{(i\backslash\bar{i})}_{w_{12}} \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; g_i(w_1), w_2\rangle = 0 \qquad \text{if } i \in \{1, \dots, n_1 - 1\}\ ,
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
&\alpha^{(i\backslash\bar{i})}_{w,w_{12}} \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle - \beta^{(i\backslash\bar{i})}_{w} \langle f; \lambda; g_i(w) | f_1, f_2; \lambda_1, \lambda_2; w_1, w_2\rangle + \\
&+\beta^{(i\backslash\bar{i})}_{w_{12}} \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; w_1, g_i(w_2)\rangle = 0 \qquad \text{if } i \in \{n_1 + 1, \dots, n - 1\}
\end{aligned}
\tag{5.40}
$$

   where

$$
\alpha^{(i\backslash\bar{i})}_{w,w_{12}} = \frac{1}{d_i(w_{12})} - \frac{1}{d_i(w)}
\tag{5.41}
$$

$$\beta_w^{(i\backslash\bar{i})} = \sqrt{1 - \frac{1}{d_i^2(w)}} \tag{5.42}$$

$$\beta_{w_{12}}^{(i\backslash\bar{i})} = \sqrt{1 - \frac{1}{d_i^2(w_{12})}}. \tag{5.43}$$

Notice that, by definition,

$$d_i(w_{12}) = \begin{cases} d_i(w_1) & \text{if } 1 \le i \le f_1 - 1 \\ d_{i-f_1}(w_2) & \text{if } f_1 + 1 \le i \le f - 1 \end{cases}, \tag{5.44}$$

where the axial distance d_i is the same of (5.12) and, given a permutation lattice $w = (w_1, \ldots, w_i, w_{i+1}, \ldots, w_f)$, the g_i action is naturally defined in the following way: consider the word $\tilde{w} = (w_1, \ldots, w_{i-1,} w_{i+1,} w_i, w_{i+2}, \ldots, w_f)$ obtained by w interchanging the elements w_i and w_{i+1}. If $\tilde{w}$ is another permutation lattice then we put $g_i(w) = \tilde{w}$, otherwise we set $g_i(w) = w$. In an analogous way, it is defined a g_i action on pairs of permutation lattices of order f_1 and f_2, respectively:

$$g_i(w_1, w_2) = \begin{cases} (g_i(w_1), w_2) & \text{if } 1 \le i \le f_1 - 1 \\ (w_1, g_i(w_2)) & \text{if } f_1 + 1 \le i \le f_1 + f_2 - 1 \end{cases}. \tag{5.45}$$

2. *Horizontal bridge*: $w \overset{\bar{i}}{\leftrightarrow} w$ and $w_{12} \overset{i\backslash\bar{i}}{\leftrightarrow} w_{12}$.

   In this case, we get the equations:

$$\alpha_{w,w_{12}}^{(i\backslash\bar{i})} \langle f; \lambda; w | f_1, f_2; \lambda_1, \lambda_2; w_{12} \rangle + \beta_{w_{12}}^{(i\backslash\bar{i})} \langle \lambda; w | f_1, f_2; \lambda_1, \lambda_2; g_i(w_{12}) \rangle = \\ - \frac{\sqrt{P_{Y(w^{(i)})}(x)}}{P_{Y(w^{(i-1)})}(x)} \sum_{u \in \bar{\Theta}_i(w)} \frac{\sqrt{P_{Y(u^{(i)})}(x)}}{\diamondsuit_i(w,u)} \langle f; \lambda; u | f_1, f_2; \lambda_1, \lambda_2; w_{12} \rangle \tag{5.46}$$

   and

$$\sum_{u \in \bar{\Theta}_i(w)} \sqrt{P_{Y(u^{(i)})}(x)} \langle f; \lambda; u | f_1, f_2; \lambda_1, \lambda_2; w_{12} \rangle = 0, \tag{5.47}$$

   where we have used the usual notation $w_{12} = (w_1, w_2)$ in the mathematical symbol of SDC.

3. *Vertical bridge*: $w \overset{i\backslash\bar{i}}{\leftrightarrow} w$ and $w_{12} \overset{\bar{i}}{\leftrightarrow} w_{12}$.

   In an analogous way as the previous case, we have:

$$\alpha^{(i\backslash\bar{i})}_{w,w_{12}}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle - \beta^{(i\backslash\bar{i})}_{w}\langle \lambda;g_i(w)|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle = \frac{\sqrt{P_{Y(w^{(i)}_{12})}(x)}}{P_{Y(w^{(i-1)}_{12})}(x)} \sum_{u_{12}\in\bar{\Theta}_i(w_{12})} \frac{\sqrt{P_{Y(u^{(i)}_{12})}(x)}}{\diamondsuit_i(w_{12},u_{12})}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle \quad (5.48)$$

   and

$$\sum_{u_{12}\in\bar{\Theta}_i(w_{12})} \sqrt{P_{Y(u^{(i)}_{12})}(x)}\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle = 0. \quad (5.49)$$

   Here, as in the definition of axial distance for pairs of permutation lattices, we have set:

$$\diamondsuit_i(w_{12},u_{12}) = \begin{cases} \diamondsuit_i(w_1,u_1) & \text{if } 1\le i\le f_1-1 \\ \diamondsuit_{i-f_1}(w_2,u_2) & \text{if } f_1+1\le i\le f-1 \end{cases} \quad (5.50)$$

   and

$$w^{(i)}_{12} = \begin{cases} w^{(i)}_1 & \text{if } 1\le i\le f_1-1 \\ w^{(i-f_1)}_2 & \text{if } f_1+1\le i\le f-1 \end{cases}. \quad (5.51)$$

4. *Singlet*: $w \overset{\bar{i}}{\leftrightarrow} w$ and $w_{12} \overset{\bar{i}}{\leftrightarrow} w_{12}$.

   In this last case, the subduction equations take the form of:

$$\alpha^{(i\backslash\bar{i})}_{w,w_{12}}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle = -\frac{\sqrt{P_{Y(w^{(i)})}(x)}}{P_{Y(w^{(i-1)})}(x)} \sum_{u\in\bar{\Theta}_i(w)} \frac{\sqrt{P_{Y(u^{(i)})}(x)}}{\diamondsuit_i(w,u)}\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle + \frac{\sqrt{P_{Y(w^{(i)}_{12})}(x)}}{P_{Y(w^{(i-1)}_{12})}(x)} \sum_{u_{12}\in\bar{\Theta}_i(w_{12})} \frac{\sqrt{P_{Y(u^{(i)}_{12})}(x)}}{\diamondsuit_i(w_{12},u_{12})}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle \quad (5.52)$$

   and

$$\frac{\sqrt{P_{Y(w^{(i)})}(x)}}{P_{Y(w^{(i-1)})}(x)} \sum_{u\in\bar{\Theta}_i(w)} \sqrt{P_{Y(u^{(i)})}(x)}\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle = \frac{\sqrt{P_{Y(w^{(i)}_{12})}(x)}}{P_{Y(w^{(i-1)}_{12})}(x)} \sum_{u_{12}\in\bar{\Theta}_i(w_{12})} \sqrt{P_{Y(u^{(i)}_{12})}(x)}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle. \quad (5.53)$$

5.3 Subduction graph

Let us now consider the three shapes $(f;\lambda;f_1,f_2;\lambda_1,\lambda_2)$ with $f_1+f_2=f$. We call *node* each ordered sequence of three permutation lattices $(w;w_1,w_2)$ such as $w\in\Xi_f^\lambda$, $w\in\Xi_{f_1}^{\lambda_1}$ and $w\in\Xi_{f_2}^{\lambda_2}$. We denote it by $\langle w;w_1,w_2\rangle$ or $\langle w;w_{12}\rangle$. The set of all nodes of $(f;\lambda;f_1,f_2;\lambda_1,\lambda_2)$ is called *subduction grid* (or simply *grid*) and it is as usual denoted by G. Building on the case of permutation lattices, the following definition extends the i-coupling relation to the nodes of the grid.

Definition 5.3.1. *Fixed the grid $(f;\lambda;f_1,f_2;\lambda_1,\lambda_2)$, and given two nodes $n=(w;w_{12})$ and $n'=(w';w'_{12})$, we say that n is* i-coupled *to n' (or that n and n' are* i-coupled*) if $w\overset{i}{\leftrightarrow}w'$ and $w_{12}\overset{i}{\leftrightarrow}w'_{12}$. Then we write $n\overset{i}{\leftrightarrow}n'$.*

Once i is fixed, it easy to see that the i-coupling is an equivalence relation on the grid. We name i-layer the partition of G which is associated to such a relation and we denote it by $G^{(i)}$. If n and n' are two distinct nodes of the grid such that $n\overset{i}{\leftrightarrow}n'$, then they are connected by an edge with a label for i.

Following the structure of explicit form for the subduction equations given in the previous section, we note that there are only four possible kinds of *i-layer configurations* beetween nodes in G:

1. *crossing* i-layer: $G^{(i\backslash\bar{i})}=\{\langle w;w_{12}\rangle\in G\mid w\overset{i\backslash\bar{i}}{\leftrightarrow}w \text{ and } w_{12}\overset{i\backslash\bar{i}}{\leftrightarrow}w_{12}\}$;
2. *horizontal bridge* i-layer: $G^{(i-\bar{i})}=\{\langle w;w_{12}\rangle\in G\mid w\overset{\bar{i}}{\leftrightarrow}w \text{ and } w_{12}\overset{i\backslash\bar{i}}{\leftrightarrow}w_{12}\}$;
3. *vertical bridge* i-layer: $G^{(\bar{i}-i)}=\{\langle w;w_{12}\rangle\in G\mid w\overset{i\backslash\bar{i}}{\leftrightarrow}w \text{ and } w_{12}\overset{\bar{i}}{\leftrightarrow}w_{12}\}$;
4. *singlet* i-layer: $G^{(\bar{i})}=\{\langle w;w_{12}\rangle\in G\mid w\overset{\bar{i}}{\leftrightarrow}w \text{ and } w_{12}\overset{\bar{i}}{\leftrightarrow}w_{12}\}$;

Clearly, $G^{(i\backslash\bar{i})}$, $G^{(i-\bar{i})}$, $G^{(\bar{i}-i)}$ and $G^{(\bar{i})}$ are disjoint sets and we have $G^{(i)}=G^{(i\backslash\bar{i})}\cup G^{(i-\bar{i})}\cup G^{(\bar{i}-i)}\cup G^{(\bar{i})}$. The crossing i-layer corresponds to the i-layer defined in chapter three for the subduction problem in symmetric groups. So, crossing, bridge and singlet configurations for the i-coupling relation are also defined in an analogous way for such a set.

Definition 5.3.2. *We call* subduction graph *the* overlap *of all i-layers obtained by identification of the corresponding nodes.*

The definition just given is a good definition of subduction graph, because there is at most *one* edge connecting two distinct nodes. This is ensured by the osservation that if n and n' are two distinct nodes which are i-coupled and j-coupled then we necessarily have $i = j$.

We remark that if the grid defined by $(f; \lambda; f_1, f_2; \lambda_1, \lambda_2)$ is such that f is equal to the number of boxes of λ, f_1 is equal to the number of boxes of λ_1 and f_3 to the number of boxes λ_3, then the definition of subduction graph just given becomes that one given for the subduction problem in symmetric groups (chapter three).

5.4 Structure of the subduction space

The solution of (5.36) can be seen as an intersection of $f - 2$ subspaces $\chi^{(i)}$ such that each one satisfies

$$\Omega^{(i)}(f_1, f_2; \lambda; \lambda_1, \lambda_2)\ \chi^{(i)} = 0. \tag{5.54}$$

Here, $\Omega^{(i)}(f_1, f_2; \lambda; \lambda_1, \lambda_2)$ is defined by the equations (5.32), (5.33), (5.34) and (5.35) written for a fixed $i \in \{1, \ldots, f_1 - 1, f_1 + 1, \ldots f - 1\}$. The definitions of grid, i-layer and the explicit form for the subduction equations, given in the previous sections, provide a suitable way to describe the solution space of (5.54) by the one-to-one corrispondence between the nodes of $(f_1, f_2; \lambda; \lambda_1, \lambda_2)$ and the SDCs for the subduction $[f_1 + f_2, \lambda] \downarrow [f_1, \lambda_1] \otimes [f_2, \lambda_2]$. To find the structure of the subduction space $\chi^{(i)}$ which is associated to the i-layer we only need to describe the structure of the spaces which are associated to $G^{(i\backslash\bar{i})}$, $G^{(i-\bar{i})}$, $G^{(\bar{i}-i)}$ and $G^{(\bar{i})}$ that we call *crossing space*, *horizontal bridge space*, *vertical bridge space* and *singlet space*, respectively.

5.4.1 Crossing space

The solution for the crossing equations was already described in chapters three and four by the subduction graph method. In fact, we observe that the structure of the two subduction systems are quite similar. For the Brauer algebras case, we only need to pay attention to use the new definition of axial distance given in (5.12) because such a definition leads to expressions for the coefficients $\alpha^{(i\backslash\bar{i})}_{w;w_{12}}$, $\beta^{(i\backslash\bar{i})}_{w}$ and $\beta^{(i\backslash\bar{i})}_{w_{12}}$ which are algebraic functions of $\mathbb{C}(x)$ instead of simple real numbers (see theorem 5.1.1). However, relations and conditions of the subduction graph method for symmetric groups still remain valid for Brauer algebras subduction issue.

5.4.2 Bridge spaces

Let us now consider the case of the horizontal bridge space. From equation (5.47), for each node $\langle w; w_{12}\rangle \in G^{(i-\bar{i})}$, we find that subduction coefficients of the horizontal bridge type, $\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle$, are the components of vectors of a vectorial space that is the kernel of the operator e_i acting on the invariant irreducible subspace defined by all the permutation lattices u which are $\bar{i}$-coupled to w. From the relation $e_i^2 = xe_i$, we note that the eigenvalues of e_i are 0 and x. Therefore $\chi^{(i-\bar{i})}$ in general is not the trivial space. So, finding such SDCs is equivalent to determind the kernel space of e_i in the explicit form given in theorem 5.1.1.

Once we now the SDCs of the form $\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle$, we can determine the coefficients $\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;g_i(w_{12})\rangle$ by using (5.46):

$$\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;g_i(w_{12})\rangle = -\frac{\alpha^{(i\backslash\bar{i})}_{w,w_{12}}}{\beta^{(i\backslash\bar{i})}_{w_{12}}}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle - \frac{1}{\beta^{(i\backslash\bar{i})}_{w}}\frac{\sqrt{P_{Y(w^{(i)})}(x)}}{P_{Y(w^{(i-1)})}(x)}\sum_{u\in\bar{\Theta}_i(w)}\frac{\sqrt{P_{Y(u^{(i)})}(x)}}{\diamondsuit_i(w,u)}\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle \tag{5.55}$$

(note that if $g_i(w_{12}) \neq w_{12}$ then $\beta^{(i\backslash\bar{i})}_{w_{12}} \neq 0$).

In an analogous way for the vertical bridge space, from equation (5.49) we find that subduction coefficients $\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle$, are the components of vectors of a vectorial space that is the kernel of the operator e_i acting on the invariant irreducible subspace defined by all pairs of the permutation lattices u_{12} which are $\bar{i}$-coupled to w_{12}.

Again, once we now the SDCs of the form $\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle$, we can determine the coefficients $\langle f;\lambda;u|f_1,f_2;\lambda_1,\lambda_2;g_i(w_{12})\rangle$ by using (5.48):

$$\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;g_i(u_{12})\rangle = \frac{\alpha^{(i\backslash\bar{i})}_{w,w_{12}}}{\beta^{(i\backslash\bar{i})}_{w}}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;w_{12}\rangle - \frac{1}{\beta^{(i\backslash\bar{i})}_{w}}\frac{\sqrt{P_{Y(w^{(i)}_{12})}(x)}}{P_{Y(w^{(i-1)}_{12})}(x)}\sum_{u_{12}\in\bar{\Theta}_i(w_{12})}\frac{\sqrt{P_{Y(u^{(i)}_{12})}(x)}}{\diamondsuit_i(w_{12},u_{12})}\langle f;\lambda;w|f_1,f_2;\lambda_1,\lambda_2;u_{12}\rangle \tag{5.56}$$

(if $g_i(w)\neq w$ then $\beta^{(i\backslash\bar{i})}_{w}\neq 0$)

5.4.3 Singlet space

To understand the structure of the solution for singlet equations, it is useful to introduce the *intertwining* operators:

$$\Omega^{(i)}_{w,w_{12}} = I_w\otimes\rho^{(i)}_{w_{12}} - \rho^{(i)}_{w}\otimes I_{w_{12}}. \tag{5.57}$$

and

$$\bar{\Omega}^{(i)}_{w,w_{12}} = I_w\otimes\bar{\rho}^{(i)}_{w_{12}} - \bar{\rho}^{(i)}_{w}\otimes I_{w_{12}} \tag{5.58}$$

Here, $\rho^{(i)}_{w}$ (resp. $\rho^{(i)}_{w_{12}}$) represents the action of the generators g_i on the invariant irreducible space defined by all permutation lattices (resp. pairs of permutation lattices) which are $\bar{i}$-coupled to w (risp. w_{12}). In an analogous way, $\bar{\rho}^{(i)}_{w}$ (resp. $\bar{\rho}^{(i)}_{w_{12}}$) represents the action of the generators e_i on the invariant irreducible spaces defined by all permutation lattices (resp. pairs of permutation lattices) which are $\bar{i}$-coupled to w (resp. w_{12}). Further, I_w and $I_{w_{12}}$ represent the identity operators on the previous invariant irreducible spaces, respectively. Solving the singlet equations is equivalent to find the kernel space of $\Omega^{(i)}_{w,w_{12}}$ and the kernel space of $\bar{\Omega}^{(i)}_{w,w_{12}}$.

The operator $\rho^{(i)}_{w}$ (resp. $\rho^{(i)}_{w_{12}}$) has eigenvalues 1 and -1, as we can see by the relation $g_i^2 = 1$. Denoted by $g^{(i)}_{w,1}$ (resp. $g^{(i)}_{w_{12},1}$) the eigenvector relative to the eigenvalue 1 and by $g^{(i)}_{w,-1}$ (resp. $g^{(i)}_{w_{12},-1}$) that one relative to the eigenvalue -1, the eigenvectors of the intertwining operator $\Omega^{(i)}_{w,w_{12}}$ are:

1. $g^{(i)}_{w,1}\otimes g^{(i)}_{w_{12},1}$ with eigenvalue 0;

2. $g^{(i)}_{w,-1}\otimes g^{(i)}_{w_{12},1}$ with eigenvalue 2;

3. $g_{w,1}^{(i)} \otimes g_{w_{12},-1}^{(i)}$ with eigenvalue -2;

4. $g_{w,-1}^{(i)} \otimes g_{w_{12},-1}^{(i)}$ with eigenvalue 0.

Therefore, the kernel space is given by $\text{span}(g_{w,1}^{(i)} \otimes g_{w_{12},1}^{(i)}, g_{w,-1}^{(i)} \otimes g_{w_{12},-1}^{(i)})$.

The operator $\bar{\rho}_w^{(i)}$ (resp. $\bar{\rho}_{w_{12}}^{(i)}$) has eigenvalues x and 0 (remember that ${e_i}^2 = xe_i$). Denoted by $e_{w,x}^{(i)}$ (resp. $e_{w_{12},x}^{(i)}$) an eigenvector relative to the eigenvalue x and by $e_{w,0}^{(i)}$ (resp. $e_{w_{12},0}^{(i)}$) one relative to the eigenvalue -1, the eigenvectors of the intertwining operator $\Omega_{w,w_{12}}^{(i)}$ have the form:

1. $e_{w,x}^{(i)} \otimes e_{w_{12},x}^{(i)}$ with eigenvalue 0;

2. $e_{w,0}^{(i)} \otimes e_{w_{12},x}^{(i)}$ with eigenvalue x;

3. $e_{w,x}^{(i)} \otimes e_{w_{12},0}^{(i)}$ with eigenvalue $-x$;

4. $e_{w,0}^{(i)} \otimes e_{w_{12},0}^{(i)}$ with eigenvalue 0.

from which we can construct the kernel space for $\bar{\Omega}_{w,w_{12}}^{(i)}$.

The singlet space is the intersaction of the two kernel spaces just given.

5.5 Orthonormalization and phase conventions

The subduction space given by (5.36) has dimension μ equal to the multiplicity of $[f, \lambda] \downarrow [f_1, \lambda_1] \otimes [f_2, \lambda_2]$. Then SDCs are not univocally determined. A choice of orthonormality between the different copies of multiplicity imposes a precise form for the multiplicity separations.

Following the notation given in chapter three, let $\{\chi_1, \ldots, \chi_\mu\}$ be a basis in the subduction space. Orthonormality implies for the scalar products:

$$(\chi_\eta, \chi_{\eta'}) = \dim(\mathfrak{B}_{f_1}(x), [f_1, \lambda_1]) \dim(\mathfrak{B}_{f_2}(x), [f_2, \lambda_2]) \; \delta_{\eta\eta'}. \tag{5.59}$$

If we denote by χ the matrix which has the basis vectors of the subduction space as columns, we may orthonormalize it by a suitable $\mu \times \mu$ matrix σ, i.e.

$$\tilde{\chi} = \chi\sigma. \tag{5.60}$$

In (5.60) $\tilde{\chi}$ is the matrix which has the orthonormalized basis vectors of the subduction space as columns. Now we can write (5.59) as

$$\sigma^t \; \tau \; \sigma = I, \tag{5.61}$$

where I is the $\mu \times \mu$ identity matrix and τ is the $\mu \times \mu$ positive defined quadratic form given by

$$\tau = \frac{1}{\dim(\mathfrak{B}_{f_1}(x), [f_1, \lambda_1]) \dim(\mathfrak{B}_{f_2}(x), [f_2, \lambda_2])} \; \chi^t \chi. \tag{5.62}$$

From (5.61) we can see σ as the Sylvester matrix of τ, i.e. the matrix for the change of basis that reduces τ in the identity form. We can express σ in terms of the orthonormal matrix O_τ that diagonalizes τ

$$\sigma = O_\tau D_\tau^{-\frac{1}{2}} O, \tag{5.63}$$

where $D_\tau^{-\frac{1}{2}}$ is the diagonal matrix with eigenvalues given by the inverse square root of the eigenvalues of τ and O a *generic* orthogonal matrix. Thus, the general form for the orthonormalized χ is

$$\tilde{\chi} = \chi O_\tau D_\tau^{-\frac{1}{2}} O. \tag{5.64}$$

We notice that in case of multiplicity-free subduction, only one choice of global phase has to be made (for example Young-Yamanouchi phase convention [99]). It derives from the trivial form of the orthogonal 1×1 matrices O and O_τ.

To fix the Young-Yamanouchi phase convention we need an ordering relation on permutation lattices (or pair of permutation lattices) and on nodes of the subduction graph. A possible natural choice is the following: given two distinct permutation lattices of order f and diagram λ, $w = (w_1, w_2, \ldots, w_f)$ and $w' = (w'_1, w'_2, \ldots, w'_f)$, we say that $w < w'$ if the first non-zero element of the word $w - w' = (w_1 - w'_1, w_2 - w'_2, \ldots, w_f - w'_f)$ is a negative number. Such a relation can be extended to pairs of permutation lattices alphabetically: given two distinct pairs of permutation lattices $w_{12} = (w_1, w_2)$ and $w'_{12} = (w'_1, w'_2)$, we say that $w_{12} < w'_{12}$ if $w_1 < w'_1$ or $w_1 = w'_1$ and $w_2 < w'_2$. Resulting from the previous ordering relations, we can provide the ordering relation for nodes of the grid $G = (f; \lambda; f_1, f_2; \lambda_1, \lambda_2)$. For two distinct nodes $n = \langle w; w_{12} \rangle \in G$, $n' = \langle w'; w'_{12} \rangle \in G$ we say $n < n'$ if $w < w'$ or $w = w'$ and $w_{12} < w'_{12}$.

Thus, the Young-Yamanouchi phase convention can be stated as follows: we fix to be *positive* the first non-zero SDC with respect to the ordering relation defined on the corresponding nodes.

We conclude by observing that, in the general case of multiplicity $\mu > 1$, $2^{\mu-1}$ phases deriving from the O_τ matrix and 1 phase from the matrix O have to be fixed. Therefore we have $2^{\mu-1}+1$ phases to choose. Furthermore we have other $\frac{\mu(\mu-1)}{2}$ degrees of freedom deriving from O. In sum we have a total of $(2^{\mu-1}+1)+\frac{\mu(\mu-1)}{2}$ choices to make.

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