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Ph. D. Thesis

Smoothing - Strichartz Estimates for Dispersive Equations Perturbed by a First Order Differential Operator

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Introduction

1. Background of Mathematical Physics

1.1. General Setting. Our main purpose is to represent in the following some new ideas for the study of semilinear dispersive equations, with particular attention on partial differential equations of hyperbolic type. Among the most important dispersive equations of mathematical physics we shall focus our attention to the **Schrödinger equation**

$$\imath u_t(t,x) + \Delta u(t,x) = F(t,x)\ ,\quad t \geq 0\ ,\ x \in \mathbb{R}^n \tag{0.1.1a}$$

$$u(0,x) = u_0(x)\ , \tag{0.1.1b}$$

and the **Wave equation**

$$\Box u(t,x) = F(t,x)\ ,\quad t \geq 0\ ,\ x \in \mathbb{R}^n \tag{0.1.2a}$$

$$u(0,x) = u_0(x)\ ,\quad \partial_t u(0,x) = u_1(x)\ . \tag{0.1.2b}$$

Both the equations (0.1.1a) and (0.1.2a) play a crucial role in Quantum Mechanics (we refer to the books of Reed and Simon [**160**], [**157**], [**159**], [**158**], and also to [**56**], [**139**] for further information). Indeed, let H_0 denote an **Hamiltonian**, defined by

$$H_0(x,p) = \tfrac{p^2}{2m}\ , \tag{0.1.3}$$

where p is the **momentum** and m is the **mass** of the particle. Up to replacing the physical observables with operators, we can consider H_0 as a self-adjoint operator acting on some suitable Hilbert space, usually $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n) \times \dot{H}^1(\mathbb{R}^n)$. Namely, $p = (p_1, \ldots, p_n)$ where each p_j corresponds to the operator $\frac{\hbar}{\imath}\frac{\partial}{\partial x_j}$, where $\hbar$ is the Plank constant, in such a way that $H_0 = -\frac{\hbar^2}{2m}\Delta$. Therefore, the homogeneous versions of (0.1.1a) and (0.1.2a) ($F = 0$) are nothing but the rescaled versions of

$$\imath\hbar\tfrac{\partial}{\partial t}\varphi(t,x) = H_0\varphi(t,x)\ , \qquad \hbar^2\tfrac{\partial^2}{\partial t^2}\varphi(t,x) = -H_0\varphi(t,x)\ .$$

Therefore, from Stone's Theorem we get

$$\varphi(t,x) = \mathrm{e}^{-\imath\frac{H_0 t}{\hbar}}\varphi_0(x)\ , \qquad \Phi(t,x) = \mathrm{e}^{\imath M_0 t}\Phi_0(x)\ ,$$

where $\Phi_0 = (\varphi_0, \varphi_1)$ and M_0 is the matrix defined by

$$M_0 = \begin{pmatrix} \mathbf{0} & -\imath\mathbf{1} \\ \imath\frac{H_0}{\hbar^2} & \mathbf{0} \end{pmatrix}\ ,$$

in such a way that we reduce the wave equation to a first order evolution problem.

Another important hyperbolic problem is the **Dirac equation**

$$i\gamma_\mu\partial_\mu\psi = 0. \tag{0.1.4}$$

Here $\psi(t,x)$ is a function defined in the Minkowski space $\mathbf{R}^{1+3}$ with values in $\mathbb{C}^4$. Usually, ψ is called a spinor. Moreover, γ_μ are the Dirac matrices defined as follows

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \ , k = 1, 2, 3.$$

The Pauli matrices σ_k are determined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The initial data are determined by

$$\psi(0,x) = f(x)$$

The Dirac matrices satisfy the relations

$$\gamma^\mu \gamma^\ni + \gamma^\ni \gamma^\mu = -2\eta^{\mu\ni} \tag{0.1.5}$$

A simple reduction of the Dirac equation to the wave equation can be done by applying the operator $i\gamma_\mu \partial_\mu$ to the Dirac equation in (0.1.4). We use the relations (0.1.5) and find

$$\partial_\mu \partial^\mu \psi = 0.$$

Several either physical and mathematical problems arise quite naturally in the study of such partial differential equations. One of the most developed topic is the investigation of the effects of perturbations of the original operators. Namely we shall consider **perturbed operators**, denoted by H_V, defined as

$$H_V = H_0 + V \ , \tag{0.1.6}$$

where V is a suitable potential operator (see [**113**], [**127**], [**150**]).

1.2. Problem Setting. Many questions may arise, some of which we itemize below.

- Local or global well-posedness of the associated problems. It is of interest to find suitable spaces for the initial data in such a way to have either local or global existence and continuous dependence of the evolution operator on the initial data. The main argument here is the use of the contraction principle, together with some a-priori estimates, which are usually given either by some conserved quantities (for instance the mass or the energy), or by some space embedding (for instance among Sobolev spaces).
- Smoothing effect for a class of hyperbolic equation. It is possible for time evolution partial differential equations which are reversible and conservative to smooth locally the initial data? For the linear wave equation, for instance, the answer is no. The study of partial differential equation in order to describe general smoothing property for dispersive equation: the solution of the initial data value problem is, locally, smoother (higher regularity) than the initial datum (see [**42**]).
- Blow-up and the control of life-span. In the case of local existence results we may investigate whether the solution may blow up in some finite time, as well as study the behavior of such life-span with respect to the parameters involved (smallness of the initial data, regularity of the initial data, ...). The main technique in this case is the reduction to ordinary differential equations of special form (see [**105**] and [**174**]).

- Asymptotic behavior of the solutions. A special interest is devoted to the study of the decay properties of the solutions, and more generally its regularity. The main arguments here can be either the use of some explicit representation of the solution (for instance via spectral theory), or the use of some special properties such as symmetries and invariance.
- Scattering theory. We want to compare the behavior of the perturbed problem checking the existence of the following two limits, called **wave operators**

$$W_{\pm} f = \lim_{t \to \pm\infty} \mathrm{e}^{\imath t H_V} \mathrm{e}^{-\imath t H_0} f \ ,$$

  for any initial data f (up to some projection operator). We refer to [**127**] and [**150**] for a deeper analysis. Indeed the compositions of these two operators (their inverse or their adjoint), as the **scattering operator** $S = W_+^* W_-$, as well as the completeness play a crucial role in this field (see [**92**]). An important tool in this field is represented by the decay of the local energy.
- Associated nonlinear problems. It is very important the case of a nonlinear source term $F = F(u)$ (for instance $F_s(u) = u\,|u|^{s-1}$ with $s > 1$). All the questions raised in the linear case may be extended to the nonlinear one. The main argument here is again the use of contraction principle. More precisely for a more general hyperbolic equation, if we are able to find suitable Banach spaces X_T, Y_T, Z_0 such that the solution to the problem u satisfies

$$\|u\|_{X_T} \leq C_1(T) \left(\|F\|_{Y_T} + \|u_0\|_{Z_0} \right) \ ,$$

  for some time $T \in (0, \infty]$ and the nonlinear term F satisfies

$$\|F(u) - F(v)\|_{Y_T} \leq C_2(T) C_3 \left(\|u_0\|_{X_T} \right) \|u - v\|_{X_T} \ ,$$

  with $\lim_{t \downarrow 0} C_2(T) = 0$, and $C_3 \left(\|u_0\|_{X_T} \right)$ increasing monotone function with respect to the time T, then a contraction argument guarantees the local well-posedness in Z_0.

The items just listed show the essential importance of the a-priori estimates for the operators H_0 and H_V in this field. Next Section will deal with the specific a-priori estimates we are going to face in the sequel.

2. A-Priori Estimates

We shall deal with three different types of a-priori estimates.

2.1. Resolvent Estimates. The first a-priori estimates we are interested in are the so-called **resolvent estimates**, which deal with the resolvent operators, $R_0(z) = (z - H_0)^{-1}$ in the free case, or $R_V(z) = (z - H_V)^{-1}$ in the perturbed case (we refer to Chapter 2). If $\sigma(H_0)$ denotes the **spectrum** of the self-adjoint operator H_0, the mapping

$$R_0 : \mathbb{C} \setminus \sigma(H_0) \longrightarrow \mathcal{B}\left(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n) \right) \ ,$$

where $\mathcal{B}\left(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n) \right)$ is the set of all bounded operators from $L^2(\mathbb{R}^n)$ to $H^2(\mathbb{R}^n)$ is analytic with respect to the complex variable z.

We first need to extend this regularity properties in serval directions. On one side we need to approach the spectrum (which lies on the real axis), taking into account $z = \lambda \pm \imath\varepsilon$ (for $\varepsilon > 0$), and looking for the existence of the upper and lower limit operators

$$R_0^{\pm} = \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm \imath\varepsilon) ,$$

in some suitable weighted Sobolev space. This technique is called limiting absorption principle, and it is very well studied (see [**5**], [**94**], [**166**], [**167**] and [**198**]). It turns out that this can be done introducing the following weighted Hilbert space $H_s^m(\mathbb{R}^n)$ defined by

$$\|u\|_{H_s^m}^2 = \sum_{|\alpha| \le m} \|\langle x\rangle^s D^\alpha u\|_{L^2}^2 , \quad \forall m \in \mathbb{N} , \ s \in \mathbb{R} ,$$

where the weight $\langle x\rangle$ is given by the smoothed norm $\langle x\rangle = \sqrt{1 + |x|^2}$. Under suitable condition on the exponent s it can be shown the existence of $R_0^{\pm}$ as operators

$$R_0^{\pm} : \mathbb{C} \setminus \sigma_p(H_0) \longrightarrow \mathcal{B}\left(L_s^2(\mathbb{R}^n), H_{-s}^2(\mathbb{R}^n)\right) ,$$

where $L_s^2(\mathbb{R}^n) = H_{-s}^0(\mathbb{R}^n)$, and $\sigma_p(H_0)$ denotes the **point spectrum** associated to the operator H_0 (see [**113**] for further details of the decomposition of the spectrum into point and continuous part). Next we shall look for some classes of real valued operators $V = V(x, D) = \sum_{|j|\le 2} a_j(x)D^j$, for which the **Schrödinger operator** H_V admits a similar construction. Namely, the existence of the operators

$$R_V^{\pm} : \mathbb{C} \setminus \sigma_p(H_V) \longrightarrow \mathcal{B}\left(L_s^2(\mathbb{R}^n), H_{-s}^2(\mathbb{R}^n)\right)$$

may be done under the same condition on s for any operator with coefficients a_j such that

$$a_j = O\left(|x|^{-1-\delta}\right) , \text{ as } |x| \to +\infty ,$$

for some $\delta > 0$. Such operator are called **Short Range** potentials (see the works [**17**], [**91**], [**96**], [**119**], [**123, 124, 125**] [**145**] for a deeper analysis).

2.2. Dispersive Estimates. The second type of a-priori estimates we are interested involving hyperbolic equations are the so-called **dispersive estimates** (we refer to Chapter 5). They are meant to denote estimates of the form

$$\|\varphi\|_{B_1} \lesssim |t|^{-\sigma} \|\varphi_0\|_{B_0} , \quad t > 0 \tag{0.2.7}$$

for some $\sigma > 0$, and suitable Banach spaces B_0 and B_1, for φ solution to some hyperbolic equation with initial data φ_0. These kind of behavior do actually hold for the equation we are dealing with, for instance we have

$$\sigma = \tfrac{n-1}{2} , \quad B_0 = L^\infty(\mathbb{R}^n) , \quad B_1 = \dot{W}^{\frac{1}{2},1}(\mathbb{R}^n) \qquad \text{(Wave Equation in odd dim. } n) ,$$

$$\sigma = \tfrac{n-1}{2} , \quad B_0 = L^\infty(\mathbb{R}^n) , \quad B_1 = B_{1,1}^{\frac{1}{2}}(\mathbb{R}^n) \qquad \text{(Wave Equation in even dim. } n \ge 3) ,$$

$$\sigma = \tfrac{n}{2} , \quad B_0 = L^\infty(\mathbb{R}^n) , \quad B_1 = L^1(\mathbb{R}^n) \qquad \text{(Schrödinger Equation in any dim.)} ,$$

where $\dot{W}^{\frac{1}{2},1}$ and $B_{1,1}^{\frac{1}{2}}(\mathbb{R}^n)$ denote the homogeneous Sobolev spaces and the Besov spaces, respectively (see [**174**], [**193, 195**]).

The main tools in the proof of such estimates is the use of the **Stationary Phase Method**, which allow the control of oscillating integrals. Indeed, such type of integrals appears in the representation of the solutions via the Fourier analysis. The important remark of that method is to underline the influence of the geometric structure of the equation. Namely, the decay rate σ depends on the rank of Hessian matrix on stationary points of the phase function.

The dispersive estimates are the heart of the **Strichartz Type Estimates** (see [**192, 194**], [**74, 75**], [**115**]). Indeed, interpolating the dispersive estimates with the energy estimates, and using some harmonic analysis results (such as Paley-Littlewood theory and TT^* method) it is possible to obtain mixed norm estimates, which distinguish space and time variables. We restrict our attention to a linear wave $\Box u = F$ in dimension $n \geq 2$, with initial data (u_0, u_1) (a similar result holds for the Schrödinger equation). In this case we have

$$\|u\|_{L^q([0,T];L^r(\mathbb{R}^n))} \lesssim \|u_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|u_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L^{\tilde{q}}([0,T];L^{\tilde{r}}(\mathbb{R}^n))} \ , \tag{0.2.8}$$

under the dimensional condition

$$\tfrac{1}{q} + \tfrac{n}{r} = \tfrac{n}{2} - \gamma = \tfrac{1}{\tilde{q}} + \tfrac{n}{\tilde{r}} - 2$$

and the assumption that the pairs $\left(\frac{1}{q}, \frac{1}{r}\right)$, $\left(\frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}\right)$ satisfy the following admissibility relation ([**193**]).

$$\tfrac{1}{p} + \tfrac{\sigma}{q} \leq \tfrac{\sigma}{2}, \quad \tfrac{1}{\tilde{p}} + \tfrac{1}{\tilde{q}} \leq \tfrac{\sigma}{2}, \quad (p,q) \neq (2,\infty), \quad (\tilde{p},\tilde{q}) \neq (2,\infty), \quad \sigma = \tfrac{n-1}{2}. \tag{0.2.9}$$

The Strichartz estimates are one of the main tools in the study of local and global existence, both for linear and nonlinear equation, because they fit the assumptions required by the contraction argument (see for instance [**109, 110, 111**], [**131, 132**], [**133**], [**115**], [**199**]). The relation between the regularity of the initial data, and the power of the nonlinear term turns out to be crucial in this field, and many problems may arise (minimal regularity, smallness of the initial data, ...).

2.3. Smoothing Estimates. Finally, the third a-priori estimates we are interested involving dispersive equations are the so-called **smoothing estimates** (we refer to Chapter 4). Smoothing properties were given, for Schrödinger equations for example, by estimates of the form

$$\|Au\|_{L^2_t(\mathbb{R};L^2_x(\mathbb{R}^n))} \leq C\,\|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{0.2.10}$$

where A is the form

(1) $A = \langle x\rangle^{-s}\,|D|^{1/2}$, $\quad s > 1/2$,

(2) $A = \langle x\rangle^{-s}\,|D|^{1/2}$, $\quad s \geq 1/2$, $\quad (s > 1/2 \text{ if } n = 2)$,

(3) $A = |x|^{\alpha-1}\,|D|^{\alpha}$, $\quad 1 - n/2 < \alpha < 1/2$.

In order to treat more general equation, the following dispersive equation were considered

$$\begin{aligned} &\imath u_t(t,x) + P(D)u(t,x) = F \ , \quad t \geq 0 \ , \ x \in \mathbb{R}^n \\ &u(0,x) = u_0(x) \ , \end{aligned} \tag{0.2.11}$$

to obtain smoothing properties requiring some regularity in the symbol $P(\xi)$.
Roughly speaking smoothing effect is a gain of regularity of the solution with respect homogeneous equation and non-homogeneous one. There is a vast literature on such kind of problem. We refer principally to the work of Kenig, Ponce, Vega [**207, 116, 117**] for the Schrödinger equation and [**118**] for problem (0.2.10). The type (1) was given by Ben-Arzi and Klainerman [**18**], $(n > 3)$, Chihara [**40**] $n \geq 2$. The type (2) $(n \geq 3)$ and the type (3) $(n \geq 3, 0 \leq \alpha < 1/2,)$ or $(n = 2, 0 < \alpha < 1/2,)$ was

given by Kato and Yajima [**114**], just to name a few. Usually, the proof of such estimates carried out by proving one of the following estimates (or their variants):

$$\left\| \widehat{A^* f}_{|\rho S^{n-1}} \right\|_{L^2(\rho S^{n-1})} \leq C\sqrt{\rho}\, \|f\|_{L^2} \qquad \text{(Restriction Theorem)}, \tag{0.2.12}$$

where, $\rho S^{n-1} = \{|\xi| = \rho\}$,$\rho > 0$

$$\sup_{\mathrm{Im}\ z>0} |(R_A(z)A^*, A^* f)| \leq C\, \|f\|_{L^2} \qquad \text{(Resolvent Estimate)}, \tag{0.2.13}$$

The smoothing estimates are one of the tools in the study the stability, both for the linear and the nonlinear equation.

3. Plan of the Thesis

The plan of the Thesis is organized as follows.

In Chapter 1 we present some useful basic tools about Spectral Theory, Interpolation Theory and Harmonic Analysis which will play an important role in the following development.

Chapter 2 is devoted to the study of resolvent estimates. Some literature is presented, about the estimates for the resolvent operators in the free case, as well as the perturbed one. A selfcontained proof for the estimate of the "gradient of the resolvent operator" is given.

In **Chapter 3** we present resolvent estimates both in free and perturbed case in dimension $n = 3$. Applications to to the wave equation, Dirac equation and Schrödinger equation will be done. New space-time estimates for such kind of equations are presented. The whole Chapter is based on our results found in the work [**200**].

In **Chapter 4** our purpose is to derive new generalized scale invariant smoothing estimates for the case of magnetic potential imposing scale invariant smallness assumptions on the magnetic potential using suitable resolvent estimates and a new equivalence norm result. We follow the work of the author (joint with V. Georgiev) [**69**].

Chapter 5 is focused on the dispersive estimates. A general setting of Strichartz type estimates for the free and the perturbed case is presented following the work of [**115**].

We end up with **Chapter 6**, where we present new Strichartz-smoothing estimate necessary to show the stability for the nonlinear Schrödinger equation perturbed by a small magnetic potential. We state also a result concerning the spectrum of Schrödinger operators perturbed by a class of differential operator of order one. These mixed estimates are obtained jointly with V. Georgiev and A. Stefanov in the work [**67**].

Clearly, the matter of this subject cannot be contained in the preliminary chapters, so we recall, as we need, the tool of advanced functional analysis, harmonic analysis and theory of partial differential equation. So, in the each Chapter where we present new result, we will give a brief introduction to facility the reader to well understand the core of the arguments.

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CHAPTER 1

Functional Analysis Background

1. Operator and Spectral Theory

In this chapter we shall make a review of some basic facts from functional analysis and we shall focus our attention to two main points.

On one hand, we shall give suitable sufficient conditions that assure that a symmetric strictly monotone operator in a Hilbert space is self-adjoint. More precisely, we consider Friedrich's extention of a symmetric strictly monotone operator. The criterion to assure that its closure is self-adjoint operator is of the type: weak solution $\Rightarrow$ strong solution. We shall apply this criterion in the next chapters.

On the other hand, we represent some of the basic interpolation theorems for the Lebesgue spaces L^p.

To get a complete information on the subject one can use [**63**], [**160**], [**157**] , [**225**].

2. Linear operators in Banach spaces

Given any couple A, B of Banach spaces we denote their corresponding norms by

$$\|a\|_A \quad , \quad \|b\|_B$$

for $a \in A, b \in B$. A linear operator

$$F : A \to B$$

is called bounded (or continuous) if there is a constant $C > 0$ such that

$$\|Fa\|_B \leq C\|a\|_B.$$

The space $\mathcal{B}(A, B)$ is the set of bounded linear operators

$$F : A \to B$$

with norm

$$\|F\| = \sup_{\|a\|_A=1} \|Fa\|_B.$$

In case $A = B$ we shall denote by $\mathcal{B}(A)$ the corresponding linear space of bounded linear operators from A in A. It is easy to see that $\mathcal{B}(A, B)$ equiped with the above norm is a Banach space.

If B is the field $\mathbb{C}$ of complex numbers, then the elements in $\mathcal{B}(A, \mathbb{C})$ are called functionals and $\mathcal{B}(A, \mathbb{C})$ itself is called dual space of A.

The dual space can be defined in a more general situation of a topological vector space. Recall that a linear vector space V is topological vector space if the topology on V is such that the addition of vectors and multiplication by constants are continuous operations. For any $v' \in V'$ we denote by

$$< v', v >$$

the action of the linear functional v' on v.

For the typical case of Hilbert space H with inner product $(\cdot,\cdot)_H$ for any element $h' \in H'$ there exists an element $h_0 \in H$ so that

$$< h', h >= (h_0, h)_H$$

for any $h \in H$. This is the classical Riesz representation theorem. On the basis of this theorem there is an isometry

$$h' \in H' \to h_0 \in H.$$

We shall denote this isometry by

$$H' \sim_{(,)_H} H.$$

It is clear that the isometry depends on the choice of the product $(\cdot,\cdot)_H$.

Sometimes it is possible to define the linear operator only on a dense domain $D \subset A$ so that

$$F : D \to B.$$

Then $D = D(F)$ is called the domain for F. The range of the operator F is

$$R(F) = \{b : b = F(a), a \in D(F)\}.$$

A linear operator

$$F : D(F) \to B$$

is an extention of the operator

$$G : D(G) \to B$$

if $D(G) \subset D(F)$ and $Ga = Fa$ for $a \in D(G)$. The operator $G : D(G) \to B$ is called closed if the conditions

$$a_n \to a\ ,\ a_n \in D(G),\ G(a_n) \to b$$

imply $a \in D(G)$ and $b = Ga$.

Let

$$F : D(F) \to B$$

be a linear operator with dense domain $D(F)$. On the product

$$A \times B$$

one can define a norm by

$$\|a\|_A + \|b\|_B$$

for $a \in A, b \in B$. Then F is a closed operator if and only if its graph

$$\Gamma(F) = \{(a, F(a)); a \in D(F)\}$$

is a closed subset in $A \times B$.

THEOREM 1.1. *(Closed graph theorem) If $F : D(F) \to B$ is a closed operator and $D(F)$ is a closed subspace of A, then there exists a constant $C > 0$ such that*

$$\|Fa\|_B \leq C\|a\|_A$$

for $a \in D(F)$.

If F has a dense domain $D(F) \subset A$

$$F : D(F) \to B,$$

then the dual operator F' is an operator between B' and A' and this operator has the domain $D(F')$ defined as follows: $b' \in D(F')$ if and only if there exists an element $a' \in A'$ so that

$$< b', Fa >=< a', a > \tag{1.2.1}$$

for any $a \in D(F)$. We put $F'(b') = a'$.

Let $b' \in D(F')$. Then there is a unique $a' \in A'$, satisfying (1.2.1).

Given any Banach space A we call

$$T : A \to \mathbf{C}$$

a conjugate linear functional if

$$T(\alpha_1 a_1 + \alpha_2 a_2) = \overline{\alpha_1} T(a_1) + \overline{\alpha_2} T(a_2)$$

and if T is bounded, i.e. there exists a constant $C > 0$ such that

$$|T(a)| \leq C\|a\|_A$$

for any $a \in D(F)$.

We denote by A^* the vector space of linear conjugate functionals on A.

Then A^* is a Banach space.

For any $v^* \in V^*$ we denote by

$$< v^*, v >$$

the action of the linear functional v^* on v.

Let F be an operator with a dense domain $D(F) \subset A$ and

$$F : D(F) \to B.$$

The conjugate operator F^* is an operator between B^* and A^* and has a domain $D(F^*)$ defined as follows: $b^* \in D(F^*)$ if and only if there exists an element $a^* \in A^*$ so that

$$< b^*, Fa >=< a^*, a > \tag{1.2.2}$$

for any $a \in D(F)$.

Let $b^* \in D(F^*)$. Then there is a unique $a^* \in A^*$, satisfying (1.2.2).

By definition $F^*(b^*) = a^*$, where the element a^* is defined according to the previous problem.

The operator F^* with dense domain $D(F^*)$ is a closed operator.

Further, we turn again to the situation of a Hilbert space H. An operator F with dense domain $D(F) \subset H$ is called symmetric if

$$(Fh, g)_H = (h, Fg)_H$$

for any $h, g \in D(F)$. Using the definition of the adjoint operator F^* we see that F^* is an extention of the operator F, when F is symmetric.

We shall say that F is self-adjoint if

$$F = F^*.$$

The following criterion for self-adjointness plays an important role.

THEOREM 1.2. *(see [**157**], [**162**]) Suppose that F is symmetric operator on a Hilbert space H with dense domain $D(F)$ and*

$$R(F-\lambda) = R(F-\bar{\lambda}) = H \tag{1.2.3}$$

for some number $\lambda \in \mathbb{C}$. Then F is self-adjoint.

The condition (1.2.3) with $\lambda = i$ is equivalent to

$$\operatorname{Ker}(F^* - i) = \operatorname{Ker}(F^* + i) = 0.$$

Let F be a symmetric operator with a dense domain $D(F) \subset H$.

A natural way to extend this operator to a closed operator is to take the closure $\overline{\Gamma(F)}$ of the graph

$$\Gamma(F) = \{(h, Fh); h \in D(F)\}$$

in $H \times H$.

If F is a symmetric operator with a dense domain $D(F)$ in H, then there exists an operator $\bar{F}$ such that

$$\overline{\Gamma(F)} = \Gamma(\bar{F}).$$

We call $\bar{F}$ a closure of F.

The importance of self-adjoint operators is connected with the possibility to use the spectral theorem. (see [**160**])

THEOREM 1.3. *(Spectral theorem - functional calculus) Let F be a self-adjoint operator in a Hilbert space H. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbf{R}$ into $L(H)$ so that*

a) $\hat{\phi}$ is an algebraic $*-$ homomorphism, i.e.

$$\hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g)\ , \hat{\phi}(\lambda f) = \lambda\hat{\phi}(f)\ ,\ \hat{\phi}(1) = I\ ,\ \hat{\phi}(\bar{f}) = (\hat{\phi}(f))^*.$$

b) $\|\hat{\phi}(f)\|_{L(H)} \le \|f\|_{L^\infty}$;

c) let $h_n(x)$ be a sequence of bounded Borel functions with

$$\lim_{n\to\infty} h_n(x) = x$$

for each x and $|h_n(x)| \le |x|$ for all x and n, then for any $\psi \in D(F)$ we have

$$\lim_{n\to\infty} \hat{\phi}(h_n)\psi = F\psi;$$

d) if $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_{L^\infty}$ is bounded, then

$$\hat{\phi}(h_n) \to \hat{\phi}(h)$$

strongly;

e) if $F\psi = \lambda\psi$, then

$$\hat{\phi}(h)\psi = h(\lambda)\psi;$$

f)if $h \ge 0$, then $\hat{\phi}(h) \ge 0$.

This spectral theorem gives us a possibility to define the function of the operator F by means of the identity

$$f(F) = \hat{\phi}(f)$$

for any measurable function f on $\mathbf{R}$.

The above spectral theorem can be rewritten in projection valued measure form (see [**160**]).

Given any Borel set $\Omega \subset \mathbf{R}$, we denote by χ_Ω the corresponding characteristic function for the set Ω. Then the functional calculus for the self-adjoint operator F enables one to consider the projection:

$$P_\Omega = \chi_\Omega(F).$$

The family $\{P_\Omega\}$ satisfies the properties:

a) P_Ω is an orthogonal projection,

b) $P_\emptyset = 0$, $P_{(-\infty,\infty)} = I$,

c) If Ω is a countable disjoint union of Borel sets $\Omega_m, m = 1, 2, ...,$ then for any $h \in H$ we have

$$P_\Omega h = \lim_{N\to\infty} \sum_{m=1}^{N} P_{\Omega_n} h,$$

d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Given any $h \in H$, we see that

$$\mu(\Omega) = (h, P_\Omega h)_H$$

is a measure. By $d(h, P_\lambda h)$ we shall denote the corresponding volume element needed for integration with respect to this measure so we have

$$\int_{-\infty}^{\infty} \chi_\Omega(\lambda) d(h, P_\lambda h) = (h, P_\Omega h)_H.$$

Now for any (eventually unbounded) Borel function g on $(-\infty, \infty)$ we consider the domain

$$D_g = \{h \in H; \int_{\mathbf{R}} |g(\lambda)|^2 d(h, P_\lambda h) < \infty\}$$

and then we define the operator (eventually unbounded) $h \in D_g \to g(F)h$ by means of the identity

$$(h, g(F)h)_H = \int_{\mathbf{R}} g(\lambda) d(h, P_\lambda h).$$

Then we have the following assertion.

THEOREM 1.4. *For any Borel function $g(\lambda)$ defined on $(-\infty, \infty)$ the operator $g(F)$ with dense domain D_g is self-adjoint.*

The functional calculus enables one to define the exponential $U(t) = \mathrm{e}^{itF}$.

THEOREM 1.5. *(see* [**160**]*) If F is a self-adjoint operator in the Hilbert space H, then $U(t) = \mathrm{e}^{itF}$ satisfies the properties:*

a) $U(t)$ is a bounded unitary operator for any $t \in \mathbf{R}$,

b) $U(t)U(s) = U(t + s)$ for any real numbers t, s,

c) $\lim_{t\to 0} U(t)h = h$ for any $h \in H$,

d) $h \in D(F)$ if and only if

$$\lim_{t\to 0} \frac{U(t)h - h}{t}$$

exists in H.

Remark A. The property a) in the above theorem means that

$$\|U(t)h\|_H = \|h\|_H.$$

Remark B. An operator-valued function $U(t)$ satisfying the above properties a),b) and c) is called a strongly continuous one-parameter unitary group.

THEOREM 1.6. *(Stone's theorem, see* [**157**]*) If* $U(t)$ *is a strongly continuous one-parameter unitary group, then we can define its generator* G *so that* $h \in D(G)$ *if and only if the limit*

$$\lim_{t\to 0} \frac{U(t)h - h}{t}$$

exists. The above limit shall be denoted Gh *for* $h \in D(G)$*. One has*

$$G = iF,$$

where F *is a self-adjoint operator in* H.

3. Symmetric strictly monotone operators on Hilbert space

In this section we shall consider the special case when a symmetric operator B is defined on a dense domain $D(B) \subset H$, where H is a real Hilbert space. For simplicity we take Hilbert space over **R**, but the results are valid also for Hilbert spaces over **C**. We shall denote by

$$(\cdot,\cdot)_H \quad , \quad \|\cdot\|_H$$

the inner product and the norm in H respectively.

Our main assumption is that B is strictly monotone, i.e. there exists a constant $C > 0$, so that

$$(Bu, u) \geq C\|u\|_H^2 \tag{1.3.1}$$

for $u \in D(B)$.

First we consider the case, when the range $R(B)$ is dense in H.

LEMMA 1.1. *If* B *is a symmetric strictly monotone operator with dense range* $R(B)$*, then the closure* $\bar{B}$ *is a self-adjoint operator.*

Proof. The operator $\bar{B}$ is also symmetric and strictly monotone. Then the inequality

$$\|\bar{B}u\|_H^2 \geq C\|u\|_H^2$$

shows that $R(\bar{B})$ is closed. Since $R(B) \subset R(\bar{B})$ and $R(B)$ is dense in H, we see that $R(\bar{B}) = H$. Applying Theorem 1.2, we see that $\bar{B}$ is self-adjoint. □

The next step is to introduce the corresponding "energetic" space (see [**225**]).

For this purpose for any $u, v \in D(B)$ we define the corresponding energy inner product

$$(u, v)_E = (Bu, v)_H. \tag{1.3.2}$$

The corresponding norm is

$$\|u\|_E = \sqrt{(u, u)_E}.$$

DEFINITION 1.1. *The space* H_E *consists of all* $u \in H$ *such that there exists a sequence* $\{u_n\}_{n=1}^{\infty}$ *with the properties:*

a) $u_n \in D(B)$*,*

b) $u_n \to u$ *in* H*,*

c) u_n *is a Cauchy sequence for the norm* $\|\cdot\|_E$*, i.e. for any* $\varepsilon > 0$ *there exists an integer* $N \geq 1$*, such that*

$$\|u_n - u_m\|_E \leq \varepsilon$$

for $n, m \geq N$*.*

We shall call the sequence $\{u_n\}$, satisfying the above properties, admissible for u. Given any $u \in H_E$, we can define its norm by

$$\|u\|_E = \lim_{n\to\infty} \|u_n\|_E. \tag{1.3.3}$$

Our first step is to show that this definition is independent of the concrete choice of admissible sequences $\{u_n\}$.

LEMMA 1.2. *Suppose* $\{u_n\}$ *is an admissible zero sequence. Then*

$$\lim_{n\to\infty} \|u_n\|_E = 0.$$

Proof. Assume that the assertion of lemma is not true. Choosing a subsequences we can reduce the proof of a contradiction to the case

$$a < \|u_n\|_E < a^{-1} \tag{1.3.4}$$

with some $a > 0$. Given any $\varepsilon > 0$, we can choose N depending on $\varepsilon > 0$ according to property c) of Definition 1.1. Then for any $n \geq N$ we have the inequalities

$$\|u_n\|_E^2 \leq |(u_n, u_N)_E| + |(u_n, u_n - u_N)_E| \leq |(u_n, u_N)_E| + a^{-1}\varepsilon.$$

On the other hand, we have the identity

$$(u_n, u_N)_E = (u_n, Bu_N)_H,$$

according to our definition of the inner product $(\cdot,\cdot)_E$ on $D(B)$. Since $\{u_n\}$ is admissible zero sequence, we have $\lim_{n\to\infty} \|u_n\|_H = 0$. Therefore, we can find $n \geq N$ so large that

$$|(u_n, u_N)_E| \leq \varepsilon.$$

Thus, for any $\varepsilon > 0$ we can find n so that

$$\|u_n\|_E^2 \leq \varepsilon(1 + a^{-1})$$

It is clear that this inequality is in contradiction with the left inequality in (1.3.4), when $\varepsilon > 0$ is sufficiently small.

Therefore we have a contradiction and this completes the proof of the lemma. □

The above lemma enables one to introduce a norm in H_E as follows:

$$\|u\|_E = \lim_{n\to\infty} \|u_n\|_E, \tag{1.3.5}$$

where $\{u_n\}$ is an admissible sequence for $u \in H_E$.

Also it is easy to define the inner product in H_E. For $u_n, v_n \in D(B)$ such that $\{u_n\}, \{v_n\}$ are admissible sequences for $u, v \in H_E$ we have the polarization identity

$$(u_n, v_n)_E = \frac{1}{4}(\|u_n + v_n\|_E^2) - \frac{1}{4}(\|u_n - v_n\|_E^2).$$

Then from (1.3.5) we see that the limit

$$\lim_{n\to\infty} (u_n, v_n)_E$$

exists and it is independent of the concrete choice of admissible sequences. For this we can introduce the inner product in H_E as follows

$$(u, v)_E = \lim_{n\to\infty} (u_n, v_n)_E.$$

The next step is of special importance to verify the fact that the space H_E is a Hilbert space.

LEMMA 1.3. *If $\{u_n\}$ is an admissible sequence for $u \in H_E$, then*

$$\lim_{n\to\infty} \|u_n - u\|_E = 0. \tag{1.3.6}$$

Proof. For any integer $m \geq 1$ the sequence

$$u_n - u_m$$

is admissible for $u - u_m$. The fact that $\{u_n\}$ is a Cauchy sequence in H_E means that for any positive number ε there exists an integer $N \geq 1$, so that

$$\|u_n - u_m\|_E \leq \varepsilon$$

for $n, m \geq N$. Then definition (1.3.5) shows that

$$\|u - u_m\|_E \leq \varepsilon$$

for $m \geq N$. This completes the proof. □

It is clear that the definition (1.3.5) guarantees that

$$\|u\|_E^2 \geq C\|u\|_H^2. \tag{1.3.7}$$

This estimate shows that $(u, u)_E = 0$ implies $u = 0$, so H_E is a pre - Hilbert space. Also it is a trivial fact that $D(B)$ is a dense subset in H_E, since any element u in H_E by the definition of H_E is such that there exists an admissible sequence $\{u_n\}$ with $u_n \in D(B)$.

Our next step is to study the space H_E.

THEOREM 1.7. *The space H_E is a Hilbert space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in H_E. Since $D(B)$ is dense in H_E, for any integer $n \geq 1$ one can find $v_n \in D(B)$, so that

$$\|v_n - u_n\|_E \leq \frac{1}{n}. \tag{1.3.8}$$

Then the estimate $\|v_n\|_E^2 \geq C\|v_n\|_H^2$ shows that $\{v_n\}$ is a Cauchy sequence in H so there exists $u \in H$, so that

$$v_n \to u \text{ in } H.$$

Applying Lemma 1.3, we conclude that

$$\lim_{n\to\infty} \|u - v_n\|_E = 0,$$

and from (1.3.8) we get

$$\lim_{n\to\infty} \|u - u_n\|_E = 0.$$

This completes the proof. □

Further, we turn to the dual space H_E^*. As usual for any linear continuous functional $f \in H_E^*$ and any $g \in H_E$ we denote by

$$< f, g >$$

the action of the functional f on g. The inclusion $H \subset H_E^*$ is such that

$$< f, g >= (f, g)_H$$

for $f \in H, g \in H_E$. The norm in H_E^* is

$$\|f\|_{H_E^*} = \sup_{g\in H_E, \|g\|_E=1} < f, g > .$$

Then H_E^* is clearly a Banach space. Later on we shall introduce on H_E^* a structure of a Hilbert space. The main preparation for this is the following

LEMMA 1.4. *The symmetric strictly monotone operator $B : D(B) \to H$ can be extended to an invertible isometry*

$$B_E : H_E \to H_E^*,$$

i.e. we have the properties
a) $B_E u = Bu$ for $u \in D(B)$,
b) B_E maps H_E onto H_E^*,
c) $\|B_E u\|_{H_E^*} = \|u\|_{H_E}$.

Proof. For any $u \in H_E$ we take an admissible sequence $\{u_n\}$, such that

$$\lim_{n\to\infty} \|u_n - u\|_E = 0.$$

On the other hand, we have the relation

$$\|Bu\|_{H_E^*} = \|u\|_E \tag{1.3.9}$$

for $u \in D(B)$. Indeed, for $u \in D(B), v \in H_E$ we have

$$| < Bu, v > | = |(Bu, v)_H| = |(u, v)_E| \leq \|u\|_E\|v\|_E. \tag{1.3.10}$$

Hence,

$$\|Bu\|_{H_E^*} \leq \|u\|_E.$$

To establish the inequality in the opposite direction we choose $v = u$ in (1.3.10) and get

$$\|u\|_E^2 \leq \|Bu\|_{H_E^*}\|u\|_E.$$

Once, the relation (1.3.9) is established, we can conclude that $\{Bu_n\}$ is a Cauchy sequence in H_E^* so it is convergent in H_E^* to an element $v \in H_E^*$, so by definition

$$B_E u = v$$

It is clear that the element v is independent of the concrete choice of the admissible sequence $\{u_n\}$ for u. Also (1.3.9) can be extended to $u \in H_E$.

Therefore, it remains to show that B_E maps the energetic space H_E onto its dual H_E^*. To do this take $v \in H_E^*$ and consider the linear continuos functional

$$h \in H_E \to < v, h > \in \mathbf{R}.$$

According to Riesz representation theorem, there exists $u \in H_E$ so that

$$< v, h >= (u, h)_E.$$

Taking an admissible sequence for u we can see that

$$(u_n, h)_E = (Bu_n, h)_H \to < B_E u, h >$$

Hence, $< B_E u, h >=< v, h >$, so $B_E u = v$. This completes the proof. □

Using the fact that $B_E : H_E \to H_E^*$ is an invertible isometry, we can define via the polarization identity an inner product on H_E^* and conclude that this is a Hilbert space.

In fact starting with the relations

$$\|Bu\|_{H_E^*}^2 = \|u\|_E^2 = (Bu, u)_H$$

for $u \in D(B)$ and using the previous lemma, we see that we can introduce the inner product in H_E^* by means of

$$(B_E u, B_E v)_{H_E^*} = (u, v)_E = (B_E u, v)_H.$$

The above relations show that B_E is a symmetric operator. It is easy to see that B_E is a strictly monotone operator on H_E^* with dense domain H_E. Applying the first lemma of this section, we conclude that

LEMMA 1.5. *The operator B_E is self-adjoint.*

Our main result in this section is the following.

THEOREM 1.8. *(see* [**225**]*) If B is a symmetric strictly monotone operator, then the operator A with dense domain*

$$D(A) = \{u \in H_E, B_E u \in H\}$$

defined by $Au = B_E u$ for $u \in D(A)$ is a self-adjoint extention of B.

Proof.
Given any $f \in H$, we can find $u \in H_E$ so that $f = B_E u$.
It is not difficult to see that the operator

$$F : f \in H \to u = F(f) \in H_E$$

is well - defined bounded, symmetric and

$$F(Bh) = h, h \in D(B).$$

In fact F is a restriction of the isometry

$$B_E^{-1} : H_E^* \to H_E$$

to H. Moreover, F is a symmetric bounded operator from H into H. Then the symmetric bounded operator F is self-adjoint. Applying the spectral theorem in the norm of Theorem 1.4 with $g(\lambda) = 1/\lambda$, we see that the operator $A = F^{-1}$ with dense domain $D(A)$ is selfadjoint.

It is an open problem if the closure of the graph of B is the graph of A. For this we introduce the following.

DEFINITION 1.2. *Given any $f \in H$, we shall say that $u \in D(B)$ is a weak solution of the equation $Bu = f$, if*

$$(Bu, v)_H = (f, v)_H$$

for any $v \in H_E$.

On the other hand, we have

DEFINITION 1.3. *Given any $f \in H$, we shall say that $u \in D(B)$ is a strong solution of $Au = f$, if there exists a sequence $\{u_k\}$ such that*
a) $u_k \in D(B)$,
b) $u_k \to u$ in H_E,
c) Bu_k tends to f in H.

For the applications of special importance is the following result.

THEOREM 1.9. *Suppose in addition to assumptions of Theorem 1.8 that any weak solution of $Bu = f$ for $f \in H$ is also a strong solution. Then the closure of the operator B is self-adjoint.*

Proof. The result follows from Theorem 1.8. □

4. Spectral Families

We consider a family of functions $\{E(t)\}_{t\in\mathbb{R}}$ defined by

$$E : \mathbb{R} \longrightarrow \mathcal{B}(H) ,$$

where H is a Hilbert space. We shall follow the approach given by Section 7.2 of [**221**] (see also Section VI.5 of [**113**] for a deeper analysis), and present the following definition.

DEFINITION 1.4. *We say that $\{E(t)\}_{t\in\mathbb{R}}$ defined above is a* **spectral family** *(see also theorem 1.3) if it satisfies the following properties:*

(i) $E(t)$ *is a projection operator on* H *for every* $t \in \mathbb{R}$*;*

(ii) for any $f \in H$ *and for any* $s, t \in \mathbb{R}$*, such that* $s \leq t$*, we have:*

$$(E(s)f, f) \leq (E(t)f, f);$$

(iii) for any $f \in H$ *and for any* $t \in \mathbb{R}$ *the following identity holds:*

$$\lim_{\varepsilon \downarrow 0} E(t+\varepsilon)f = E(t)f;$$

(iv) for any $f \in H$ *we have:*

$$\lim_{t \to \infty} E(t)f = f \ , \qquad \lim_{t \to -\infty} E(t)f = 0 \ .$$

We observe that the right continuity in (iii) is guaranteed by the monotonicity property (ii). It is also possible to show using the monotonicity that if $E(t)$ is a spectral family, then the following limits exist and are projection operators

$$E(\lambda^{\pm}) = \lim_{\varepsilon \downarrow 0} E(\lambda \pm \varepsilon) \ ,$$

for any $\lambda \in \mathbb{R}$; clearly we have that $E(\lambda^{+}) = E(\lambda)$.

We give below an important example of a spectral family in the case of H represented by the Lebesgue space $L^2(\mathbb{R}^n)$.

EXAMPLE 1.1. *Let* $M : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a measurable function. We associate to* M *the following subsets of* $\mathbb{R}^n$*:*

$$M_t = \{x \in \mathbb{R}^n | M(x) \leq t\} \ ,$$

for any $t \in \mathbb{R}$*, and let* χ_{M_t} *denote the characteristic function of the set* M_t*.*

It turns out that the family of linear operators

$$E^M(t) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \ ,$$

defined by

$$E^M(t)f(x) = \chi_{M_t}(x)f(x) \ , \ \ \forall f \in L^2(\mathbb{R}^n) \ , \forall t \in \mathbb{R} \ ,$$

for almost every $x \in \mathbb{R}^n$ *is a spectral family on* $L^2(\mathbb{R}^n)$*.*

Indeed properties (i) *and* (ii) *are satisfied. If we fix* $t \in \mathbb{R}$*,* $f \in L^2(\mathbb{R}^n)$ *and a sequence* $\varepsilon_k \downarrow 0$*, then*

$$\left\| \left[E^M(t+\varepsilon_k) - E^M(t) \right] f \right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left[\chi_{M_{t+\varepsilon_k}}(x) - \chi_{M_t}(x) \right] |f(x)|^2 \ \mathrm{d}x \ .$$

Since $0 \leq \chi_{M_{t+\varepsilon_k}}(x) - \chi_{M(t)}(x) \leq 1$ *and* $\chi_{M_{t+\varepsilon_k}}(x) - \chi_{M(t)}(x) \downarrow 0$ *as* $k \to \infty$ *the property* (iii) *follows by Lebesgue's Theorem. Similarly we obtain the property* (iv) *observing that*

$$\lim_{t \to \infty} E(t) = \mathbf{1} \ , \qquad \lim_{t \to -\infty} E(t) = \mathbf{0} \ .$$

The importance of the Example 1.1 relies on the strict connection between M seen as a multiplicative operator on $L^2(\mathbb{R}^n)$ and its associated spectral family $E^M(t)$. Namely the following operator identity holds:

$$M = \int_{\mathbb{R}} t \,\mathrm{d}E^M(t) \; .$$

Indeed it is of great interest to investigate the relations between all the possible spectral families E^A associated to any operator A on H, and the operator associated to them via that functional integral.

5. Integration with Respect to a Spectral Family

In this section we shall show how a given spectral family $E(t)$ may be used to introduce an operator, denoted on an Hilbert space H via integration with respect to the t variable. Namely we have the following.

THEOREM 1.10. *For any given spectral family $E(t)$ on H it is well defined a map $\hat{E}$ associating to any Borel function $F : \mathbb{R} \longrightarrow \mathbb{C}$ an normal operator $\hat{E}(F)$ defined by*

$$\hat{E}(F) : D(\hat{E}(F)) \subseteq H \longrightarrow H \; , \tag{1.5.11}$$

such that

$$\hat{E}(F) = \int_{\mathbb{R}} F(t) \,\mathrm{d}E(t) \; . \tag{1.5.12}$$

Proof. Let I be a real interval. Let the numbers $-\infty \le a \le b \le \infty$ be given, then using the notation given in (1.5.11) we define

$$\int_{\mathbb{R}} \chi_{(a,b)}(t) \,\mathrm{d}E(t) = E(b^-) - E(a) \; , \qquad \int_{\mathbb{R}} \chi_{[a,b]}(t) \,\mathrm{d}E(t) = E(b) - E(a^-);$$

$$\int_{\mathbb{R}} \chi_{(a,b]}(t) \,\mathrm{d}E(t) = E(b) - E(a) \; , \qquad \int_{\mathbb{R}} \chi_{[a,b)}(t) \,\mathrm{d}E(t) = E(b^-) - E(a^-),$$

where as usual χ_A denotes the characteristic function of the set A.

By a linearity argument we can define $\int_{\mathbb{R}} F(t) \,\mathrm{d}E(t)$ for **step functions**, that is, for functions of the form

$$F(t) = \sum_{j=1}^{n} c_j \chi_{J_j}(t) \; ,$$

where $c_j \in \mathbb{R}$ and J_j are disjoint intervals as

$$\int_{\mathbb{R}} F(t) \,\mathrm{d}E(t) = \sum_{j=1}^{n} c_j E(J_j) \; . \tag{1.5.13}$$

In order to let $\int_{\mathbb{R}} v(t) \,\mathrm{d}E(t)$ make sense for a general Borel function $v : \mathbb{R} \longrightarrow \mathbb{R}$, we introduce the following family of non-decreasing functions

$$\rho_h^E(t) = (h, E(t)h)_H = \|E(t)h\|_H^2 \; , \tag{1.5.14}$$

where $h \in H$. We define the domain of the operator $\mathcal{D}\left(\hat{E}(F)\right)$ as follows:

$$D\left(\hat{E}(F)\right) = \left\{h \in H \mid F \in L^2\left(\mathbb{R},\ \mathrm{d}\rho_h^E\right)\right\} .$$

We observe that for the step function $F = \sum_{j=1}^{n} c_j \chi_{J_j}$ we have

$$\left\| \left(\int_{\mathbb{R}} \sum_{j=1}^{n} c_j \chi_{J_j}\ \mathrm{d}E(t) \right) h \right\|_H^2 = \left\| \left(\sum_{j=1}^{n} c_j E(J_j) \right) h \right\|_H^2 = \sum_{j=1}^{n} c_j^2 \left\| E(J_j)h \right\|_H^2 = \sum_{j=1}^{n} c_j^2 \rho_h^E(t) ,$$

by the definition and the orthogonality of the projection operators $E(J_j)$. On the other hand

$$\int_{\mathbb{R}} \left| c_j \chi_{J_j} \right|^2\ \mathrm{d}\rho_h^E(t) = \sum_{j=1}^{n} \int_{J_j} c_j^2\ \mathrm{d}\rho_h^E(t) = \sum_{j=1}^{n} c_j^2 \rho_h^E(t) ,$$

hence the following relation

$$\left\| \hat{E}(F)h \right\|_H^2 = \left\| \left(\int_{\mathbb{R}} F(t)\ \mathrm{d}E(t) \right) h \right\|_H^2 = \int_{\mathbb{R}} |F(t)|^2\ \mathrm{d}\rho_h^E(t) , \tag{1.5.15}$$

holds for any $h \in D\left(\hat{E}(F)\right)$. Relation (1.5.15) may be extended to any $F \in L^2\left(\mathbb{R},\ \mathrm{d}\rho_h^E\right)$ where $h \in D\left(\hat{E}(F)\right)$ by a density argument. □

If $F : \mathbb{R} \longrightarrow \mathbb{C}$ is a bounded function (ρ_h^E-measurable for any $h \in H$), and all $f, g \in H$ according to the polarization identity, we can define

$$\int_{\mathbb{R}} F(t)\ \mathrm{d}\left(g, E(t)f\right) = \tfrac{1}{4}\left[\int_{\mathbb{R}} F(t)\ \mathrm{d}\rho_{g+f}^E(t) - \int_{\mathbb{R}} F(t)\ \mathrm{d}\rho_{g-f}^E(t) + \imath \int_{\mathbb{R}} F(t)\ \mathrm{d}\rho_{g-\imath f}^E(t) - \imath \int_{\mathbb{R}} F(t)\ \mathrm{d}\rho_{g+\imath f}^E(t) \right] .$$

With this definition we obtain that for any couple F and G as above

$$\int_{\mathbb{R}} G(t)^* F(t)\ \mathrm{d}\left(g, E(t)f\right) = \left(\int_{\mathbb{R}} G(t)\ \mathrm{d}E(t)g, \int_{\mathbb{R}} F(t)\ \mathrm{d}E(t)f \right) .$$

The next proposition describes the most important properties of the map $\hat{E}$ associated to a spectral family $E(t)$ build in the proof of Theorem 1.10.

PROPOSITION 1.1. *Let $E(t)$ be a spectral family on the Hilbert space H. Let $F, G : \mathbb{R} \longrightarrow \mathbb{C}$ be two Borel functions, and $\hat{E}(F), \hat{E}(G)$ the corresponding operators defined by Theorem 1.10 and let*

$$\varphi_k(t) = \begin{cases} 1 & \text{if } |F(t)| \le k \\ 0 & \text{otherwise} \end{cases} , \qquad \psi_k(t) = \begin{cases} 1 & \text{if } |G(t)| \le k \\ 0 & \text{otherwise} \end{cases} .$$

Then the following properties hold true.

(i) For all $f \in D\left(\hat{E}(F)\right)$ and $g \in D\left(\hat{E}(G)\right)$ we have

$$(\hat{E}(G)g, \hat{E}(F)f) = \lim_{k \to \infty} \int_{\mathbb{R}} \psi_k(t) G(t)^* \varphi_k(t) F(t)\ \mathrm{d}(g, E(t)f) .$$

(ii) For any $h \in D\left(\hat{E}(F)\right)$ *we have*

$$\left\|\hat{E}(F)h\right\|_H^2 = \int_{\mathbb{R}} |F(t)|^2 \, \mathrm{d}\rho_h^E(t) \ ,$$

where $\rho_h^E(t)$ *is defined in* (1.5.14).

(iii) If $F : \mathbb{R} \longrightarrow \mathbb{C}$ *is a bounded function, then* $\hat{E}(F) \in \mathcal{B}(H)$ *and*

$$\left\|\hat{E}(F)\right\|_{\mathcal{B}(H)} \leq \|F\|_{L^\infty(\mathbb{R})} \ .$$

(iv) If $F(t) = 1$ *for any* $t \in \mathbb{R}$*, then* $\hat{E}(F) = \mathbf{1}$*;*

(v) For every $f \in D\left(\hat{E}(F)\right)$ *and all* $g \in H$ *we have*

$$(g, \hat{E}(F)f) = \int_{\mathbb{R}} F(t) \, \mathrm{d}(g, E(t)f) \ .$$

(vi) If $F(t) \geq c$ *for all* $t \in \mathbb{R}$ *we have*

$$(h, \hat{E}(F)h) \geq c \, \|h\|_H \ ,$$

for all $h \in H$.

(vii) For all $a, b \in \mathbb{R}$ *we have* $a\hat{E}(G)+b\hat{E}(G) = \hat{E}(aF+bG)$*, and* $D\left(\hat{E}(F) + \hat{E}(G))\right) = D\left(\hat{E}(|F| + |G|)\right)$.

(viii) $\hat{E}(FG) = \hat{E}(F)\hat{E}(G)$ *and* $D\left(\hat{E}(F)\hat{E}(G)\right) = D\left(E\hat{(G)}\right) \cap D\left(\hat{E}(FG)\right)$.

Proof. (i) follows from density arguments, since the relation is true for F and G step functions and observing that by definition

$$(\hat{E}(G)g, \hat{E}(F)f) = \lim_{k\to\infty} \left(\hat{E}(\psi_k G)g, \hat{E}(\varphi_k F)f\right) = \lim_{k\to\infty} \int_{\mathbb{R}} \psi_k(t)G(t)^* \varphi_k(t)F(t) \, \mathrm{d}(g, E(t)f) \ .$$

(ii) follows from (1.5.15). If $F : \mathbb{R} \longrightarrow \mathbb{C}$ is a bounded function, then $F \in L^2\left(\mathbb{R}, \, \mathrm{d}\rho_h^E\right)$ for all $h \in H$, consequently $\hat{E}(F) \in \mathcal{B}(H)$. From the property (iv) of the Definition (1.4) we have that

$$\lim_{t\to\infty} \rho_h^E(t) = \|h\|_H \ , \qquad \lim_{t\to-\infty} \rho_h^E(t) = 0 \ .$$

Hence by the relation in (ii) we have

$$\left\|\hat{E}(F)h\right\|_H^2 = \int_{\mathbb{R}} |F(t)|^2 \, \mathrm{d}\rho_h^E(t) \leq \|F\|_{L^\infty(\mathbb{R})}^2 \|h\|_H^2 \ ,$$

since $\int_{\mathbb{R}} \mathrm{d}\rho_h^E(t) = \|h\|_H^2$. Hence (iii) is proved. Property (iii) provides that $D\left(\hat{E}(1)\right) = H$. If we consider the sequence $\chi_{[-k,k]} = 1$, therefore

$$\hat{E}(F)h = \lim_{k\to\infty} \hat{E}\left(\chi_{[-k,k]}\right)h = \lim_{k\to\infty} (E(k)h - E(-k)h) = h \ ,$$

for any $h \in H$, which gives (iv). From (i) with $G = 1$, and taking (iv) into account, the following relation

$$(g, \hat{E}(F)f) = \lim_{k\to\infty} \int_{\mathbb{R}} \varphi_k(t)F(t) \, \mathrm{d}(g, E(t)f)$$

holds true. That concludes the proof of (v). (vi) follows immediately from (v), whereas (vii) follows from the definition and the properties of the integrals. By (i) and (iv) we have that for all pairs of bounded functions F, G and for all $f, g \in H$ we have that

$$(g, \hat{E}(F)f) = (\hat{E}(\mathbf{1})g, \hat{E}(F)f) = (\hat{E}(F^*)g, \hat{E}(\mathbf{1})f) = (\hat{E}(F^*)g, f) \ ,$$

consequently,

$$(g, \hat{E}(F)\hat{G}f) = (\hat{E}(F^*)g, \hat{E}(G)f) = \int_{\mathbb{R}} F(t)G(t) \,\mathrm{d}\,(g, E(t)f) = \left(g, \hat{E}(FG)f\right) \ .$$

Let $h \in D\left(\hat{E}(F)\hat{E}(G)\right)$. As $\varphi_k F$ is bounded for fixed $k \in \mathbb{N}$, it follows that $\varphi_k F \psi_l G \to \varphi_k FG$ in $L^2\left(\mathbb{R}, \rho_h^E\right)$ as $l \to \infty$. Thus

$$\begin{aligned} \hat{E}(F)\hat{E}(G)h &= \lim_{k\to\infty} \hat{E}(\varphi_k F)\left[\lim_{l\to\infty} \hat{E}(\psi_l G)h\right] = \lim_{k\to\infty}\lim_{l\to\infty} \hat{E}(\varphi_k F)\hat{E}(\psi_l G)h = \\ &= \lim_{k\to\infty}\lim_{l\to\infty} \hat{E}(\varphi_k F \psi_l G)h = \lim_{k\to\infty} \hat{E}(\varphi_k FG)h \ . \end{aligned}$$

The existence of the limit means that $\varphi_k FG$ is a Cauchy sequence in $L^2\left(\mathbb{R}, \rho_h^E\right)$. Since, moreover, $\varphi_k(t)F(t)G(t) \to F(t)G(t)$ for all $t \in \mathbb{R}$, it follows that FG belongs to $L^2\left(\mathbb{R}, \rho_h^E\right)$; consequently, $h \in D\left(\hat{E}(FG)\right)$ and $\hat{E}(FG) = \hat{E}(F)\hat{E}(G)$. On the other hand if $h \in D\left(\hat{E}(G)\right) \cup D\left(\hat{E}(FG)\right)$, then

$$\begin{aligned} \hat{E}(FG)h &= \lim_{k\to\infty}\lim_{l\to\infty} \hat{E}(\varphi_k F \psi_l G)h = \lim_{k\to\infty}\lim_{l\to\infty} \hat{E}(\varphi_k F)\hat{E}(\psi_l G)h = \\ &= \lim_{k\to\infty}\lim_{l\to\infty} \hat{E}(\varphi_k F \psi_l G)h = \lim_{k\to\infty} \hat{E}(\varphi_k F)\hat{E}(g)h \ . \end{aligned}$$

The existence of the limit means that $F \in L^2\left(\mathbb{R}, \rho_h^E\right)$; consequently, $\hat{E}(F)h \in D\left(\hat{E}(F)\right)$, and thus $h \in D\left(\hat{E}(F)\hat{E}(G)\right)$. That concludes the proof of (viii). □

6. Some facts about holomorphic functions

Let $\mathbb{C}$ be the complex plane and let $U \subseteq \mathbb{C}$ be an open domain in this plane. Any point $z \in U$ can be represented as

$$z = x + iy,$$

where x, y are real numbers. A function

$$f : U \to \mathbb{C}$$

is $C^1(U)$ if the partial derivatives

$$\partial_x f(x+iy), \partial_y f(x+iy)$$

exist and are continuous functions. Of special interest are the vector fields

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

If $f \in C^1(U)$, then f is called holomorphic in U, if it satisfies the equation

$$\partial_{\bar{z}} f(z) = 0, \quad z \in U.$$

One can see that a function $f : U \to \mathbb{C}$ is holomorphic in U if and only if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for any $z \in U$.

The most important formula in the elementary theory of holomorphic functions is the Cauchy theorem and the Cauchy formula.

Let Γ be a closed path in U and let $z \in \mathbb{C}$ be a point such that Γ does not pass through z. Then the index of z with respect to Γ is

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}.$$

The Cauchy theorem states that if Γ is a closed path in U such that $\mathrm{Ind}_\Gamma(w) = 0$ for any w outside U, then

$$\int_\Gamma f(\zeta) d\zeta = 0 \tag{1.6.16}$$

for any function holomorphic in $\mathbb{C}$. The corresponding Cauchy formula is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{z - \zeta} d\zeta. \tag{1.6.17}$$

The condition $\mathrm{Ind}_\Gamma(w) = 0$ for w outside U is fulfilled for the case U is simply connected.

Also in the case of a simply connected domain U with smooth boundary ∂U for any function holomorphic in U and continuous on the closure of U we have the corresponding Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{z - \zeta} d\zeta. \tag{1.6.18}$$

Applying for example the above formula for $\{z, |z - z_0| < \delta\} \subset U$, we obtain the estimate

$$|\partial_z^k f(z_0)| \leq \frac{M k!}{\delta^k}, \tag{1.6.19}$$

where

$$M = \sup_{|z - z_0| = \delta} |f(z)|.$$

This estimate guarantees that the formal Taylor series

$$\sum_{k=0}^{\infty} \partial_z^k f(z_0)(z - z_0)^k / k!$$

converges absolutely and uniformly for $|z - z_0|$ sufficiently small and moreover the series coincides with $f(z)$ for z sufficiently close to z_0.

Our next step is the study of holomorphic functions in the strip

$$S = \{z; 0 < \mathrm{Re} z < 1\}.$$

More precisely, given any real number γ we consider the class $F(\gamma)$ of all functions $f \in C^1(\bar{S})$ holomorphic in S and satisfying the estimate

$$|f(z)| \leq C e^{\gamma |\mathrm{Im} z|}. \tag{1.6.20}$$

LEMMA 1.6. *(Three lines Lemma.) If $f \in F(\gamma)$, then for any $\theta \in (0,1)$ we have*

$$|f(\theta)| \leq \|e^{\delta(i\ \cdot)^2} f(i\ \cdot)\|^{1-\theta}_{L^\infty(\mathbf{R})} \|e^{\delta(1+i\ \cdot)^2} f(1+i\ \cdot)\|^{\theta}_{L^\infty(\mathbf{R})}.$$

Proof. If $f \in F(\gamma)$, then we can consider the function

$$g(z) = e^{\delta z^2} f(z) a_0^{z-1} a_1^{-z},$$

where

$$a_j = \|e^{\delta(j+i\ \cdot)^2} f(j+i\cdot)\|_{L^\infty(\mathbf{R})},\ j = 0, 1.$$

It is clear that we can assume that a_j are positive numbers. Otherwise, if $a_1 = 0$, then we can replace a_1 by $a_1 + \varepsilon$. Then it is easy to see that $g \in F(-\delta_1)$ with $0 < \delta_1 < \delta$ so we have the estimate

$$|g(z)| \leq C e^{-\delta_1 |\mathrm{Im} z|}$$

for $\mathrm{Re} z \in [0,1]$. Then this estimate enables us to extend the Cauchy formula (**1.6.18**) for the strip S.

From the Cauchy formula it follows the maximum principle, i.e.

$$\sup_{z \in S} |g(z)| \leq \max(\sup_{t \in \mathbf{R}} |g(it)|, \sup_{t \in \mathbf{R}} |g(1+it)|).$$

Since, $|g(it)| \leq 1$ and $|g(1+it)| \leq 1$, we get

$$|g(\theta + iy)| \leq 1.$$

Taking $y = 0$, we complete the proof of the lemma. □

REMARK 1.1. *From the proof it is clear that we have the estimate*

$$|f(\theta+iy)| \leq e^{-\delta|\theta+iy|^2} \|e^{\delta(i\ \cdot)^2} f(i\ \cdot)\|^{1-\theta}_{L^\infty(\mathbf{R})} \|e^{\delta(1+i\ \cdot)^2} f(1+i\ \cdot)\|^{\theta}_{L^\infty(\mathbf{R})}.$$

For the case of a function

$$f : U \to V,$$

where $U \subseteq \mathbb{C}$ is an open domain and V is a topological vector space, we shall say that f is weakly holomorphic if Λf is a holomorphic function for any $\Lambda \in V'$. Then f is also strongly holomorphic in the sense that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in V for any $z \in U$. (see [**162**], Chapter 3)

7. Spectral Theorem for Self-Adjoint Operators

Given any operator A on a (complex) Hilbert space H there are many ways to associate a spectral family $E^A(t)$. The corresponding operator $\hat{E}^A$ defined by Theorem 1.10 in general may not be related to the initial operator A. Nevertheless the following proposition states that whenever A is self-adjoint there is a very precise relation.

We shall need some results the resolvent operator $R_A(z) = (A - z)^{-1}$ associated to A. The first we present shows that $R_A(\cdot)$ and other complex functions associated to it are holomorphic (on some suitable domain of $\mathbb{C}$).

PROPOSITION 1.2. *Let A be a closed operator on H, and $f, g \in H$. Then the functions*

$$\begin{aligned} R_A(\cdot) : \rho(A) &\longrightarrow \mathcal{B}(H) \ , \quad z \mapsto R_A(z) \ , \\ R_A(\cdot)f : \rho(A) &\longrightarrow H \ , \qquad z \mapsto R_A(z)f \ , \\ (g, R_A(\cdot)f) : \rho(A) &\longrightarrow \mathbb{C} \ , \qquad z \mapsto (g, R_A(z)f) \ , \end{aligned}$$

are holomorphic functions on the resolvent set $\rho(A)$.

Proof. Let $z_0 \in \rho(T)$, and let $r = \|R_A(z_0)\|_H^{-1}$. Then for any $z \in \mathbb{C}$ such that $|z - z_0| < r$ we have

$$R_A(z) = \sum_{n=0}^{\infty} (z_0 - z)^n R_A(z_0)^{n+1} \ ,$$

with respect to the operator norm of $\mathcal{B}(H)$,

$$R_A(z)f = \sum_{n=0}^{\infty} (z_0 - z)^n R_A(z_0)^{n+1} f \ ,$$

with respect to the operator norm of H and

$$(g, R_A(z)f) = \sum_{n=0}^{\infty} (z_0 - z)^n \left(g, R_A(z_0)^{n+1} f\right) \ .$$

Therefore by definition the three functions above introduced are holomorphic. □

Now we restrict our attention to Hermitian operators, $A : H \longrightarrow H$. The following result describes the relation between the norm of $R_A(z)$ and the distance of z from the real axis $\text{Re}\,(z)$.

PROPOSITION 1.3. *Let A be Hermitian operator on H. Then all the eigenvalues of A are real and eigenvectors corresponding to different eigenvalues are orthogonal. Moreover for any $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $A - z$ is invertible, its inverse is continuous, and the following estimate holds*

$$\|R_A(z)\|_{\mathcal{B}(H)} \leq \tfrac{1}{|\text{Im}\,(z)|} \ .$$

Proof. Let $z = x + \imath y$ be an eigenvalue of A. By assumption it turns out that $z = \overline{z}$, that is $z \in \mathbb{R}$. Orthogonality between eigenvectors of distinct eigenvalues is also straightforward. Now for any $f \in D(A)$ we have

$$\|(A-z)f\|_H^2 = \|(A-x)f - \imath y f\|_H = \|(A-x)f\|_H^2 + |y|^2 \|f\|_H^2 \geq |\ \mathrm{Im}\ (z)|^2 \|f\|_H^2 \ .$$

If $z \in \mathbb{C} \setminus \mathbb{R}$ then $A - z$ is injective, and $g = (z - A)f \in D(R_A(z))$. We have

$$\|R_A(z)g\|_H = \|f\|_H \leq \tfrac{1}{|\ \mathrm{Im}\ (z)|} \|(z-A)f\|_H = \tfrac{1}{|\ \mathrm{Im}\ (z)|} \|g\|_H \ ,$$

which gives the statement. □

We shall also need some general properties and representations of complex holomorphic functions. We recall that a function $w : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be **of bounded variation** if and only if there exists a constant $C \geq 0$ such that $\sum_k |w(b_k) - w(a_k)| \leq C$ for every sequence $\{(a_k, b_k]\}$ of disjoint intervals. The smallest C of this kind is called the **variation** of w. Whenever w is a right continuous function of bounded variation the integral

$$\int_{-\infty}^{\infty} \tfrac{1}{z-t}\ \mathrm{d}w(t)$$

for $z \in \mathbb{C} \setminus \mathbb{R}$ can be considered as a Riemann-Stieltjes integral. The properties of such an integral function will be described in the following two propositions.

PROPOSITION 1.4. *Let $w : \mathbb{R} \longrightarrow \mathbb{R}$ be a right continuous function of bounded variation, such that* $\lim_{t \to -\infty} w(t) = 0$. *Then*

$$f(z) = \int_{-\infty}^{\infty} \tfrac{1}{z-t}\ \mathrm{d}w(t)\ , \quad \textit{for } z \in \mathbb{C}_+ \ .$$

(i) For all $t \in \mathbb{R}$ the **Stieltjes inversion formula**

$$w(t) = -\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \tfrac{1}{\pi} \int_{-\infty}^{t+\delta} \mathrm{Im}\ (f(s + \imath\varepsilon))\ \mathrm{d}s \ . \tag{1.7.21}$$

holds.

(ii) If $f(z) = 0$ for all $z \in \mathbb{C}_+$, then $w(t) = 0$ for all $t \in \mathbb{R}$.

Proof. Since w is real valued we have

$$\mathrm{Im}\ (f(s+\imath\varepsilon)) = \int_{-\infty}^{\infty} \mathrm{Im}\ \left[(s + \imath\varepsilon - u)^{-1}\right]\ \mathrm{d}w(u) = -\varepsilon \int_{-\infty}^{\infty} \left[(s+u)^2 + \varepsilon^2\right]^{-1}\ \mathrm{d}w(u)\ ,$$

for every $\varepsilon > 0$. By Fubini's Theorem it follows that

$$\int_{-\infty}^{r} \mathrm{Im}\ (f(s+\imath\varepsilon))\ \mathrm{d}s = -\int_{-\infty}^{\infty} \int_{-\infty}^{r} \tfrac{\varepsilon}{(s-u)^2 - \varepsilon^2}\ \mathrm{d}s\ \mathrm{d}w(u) = -\int_{-\infty}^{\infty} \left[\arctan\left(\tfrac{s-u}{\varepsilon}\right) + \tfrac{\pi}{2}\right]\ \mathrm{d}w(u)\ .$$

Since $\left|\arctan\left(\frac{r-u}{\varepsilon}\right)+\frac{\pi}{2}\right| \leq \pi$ for all $r \in \mathbb{R}$ and

$$\lim_{\varepsilon \downarrow 0} \arctan\left(\tfrac{r-u}{\varepsilon}\right)+\tfrac{\pi}{2} = \begin{cases} \pi & \text{per } r > u, \\ \frac{\pi}{2} & \text{per } r = u, \\ 0 & \text{per } r < u, \end{cases}$$

recalling the definition of the integration with respect to a right continuous function on a single point set in the sense of of the Riemann-Stieltjes, we have from Lebesgue's Theorem

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{r} \operatorname{Im}\ (f(s+\imath\varepsilon))\ \mathrm{d}s = -\int_{(-\infty,r)} \pi\ \mathrm{d}w(u) - \int_{\{r\}} \tfrac{\pi}{2}\ \mathrm{d}w(u) - \int_{(r,\infty)} 0\ \mathrm{d}w(u) =$$
$$= -\pi w(r_-) - \tfrac{\pi}{2}\left[w(r) - w(r_-)\right] = -\tfrac{\pi}{2}\left[w(r) + w(r_-)\right]\ .$$

If we set $r = t + \delta$ with $\delta > 0$ and let δ tend to zero, then the assertion follows. Property (ii) follows from (i). □

PROPOSITION 1.5. *Let $w : \mathbb{R} \longrightarrow \mathbb{C}$ be right continuous, and of bounded variation such that* $\lim_{t\to-\infty} w(t) = 0$. *If*

$$\int_{\infty}^{-\infty} \tfrac{1}{z-t}\ \mathrm{d}w(t) = 0\ ,$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ then $w(t) = 0$ for all $t \in \mathbb{R}$.

Proof. For $z \in \mathbb{C}_+$ the hypothesis gives

$$\int_{\infty}^{-\infty} \tfrac{1}{z-t}\ \mathrm{d}\overline{w}(t) = \overline{\int_{\infty}^{-\infty} \tfrac{1}{\overline{z}-t}\ \mathrm{d}w(t)} = 0\ .$$

Therefore

$$\int_{-\infty}^{\infty} \tfrac{1}{z-t}\ \mathrm{d}\left[\ \operatorname{Re} w(t)\right] = \int_{-\infty}^{\infty} \tfrac{1}{z-t}\ \mathrm{d}\left[\ \operatorname{Im}\ (w(t))\right]\ ,$$

and the thesis follows from (ii) of Proposition 1.4. □

We are ready to present the main result, which states under which conditions the existence and the uniqueness of such a function w is guaranteed, once the function f is given (see Theorem B3 in [**221**]).

THEOREM 1.11 (Herglotz). *Let $f : \mathbb{C}_+ \longrightarrow \mathbb{C}$ be an holomorphic function such that* $\operatorname{Im}\ (f(z)) \geq 0$ *and $|f(z) \operatorname{Im}\ (z)| \leq M$ for all $z \in \mathbb{C}_+$. Then there exists a unique right continuous non-decreasing function $w : \mathbb{R} \longrightarrow \mathbb{R}$, for which $\lim_{t\to-\infty} w(t) = 0$ and*

$$f(z) = \int_{-\infty}^{\infty} \tfrac{1}{z-t}\ \mathrm{d}w(t)\ ,$$

for all $z \in \mathbb{C}_+$*. For any* $t \in \mathbb{R}$ *we have that* $w(t) \leq M$ *and*

$$w(z) = -\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \tfrac{1}{\pi} \int_{\delta}^{t+\delta} \text{ Im } (f(s+\imath\varepsilon)) \ \mathrm{d}s \ .$$

Proof. The last identity will follow from Proposition 1.4 once we prove the existence of a function w having the remaining properties. For $0 < \varepsilon < r$ let the path $\Gamma_{\varepsilon,R} = \partial D_{\varepsilon,R}$ be defined as the Figure on the right,

$$\Gamma_{\varepsilon,R} = \Gamma_R \cup \Gamma_{\varepsilon,R} \ ,$$

where

$$\Gamma_R = \{z \in \mathbb{C} | \, |z - \imath\varepsilon| = R \ , \ \ \text{Im } (z) > \varepsilon\} \ ,$$
$$\Gamma_\varepsilon = \{z \in \mathbb{C} | -R \leq \text{ Re } (z) \leq R \ , \ \ \text{Im } (z) = \varepsilon\} \ .$$

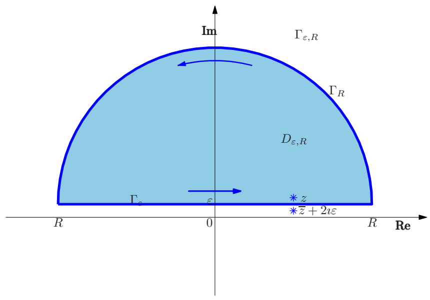

FIGURE 1.1 Path for the Cauchy integral.

For $z = x + \imath y \in D_{\varepsilon,R}$, then $\overline{z} + 2\imath\varepsilon$ lies outside $D_{\varepsilon,R}$. Therefore, by the Cauchy integral formula we have

$$f(z) = \tfrac{1}{2\pi\imath} \int_{\Gamma_{\varepsilon,R}} \tfrac{f(\zeta)}{\zeta - z} \ \mathrm{d}\zeta = \tfrac{1}{2\pi\imath} \int_{\Gamma_{\varepsilon,R}} \left[\tfrac{1}{\zeta - z} - \tfrac{1}{\zeta - (\overline{z} + 2\imath\varepsilon)} \right] f(\zeta) \ \mathrm{d}\zeta =$$
$$= \tfrac{1}{2\pi\imath} \int_{\Gamma_{\varepsilon,R}} \tfrac{z - \overline{z} - 2\imath\varepsilon}{(\zeta - z)(\overline{z} - 2\imath\varepsilon)} f(\zeta) \ \mathrm{d}\zeta = \tfrac{1}{\pi} \int_{\Gamma_{\varepsilon,R}} \tfrac{y - \varepsilon}{(\zeta - z)(\zeta - \overline{z} - 2\imath\varepsilon)} f(\zeta) \ \mathrm{d}\zeta \ .$$

For $\zeta \in \Gamma_R$ we have for fixed z that $|f(\zeta)| \leq \varepsilon^{-1} M$, and

$$f(z) = \tfrac{1}{2\pi\imath} \int_{\Gamma_{\varepsilon,R}} \tfrac{f(\zeta)}{\zeta - z} \ \mathrm{d}\zeta$$

thus the term over $\zeta \in \Gamma_R$ tends to 0 as $r \to \infty$. As far as the remainder term is concerned we have that

$$f(z) = \tfrac{1}{\pi} \int_{\Gamma_\varepsilon} \tfrac{y - \varepsilon}{(\zeta - z)(\zeta - \overline{z} - 2\imath\varepsilon)} f(\zeta) \ \mathrm{d}\zeta = \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y - \varepsilon}{(t + \imath\varepsilon - z)(t - \imath\varepsilon - \overline{z})} f(t + \imath\varepsilon) \ \mathrm{d}\zeta =$$
$$= \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y - \varepsilon}{(x - t)^2 + (y - \varepsilon)^2} f(t + \imath\varepsilon) \ \mathrm{d}\zeta \ .$$

If we set $v(z) = \text{ Im } (f(z))$, then it follows for $\text{ Im } (z) = y > \varepsilon > 0$ that

$$v(z) = \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y - \varepsilon}{(x - t)^2 + (y - \varepsilon)^2} v(t + \imath\varepsilon) \ \mathrm{d}\zeta \ .$$

By the hypothesis $|v(z) y| \leq |f(z) \text{ Im } (z)| \leq M$, hence for $y - \varepsilon > 0$ we have

$$\left| \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{(y - \varepsilon)^2}{(x - t)^2 + (y - \varepsilon)^2} v(t + \imath\varepsilon) \ \mathrm{d}\zeta \right| \leq |(y - \varepsilon) v(z)| \leq M \ .$$

By letting $y \to \infty$, we obtain by Fatou's Lemma (since $v \leq 0$ by hypothesis) that $v(\cdot + \imath\varepsilon) \in L^1(\mathbb{R})$ and

$$0 \leq -\tfrac{1}{\pi} \int_{-\infty}^{\infty} v(t + \imath\varepsilon)\, \mathrm{d}t \leq M \ , \quad \forall \varepsilon > 0 \ .$$

Since for all $y > \varepsilon > 0$

$$\left| \tfrac{y-\varepsilon}{(x-t)^2+(y-\varepsilon)^2} - \tfrac{y}{(x-t)^2+y^2} \right| \leq \varepsilon \left(\tfrac{1}{y(y-\varepsilon)} + \tfrac{1}{y^2} \right) \ ,$$

it follows that

$$\int_{-\infty}^{\infty} \left[\tfrac{y-\varepsilon}{(x-t)^2+(y-\varepsilon)^2} - \tfrac{y}{(x-t)^2+y^2} \right] v(t + \imath\varepsilon)\, \mathrm{d}t \to 0$$

as $\varepsilon \downarrow 0$. Therefore, for all $z \in \mathbb{C}_+$,

$$v(z) = \lim_{\varepsilon \uparrow 0} \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2} v(t + \imath\varepsilon)\, \mathrm{d}t \ .$$

Given $t \in \mathbb{R}$ and $\varepsilon > 0$ we can define

$$\theta_\varepsilon(t) = -\tfrac{1}{\pi} \int_{-\infty}^{t} v(s + \imath\varepsilon)\, \mathrm{d}s \ .$$

The functions θ_ε are all non-decreasing and bounded, $0 \leq \theta_\varepsilon(t) \leq M$ for all $t \in \mathbb{R}$. Let us construct, with the aid of the diagonal process, a positive null sequence $\varepsilon_n \to 0$ such that $\{\theta_{\varepsilon_n}(t)\}$ is convergent for all $t \in \mathcal{Q}$. If we set

$$\theta(t) = \lim_{n \to \infty} \theta_{\varepsilon_n}(t) \ , \quad \forall t \in \mathcal{Q} \ ,$$

then $\theta(s) \leq \theta(t)$ for any rational pair $s \leq t$. If we extend θ on $\mathbb{R}$ defining

$$\theta(t) = \inf_{\substack{s>t \\ s \in \mathcal{Q}}} \{\theta(s)\} \ , \quad \forall t \in \mathbb{R} \ ,$$

then θ is non-decreasing as well, and $\lim_{t \to \infty} (\theta(t) - \theta(-t)) \leq M$. We show that in the sense of the Riemann-Stieltjes integral

$$v(z) = -\int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2} v(t + \imath\varepsilon)\, \mathrm{d}\theta(t) \ , \quad \forall z \in \mathbb{C}_+ \ .$$

Since

$$\begin{aligned} v(z) &= \lim_{\varepsilon \uparrow 0} \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2} v(t + \imath\varepsilon)\, \mathrm{d}t = -\lim_{\varepsilon \uparrow 0} \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2} \tfrac{\mathrm{d}}{\mathrm{d}t} \theta_\varepsilon(t)\, \mathrm{d}t = \\ &= -\lim_{\varepsilon \uparrow 0} \tfrac{1}{\pi} \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta_\varepsilon(t) \ , \end{aligned}$$

the assertion is equivalent to the equality

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta_{\varepsilon_n}(t) = \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta(t) \ .$$

For the proof of this equality we notice that if we wish to approximate this Riemann-Stieltjes (with a continuous one) by a Riemann sums, then it is enough to consider only partitions involving rational points. For every rational partition P and for fixed $z = x + \imath\varepsilon$ let U_P, L_P, $U_{P,n}$ and $L_{P,n}$ be the upper and lower sums of the integrals

$$J = \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta(t)\ , \quad J_n = \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta_{\varepsilon_n}(t)\ ,$$

that correspond to P. For every rational partition P we have that $U_{P,n} \to U_P$ and $L_{P,n} \to L_P$. For every $\delta > 0$ there exists a rational partition P for which $U_P - L_P \leq \frac{\delta}{2}$. For such P there is an $n_0 \in \mathbb{N}$ such that $|U_{P,n} - U_P| \leq \frac{\delta}{2}$ and $|L_{P,n} - L_P| \leq \frac{\delta}{2}$ for all $n \geq n_0$. Since $L_{P,n} \leq J_n \leq U_{P,n}$ and $L_P \leq J \leq U_P$, it follows that $|J - J_n| \leq \delta$ for $n \geq n_0$. Therefore $J_n \to J$ as $n \to \infty$. Consequently, we have shown that for $z \in \mathbb{C}_+$

$$\mathrm{Im}\ f(z) = v(z) = \int_{-\infty}^{\infty} \tfrac{y}{(x-t)^2+y^2}\, \mathrm{d}\theta(t) = \ \mathrm{Im}\ \int_{-\infty}^{\infty} \tfrac{1}{z-t}\, \mathrm{d}\theta(t)\ .$$

Since f and $z \mapsto \int_{-\infty}^{\infty} (z-t)^{-1}\, \mathrm{d}\theta(t)$ are holomorphic in $\mathbb{C}_+$, we have

$$f(z) = \int_{-\infty}^{\infty} \tfrac{1}{z-t}\, \mathrm{d}\theta(t) + C\ ,$$

with some $C \in \mathbb{R}$. Because $|f(z)\ \mathrm{Im}\ (z)| \leq M$ and

$$\left| \mathrm{Im}\ (z) \int_{-\infty}^{\infty} \tfrac{1}{z-t}\, \mathrm{d}\theta(t) \right| \leq \int_{-\infty}^{\infty} \mathrm{d}\theta(t) \leq M\ , \quad \forall z \in \mathbb{C}_+\ ,$$

we must have $|C\ \mathrm{Im}\ (z)| \leq 2M$, and thus $C = 0$. If we now define

$$\tilde{\theta} = \lim_{\delta \downarrow 0} \theta(t+\delta)\ , \qquad w(t) = \tilde{\theta} - \lim_{s \to -\infty} \tilde{\theta}(s)\ ,$$

for $t \in \mathbb{R}$, then w satisfies the required properties. The passage from θ to $\tilde{\theta}$ does not prevent the integral formula above from holding, since $\tilde{\theta}$ has at most countably many points of discontinuity and they can be avoided during the formation of the partitions. The passage from $\tilde{\theta}$ to w does not influence the integral formula as well. □

Now we can state the main spectral theorem about a self-adjoint operator A on a Hilbert space H. Its importance relies on the explicit representation formula presented for $E^A(b)f - E^A(a)$ (for any $a \leq b$ real), where $E^A(t)$ is the unique spectral family associated to A such that $A = \int_{\mathbb{R}} t\, \mathrm{d}E^A(t)$.

THEOREM 1.12. *For any self-adjoint operator $A : D(A) \subseteq H \longrightarrow H$, there exists one and only one spectral family $E^A(t)$ such that*

$$A = \int_{\mathbb{R}} t\, \mathrm{d}E^A(t)\ .$$

Namely the family $E^A(t)$ is given by

$$\big(g, E^A(b)f - E^A(a)f\big) = \tfrac{1}{2\pi\imath} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{a+\delta}^{b+\delta} \sum_{\pm} \Big(g, \left[A - (\lambda \pm \imath\varepsilon)\right]^{-1} f\Big)\ \mathrm{d}\lambda\ , \qquad (1.7.22)$$

for all f, $g \in H$ and $-\infty < a \leq b < \infty$.

Proof. We shall prove uniqueness first. If $A = \hat{1}$, then $A - z = \widehat{1-z}$ by Proposition 1.1(vii). Then for all $z \in \mathbb{C}$ such that $\text{Im}\,(z) \neq 0$ we have by Proposition 1.1(viii) that

$$(A-z)\hat{E}\left((t-z)^{-1}\right)h = \hat{E}\left((1-z)(t-z)^{-1}\right)h = h\ ,\quad \forall h \in H\ ,$$
$$\hat{E}\left((t-z)^{-1}\right)(A-z)h = \hat{E}\left((t-z)^{-1}(1-z)\right)h = h\ ,\quad \forall h \in D(A)\ .$$

Consequently, $\hat{E}\left((t-z)^{-1}\right) = R_A(z)$ for all such z. This implies via Proposition 1.1(v) that

$$(h, R_A(z)h) = \int_{\mathbb{R}} \tfrac{1}{t-z}\,\mathrm{d}\rho_h^{E^A}(t)\ ,$$

for all $h \in H$. We can use Proposition 1.4, and obtain for all $t \in \mathbb{R}$

$$\left\|E^A(t)h\right\|_H^2 = \left(h, E^A(t)h\right) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} -\tfrac{1}{\pi}\int_{-\infty}^{t+\delta} \text{Im}\ (f, R_A(s+\imath\varepsilon)h)\ \mathrm{d}s =$$
$$= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \tfrac{1}{2\pi\imath}\int_{-\infty}^{t+\delta} (f, [R_A(s-\imath\varepsilon) - R_A(s+\imath\varepsilon)]\,h)\ \mathrm{d}s\ ,$$

which gives (1.7.22) with the aid of the polarization formula. Since (1.7.22) holds for all $f, g \in H$ the uniqueness has been proven.

Now we focus our attention to the existence. If there exists a spectral family $E^A(t)$, such that $A = \int_{\mathbb{R}} t\ \mathrm{d}E^A(t)$, then (1.7.22) must hold. Therefore we study whether (1.7.22) defines a spectral family with such a property. For every $h \in H$ the function $F_h(z) = (h, R_A(z)h)$ satisfies the assumptions of Theorem 1.11, since F_h is holomorphic for $\text{Im}\,(z) > 0$ by Proposition 1.2 and we have

$$\text{Im}\ (F_h(z)) = \text{Im}\ (h, R_A(z)h) = \text{Im}\ ((A-z)R_A(z)h, R_A(z)h) = \|R_A(z)h\|_H^2\ \text{Im}\,(\overline{z}) < 0\ ,$$

for $\text{Im}\,(z) > 0$, and by Proposition 1.3

$$|\ \text{Im}\ F_h(z)\ \text{Im}\,(z)| \leq \tfrac{1}{\text{Im}\,(z)}\,\|h\|_H^2\,|\ \text{Im}\,(z)| = \|h\|_H^2\ .$$

Consequently

$$(h, R_A(z)h) = \int_{\mathbb{R}} \tfrac{1}{z-t}\,\mathrm{d}w(h;t)\ , \tag{1.7.23}$$

where

$$w(h;t) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \tfrac{1}{2\pi\imath}\int_{-\infty}^{t+\delta} (h, [R_A(s-\imath\varepsilon) - R_A(s+\imath\varepsilon)]\,h)\ \mathrm{d}s\ .$$

We notice that $w(h;t)$ is a non-decreasing and right continuous function of t, and $w(h;t) \to 0$ as $t \to -\infty$, $w(h;t) \leq \|h\|_H^2$ for all $t \in \mathbb{R}$. Equation (1.7.23) holds for all $z \in \mathbb{C}\setminus\mathbb{R}$ since $(h, R_A(\overline{z})h) \leq \overline{(h, R_A(z)h)}$. Furthermore, we define

$$w(g,h;t) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \tfrac{1}{2\pi\imath}\int_{-\infty}^{t+\delta} (g, [R_A(s-\imath\varepsilon) - R_A(s+\imath\varepsilon)]\,h)\ \mathrm{d}s\ ,$$

whose existence follows by means of the polarization identity for the sesquilinear forme

$$(g,h) \mapsto (g, [R_A(s-\imath\varepsilon) - R_A(s+\imath\varepsilon)]\,h)\ .$$

The mapping $(g,h) \mapsto w(g,h;t)$ is a bounded non-negative sesquilinear form on H. The sesquilinearity is clear from the definition; moreover $w(h,h;t) = w(h;t) \geq 0$ for all $t \in \mathbb{R}$. The Schwarz inequality and the inequality $w(h;t) \leq \|f\|_H^2$ imply for all $g,h \in H$ and $t \in \mathbb{R}$ that

$$|w(g,h;t)|^2 \leq w(g;t)w(h;t) \leq \|g\|_H^2 \, \|h\|_H^2 \ .$$

Therefore, by Riesz Representation Theorem there exists for every $t \in \mathbb{R}$ an operator $E^A(t) \in \mathcal{B}(H)$ such that $\left\|E^A(t)\right\|_{\mathcal{B}(H)} \leq 1$ and

$$\left(g, E^A(t)h\right) = w(g,h;t) \quad \forall g,h \in H \ .$$

$E^A(t)$ is self-adjoint, and $E^A(t) \geq 0$.

Now we show that $E^A(t)$ is a spectral family. For this aim we first show that $E^A(s)E^A(t) = E^A(\min(s,t))$ for any pair $s,t \in \mathbb{R}$. From polarization identity it follows that for all $z \in \mathbb{C} \setminus \mathbb{R}$ and for all $h \in H$

$$(g, R_A(z)h) = \int_{\mathbb{R}} \tfrac{1}{t-z} \, \mathrm{d}w(g,h;t) = \int_{\mathbb{R}} \tfrac{1}{t-z} \, \mathrm{d}\left(g, E^A(t)h\right) \ . \tag{1.7.24}$$

Consequently, the resolvent identity implies for all $z, z' \in \mathbb{C} \setminus \mathbb{R}$ with $z \neq z'$

$$\begin{aligned} &\int_{\mathbb{R}} \tfrac{1}{z-t} \, \mathrm{d}\left(R_A\left(\overline{z'}\right) g, E(t)h\right) = \left(R_A(\overline{z'})g, R_A(z)f\right) = \left(g, R_A\left(z'\right) R_A(z)f\right) = \\ &= \tfrac{1}{z-z'} \left[(g, R_A(z)h) - \left(g, R_A\left(z'\right)h\right)\right] = \tfrac{1}{z-z'} \int_{\mathbb{R}} \left[\tfrac{1}{t-z} - \tfrac{1}{t-z'}\right] \, \mathrm{d}\left(g, E(t)h\right) = \\ &= \int_{\mathbb{R}} \tfrac{1}{(t-z)(t-z')} \, \mathrm{d}\left(g, E(t)h\right) = \int_{\mathbb{R}} \tfrac{1}{z-t} \, \mathrm{d}_t \int_{-\infty}^{t} \tfrac{1}{z'-s} \, \mathrm{d}\left(g, E^A(s)h\right) \ . \end{aligned}$$

From Proposition 1.4 it follows that

$$\begin{aligned} &\int_{\mathbb{R}} \tfrac{1}{(s-z')} \, \mathrm{d}_s \left(g, E^A(s)E^A(t)h\right) = \left(g, R_A(z')E^A(t)h\right) = \\ &= \left(R_A(z'^{*})g, E^A(t)h\right) = \int_{-\infty}^{t} \tfrac{1}{s-z'} \, \mathrm{d}\left(g, E(s)h\right) \ . \end{aligned}$$

Hence we get that for all $f,g \in H$ and $s,t \in \mathbb{R}$ again by Proposition 1.4

$$(g, E(s)E(t)h) = \begin{cases} \left(g, E^A(s)h\right) & \text{if } s \leq t \\ \left(g, E^A(t)h\right) & \text{if } s \geq t \end{cases} ,$$

which shows that $E^A(s)E^A(t) = E^A(\min(s,t))$. In particular $E^A(t)^2 = E^A(t)$. Therefore, the $E^A(t)$'s are orthogonal projections for all $t \in \mathbb{R}$, and $E^A(s) \leq E^A(t)$ for $s \leq t$ (where the inequality in meant to be in the sense of self-adjoint operators). Thus 1.4 (i) and 1.4(ii) are satisfied. The right continuity 1.4(iii) follows from the formula

$$\left\|E^A(t+\varepsilon)h - E^A(t)h\right\|_H^2 = \left\|E^A(t+\varepsilon)h\right\|_H^2 - \left\|E^A(t)h\right\|_H^2 = w(h;t+\varepsilon) - w(h;t) \to 0$$

as $\varepsilon \to 0$ since $w(h;\cdot)$ is right continuous. Moreover, $\left\|E^A(t)h\right\|_H^2 = w(h;t) \to 0$ as $t \to -\infty$, hence $E^A(t) \to \mathbf{0}$ as $t \to -\infty$. As $E^A(\cdot)$ is monotone, it must converge to an orthogonal projection $E^A(\infty)$ as $t \to \infty$. We have

$$\left(h, E^A(\infty)h\right) = \lim_{s \to -\infty} \left(h, E^A(s)h\right) \geq \left(h, E^A(t)h\right) \ .$$

Consequently, $E^A(\infty) \geq E^A(t)$ for all $t \in \mathbb{R}$. Let $F = \mathbf{1} - E^A(\infty)$. Then

$$E^A(t)F = E(t)\left(\mathbf{1} - E^A(\infty)\right) = E^A(t) - E^A(t) = 0 \,.$$

It follows from this for all $g, h \in H,\ \mathrm{Im}\ (z) \neq 0$ that

$$(g, R_A(z)Fh) = \int_{\mathbb{R}} \tfrac{1}{t-z}\ \mathrm{d}\,(g, R_A(t)Fh) = 0 \,.$$

Hence $R_A(z)Fh = 0$ for all $h \in H$, and thus $F = 0$. That is $E^A(\infty) = \mathbf{1}$ and proves 1.4(iv).

We conclude our proof observing that $R_A(z) = \hat{E}((t-z)^{-1})$ by (1.7.24), and this implies that $\hat{E}(1-z) = z - A$ and $\hat{E}(1) = A$ (Proposition 1.1(viii) and 1.1(vii) respectively). □

The Spectral Theorem 1.12 is the key tool in the introduction of the **functional calculus** on an Hilbert space H. There exist several methods to introduce the functional calculus associated to a given self-adjoint operator, but we keep on following [**221**] (see Chapter VII in [**160**] for an alternative approach).

It is possible to show that the operators $F(A)$ depend only on the values of F on the spectrum of the operator A. For this reason in the sequel we will use the following equivalent notations:

$$F(A) = \int_{\mathbb{R}} F(t)\ \mathrm{d}E^A(t) = \int_{\sigma(A)} F(t)\ \mathrm{d}E^A(t) \,,$$

where $\sigma(A) \subseteq \mathbb{R}$ denotes the spectrum of the self-adjoint operator A.

8. Some Interpolation Results

We start this section by a theorem important in the complex interpolation consequence of Three Lines Lemma 1.6. To formulate this theorem we shall denote by $\mathcal{L}(A, B)$ the Banach space of bounded operators from a Banach space A into the Banach space B.

Given any positive real numbers p_0, p_1 with $1 \leq p_0 < p_1 \leq \infty$, we denote by $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ the linear space

$$\{f : f = f_0 + f_1, f_0 \in L^{p_0}(\mathbf{R}^n), f_1 \in L^{p_1}(\mathbf{R}^n)\}.$$

The norm in this space we define as follows

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}.$$

Here the infimum is taken over all representations $f = f_0 + f_1$, where $f_0 \in L^{p_0}(\mathbf{R}^n)$ and $f_1 \in L^{p_1}(\mathbf{R}^n)$.

It is easy to see that $L^{p_0} + L^{p_1}$ is a Banach space.

THEOREM 1.13. *(Stein interpolation theorem, see* [**160**]*)*

Suppose $1 \leq p_0, p_1, q_0, q_1 \leq \infty, T(z)$ *is a continuous function from the strip* $0 \leq \mathbf{Re}z \leq 1$ *into* $\mathrm{L}(L^{p_0} + L^{p_1}; L^{q_0} + L^{q_1})$ *, holomorphic for* $0 < \mathbf{Re}z < 1$ *and satisfying the properties*

$$\|T(z)\|_{\mathrm{L}(L^{p_0};L^{q_0})} \leq C\exp(C|\mathbf{Im}z|) \ \text{ for } \ \mathbf{Re}z = 0, \tag{1.8.25}$$

$$\|T(z)\|_{\mathrm{L}(L^{p_1};L^{q_1})} \leq C\exp(C|\mathbf{Im}z|) \ \text{ for } \ \mathbf{Re}z = 1. \tag{1.8.26}$$

Then for any $\theta \in (0,1)$ *we have*

$$\|T(\theta)\|_{\mathrm{L}(L^p;L^q)} \leq C,$$

where

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} \ , \ \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}. \tag{1.8.27}$$

The **convolution** of f and g, defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\ \mathrm{d}y\ , \tag{1.8.28}$$

plays a crucial role in the study of partial differential equation, therefore it turns out to be very important to investigate for some a-priori estimates for this kind of operation.

A standard tool at this aim are the interpolation theory for Lebesgue spaces, whose main result is represented by the following theorems: (see [**20**, Theorem 1.1.1]).

THEOREM 1.14 (Riesz-Thorin). *Assume* $p_1 \neq p_2$ *and* $q_1 \neq q_2$*, and that* T *is a bounded operator acting on the following spaces:*

$$T : L^{p_1}(\mathbb{R}^n) \longrightarrow L^{q_1}(\mathbb{R}^n)\ ,$$
$$T : L^{p_2}(\mathbb{R}^n) \longrightarrow L^{q_2}(\mathbb{R}^n)\ .$$

with operator norm respectively M_1*, and* M_2 *respectively. Then* T *is a bounded operator for the spaces*

$$T : L^{p}(\mathbb{R}^n) \longrightarrow L^{q}(\mathbb{R}^n)\ , \tag{1.8.29}$$

with norm $M = M_1^{\theta} M_2^{1-\theta}$*, provided that* $0 < \theta < 1$*, and* $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$*,* $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$.

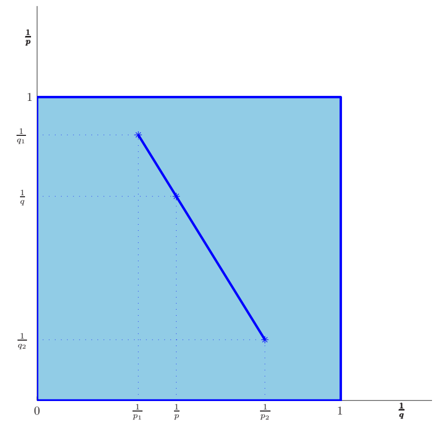

FIGURE 1.2 Interpolation exponents.

Note that Riesz-Thorin interpolation theorem is a trivial corollary of this complex interpolation theorem. Indeed an easy consequence of this result is the **Hausdorff-Young estimate**, that we state as follows. If $p, q, r \in [1, \infty]$ are such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then

$$\|f * g\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}\ , \tag{1.8.30}$$

for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Using duality arguments, one can obtain a bilinear version of the Hausdorff-Young estimate, namely for $p, q \in [1, \infty]$

$$\left| \int_{\mathbb{R}^n} (f * g)(x) h(x)\ \mathrm{d}x \right| \lesssim \|f\|_{L^s} \|g\|_{L^q} \|h\|_{L^p}\ , \tag{1.8.31}$$

for any $f \in L^s(\mathbb{R}^n)$, $h \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where the exponent s is defined by $n + \frac{n}{p'} = \frac{n}{s} + \frac{n}{q}$.

8.1. Interpolation for sequences with values in Banach spaces. (see [**20**], [**204**])

Of special interest for applications is the abstract interpolation for the space $l_q(A)$. Given any Banach space A, we denote by $l_q(A)$ the linear space of all sequences $(a_k)_{k=0}^{\infty}, a_k \in A$, such that the norm

$$\|(a_k)\|_{l_q(A)} = (\sum_{k=0}^{\infty} \|(a_k)\|_A^q)^{1/q} \tag{1.8.32}$$

is bounded. For $q = \infty$, the corresponding norm is

$$\|a_k\|_{l_\infty(A)} = \sup_k \|a_k\|_A. \tag{1.8.33}$$

For $1 \leq q \leq \infty$ the space $l_q(A)$ is a Banach space.

The main result of this section is the following interpolation result for spaces of sequences.

THEOREM 1.15. *(see Section 5.6 in* [**20**]*) Let* $A_1 \subset A_0$ *be dense in* A_0. *Then for* $1 < q, q_0, q_1 < \infty$, *satisfying*

$$1/q = (1-\theta)/q_0 + \theta/q_1$$

with some $\theta \in (0,1)$, *we have*

$$(l_{q_0}(A_0), l_{q_1}(A_1))_\theta = l_q((A_0, A_1)_\theta), \tag{1.8.34}$$

For the proof and more details we refer to [**63**].

Further, given any real number s, we denote by $l_q^s(A)$ the linear space of all sequences $(a_k)_{k=0}^{\infty}, a_k \in A$, such that the norm

$$\|(a_k)\|_{l_q^s(A)} = (\sum_{k=0}^{\infty} 2^{ksq}\|a_k\|_A^q)^{1/q} \tag{1.8.35}$$

is bounded. For $1 \leq q < \infty$ the space $l_q^s(A)$ is a Banach space.

Then we have the following result for the complex interpolation (see Section 5.6 in [**20**]).

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_\theta = l_q^s((A_0, A_1)_\theta), \tag{1.8.36}$$

where

$$1/q = (1-\theta)/q_0 + \theta/q_1 \quad , \quad s = (1-\theta)s_0 + \theta s_1$$

and moreover $1 \leq q_0, q_1 < \infty$.

9. Hardy-Littlewood-Sobolev Inequality

The convolutions we are interested in involve singular kernels as functions g of the form $g(x) = |x|^{-\gamma}$, which do not belong to any Lebesgue space $L^q(\mathbb{R}^n)$ for any $q \in [1, \infty]$. Nevertheless a generalization of this kind of estimate can be achieved and it is given by the following result.

THEOREM 1.16 (Hardy-Littlewood-Sobolev Inequality). *Let* $0 < \gamma < n$ *and* $1 < p < q < \infty$ *be such that*

$$n - \gamma + \frac{n}{q} = \frac{n}{p} . \tag{1.9.37}$$

Then there exists a constant $C = C(n,p,q,\gamma)$ *such that the following estimate*

$$\left\| |\cdot|^{-\gamma} * f \right\|_{L^q} \leq C \left\| f \right\|_{L^p} \tag{1.9.38}$$

holds for any $f \in L^p(\mathbb{R}^n)$.

The Hardy-Littlewood-Sobolev inequality was proved in one-dimension in [**80**] and [**81**], and then extended to any dimension in [**180**]. Sharp values for the constant $C(n,p,q,\gamma)$ have been studied in [**129**]. In the literature there exist many different proofs of such estimate, and their techniques apparently look quite different from each other (see [**185**], [**82**] and [**130**]).

It is possible to have a bilinear version of the Hardy-Littlewood-Sobolev inequality using standard dual arguments,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\gamma}\,\mathrm{d}y\,\mathrm{d}x \lesssim \|f\|_{L^p}\|g\|_{L^q} \tag{1.9.39}$$

whenever $n - \gamma + \frac{n}{q'} = \frac{n}{p}$ and $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.

To complete this section we state also the following Hardy, Littlewood inequality involved the Fourier transform (see [**63**]).

THEOREM 1.17. *(Hardy , Littlewood inequality) Let* $m(\xi) = c|\xi|^{-\gamma}$ *and consider the operator*

$$I(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} m(\xi)\hat{f}(\xi)d\xi.$$

Prove that for $1 < p \leq 2 \leq q < \infty$ *and* $1/p - 1/q = \gamma/n$ *this operator can be extended as a bounded operator from* L^p *to* L^q.

10. TT^* Method

Suppose we have an operator $T : D(T) \subset \mathcal{B} \longrightarrow H$, where H is an Hilbert space, $\mathcal{B}$ is a Banach space, and $\mathcal{D}(T)$ the domain of T densely contained in $\mathcal{B}$. The adjoint operator $T^* : H \longrightarrow \mathcal{D}(T)^*$ is defined by

$$\langle T^*h, F\rangle = (h, TF) \ , \ \ \forall h \in H \ , \ \ \forall F \in \mathcal{D}(T) \ ,$$

where the $\langle\cdot,\cdot\rangle$ denotes the action of $\mathcal{B}^*$ on $\mathcal{B}$, and $(\cdot,\cdot)$ the inner product of H. We observe here that in this case it makes sense to consider the composition operator $T^*T : \mathcal{D}(T) \longrightarrow \mathcal{B}^*$. We shall also introduce $B : \mathcal{D}(T) \times \mathcal{D}(T) \longrightarrow \mathbb{R}$, the bilinear operator associated to T, defined by

$$B(F,G) = (TF, TG) \ , \ \ \forall F, G \in \mathcal{B} \ .$$

The so-called ***TT* method*** is indeed the equivalence of the boundedness of the operators T, T^*, and T^*T, and B, defined above. The results is stated in the following theorem (see Lemma 2.1 in [**72**] or see [**75**]).

THEOREM 1.18. *Let* T, T^*, T^*T *and* B *the operators defined as above. Then the following conditions are equivalent:*

(1) there exists a constant $M > 0$ *such that* $\|TF\|_H \leq M\|F\|_{\mathcal{B}}$, *for all* $F \in \mathcal{D}(T)$,
(2) $\mathcal{R}(T^*) \subset \mathcal{B}^*$ *and there exists a constant* $M > 0$ *such that* $\|T^*h\|_{\mathcal{B}^*} \leq M\|h\|_H$, *for all* $h \in H$,
(3) $\mathcal{R}(T^*) \subset \mathcal{B}^*$ *and there exists a constant* $M > 0$ *such that* $\|T^*TF\|_{\mathcal{B}^*} \leq M^2\|F\|_{\mathcal{B}}$, *for all* $F \in \mathcal{B}$,

(4) there exists a constant $M > 0$ such that $|B(F,G)| \le M^2 \|F\|_{\mathcal{B}} \|G\|_{\mathcal{B}}$, for all $F, G \in \mathcal{B}$.

If one of (all) those conditions is (are) satisfied, the operators T, T^*T and B extend by continuity to bounded operators from B and from $\mathcal{B} \times \mathcal{B}$ to H respectively.

Proof. (1) $\Longrightarrow$ (2). From the fact that $\mathcal{D}(T)$ is densely contained in $\mathcal{B}$, it follows that $\mathcal{B}^*$ is a subspace of $\mathcal{D}(T)^*$. By Cauchy-Schwarz inequality and (1) we have for any $F \in \mathcal{D}(T)$

$$|\langle T^*h, F\rangle| = |(h, TF)| \le \|h\|_H \|TF\|_{\mathcal{B}} \le \|h\|_H M \|F\|_{\mathcal{B}} \ .$$

(2) $\Longrightarrow$ (1). If $F \in \mathcal{D}(T)$ by (2) we have

$$|(h, TF)| = |\langle T^*h, F\rangle| \le \|T^*h\|_{\mathcal{B}^*} \|F\|_{\mathcal{B}} \le M \|h\|_H \|F\|_{\mathcal{B}} \ , \quad \forall h \in H \ .$$

(1) $\Longrightarrow$ (3). We have already showed that (1) implies (2). Then it is clear that the composition operator T^*T has norm bounded by M^2.

(3) $\Longrightarrow$ (1). Let $F \in \mathcal{D}(T)$. Then (3) implies

$$\|TF\|_H^2 = (TF, TF) = \langle F, T^*TF\rangle \le M^2 \|F\|_{\mathcal{B}}^2 \ .$$

(3) $\Longrightarrow$ (4). Let $F, G \in \mathcal{B}$. Then by (3) we have

$$|B(F,G)| = |(TF, TG)| = |\langle T^*TF, G\rangle| \le \|TT^*F\|_{\mathcal{B}^*} \|G\|_{\mathcal{B}} \le M^2 \|F\|_{\mathcal{B}} \|G\|_{\mathcal{B}} \ .$$

(4) $\Longrightarrow$ (3). Let $F \in \mathcal{B}$ By (3) we have that for any $G \in \mathcal{B}$

$$|\langle T^*TF, G\rangle| = |(TF, TG)| = |B(F,G)| \le M^2 \|F\|_{\mathcal{B}} \|G\|_{\mathcal{B}} \ ,$$

that concludes the proof of the Theorem. □

11. Paley-Littlewood Partition of Unity

A very important tool in harmonic analysis is represented by localizing functions in the phase space in dyadic annuluses. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\eta(\xi) = 1$ whenever $|\xi| \le 1$, and $\eta(\xi) = 0$ when $|\xi| \ge 2$. Thus we define $\phi(\xi) = \eta(\xi) - \eta(2\xi)$, in such a way that $\mathrm{supp}\,(\phi) \subset \left\{2^{-1} \le |\xi| \le 2\right\}$. Then for any integer number j we set

$$\phi_j(\xi) = \phi\left(\frac{\xi}{2^j}\right) \ . \tag{1.11.40}$$

It turns out that the ϕ_j's are $C_0^\infty(\mathbb{R}^n)$ functions, supported in $2^{j-1} \le |\xi| \le 2^{j+1}$, and moreover $\sum_{j\in\mathbb{Z}} \phi_j(\xi) = 1$ for any $\xi \ne 0$. The resulting dyadic partition of unity $\{\phi_j\}_{j\in\mathbb{Z}}$ is called the **Paley-Littlewood partition of unity**. We shall also call **dyadic multiplicative operators S_j** the multiplicative operators defined by

$$\widehat{S_j f}(\xi) = \phi_j(\xi)\hat{f}(\xi) \ , \tag{1.11.41}$$

in such a way that the following dyadic decomposition $f = \sum_{j\in\mathbb{Z}} S_j f$ holds for any tempered distribution f.

We observe here that for any integer j, the operator S_j is bounded from L^p into itself for all $1 \leq p \leq \infty$, since S_j is a convolution operator with kernel $\check{\phi}_j$, and from Haussdorff-Young inequality (1.8.30) we have

$$\|S_j f\|_{L^p} \leq C_p \|f\|_{L^p} \ ,$$

where the constant $C_p = \|\check{\phi}_j\|_{L^1}$ does not depend on j (since we have $\|\check{\phi}_j\|_{L^1} \leq \|\check{\phi}\|_{L^1}$ for any $j \in \mathbb{Z}$).

By Plancherel's Theorem we have for any $f \in L^2(\mathbb{R}^n)$ the following identity:

$$\|f\|_{L^2}^2 = \sum_{j \in \mathbb{Z}} \|S_j f\|_{L^2}^2 \, . \tag{1.11.42}$$

Therefore our main goal is to establish the analogous of (1.11.42) in any $L^p(\mathbb{R}^n)$ space. We shall see that the key role is played by the function defined below.

DEFINITION 1.5. *We shall call* $\boldsymbol{l^2}$ **norm function** *the (non-linear) operator*

$$Sf(x) = \left(\sum_{j \in \mathbb{Z}} |S_j f(x)|^2 \right)^{\frac{1}{2}} \ ,$$

The key result of this Section indeed states that Sf and f itself are equivalent with respect to the L^p norm for any $p \in (1, \infty)$. The proof of such an equivalence relies on the so-called **Calderon-Zygmund operators**, which turn out to map any $L^p(\mathbb{R}^n)$ into itself. The precise definition of such operators can be found in [**185**] (the original paper is due to [**34**]), nevertheless we shall only need the following sufficient conditions

(1) K is a convolution operator $K : L^2(\mathbb{R}^n; H_1) \longrightarrow L^2(\mathbb{R}^n; H_2)$ where H_1, H_2 are Hilbert spaces, in the form

$$Kf = k * f \ , \ \forall f \in H_1 \ ,$$

where the kernel k is a measurable function $k : \mathbb{R}^n \longrightarrow H_2$;

(2) the kernel k of K is a Fourier multiplier of class $C^l(\mathbb{R}^n \setminus 0)$, with $l > \frac{n}{2}$, and it satisfies

$$\left\| \partial_\xi^\alpha \hat{k} \right\|_{H_2} \lesssim |\xi|^{-|\alpha|} \ , \ \forall |\alpha| \leq l. \tag{1.11.43}$$

Now we are ready to state the main theorem.

THEOREM 1.19. *For any* $1 < p < \infty$ *there exists a constant* $C_p > 0$ *such that*

$$C_p^{-1} \|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C_p \|f\|_{L^p} \ , \quad \forall f \in L^p(\mathbb{R}^n) \, . \tag{1.11.44}$$

Proof. We first show by duality arguments that the first inequality in (1.11.44) follows from the second one. Indeed using Plancherel Theorem and Cauchy-Schwarz inequality, and observing that

$\phi_j\phi_k = 0$ unless $j \approx k$ we get

$$\int f(x)g(x)\,\mathrm{d}x \approx \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \sum_{j,k\in\mathbb{Z}} \widehat{S_jf}(\xi)\widehat{S_kg}(\xi)\,\mathrm{d}\xi \approx$$

$$\approx \int_{\mathbb{R}^n} \sum_{j\approx k} S_jf(x)S_kg(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |S_jf(x)|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} |S_kg(x)|^2\right)^{\frac{1}{2}} \mathrm{d}x \leq$$

$$\leq \|Sf\|_{L^p} \|Sg\|_{L^{p'}} \leq C_{p'} \|Sf\|_{L^p} \|g\|_{L^{p'}} \ .$$

We claim that the operator $K : L^2(\mathbb{R}^n;\mathbb{C}) \longrightarrow L^2(\mathbb{R}^n; l^2(\mathbb{C}))$ defined by

$$(K)_j = S_jf \ ,$$

is a Calderon-Zygmund operator. Indeed the condition (1) is fulfilled choosing $H_1 = \mathbb{C}$, $H_2 = l^2(\mathbb{C})$, and observing that $Kf = k * f$, where $k : \mathbb{R}^n \longrightarrow l^2(\mathbb{C})$ is defined by $k_j = \phi_j$. Condition (2) is guaranteed by the smoothness of the localizing functions ϕ_j's, and the following estimate. For any $\xi \neq 0$ actually only three dyadic components are not null, and we have

$$\left\|\partial_\xi^\alpha m\right\|_{l^2} = \left(\sum_{j\in\mathbb{Z}} \left|\partial_\xi^\alpha \phi\left(\frac{\xi}{2^j}\right)\right|^2\right)^{\frac{1}{2}} = \left(\sum_{j\in\mathbb{Z}} \left(2^{-j|\alpha|}\left|\partial_\xi^\alpha \phi(\xi)\right|\right)^2\right)^{\frac{1}{2}} \lesssim |\xi|^{-|\alpha|}\, 3 \left\|\partial_\xi^\alpha \phi\right\|_{L^\infty} \ ,$$

that is exactly (1.11.43) with $l = \infty$ (using $|xi| \approx 2^j$ in the support of ϕ_j). Hence, as we remarked above, K is a bounded operator from $L^p(\mathbb{R}^n;\mathbb{C})$ to $L^p(\mathbb{R}^n; l^2(\mathbb{C}))$, in other words

$$\|Kf\|_{L^p(\mathbb{R}^n;l^2(\mathbb{C}))} \lesssim \|f\|_{L^p} \ .$$

Our thesis follows observing that $\|Kf\|_{L^p(\mathbb{R}^n;l^2(\mathbb{C}))} = \|\|Kf\|_{l^2}\|_{L^p} = \|Sf\|_{L^p}$. □

The following corollary shows how the norms of $\sum_{j\in\mathbb{Z}} S_jf$ and f itself are related to $L^p(\mathbb{R}^n)$ space. The importance of that kind of relations relies in the fact that it influences the embedding of spaces built throughout the dyadic multiplicative operators (as the Besov, or the homogeneous Sobolev spaces) in usual Lebesgue spaces, as we shall see in the sequel.

COROLLARY 1.1. *Let* $2 \leq p < \infty$. *Then we have*

$$\|f\|_{L^p}^2 \leq C_p \sum_{j\in\mathbb{Z}} \|S_jf\|_{L^p}^2 \ . \tag{1.11.45}$$

Let $1 < p \leq 2$. *Then we have*

$$\sum_{j\in\mathbb{Z}} \|S_jf\|_{L^p}^2 \leq C_p \|f\|_{L^p}^2 \ . \tag{1.11.46}$$

Proof. In the first case, if $\frac{p}{2} \geq 1$, from Theorem 1.19 and Minkowski inequality we have

$$\|f\|_{L^p}^2 \leq C_p \left\|\sum_{j\in\mathbb{Z}} |S_jf|^2\right\|_{L^{\frac{p}{2}}} \leq \sum_{j\in\mathbb{Z}} \left\||S_jf|^2\right\|_{L^{\frac{p}{2}}} = \sum_{j\in\mathbb{Z}} \|S_jf\|_{L^p}^2 \ .$$

On the other hand, if $\frac{1}{2} < \frac{p}{2} \leq 1$, Theorem 1.19 and the reversed Minkowski inequality give

$$\|f\|_{L^p}^2 \geq C_p^{-1} \left\| \sum_{j\in\mathbb{Z}} |S_j f|^2 \right\|_{L^{\frac{p}{2}}} \geq \sum_{j\in\mathbb{Z}} \left\| |S_j f|^2 \right\|_{L^{\frac{p}{2}}} = \sum_{j\in\mathbb{Z}} \|S_j f\|_{L^p}^2 \ .$$

□

REMARK 1.2. *The* **reversed Minkowski inequality** *involved in the last proof is meant as the following inequality*

$$\left\| \sum_{j\in\mathbb{Z}} |\phi_j| \right\|_{L^q} \geq \sum_{j\in\mathbb{Z}} \|\phi_j\|_{L^q} \ , \ \ \forall\, 0 < q < 1. \tag{1.11.47}$$

Proof. *In order to prove* (1.11.47)*, we set* $p = \frac{1}{q} \geq 1$ *and* $\psi_j = |\phi_j|^q$*. Then by Minkowski inequality we get*

$$\left(\sum_{j\in\mathbb{Z}} \|\phi_j\|_{L^q} \right)^q = \left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}} |\psi_j| \ \mathrm{d}x \right)^p \right)^{\frac{1}{p}} \geq \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} |\psi_j|^p \right)^{\frac{1}{p}} \ \mathrm{d}x = \left\| \sum_{j\in\mathbb{Z}} |\phi_j| \right\|_{L^q}^q \ .$$

□

The final proposition shows how we do control the norm of a function f in the homogeneous Sobolev spaces $\dot{H}_p^\gamma$ in terms of the sum of the L^p norm of the dyadic components $S_j f$.

PROPOSITION 1.6. *Let* $p \in [2, \infty)$ *and* $\gamma \in \mathbb{R}$*. Then there exists a constant* $C_{p,\gamma}$ *such that*

$$C_{p,\gamma}^{-1} \left(\sum_{j\in\mathbb{Z}} 2^{pj\gamma} \|S_j f\|_{L^p}^p \right)^{\frac{1}{p}} \leq \|f\|_{\dot{H}_p^\gamma} \leq C_{p,\gamma} \left(\sum_{j\in\mathbb{Z}} 2^{2j\gamma} \|S_j f\|_{L^p}^2 \right)^{\frac{1}{2}} \ .$$

Let $p \in (1, 2]$ *and* $\gamma \in \mathbb{R}$*. Then there exists a constant* $C_{p,\gamma}$ *such that*

$$C_{p,\gamma}^{-1} \left(\sum_{j\in\mathbb{Z}} 2^{2j\gamma} \|S_j f\|_{L^p}^2 \right)^{\frac{1}{2}} \leq \|f\|_{\dot{H}_p^\gamma} \leq C_{p,\gamma} \left(\sum_{j\in\mathbb{Z}} 2^{pj\gamma} \|S_j f\|_{L^p}^p \right)^{\frac{1}{p}} \ .$$

If we are in the situation described in the proof of Corollary 5.1, once a homogeneous estimate on the dyadic components of the form

$$\|S_j u\|_{L_t^q L_x^r} \lesssim 2^{j\gamma} \|S_j u_1\|_{L^2}$$

has been achieved, then by observing that $2^{j\gamma}\left\|S_j u_1\right\|_{L^2} = \left\|\partial^\gamma S_j u_1\right\|_{L^2} = \left\|S_j u_1\right\|_{\dot{H}^\gamma}$, Corollary 1.1 and Proposition 1.6 give precisely

$$\|u\|_{L_t^q L_x^r} \lesssim \left(\sum_{j\in\mathbb{Z}} \|S_j u\|_{L_t^q L_x^r}^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{j\in\mathbb{Z}} \|S_j u_1\|_{\dot{H}^\gamma}^2\right)^{\frac{1}{2}} \approx \|u_1\|_{\dot{H}^\gamma} \ , \ \ \forall r \in [2,\infty).$$

A similar argument can be used for the estimate with respect to the source term F. Notice that the failure of the Proposition 1.6 for $q=\infty$ explains also the lack of $r,\tilde{r}=\infty$ in the hypothesis of the Corollary 5.1 and the remark pointed out in Footnote 1.

CHAPTER 2

Resolvent Estimates

In this chapter we shall focus our attention to the free Laplace operator $H_0 = -\Delta$ and some suitable perturbation of H_0, that we shall denote $H_V = -\Delta + V(x, D)$. It will be interesting to investigate which kind of a-priori estimates these operators satisfy, and consequently their domains and their ranges.

It will be also interesting to study how the resolvent estimates for H_V may be derived from those of H_0.

1. The Limiting Absorption Principle

1.1. The Free Case. We denote by $H_0 = -\Delta$ the free Laplacian on the whole space $\mathbb{R}^n$. We observe here that H_0 is a self-adjoint operator on the domain $\mathcal{D}(H_0) = H^2(\mathbb{R}^n)$; on the other hand, as far as its (real) spectrum is concerned, we have that $\sigma(H_0) = \sigma_c(H_0) = \overline{\mathbb{R}}_+$ (see [**113**]). We shall also denote by $R_0(z) = (H_0 - z)^{-1}$ the resolvent operator associated to H_0, in such a way that we can define the mapping

$$R_0 : \mathbb{C} \setminus \overline{\mathbb{R}}_+ \longrightarrow \mathcal{B}\left(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n)\right) , \tag{2.1.1}$$

where $\mathcal{B}\left(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n)\right)$ is the set of all bounded operators from $L^2(\mathbb{R}^n)$ to $H^2(\mathbb{R}^n)$. The operator (2.1.1) turns out to be analytic with respect to the complex variable z outside $\sigma(H_0)$. In order to extend it on $\sigma(H_0)$ we have to weaken its definition, namely the spaces involved in the target operator space.

At this aim we introduce the weighted Hilbert space $H_r^m(\mathbb{R}^n)$ defined by

$$\|u\|_{H_r^m}^2 = \sum_{|\alpha| \le m} \|\langle x\rangle^r D^\alpha u\|_{L^2}^2 , \quad \forall m \in \mathbb{N} ,\ r \in \mathbb{R} , \tag{2.1.2}$$

where the weight $\langle x\rangle$ is given by the norm $\langle x\rangle = \sqrt{1 + |x|^2}$. From the definition of the norms (2.1.2) we get the following imbedding

$$H_{r_1}^m(\mathbb{R}^n) \subseteq H_{r_2}^m(\mathbb{R}^n) , \quad \text{if } r_2 \le r_1 , \tag{2.1.3}$$

therefore the map $z \mapsto R_0(z)$ wich is defined above keeps on being analytic even if we re-define it in the spaces

$$R_0 : \mathbb{C} \setminus \overline{\mathbb{R}}_+ \longrightarrow \mathcal{B}\left(L_s^2(\mathbb{R}^n), H_{-s}^2(\mathbb{R}^n)\right) , \qquad s > \tfrac{1}{2}. \tag{2.1.1$_E$}$$

Now it makes sense to investigate whether it may be possible to extend the resolvent mapping (2.1.1$_E$) on the real positive half-line, that is on values of the continuous spectrum of H_0 in the limit as $z \to \lambda \in \overline{\mathbb{R}}_+$ through one of the two half-planes $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im}(z) > 0\}$. We shall show that the answer is positive, and such a procedure is called **limiting absorption principle**.

The main tool is represented by L^2-weighted a priori estimates for the H^2_{-s} norm of a function u in terms of the L^2_s norm of $(-\Delta - z)\,u$ provided z is in some compact set, and $s > \frac{1}{2}$. The following Theorem (see Appendix A in [**5**]) gives such a result for a general **elliptic** operator $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ wich is of **principal type** (see, for example, [**87**] for the definition), and we shall apply it to $-\Delta$. We recall here that λ is said to be a **critical value** of P if for some $\xi_0 \in \mathbb{R}^n$, $P(\xi_0) = \lambda$ and $\nabla P(\xi_0) = 0$ and that the set of the critical value of P, here denoted by $\Lambda(P)$, is finite. We also denote by $P_m = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be the principal part of P.

THEOREM 2.1. *Let $P(D)$ be an elliptic differential operator with constant coefficients of order m and of principal type. Denote by $\Lambda_c(P)$ the set of critical values of P, and let K be a compact set in $\mathbb{C} \setminus \Lambda_c(P)$. Then there exists a constant C such that the following estimate holds*

$$\|u\|_{H^m_{-s}} \leq C \,\|(P(D) - z)\, u\|_{L^2_s} \ , \tag{2.1.4}$$

for any $z \in K$, and any $u \in H^m_{-s}(\mathbb{R}^n)$, provided $s > \frac{1}{2}$.

We shall also need the introduction of the trace operator of a function in some Sobolev space $H^r(\mathbb{R}^n)$ over the sphere $\mathbb{S}^n$. The following result guaranties then existence of such an operator under the assumption $r \geq \frac{1}{2}$ for any sufficiently smooth $n-1$ dimensional manifold embedded in $\mathbb{R}^n$ (see [**185**] or [**135**] p. 44 for the case $r > \frac{1}{2}$).

THEOREM 2.2. *Let Γ be a compact $n-1$ dimensional C^∞ manifold imbedded in $\mathbb{R}^n$. Let $\mathrm{d}\sigma$ be the measure induced on Γ by the Lebesgue measure $\mathrm{d}x$, and denote by $L^2(\Gamma)$ the class of L^2 functions on Γ with respect to the measure $\mathrm{d}\sigma$. For any given $r \geq \frac{1}{2}$, there exists a bounded linear map*

$$\tau : H^r(\mathbb{R}^n) \longrightarrow L^2(\Gamma) \tag{2.1.5}$$

such $\tau u = u|_\Gamma$ for any $u \in H^r(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

We now present the main result (we shall refer to [**5**], but a different proof can be found in [**94**] and [**17**]), which states the existence of the lower and the upper limits of our resolvent operator $R_0(z)$ on the positive real half-line.

THEOREM 2.3. *Let $s > \frac{1}{2}$ be a real number. Then*

(i) for any $\lambda^2 \in \mathbb{R}_+$ the following limits

$$\lim_{\varepsilon \downarrow 0} R_0(\lambda^2 \pm \imath\varepsilon) = R_0^{\pm}(\lambda^2) \tag{2.1.6}$$

exist in the space $\mathcal{B}\left(L^2_s(\mathbb{R}^n), H^2_{-s}(\mathbb{R}^n)\right)$ equipped with the uniform operator topology;

(ii) for any $f \in L^2_s(\mathbb{R}^n)$, and $\lambda^2 \in \mathbb{R}_+$, the function $u = R_0^{\pm}(\lambda^2) f$ satisfies the differential equation

$$\left(-\Delta - \lambda^2\right)\, u = f \ ; \tag{2.1.7}$$

(iii) the following formula

$$\mathrm{Im}\ \left(\langle\, R_0^{\pm}(\lambda^2) f, f\rangle\right) = \pm\tfrac{\pi}{2\lambda} \int_{|\xi|=\lambda} \left|\left(\tau \hat{f}\right)(\xi)\right|^2 \,\mathrm{d}\sigma_\lambda \ . \tag{2.1.8}$$

holds for any function $f \in L^2_s(\mathbb{R}^n)$, where τ is the trace operator (given by Theorem 2.2) on the sphere $|\xi| = \lambda$.

Proof. Let $s > \frac{1}{2}$, and $f, g \in L^2_s(\mathbb{R}^n)$. We shall show that the analytic function $F : \mathbb{C} \setminus \overline{\mathbb{R}}_+ \longrightarrow \mathbb{C}$ defined by $F(z) = \langle R_0(z)f, g\rangle$ admits unique continuous (upper and lower) extensions on $\mathbb{R}_+$. From Theorem 2.1 we have (for $K > 1$)

$$\|R_0(z)f\|_{H^2_{-s}} \lesssim \|f\|_{L^2_s} \ , \ \ \forall f \in L^2_s(\mathbb{R}^n) \ , \ \tfrac{1}{K} \leq |z| \leq K \ ,$$

hence by Cauchy-Schwarz inequality and the natural imbedding of $H^2_{-s}(\mathbb{R}^n)$ in $L^2_{-s}(\mathbb{R}^n)$ we get

$$|\langle R_0(z)f, g\rangle| \leq \|R_0(z)f\|_{L^2_{-s}} \|g\|_{L^2_s} \leq \|R_0(z)f\|_{H^2_{-s}} \|g\|_{L^2_s} \lesssim \|f\|_{L^2_s} \|g\|_{L^2_s} \ ,$$

for any couple f and g in $L^2_s(\mathbb{R}^n)$, in the annulus $\frac{1}{K} \leq |z| \leq K$. By a density argument we can restrict our attention to f and $g \in C_0^\infty(\mathbb{R}^n)$. By Parseval's Theorem we have for $z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$

$$\langle R_0(z)f, g\rangle = \int_{\mathbb{R}^n} \tfrac{\hat{f}(\xi)}{|\xi|^2 - z} \overline{\hat{g}}(\xi) \ \mathrm{d}\xi = \int_0^\infty \tfrac{t^{n-1}}{t^2 - z} \int_{\mathbb{S}^n} \hat{f}(t\omega)\overline{\hat{g}}(t\omega) \ \mathrm{d}\omega \ \mathrm{d}t \ ,$$

switching to polar coordinates $\xi = t\omega$, where $t = |\xi| > 0$ and $\omega = \frac{\xi}{|\xi|} \in \mathbb{S}^n$.

Residue calculus (see [**9**]) can now be used to consider the complex integral

$$I^\pm_\varepsilon = \int_{-\infty}^{\infty} F^\pm_\varepsilon(\eta) \ \mathrm{d}\eta \ ,$$

where

$$F^\pm_\varepsilon(\eta) = \tfrac{\eta^{n-1}}{\eta^2 - (\lambda^2 \pm \imath\varepsilon)} \int_{\mathbb{S}^n} \hat{f}(\eta\omega)\overline{\hat{g}}(\eta\omega) \ \mathrm{d}\omega \ .$$

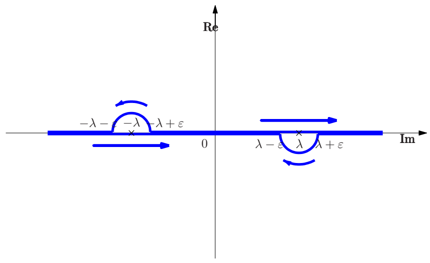

FIGURE 2.1 Curves and poles.

We can restrict our attention to the $+$ case ($z = \lambda^2 + \imath\varepsilon$), and decompose I^+_ε into five parts (see the Figure 1).

If $z_\pm(\lambda, \varepsilon)$ denote the roots of $\lambda^2 + \imath\varepsilon$ with positive and negative real part respectively, and if ε tend to zero, then we have

$$\lim_{\varepsilon \downarrow 0} I^\pm_\varepsilon(R) = \pi\imath \lim_{\varepsilon \downarrow 0} \sum_\pm (\pm)\mathrm{Res}\left(F^+_\varepsilon(\eta), z_\pm(\lambda, \varepsilon)\right) + \mathrm{PV}\int_{-\infty}^{\infty} F^\pm_0(t) \ \mathrm{d}t \ ,$$

Observing that the residues of the meromorphic function $F^\pm_\varepsilon(\eta)$ in the poles $z_\pm(\lambda, \varepsilon)$ are given by

$$\mathrm{Res}\left(F^+_\varepsilon(\eta), z_\pm(\lambda, \varepsilon)\right) = \tfrac{\lambda^{n-1}}{2z_\pm(\lambda,\varepsilon)} \int_{\mathbb{S}^n} \hat{f}(\lambda\omega)\overline{\hat{g}}(\lambda\omega) \ \mathrm{d}\omega \ .$$

we achieve the existence of the continuous (upper) boundary values, that is,

$$\lim_{\varepsilon \downarrow 0} \langle R_0(\lambda^2 + \imath\varepsilon)f, g\rangle = \tfrac{\pi\imath}{2\lambda} \int_{|\xi| = \lambda} \hat{f}(\xi)\overline{\hat{g}}(\xi) \ \mathrm{d}\sigma_\lambda + \mathrm{PV}\int_{\mathbb{R}^n} \tfrac{\hat{f}(\xi)}{|\xi|^2 - \lambda^2} \overline{\hat{g}}(\xi) \ \mathrm{d}\xi \ , \tag{2.1.9}$$

since $z_\pm(\lambda, \varepsilon) \to \pm\lambda$ as $\varepsilon \downarrow 0$. In other words, we can state that there exist the two weak limits

$$R_0(\lambda^2 \pm \imath\varepsilon)f \rightharpoonup R^\pm_0(\lambda^2)f \ , \tag{2.1.10}$$

for any $f \in L^2_s(\mathbb{R}^n)$. We observe here that $R_0(z)f$ is bounded near λ^2 if it is considered as a function with values in $f \in L^2_s(\mathbb{R}^n)$, hence from Banach-Alaoglu Theorem (see [**162**]) we can conclude that (2.1.10) holds for any $H^2_{-s}(\mathbb{R}^n)$. It turns out that $R^\pm_0(\lambda)f$ is in $\mathcal{B}\left(L^2_s(\mathbb{R}^n), H^2_{-s}(\mathbb{R}^n)\right)$ for $s > \frac{1}{2}$.

We prove now that the limits exists in a stronger sense, as well. We denote with $\widetilde{\mathbb{C}}_\pm$ the domains defined by $\widetilde{\mathbb{C}}_\pm = \mathbb{C}_\pm \cup \mathbb{R}_+$, and we keep on denoting with $R_0^\pm : \widetilde{\mathbb{C}}_\pm \longrightarrow \mathcal{B}\left(L^2_s(\mathbb{R}^n), H^2_{-s}(\mathbb{R}^n)\right)$ the map defined by

$$R_0^\pm(z) = \begin{cases} R_0(z) & \text{if} \quad \pm \operatorname{Im} z > 0 \\ R_0^\pm(\lambda) & \text{if} \quad z = \lambda \in \mathbb{R}_+ \end{cases}$$

Using the norm equivalence $\|u\|_{H^2_{-s}} \simeq \|(\mathbf{1} - \Delta)\, u\|_{L^2_{-s}} \simeq \|u\|_{L^2_{-s}} + \|\Delta u\|_{L^2_{-s}}$ and the relation $-\Delta R_0^\pm(\lambda) = \mathbf{1} + z R_0^\pm(\lambda)$, we are able to restrict our attention to the space $\mathcal{B}\left(L^2_s(\mathbb{R}^n), L^2_{-s}(\mathbb{R}^n)\right)$ without loss of generality.

Let $f \in L^2_s(\mathbb{R}^n)$ with $s > \frac{1}{2}$, and $z_0 \in \widetilde{\mathbb{C}}_\pm$. We first observe that (2.1.10) holds also in the strong topology of $\mathcal{B}\left(L^2_s(\mathbb{R}^n), L^2_{-s}(\mathbb{R}^n)\right)$, using RellichÐKondrachov Compactness Theorem (see Section 5.7 in [**57**]). Next we observe that if $\{z_j\} \subseteq \mathbb{C}_\pm$ and $\{f_j\} \subseteq L^2_s(\mathbb{R}^n)$ are sequences such that $z_j \to z_0$ and $f_j \rightharpoonup f \in L^2_{-s}(\mathbb{R}^n)$, then

$$R_0^\pm(z_j) f_j \to R_0^\pm(z_0) f \quad \text{in } L^2_{-s}(\mathbb{R}^n) \ . \tag{2.1.11}$$

Indeed, for any $g \in L^2_s(\mathbb{R}^n)$

$$\lim_{j\to\infty} \langle R_0^\pm(z_j) f_j, g\rangle = \lim_{j\to\infty} \langle f_j, R_0^\mp(\overline{z}_j) g\rangle = \langle f, R_0^\mp(\overline{z}_0) g\rangle = \langle R_0^\pm(z) f, g\rangle \ ,$$

hence $R_0^\pm(z_j) f_j \rightharpoonup R_0^\pm(z_0) f$ in $L^2_{-s}(\mathbb{R}^n)$. It follows that $R_0^\pm(z)$ is continuous on $\widetilde{\mathbb{C}}_\pm$ with respect to the uniform operator topology in $\mathcal{B}\left(L^2_s(\mathbb{R}^n), L^2_{-s}(\mathbb{R}^n)\right)$ by the following contradiction argument. Suppose thus that this is not true. Then there exist two sequences $\{z_j\} \subseteq \widetilde{\mathbb{C}}_\pm$ and $\{f_j\} \subseteq L^2_s(\mathbb{R}^n)$ such that $z_j \to z_0 \in \widetilde{\mathbb{C}}_\pm$, $\|f_j\|_{L^2_s} = 1$ and

$$\liminf_{j\to\infty} \left\| \left(R_0^\pm(z_j) - R_0^\pm(z_0)\right) f_j \right\|_{L^2_{-s}} > 0 \ .$$

Up to extracting a subsequence we can state that $f_j \rightharpoonup f$ in $L^2_s(\mathbb{R}^n)$, and now (2.1.11) gives

$$\lim_{j\to\infty} R_0^\pm(z_j) f_j = R_0^\pm(z_0) f = \lim_{z\to z_0} R_0^\pm(z) f \ ,$$

since $R_0^\pm(\cdot) f$ in continuous on $\widetilde{\mathbb{C}}_\pm$. That yields a contradiction, and complete the proof of (i).

We shall prove now (ii) and (iii). Let $u = R_0^\pm(\lambda^2)$, for $f \in L^2_s(\mathbb{R}^n)$, and $\lambda^2 > 0$. The equation (2.1.7) follows noticing that for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $z \notin \overline{\mathbb{R}}_+$ we have

$$\langle R_0(\lambda^2 \pm \imath\varepsilon) f, \left(-\Delta - (\lambda^2 \pm \imath\varepsilon)\right) \varphi\rangle = \langle f, \varphi\rangle \ .$$

As $\varepsilon \downarrow 0$ in the left hand side we have

$$\lim_{\varepsilon\downarrow 0} \langle R_0(\lambda^2 \pm \imath\varepsilon) f, \left(-\Delta - (\lambda^2 \pm \imath\varepsilon)\right) \varphi\rangle = \langle u, \left(-\Delta - \lambda^2\right) \varphi\rangle \ ,$$

thus $\langle \left(-\Delta - \lambda^2\right) u, \varphi\rangle = \langle f, \varphi\rangle$ by the symmetry of the operator $-\Delta - \lambda^2$. We conclude the proof observing that (2.1.8) follows from (2.1.9) evaluated for $f = g \in C_c^\infty(\mathbb{R}^n)$, and using the continuity properties of the trace operator τ. □

By Theorem 2.3 the following definition makes sense (see also [**127**] for an alternative definition).

DEFINITION 2.1. *Let $u \in H^2_{loc}(\mathbb{R}^n)$. We say that u is a* **λ-outgoing function** *(resp.* **λ-incoming function***) if for $\lambda > 0$*

$$u = \mathbb{R}_0^+\left(\lambda^2\right) f\ , \quad (\text{resp. } u = \mathbb{R}_0^-\left(k^2\right) f\ )\ . \tag{2.1.12}$$

for some $f \in L^2_s(\mathbb{R}^n)$ and $s > \frac{1}{2}$.

1.2. The Perturbed Case. The next step is to take into account the Schrödinger operators H_V, defined as perturbations of the free Laplace operator by an operator $V = V(x,D)$, that is $H_V = H_0 + V$. We shall try to achieve for $R(z) = (H_V - z)^{-1}$, the resolvent operator associated to H_V, the same results obtained in the free case. We shall first need to specify the class of potential we want to deal with.

DEFINITION 2.2. *An operator $V(x,D) = \sum_{|j|\le 2} a_j(x)D^j$ is said to belong to the* **class of short range** *if, for some $\varepsilon > 0$ the multiplicative mapping*

$$u \longmapsto \langle x\rangle^{1+\varepsilon} V(x,D)u$$

defines a compact operator from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

REMARK 2.1. *If V is a short range operator, then for some $\varepsilon > 0$ and any real s, the multiplicative operator $V(x,D) : H^2_s(\mathbb{R}^n) \longrightarrow L^2_{s+1+\varepsilon}(\mathbb{R}^n)$, defined by $V(x,D)(u) = V(x,D)u$ is compact.*

EXAMPLE 2.1. *If the coefficients of $V(x,D)$ satisfiy the following decay at infinity:*

$$a_j(x) = O\left(|x|^{-1-\varepsilon}\right)\ , \quad \text{as } |x| \to +\infty\ , \tag{2.1.13}$$

then $V(x,D)$ is a short range potential (see **[5]** *or* **[86]**, XIV, Scattering Theory]*)*

We shall need some more information about the behavior of functions from Sobolev spaces as traces on some sphere (see Theorem 3.2 in [**5**]) and about decay property for certain improper eigenfunctions of H_V, which first are assumed to be not in the domain of H_V (see Theorem 3.3 in [**5**]). The general results are the followings:

THEOREM 2.4. *Let $h \in H^r(\mathbb{R}^n)$ for some $r > \frac{1}{2}$. Suppose that $h(\xi) = 0$ on a sphere $|\xi| = \lambda$ in the trace sense, and let $\frac{1}{K} \le \lambda \le K$ for some positive constant K. For any multi-index $0 \le |\alpha| \le 2$, set*

$$v_\alpha(\xi) = \tfrac{\xi^\alpha h(\xi)}{|\xi|^2 - \lambda^2}. \tag{2.1.14}$$

Then $v_\alpha \in H^{r-1}(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$, and moreover, there exists a constant $C = C(r,K)$ such that

$$\|v_\alpha\|_{H^{r-1}} \le C\, \|h\|_{H^r}$$

is satisfied for any $h \in H^r$.

THEOREM 2.5. *Let $V = V(x,D)$ a perturbation operator on $\mathbb{R}^n$ such that $\langle x\rangle^{1+\delta}V(x,D)$ belongs to the class of short range operator for some $\delta > 0$. Let $u \in H^2_{loc}(\mathbb{R}^n)$ be a solution of the differential equation*

$$-\Delta u + Vu = \lambda u\ ,$$

λ is a positive number. Suppose that $u \in H^2_{s_0}(\mathbb{R}^n)$ for some $s_0 > -\frac{1}{2} - \delta$. Consider u as a tempered distribution acting on $S(\mathbb{R}^n)$, and let $\hat{u}$ be a distributional Fourier transform of u ($\hat{u} \in S'(\mathbb{R}^n)$). If $\hat{u} \in L^1(\mathbb{R}^n)$, then $u_\alpha \in H^2_s(\mathbb{R}^n)$, for any real s.

The last theorem will give the following lemma, that will play a key role in the generalization of the limiting absorption principle to H_V.

LEMMA 2.1. *Let $u \in H^2_{loc}(\mathbb{R}^n)$, be a λ-outgoing function (resp. λ-incoming) satisfying a differential equation of the form*

$$-\Delta u + Vu = \lambda^2 u \ ,$$

where V is a short range potential. Then $u \in \bigcap_{s\in\mathbb{R}} H^2_s(\mathbb{R}^n)$.

Proof. We can suppose u to be outgoing (the case of u incoming can be handled with the same procedure). We have that $u = R^+_0(\lambda^2)f$ for some $f \in L^2_{s_0}(\mathbb{R}^n)$, where $s_0 > \frac{1}{2}$, that is $f = \left(-\Delta - \lambda^2\right)u$. It turns out that $f = -Vu$. Applying (2.1.8), using the fact that V is real, we obtain

$$\int_{|\xi|=\lambda} \left|\left(\tau\hat{f}\right)(\xi)\right|^2 \mathrm{d}\sigma = \tfrac{2\lambda}{\pi}\,\mathrm{Im}\,\left(\langle\mathbb{R}^+_0(\lambda^2)f, f\rangle\right) = -\tfrac{2\lambda}{\pi}\,\mathrm{Im}\,\left(\langle u, Vu\rangle\right) = 0 \ ,$$

that implies that $\hat{f}(\xi) = 0$ on the sphere $|\xi| = \lambda$ (in the sense of trace). From Theorem 2.4 with $h = \hat{f}$ we get that

$$\hat{f}(\xi)\left(|\xi|^2 - \lambda^2\right)^{-1} \in L^1_{loc}(\mathbb{R}^n) \ .$$

Next we show that $\hat{u}(\xi) = \hat{f}(\xi)\left(|\xi|^2 - \lambda^2\right)^{-1}$ in the sense of tempered distribution. Let $g \in \mathcal{S}(\mathbb{R}^n)$; we have

$$\langle u, g\rangle = \lim_{\varepsilon\downarrow 0}\langle R_0(\lambda^2 + \imath\varepsilon)f, g\rangle = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} \tfrac{\hat{f}(\xi)}{|\xi|^2-\lambda^2}\overline{\hat{g}}(\xi)\tfrac{|\xi|^2-\lambda^2}{|\xi|^2-\lambda^2-\imath\varepsilon}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \tfrac{\hat{f}(\xi)}{|\xi|^2-\lambda^2}\overline{\hat{g}}(\xi)\,\mathrm{d}\xi \ ,$$

from Parseval's Theorem and Lebesgue Convergence Theorem. Therefore our claim on $\hat{u}$ is true, and $\hat{u} \in L^1_{loc}(\mathbb{R}^n)$. Now from Theorem 2.5 it follows that $u \in H^2_s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. □

Now we shall consider the Schrödinger operator $H_V = -\Delta + V$, where $V = V(x, D)$ is in the class of short range operators. In this case H_V turns out to be a self-adjoint operator on the domain $\mathcal{D}(H_V) = H^2(\mathbb{R}^n)$ (see [**113**]); on the other hand, as far as its (real) spectrum is concerned, we have that $\sigma(H_V) = \sigma_c(H_V) \cup \sigma_p(H_V)$ (continuous and point spectrum), where $\sigma_c(H_V) = [0, +\infty)$, and $\sigma_p(H_V) = \{h_j\}$, a discrete set of eigenvalues with finite multiplicity, having zero as its only limit point.

We can now extend the limiting absorption principle to these operators, wich are self-adjoint realizations of higher order elliptic operator (see [**5**], section 1 and Theorem 4.2). For simplicity we use the notation $V = V(x, D)$ in the following.

THEOREM 2.6. *Let $s > \frac{1}{2}$, and let $\sigma^+_p(H_V) = \sigma_p(H_V) \cap \mathbb{R}_+$. Consider $\lambda^2 \in \mathbb{R}_+ \setminus \sigma^+_p(H_V)$. Then*

$$\exists \lim_{\varepsilon\downarrow 0} R_V\left(\lambda^2 \pm \imath\varepsilon\right) = R^\pm_V\left(\lambda^2\right) \ . \tag{2.1.15}$$

Moreover, for any $f \in L^2_s(\mathbb{R}^n)$ the following identity holds:

$$R^\pm_V\left(\lambda^2\right)f = \mathbb{R}^\pm_0\left(\lambda^2\right)f - \mathbb{R}^\pm_0\left(\lambda^2\right)V\mathbb{R}^\pm_V\left(\lambda^2\right)f \ . \tag{2.1.16}$$

In particular, $u^+ = R_V^+ \left(\lambda^2\right) f$ *is a* λ*-outgoing solution, and* $u^- = R_V^- \left(\lambda^2\right) f$ *is a* λ*-incoming solution of the differential equation*

$$\left(-\Delta + V - \lambda^2\right) u = f \ , \tag{2.1.17}$$

in $\mathbb{R}^n$.

Proof. We can restrict our attention to the case $R_V^+ \left(\lambda^2\right)$, and $s \in (\frac{1}{2}, \frac{1}{2} + \varepsilon]$, for ε small enough, in such a way that

$$V : H^2_{-\frac{1}{2}-\varepsilon} \left(\mathbb{R}^n\right) \longrightarrow H^2_{\frac{1}{2}+\varepsilon} \left(\mathbb{R}^n\right)$$

is a compact operator (as stressed out in Remark 2.1). From such compactness, and Theorem 2.3 it follows that the composition $R_0^+(z)V$ is compact for any $z \in \widetilde{\mathbb{C}}_+$, and

$$R_0^+(\cdot)V : \widetilde{\mathbb{C}}_+ \longrightarrow \mathcal{B}\left(H^2_{-s}\left(\mathbb{R}^n\right), H^2_{-s}\left(\mathbb{R}^n\right)\right)$$

is continuous. We claim that the existence of the inverse operator $\left(\mathbf{1} + R_0^+(z)V\right)^{-1}$ is equivalent to $z \in \widetilde{\mathbb{C}}_+ \setminus \sigma_p^+\left(H_V\right)$. Indeed two cases may occur.

$\mathrm{Im}\,(z) > 0$ The resolvent identity $R_V(z) + R_0(z)VR_V(z) = R_0(z)$ holds true, that is

$$\left(Id + R_0(z)V\right) u = R_0(z) f \ ,$$

for any $f \in L^2\left(\mathbb{R}^n\right)$ and $u = R_V(z)f \in H^2\left(\mathbb{R}^n\right)$. This implies $H^2\left(\mathbb{R}^n\right) \subseteq Im\left(Id + R_0(z)V\right)$, hence

$$\overline{Im\left(\mathbf{1} + R_0(z)V\right)} = H^2_{-s}\left(\mathbb{R}^n\right) \ .$$

The Fredholm-Riesz theory gives the existence of the inverse operator $\left(\mathbf{1} + R_0^+(z)V\right)^{-1}$ in the operator space $\mathcal{B}\left(H^2_{-s}\left(\mathbb{R}^n\right), H^2_{-s}\left(\mathbb{R}^n\right)\right)$.

$z = \lambda^2$ Applying again Fredholm-Riesz theory we have that $\mathbf{1} + R_0^+\left(\lambda^2\right) V$ is invertible if and only if -1 is not an eigenvalue for $\mathbb{R}_0\left(\lambda^2\right) V$. Suppose that

$$R_0\left(\lambda^2\right) Vu + u = 0, \quad u \in H^2_{-s}\left(\mathbb{R}^n\right) \ .$$

Observe that $u = -R_0^+\left(\lambda^2\right) Vu$ implies that u is a λ-outgoing solution for the differential equation $\left(-\Delta + V\right) u = \lambda^2 u$. From Lemma 2.1 we have that $u \in \mathcal{D}\left(H_V\right)$, hence λ^2 is an eigenvalue of H_V. Viceversa whenever $\lambda^2 > 0$ is an eigenvalue of H_V with eigenfunction $u \in \mathcal{D}\left(H_V\right)$, then passing through the limit as $z \to \lambda^2$ in

$$u + R_0(z)Vu = \left(\lambda^2 - z\right) R_0(z) u$$

(holding for $z \in \mathbb{C}_+$) we have $u + R_0^+\left(\lambda^2\right) Vu = 0$, that is -1 is an eigenvalue for $R_0^+\left(\lambda^2\right) V$.

We conclude observing that $R_V(z) = \left(\mathbf{1} + R_0^+(z)V\right)^{-1} R_0(z)$ for $\mathrm{Im}\,(z) > 0$. Using continuity of the operators $\left(\mathbf{1} + R_0^+(z)V\right)^{-1}$ and $R_0(z)$ it follows that

$$\lim_{\varepsilon \downarrow 0} R_V\left(\lambda^2 \pm \imath\varepsilon\right) = \left(\mathbf{1} + R_0^+\left(\lambda^2\right) V\right)^{-1} R_0\left(\lambda^2\right) \ .$$

That will also imply identity (2.1.16) in $\widetilde{\mathbb{C}}_+$. □

2. The Resonances

We consider a closed linear operator A in a given Banach space B. We assume that $\sigma(A) \neq \mathbb{C}$. We denote by D a domain in $\mathbb{C}$ such that $D \cap \sigma(A) \subset \sigma_p(A)$. Thus the resolvent $R_A(z) = (A - z)^{-1}$ is a well defined meromorphic operator function in D with values in $\mathcal{L}(B)$. Its poles in D (the isolated eigenvalues of A) are of finite rank. Next we introduce a notion of a generalized resolvent. To this end we assume that in addition to B there are given two Banach spaces B_0 and B_1 with $B_0 \subset B \subset B_1$ such that the injections:

$$J_0 : B_0 \to B \qquad \text{and} \qquad J : B \to B_1, \tag{2.2.18}$$

are continuous. For $z \in D \setminus \sigma_p(A)$, we set

$$\widetilde{R}_A(z) = J R_A(z) J_0. \tag{2.2.19}$$

Clearly $\widetilde{R}_A(z)$ is a meromorphic operator function in D with values in $\mathcal{L}(B_0, B_1)$. We refer to $\widetilde{R}_A(z)$ as the **generalized resolvent** of A. We shall assume that the following basic conditions holds:

HYPOTESIS 2.1. *The operator function $\widetilde{R}_A(z)$ admits a meromorphic continuation with finite rank poles from D to a domain $D_+ \supset D$, where*

$$D_+ \cap \sigma(A) \supset \sigma_p(A)$$

, (The last restriction is of course the statement that $\widetilde{R}_A(z)$ does not admit such a meromorphic continuation to D_+.)

DEFINITION 2.3. *A resonance of A is a pole z_0 of $\widetilde{R}_A(z)$, $z_0 \in D_+ \setminus D$, which verifies one of the following conditions. Either*

$$z_0 \notin \sigma_p(A) \quad \textit{or}, \tag{2.2.20}$$

$$z_0 \in \sigma_p(A), \tag{2.2.21}$$

in this case the relation (2.2.19) *does not hold (identically) in any deleted neighborhood of z_0*

Here we followed [**6**]. For other details and an analysis we refer also to [**123, 124, 125**]. In this section we stated a abstract definition of resonaces. Namely, we find a mathematical quantity to analyze, that is the (generalized) resolvent $\widetilde{R}_A(z)$). The problem of resonances is a phenomenon involved in many physical process (an example could be the analysis of a vibrating strings or of the states of particles in the atom, just to name a few). For a more extended exposition we refer to [**158**, Chapter XII].

3. The Resolvent Estimates

First we shall restrict to estimates involving the resolvent itself.

Theorem 2.3 states the existence of the two continuous operators

$$R_0^{\pm} : \mathbb{C} \longrightarrow \mathcal{B}\left(L^2_s(\mathbb{R}^n), H^2_{-s}(\mathbb{R}^n)\right) ,$$

for $s > \frac{1}{2}$. The following results (see [**166**] and [**167**]) describe a-priori estimates which are satisfied by such operators $R_0^{\pm}$. There are two key points in those results. The first one is the following **Interior Estimate**. For any fixed $0 < t < r$ there exists a constant $C = C(t, r)$ such that

$$\|\nabla u\|_{L^2(B_t)} \le C \|f\|_{L^2(B_r)} + \|u\|_{L^2(B_r)} , \ \forall\, 0 < t < r,\ \lambda \ge 0 , \tag{2.3.22}$$

where B_t is the ball of radius t and u solves the equation $\left(-\Delta-\lambda^2\right)u = f$. The latter is the introduction of the so called **radiation condition** for the equation $\left(-\Delta-\lambda^2\right)u=f$, defined by

$$\int_{|x|\geq 1} \langle x\rangle^{2s} |\mathcal{D}u|^2 \, \mathrm{d}x < \infty \,, \tag{2.3.23}$$

where $|\mathcal{D}u|^2 = \sum_{j=1}^n |\mathcal{D}_j u|^2$, and $\mathcal{D}_j u = \partial_j u + \frac{n-1}{2|x|}\frac{x_j}{|x|}u - \imath\kappa\frac{x_j}{|x|}u$.

THEOREM 2.7. *Let $s>\frac{1}{2}$. Then for any $\lambda_0\in\mathbb{R}^+$, there exists a constant $C=C(\lambda_0)>0$ such that*

$$\left\|R_0^{\pm}\left(\lambda^2\right)f\right\|_{-s} \leq \frac{C}{\lambda}\|f\|_s \,, \tag{2.3.24}$$

for any $\lambda>\lambda_0$ and any $f\in L^2_s(\mathbb{R}^n)$.

Proof. Suppose that (2.3.24) is false. Then we can find sequences $\varepsilon_k \downarrow 0$ and $\{u_k\}$ such that

$$\|u_k\|_{-s}=1\,, \qquad \frac{1}{|\lambda^2\pm\varepsilon_k|}\|f\|_s \leq \frac{1}{k}\,, \qquad \left[-\Delta-\left(\lambda^2\pm\varepsilon_k\right)\right]u_k = f_k\,, \tag{2.3.25}$$

for any $k\geq 0$. From the interior estimate (2.3.22) we obtain the boundedness of the sequence $\{u_k\}$ in $H^1(B_t)$ for any $t>0$. The compact imbedding properties of Sobolev spaces give the existence of the limit

$$\lim_{k\to\infty} u_k = u$$

in $H^1_{loc}(\mathbb{R}^n)$. On the other side from the second equation in (2.3.25) we get

$$\lim_{k\to\infty} f_k = 0\,,$$

which, together with the previous limit implies that u is a weak solution of the equation $\left(-\Delta-\lambda^2\right)u = 0$. Hence we have that u is in $H^2_{loc}(\mathbb{R}^n)$ and it is a strong solution as well (see [**93**]). In this case we have the following estimate:

$$\|u_k\|_{L^2_s(B_\rho)} = O\left(\rho^{s-\frac{1}{2}}\right)$$

uniformly with respect to k. Hence $u_k\to u$ in $L^2_s(\mathbb{R}^n)$ and $\|u_k\|_s=1$. On the other hand, since the sequence $\|Du_k\|_{L^2_s(B_1)}$ is bounded and $u_k\to u$ in $H^1_{loc}(\mathbb{R}^n)$, we have $u\in L^2_s(B_1)$. Therefore u is a solution of the equation $\left(-\Delta-\lambda^2\right)u=0$ in $H^1_{loc}(\mathbb{R}^n)$. Thus u solves $\left(-\Delta-\lambda^2\right)u=0$, whence $u=0$ (see [**94**]) which is a contradiction since $\|u\|_s=1$. □

In order to achieve the point 0 we have to relax the condition on the parameter s to be $s>1$. Namely, that can be done using some Hölder continuity property related to the resolvent operators $R_0^{\pm}$, thus we shall need some definitions (see [**17**], and the notations thereby).

DEFINITION 2.4. *Let $U\subseteq\mathbb{R}$ be open, $\alpha\in(0,1)$, and $X,Y\subseteq H$ two Banach spaces, both dense and continuously embedded in H. If A is a self-adjoint operator on H, and $E^A(t)$ is the spectral family associated to A (by the Spectral Theorem 1.12), we say that A is* **of type (X,Y,α,U)** *if the operator-valued function $E^A(t)\in\mathcal{B}(X,Y^*)$ is weakly differentiable in U with Hölder continuous derivative, that is there exists a function $E^A_\lambda(\lambda)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g,E^A_\lambda f\right) = \left(g,E^A_\lambda(\lambda)f\right)\,,\ \forall f\in X, g\in Y, \lambda\in U\,, \tag{2.3.26}$$

for every compact interval $K \subset U$ *there exists a constant* $C_K > 0$ *such that*

$$\left\|E^A_\lambda(\lambda_2) - E^A_\lambda(\lambda_1)\right\|_{\mathcal{B}(X,Y^*)} \le C_K \left|\lambda_1 - \lambda_2\right|^\alpha \ , \ \forall \lambda_1, \lambda_2 \in K \ . \tag{2.3.27}$$

If we consider the case of a constant real coefficient operator of order m of the form

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

we have that P is a self-adjoint operator in $L^2(\mathbb{R}^n)$, unitarily equivalent to multiplication by $P(\xi)$, $\xi \in \mathbb{R}^n$ by Fourier transform. Let $E^P(\lambda)$ be the spectral family associated with P. It is well-known (see [**170**]) that P is absolutely continuous and $E^P(\lambda)$ is represented in the phase space by the multiplication operator $\chi_\lambda(\xi)$, the characteristic function of the set $\{P(\xi) \le \lambda\}$.

Moreover, we have that if P is of principal type the following estimate holds

$$|\nabla P(\xi)| \gtrsim (1 + |\xi|)^{m-1} \ ,$$

for ξ large enough. We shall need two result about such an operator P. The first result concerns the Hölder continuity of P in suitable interpolation spaces. Namely, let $X_0 \subseteq X_1$, $Y_0 \subseteq Y_1$ be Banach spaces, where the embeddings are dense and continuous, and for $\theta \in [0,1]$ let $X_\theta = [X_0, X_1]$, $Y_\theta = [Y_0, Y_1]$.

LEMMA 2.2. *Let* $U \subseteq \mathbb{R}$ *be open and let* $T(\lambda)$, $\lambda \in U$, *be an operator-valued continuous function,*

$$T(\lambda) \in \mathcal{B}(X_0, Y_0) \cap \mathcal{B}(X_1, Y_1) \ , \ \lambda \in U \ ,$$

so that, for some $\alpha > 0$ *and every compact* $K \subset U$, *there exists a constant* $M_K > 0$ *such that*

$$\|T(\lambda_2) - T(\lambda_1)\|_{\mathcal{B}(X_0,Y_0)} \le M_K \left|\lambda_2 - \lambda_1\right|^\alpha \ , \ \lambda_1, \lambda_2 \in K$$
$$\|T(\lambda)\|_{\mathcal{B}(X_1,Y_1)} \le M_K \ .$$

Then $T(\lambda)$ *is uniformly locally Hölder continuous in* $\mathcal{B}(X_\theta, Y_\theta)$ *for* $\theta \in (0,1)$.

The second result concerns the Hölder continuity of P in the sense specified in the Definition 2.4.

THEOREM 2.8. *Let* $P(D)$ *be a self-adjoint constant coefficient differential operator of principal type. Then for every* $s \ge \frac{1}{2}$ *there is a constant* $\alpha \in (0,1)$ *such that* $P(D)$ *is of type* $\left(L^2_s, L^2_s, \alpha, \mathbb{R} \setminus \Lambda(P)\right)$. *Furthermore, if* $R_\delta = \{\lambda \in \mathbb{R} | \operatorname{dist}(\lambda, \Lambda(P)) > \delta\}$, *where* $\delta > 0$, *then with some constant* $M_\delta > 0$,

$$\left\|E^P_\lambda(\lambda)\right\|_{\mathcal{B}(X,Y^*)} \le M_\delta (1 + |\lambda|)^{-\frac{m-1}{m}} \ ,$$

$$\sup_{\lambda_1 \ne \lambda_2 \in R_\delta} \frac{\left\|E^P_\lambda(\lambda_2) - E^P_\lambda(\lambda_1)\right\|_{\mathcal{B}(X,Y^*)}}{\left|\lambda_2 - \lambda_1^\alpha\right|} \le M_\delta \left[(1 + |\lambda_1|)^{-\frac{m-1}{m}} + (1 + |\lambda_2|)^{-\frac{m-1}{m}}\right] \ .$$

The proof of the previous theorem is based on the explicit representation of the derivative E^P_λ, given by

$$\left(g, E^P_\lambda(\lambda) f\right) = \int_{P(\xi)=\lambda} \hat{g}(\xi) \overline{\hat{f}(\xi)} \frac{\mathrm{d}\sigma}{|\nabla P(\xi)|} \ , \tag{2.3.28}$$

where $\mathrm{d}\sigma$ is the Lebesgue surface measure on $P(\xi) = \lambda$. That is also the reason why the condition $s \ge \frac{1}{2}$, required by the Trace Theorem 2.2, is needed.

Specializing the last case to Laplacian a stronger result can be obtained.

THEOREM 2.9. *Let $H_0 = -\Delta$ be in $L^2(\mathbb{R}^n)$, for $n \geq 2$, let $E^{H_0}(\lambda)$ be the associated spectral family, and $E^{H_0}_\lambda(\lambda)$ its derivate with respect to λ. Let $s, s' > \frac{1}{2}$ ($s, s' > 1$ for $n = 2$). Then $E^{H_0}_\lambda(\lambda)$ is uniformly continuous in the operator norm of $\mathcal{B}\left(L^2_s(\mathbb{R}^n), L^2_{-s'}(\mathbb{R}^n)\right)$ for $\lambda \in [0, \infty)$ and*

$$\left\| E^{H_0}_\lambda\left(\lambda^2\right)\right\|_{\mathcal{B}\left(L^2_s(\mathbb{R}^n), L^2_{-s'}(\mathbb{R}^n)\right)} = O(\lambda)\ , \quad \text{as } \lambda \to \infty\ . \tag{2.3.29}$$

In the case $n \geq 3$ we have also $E^{H_0}_\lambda(0) = 0$, so that $E^{H_0}_\lambda$ is uniformly Hölder continuous on $\mathbb{R}$ (being null for $\lambda < 0$).

Proof. We observe that $E(\mu) = 0$ for $\mu < 0$. The representation (2.3.28) gives on the positive half-line ($\lambda > 0$) the following formula

$$\left(g, E^{H_0}_\lambda\left(\lambda^2\right) f\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g, E^{H_0}\left(\lambda^2\right) f\right) = \frac{1}{2\lambda}\int_{|\xi|=\lambda} \hat{g}(\xi)\overline{\hat{f}(\xi)}\,\mathrm{d}\sigma\ , \tag{2.3.30}$$

for any $f, g \in L^2(\mathbb{R}^n)$ such that $\hat{f}, \hat{g} \in C^\infty_0(\mathbb{R}^n)$. All the statements made here, including (2.3.29), follow from Theorem 2.8 for λ large. Therefore, we only need to discuss the neighborhood of 0. Let $\hat{\varphi} \in H^s(\mathbb{R}^n)$, for $s > \frac{1}{2}$. If τ denotes the trace operator defined in Theorem 2.2, then $\tau\hat{\varphi}(r)$ is the trace of $\hat{\varphi}$ on the sphere of radius $r \in (0, +\infty)$. Namely, we have

$$\tau\hat{\varphi}(r) = r^{n-1}\hat{\varphi}(r\omega) \in L^2\left(\mathbb{S}^{n-1}\right)\ .$$

Assume first that $n \geq 3$. From the Trace Theorem we get

$$\|\tau\hat{\varphi}(r)\|_{L^2(\mathbb{S}^{n-1})} \lesssim \|\hat{\varphi}\|_{H^s(\mathbb{R}^n)}$$

for any $r \in (0, +\infty)$, while the standard Sobolev Imbedding yelds Theorem

$$\|\tau\hat{\varphi}(r)\|_{L^\infty(\mathbb{S}^{n-1})} \lesssim \|\hat{\varphi}\|_{H^s(\mathbb{R}^n)}\ .$$

Interpolating the last two inequalities we have, for arbitrarily small $\theta > 0$,

$$\|\tau\hat{\varphi}(r)\| \leq Cr^{s-\frac{1}{2}-\theta}\,\|\hat{\varphi}\|_{H^s(\mathbb{R}^n)}\ , \tag{2.3.31}$$

for any $\hat{\varphi} \in H^s(\mathbb{R}^n)$, with $s \in \left(\frac{1}{2}, \frac{n}{2}\right]$ and $C = C(n, s, \theta)$. Returning now to (2.3.30) with $f \in L^2_s(\mathbb{R}^n)$, $g \in L^2_{s'}(\mathbb{R}^n)$ we get from (2.3.31)

$$\left|\left(g, E^{H_0}_\lambda\left(\lambda^2\right) f\right)\right| \lesssim \lambda^{-1}\left\|\tau\hat{f}(\lambda)\right\|_{L^2(\mathbb{S}^{n-1})}\|\tau\hat{g}(\lambda)\|_{L^2(\mathbb{S}^{n-1})} \lesssim \|f\|_{L^2_s(\mathbb{R}^n)}\|g\|_{L^2_{s'}(\mathbb{R}^n)}\ ,$$

since $s + s' > 1$. Observe that (2.3.31) is not always optimal. Thus, it is well known (see Proposition 1.1 in [**168**]) that for $n \geq 3$ we have

$$\|\tau\hat{\varphi}(r)\| \leq Cr^{\frac{1}{2}}\|\hat{\varphi}\|_{H^1(\mathbb{R}^n)}\ ,\ \forall\hat{\varphi} \in H^1(\mathbb{R}^n)\ .$$

Therefore, for large s, s', the right-hand side of (2.3.30) is Hölder continuous up to 0, so that Hölder continuity is proved in view of the interpolation Lemma 2.2.

As for the last part, still with $n \geq 3$, take $\hat{f}, \hat{g} \in C^\infty_0(\mathbb{R}^n)$. Then, from (2.3.30) we get

$$\left(g, E^{H_0}_\lambda(0) f\right) = \frac{1}{2}\lim_{\lambda\to 0}\lambda^{n-2}\int_{\mathbb{S}^n}\hat{g}(\lambda\omega)\overline{\hat{f}(\lambda\omega)}\,\mathrm{d}\omega\ ,$$

By the density of $C^\infty_c(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, we get $E^{H_0}_\lambda(0)$.

Let now $n = 2$ and $s > 1$, $s' > 1$. By the Sobolev embedding both $\hat{f}, \hat{g}$ are then continuous function, and from (2.3.30) we have

$$\left(g, E_\lambda^{H_0}\left(\lambda^2\right) f\right) \leq \tfrac{1}{2}\lambda \sup_{\xi\in\mathbb{R}^2} \left|\hat{f}(\xi)\right| \sup_{\xi\in\mathbb{R}^2} |\hat{g}(\xi)|\, 2\pi\lambda \lesssim \|f\|_{L^2_s(\mathbb{R}^n)} \|g\|_{L^2_{s'}(\mathbb{R}^n)} .$$

The Hölder continuity is now proved as before. Note, however, that since $\left(g, E_\lambda^{H_0}(0)f\right) = \pi\hat{g}(0)\overline{\hat{f}(0)}$, it follows that $E_\lambda^{H_0}(0) \neq 0$. □

A corollary of the previous result is the following extension of Theorem 2.7.

COROLLARY 2.1. *Let $s > 1$. Then for all $\lambda^2 \in \mathbb{R}^+$ we have*

$$\left\|R_0^{\pm}\left(\lambda^2\right) f\right\|_{L^2_{-s}} \lesssim \tfrac{1}{\sqrt{1+\lambda^2}} \|f\|_{L^2_s}$$

for any $f \in L^2_s\left(\mathbb{R}^n\right)$.

4. Resolvent estimate for $\nabla R_0^{\pm}\left(\lambda^2\right)$

In this section we shall recall a basic estimate involving the derivative of order one of the resolvent.

More precisely we shall give a selfcontained proof of the following theorem (see also Theorem 2.1)

THEOREM 2.10. *Let n be fixed and $\varepsilon > 0$. Then there exists a constant C such that the following estimate*

$$\|\nabla u\|_{L^2_{-s}} \leq C \left\|(-\Delta - z)\, u\right\|_{L^2_s} , \tag{2.4.32}$$

holds for any $z \in \mathbb{C}$, and any $u \in H_{-s}\left(\mathbb{R}^n\right)$, provided $s > 1/2$.

Proof.

We can consider the case of $n = 1$, since a similar argument works for $n > 1$. Let $u \in S(\mathbb{R})$, first we have

$$\|u\|_{L^1} \leq C \|u\|_{L^2_s} . \tag{2.4.33}$$

In fact from the Cauchy-Schwartz inequality we obtain

$$\|u\|_{L^1} \leq \left(\langle x\rangle^{-s}, \langle x\rangle^{s} u\right), \tag{2.4.34}$$

and from the bound

$$\|\langle x\rangle^{-s}\|_{L^2} < C,$$

we arrive at (2.4.33).

Dualizing the estimate (2.4.33), we have

$$\|u\|_{L^2_{-s}} \leq C\|u\|_{L^\infty}. \tag{2.4.35}$$

Now we need the following two lemmas.

LEMMA 2.1. *There exists a constant $C \geq 0$ so that for all $\lambda \in \mathbb{C}$ and all $v \in S(\mathbb{R})$ we have*

$$\|v(x)\|_{L^\infty} \leq C \left\| \left(\frac{d}{dx} - \lambda\right) v(x) \right\|_{L^1}. \tag{2.4.36}$$

Proof. Suppose that $Re\lambda \leq 0$. Let $w(x) = (d/dx - \lambda)v(x) \in S(\mathbb{R})$. Then

$$v(x) = \int_{-\infty}^{x} e^{\lambda(x-y)} w(y) dy. \tag{2.4.37}$$

Thus

$$\|v(x)\|_{L^\infty} \leq C \|w(x)\|_{L^1}. \tag{2.4.38}$$

The estimate (2.4.36) follows for $Re\lambda \leq 0$. A similar argument works for $Re\lambda \geq 0$. In fact, we have $w(x) = (d/dx - \lambda)v(x)$ so

$$v(x) = \int_{x}^{\infty} e^{\lambda(x-y)} w(y) dy. \tag{2.4.39}$$

and we obtain, as in the previous case,

$$\|v(x)\|_{L^\infty} \leq C \|w(x)\|_{L^1}. \tag{2.4.40}$$

□

We are able to prove the following statement:

LEMMA 2.2. *For $n \geq 1$ there exists a constant $C \geq 0$ so that for all $\lambda \in \mathbb{C}$ and all $v \in S(\mathbb{R}^n)$ we have*

$$\|\partial_1 v(x_1, x')\|_{L^\infty_{x_1} L^2_{x'}} \leq C \|(-\Delta - \lambda) v(x_1, x')\|_{L^1_{x_1} L^2_{x'}}, \tag{2.4.41}$$

where $x = (x_1, x')$ and $x' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$.

Proof. Consider first the case $n = 1$. Let be $\lambda = -\mu^2$. Then by Lemma 2.1,

$$\begin{aligned} \left\| \frac{d}{dx} v(x) \right\|_{L^\infty} &\leq \frac{1}{2} \left\| \left(\frac{d}{dx} - \mu\right) v(x) \right\|_{L^\infty} + \frac{1}{2} \left\| \left(\frac{d}{dx} + \mu\right) v(x) \right\|_{L^\infty} \leq \\ &\leq C \left\| \left(-\frac{d^2}{dx^2} - \lambda\right) v(x) \right\|_{L^1}. \end{aligned} \tag{2.4.42}$$

Consider now the case $n > 1$. Given any $v \in S(\mathbb{R}^n)$ we denote by

$$\widetilde{v}(x_1, k'),\ k' = (k_2, ..., k_n)$$

its partial Fourier transform with respect to x', i.e.

$$\widetilde{v}(x_1, k') = (2\pi)^{-(n-1)} \int e^{-ik'x'} v(x_1, x') dx'. \tag{2.4.43}$$

Using the one-dimensional result (2.4.42), for each fixed k', we obtain

$$|\partial_1 \widetilde{v}(x_1, k')|^2 \leq C \int \left|(-\partial_1^2 + |k'|^2)\widetilde{v}\right|^2 dx_1. \tag{2.4.44}$$

Integrating with respect k' and using the Plancherel identity, we derive

$$\|\partial_1 v(x_1, x')\|_{L^\infty_{x_1} L^2_{x'}} \leq C \|(-\Delta - \lambda) v(x_1, x')\|_{L^1_{x_1} L^2_{x'}}. \tag{2.4.45}$$

This completes the proof. □

The L^2-weighted bounds (2.4.33) and (2.4.35) give Lemma 2.2 and the fact that the norms used in the proof are invariant under the action of the group of rotations $SO(n)$ give the desired estimate (2.4.32) .

□

REMARK 2.2. *The limiting absorption principle from Theorem 2.6 introduced in the Section 1, Chapter 2 allow us to obtain the following theorem:*

THEOREM 2.11. *We have, for all* $\lambda^2 \in \mathbb{R}^+$

$$\left\|\nabla R_0^{\pm}\left(\lambda^2\right) f\right\|_{L^2_{-s}} \lesssim C \left\|f\right\|_{L^2_s},$$

with $C > 0$ *and* $s > 1/2$.

REMARK 2.3. *In this Chapter we used, in the definitions and proofs, spaces with smooth weights, namely* $\langle x\rangle$ *raised to some power* s. *This is done for the aim of simplicity and because we want to be close to the traditional formalism of the scattering theory (*[**4, 5**]*). However, in the follows we introduce some perturbations belonging to the* **short range class** *(see Definition 2.2). Such kind of perturbations may have some "slight" singularities (see* [**86**, Chapter XIV, Scattering Theory]*), as in the case when we consider some mathematical descriptions of physical phenomena (an example could be* **the wave equation perturbed by an "electric" potential or by a "magnetic" potential***). So it is natural replaces the spaces with smooth weights by others with singularities in the weights suggested by the nature of the perturbation operator. Roughly speaking, we will be able to show that the estimates obtained in this chapter remain valid using Sobolev spaces with singular weights.*
This extension will be presented in the Chapter 3.

CHAPTER 3

Applications to a Class of Dispersive Equations

In this chapter we prove some estimates for the resolvent of the operator $-\Delta$ perturbed by the differential operator

$$V(x, D) = ia(x) \cdot \nabla + V(x) \quad \text{in } \mathbb{R}^3 .$$

This differential operator is of short range type and a compact perturbation of the Laplacian on $\mathbb{R}^3$. Also we find estimates in the space-time norm for the solution of the wave equation with such perturbation. Here we will use the explicit representation of the resolvent $R_0^{\pm}$ in the case when the dimension of the space is $n = 3$ to obtain new resolvent estimates. As said in the introductory part we follow the work [**200**].

1. The Setup

We study the following perturbation of the classical wave equation (0.1.2), classical Schrödinger equation (0.1.1) and the classical Dirac equation (0.1.4). More precisely we consider the following Cauchy problems:

$$\begin{cases} \Box u + ia(x) \cdot \nabla u + V(x)u = F, \\ u(0, x) = 0, \partial_t u(0, x) = 0, \end{cases} \tag{3.1.1}$$

$$\begin{cases} i\partial_t u - \Delta u + ia(x) \cdot \nabla u + V(x)u = F, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \\ u(0, x) = 0, \end{cases} \tag{3.1.2}$$

and

$$\begin{cases} i\gamma_\mu \partial_\mu u + ia(x) \cdot \nabla u + V(x)u = F, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \\ u(0, x) = 0. \end{cases} \tag{3.1.3}$$

The solution of (3.1.3) is called spinor and γ_μ are the Dirac matrices definied as in (0.1.5).

If we introduce the corresponding 1-form $a = \sum_{j=1}^{3} a_j dx^j$ for the magnetic potential, by the Poincaré Lemma we know that if we have two magnetic potential a', a with $da = da'$, then $a = a' + d\phi$, where $\phi \in C^\infty$. The operators $(-\Delta + ia' \cdot \nabla + V)$ and $(-\Delta + ia \cdot \nabla + \widetilde{V})$ are related by

$$(-\Delta + ia' \cdot \nabla + V) = e^{-i\phi}(-\Delta + ia \cdot \nabla + \widetilde{V})e^{i\phi}, \tag{3.1.4}$$

where $V = V_1 - i \cdot \nabla a' + (a')^2$ and $\widetilde{V} = \widetilde{V}_1 - i \cdot \nabla a - \Delta\phi + a^2 + \phi^2$. So we will assume that $a = (a_1, a_2, a_3)$ are measurable functions, such that ∇a_j exists (in distributional sense) and it is measurable, defined as $a_j = a'_j + \partial_j \phi$ for $j = 1, 2, 3$, where the functions a'_j and $\partial_j \phi$ satisfy the inequalities

$$|a'_j(x)| + ||x|\nabla a'_j(x)| \leq \frac{\delta}{|x|\, W_{\epsilon_0}(x)},\ a.e.\ x \in \mathbb{R}^3, \delta > 0,$$
$$|\partial_j \phi(x)| + ||x|\nabla \partial_j \phi(x)| \leq \frac{C_0}{|x|\, W_{\epsilon_0}(x)},\ a.e.\ x \in \mathbb{R}^3. \tag{3.1.5}$$

The potentials V (resp. V_1, $\widetilde{V}_1$) is a non-negative measurable function satisfying the inequality

$$|V(x)| \leq \frac{C_1}{|x|^2\, W_{\epsilon_0}(x)},\ a.e.\ x \in \mathbb{R}^3, \tag{3.1.6}$$

where $\epsilon_0, C_0 > 0, C_1 > 0$ are constants, and

$$W_\epsilon(|x|) := |x|^\epsilon + |x|^{-\epsilon},\ \forall x \in \mathbb{R}^3. \tag{3.1.7}$$

We see that the **potential $a_j(x)$ is bounded from above by $C\delta|x|^{-1-\varepsilon_0}$ if $|x| \geq 1$, while $a_j(x) \leq$ $\leq C\delta|x|^{-1+\varepsilon_0}$ if $|x| \leq 1$, and the potential $V(x)$ is bounded from above by $C|x|^{-2-\varepsilon_0}$ if $|x| \geq 1$, while $V(x) \leq \frac{C}{|x|^{-2+\varepsilon_0}}$ if $|x| \leq 1$.** The last estimate shows that we admit singularities of a_j and V, such that a_j is in $L^2_{loc}(\mathbb{R}^3)$, while V is not in $L^2_{loc}(\mathbb{R}^3)$. In the papers [**4, 5**], Agmon showed how scattering theory could be developed for general elliptic operator with perturbations $O(|x|^{-1-\epsilon})$ at infinity and Agmon-Hörmander generalized the techniques required to study the perturbation of simply characteristic operators (see [**86**]). In [**70**] one can find a perturbation theory for potentials decaying as $|x|^{-2-\epsilon}$ at infinity.
In [**207**] the free wave equation and Schrödinger equation (i.e. $a = 0, V = 0$) are studied and for both the following estimate are obtained (in [**207**] some sharper estimates are proved):

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \leq C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L^2_t L^2_x}. \tag{3.1.8}$$

Similar estimates hold for other dispersive equations of mathematical physics. The equation (3.1.8) is known as smoothing estimate for the Schrödinger equation.
In this work we shall establish the same estimate (3.1.8) for potential perturbation of the wave and the Schrödinger equations. Namely, we have:

THEOREM 3.1. *If $u(x,t)$ is the solution of the Cauchy problem* (3.1.1) *with* $(-\Delta + ia \cdot \nabla + V)$ *satisfying* (3.1.5) *and* (3.1.6) *then, for any $\delta, \delta' > 0$:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \leq C \||x|^{\frac{1}{2}} W_{\delta'} F(x,t)\|_{L^2_t L^2_x}, \tag{3.1.9}$$

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} u(x,t)\|_{L^2_t L^2_x} \leq C \|F(x,t)\|_{L^2_t L^1_x}, \tag{3.1.10}$$

$$\||x|^{\frac{1}{2}} W_\delta V(x,D) u(x,t)\|_{L^2_t L^2_x} \leq C \||x|^{\frac{1}{2}} W_{\delta'} F(x,t)\|_{L^2_t L^2_x}. \tag{3.1.11}$$

For (3.1.2) we have:

THEOREM 3.2. *If $u(x,t)$ is the solution of the problem* (3.1.2) *or* (3.1.3) *with* $(-\Delta + ia \cdot \nabla + V)$ *satisfying* (3.1.5) *and* (3.1.6) *then, for any $\delta, \delta' > 0$:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \leq C \||x|^{\frac{1}{2}} W_{\delta'} F(x,t)\|_{L^2_t L^2_x}, \tag{3.1.12}$$

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} u(x,t)\|_{L^2_t L^2_x} \leq C \|F(x,t)\|_{L^2_t L^1_x}, \tag{3.1.13}$$

$$\||x|^{\frac{1}{2}}W_{\delta}V(x,D)u(x,t)\|_{L_t^2L_x^2} \leq C\||x|^{\frac{1}{2}}W_{\delta'}F(x,t)\|_{L_t^2L_x^2}. \tag{3.1.14}$$

For the corresponding homogeneous problem

$$\begin{cases} i\partial_t u - \Delta u + ia(x)\cdot\nabla u + V(x)u = 0, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^3, \\ u(0,x) = f, \end{cases} \tag{3.1.15}$$

we have

THEOREM 3.3. *If $u(x,t)$ is the solution of* (3.1.15) *then, for any δ :*

$$\||x|^{-\frac{1}{2}}W_{\delta}^{-1}\nabla u(x,t)\|_{L_t^2L_x^2} \leq C\|f\|_{\dot{H}_V^{1/2}}, \tag{3.1.16}$$

where $\dot{H}_V^s(\mathbb{R}^3)$ is the perturbed homogeneous Sobolev space.

Recall that $\dot{H}_V^s(\mathbb{R}^3)$ is defined, for any $p,q\geq 1$ and for any $s\in\mathbb{R}$, as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the following norm:

$$\|f\|^2_{\dot{H}_V^s} := \sum_{j\in\mathbb{Z}} 2^{2js}\|\varphi_j(\sqrt{-\Delta_V})f\|^2_{L^2}, \forall f\in C_0^\infty(\mathbb{R}^3), \tag{3.1.17}$$

where $-\Delta_V$ is the operator defined by

$$-\Delta_V := -\Delta + V(x,D), \tag{3.1.18}$$

with

$$V(x,D) = ia(x)\cdot\nabla + V(x) = i\sum_{j=1}^{3} a_j(x)\partial_j + V(x) \tag{3.1.19}$$

and

$$\sum_{j\in\mathbb{Z}}\varphi_j(\lambda) = 1,$$

with $\varphi_j(\lambda) = \varphi(\frac{\lambda}{2^j})$, $\varphi\in C_0^\infty(\mathbb{R})$, $supp\varphi\subset[\frac{1}{2},2]$.

REMARK 3.1. *We can use the perturbed homogeneous Sobolev space in* (3.1.17) *because, the assumptions* (3.1.5) *and* (3.1.6) *implies that $\sigma_{sing}(-\Delta+V(x,D))=\varnothing$ so the wave operators exist and are complete (see* [**123**]*,* [**124**]*,* [**159**] *).*

The key point in this chapter is the use of appropriate estimates of the resolvent $R_V(\lambda^2\pm i0)$ defined as follows:

$$R_V(\lambda^2\pm i0)f = \lim_{\varepsilon\to 0^+} R_V(\lambda^2\pm i\varepsilon)f, \tag{3.1.20}$$

where

$$R_V(\lambda^2\pm i\varepsilon) = [(\lambda^2\pm i\varepsilon)+\Delta_V)]^{-1}. \tag{3.1.21}$$

The operator in (3.1.18) has to be understood in the sense of the classical Friedrich's extension defined by the quadratic form (see [**128**])

$$
\begin{aligned}
(-\Delta_V f, f) &= \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx + \int_{\mathbb{R}^3} V(x) \, |f(x)|^2 \, dx + \\
&+2 \sum_{j=1}^{3} \int_{\mathbb{R}^3} i a_j(x) f(x) \overline{\partial_j f(x)} \, dx, f \in C_0^\infty(\mathbb{R}^3),
\end{aligned} \tag{3.1.22}
$$

and the limit in (3.1.20) is taken in a suitable L^2 weighted sense.
More precisely, given any real a and $\delta > 0$, we define the spaces $L^2_{a,\delta}$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the following norms:

$$
\|f\|^2_{L^2_{a,\delta}} := \int_{\mathbb{R}^3} |f|^2 |x|^{2a} W_\delta^2(|x|) dx, \text{ if } a > 0,
$$

and

$$
\|f\|^2_{L^2_{a,\delta}} := \int_{\mathbb{R}^3} |f|^2 |x|^{2a} W_\delta^{-2}(|x|) dx, \text{ if } a < 0,
$$

where the weights $W_\delta(|x|)$ are defined in (3.1.7).
The existence of the limit in (3.1.20) (known as limiting absorption principle, see Section 1, Chapter 2) can be established in the uniform operator norm

$$
\mathcal{B}(L^2_{1/2,\delta}, L^2_{-1/2,\delta}) \text{ , } \forall \delta > 0.
$$

To verify the limiting absorption principle we use the following resolvent identities:

$$
\begin{aligned}
R_V(\lambda^2 \pm i\varepsilon) &= R_0(\lambda^2 \pm i\varepsilon) + i R_0(\lambda^2 \pm i\varepsilon) a \cdot \nabla R_V(\lambda^2 \pm i\varepsilon) + \\
&+ R_0(\lambda^2 \pm i\varepsilon) V R_V(\lambda^2 \pm i\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
R_V(\lambda^2 \pm i\varepsilon) &= R_0(\lambda^2 \pm i\varepsilon) + i R_V(\lambda^2 \pm i\varepsilon) a \cdot \nabla R_0(\lambda^2 \pm i\varepsilon) + \\
&+ R_V(\lambda^2 \pm i\varepsilon) V R_0(\lambda^2 \pm i\varepsilon).
\end{aligned}
$$

The previous identities combined with the classical limiting absorption principle for the free resolvent imply the following ones:

$$
\begin{aligned}
R_V^{\pm}\left(\lambda^2\right) &= R_0^{\pm}\left(\lambda^2\right) + i R_0^{\pm}\left(\lambda^2\right) a \cdot \nabla R_V^{\pm}\left(\lambda^2\right) \\
&+ R_0^{\pm}\left(\lambda^2\right) V R_V^{\pm}\left(\lambda^2\right),
\end{aligned} \tag{3.1.23}
$$

and

$$
\begin{aligned}
R_V^{\pm}\left(\lambda^2\right) &= R_0^{\pm}\left(\lambda^2\right) + i R_V^{\pm}(\lambda^2) a \cdot \nabla R_0^{\pm}(\lambda^2) \\
&+ R_V^{\pm}(\lambda^2) V R_0^{\pm}(\lambda^2).
\end{aligned} \tag{3.1.24}
$$

REMARK 3.1. *In the proof of the limiting absorption principle in Section 1 we used smooth weights. Here, the presence of singularities in the perturbation operator suggests to substitute the smooth weight by weights with singularities. The Limiting absorption principle remain valid, in fact one can also allow sufficiently mild singularities, as suggested in the work* [**5**] *and* [**86**, XIV, Scattering Theory]. *To more details we refer also to* [**70**].

Several papers have treated the potential type perturbation of the free wave operator. The case of purely potential perturbation $V(x)$ is considered in [**15**] under the following decay assumption:

$$|V(x)| \leq \frac{C}{|x|^{4+\delta_0}}, \quad |x| \geq 1,$$

for some $C, \delta_0 > 0$. In [**44**] the previous assumption is weaken and the decay required at infinity is the following one: $|V(x)| \leq \frac{C}{|x|^{3+\delta_0}}$. The family of radial potentials $V(x) = \frac{c}{|x|^2}$, where $c \in \mathbb{R}^+$, are treated in the papers [**152**] and [**32**]. More precisely, the first paper treats the case of radial initial data, while in the second one general initial data are considered. In these papers dispersive estimates for the corresponding perturbed wave equations are established. In [**70**] the assumption (3.1.6) means that at infinity the potential is bounded from above by $C|x|^{-2-\varepsilon_0}$, while its behavior near $x = 0$ is dominated by constant times $|x|^{-2+\varepsilon_0}$. In this paper Strichartz type estimates for the corresponding perturbed wave equation are established. In this work we introduce a "short range" perturbation with symbol of order one and (3.1.5) means that at infinity our potential is bounded from above by $C|x|^{-1-\varepsilon_0}$, while its behavior near $x = 0$ is dominated by constant times $|x|^{-1+\varepsilon_0}$. It is clear that the assumption (3.1.5), (3.1.6) are quite general and allow one to consider non radially symmetric potentials.
This chapter is organized as follows. In Section 2 we prove some estimates for the operators $R_0^{\pm}(\lambda^2)$. In Section 3 we give some estimates for the perturbed resolvent $R_V^{\pm}(\lambda^2)$. In Section 4 we prove theorem 3.1, 3.2, 3.3.

2. Free Resolvent Estimates for $n = 3$.

This section is devoted to prove some estimates satisfied by the free resolvent operator $R_0^{\pm}(\lambda^2)$. First of all we will prove the representation formula for the operators $R_0^{\pm}$ for $n = 3$.

PROPOSITION 3.1. *Assume that $n = 3$, then the following representation formula*

$$R_0^{\pm}\left(\lambda^2\right) f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm \imath |\lambda| |x-y|} f(y)}{|x-y|} \, dy \,, \tag{3.2.25}$$

holds for any $\lambda \in \mathbb{R}, x \in \mathbb{R}^3$, $f \in C_0^{\infty}(\mathbb{R}^3)$.

Proof. We shall prove the identity (3.1) only for $R_0^{+}\left(\lambda^2\right)$. The same procedure work for $R_0^{-}\left(\lambda^2\right)$. Let $\lambda, \varepsilon > 0$ be fixed, and let $\mu_+(\varepsilon)$ be defined by $\mu_{\pm}(\varepsilon) = \lambda^2 \pm \imath\varepsilon$. The Fourier transform implies that

$$R_0^{+}\left(\lambda^2\right) f = \frac{1}{(2\pi)^{\frac{3}{2}}} K_{\varepsilon,\lambda} * f,$$

where the kernel $K_{\varepsilon,\lambda}$ is defined by

$$K_{\varepsilon,\lambda}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{e^{-\imath x\xi}}{|\xi|^2 - \mu_+(\varepsilon)^2} \, d\xi.$$

Since the function $\left(|\xi|^2 - \mu_+(\varepsilon)^2\right)^{-1}$ is radial, then also its Fourier transform must like that, i. e. $K_{\varepsilon,\lambda}(x) = K_{\varepsilon,\lambda}(|x|)$. Switching to polar coordinates

$$(0,\infty) \times (0,\pi) \times (0,2\pi) \ni (r,\theta,\phi) \mapsto (r sin\phi \sin\theta, r\cos\phi\sin\theta, r\cos\theta) \in \mathbb{R}^3 \,,$$

the previous integral gets into the form

$$K_{\varepsilon,\lambda}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{r^2}{r^2-\mu_\pm(\varepsilon)^2} \int_0^\pi \mathrm{e}^{-\imath|x|r\cos\theta} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}r \, .$$

Integrating with respect to θ we obtain

$$K_{\varepsilon,\lambda}(x) = \frac{\sqrt{2}}{\sqrt{\pi}|x|} \int_0^\infty \frac{\sin|x|r}{r^2-\mu_+(\varepsilon)^2} r \, \mathrm{d}r = \frac{1}{\sqrt{\pi}|x|} \int_{\mathbb{R}} \frac{\mathrm{e}^{\imath|x|r}}{r^2-\mu_+(\varepsilon)^2} r \, \mathrm{d}r \, .$$

Applying residue theorem it implies

$$K_{\varepsilon,\lambda}(x) = \frac{\sqrt{2\pi}}{|x|} \mathrm{Res}\left(z\mathrm{e}^{\imath|x|z} \left(z^2 - \mu_+(\varepsilon)^2\right)^{-1}, \mu_+(\varepsilon)\right) = \frac{\sqrt{\pi}\mathrm{e}^{\imath|x|\mu_+(\varepsilon)}}{\sqrt{2}x} \, , \tag{3.2.26}$$

as we did for the proof of Theorem 2.3 (see [**9**] for more details).

By using the definition of $\mu_\pm(\varepsilon)$ we deduce that if $\varepsilon \downarrow 0$, then $\mu_+(\varepsilon) \to |\lambda|$, that implies

$$\lim_{\varepsilon\downarrow 0} K_{\varepsilon,\lambda} = \frac{\sqrt{\pi}\mathrm{e}^{\imath|\lambda||x|}}{\sqrt{2|x|}}$$

by (3.2.26). That completes the proof of the proposition. □

LEMMA 3.1. *The family of operators $R_0^\pm\left(\lambda^2\right)$ satisfies the following estimates:*
(i) for any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta,\delta') > 0$ such that for any $\lambda > 0$:

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq \frac{C}{\lambda} \||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2}; \tag{3.2.27}$$

(ii) for any $\delta, \delta', \epsilon > 0$ that satisfy $0 < \epsilon < 2\delta'$, there exists $C = C(\delta,\delta',\epsilon) > 0$ such that for any $\lambda > 0$:

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq C \||x|^{\frac{3+\epsilon}{2}} W_{\delta'} f\|_{L^2}; \tag{3.2.28}$$

(iii) for any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta,\delta') > 0$ such that for any $\lambda > 0$:

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{3}{2}} W_{\delta'} f\|_{L^2}; \tag{3.2.29}$$

(iv) for any $\delta, \delta' > 0$ and for $s \in [1/2, 3/2]$, there exists a real constant $C = C(\delta,\delta') > 0$ such that for any $\lambda \in \mathbb{R}$:

$$\||x|^{-s} W_\delta^{-1} R_0^\pm(\lambda^2) f\|_{L^2} \leq C \||x|^{2-s} W_{\delta'} f\|_{L^2}; \tag{3.2.30}$$

(v) for any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta,\delta') > 0$ such that for any $\lambda > 0$:

$$\||x|^{-\frac{3}{2}} W_\delta^{-1} R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2}; \tag{3.2.31}$$

(vi) for any $\delta > 0$ there exists a real constant $C = C(\delta) > 0$ such that for any $\lambda \geq 0$:

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq C \|f\|_{L^1}; \tag{3.2.32}$$

(vii) for any $\delta, \delta' > 0$ and for $s \in [1/2, 3/2]$, there exists a real constant $C = C(\delta,\delta') > 0$ such that for any $\lambda > 0$:

$$\||x|^{-s} W_\delta^{-1} \nabla R_0^\pm\left(\lambda^2\right) f\|_{L^2} \leq C \, \||x|^s W_{\delta'} f\|_{L^2}. \tag{3.2.33}$$

Proof.
Proof of (3.2.27). The proof can be found in [**5**] and [**14**] for example.
Now to prove the others estimates we follow [**70**].
Proof of (3.2.28): The identity (3.2.25) implies:

$$\|R_0^{\pm}\left(\lambda^2\right) f\|^2_{L^2_{-\frac{1}{2},\delta}} \leq C \int \left| \int \frac{|f(y)|}{|x-y|} dy \right|^2 |x|^{-1} W_\delta^{-2} dx,$$

that combined with the Cauchy–Schwartz inequality gives:

$$\|R_0^{\pm}\left(\lambda^2\right) f\|^2_{L^2_{-\frac{1}{2},\delta}} \leq$$
$$\leq C \left(\int |f(y)|^2 |y|^{3+\epsilon} W^2_{\delta'}(y) dy \right) \left(\int \int \frac{|x|^{-1} W_\delta^{-2}(|x|)}{|y|^{3+\epsilon} W^2_{\delta'}(y) |x-y|^2} dx dy \right) \leq$$
$$\leq C \|f\|^2_{L^2_{\frac{3+\epsilon}{2},\delta'}},$$

where

$$C := C(\epsilon, \delta) = \int \int \frac{|x|^{-1} W_\delta^{-2}(x)}{|y|^{3+\epsilon} W^2_{\delta'}(y) |x-y|^2} dx dy < \infty.$$

Proof of (3.2.29): It is sufficient to to interpolate between (3.2.28) with $\epsilon = \delta'$ and (3.2.27).
Proof of (3.2.31): It is tsufficient to interpolate between (3.2.29) and its dual estimate.

The main tool that we will use to prove the following estimate is the following inequality,

$$\left\| \int K(y) dy \right\|_B \leq \int \|K(y)\|_B dy, \tag{3.2.34}$$

where $K(y)$ is a measurable function defined over $\mathbb{R}^n$ and valued in a Banach space $(B, \|.\|_B)$.
Proof of (3.2.32). If we combine (3.2.34), where the function K and the space B are chosen as follows:

$$K(y,.) = \frac{e^{i\lambda|.-y|}}{|.-y|} f(y), B = L^2_{-\frac{1}{2},\delta},$$

with the identity (3.2.25), then we obtain,

$$\|R_0(\lambda^2 \pm i0)\|_{L^2_{-\frac{1}{2},\delta}} = |c| \left\| \int \frac{e^{i\lambda|x-y|}}{|x-y|} f(y) dy \right\|_{L^2_{-\frac{1}{2},\delta}}$$
$$\leq |c| \int \left\| \frac{e^{i\lambda|.-y|}}{|.-y|} \right\|_{L^2_{-\frac{1}{2},\delta}} |f(y)| dy \leq C \|f\|_{L^1} \tag{3.2.35}$$

where we used a result from [**70**] (Appendix I) to obtain the following uniform bound,

$$C = C(\delta) := sup_x \left\| \frac{e^{i\lambda|.-y|}}{|.-y|} \right\|_{L^2_{-\frac{1}{2},\delta}} < \infty.$$

Proof of (3.2.33). The proofs can be found in Chapter 2, Section 4, where slightly different spaces have been used (see also the Remark 2.3).We follow the proof of Theorem 2.10. In a crucial step, to prove the bound (2.4.33), we used Cauchy-Schwartz inequality

$$\|u\|_{L^1} \leq \left(\langle x \rangle^{-s}, \langle x \rangle^s u \right),$$

and the fact that $\|\langle x\rangle^{-s}\|_{L^2} < C$, provided $s > 1/2$. Now we want to do the same using the singular weight $|x|^a W_\delta$ instead of $\langle x\rangle^s$. We have

$$\|u\|_{L^1} \leq \left(|x|^{-a} W_\delta^{-1}, |x|^a W_\delta u\right),$$

and in order to apply the Cauchy-Schwartz inequality we have to check for which values of a the function $|x|^{-a} W_\delta^{-1}$ is $L^2(\mathbb{R})$–integrable. A simple calculation shows that we have $\| |x|^{-a} W_\delta^{-1}\|_{L^2} < C$, whenever $1/2 \geq a$, this completes the proof (in this case we restrict the range to $1/2 \leq a \leq 3/2$ because in the following we need only this domain for the exponent). □

LEMMA 3.2. *Assume that the perturbation $V(x,D)$ satisfies the assumption* (3.1.5),(3.1.6). *Then the following estimates are satisfied:*
for any $\delta, \delta' > 0$ there exists a real constant $C := C(\delta, \delta') > 0$, such that for any $\lambda \geq 0$

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0^{\pm}\left(\lambda^2\right) V(x,D) f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}; \tag{3.2.36}$$

$$\||x|^{\frac{1}{2}} W_\delta V(x,D) R_0^{\pm}\left(\lambda^2\right) f\|_{L^2} \leq C\||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2}. \tag{3.2.37}$$

Proof.
Proof of (3.2.36). We split the proof into two step.
Step 1. Estimate of

$$iR_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f. \tag{3.2.38}$$

We have the following formula,

$$\begin{aligned} iR_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f &= iR_0^{\pm}\left(\lambda^2\right) \nabla(a \cdot f) - \\ &- iR_0^{\pm}\left(\lambda^2\right) (\nabla a) \cdot f. \end{aligned} \tag{3.2.39}$$

From the functional calculus we have $[\nabla, R_0^{\pm}\left(\lambda^2\right)] = 0$, so we rewrite (3.2.39) as

$$\begin{aligned} iR_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f &= i\nabla R_0^{\pm}\left(\lambda^2\right))(a \cdot f) - \\ &- iR_0^{\pm}\left(\lambda^2\right) (\nabla a) \cdot f. \end{aligned} \tag{3.2.40}$$

We have

$$\begin{aligned} &\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f\|_{L^2} \leq \\ &\leq C\||x|^{-\frac{1}{2}} W_\delta^{-1} i\nabla R_0^{\pm}\left(\lambda^2\right) af\|_{L^2} + \\ &+ C\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0^{\pm}\left(\lambda^2\right) (\nabla a) f\|_{L^2}. \end{aligned} \tag{3.2.41}$$

Now we can estimate now the first term in the right hand side of the inequality (3.2.41), using (3.2.33) and obtain

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} i\nabla R_0^{\pm}\left(\lambda^2\right) af\|_{L^2} \leq C\|||x|^{-\frac{1}{2}} W_{\delta''}^{-1} af\|_{L^2}. \tag{3.2.42}$$

By the assumption (3.1.5) and choosing $0 < \delta'' < \epsilon_0$, $\delta_a \leq \epsilon_0 - \delta''$ we have

$$\begin{aligned} \||x|^{-\frac{1}{2}} W_{\delta}^{-1} i R_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f\|_{L^2} &\leq C\||x|^{\frac{1}{2}} W_{\delta''}^{-1} a f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\epsilon_0-\delta''}^{-1} f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\delta_a}^{-1} f\|_{L^2}. \end{aligned} \quad (3.2.43)$$

For the second term in the right side of (3.2.41) we use the estimates (3.2.30) and obtain

$$\||x|^{-\frac{1}{2}} W_{\delta}^{-1} i R_0^{\pm}\left(\lambda^2\right) (\nabla a) f\|_{L^2} \leq C\||x|^{\frac{3}{2}} W_{\delta''} \nabla a f\|_{L^2}. \quad (3.2.44)$$

By the (3.1.5), choosing $0 < \delta'' < \epsilon_0$, $\delta_b \leq \epsilon_0 - \delta''$ we have

$$\begin{aligned} \||x|^{-\frac{1}{2}} W_{\delta}^{-1} i R_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f\|_{L^2} &\leq C\||x|^{\frac{3}{2}} W_{\delta''} (\nabla a) f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\epsilon_0-\delta''} f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\delta_b}^{-1} f\|_{L^2}. \end{aligned} \quad (3.2.45)$$

From the fact that $\delta_b < \delta_a$ we put $\delta' \leq \delta_b$ and (3.2.41) becomes

$$\||x|^{-\frac{1}{2}} W_{\delta}^{-1} i R_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}. \quad (3.2.46)$$

Step 2. Estimate of

$$R_0^{\pm}\left(\lambda^2\right) V f. \quad (3.2.47)$$

From the assumption (3.1.5) we see that $|\nabla a_j(x)| \leq \frac{C_0 \delta}{|x|^2 W_{\epsilon_0}(x)}$, so we proceed as in Step 1 and obtain

$$\||x|^{-\frac{1}{2}} W_{\delta}^{-1} R_0^{\pm}\left(\lambda^2\right) V f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}. \quad (3.2.48)$$

Taking into account the estimates (3.2.46) and (3.2.48), we arrive at

$$\begin{aligned} &\||x|^{-\frac{1}{2}} W_{\delta}^{-1} R_0^{\pm}\left(\lambda^2\right) V(x,D) f\|_{L^2} \leq \||x|^{-\frac{1}{2}} W_{\delta}^{-1} i R_0^{\pm}\left(\lambda^2\right) a \cdot \nabla f\|_{L^2} + \\ &+ \||x|^{-\frac{1}{2}} W_{\delta}^{-1} R_0^{\pm}\left(\lambda^2\right) V f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2} \end{aligned}$$

and (3.2.36) is established.
Proof of (3.2.37). It is the dual to estimate (3.2.36).

□

3. Perturbed Resolvent Estimates

In this section we prove some estimates for the perturbed resolvent $R_V^{\pm}\left(\lambda^2\right)$.

THEOREM 3.4. *Assume that the perturbation $V(x,D)$ satisfies the assumptions* (3.1.5) *and* (3.1.6). *Then for any $0<\delta<\epsilon_0/2$ there exists a family of operators $A_\lambda^{\pm}\in\mathcal{B}(L^2_{-\frac12,\delta},L^2_{-\frac12,\delta})$ such that,*

$$A_\lambda^{\pm}\circ[I-R_0^{\pm}\left(\lambda^2\right)V(x,D)]=I=[I-R_0^{\pm}\left(\lambda^2\right)V(x,D)]\circ A_\lambda^{\pm}.$$

Moreover, there exists a constant $C=C(\delta)>0$ such that,

$$\|A_\lambda^{\pm}f\|_{L^2_{-\frac12,\delta}}\leq C\|f\|_{L^2_{-\frac12,\delta}},\ \forall\lambda\in\mathbb{R}.$$

THEOREM 3.5. *Assume that the perturbation $V(x,D)$ satisfies the assumptions* (3.1.5) *and* (3.1.6). *Then for any $0<\delta<\epsilon_0/2$ there exists a family of operators $B_\lambda^{\pm}\in\mathcal{B}(L^2_{\frac12,\delta},L^2_{\frac12,\delta})$ such that,*

$$B_\lambda^{\pm}\circ[I-V(x,D)R_0^{\pm}\left(\lambda^2\right)]=I=[I-V(x,D)R_0^{\pm}\left(\lambda^2\right)]\circ B_\lambda^{\pm}.$$

Moreover, there exists a constant $C=C(\delta)>0$ such that,

$$\|B_\lambda^{\pm}f\|_{L^2_{\frac12,\delta}}\leq C\|f\|_{L^2_{\frac12,\delta}},\ \forall\lambda\in\mathbb{R}.$$

We have

$$R_0^{\pm}\left(\lambda^2\right)V(x,D)=iR_0^{\pm}\left(\lambda^2\right)a\cdot\nabla+R_0^{\pm}\left(\lambda^2\right)V. \tag{3.3.49}$$

Now we need the following lemmas.

LEMMA 3.3. *Assume that the potential V satisfies the assumptions* (3.1.6) .
(1). The operators $R_0(\lambda^2)V$ are compact in the space $\mathcal{B}(L^2_{-\frac12,\delta},L^2_{-\frac12,\delta'})$, provided that δ,δ' are small. Moreover the following estimate is satisfied:

$$\|R_0^{\pm}\left(\lambda^2\right)V\|_{\mathcal{B}(L^2_{-\frac12,\delta},L^2_{-\frac12,\delta'})}\to 0,$$

as $\lambda\to\infty$.
(2). The operators $VR_0^{\pm}\left(\lambda^2\right)$ are compact in the space $\mathcal{B}(L^2_{\frac12,\delta},L^2_{\frac12,\delta'})$, provided that δ,δ' are small. Moreover the following estimate is satisfied:

$$\|VR_0^{\pm}\left(\lambda^2\right))\|_{\mathcal{B}(L^2_{\frac12,\delta},L^2_{\frac12,\delta})}\to 0,$$

as $\lambda\to\infty$.

Proof. *Proof of (1)*: The proof can be found in [**70**] (Theorem III.1 and Lemma III.1).
Proof of (2). It is the dual of (1) where

$$VR_0^{\pm}\left(\lambda^2\right)=(R_0^{\mp}\left(\lambda^2\right)V)^*.$$

LEMMA 3.4. *Assume that the potential $ia\cdot\nabla$ satisfies the assumptions* (3.1.5).
(1). The operators $iR_0^{\pm}\left(\lambda^2\right)a\cdot\nabla$ are compact in the space $\mathcal{B}(L^2_{-\frac12,\delta},L^2_{-\frac12,\delta'})$, provided that δ,δ' are small.
(2). The operators $ia\cdot\nabla R_0^{\pm}\left(\lambda^2\right)$ are compact in the space $\mathcal{B}(L^2_{\frac12,\delta},L^2_{\frac12,\delta'})$, provided that δ,δ' are small.

Proof.*Proof of (1).* We will follow the proof in [**70**]. Let $\{f_n\}$ be a sequence bounded in $L^2_{-\frac{1}{2},\delta}$ and let $g_n := iR_0^{\pm}\left(\lambda^2\right) a\cdot\nabla\, f_n$. We split the proof in two cases:
Case 1. Compactness in $B_{2r}\setminus B_{\frac{1}{2r}}$, for $1<r<\infty$.
The estimate (3.2.36) implies that if δ,δ' are small, then

$$iR_0(\lambda^2)a\cdot\nabla\ \in\mathcal{B}(L^2_{-\frac{1}{2},\delta},L^2_{-\frac{1}{2},\delta'}). \tag{3.3.50}$$

In the proceeding of the proof we use the representation (3.2.40) for the operator (3.2.38) acting on $L^2_{-\frac{1}{2},\delta}$. The property (3.3.50) implies that $\|g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\le C(r)$.
We have

$$\begin{aligned}&\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\le C\|i(\Delta+\lambda^2)R_0^{\pm}\left(\lambda^2\right)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}+\\&+C\lambda^2\||x|^{-\frac{1}{2}}W_\delta^{-1}iR_0^{\pm}\left(\lambda^2\right)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}+\\&+C\||x|^{-\frac{1}{2}}W_\delta^{-1}i\nabla R_0^{\pm}\left(\lambda^2\right)(\nabla a)f\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\le C\|af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}+\\&+C\lambda^2\||x|^{-\frac{1}{2}}W_\delta^{-1}iR_0^{\pm}\left(\lambda^2\right)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}+\\&+C\||x|^{-\frac{3}{2}}W_\delta^{-1}i\nabla R_0^{\pm}\left(\lambda^2\right)(\nabla a)f\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}.\end{aligned} \tag{3.3.51}$$

With the aim of the estimates (3.2.27), (3.2.33) and the assumption (3.1.5) we obtain

$$\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\le C(r,\lambda)\||x|^{-\frac{1}{2}}W_\delta^{-1}f_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}$$

and from the boundness of $\{f_n\}$, $\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\le C(r,\lambda)$. So we have

$$\|\nabla g_n\|_{H^1(B_{2r}\setminus B_{\frac{1}{2r}})}\le C(r,\lambda).$$

The compactness of the Sobolev embedding due to Rellich-Kondrachov theorem implies that $\{g_n\}$ is compact in $L^2(B_r\setminus B_{\frac{1}{r}})$ for any $1<r<\infty$.
Case 2. Compactness in $(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}$
To study the compactness in $(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}$ we use the following inequality:

$$\begin{aligned}&\int_{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}}g_n^2(|x|)W_\delta^{-2}(|x|)|x|^{-1}dx\le\\&\le(sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}}W_\delta^{-1}(|x|))\int_{\mathbb{R}^3}g_n^2(|x|)W_\delta^{-1}(|x|)|x|^{-1}dx.\end{aligned} \tag{3.3.52}$$

The definition of the weights $W_\delta(|x|)$ guarantees that for $\delta>0$ there exist real constants $c_1(\delta),c_2(\delta)$ such that $c_1(\delta)W_\delta\le W^2_{\frac{\delta}{2}}\le c_2(\delta)W_\delta$. This property combined with (3.3.52),where we chose $\delta'=\frac{\delta}{2}$, implies:

$$\int_{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}} g_n^2(|x|)W_\delta^{-2}(|x|)|x|^{-1}dx \leq$$
$$\leq C(sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}}W_\delta^{-1}(|x|))\int_{\mathbb{R}^3} g_n^2(|x|)W_{\frac{\delta}{2}}^{-2}(|x|)|x|^{-1}dx$$
$$\leq C'(sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}}W_\delta^{-1}(|x|))\|f|_{L^2_{-\frac{1}{2},\delta}}.$$

Moreover, $(sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}}W_\delta^{-1}(|x|)) \to 0$ if $r \to \infty$ and it implies with an easy diagonal argument the compactness of the sequence $\{g_n\}$ in the space $L^2_{-\frac{1}{2},\delta}$.

Proof of (2) It is the dual to the part (1) of the theorem. We can also proceed independently following [**5**], [**86**, Chapter XIV, Scattering Theory, Lemma 14.5.1] or [**221**].

□

Proof of Theorem 3.4. Lemmas 3.3, 3.4 and the choice of δ (small perturbation) in the coefficients of the pertubing term (3.1.5) imply that the operators $[I - R_0^{\pm}(\lambda^2)V(x,D)]$ are injective in $\mathcal{B}(L^2_{-\frac{1}{2},\delta})$ and are compact perturbation of the invertible operator I. We can apply the Fredholm Alternative Theorem to obtain the existence of the operators $A_\lambda^{\pm}$. To prove the uniform bound $\|A_\lambda^{\pm}\|_{B(L^2_{-\frac{1}{2},\delta})} \leq C$ we consider two cases.

(1) λ large.

As a consequence of Lemmas 3.3, 3.4 there exists $\bar{\lambda} > 0$ such that if $\lambda > \bar{\lambda}$ then we have $\|R_0^{\pm}(\lambda^2)V(x, D)\|_{B(L^2_{-\frac{1}{2},\delta})} \leq \frac{1}{2}$ and this implies that $\|[Id - R_0^{\pm}(\lambda^2)V(x,D)]\|_{B(L^2_{-\frac{1}{2},\delta})} \geq \frac{1}{2}$ provided that $\lambda > \bar{\lambda}$. This uniform bound from below for the operators implies an uniform bound from above for their corresponding inverse operators $A_\lambda^{\pm}$.

(2) λ small.

The boundedness for $\lambda < \bar{\lambda}$ of the norms $\|A_\lambda^{\pm}\|_{B(L^2_{-\frac{1}{2},\delta})}$ is a consequence of the continuity of the family of operators $A_\lambda^{\pm}$ in the space $\mathcal{B}(L^2_{-\frac{1}{2},\delta})$ with respect to the parameter $\lambda \in [0,\infty)$ and of the compactness of the interval $[0, \bar{\lambda}_p]$.

□

Proof of Theorem 3.5. It is analogous to the one of Theorem 3.4.

□

REMARK 3.2. *The problem of resonances (for more details and definitions see Chapter 2, Section 2) arise in mathematical physics and in other field such as geometry. In our case this problem arises when we have perturbation of operator acting in some Banach spaces. Several works have treated the theory of resonances, we refer to* [**84**], [**155**], [**101**] *and* [**212**] *for example. The remark suggest that* *resonances may exist in the case of electromagnetic perturbation of type* $V(x,D) = ia(x)\cdot\nabla + V(x)$. *To assure that resonances cannot exist we impose a smallness assumption* (3.1.5) *on* a.

THEOREM 3.6. *Assume that the perturbation* $V(x,D)$ *satisfies* (3.1.5) *and* (3.1.6). *For each* $0 < \delta < \epsilon_0/2$ *we have*
(i) there exists a real constant $C = C(\delta) > 0$ *such that for any* $\lambda > 0$*:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le \frac{C}{\lambda} \||x|^{\frac{1}{2}} W_\delta f\|_{L^2}; \tag{3.3.53}$$

(ii) for any $\epsilon > 0$ *that satisfy* $0 < \epsilon < 2\delta$*, there exists* $C = C(\delta,\epsilon) > 0$ *such that for any* $\lambda \in \mathbb{R}$*:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le C\||x|^{\frac{3+\epsilon}{2}} W_{\delta'} f\|_{L^2}; \tag{3.3.54}$$

(iii) there exists a real constant $C = C(\delta) > 0$ *such that for any* $\lambda > 0$*:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{3}{2}} W_{\delta'}\|_{L^2}; \tag{3.3.55}$$

(iv) for any $\delta, \delta' > 0$ *and for* $s \in [1/2, 3/2]$*, there exists a real constant* $C = C(\delta,\delta') > 0$ *such that for any* $\lambda \in \mathbb{R}$*:*

$$\||x|^{-s} W_\delta^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le C\||x|^{2-s} W_{\delta'} f\|_{L^2}; \tag{3.3.56}$$

(v) there exists a real constant $C = C(\delta) > 0$ *such that for any* $\lambda > 0$*:*

$$\||x|^{-\frac{3}{2}} W_{\delta'}^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{1}{2}} W_\delta f\|_{L^2}; \tag{3.3.57}$$

(vi) for any $\delta > 0$ *there exists a real constant* $C = C(\delta) > 0$ *such that for any* $\lambda \in \mathbb{R}$*:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le C\|f\|_{L^1}. \tag{3.3.58}$$

(vi) for any $\delta, \delta' > 0$ *and for* $s \in [1/2, 3/2]$*, there exists a real constant* $C = C(\delta,\delta') > 0$ *such that for any* $\lambda \in \mathbb{R}$*:*

$$\||x|^{-s} W_\delta^{-1} \nabla R_V^{\pm}\left(\lambda^2\right) f\|_{L^2} \le C\, \||x|^{s} W_\delta f\|_{L^2}. \tag{3.3.59}$$

Theorem 3.4 implies that the identity (3.1.23) can be written as:

$$[I - R_0(\lambda^2 \pm i0)V(x,D)]R_V^{\pm}\left(\lambda^2\right) = R_0^{\pm}\left(\lambda^2\right),$$

and so the following identity,

$$R_V^{\pm}\left(\lambda^2\right) = A_\lambda^{\pm} R_0^{\pm}\left(\lambda^2\right). \tag{3.3.60}$$

Theorem 3.5 implies that the identity (3.1.24) can be written now as:

$$R_V^{\pm}\left(\lambda^2\right)[I - V(x,D)R_0^{\pm}\left(\lambda^2\right)] = R_0^{\pm}\left(\lambda^2\right),$$

and so the following identity,

$$R_V^{\pm}\left(\lambda^2\right) = R_0^{\pm}\left(\lambda^2\right) B_\lambda^{\pm}. \tag{3.3.61}$$

Proof.
Proof of (3.3.53). The estimate can be proved combining the identity (3.3.60) with the theorem 3.4 and estimate (3.2.27) in the following way:

$$\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V^{\pm}\left(\lambda^2\right)f\|_{L^2} \leq \||x|^{-\frac{1}{2}}W_\delta^{-1}A_\lambda^{\pm}R_0^{\pm}\left(\lambda^2\right)f\|_{L^2} \leq$$
$$C\||x|^{-\frac{1}{2}}W_\delta^{-1}R_0^{\pm}\left(\lambda^2\right)f\|_{L^2} \leq \frac{C}{\lambda}\||x|^{\frac{1}{2}}W_\delta f\|_{L^2}.$$

Proof of (3.3.54). The estimate can be proved combining the identity (3.3.60) with the Theorem 3.4 and estimate (3.2.28) as before.
Proof of (3.3.55). The estimate can be proved combining the identity (3.3.60) with the Theorem 3.4 and estimate (3.2.29) as before.
Proof of (3.3.56). The estimate can be proved combining the identity (3.3.60) with the Theorem 3.4 and estimate (3.2.30) as before.
Proof of (3.3.57). The estimate can be proved combining the identity (3.3.60) with the Theorem 3.4 and estimate (3.2.31) as before.
Proof of (3.3.58). The estimate can be proved combining the identity (3.3.60) with the Theorem 3.4 and estimate (3.2.32) as before.
Proof of (3.3.59). The estimate can be proved combining the identity (3.3.61) with the Theorem 3.5 and estimate (3.2.33) .

□

THEOREM 3.7. *Assume that the perturbation $V(x,D)$ satisfies* (3.1.5), (3.1.6). *For each $0<\delta<\epsilon_0/2$ we have for any $\lambda \in \mathbb{R}$*

$$\||x|^{\frac{1}{2}}W_\delta V(x,D)R_V^{\pm}\left(\lambda^2\right)f\|_{L^2} \leq C\||x|^{\frac{1}{2}}W_\delta f\|_{L^2}. \tag{3.3.62}$$

Proof. The resolvent identity implies the following one:

$$V(x,D)R_V^{\pm}\left(\lambda^2\right) = V(x,D)R_0^{\pm}\left(\lambda^2\right) +$$
$$+ V(x,D)R_0^{\pm}\left(\lambda^2\right)V(x,D)R_V^{\pm}\left(\lambda^2\right),$$

and from this we have

$$[I - V(x,D)R_0^{\pm}\left(\lambda^2\right)]V(x,D)R_V(\lambda^2 \pm i0) = V(x,D)R_0^{\pm}\left(\lambda^2\right). \tag{3.3.63}$$

Following the theorem 3.5 (2) we have

$$V(x,D)R_V^{\pm}\left(\lambda^2\right) = B_\lambda^{\pm}V(x,D)R_0^{\pm}\left(\lambda^2\right)$$

and from combining this with estimate (3.2.37) obtain

$$\|V(x,D)R_V^{\pm}\left(\lambda^2\right)f\|_{L^2_{\frac{1}{2},\delta}} \leq C\|B_\lambda^{\pm}V(x,D)R_0^{\pm}\left(\lambda^2\right)f\|_{L^2_{\frac{1}{2},\delta}} \leq$$
$$\leq C\|V(x,D)R_0^{\pm}\left(\lambda^2\right)f\|_{L^2_{\frac{1}{2},\delta}} \leq C\|f\|_{L^2_{\frac{1}{2},\delta}}.$$

□

4. Weighted Space-Time Estimates

In this section we will prove the main Theorems 3.1, 3.2, 3.3. We use the techniques of [**117**] and of [**207**].

Proof of Theorem 3.1.

Case 1. Wave equation.

Proof of 3.1.9. Formally taking the Fourier Transform in time variable in (3.1.1), we get

$$(\lambda^2 + \Delta_V)\hat{u}(\lambda, x) = -\hat{F}(\lambda, x). \tag{3.4.64}$$

Using (3.1.20) and the limit absorption principle, we get

$$\hat{u}(\lambda, x) = -R_V^{\pm}\left(\lambda^2\right)\hat{F}(\lambda, x), \tag{3.4.65}$$

and, consequently,

$$\nabla\hat{u}(\lambda, x) = -\nabla R_V^{\pm}\left(\lambda^2\right)\hat{F}(\lambda, x). \tag{3.4.66}$$

Now we can use (3.3.59) and obtain

$$\||x|^{-\frac{1}{2}}W_\delta^{-1}\nabla\hat{u}(\lambda, x)\|_{L^2}^2 \le C\||x|^{\frac{1}{2}}W_\delta\hat{F}(\lambda, x)\|_{L^2}^2. \tag{3.4.67}$$

Integrating over λ and using the Plancherel identity in time variable, we have

$$\||x|^{-\frac{1}{2}}W_\delta^{-1}\nabla u(x, t)\|_{L_t^2L_x^2} \le C\||x|^{\frac{1}{2}}W_\delta F(x, t)\|_{L_t^2L_x^2}. \tag{3.4.68}$$

Proof of (3.1.10)*.* We use, after the Fourier transform, the identity (3.4.65), the Theorem 3.4 and the perturbed resolvent estimate (3.3.58).

Proof of (3.1.11)*.* The application of the Fourier transform yelds

$$V(x, D)\hat{u}(\lambda, x) = V(x, D)R_V(\lambda^2)\hat{F}(\lambda, x), \tag{3.4.69}$$

and using the estimate (3.3.62) we have

$$\||x|^{\frac{1}{2}}W_\delta V(x, D)\hat{u}(\lambda, x)\|_{L^2} \le C\||x|^{\frac{1}{2}}W_\delta\hat{F}(\lambda, x)\|_{L^2}, \tag{3.4.70}$$

and, consequently,

$$\||x|^{\frac{1}{2}}W_\delta V(x, D)u(x, t)\|_{L_t^2L_x^2} \le C\||x|^{\frac{1}{2}}W_\delta F(x, t)\|_{L_t^2L_x^2}. \tag{3.4.71}$$

REMARK 3.3. *The constants in* (3.1.9)*,* (3.1.10)*,* (3.1.11) *are all independent of* λ*.*

Case 2. Dirac equation.

The Dirac equation can be treated as the wave equation. In fact we write the solution of (3.1.3) as the following integral equation:

$$u = \int_0^t U(t-s)F(u(s), V(x, D))ds, \tag{3.4.72}$$

where $F(u(s), V(x, D)) = -a\cdot\nabla u + F(t, x)$ and $U(t)$ denote the propagator of the free Dirac equation given by

$$U(t) = cos(t\sqrt{-\Delta}) - \gamma_0(\gamma^j\partial_j)\frac{sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}. \tag{3.4.73}$$

A reduction to the wave equation can be done by applying the operator $\Box$ to the solution (3.4.72) and using the relation

$$\partial_\mu \partial^\mu u = 0. \tag{3.4.74}$$

So the estimates (3.1.9), (3.1.10) and (3.1.11) remain valid.

Proof of Theorem 3.2.

The proof of non-homogeneous case (3.1.2) is the analogous to that one for the perturbed wave equation (3.1.1). However one has to replace λ^2 by $\lambda > 0$ in the definitions (3.1.20), (3.1.21), (3.2.25) and in the estimates for free and perturbed resolvent in the Section 2 and 3 of this chapter.

Proof of Theorem 3.3.

For the homogeneous case the TT^* argument, see [**72, 75**] and [**115**], combined with the estimates (3.1.9) implies (3.1.16).

$\Box$

REMARK 3.4. *By the definition of the perturbed Besov space we have* $\dot{H}^s_V = \dot{B}^s_{V,2,2}$, *for any* $s \in \mathbb{R}$, *so we can replace* $\dot{H}^{1/2}_V$ *by* $\dot{B}^{1/2}_{V,2,2}$ *in the* (3.1.16).

REMARK 3.5. *One can also consider the following Cauchy problems for the perturbed wave equation and Dirac equation:*

$$\begin{cases} \Box u + ia(x) \cdot \nabla u + V(x)u = 0, \\ u(0,x) = f, \partial_t u(0,x) = g, \end{cases} \tag{3.4.75}$$

and

$$\begin{cases} i\gamma_\mu \partial_\mu u + ia(x) \cdot \nabla u + V(x)u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \\ u(0,x) = f. \end{cases} \tag{3.4.76}$$

As in the case of Schödinger equation, the TT^* *argument combined with the estimates* (3.1.9) *implies for the problem* (3.4.75), *for any* $\delta > 0$, *the following estimate:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \le C(\|f\|_{\dot{H}^1_V} + \|g\|_{L^2}),$$

and for the problem (3.4.76), *for any* $\delta > 0$, *the following estimate:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \le C\|f\|_{\dot{H}^1_V},$$

where, in the previous estimates, we used the $L^2 - L^2$ *boundness of the operator* $\frac{\nabla}{\sqrt{-\Delta_V}}$ *given by the Corollary 3.2 from the Appendix.*

5. Appendix

In the following we prove a relevant equivalence result. Namely,

PROPOSITION 3.2. *Under the assumptions* (3.1.5) *and* (3.1.6) *for the operator* $V = V(x,D)$, *there are constants constants* $c_1, c_2 > 0$ *such that*

$$c_1 \|\nabla u\|^2_{L^2} \le (-\Delta_V u, u) \le c_2 \|\nabla u\|^2_{L^2},$$

where $(-\Delta_V u, u)$ *is the quadratic form* (3.1.22).

Proof. First of all we notice that by the assumptions (3.1.5) and (3.1.6) we immediately have

$$|a_j(x)| \in L^n(\mathbb{R}^n),\ j = 1, \dots, n \qquad |V(x)| \in L^{\frac{n}{2}}(\mathbb{R}^n),$$

(we prove the statement without restriction on n because it remains valid for any dimension of the space $\mathbb{R}^n$). The quadratic form (3.1.22) is well defined on $\dot{H}^1(\mathbb{R}^n)$. In fact using the embedding $\dot{H}^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}$ and the Hölder inequality we have

$$\begin{aligned}(-\Delta_V u, u) \quad &\leq \quad \|ia \cdot \nabla u\|_{L^2} + \|V(x)u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \|a\|_{L^n} \|(\nabla u, u)\|_{L^{\frac{n}{n-1}}} + \|V(x)u\|_{L^2}^2 \leq \\ &\leq \quad C\|\nabla u\|_{L^2}^2 + C\delta \|\nabla u\|_{L^2} \|u\|_{L^{\frac{2n}{n-2}}} + \|V(x)u\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^2 + C\|V(x)u\|_{L^2}^2\end{aligned} \tag{3.5.77}$$

and the quadratic form is that $q(u,u) := \|\nabla u\|_{L^2}^2 + \|V(x)u\|_{L^2}^2 \approx \|\nabla u\|_{L^2}^2$ by [**70**, Theorem 7.3, 7.4]. So we obtain

$$(-\Delta_V u, u) \leq C\|\nabla u\|_{L^2}^2 + C\|V(x)u\|_{L^2}^2. \tag{3.5.78}$$

The form (3.5.77) is symmetric and to prove that $(-\Delta_V u, u)$ is associated to a (unique) self-adjoint operator, it will be sufficient to show that it is **closed**, namely its domain is complete with respect the norm $(-\Delta_V u, u)$, and **semibounded**, namely

$$(-\Delta_V u, u) \geq -C\|\nabla u\|_{L^2}^2$$

with $C > 0$ (see [**157**, Theorem VIII.15]). Both properties follow from the definition of (3.1.22) (see [**128**]), moreover by estimating as (3.1.22) and using again the result proved in [**70**] cited above we obtain

$$(-\Delta_V u, u) \geq C\|\nabla u\|_{L^2}^2 - C\delta\|\nabla u\|_{L^2}^2. \tag{3.5.79}$$

In particular this implies that the norm $(-\Delta_V u, u)$ in **equivalent** to the norm in $\dot{H}^1(\mathbb{R}^n)$. This completes the proof. □

An important consequence of the above proposition is the following result:

COROLLARY 3.1. *Let $\dot{H}^s(\mathbb{R}^n)$ denote the scale of homogeneous Sobolev spaces based on the powers of the operator $V = V(x, D)$, i.e., the completion of $C_0^\infty(\mathbb{R})$ with respect to the seminorm*

$$\|u\|_{\dot{H}_V^s} := \left\|(-\Delta_V)^{1/2}u\right\|_{L^2}.$$

If $V = V(x, D)$ satisfies (3.1.5) *and* (3.1.6) *then the spaces are equivalent to the standard Sobolev spaces (based on the powers of $-\Delta$) for $|s| \leq 1$.*

Proof. For $s = 1$ this follows immediately from the above Proposition, noting that $(-\Delta_V u, u) = \left\|(-\Delta_V)^{1/2}u\right\|_{L^2}^2$. The case $s = -1$ then follows by duality, and by interpolation we get the statement for $s \in [-1, 1]$ in between. □

We have also the following corollary

COROLLARY 3.2. *The operator* $\frac{\nabla}{\sqrt{-\Delta_V}}$, *where* ∇ *is the gradient on* $\mathbb{R}^n$ *and* $-\Delta_V$ *is defined by* (3.1.18)*, satisfies the following estimate:*

$$\left\| \frac{\nabla}{\sqrt{-\Delta_V}} f \right\|_{L^2} \leq C\|f\|_{L^2}, \; f \in L^2. \tag{3.5.80}$$

Proof. One can rewrite the left side of the inequality (3.5.80) as

$$\left(\frac{\nabla}{\sqrt{-\Delta_V}} f, \frac{\nabla}{\sqrt{-\Delta_V}} f \right)^{1/2}. \tag{3.5.81}$$

Setting the (3.5.81) $g = \frac{1}{\sqrt{-\Delta_V}} f$, we obtain, using (3.5.79)

$$(\nabla g, \nabla g) \quad \leq \quad C(-\Delta_V f, f).$$

So (3.5.80) is established. □

REMARK 3.2. *The Corollary 3.1 shows also the fact that we can replace in the estimate* (3.1.16) *of Theorem 3.3 and in the estimates in Remark 3.5 the homogeneous perturbed Sobolev space* $\dot{H}^s_V$ *by the equivalent space* $\dot{H}^s$. *So in the following chapter we will use only the* $\dot{H}^s$ *norms.*

CHAPTER 4

Scale invariant energy smoothing estimates for the Schrödinger Equation and applications

1. Introduction

In this chapter we study smoothing properties of the Schrödinger equation with magnetic potential

$$A = (A_1(t,x), \cdots, A_n(t,x)), \quad x \in \mathbb{R}^n.$$

Here $A_j(t,x), j = 1, \cdots, n$, are real valued functions, $n \geq 3$ and the corresponding Cauchy problem for the Schrödinger equation has the form

$$\begin{cases} \partial_t u - i\Delta_A u = F, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(0,x) = f(x), \end{cases} \tag{4.1.1}$$

where

$$\Delta_A = \sum_{j=1}^{n} (\partial_{x_j} - iA_j)(\partial_{x_j} - iA_j). \tag{4.1.2}$$

The energy type estimates and the well – posedness of the Cauchy problem (4.1.1) in the energy space are studied in the works [**54**] and [**55**] of Doi.

Since the smoothing properties of this evolution problem are closely connected with suitable resolvent estimates for the solution $U = U(x)$ of the elliptic problem

$$\Big\{ \varepsilon U - i\Delta_A U - i\tau U = H, \quad \varepsilon > 0, \tau > 0, \quad x \in \mathbb{R}^n, H = H(x), \tag{4.1.3}$$

we can use as a starting point the scale invariant smoothing estimate obtained in the works of Kenig, Ponce, Vega [**117**] and Perthame, Vega [**149**]. This estimate extends earlier works of Agmon, Hörmander [**8**] and P. Constantin and J.-C. Saut [**42**].

The scale invariant estimate for (4.1.3) with $A = 0$ has the form

$$|||\nabla_x U||| \leq CN(H), \tag{4.1.4}$$

where $C > 0$ is independent of $\varepsilon > 0, \tau > 0$,

$$|||G|||^2 = \sup_{R>0} \frac{1}{R} \int_{|x| \leq R} |G(y)|^2 dy$$

is the Morrey - Campanato norm, while

$$N(H) = \sum_{k \in \mathbb{Z}} 2^{k/2} \|H\|_{L^2(2^{k-1} \leq |x| \leq 2^{k+1})}.$$

From this estimate one can derive the estimate

$$\sup_{k\in\mathbb{Z}} 2^{-k/2}\|G\|_{L^2(2^{k-1}\le|x|\le 2^{k+1})} \le C\|G\|$$

and the following smoothing scale invariant estimate for the solution $u(t,x)$ to **(4.1.1)** with $A=0$ and $f=0$:

$$\int_{\mathbb{R}}\left(\sup_{k\in\mathbb{Z}} 2^{-k/2}\|\nabla_x u(t,\cdot)\|_{L^2(2^{k-1}\le|x|\le 2^{k+1})}\right)^2 dt \le \tag{4.1.5}$$
$$\le C\int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}} 2^{k/2}\|F(t,\cdot)\|_{L^2(2^{k-1}\le|x|\le 2^{k+1})}\right)^2 dt.$$

Our purpose in this Chapter is to derive similar scale invariant smoothing estimates for the case of magnetic potential imposing scale invariant smallness assumptions on the magnetic potential $A(x)$. As in the Chapter 3 we treat only small magnetic potential. This connected with the necessity to avoid resonances phenomena (see Chapter 1, Section 2 and Chapter 3 for more details) (for magnetic perturbation see [**45**], [**47**], [**146**], [**155**], [**179**], [**183**], [**200**]). The absence of eigenvalues of Δ_A with magnetic potential decaying as $(1+|x|)^{-1-\delta}$ is discussed in [**23**]. However, even the remarkable result in [**23**] can not guarantee that 0 is not an eigenvalue of the Hamiltonian Δ_A. The result in [**84**] shows that even nontrivial smooth compactly supported magnetic field can create resonances. Here we follow our work [**69**].

To avoid possible eigenvalues or resonances of Δ_A we impose the following assumption on the potential A.

ASSUMPTION 1.1. *There exists $\varepsilon>0$, such that we have*

$$\max_{1\le j\le n}\sum_{k\in\mathbb{Z}}\sum_{|\beta|\le 1} 2^{k(1+|\beta|)}\ \|D_x^\beta A_j(t,x)\|_{L_t^\infty L^\infty(|x|\sim 2^k)} \le \varepsilon. \tag{4.1.6}$$

Our main smoothing estimate is the following one.

THEOREM 4.1. *There exists $\varepsilon>0$ so that for any potential $A(x)$ satisfying* (4.1.6) *there exists $C>0$, so that for any $f\in S(\mathbb{R}^n)$ and any $F(t,x)\in C_0^\infty(\mathbb{R}\times(\mathbb{R}^n\setminus 0))$ the solution $u(t,x)$ to* (4.1.1) *satisfies the estimate*

$$\int_{\mathbb{R}}\left(\sup_{k\in\mathbb{Z}}\||x|_k^{-1/2}u(t,\cdot)\|_{\dot H_x^{1/2}}\right)^2 dt \ \le\ C\|f\|^2_{L^2_x} +$$
$$+\ C\int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}}\||x|_k^{1/2}F(t,\cdot)\|_{\dot H_x^{-1/2}}\right)^2 dt, \tag{4.1.7}$$

where $\dot H^s_x=\dot H^s(\mathbb{R}^n)$ is the classical homogeneous Sobolev space and $|x|_k^{\pm 1/2}=|x|^{\pm1/2}Q_k(x)$ and the Paley - Littlewood partition of unity

$$1=\sum_{k\in\mathbb{Z}} Q_k(x), \tag{4.1.8}$$

is defined as follows

$$Q_k(x)=\varphi\left(\frac{|x|}{2^k}\right),$$

where $\varphi(s)\in C_0^\infty((1/2,2))$ is a non - negative function.

The key point to derive this estimate is a suitable scale and time invariant smoothing estimate for the free Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta u = F, \quad t>0, \quad x\in\mathbb{R}^n, \\ u(0,x) = f(x). \end{cases} \tag{4.1.9}$$

To be more precise, we introduce the following norms motivated by the statement of the main result in Theorem 4.1. Take

$$Y = L^2_t(\ell^{1,1/2}_x \dot{H}^{-1/2}), \ \ Y' = L^2_t(\ell^{\infty,-1/2}_x \dot{H}^{1/2}), \tag{4.1.10}$$

where the spaces $\ell^{q,\alpha}_x B$ for any Banach space B is introduced in Section 2. Note that Y is not reflexive ($(\ell^{1,1/2}_x)' = \ell^{\infty,-1/2}_x$, but $(\ell^{\infty,-1/2}_x)' \neq \ell^{1,1/2}_x$).

Then the estimate of the previous theorem can be rewritten in the form

$$\|u\|^2_{Y'} \leq C\|f\|^2_{L^2} + \|F\|^2_Y, \tag{4.1.11}$$

and we shall call Y' smoothing space.

Then the main point in the proof of Theorem 4.1 is to establish first the following energy smoothing estimate for the case $A=0$.

THEOREM 4.2. *There exists $C>0$, such that for any $f\in S(\mathbb{R}^n)$ and any $F(t,x)\in C_0^\infty(\mathbb{R}\times(\mathbb{R}^n\setminus 0))$ the solution $u(t,x)$ to* (4.1.9) *satisfies the estimate*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{Y'} \leq C\|f\|_{L^2_x} + C\left(\min_{F=F_1+F_2} \|F_1\|_Y + \|F_2\|_{L^1_t L^2_x}\right). \tag{4.1.12}$$

On the basis of the estimate in Theorem 4.2 we shall derive a slightly stronger estimate for the perturbed Schrödinger equation.

COROLLARY 4.1. *There exists $\varepsilon>0$ so that for any potential $A(x)$ satisfying* (4.1.6) *there exists $C>0$, so that for any $f\in S(\mathbb{R}^n)$ and any $F(t,x)\in C_0^\infty(\mathbb{R}\times(\mathbb{R}^n\setminus 0))$ the solution $u(t,x)$ to* (4.1.1) *satisfies the estimate*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{Y'} \leq C\|f\|_{L^2_x} + C\left(\min_{F=F_1+F_2} \|F_1\|_Y + \|F_2\|_{L^1_t L^2_x}\right). \tag{4.1.13}$$

As an application we consider the following semilinear Schrödinger equation:

$$\begin{cases} \partial_t u - i\Delta_A u = |V(t,x)u|^p, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n \\ u(0,x) = f(x), \end{cases} \tag{4.1.14}$$

where $p>1$ and $V(t,x)$ is a measurable function satisfying the inequality

$$\sum_{k\in\mathbb{Z}} 2^{ka} \|V(t,x)\|_{L^\infty_t L^\infty(|x|\sim 2^k)} \leq C < \infty. \tag{4.1.15}$$

Then we have the following global existence result with initial data having small L^2- norm only.

THEOREM 4.3. *Suppose that the potential $A(x)$ satisfies* (4.1.6)*, V obeys* (4.1.15) *with $a\in[1,2)$ and*

$$p = \frac{n+4}{n+2a}. \tag{4.1.16}$$

Then there exists $\delta > 0$, *so that for any* $f \in L^2(\mathbb{R}^n)$ *with*

$$\|f\|_{L^2} \leq \delta$$

the problem (4.1.14) *has a unique global solution*

$$u(t,x) \in C(\mathbb{R}, L^2(\mathbb{R}^n)) \cap Y'.$$

The proof of Theorem 4.2 is based on the estimate (4.1.5) due to Kenig, Ponce, Vega [**117**], [**118**]. In order to have a self contained presentation we give an alternative proof of this result due to Konig, Ponce, Vega in Section 8.

The key step to derive the estimate (4.1.12) from the estimate (4.1.5) is the following equivalence norm result.

THEOREM 4.4. *For* $n \geq 3, 1 < q < \infty$, *for* $s \in [-1,1]$ *and* $a \in \mathbb{R}$ *that satisfy*

$$|a| + |s| < \frac{n}{2}, \tag{4.1.17}$$

the following norms are equivalent

$$\left(\sum_{k\in\mathbb{Z}} 2^{qka} \|Q_k |D|^s f\|^q_{L^2}\right)^{1/q},$$
$$\left(\sum_{k\in\mathbb{Z}} 2^{qka} \| \, |D|^s Q_k f\|^q_{L^2}\right)^{1/q}, \tag{4.1.18}$$
$$\left(\sum_{k\in\mathbb{Z}} \| \, |D|^s |x|^a_k f\|^q_{L^2}\right)^{1/q},$$

where $|x|^a_k = |x|^a Q_k(x)$ *and the Paley - Littlewood partition of unity* $Q_k(x)$ *is defined in* (4.1.8). *For* $q = \infty$ *the result is still valid with obvious modification in* (4.1.18).

The main idea to establish the theorem is similar to the approach developed in [**63**], [**48**] and [**66**] for the case of nonhomogeneous Sobolev spaces and non homogeneous weights. Therefore, we shall make a localization in the coordinate space and we shall use the Paley Littlewood partition (4.1.8). The key point in this approach is to evaluate the norm of the operator of type $Q_k|D|^{-s}Q_m|D|^s$ with $|k-m|$ large enough.

The proof of Theorem 4.2 can be obtained from the estimate for the Cauchy problem with initial data $f = 0$ and the following theorems (see section 7 for the definition of the spaces $\ell^{r,s}_D B$ for any Banach space B).

THEOREM 4.5. *If* $q \in [1,2]$ *and* $a, s \in \mathbb{R}$ *satisfy*

$$\begin{cases} |s| \leq 1, \\ |a| + |s| < \frac{n}{2}, \end{cases} \tag{4.1.19}$$

then

$$\|f\|_{\ell^{2,0}_D \ell^{q,a}_x \dot{H}^s} \leq C \|f\|_{\ell^{q,a}_x \dot{H}^s}. \tag{4.1.20}$$

THEOREM 4.6. *If* $q \in [2,\infty]$ *and* $a, s \in \mathbb{R}$ *satisfy* (4.1.19)*, then*

$$\|f\|_{\ell_x^{q,a}\dot{H}^s} \leq C\|f\|_{\ell_D^{2,0}\ell_x^{q,a}\dot{H}^s}. \tag{4.1.21}$$

The plan of this chapter is the following. The proof of the free smoothing estimate of Theorem 4.2 is given in Section 2. The proof of the main scale invariant smoothing estimate of Theorem 4.1 is done in Section 3. In Section 4 we treat the commutator estimates needed in the proof the equivalence of the norms in Theorem 4.4. Some convolution type inequalities needed in the proofs of Theorem 4.4 are included in Section 5. The concluding steps in the proof of Theorem 4.4 are presented in Section 6. Finally the phase localization and the proofs of Theorems 4.5 and (4.6) are given in the last Section 7. The proof of the estimate due to Kenig, Ponce, Vega is presented in Section 8 for self contained completeness.

2. Weighted Sobolev space estimates for the free Schrödinger equation and proof of Theorem 4.2

Given any Banach space $B \subset D'(\mathbb{R}^n)$ satisfying the property

$$\text{for any } Q(x) \in C_0^\infty(\mathbb{R}^n),\ \ f \in B \Rightarrow Q(x)f \in B, \tag{4.2.22}$$

we can define for any $q \in [1,\infty]$ and for any $\alpha \in \mathbb{R}$ the space $\ell_x^{q,\alpha}B$ as follows

$$\|f\|_{\ell_x^{q,\alpha}B} = \left(\sum_{k\in\mathbb{Z}} \|Q_k f\|_B^q 2^{kq\alpha}\right)^{1/q}, \tag{4.2.23}$$

with obvious modification for $q=\infty$. Note that for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$\|f\|_{\ell_x^{q,\alpha}B} < \infty.$$

So $\ell_x^{q,\alpha}B$ can be defined as the closure of $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to the norm (4.2.23). An alternative definition is based on the map

$$J : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset B \to J_B(f)_k = \|Q_k f\|_B \in \ell^{q,\alpha}, \tag{4.2.24}$$

where $\ell^{q,\alpha}$ is the space of all sequences $a = (a_k)_{k\in\mathbb{Z}}$ such that

$$\|a\|_{\ell^{q,\alpha}} = \left(\sum_{k\in\mathbb{Z}} \|a_k\|^q 2^{kq\alpha}\right)^{1/q} < \infty, \tag{4.2.25}$$

with obvious modification for $q=\infty$ (see Chapter 1, Section 9 for more details). Then

$$\|f\|_{\ell_x^{q,\alpha}B} = \|J_B(f)\|_{\ell^{q,\alpha}}. \tag{4.2.26}$$

The space $\ell_x^{q,\alpha}B$ is independent of the concrete choice of Paley-Littlewood decomposition

$$\sum_{j\in\mathbb{Z}} Q_j(x) = 1 \tag{4.2.27}$$

satisfying

$$\begin{cases} Q_j(x) \geq 0, \\ supp\, Q_j(x) \in \{|x| \sim 2^j\}. \end{cases} \tag{4.2.28}$$

A typical example, wich is needed for the smoothing resolvent type estimates, is the case $B = \dot{H}^s_p$ where $s \in (-1,1),\ 1 < p < \infty$. For $s > -\frac{n}{p}$ we have $\dot{H}^s_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ (see [**48**]) and the norm is defined by

$$\|f\|_{\dot{H}^s_p} = \||D|^s f\|_{L^p}. \tag{4.2.29}$$

After this preparation we can turn to the proof of Theorem 4.2. Starting with the estimate (4.1.5), we use Lemma 4.12 (see Section 6) and find

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}\nabla u\|_{L^2_tL^2_x} \sim \sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|u\|_{L^2_tL^2_x}$$

so (4.1.5) can be rewritten as

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|u\|_{L^2_tL^2_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}F\|_{L^2_tL^2_x}\right). \tag{4.2.30}$$

Using the fact that $|D|^s$ commutes with Δ one can obtain the following consequence of this estimate

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|^{1-\sigma}u\|_{L^2_tL^2_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}|D|^{\sigma}F\|_{L^2_tL^2_x}\right) \tag{4.2.31}$$

for any $\sigma \in [0,1]$. In particular for $\sigma = 1/2$ we get

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|^{1/2}u\|_{L^2_tL^2_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}|D|^{1/2}F\|_{L^2_tL^2_x}\right). \tag{4.2.32}$$

To this end, we are in position to apply the result of Proposition 4.1 (see Section 6) and derive that

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|^{1/2}u\|_{L^2_tL^2_x} \sim \|u\|_{L^2_t\ell^{\infty,-1/2}_x\dot{H}^{1/2}_x}, \tag{4.2.33}$$

so

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}|D|^{1/2}u\|_{L^2_tL^2_x} \sim \sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}u\|_{L^2_t\dot{H}^{1/2}_x}.$$

In a similar way Proposition 4.1 implies

$$\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}|D|^{-1/2}F\|_{L^2_tL^2_x} \sim \|f\|_{L^2_t\ell^{1,1/2}_x\dot{H}^{-1/2}_x}, \tag{4.2.34}$$

so

$$\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}|D|^{-1/2}F\|_{L^2_tL^2_x} \sim \sum_{k\in\mathbb{Z}} \||x|_k^{1/2}F\|_{L^2_t\dot{H}^{-1/2}_x}.$$

The estimate (4.2.32) reads as

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2}u\|_{L^2_t\dot{H}^{1/2}_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2}F\|_{L^2_t\dot{H}^{-1/2}_x}\right) \tag{4.2.35}$$

or using the notations of this section (see (4.1.10) and the definition (4.2.26)) as

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)ds\right\|_{Y'} \le C\|F\|_Y. \tag{4.2.36}$$

It is easy to derive a similar estimate

$$\left\|\int_t^\infty e^{i(t-s)\Delta}F(s)ds\right\|_{Y'} \le C\|F\|_Y, \tag{4.2.37}$$

by the aid of (4.2.36) and a duality argument for the quadratic form

$$Q(F,G) = \int\int_{t>s} \langle e^{i(t-s)\Delta}F(s), G(t)\rangle_{L^2(\mathbb{R}^n)} ds dt.$$

Further, we have to derive the estimate

$$\left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{L^\infty_t L^2_x} \leq C\|F\|_Y. \tag{4.2.38}$$

For that purpose set $u(t) = \int_0^t e^{i(t-s)\Delta}F(s)ds$. Then $u = u(t,x)$ is a solution to

$$\begin{cases} \partial_t u - i\Delta u = F, \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0 \end{cases} \tag{4.2.39}$$

Multiplying by u and integrating over $\{0 \leq t \leq T, x \in \mathbb{R}^n\}$ we get

$$\|u(T)\|^2_{L^2(\mathbb{R}^n)} \leq \int_0^T \langle F(t), u(t)\rangle_{L^2(\mathbb{R}^n)} dt \leq \|F\|_Y \|u\|_{Y'}.$$

Applying (4.2.36), we arrive at (4.2.38). In a similar way we get

$$\left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{Y'} \leq C\|F\|_{L^1_t L^2_x}. \tag{4.2.40}$$

Finally, it remains to prove

$$\left\| e^{it\Delta}f \right\|_{Y'} \leq C\|f\|_{L^2_x}. \tag{4.2.41}$$

Consider the operator L defined by

$$L : f \in L^2_x \implies e^{it\Delta}f.$$

Our goal is to show that L is bounded from L^2_x to $L^2_t \ell^{\infty,-1/2}_x \dot{H}^{1/2}_x$. But the continuity of L from L^2_x to $L^2_t \ell^{\infty,-1/2}_x \dot{H}^{1/2}_x$ is follows from the continuity of its (formally) adjoint

$$L^* f = \int_0^\infty e^{-i\tau\Delta} f(\tau) d\tau,$$

from Y to L^2_x, which in turns follows from (4.2.38) and the fact that $e^{it\Delta}$ is a unitary operator in L^2.

From (4.2.35), (4.2.38), (4.2.40), (4.2.41) and standard energy estimate, we get (4.1.13) and the proof of Theorem 4.2 is completed.

3. Proof of Theorem 4.1

In this section we will prove the Theorem 4.1, so we shall prove the estimate (4.1.7), where u is the solution of the problem (4.1.1). First of all we have the identities

$$\begin{aligned} \Delta_A u &= \sum_{j=1}^n (\partial_{x_j} - iA_j)(\partial_{x_j} - iA_j)u \\ &= \Delta u - 2i\nabla \cdot (Au) + Wu, \end{aligned} \tag{4.3.42}$$

where

$$W(t,x) = |A(t,x)|^2 - i\nabla \cdot A$$

satisfies

$$\sum_{k\in\mathbb{Z}} 2^{2k} \|W(t,x)\|_{L_t^\infty L^\infty_{\{|x|\sim 2^k\}}} \leq \varepsilon$$

due to (4.1.6).

So, after substitution in the equation of (4.1.1), we obtain

$$\begin{cases} i\partial_t u - \Delta u = -2i\nabla\cdot(Au) + Wu + F \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n \\ u(0,x) = f(x). \end{cases} \tag{4.3.43}$$

First of all, we observe that the term Wu, thanks to the smallness assumption (1.1), can be absorbed in the left hand side of the estimate (4.1.7). This fact suggests to localize our attention to the reduced problem

$$\begin{cases} \partial_t u - i\Delta u = -2i\nabla\cdot(Au) + F, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n \\ u(0,x) = f(x). \end{cases} \tag{4.3.44}$$

So, using the norms in the spaces $\ell_x^{p,\alpha}B$ introduced in (4.2.23) we apply the estimate (4.1.7) and obtain

$$\|u\|_{L_t^2\ell_x^{\infty,-1/2}\dot H_x^{1/2}} \leq C\|\nabla\cdot(Au)\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{-1/2}} + C\|F\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{-1/2}} + C\|f\|_{L^2}. \tag{4.3.45}$$

From the equivalent norm estimates in Proposition 4.1 (see Section 6 and the equivalence norms relations in (4.2.33), (4.2.34) also) we have

$$\|\nabla\cdot(Au)\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{-1/2}} \sim \|Au\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{1/2}}. \tag{4.3.46}$$

From Proposition 4.2 we have

$$\|Au\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{1/2}} \lesssim \|A\|_{L_t^\infty\ell_x^{1,1}\dot H_{2n}^{1/2}}\|u\|_{L_t^2\ell_x^{\infty,-1/2}L_x^{2n/(n-1)}} +$$
$$+\|A\|_{L_t^\infty\ell_x^{1,1}L_x^\infty}\|u\|_{L_t^2\ell_x^{\infty,-1/2}\dot H_x^{1/2}}.$$

From the Sobolev embedding $\dot H_x^{1/2}\subset L_x^{2n/(n-1)}$, we obtain

$$\|u\|_{L_t^2\ell_x^{\infty,-1/2}L_x^{2n/(n-1)}} \lesssim \|u\|_{L_t^2\ell_x^{\infty,-1/2}\dot H_x^{1/2}},$$

while the interpolation inequality of Proposition 4.3 (see Section 6) guarantees that

$$\|A\|^2_{L_t^\infty\ell_x^{1,1}\dot H_{2n}^{1/2}} \lesssim \|\nabla A\|_{L_t^\infty\ell_x^{1,3/2}L^{2n}}\|A\|_{L_t^\infty\ell_x^{1,1/2}L^{2n}}$$

so applying the Hölder inequality

$$\|g\|_{\ell_x^{1,a}L^p} \lesssim \|g\|_{\ell_x^{1,a+n/p}L^\infty},$$

we get

$$\|A\|^2_{L_t^\infty\ell_x^{1,1}\dot H_{2n}^{1/2}} \lesssim \|\nabla A\|_{L_t^\infty\ell_x^{1,2}L^\infty}\|A\|_{L_t^\infty\ell_x^{1,1}L^\infty} \leq \varepsilon^2$$

due to assumption on A. The above observation implies

$$\|Au\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{1/2}} \lesssim \varepsilon\|u\|_{L_t^2\ell_x^{\infty,-1/2}\dot H_x^{1/2}}.$$

Using again the estimates (4.3.45), we obtain

$$\sup_{k\in\mathbb{Z}}\||x|_k^{-1/2}u(t,\cdot)\|_{L_t^2\dot H_x^{1/2}} \leq C\|F\|_{L_t^2\ell_x^{1,1/2}\dot H_x^{-1/2}} + C\|f\|_{L_x^2}. \tag{4.3.47}$$

This concludes the proof of the theorem.

4. Estimate of the Operator $Q_k|D|^{-s}Q_m|D|^s$.

Our goal is to compare the norms

$$\| \, |D|^{-s}f\|_{\ell^{q,a}_x L^p} = \left(\sum_{k\in\mathbb{Z}} 2^{kqa} \|Q_k|D|^{-s}f\|^q_{L^p} \right)^{1/q}$$

and

$$\|f\|_{\ell^{q,a}_x \dot{H}^s_p} = \left(\sum_{k\in\mathbb{Z}} 2^{kqa} \||D|^{-s}Q_k f\|^q_{L^p} \right)^{1/q}$$

(see Section 8.1 in the Chapter 1 or Section 2 of this chapter for the definition of the spaces $\ell^{q,\alpha}_x B$, where B is any Banach space such that $B \subset D'(\mathbb{R}^n)$). The key point in the proof that these norms are equivalent is the following estimate for the operator

$$Q_k|D|^{-s}Q_m|D|^s \quad \text{for } |k-m| > 2. \tag{4.4.48}$$

LEMMA 4.1. *For any $s \in \mathbb{R}, |s| < 1$, any $p, 1 < p < n$ and any $k, m \in \mathbb{Z}, |k-m| \geq 3$ we have the estimate*

$$\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C \; 2^{t(k,m,s,p)} \; \|f\|_{L^p}, \tag{4.4.49}$$

where $C = C(s,p)$ independent of $k, m \in \mathbb{Z}$, and

$$t(k,m,s,p) = k\frac{n}{p} + m\frac{n}{p'} - (n - (s \vee 0))(k \vee m) - (s \vee 0))(k \wedge m), \tag{4.4.50}$$

$\frac{1}{p'} = 1 - \frac{1}{p}$, $k \wedge m = min(k,m)$, $k \vee m = max(k,m)$.

Proof: First we shall prove the lemma for $s \in \mathbb{C}$ with $\operatorname{Re} s \in [0,1]$. For that purpose consider the family of operators

$$T^z = e^{z^2} Q_k|D|^{-z}Q_m|D|^z. \tag{4.4.51}$$

If $\operatorname{Re} z = 0$, then $z = i\sigma$, $\sigma \in \mathbb{R}$ and

$$T^{i\sigma} = e^{-\sigma^2} Q_k|D|^{-i\sigma}Q_m|D|^{i\sigma}. \tag{4.4.52}$$

Applying stationary phase method (in this case simply integration by parts), we see that the operator $Q_k|D|^{-i\sigma}Q_m$ has the kernel

$$K_{k,m,\sigma}(x,y)$$

satisfying

$$|K_{k,m,\sigma}(x,y)| \leq C\frac{Q_k(x)Q_m(y)}{2^{n(k\vee m)}}(1+\sigma)^{n+1}. \tag{4.4.53}$$

This estimate implies

$$\|Q_k|D|^{-i\sigma}Q_m g\|_{L^p} \leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}\|g\|_{L^p}. \tag{4.4.54}$$

Further we apply this inequality with $g = |D|^{i\sigma}f$ and using the following one (see Theorem 1, Section 2.2 in **[185]**)

$$\||D|^{i\sigma}f\|_{L^p} \leq C\|f\|_{L^p}(1+\sigma)^{n+1}, \tag{4.4.55}$$

we get

$$\begin{aligned}\|T^{i\sigma}(f)\|_{L^p} &\leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}e^{-\sigma^2}(1+\sigma)^{2(n+1)}\|f\|_{L^p}\leq \\ &\leq C_1\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}\|f\|_{L^p} \end{aligned} \tag{4.4.56}$$

$\forall\sigma\in\mathbb{R}$ with C_1 independent of k, m and σ. If $z = 1 + i\sigma$, then

$$T^{1+i\sigma} = e^{1-\sigma^2+2i\sigma}Q_k|D|^{-1-i\sigma}Q_m\nabla|D|^{i\sigma}\frac{\nabla}{|D|}. \tag{4.4.57}$$

So it is sufficient to estimate the operator

$$S^\sigma = Q_k|D|^{-1-i\sigma}Q_m\nabla. \tag{4.4.58}$$

Note that

$$S^\sigma = Q_k(|D|^{-1-i\sigma}\nabla)Q_m - Q_k|D|^{-1-i\sigma}Q'_m,$$

where $Q'_m = \nabla Q_m$. The operator $Q_k(|D|^{-1-i\sigma}\nabla)Q_m$ has kernel $K'_{k,m}$ satisfying

$$|K'_{k,m}| \leq C\frac{Q_k(x)Q_m(y)}{2^{n(k\vee m)}}(1+\sigma)^{n+1}. \tag{4.4.59}$$

This estimate is verified in the same way as (4.4.53). The operator $Q_k|D|^{-1-i\sigma}Q'_m$ has the kernel $K''_{k,m}$ that satisfies the estimate

$$|K''_{k,m}| \leq C\frac{Q_k(x)Q_m(y)}{2^{(n-1)(k\vee m)}2^m}(1+\sigma)^{n+1}. \tag{4.4.60}$$

From (4.4.59) and (4.4.60) together with (4.4.55) we find

$$\|T^{1+i\sigma}(f)\|_{L^p} \leq C\ \frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{(n-1)(k\vee m)}2^{k\wedge m}}\|f\|_{L^p}. \tag{4.4.61}$$

Applying the complex interpolation argument of Stein (see [**186**]), we get (4.4.49) for $0 < s < 1$. This complete the proof for $s \in (0, 1)$.

Next we take $z = -1 + i\sigma$. Then

$$T^{1+i\sigma} = e^{1-\sigma^2-2i\sigma}Q_k|D|^{i\sigma}\frac{\nabla}{|D|}\nabla Q_m|D|^{-1+i\sigma}. \tag{4.4.62}$$

Then we use the relation

$$\begin{aligned}Q_k|D|^{i\sigma}\frac{\nabla}{|D|}\nabla Q_m|D|^{-1+i\sigma} &= Q_k|D|^{i\sigma}\frac{\nabla}{|D|}Q_m\nabla|D|^{-1+i\sigma} + \\ &+Q_k|D|^{i\sigma}\frac{\nabla}{|D|}(\nabla Q_m)|D|^{-1+i\sigma}.\end{aligned} \tag{4.4.63}$$

The kernel of $Q_k|D|^{i\sigma}\frac{\nabla}{|D|}(\nabla Q_m)$ is $K'''_{k,m}$ and satisfies

$$|K'''_{k,m}| \leq C\frac{Q_k(x)Q_m(y)}{2^{(n-1)(k\vee m)}2^m}(1+\sigma)^{n+1}, \tag{4.4.64}$$

then we obtain

$$\|Q_k|D|^{i\sigma}\frac{\nabla}{|D|}(\nabla Q_m)|D|^{-1+i\sigma}g\|_{L^p} \leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{r'}}}{2^{n(k\vee m)}2^m}\|g\|_{L^r}. \tag{4.4.65}$$

Taking $g = |D|^{-1}f$, we get (Hardy-Sobolev)

$$\||D|^{-1}f\|_{L^r} \leq \|f\|_{L^p}, \quad \frac{1}{p} - \frac{1}{r} = \frac{1}{n}.$$

From the fact that $\frac{1}{r} = \frac{1}{p} - \frac{1}{n}$ we have $1 - \frac{1}{r} = \frac{1}{r'} = 1 - \frac{1}{p} + \frac{1}{n} = \frac{1}{p'} + \frac{1}{n}$ and we arrive at

$$\|Q_k|D|^{i\sigma}\frac{\nabla}{|D|}(\nabla Q_m)|D|^{-1+i\sigma}f\|_{L^p} \leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}\|f\|_{L^p}, \tag{4.4.66}$$

provided $p < n$. Since

$$\|Q_k|D|^{i\sigma}\frac{\nabla}{|D|}Q_m\nabla|D|^{-1+i\sigma}\|_{L^p} \leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}\|f\|_{L^p}, \tag{4.4.67}$$

from (4.4.63) and (4.4.66) we get

$$\|T^{1+i\sigma}(f)\|_{L^p} \leq C\frac{2^{k\frac{n}{p}}2^{m\frac{n}{p'}}}{2^{n(k\vee m)}}\|f\|_{L^p}. \tag{4.4.68}$$

The application of the Stein interpolation argument for z; $\mathrm{Re}\ z \in [-1, 0]$ combined with the above estimate and (4.4.56) guarantees that (4.4.49) is fulfilled for $s \in (-1, 0]$ and this complete the proof of the lemma.

□

It is not difficult to extend the result of Lemma 4.1 for $|k - m| \leq 3$. Note that a formal calculus of $t(k, m, s, p)$ for $|k - m| \leq 3$ in (4.4.50) gives $2^{t(k,m,s,p)} \sim 1$. To verify

$$\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C\|f\|_{L^p}, \tag{4.4.69}$$

for $|s| < 1, 1 < p < n$, it is sufficient to use a scale argument and to show (4.4.69) for $k = m = 0$ so we shall verify the inequality

$$\|Q_0|D|^{-s}Q_0|D|^s f\|_{L^p} \leq C\|f\|_{L^p}. \tag{4.4.70}$$

Here we can use an interpolation argument as in the proof of Lemma 4.1. Then we have to show that $L_\sigma = Q_0|D|^{-1+i\sigma}Q_0\nabla$ is L^p-bounded. But

$$L_\sigma = Q_0(|D|^{-1+i\sigma}\nabla)Q_0 + Q_0|D|^{i\sigma}|D|^{-1}(\nabla Q_0). \tag{4.4.71}$$

Since $|D|^{-1}\nabla$ is L^p-bounded and $|D|^{i\sigma}$ is also L^p -bounded as Riesz potential, we see that $Q_0(|D|^{-1+i\sigma}\nabla)Q_0$ is L^p -bounded. From the property

$$|D|^{-1} : L^r \to L^p, \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{n}, \tag{4.4.72}$$

and

$$\nabla Q_0 : L^p \to L^r, \tag{4.4.73}$$

we see that $|D|^{-1}(\nabla Q_0) : L^p \to L^p$ so L_σ is L^p -bounded. This observation and a Stein interpolation argument implies (4.4.69) for $s \in (0, 1)$. To cover the case $s \in (-1, 0)$ we have to show that

$$L'_\sigma = Q_0|D|^{i\sigma}\frac{\nabla}{|D|}\nabla Q_0|D|^{-1+i\sigma}, \quad \text{is } L^p\text{-bounded} \tag{4.4.74}$$

But

$$L'_\sigma = Q_0|D|^{i\sigma}\frac{\nabla}{|D|}(\nabla Q_0)|D|^{-1}|D|^{i\sigma} + Q_0|D|^{i\sigma}\frac{\nabla}{|D|}\nabla Q_0\frac{\nabla}{|D|}|D|^{i\sigma}, \tag{4.4.75}$$

and again we can show that (4.4.72) and (4.4.73) imply that the operator on the right hand side of the (4.4.75) is L^p-bounded (with norm $\leq C(1+\sigma)^{n+1}$). Since the second operator is also L^p -bounded, we see that L'_σ is also L^p -bounded and this argument implies

LEMMA 4.2. *For any $s \in \mathbb{R}$, $|s| < 1$ any $p, 1 < p < n$ there exists a constant $C = C(s,p,n) > 0$ so that for any $k, m \in \mathbb{Z}$, and for $f \in S(\mathbb{R}^n)$ we have*

$$\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C\ 2^{t(k,m,s,p)}\ \|f\|_{L^p}, \tag{4.4.76}$$

where $t(k,m,s,p)$ is defined as (4.4.50).

Finally we use a duality argument and find :

LEMMA 4.3. *For any $s \in \mathbb{R}$ such that $|s| < 1$, and for any p, $\frac{n}{n-1} < p < n$, there exists a constant $C = C(s,p,n) > 0$ so that for $f \in S(\mathbb{R}^n)$ we have*

$$\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C\ 2^{t(k,m,s,p)}\ \|f\|_{L^p}, \tag{4.4.77}$$

where $t(k,m,s,p)$ is defined as (4.4.50).

Proof: For any $f, g \in S(\mathbb{R}^n)$ we have

$$\begin{aligned} |(g, |D|^{-s}Q_k|D|^sQ_m f)| &= |(Q_m|D|^{-s}Q_k|D|^s g, f)| \\ &\leq \|f\|_{L^p}\||D|^{-s}Q_k|D|^sQ_m g\|_{L^{p'}}. \end{aligned} \tag{4.4.78}$$

Applying for p' the estimate of (4.2), we find

$$\||D|^{-s}Q_k|D|^sQ_m g\|_{L^{p'}} \leq 2^{t(m,k,s,p')}\ \|g\|_{L^p}, \tag{4.4.79}$$

where

$$t(m,k,s,p') = k\frac{n}{p} + m\frac{n}{p'} - (n-(s\vee 0))(k\vee m) - (s\vee 0))(k\wedge m) = t(k,m,s,p).$$

This complete the proof. □

5. Discrete Estimate

Consider the operator

$$T : a = \{a_k\}_{k\in\mathbb{Z}} \to Ta = b_m = \sum_{k\in\{|k-m|\geq 4\}} t_{k,m}a_k, \tag{4.5.80}$$

where

$$t_{k,m} = 2^{m\lambda}\ 2^{\mu k}\ 2^{-\beta(m\vee k)},\quad m\vee k = max(m,k), \tag{4.5.81}$$

$$\lambda > 0, \mu > 0,\quad \beta = \lambda + \mu. \tag{4.5.82}$$

LEMMA 4.4. *If $\lambda, \mu > 0$, and $\beta = \lambda + \mu$, then the operator*

$$T : \ell^q \to \ell^q$$

is bounded for any $q \in [1, \infty]$.

Proof: First we consider the cases $q = \infty$ and $q = 1$, then we apply the interpolation argument. We represent Ta as

$$Ta = T_1 a + T_2 a, \tag{4.5.83}$$

where

$$(T_1 a)_m = \sum_{k=m+1}^{\infty} t_{k,m} a_k, \tag{4.5.84}$$

$$(T_2 a)_m = \sum_{-\infty}^{m} t_{k,m} a_k. \tag{4.5.85}$$

From (4.5.81) we find for $T_1 a$

$$\begin{aligned} \|T_1 a\|_\infty &\leq C sup_{m\in\mathbb{Z}} \left(\sum_{k=m+1}^{\infty} t_{k,m} \right) \|a\|_\infty \leq \\ &\leq C sup_{m\in\mathbb{Z}} \left(\sum_{k=m+1}^{\infty} 2^{m\lambda} 2^{k\mu} 2^{-\beta m} \right) \|a\|_\infty, \end{aligned} \tag{4.5.86}$$

so

$$\|T_1 a\|_\infty \leq C \|a\|_\infty. \tag{4.5.87}$$

From (4.5.81) we have for $T_2 a$ the following estimate

$$\begin{aligned} \|T_2 a\|_\infty &\leq C sup_{m\in\mathbb{Z}} \left(\sum_{k=-\infty}^{m} 2^{m\lambda} 2^{\mu k} 2^{-\beta k} \right) \|a\|_\infty = \\ &= C sup_{m\in\mathbb{Z}} \left(\sum_{k=-\infty}^{m} 2^{m\lambda} 2^{-\lambda k} \right) \|a\|_\infty = C \|a\|_\infty. \end{aligned} \tag{4.5.88}$$

This estimate and (4.5.87) imply

$$\|Ta\|_\infty \leq C \|a\|_\infty. \tag{4.5.89}$$

For $q = 1$ we have

$$\begin{aligned} \|T_1 a\|_1 &\leq C sup_{k\in\mathbb{Z}} \left(\sum_{\{m\in\mathbb{Z}; m\geq k\}} t_{k,m} \right) \|a\|_1 \leq \\ &\leq sup_{k\in\mathbb{Z}} \left(\sum_{m=k}^{\infty} 2^{m\lambda} 2^{k\mu} 2^{-\beta m} \right) \|a\|_1 = \\ &= C sup_{k\in\mathbb{Z}} 2^{k\mu} \left(\sum_{m=k}^{\infty} 2^{-m\mu} \right) \|a\|_1 \leq 2C \|a\|_1. \end{aligned} \tag{4.5.90}$$

In a similar way we estimate $T_2 a$,

$$\|T_2 a\|_1 \leq Csup_{k\in\mathbb{Z}}\left(\sum_{m=-\infty}^{k-1} 2^{m\lambda}2^{k\mu}2^{-\beta k}\right)\|a\|_1 =$$
$$= Csup_{k\in\mathbb{Z}}\left(\sum_{m=-\infty}^{k-1}\right)2^{m\lambda}2^{-k\lambda}\|a\|_1 \leq C\|a\|_1. \tag{4.5.91}$$

Thus we get

$$\|Ta\|_1 \leq C\|a\|_1, \tag{4.5.92}$$

and this completes the proof of the lemma.

□

It easy to obtain a corresponding weighted version of Lemma 4.4 in terms of the weighted ℓ^q spaces

$$\ell^{q,\alpha} = \{a = (a)_{k\in\mathbb{Z}}; \sum_k 2^{kq\alpha}|a_k|^q < \infty\}. \tag{4.5.93}$$

For that purpose consider the operator

$$J^\alpha : a \to b = J^\alpha a,$$

defined as follows:

$$b_k = 2^{k\alpha}a_k. \tag{4.5.94}$$

We have the following two Lemmas:

LEMMA 4.5. *The application* $J^\alpha : \ell^q \to \ell^{q,\alpha}$ *is an isomorphism for any* $\alpha \in \mathbb{R}$ *and any* $q \in [1,\infty]$.

LEMMA 4.6. *If* $\sigma, \nu, \lambda, \mu$ *are real numbers such that*

$$\begin{cases} \lambda + \sigma > 0, \\ \mu - \nu > 0, \end{cases} \tag{4.5.95}$$

then for $\beta = \lambda + \sigma + \mu - \nu$ *we have*

$$T : \ell^{q,\sigma} \to \ell^{q,\nu}$$

where T *is defined by* (4.5.80) *and* (4.5.81).

Proof: Let

$$\widetilde{T} : J_\sigma T J_\nu^{-1}.$$

Then Lemma 4.5 guarantees that $T : l^{q,\sigma} \to \ell^{q,\nu}$ if and only if $\widetilde{T} : \ell^q \to \ell^q$. Note that $\widetilde{T}$ is defined by

$$t_{m,k} = 2^{m(\lambda+\sigma)}2^{k(\mu-\nu)}2^{-\beta(m\vee k)}. \tag{4.5.96}$$

So applying Lemma 4.4 with $\widetilde{\lambda} = \lambda + \sigma$ and $\widetilde{\mu} = \mu - \nu$, we complete the proof.

□

A slight generalization of Lemma 4.4 can be obtained for the case when λ, μ, β are vectors in $\mathbb{R}^2$, that is

$$\begin{cases} \lambda = (\lambda_1, \lambda_2), \\ \mu = (\mu_1, \mu_2), \\ \beta = (\beta_1, \beta_2). \end{cases}$$

Then (4.5.80)

$$\begin{cases} Ta = b, \quad \text{where } \ b_m = \sum_{k\in\mathbb{Z}^2} t_{m,k} a_k, \ m \in \mathbb{Z}^2 \\ a = \{a_k\}_{k\in\mathbb{Z}^2}, \end{cases} \tag{4.5.97}$$

where

$$t_{m,k} = 2^{\sum_{j=1}^{2} m_j\lambda_j + k_j\mu_j - \beta_j(m_j \vee k_j)}. \tag{4.5.98}$$

The assumption (4.5.81) can be replaced again by the following one

$$\lambda_j > 0, \mu_j > 0, \ j = 1, 2. \tag{4.5.99}$$

LEMMA 4.7. *If $\lambda, \mu, \beta \in \mathbb{R}^2$ satisfy $\beta_j = \lambda_j + \mu_j, i = 1, 2$ and if $t_{m,k}$ is chosen as in* (4.5.96) *then*

$$T : \ell^{q_1}_{k_1} \ell^{q_2}_{k_2} \to \ell^{q_1}_{k_1} \ell^{q_2}_{k_2}, \tag{4.5.100}$$

is bounded for $q = (q_1, q_2), 1 \leq q_j \leq \infty$.

REMARK 4.1. *Given any sequence $a = \{a_{k_1k_2}\}_{k=(k_1,k_2)\in\mathbb{Z}^2}$ we can consider the norm*

$$\|a\|_{\ell^{q_1}_{k_1}\ell^{q_2}_{k_2}} = \left(\sum_{k_1\in\mathbb{Z}} \left(\sum_{k_2\in\mathbb{Z}} |a_{k_1k_2}|^{q_2} \right)^{q_1/q_2} \right)^{1/q_1}, \tag{4.5.101}$$

(with obvious modifications if $q_1 = \infty$ or $q_2 = \infty$), and the corresponding Banach space $\ell^{q_1}_{k_1}\ell^{q_2}_{k_2}$. Note that

$$\ell^{q_1}_{k_1}\ell^{q_2}_{k_2} \neq \ell^{q_2}_{k_2}\ell^{q_1}_{k_1},$$

but the assertion of Lemma 4.7 is still true if we replace $\ell^{q_1}_{k_1}\ell^{q_2}_{k_2}$ by $\ell^{q_2}_{k_2}\ell^{q_1}_{k_1}$. The corresponding generalization of Lemma 4.6 is the following,

LEMMA 4.8. *If $\sigma, \nu, \lambda, \mu \in \mathbb{R}^2$ satisfy*

$$\begin{cases} \lambda_j + \sigma_j > 0, \\ \mu_j - \nu_j > 0, \end{cases} \tag{4.5.102}$$

then for $\beta_j = \lambda_j + \sigma_j + \mu_j - \nu_j, j = 1, 2$ the operator T defined by (4.5.97) *and* (4.5.98) *is in $B(\ell^{q_1,\sigma_1}_{k_1}\ell^{q_2,\sigma_2}_{k_2}, \ell^{q_1,\nu_1}_{k_1}\ell^{q_2,\nu_2}_{k_2})$.*

6. Space localization

Given any Banach space $B \subset D'(\mathbb{R}^n)$ satisfying the property

$$\text{for any } Q(x) \in C_0^\infty(\mathbb{R}^n), \ f \in B \Rightarrow Q(x)f \in B, \tag{4.6.103}$$

we can define for any $p \in [1, \infty]$ and for any $\alpha \in \mathbb{R}$ the space $\ell^{q,\alpha}_x B$ as follows

$$\|f\|_{\ell^{q,\alpha}_x B} = \left(\sum_{k\in\mathbb{Z}} 2^{kq\alpha} \, \|Q_k f\|^q_B \right)^{1/q}, \tag{4.6.104}$$

with obvious modification for $q=\infty$. Note that for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$\|f\|_{\ell_x^{q,\alpha}B} < \infty.$$

So $\ell_x^{q,\alpha}B$ can be defined as the closure of $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to the norm (4.6.104). An alternative definition is based on the map

$$J : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset B \to J_B(f)_k = \|Q_k f\|_B \in \ell^{q,\alpha}, \tag{4.6.105}$$

where $\ell^{q,\alpha}$ is the space of all sequences $a=(a_k)_{k\in\mathbb{Z}}$ such that

$$\|a\|_{l^{q,\alpha}} = \left(\sum_{k\in\mathbb{Z}} \|a_k\|^q 2^{kq\alpha}\right)^{1/q}, \tag{4.6.106}$$

with obvious modification for $p=\infty$. Then

$$\|f\|_{l^{q,\alpha}B} = \|J_B(f)\|_{l^{q,\alpha}}. \tag{4.6.107}$$

The space $l^{q,\alpha}B$ is independent of the concrete choice of Paley-Littlewood decomposition

$$\sum_{j\in\mathbb{Z}} Q_j(x) = 1 \tag{4.6.108}$$

satisfying

$$\begin{cases} Q_j(x) \geq 0, \\ supp\ Q_j(x) \in \{|x| \sim 2^j\}. \end{cases} \tag{4.6.109}$$

A typical example is the case $B = \dot{H}_p^s$, where $s \in (-1,1), 1<p<\infty$. For $s > -\frac{n}{p}$ we have $\dot{H}_p^s(\mathbb{R}^n) \subset D'$ (see [**48**]) and the norm is defined by

$$\|f\|_{\dot{H}_p^s} = \||D|^s f\|_{L^p}. \tag{4.6.110}$$

Our next goal is to show the equivalence of the norms

$$\|\ |D|^{-s} f\|_{\ell_x^{q,\alpha}L^p} = \left(\sum_{k\in\mathbb{Z}} 2^{kq\alpha} \|Q_k |D|^{-s} f\|_{L^p}^q\right)^{1/q}$$

and

$$\|f\|_{\ell_x^{q,\alpha}\dot{H}_p^s} = \left(\sum_{k\in\mathbb{Z}} 2^{kq\alpha} \||D|^{-s} Q_k f\|_{L^p}^q\right)^{1/q}$$

The proof of the equivalence of the norm from Theorem 4.4 is a direct consequence (taking $p=2$) of the following estimates.

PROPOSITION 4.1. *For* $p \in (n/(n-1), n), q \in [1,\infty]$*, for* $s \in [-1,1]$ *and* $a \in \mathbb{R}$ *that satisfy*

$$|a| + |s| < min\left(\frac{n}{p}, \frac{n}{p\prime}\right) \tag{4.6.111}$$

one can find a constant $C = C(n,s,p,q,a) > 0$ *so that*

$$C^{-1}\|\ |D|^{-s} f\|_{\ell_x^{q,a}L^p} \leq \|f\|_{\ell_x^{q,a}\dot{H}_p^s} \leq C\|\ |D|^{-s} f\|_{\ell_x^{q,a}L^p}. \tag{4.6.112}$$

Proof: The left inequality in (4.6.112) is equivalent to

$$\left(\sum_{k\in\mathbb{Z}} 2^{kq\alpha}\|Q_k|D|^{-s}f\|^q_{L^p}\right)^{1/q} \leq C\left(\sum_{k\in\mathbb{Z}} 2^{kq\alpha}\||D|^{-s}Q_kf\|^q_{L^p}\right)^{1/q}. \tag{4.6.113}$$

Indeed, given any integers $k, m \in \mathbb{Z}$ with $|k-m| > 2$ we have the identity

$$Q_k|D|^{-s}Q_mf = Q_k|D|^{-s}Q_m|D|^s|D|^{-s}\widetilde{Q}_mf, \tag{4.6.114}$$

where $\widetilde{Q}_m = \frac{1}{3}(Q_{m-1}+Q_m+Q_{m+1})$ is another Paley-Littlewood partition of unity

$$\sum_{m\in\mathbb{Z}} \widetilde{Q}_m = 1,$$

such that $\widetilde{Q}_m(s) = 1$ for $s \in supp\, Q_m$. To verify (4.6.113) it is sufficient to show that

$$\left(\sum_{k\in\mathbb{Z}} 2^{kqa}\|Q_k|D|^{-s}f\|^q_{L^p}\right)^{1/q} \leq C\left(\sum_{m\in\mathbb{Z}} 2^{mqa}\||D|^{-s}\widetilde{Q}_mf\|^q_{L^p}\right)^{1/q}. \tag{4.6.115}$$

From the estimate of Lemma 4.1 we have

$$\|Q_k|D|^{-s}Q_m|D|^sf\|_{L^p} \leq C2^{t(k,m,s,p)}\,\|f\|_{L^p}, \tag{4.6.116}$$

where $t(k,m,s,p)$ is defined in (4.4.50). Applying the above estimate with

$$g = |D|^{-s}\widetilde{Q}_mf$$

together with Lemma 4.6, we complete the proof of (4.6.115).

To verify the right inequality in (4.6.113) it is sufficient to show

$$\left(\sum_{k\in\mathbb{Z}} 2^{kqa}\||D|^{-s}Q_kf\|^q_{L^p}\right)^{1/q} \leq C\left(\sum_{k\in\mathbb{Z}} 2^{kqa}\|Q_k|D|^{-s}f\|^q_{L^p}\right)^{1/q}. \tag{4.6.117}$$

To this end we use the relation

$$\begin{aligned}|D|^{-s}Q_kf &= |D|^{-s}Q_k|D|^s|D|^{-s}f = \\ &= \sum_{m\in\mathbb{Z}} |D|^{-s}Q_k|D|^sQ_m\widetilde{Q}_m|D|^{-s}f.\end{aligned}$$

From Lemma 4.3 we have

$$\|Q_k|D|^{-s}Q_m|D|^sf\|_{L^p} \leq C2^{t(k,m,s,p)}\,\|f\|_{L^p}, \tag{4.6.118}$$

where $t(k,m,s,p)$ is defined as in (4.4.50), so applying Lemma 4.6, we obtain (4.6.117) and complete the proof of the Proposition.

□

PROPOSITION 4.2. *For* $p \in (n/(n-1), n), q \in [1,\infty]$*, for* $s \in [0,1]$ *and* $a > 0$ *that satisfy* (4.6.111) *one can find a constant* $C = C(n,s,p,q,a) > 0$ *so that*

$$\|fg\|_{\ell^{q,a}_x\dot{H}^s_p} \leq C\left(\|f\|_{\ell^{q_1,a_1}_x\dot{H}^s_{p_1}}\|g\|_{\ell^{q_2,a_2}_xL^{p_2}} + \|f\|_{\ell^{q_3,a_3}_xL^{p_3}}\|g\|_{\ell^{q_4,a_4}_x\dot{H}^s_{p_4}}\right), \tag{4.6.119}$$

provided $a_1,a_2,a_3,a_4 \geq 0$ *and* $1 \leq p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \leq \infty$ *satisfy*

$$a_1+a_2 = a_3+a_4 = a, \frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_3}+\frac{1}{p_4}, \frac{1}{q} = \frac{1}{q_1}+\frac{1}{q_2} = \frac{1}{q_3}+\frac{1}{q_4}.$$

Proof: The proof uses the previous proposition and the standard multiplicative Sobolev inequality

$$\|fg\|_{\dot H^s_p} \le C\left(\|f\|_{\dot H^s_{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot H^s_{p_4}}\right),$$

so we omit the details.

□

Using the interpolation property

$$(H^{s_1}_{p_1}, H^{s_2}_{p_2})_\theta = H^s_p,\ s = (1-\theta)s_1 + \theta s_2,\ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

we arrive at

PROPOSITION 4.3. *For $p \in (n/(n-1), n), q \in [1,\infty]$, for $s \in (0,1)$ and $a > 0$ that satisfy* (4.6.111) *one can find a constant $C = C(n,s,p,q,a) > 0$ so that*

$$\|f\|_{\ell^{q,a}_x \dot H^s_p} \le C\left(\|f\|_{\ell^{q_1,a_1}_x \dot H^1_{p_1}}\right)^{1-\theta}\left(\|f\|_{\ell^{q_2,a_2}_x L^{p_2}}\right)^{\theta}, \tag{4.6.120}$$

provided $a_1, a_2 \ge 0$ and $1 < p_1, p_2, q_1, q_2 < \infty$ satisfy

$$a = a_1(1-\theta) + a_2\theta, s = 1-\theta, \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

REMARK 4.2. *Note that, using the norms introduced in* (4.6.104)*, the estimate* (4.1.12) *can be written as*

$$\|u\|_{L^2_t \ell^{\infty,-1/2}_x \dot H^{1/2}} \le C\|f\|_{L^2_x} + C\|F\|_{L^2_t \ell^{1,1/2} \dot H^{-1/2}_x}. \tag{4.6.121}$$

7. Phase localization

Given any Banach space $B \subset D'(\mathbb{R}^n)$ satisfying the property

$$\text{for any } P(\xi) \in C_0^\infty(\mathbb{R}^n),\ f \in B \Rightarrow P(D)f \in B, \tag{4.7.122}$$

we can define for any $r \in [1,\infty]$ and for any $s \in \mathbb{R}$ the space $\ell^{r,s}_D B$ as follows

$$\|f\|_{\ell^{r,s}_D B} = \left(\sum_{k\in\mathbb{Z}} \|P_k(D)f\|_B^r 2^{krs}\right)^{1/r}, \tag{4.7.123}$$

with obvious modification for $r = \infty$. Here $\{P_k(\xi)\}$ is a Paley-Littlewood decomposition.

Our goal is to find some concrete examples of Banach spaces B satisfying the embedding

$$B \subset \ell^{r,0}_D B. \tag{4.7.124}$$

Therefore we look for estimate of the type

$$\left(\sum_{k\in\mathbb{Z}} \|P_k(D)f\|_B^r\right)^{1/r} \le C\|f\|_B. \tag{4.7.125}$$

A typical example for a Banach space B satisfying (4.7.125) is $B = L^p$ with $1 < p \le 2$, so

$$\left(\sum_{k\in\mathbb{Z}} \|P_k(D)f\|_{L^p}^2\right)^{1/2} \le C\|f\|_{L^p},\quad 1 < p \le 2. \tag{4.7.126}$$

Having in mind that the spaces $\ell_x^{r,a}\dot{H}_p^s$ are natural candidate for estimate of type (4.7.125), we shall verify that the conditions

$$\begin{cases} 1 \le q \le 2, \ p = 2 \\ |a| + |s| < \frac{n}{2}, \end{cases} \tag{4.7.127}$$

imply (4.7.125). More precisely we have

LEMMA 4.9. *If* $q \in [1,2]$ *and* $a, s \in \mathbb{R}$ *satisfy*

$$\begin{cases} |s| \le 1, \\ |a| + |s| < \frac{n}{2}, \end{cases} \tag{4.7.128}$$

then

$$\|f\|_{\ell_D^{2,0}\ell_x^{q,a}\dot{H}^s} \le C\|f\|_{\ell_x^{q,a}\dot{H}^s}. \tag{4.7.129}$$

Proof: For any $f \in S(\mathbb{R}^n)$ we have (for any $r, q \in (1,\infty)$)

$$\|f\|_{\ell_D^{2,0}\ell_x^{q,a}\dot{H}^s} \cong \left(\sum_{k_2\in\mathbb{Z}}\left(\sum_{k_1\in\mathbb{Z}} 2^{k_1qa}\||D|^sQ_{k_1}P_{k_2}(D)f\|_{L^2}^q\right)^{2/q}\right)^{1/2} \cong \tag{4.7.130}$$

$$\cong \left(\sum_{k_2\in\mathbb{Z}}\left(\sum_{k_1\in\mathbb{Z}} 2^{k_1qa}\|Q_{k_1}|D|^sP_{k_2}(D)f\|_{L^2}^q\right)^{2/q}\right)^{1/2} \cong \|\|Q_{k_1}P_{k_2}(D)f\|_{L^2}\|_{\ell_{k_2}^{2,s}\ell_{k_1}^{q,a}},$$

where here and below we use the discrete norm in $l_{k_1}^{q,a}l_{k_2}^{2,s}$ introduced in (4.5.101). In the second equivalence relation we have used Proposition 4.1. Further we have

$$Q_{k_1}P_{k_2}(D)f = \sum_{m_1\in\mathbb{Z}}\sum_{m_2\in\mathbb{Z}} Q_{k_1}P_{k_2}(D)Q_{m_1}P_{m_2}(D)\widetilde{P}_{m_2}(D)\widetilde{Q}_{m_1}f. \tag{4.7.131}$$

It is not difficult, using again an integration by parts argument, to see that for $|k_1 - m_1| \ge 3$ we have

$$\|Q_{k_1}P_{k_2}(D)Q_{m_1}g\|_{L^2} \le C\frac{2^{(k_1+m_1)n/2}}{2^{(k_1\vee m_1)n}}\|g\|_{L^2}, \tag{4.7.132}$$

so

$$\|Q_{k_1}P_{k_2}(D)Q_{m_1}P_{m_2}(D)f\|_{L^2} \le C\frac{2^{(k_1+m_1)n/2}}{2^{(k_1\vee m_1)n}}\|f\|_{L^2}. \tag{4.7.133}$$

In a similar way, using the same integration by parts argument, we find for $|k_2 - m_2| \ge 3$

$$\|P_{k_2}(D)Q_{m_1}P_{m_2}(D)g\|_{L^2} \le C\frac{2^{(k_2+m_2)n/2}}{2^{(k_2\vee m_2)n}}\|g\|_{L^2}, \tag{4.7.134}$$

so

$$\|Q_{k_1}P_{k_2}(D)Q_{m_1}P_{m_2}(D)f\|_{L^2} \le C\frac{2^{(k_2+m_2)n/2}}{2^{(k_2\vee m_2)n}}\|f\|_{L^2}. \tag{4.7.135}$$

Interpolation between (4.7.133) and (4.7.135) gives

$$\|Q_{k_1}P_{k_2}(D)Q_{m_1}P_{m_2}(D)f\|_{L^2} \le Ct_{k,m}^{(\theta)}\|f\|_{L^2}, \tag{4.7.136}$$

where $k = (k_1, k_2) \in \mathbb{Z}^2, m = (m_1, m_2) \in \mathbb{Z}^2$, $\theta \in [0,1]$ will be chosen later on and

$$t_{k,m}^{(\theta)} = \frac{2^{(k_1+m_1)\theta n/2}}{2^{(k_1\vee m_1)\theta n}}\frac{2^{(k_2+m_2)(1-\theta)n/2}}{2^{(k_2\vee m_2)(1-\theta)n}}. \tag{4.7.137}$$

If $a, s \in \mathbb{R}$ satisfy

$$|a| + |s| < \frac{n}{2}, \tag{4.7.138}$$

then we can choose $\theta \in [0,1]$ so that

$$\begin{cases} |a| \le \frac{n}{2}(1-\theta), \\ |s| < \frac{n}{2}\theta. \end{cases} \tag{4.7.139}$$

Using the argument of the proof from Lemma 4.1, we see that (4.7.135) is fulfilled without the restrictions $|k_1 - m_1| \ge 3$, $|k_2 - m_2| \ge 3$. Applying Lemma 4.8, we get

$$\| \| \| Q_{k_1} P_{k_2}(D) f \|_{L^2} \|_{\ell^{2,s}_{k_2} \ell^{q,a}_{k_1}} \le C \| \| \| \widetilde{P}_{m_2}(D) \widetilde{Q}_{m_1} f \|_{L^2} \|_{\ell^{2,s}_{m_2} \ell^{q,a}_{m_1}}. \tag{4.7.140}$$

For $1 \le q \le 2$, we have the inequality

$$\| \| \| \widetilde{P}_{m_2}(D) \widetilde{Q}_{m_1} f \|_{L^2} \|_{\ell^{2,s}_{m_2} \ell^{q,a}_{m_1}} \le \| \| \| \widetilde{P}_{m_2}(D) \widetilde{Q}_{m_1} f \|_{L^2} \|_{\ell^{q,a}_{m_1} \ell^{2,s}_{m_2}}.$$

From relation

$$\| \| \| \widetilde{P}_{m_2}(D) g \|_{L^2} \|_{\ell^{2,s}_{m_2}} \cong \| g \|_{\dot{H}^s}$$

we get

$$\begin{cases} \| \| \| Q_{k_1} P_{k_2}(D) f \|_{L^2} \|_{\ell^{2,s}_{k_2} \ell^{q,a}_{k_1}} \le \\ \le C \| \| \| \widetilde{P}_{m_2}(D) \widetilde{Q}_{m_1} f \|_{L^2} \|_{\ell^{q,a}_{m_1} \ell^{2,s}_{m_2}} \cong \\ C \ \| \| \| |D|^s Q_{m_1} f \|_{L^2} \| \|_{\ell^{q,a}_{m_1}} \cong \| f \|_{\ell^{q,a}_x \dot{H}^s}. \end{cases} \tag{4.7.141}$$

This inequalities and (4.7.130) imply

$$\| f \|_{\ell^{2,0}_D l^{q,\alpha}_x \dot{H}^s} \le C \| f \|_{\ell^{q,\alpha}_x \dot{H}^s}. \tag{4.7.142}$$

This completes the proof.

□

Further, we obtain in a similar way the following:

LEMMA 4.10. *If $q \in [2, \infty]$ and $a, s \in \mathbb{R}$ satisfy*

$$\begin{cases} |s| \le 1, \\ |a| + |s| < \frac{n}{2}, \end{cases} \tag{4.7.143}$$

then

$$\| f \|_{\ell^{q,a}_x \dot{H}^s} \le C \| f \|_{\ell^{2,0}_D \ell^{q,a}_x \dot{H}^s}. \tag{4.7.144}$$

In a similar way we can verify the following statement:

LEMMA 4.11. *If $q \in [1, 2]$ and $a, s \in \mathbb{R}$ satisfy*

$$\begin{cases} |s| \le 1, \\ |a| + |s| < \frac{n}{2} \end{cases} \tag{4.7.145}$$

and R is a pseudo differential operator with convolution type symbol which is homogeneous of degree 0*, then*

$$\|Rf\|_{l^{q,a}_x \dot{H}^s} \le C\|f\|_{l^{q,a}_x \dot{H}^s}. \tag{4.7.146}$$

By using a duality argument one can relax the assumptions on q and obtain the following statement:

LEMMA 4.12. *If* $q \in [2,\infty]$ *and* $a, s \in \mathbb{R}$ *satisfy*

$$\begin{cases} |s| \le 1, \\ |a| + |s| < \frac{n}{2} \end{cases} \tag{4.7.147}$$

and R is a pseudo differential operator with convolution type symbol which is homogeneous of degree 0*, then*

$$\|Rf\|_{\ell^{q,a}_x \dot{H}^s} \le C\|f\|_{\ell^{q,a}_x \dot{H}^s}. \tag{4.7.148}$$

8. Proof of the Smoothing Estimate (4.1.5)

In this section we shall recall the basic scale invariant smoothing estimate due to Kenig, Ponce, Vega.

One possible selfcontained proof of the Konig, Ponce, Vega estimate (4.1.5) is based on the following lemmas:

LEMMA 4.13. *For any* $u \in S(\mathbb{R}^n)$ *we have*

$$\|u(x_1, x')\|_{L^1_{x_1} L^2_{x'}} \le C \left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2} u(x)\|_{L^2_x} \right), \tag{4.8.149}$$

where $x = (x_1, x')$ *with* $x_1 \in \mathbb{R}$ *and* $x' \in \mathbb{R}^{n-1}$.

Proof. We can consider the case of $n = 1$, since a similar argument works for $n > 1$. Let $u \in S(\mathbb{R})$, we have

$$\|u(x)\|_{L^1} \le C \left\| \sum_{k\in\mathbb{Z}} |x|^{1/2} Q_k(x) u(x) \right\|_{L^2}. \tag{4.8.150}$$

From the Cauchy-Schwartz inequality and the fact that for the functions $Q_k(x)$ we have $\mathrm{supp}_s Q_k(x) \subset \{2^{k-1} \le |x| \le 2^{k+1}\}$, we obtain

$$\left\| \sum_{k\in\mathbb{Z}} |x|^{1/2} Q_k(x) u(x) \right\|_{L^2} \le C \left(\sum_{k\in\mathbb{Z}} \| \, |x|_k^{1/2} u(x)\|_{L^2} \right), \tag{4.8.151}$$

so

$$\|u(x)\|_{L^1} \le C \left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2} u(x)\|_{L^2} \right). \tag{4.8.152}$$

□

Similarly, we have

LEMMA 4.14. *For any* $u \in S(\mathbb{R}^n)$ *we have*

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2} u(x)\|_{L^2_x} \le C\|u(x_1,x')\|_{L^\infty_{x_1} L^2_{x'}}, \tag{4.8.153}$$

where $x=(x_1,x')$ *with* $x_1 \in \mathbb{R}$ *and* $x' \in \mathbb{R}^{n-1}$.

The key point in the proof of (4.1.5) is to establish the estimate

$$\|\partial_1 u(t,x_1,x')\|_{L^\infty_{x_1} L^2_{t,x'}} \le C\|F(t,x_1,x')\|_{L^1_{x_1} L^2_{t,x'}}. \tag{4.8.154}$$

Now we shall show now that this estimate completes the proof of (4.1.5).
From (4.8.154), (4.8.149) and (4.8.153) we get

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2} \partial_1 u(t,x)\|_{L^2_t L^2_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2} F(t,x)\|_{L^2_t L^2_x}\right). \tag{4.8.155}$$

Using the fact that the Schrödinger equation in (4.1.9) and the norm in the right hand side of (4.8.155) are invariant under the action of the group of rotations $SO(n)$, we obtain

$$\sup_{k\in\mathbb{Z}} \||x|_k^{-1/2} \partial_j u(t,x)\|_{L^2_t L^2_x} \le C\left(\sum_{k\in\mathbb{Z}} \||x|_k^{1/2} F(t,x)\|_{L^2_t L^2_x}\right), \quad \forall j=1,\cdots,n. \tag{4.8.156}$$

The estimate (4.8.154) follows from the previous lemmas and the Lemmas 2.1, 2.2 of Chapter 2, Section 4

In fact the basic idea of the proof of (4.8.154) is to compute the Fourier transform with respect the time variable of equation (4.1.9) and obtain

$$-i\Delta\hat{u}(\lambda,x) - i\lambda\hat{u}(\lambda,x) = \widehat{F}(\lambda,x). \tag{4.8.157}$$

Now if we split $x=(x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1}$ and indicate with $v_\lambda(x_1,x') = \hat{u}(\lambda,x)$, we can apply Lemma 2.2 and have

$$\|\partial_1 v_\lambda(x_1,x')\|_{L^\infty_{x_1} L^2_{x'}} \le C\,\|(-\Delta-\lambda)\,v_\lambda(x_1,x')\|_{L^1_{x_1} L^2_{x'}}. \tag{4.8.158}$$

This estimate with the equation (4.8.157) give the following other one:

$$\begin{aligned} \|\partial_1 \hat{u}(\lambda,x_1,x')\|_{L^\infty_{x_1} L^2_{x'}} &\le C\,\|(-\Delta-\lambda)\,\hat{u}(\lambda,x_1,x')\|_{L^1_{x_1} L^2_{x'}} \le \\ &\le \left\|\widehat{F}(\lambda,x_1,x')\right\|_{L^1_{x_1} L^2_{x'}}. \end{aligned} \tag{4.8.159}$$

The application of Plancherel's theorem with respect to time variable gives the estimate (4.8.154) and this completes the proof of (4.1.5).

9. Application to semilinear Schrödinger equations

The last section of this chapter is devoted to present an application of the previous results. Turning to the semilinear Schrödinger equation

$$\partial_t u - i\Delta_A u = |Vu|^p, \tag{4.9.160}$$

we note that the class of potentials $V = V(t,x)$, satisfying (4.1.15), obeys certain rescaling property, thus one can compute the scaling critical regularity

$$s = \frac{n}{2} - \frac{2-ap}{p-1}$$

and one can expect a well posedness for initial data $f \in L^2$ if

$$p = \frac{n+4}{n+2a}.$$

To verify this we shall construct a sequence $u_k(t,x)$ of functions defined as follows: $u_{-1}(t,x) = 0$, then we define the recurrence relation

$$u_k \to u_{k+1}(t,x)$$

so that

$$\begin{cases} \partial_t u_{k+1} - i\Delta_A u_{k+1} = V(t,x)u_k|u_k|^{p-1}, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u_{k+1}(0,x) = f(x). \end{cases} \tag{4.9.161}$$

The estimate (4.1.13) suggests to show the convergence of the sequence u_k in the Banach space

$$Z = L_t^\infty L_x^2 \cap Y'.$$

The definition of the recurrence relation (4.9.161) shows that we have to show first the property: the map

$$u \in Z = L_t^\infty L_x^2 \cap Y' \to V(t,x)u|u|^{p-1} \in L_t^1 L_x^2 + Y$$

is a well defined continuous operator provided V satisfies (4.1.15). Our goal is to show

$$\|u_{k+1}\|_{L_t^\infty L_x^2} + \|u_{k+1}\|_{Y'} \leq C\|f\|_{L^2} + C\left(\|u_k\|_{L_t^\infty L_x^2} + \|u_k\|_{Y'}\right)^p$$

or shortly

$$\|u_{k+1}\|_Z \leq C\|f\|_{L^2} + C\|u_k\|_Z^p. \tag{4.9.162}$$

To apply a contraction argument we need also the inequality

$$\|u_{k+1} - u_k\|_Z \leq C\|u_k - u_{k+1}\|_Z \left(\|u_k\|_Z + \|u_{k-1}\|_Z\right)^{p-1}. \tag{4.9.163}$$

Combining (4.9.162) and (4.9.163), taking $\|f\|_{L^2}$ sufficiently small, we can show via contraction argument that u_k converges in Z to the unique solution of (4.9.160) with initial data $u(0) = f$. Since the proofs of (4.9.162) and (4.9.163) are similar, we treat (4.9.162) only. We need actually to verify

$$\||Vu|^p\|_Y \leq C\|u\|_Z^p \tag{4.9.164}$$

To verify this inequality, we start with the definition of the space Y

$$\|g\|_Y \sim \sum_m 2^{m/2}\||D|^{-1/2}\varphi\left(\frac{\cdot}{2^m}\right) g\|_{L_t^2 L_x^2}. \tag{4.9.165}$$

We apply this relation with $g = |Vu|^p$, combined with the Sobolev embedding $\dot{H}^{1/2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n$, with

$$1/q - 1/2 = 1/(2n), \tag{4.9.166}$$

and get

$$\||Vu|^p\|_Y \leq \sum_m 2^{m/2} \|\varphi\left(\frac{\cdot}{2^m}\right) Vu\|^p_{L_t^{2p} L_x^{pq}} \tag{4.9.167}$$

We can apply now the interpolation inequality

$$\|\phi\|_{L_t^{2p} L_x^{pq}} \leq C \left(\|\phi\|_{L_t^\infty L_x^{r_1}}\right)^\theta \left(\|\phi\|_{L_t^2 L_x^{r_2}}\right)^{1-\theta}, \tag{4.9.168}$$

where $\theta \in (0,1)$ satisfy the relations

$$\frac{1}{2p} = \frac{1-\theta}{2}, \quad \frac{1}{pq} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}. \tag{4.9.169}$$

Hence $p(1-\theta) = 1$.

From (4.9.167) and (4.9.168) we get

$$\||Vu|^p\|_Y \leq \sum_m 2^{m/2} \|Vu\|^{p\theta}_{L_t^\infty L^{r_1}_{|x|\sim 2^m}} \|Vu\|^{p(1-\theta)}_{L_t^2 L^{r_2}_{|x|\sim 2^m}}. \tag{4.9.170}$$

We choose $r_1 \in (1,2]$ so that

$$\|V\psi\|_{L^{r_1}_{|x|\sim 2^m}} \leq C \|\psi\|_{L^2_{|x|\sim 2^m}} \tag{4.9.171}$$

so taking into account the assumption on V we see that

$$\frac{a}{n} = \frac{1}{r_1} - \frac{1}{2}, \tag{4.9.172}$$

while $r_2 \in (1,2]$ is chosen so that

$$2^{m/(2p(1-\theta))} \|V\psi\|_{L^{r_2}_{|x|\sim 2^m}} \leq C 2^{-m} \|\psi\|_{L^2_{|x|\sim 2^m}},$$

i.e.

$$\frac{1}{r_2} = \frac{1}{2} + \frac{a-1}{n} - \frac{1}{2np(1-\theta)} = \frac{1}{2} + \frac{a}{n} - \frac{3}{2n}, \tag{4.9.173}$$

since $p(1-\theta) = 1$. From (4.9.169) we find

$$\frac{1}{pq} = \frac{1}{2} + \frac{a}{n} - \frac{3}{2pn},$$

so this relation and (4.9.166) implies that

$$p = \frac{n+4}{n+2a}.$$

From (4.9.170), (4.9.171) and (4.9.172) we get

$$\||Vu|^p\|_Y \leq \|u\|^{p-1}_{L_t^\infty L_x^2} \||x|^{-1} u\|_{L_t^2 L_x^2}.$$

From the Hardy inequality we have

$$\||x|^{-1} u\|_{L_t^2 L_x^2} \leq C \|u\|_{Y'}$$

so

$$\||Vu|^p\|_Y \leq \|u\|^{p-1}_{L_t^\infty L_x^2} \|u\|_{Y'}$$

and this completes the proof of (4.9.164).

This completes the proof of Theorem 4.3.

CHAPTER 5

Dispersive Estimate

1. General Strichartz Estimates

Following the proof of the Strichartz estimates given in [**115**], we shall set the more abstract environment of an evolution operator defined as follows.

- ⋄ $(X,\ \mathrm{d}x)$ is a measurable space;
- ⋄ H is an Hilbert space wich is dense in $L^2(X)$;
- ⋄ $U(t) : H \longrightarrow L^2(X)$ a family of linear operators such that
  - ∘ that family is equi-continuous, that is $\forall\ f \in H$ we have

$$\|U(t)f\|_{L^2} \lesssim \|f\|_H \quad \forall\ t \ \ \text{(Energy estimate)}\ , \tag{5.1.1}$$

  - ∘ $\exists\ \sigma > 0$ such that $\forall\ g \in L^1_x(X) \cap L^2_x(X)$ we have either

$$\left\|U(s)\left(U(t)\right)^* g\right\|_{L^\infty_x} \lesssim |t-s|^{-\sigma} \|g\|_{L^1_x} \quad \forall\ t \neq s \qquad \text{(Untruncated decay)}\ , \tag{5.1.2a}$$

or

$$\left\|U(s)\left(U(t)\right)^* g\right\|_{L^\infty_x} \lesssim (1+|t-s|)^{-\sigma} \|g\|_{L^1_x} \quad \forall\ t, s \qquad \text{(Truncated decay)}\ . \tag{5.1.2b}$$

In the applications it is natural to regard the equi-continuous family $U(t) : H \longrightarrow L^2(X)$ as the evolution operator associated to the Cauchy problem for a partial differential equation, and to consider H as the space of initial data. Observe that $(U(t))^* : L^2(X) \longrightarrow H$ and we can define the operator $T : L^1\left(\mathbb{R}; L^2(X)\right) \longrightarrow H$ by

$$TF = \int_{\mathbb{R}} (U(s))^* F(s,\cdot)\ \mathrm{d}s\ ,\ \ \forall F \in L^1\left(\mathbb{R}; L^2(X)\right)\ . \tag{5.1.3}$$

As far as its adjoint is concerned we have that $T^* : H \longrightarrow L^\infty\left(\mathbb{R}; L^2(X)\right)$ and

$$\begin{aligned}\langle T^*h, F\rangle = (h, TF) &= \left(h, \int_{\mathbb{R}} (U(s))^* F(s,\cdot)\ \mathrm{d}s\right) = \\ &= \int_{\mathbb{R}} \left(h, (U(s))^* F(s,\cdot)\right)\ \mathrm{d}s = \int_{\mathbb{R}} \langle U(s)h, F(s,\cdot)\rangle\ \mathrm{d}s = \\ &= \int_{\mathbb{R}} \langle U(s)h, F(s,\cdot)\rangle\ \mathrm{d}s = \int_{\mathbb{R}}\int_X \left(U(s)h\right)(y)\overline{F(s,y)}\ \mathrm{d}y\ \mathrm{d}s\ ,\end{aligned}$$

for any $h \in H$ and $F \in L^1\left(\mathbb{R}; L^2(X)\right)$, thus T^* is defined as

$$\left(T^*h\right)(t,x) = \left(U(t)h\right)(x)\ ,\ \forall h \in H\ , \tag{5.1.4}$$

for any time $t \in \mathbb{R}$, any $x \in X$.

If that is the case, then the estimate (5.1.1) looks like an energy estimate, and it states the boundedness of T^*, and consequently, that one of T because of the TT^* method (see Section 10 in Chapter 1). On the other hand the estimates (5.1.2) play the role of a dispersive estimate with respect to the time variable, where the σ has a meaning of a decay rate. The composition T^*T : $L^1\left(\mathbb{R}; L^2(X)\right) \longrightarrow L^\infty\left(\mathbb{R}; L^2(X)\right)$ is the operator

$$(T^*T)\, F(t,\cdot) = \int_{\mathbb{R}} U(t)\,(U(s))^*\, F(s,\cdot)\;\mathrm{d}s\ , \quad \forall F \in L^1\left(\mathbb{R}; L^2(X)\right)\ , \tag{5.1.5}$$

which can be decomposed as a sum of its retarded and advanced parts

$$(T^*T)_R\, F(t,\cdot) = \int_{s<t} U(t)\,(U(s))^*\, F(s,\cdot)\;\mathrm{d}s\ , \tag{5.1.5$_R$}$$

$$(T^*T)_A\, F(t,\cdot) = \int_{s>t} U(t)\,(U(s))^*\, F(s,\cdot)\;\mathrm{d}s\ . \tag{5.1.5$_A$}$$

Observe that $(T^*T)_R$ solves the corresponding inhomogeneous problem with zero initial data (Duhamel's principle).

1.1. Main Theorem. This section is devoted to investigate precisely the properties of the operators $U(t)$, $U(t)^*$, T, T^* and their compositions involved in the previous setting. We shall see that actually more general $L^q_t L^r_x$ spaces will enter in the domain of T^*T. In order to describe all such possible pairs (r,q) (once the decay σ is given) we shall need for the sequel the following definition.

DEFINITION 5.1. *Let the decay rate, $\sigma > 0$, be given. We say that the exponent pair (r,q) is* ***σ-admissible*** *if $r, q \geq 2$, $(r,q,\sigma) \neq (\infty, 2, 1)$ and*

$$\tfrac{1}{q} \leq \sigma\left(\tfrac{1}{2} - \tfrac{1}{r}\right)\ . \tag{5.1.6}$$

If equality holds in (5.1.6)*, then we say that (r,q) is* ***sharp σ-admissible****, otherwise we say that (r,q) is* ***non-sharp σ-admissible****. When $\sigma > 1$ the sharp σ-admissible point*

$$P = \left(\tfrac{2\sigma}{\sigma-1}, 2\right) \tag{5.1.7}$$

is called ***endpoint****.*

We are able now to state the main result.

THEOREM 5.1 (Keel and Tao). *If $U(t)$ obeys* (5.1.1) *and* (5.1.2a)*, then the following estimates*

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_H\ , \tag{5.1.8}$$

$$\left\| \int_{s<t} (U(s))^*\, F(s)\;\mathrm{d}s \right\|_H \lesssim \|F\|_{L^{q'}_t L^{r'}_x}\ , \tag{5.1.9}$$

$$\left\| \int_{s<t} U(t)\,(U(s))^*\, F(s)\;\mathrm{d}s \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}\ , \tag{5.1.10}$$

hold for all sharp-admissible exponent pairs (r,q), $(\tilde{q},\tilde{r})$.

Furthermore, if the decay hypothesis is strengthened to (5.1.2b)*, then* (5.1.8)*,* (5.1.9) *and* (5.1.10) *hold for all σ-admissible exponent pairs (r,q), $(\tilde{r},\tilde{q})$.*

The proof will be given in the following subsections, tough even the inhomogeneous case, and the endpoint cases will be only sketched.

There are several advantages to formulating Theorem 5.1 in this level of generality. First, it allows both wave equation and Schrödinger equation estimates to be treated in a unified manner. Second, it eliminates certain distractions and unnecessary assumptions (e.g. group structure on the $U(t)$). Finally, there is a natural scaling to these estimates which is only apparent in this setting. More precisely, the sharp statement of the theorem is invariant under the scaling

$$U(t) \leftarrow U(\tfrac{t}{\lambda}) \;, \qquad dx \leftarrow \lambda^\sigma dx \;, \qquad \langle f, g\rangle \leftarrow \lambda^\sigma \langle f, g\rangle \;.$$

In other words, for scaling purposes time behaves like $\mathbb{R}$, X behaves like $\mathbb{R}^\sigma$, H behaves like $L^2(\mathbb{R}^\rho)$, and $U(t)$ is dimensionless. In practice the scaling dimension differs from the Euclidean dimension; for instance, in the wave equation $\sigma = \frac{n-1}{2}$, and in the Schrödinger equation $\sigma = \frac{n}{2}$.

1.2. The Non-Endpoint Homogeneous Case. First we shall prove the homogeneous estimates (5.1.8) and (5.1.9) for $(q, r) \neq P$. By duality (5.1.8) is equivalent to (5.1.9), who is in turn equivalent to the bilinear estimate

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle \, \mathrm{d}s \, \mathrm{d}t\right| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}} \;. \tag{5.1.11}$$

by the so-called TT^* method, namely by the statement 4 from Theorem (1.18) with $\mathcal{B} = L^q\left(\mathbb{R}; L^r\left(\mathbb{R}^n\right)\right)$. By real interpolation between the bilinear version of (5.1.1)

$$\left|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle\right| \lesssim \|F(s)\|_{L_x^2} \|G(t)\|_{L_x^2}$$

and the bilinear version of (5.1.2a)

$$\left|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle\right| \lesssim |t-s|^{-\sigma} \|F(s)\|_{L_x^1} \|G(t)\|_{L_x^1} \;,$$

we obtain

$$\left|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle\right| \lesssim |t-s|^{-\sigma\left(1-\frac{2}{r}\right)} \|F(s)\|_{L_x^{r'}} \|G(t)\|_{L_x^{r'}} \;, \tag{5.1.12}$$

by Theorem 1.14, interpolating the previous estimates, and considering $\frac{1}{r'} = 1 - \frac{\theta}{2}$. Recalling that in the sharp σ-admissible case $\frac{1}{q} = \frac{\sigma}{2} - \frac{\sigma}{r}$ and integrating over $\mathbb{R} \times \mathbb{R}$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle\right| \, \mathrm{d}s \, \mathrm{d}t \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} |t-s|^{-\frac{2}{q}} \|F(s)\|_{L_x^{r'}} \|G(t)\|_{L_x^{r'}} \, \mathrm{d}s \, \mathrm{d}t \;,$$

and the (5.1.11) follows from the Hardy-Littlewood-Sobolev inequality (see (1.9.39) with $n = 1$ and $p = q$), provided that $\gamma = \frac{2}{q} < 1$, that is $q > 2$, equivalent to $(q, r) \neq P$. If we are assuming the truncated decay (5.1.2b), then the estimate (5.1.12) can be improved to

$$\left|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle\right| \lesssim (1 + |t-s|)^{-\sigma\left(1-\frac{2}{r}\right)} \|F(s)\|_{L_x^{r'}} \|G(t)\|_{L_x^{r'}} \;,$$

and now it suffices to apply Hausdorff-Young inequality (see (1.8.31)) with $n = 1$ and $p = q$), whenever the function $(1 + |\cdot|)^{-\sigma\left(1-\frac{2}{r}\right)}$ is in the space $L^s(\mathbb{R})$, with $1 + \frac{1}{q'} = \frac{1}{s} + \frac{1}{q}$, that is,

$$\sigma\left(1 - \tfrac{2}{r}\right)\tfrac{1}{s} = \sigma\left(1 - \tfrac{2}{r}\right)\tfrac{2}{q} > 1 \;,$$

or in other words whenever the pair (r, q) is non-sharp σ-admissible condition. This concludes the proof of the homogeneous estimates (5.1.8) and (5.1.9) in the non-endpoint case, $(r, q) \neq P$.

1.3. The Endpoint Homogeneous Case. We deal now with $(r, q) = P = \left(\frac{2\sigma}{\sigma-1}, 2\right)$, for $\sigma > 1$, therefore we can assume only the untruncated decay (5.1.2a).

By symmetry it suffices to restrict our attention to the retarded version of (5.1.11),

$$|B_R(F,G)| = \left| \iint_{s<t} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle \, \mathrm{d}s \, \mathrm{d}t \right| \lesssim \|F\|_{L_t^2 L_x^{r'}} \|G\|_{L_t^2 L_x^{r'}} , \tag{5.1.13}$$

nevertheless it will be not sufficient in this case to obtain (by interpolation) a one-parameter family of estimates, and a wider two-parameter family of estimates will be required. For this purpose we consider a further dyadic decomposition of the retarded bilinear form B_R, defined in (5.1.13), as $B_R(F,G) = \sum_j B_{R,j}(F,G)$, where

$$B_{R,j}(F,G) = \iint_{t-2^{j+1}<s\leq t-2^j} \langle (U(s))^* F(s), (U(t))^* G(t) \, \mathrm{d}s \, \mathrm{d}t \rangle , \tag{5.1.14}$$

so that (5.1.9) is reduced to the estimate

$$\sum_{j\in\mathbb{Z}} |B_{R,j}(F,G)| \lesssim \|F\|_{L_t^2 L_x^{r'}} \|G\|_{L_t^2 L_x^{r'}} , \tag{5.1.15}$$

where $\frac{1}{r'} = \frac{\sigma+1}{2\sigma}$ is the exponent conjugated to $\frac{1}{r}$ in the endpoint P.

The first step is to obtain an estimate for (5.1.14). This can be done in a whole neighborhood of $\left(\frac{1}{r}, \frac{1}{r}\right)$, denoted as D_r, (see Figure 5.1), and it takes the following form,

$$|B_{R,j}(F,G)| \lesssim 2^{-j\sigma\left(1-\frac{1}{a}-\frac{1}{b}\right)} \|F\|_{L_t^2 L_x^{a'}} \|G\|_{L_t^2 L_x^{b'}} , \tag{5.1.16}$$

for all $j \in \mathbb{Z}$, and all $\left(\frac{1}{a}, \frac{1}{b}\right) \in D_r$.

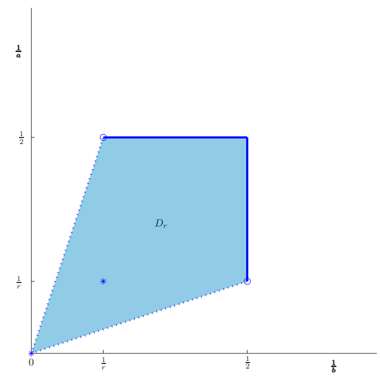

FIGURE 5.1 Neighborhood D_r of $\left(\frac{1}{r}, \frac{1}{r}\right)$.

This result can be achieved interpolating the following three cases:

(i) $a = b = \infty$;
(ii) $a \in [2, \infty)$, and $b = 2$;
(iii) $b \in [2, \infty)$ and $a = 2$.

We shall also remark that this argument breaks down, when $\sigma = 1$, then $r = \infty$. Once we have (5.1.16), the second step is to apply it to some special F and G in the form

$$F(t) = 2^{-\frac{k}{r'}} f(t) \chi_{E(t)} , \qquad G(s) = 2^{-\frac{\tilde{k}}{r'}} g(s) \chi_{\tilde{E}(s)} , \tag{5.1.17}$$

where f and g are scalar functions, $k, \tilde{k} \in \mathbb{Z}$ and $E(t), \tilde{E}(s)$ are sets of measure 2^k and $2^{\tilde{k}}$ respectively for each t and s. For such a case we may choose a and b in such a way that

$$|B_{R,j}(F,G)| \lesssim 2^{-\varepsilon(|k-j\sigma|+|\tilde{k}-j\sigma|)} \|f\|_{L_t^2} \|g\|_{L_t^2} \ , \tag{5.1.18}$$

holds for any $\varepsilon > 0$, and this will imply (5.1.15). For the case of general F and G, we can make profit of the so-called **atomic decomposition of L^p**, stated in the following result.

LEMMA 5.1. *Let $0 < p < \infty$. Then any $f \in L^p(\mathbb{R}^n)$ can be written as*

$$f(x) = \sum_{j\in\mathbb{Z}} c_k \chi_k(x) \ ,$$

where each χ_k is a function bounded by $O\left(2^{-\frac{k}{p}}\right)$ and supported on a set of measure $O\left(2^k\right)$, and the c_k's are non-negative constants such that $\|\{c_k\}\|_{l^p} \lesssim \|f\|_{L^p}$.

Applying Lemma 5.1 with $p = r'$ to $F(t)$ and $G(s)$ we have the atomic decompositions

$$F(t) = \sum_{k\in\mathbb{Z}} f_k(t)\chi_k(t) \ , \qquad G(s) = \sum_{\tilde{k}\in\mathbb{Z}} g_{\tilde{k}}(s)\tilde{\chi}_{\tilde{k}} \ , \tag{5.1.19}$$

and (5.1.15) follows from the following argument. Using the arguments for the estimate (5.1.18), and Young's inequality we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |B_{R,j}(F,G)| &\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{\tilde{k}\in\mathbb{Z}} 2^{-\varepsilon(|k-j\sigma|+|\tilde{k}-j\sigma|)} \|f_k\|_{L_t^2} \|g_{\tilde{k}}\|_{L_t^2} \lesssim \\ &\lesssim \sum_{k\in\mathbb{Z}}\sum_{\tilde{k}\in\mathbb{Z}} \left(1 + \left|k-\tilde{k}\right|\right) 2^{-\varepsilon\left|k-\tilde{k}\right|} \|f_k\|_{L_t^2} \|g_{\tilde{k}}\|_{L_t^2} \lesssim \\ &\lesssim \left(\sum_{k\in\mathbb{Z}} \|f_k\|_{L_t^2}^2\right)^{\frac{1}{2}} \left(\sum_{\tilde{k}\in\mathbb{Z}} \|g_{\tilde{k}}\|_{L_t^2}^2\right)^{\frac{1}{2}} \ ,\end{aligned}$$

and the statement (5.1.15) follows after interchanging L^2 and l^2, and including $l^{r'}$ in l^2.

Now we focalize our attention on the retarded estimate (5.1.10),

As for the (5.1.13), the **retarded estimate** (5.1.10) is equivalent to

$$|B_R(F,G)| \lesssim \|F\|_{L_t^{q'}L_x^{r'}} \|G\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}} \ , \tag{5.1.20}$$

and one can follow the same procedure used in the previous subsection, modifying the estimate 5.1.16 for the dyadic component of B_R in a suitable way. This concludes the proof of the main theorem. □

2. Strichartz Estimates for Free Wave and Schrödinger Equations

As a consequence of Theorem 5.1 we can prove the endpoint Strichartz estimates for the free wave and Schrödinger Equation in higher dimension. This completely settles the problem of determining the possible homogeneous Strichartz estimates for those problems (as far as the retarded ones are concerned it is still open). We shall need the following definition for the parameter σ.

DEFINITION 5.2. *For a given dimension n, we say that a pair (r,q) is* **wave admissible** *if $n \geq 2$, and (r,q) is $\frac{n-1}{2}$-admissible, and* **Schrödinger admissible** *if $n \geq 1$ and (r,q) is sharp $\frac{n}{2}$-admissible. In particular,*

$$P = \left(\frac{2(n-1)}{n-3}, 2\right) \text{ is wave admissible for } n > 3 \,, \tag{5.2.21a}$$

and

$$P = \left(\frac{2n}{n-2}, 2\right) \text{ is Schrödinger admissible for } n > 2 \,, \tag{5.2.21b}$$

as it is sketched in the Figures 5.1 and 5.2 below.

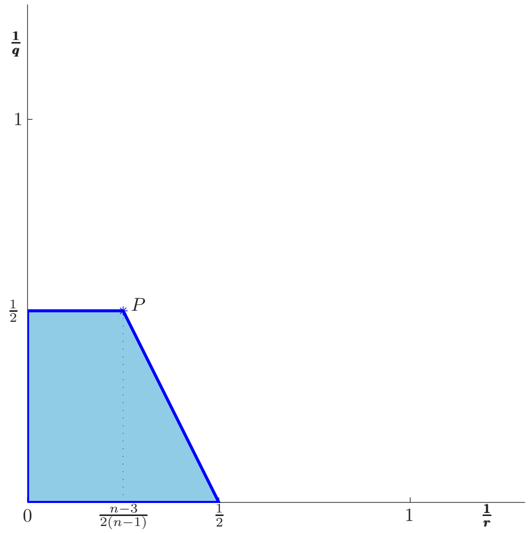

FIGURE 5.1. Wave-admissible pairs (for $n > 3$)

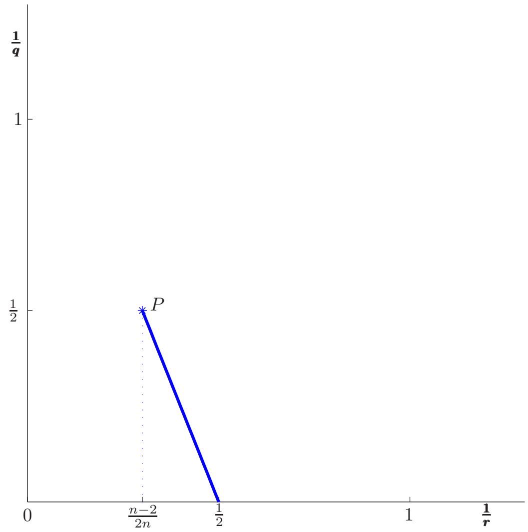

FIGURE 5.2. Schrödinger admissible pairs (for $n > 2$)

2.1. The Linear Case. In this subsection we shall focus at first our attention to the free linear wave equation first. The following Corollary can be achieved from Theorem 5.1, and it extends a long line of investigation going back to a specific space-time estimate for the linear Klein-Gordon equation in [**173**] and the fundamental paper of Strichartz [**194**] drawing the connection to the restriction theorem of Tomas and Stein.

COROLLARY 5.1. *Suppose $n \geq 2$ and (q,r) and $(\tilde{q},\tilde{r})$ are wave admissible pairs with $r,\tilde{r} < \infty$* [1]. *If u is a (weak) solution to the problem*

$$\Box u(t,x) = F(t,x) \,, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{5.2.22a}$$

$$u(0,x) = u_0(x) \,, \quad \partial_t u(0,x) = u_1(x) \,, \tag{5.2.22b}$$

[1]When $r = \infty$ the estimate (5.2.23) is still true, provided replacing the Lebesgue space $L^r(\mathbb{R}^n)$ with the Besov space $\dot{B}^0_{r,2}(\mathbb{R}^n)$, and similarly for $\tilde{r} = \infty$ (see [**73**])

for some data u_0, u_1 and F and some time $T > 0$, then

$$\|u\|_{L^q([0,T];L^r_x)} + \|u\|_{C\left([0,T];\dot{H}^{\gamma}\right)} + \|u_t\|_{C\left([0,T];\dot{H}^{\gamma-1}\right)} \lesssim \|u_0\|_{\dot{H}^{\gamma}} + \|u_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{q}'}\left([0,T];L^{\tilde{r}'}_x\right)} \;, \quad (5.2.23)$$

provided the gap condition

$$\tfrac{1}{q} + \tfrac{n}{r} = -\gamma + \tfrac{n}{2} = \tfrac{1}{\tilde{q}'} + \tfrac{n}{\tilde{r}'} - 2 \;, \quad (5.2.24)$$

holds. Conversely, if (5.2.23) *holds for all u_0, u_1, F and T, then (q, r) and $(\tilde{q}, \tilde{r})$ must be wave admissible and the gap condition must hold.*

Proof. We start with showing the necessity of the various conditions on the parameters involved. The gap condition follows from dimensional analysis (scaling considerations), whereas the admissibility conditions follow from the Knapp counterexample for the cone and its adjoint . The inadmissibility of $(q, r) = (2, \infty)$, or $(q, r) = (2, \infty)$ in the three-dimensional case was shown in [**120**]. At last the remaining conditions $q \geq q'$, and $\tilde{q} \geq \tilde{q}'$ follow from a translation invariant argument, since in the limiting case $T = \infty$ the homogeneous part of the estimate can be viewed as a time-translation invariant operator from $L^{q'}_t\left([0,\infty); L^{r'}_x(\mathbb{R}^n)\right)$ to $L^q_t\left([0,\infty); L^r_x(\mathbb{R}^n)\right)$.

Now we suppose that q and r satisfy the assumptions of the corollary, and that u is a solution of (5.2.22). We use Duhamel's principle to write u as

$$u(t,\cdot) = U(t)(u_1) + U'(t)(u_0) + (GF)(t,\cdot) \;, \quad (5.2.25)$$

where the operators U, and G are defined by functional calculus as

$$U(t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \;, \quad (5.2.26a)$$

$$GF = \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s,\cdot)\,\mathrm{d}s \;. \quad (5.2.26b)$$

Paley-Littlewood theory (see Section 11 of Chapter 1, in particular Proposition 1.6 and the following discussion) allows us to restrict our attention to the case of spatial Fourier transform of u_0, u_1 and F (and consequently u) localized in the annulus $\left\{|\xi| \approx 2^j\right\}$. By the gap condition the estimate is scale invariant, and so we may assume that $j = 0$. Combining these two reductions with (5.2.25) and (5.2.26) we see that (5.2.23) is equivalent to

$$\|U_\pm(t)u_1\|_{C(L^2_x)} \lesssim \|u_1\|_{L^2} \;, \qquad \|G_\pm F\|_{C(L^2_x)} \lesssim \|F\|_{L^{q'}_t L^{r'}_x} \;,$$

$$\|U_\pm(t)u_1\|_{L^q_t L^r_x} \lesssim \|u_1\|_{L^2} \;, \qquad \|G_\pm F\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x} \;,$$

where the localized wave evolution operator $U_\pm$, and the operator $G_\pm$ are defined as

$$\widehat{U_\pm(t)} = \chi_{[0,T]}\phi_0(\xi)\mathrm{e}^{\pm\imath t|\xi|} \;, \qquad G_\pm F = \int_{t>s} U_\pm(t)\left(U_\pm(s)\right)^* F(s,\cdot)\,\mathrm{d}s \;,$$

where ϕ_0 is the partition function of the phase space localized in $|\xi| \approx 1$.

Our firs step is to consider the $L^\infty_t L^2_x$ norms. All above estimates will follow from Theorem 5.1 with $H = L^2(\mathbb{R}^n)$ and $X = \mathbb{R}^n$, and $\sigma = \frac{n-1}{2}$, once we show that $U_\pm$ obeys the energy estimate (5.1.1) and the truncated decay estimate (5.1.2b). The former is immediate from Plancherel's Theorem, and the latter can be achieved using the stationary phase method on the kernel of $U_\pm(t)\left(U_\pm(s)\right)^*$ (see either [**181**], or [**87**] or [**186**]).

Continuity of $U_{\pm}u_1$ will follow from Plancherel's Theorem, while for $G_{\pm}F$ we can use the identity

$$(G_{\pm}F)(t+\varepsilon,\cdot) = \mathrm{e}^{\imath\varepsilon\sqrt{-\Delta}}(G_{\pm}F)(t,\cdot) + \left(G_{\pm}\chi_{[t,t+\varepsilon]}F\right)(t,\cdot)\ ,$$

the continuity of the operator $\mathrm{e}^{\imath\varepsilon\sqrt{-\Delta}}$ on L^2, and the fact that $\left\|\chi_{[t,t+\varepsilon]}F\right\|_{L_t^{q'}L_x^{r'}} \to 0$ as $\varepsilon \to 0$. □

Now we shall turn our attention to the free linear Schrödinger equation. In this case Theorem 5.1 gives the following corollary.

COROLLARY 5.2. *Suppose $n \geq 1$ and (q,r) and $(\tilde{q},\tilde{r})$ are Schrödinger admissible pairs. If u is a (weak) solution to the problem*

$$\imath u_t(t,x) - \Delta u(t,x) = F(t,x)\ ,\quad (t,x) \in [0,T]\times\mathbb{R}^n \tag{5.2.27a}$$

$$u(0,x) = u_0(x)\ , \tag{5.2.27b}$$

for some data u_0, and F and some time $0 < T < \infty$, then

$$\|u\|_{L^q([0,T];L_x^r)} + \|u\|_{C([0,T];L^2)} \lesssim \|u_0\|_{L^2} + \|F\|_{L^{\tilde{q}'}\left([0,T];L_x^{\tilde{r}'}\right)}\ . \tag{5.2.28}$$

Conversely, if (5.2.28) *holds for all u_0, F and T, then (q,r) and $(\tilde{q},\tilde{r})$ must be Schrödinger admissible.*

Proof. The proof is similar to the one of Corollary 5.1, but it is simpler, since no localization is involved. Conditions on q and r are necessary because of scaling invariance, whereas the negative result for $(r,q) = (\infty,2)$ in the two-dimensional case is proved in [**140**]. For sufficiency as in (5.2.25), we decompose u as

$$u(t,\cdot) = U(t)(u_0) + (GF)(t,\cdot)\ , \tag{5.2.29}$$

where the operators U, and G are defined by functional calculus as

$$U(t) = \chi_{[0,T]}(t)\mathrm{e}^{\imath t\sqrt{-\Delta}}\ , \tag{5.2.30a}$$

$$GF = \int_{t>s} U(t)\left(U(s)\right)^{*} F(s,\cdot)\,\mathrm{d}s\ , \tag{5.2.30b}$$

and we apply Theorem 5.1 with $H = L^2(\mathbb{R}^n)$ and $X = \mathbb{R}^n$, and $\sigma = \frac{n}{2}$. The energy estimate follows from Plancherel's Theorem, while the untruncated decay estimate can be obtained directly from the explicit representation of the solution

$$u(t,x) = \mathrm{e}^{\imath t\sqrt{-\Delta}}u_0(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\mathrm{e}^{-\frac{|x-y|^2}{2\imath t}}u_0(y)\,\mathrm{d}y\ .$$

□

CHAPTER 6

Strong endpoint Strichartz Estimates for the Schrödinger Equation with small Magnetic Potential

In this chapter we present some new results regarding the Schrödinger equation perturbed by a potential assumed to be small with respect some suitable norms. We present also a result involving the spectrum of Δ_A. The author follows, as said before, his work (joint with V. Georgiev and A. Stefanov) [**67**].

1. Introduction and statement of results

Let $A = (A_1(t,x), \cdots, A_n(t,x)), x \in \mathbb{R}^n, n \geq 3$ be a magnetic potential, such that $A_j(t,x), j = 1, \cdots, n$, are real valued functions, and let the magnetic Laplacian operator be

$$\Delta_A = \sum_j (\partial_j + iA_j)^2 = \Delta + 2iA\nabla + idiv(A) - (\sum_j A_j^2).$$

Our goal is to study the dispersive properties of the corresponding Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta_A u = F(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(0,x) = f(x). \end{cases} \tag{6.1.1}$$

In this chapter, we will be concerned with the Strichartz and smoothing estimates for (4.1.1), when the vector potential A is small in a certain sense. In fact, we aim at obtaining global scale invariant Strichartz and smoothing estimates, under appropriate scale invariant smallness assumptions on A.

In the "free" case $A = 0$, there exists vast literature on this subject. Let us introduce the mixed space-time norms

$$\|u\|_{L^q_t L^r_x} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

We say that a pair of exponents (q,r) is Strichartz admissible, if $2 \leq q, r \leq \infty$, $2/q + n/r = n/2$ and $(q,r,n) \neq (2,\infty,2)$. Then, by result of Strichartz, Ginibre-Velo, and Keel-Tao,

$$\|e^{it\Delta} f\|_{L^q_t L^r_x} \leq C\|f\|_{L^2} \tag{6.1.2}$$

$$\left\| \int_0^t e^{is\Delta} F(s,\cdot) ds \right\|_{L^2_x} \leq C \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{6.1.3}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s,\cdot) ds \right\|_{L^q_t L^r_x} \leq C \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{6.1.4}$$

where $(\tilde{q},\tilde{r})$ is another Strichartz admissible pair and $q' = q/(q-1)$. Note that for $n \geq 3$ the set of admissible pairs (q,r) can be represented equivalently as $(1/q,1/r) \in AB$, where AB is the segment with end points $A(0,1/2)$, $B(1/2,2n/(n-2))$ and we can rewrite the estimate **(6.1.3)** as

$$\left\| \int_0^t e^{is\Delta} F(s,\cdot) ds \right\|_{L^2_x} \leq C \left(\inf_{F=F_1+F_2} \|F_1\|_{L^1_t L^2_x} + \|F_2\|_{L^2_t L^{2n/(n+2)}_x} \right). \tag{6.1.5}$$

On the other hand, the smoothing estimates were established by Kenig-Ponce-Vega in the seminal paper, [**117**], see also Ruiz-Vega [**164**] . These were later extended to more general second order Schrödinger equations in [**118**]. Some possible scale and rotation invariant smoothing estimates similar to (6.1.2), (6.1.3) and (6.1.4) can be written as (see Corollary 6.1 below)

$$\sup_{m\in\mathbb{Z}} \left(2^{-m/2} 2^{k/2} \left\| e^{it\Delta} f_k \right\|_{L^2_t L^2(|x|\sim 2^m)} \right) \leq C \|f_k\|_{L^2}, \tag{6.1.6}$$

$$\left\| \int_0^t e^{is\Delta} F_k(s,\cdot) ds \right\|_{L^2_x} \leq C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} 2^{-k/2} \|F_k\|_{L^2_t L^2(|x|\sim 2^m)} \right), \tag{6.1.7}$$

$$\sup_{m\in\mathbb{Z}} \left(2^{-m/2} 2^{k/2} \left\| \int_0^t e^{i(t-s)\Delta} F_k(s,\cdot) ds \right\|_{L^2_t L^2(|x|\sim 2^m)} \right) \leq \tag{6.1.8}$$

$$\leq C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} 2^{-k/2} \|F_k\|_{L^2_t L^2(|x|\sim 2^m)} \right),$$

where k is any integer, $\phi_k := P_k\phi$ is the k^{th} Littlewood-Paley piece of ϕ (see Section 2.1 below).

Motivated by these estimates, given any integer $k \in \mathbb{Z}$ introduce the spaces Y_k, defined by the norms[1]

$$\|\phi\|_{Y_k} = 2^{-k/2} \sum_m 2^{m/2} \|\phi_k\|_{L^2_t L^2(|x|\sim 2^m)}.$$

Now we can define the Banach spaces Y as a closure of the functions

$$\phi(t,x) \in C_0^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus 0))$$

with respect to the norm

$$\|\phi\|_Y := \left(\sum_k \|\phi\|^2_{Y_k} \right)^{1/2}. \tag{6.1.9}$$

Its dual space Y' consists of tempered distributions $S'(\mathbb{R}\times\mathbb{R}^n)$, having finite norm

$$\|\phi\|_{Y'} := \left(\sum_k \|\phi\|^2_{Y'_k} \right)^{1/2},$$

where

$$\|\phi\|_{Y'_k} = 2^{k/2} \sup_m 2^{-m/2} \|\phi_k\|_{L^2_t L^2(|x|\sim 2^m)}.$$

[1]The expressions $\phi \to \|\phi\|_{Y_k}$ are not faithfull norms, in the sense that may be zero, even for some $\phi \neq 0$. On the other hand, they satisfy all the other norm requirements and $\phi \to (\sum_k \|\phi_k\|^2_{Y_k})^{1/2}$ is a norm!

Then the smoothing estimates (6.1.6), (6.1.7) and (6.1.8) imply

$$\left\|e^{it\Delta}f\right\|_{Y'} \leq C\left\|f\right\|_{L^2}\ ,\quad \left\|\int_0^t e^{is\Delta}F(s,\cdot)ds\right\|_{L^2_x} \leq C\left\|F\right\|_{Y}. \tag{6.1.10}$$

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds\right\|_{Y'} \leq C\left\|F\right\|_{Y}. \tag{6.1.11}$$

Motivated by the Strichartz estimates and Besov versions of "local smoothing" norms, we introduce the spaces

$$X = L^1_tL^2_x + L^2_tL^{2n/(n+2)}_x + Y$$

with norm

$$\|F\|_X = \inf_{F=F^{(1)}+F^{(2)}+F^{(3)}} \left\|F^{(1)}\right\|_{L^1_tL^2_x} + \left\|F^{(2)}\right\|_{L^2_tL^{2n/(n+2)}_x} + \left\|F^{(3)}\right\|_{Y}.$$

The dual space to X space is X' and the norm in this space is defined in a similar way:

$$\|\phi\|_{X'} := \left(\sum_k \|\phi\|^2_{X'_k}\right)^{1/2}, \tag{6.1.12}$$

where

$$\|\phi\|_{X'_k} = \sup_{(q,r)-Str.} \|\phi_k\|_{L^q_tL^r_x} + 2^{k/2}\sup_m 2^{-m/2}\|\phi_k\|_{L^2_tL^2(|x|\sim 2^m)}.$$

The main result of this chapter is the following:

THEOREM 6.1. *If $n \geq 3$, then one can find a positive number $\varepsilon > 0$ so that for any (vector) potential $A = A(t,x)$ satisfying*

$$\|A\|_{L^\infty L^n} + \|\nabla A\|_{L^\infty L^{n/2}} + \sup_k\left(\sum_m 2^m\|A_{<k}\|_{L^\infty L^\infty(|x|\sim 2^m)}\right) \leq \varepsilon, \tag{6.1.13}$$

there exists $C > 0$, such that for any $F(t,x) \in S(\mathbb{R}\times\mathbb{R}^n)$ we have the estimate

$$\left\|\int_{t>s} e^{i(t-s)\Delta_A}F(s,\cdot)ds\right\|_{X'} \leq C\|F\|_X.$$

In particular, the solutions to (4.1.1) *satisfy the smoothing - Strichartz estimate*

$$\|u\|_{X'} \lesssim \|f\|_{L^2} + \|F\|_X. \tag{6.1.14}$$

REMARK 6.1. *The estimate* (6.1.14) *implies various interesting inequalities. For example we have the classical Strichartz estimate*

$$\sup_{(q,r)-Str.}\|u\|_{L^qL^r} \lesssim \|f\|_{L^2} + \inf_{F=F_1+F_2}\|F_1\|_{L^1_tL^2_x} + \|F_2\|_{L^2_tL^{2n/(n+2)}_x}$$

as well as the smoothing - Strichartz estimates

$$\|u\|_{Y'} \lesssim \|f\|_{L^2} + \inf_{F=F_1+F_2}\|F_1\|_{L^1_tL^2_x} + \|F_2\|_{L^2_tL^{2n/(n+2)}_x},$$

$$\sup_{(q,r)-Str.}\|u\|_{L^qL^r} \lesssim \|f\|_{L^2} + \|F\|_Y.$$

The main idea to prove this theorem is to apply appropriate scale invariant estimate for the free Schrödinger equation involving Strichartz and smoothing type norms.

Estimates of this type have been obtained earlier in [**163**] and [**164**] with Strichartz type norms of the form $\|F\|_{L_x^{2n/(n+2)}L_t^2}$. Recently, we found a similar estimate in the work [**95**] and this estimate has the form

$$\|D_x^{1/2}\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds\|_{L^\infty_{x_1}L^2_{x'}L^2_t} \leq C\|F\|_{L^2_tL_x^{2n/(n+2)}}. \tag{6.1.15}$$

On one hand, this estimate can be used to derive the Strichartz estimate for the perturbed Schrödinger equation provided its (formally) "dual" version

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds\right\|_{L^2_tL_x^{2n/(n-2)}} \leq C\|D_x^{-1/2}F\|_{L^1_{x_1}L^2_{x'}L^2_t}. \tag{6.1.16}$$

is verified . The properties $(L^\infty)' \neq L^1$ and $(L^1)' = L^\infty$ show that (6.1.16) implies (6.1.15), but not viceversa. However, we establish (6.1.16) and show that these estimates are stable under small magnetic perturbations satisfying (6.1.13).

2. Preliminaries

2.1. Fourier transform and Littlewood-Paley projections. Define the Fourier transform and its inverse by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$$

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

Introduce a positive, decreasing, smooth away from zero function $\chi : \mathbf{R}^1_+ \to \mathbf{R}^1_+$, supported in $\{\xi : 0 \leq \xi \leq 2\}$ and $\chi(\xi) = 1$, for all $0 \leq \xi \leq 1$. Define $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$, which is positive and supported in the annulus $\{1/2 \leq |\xi| \leq 2\}$. We have that φ is smooth and $\sum_{k\in\mathcal{Z}}\varphi(2^{-k}\xi) = 1$ for all $\xi \neq 0$. In higher dimensions, we slightly abuse the notations and denote a function with similar properties by the same name, i.e. $\varphi(\xi) = \varphi(|\xi|)$, $\chi(\xi) = \chi(|\xi|)$ etc. Note that for $n > 1$, $\chi(\xi) : \mathbf{R}^n \to \mathbf{R}^1$ is a smooth function even at zero.

The k^{th} Littlewood-Paley projection (see also Section 11, Chapter 1 for more details) is defined as a multiplier type operator by $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Note that the kernel of P_k is integrable, smooth and real valued for every k. In particular, it is bounded on every $L^p : 1 \leq p \leq \infty$ and it commutes with differential operators. Another helpful observation is that for the differential operator D_x^s defined via the multiplier $|\xi|^s$, one has

$$D_x^s P_k u = 2^{ks}\tilde{P}_k u,$$

where $\tilde{P}_k$ is given by the multiplier $\tilde{\varphi}(2^{-k}\xi)$, where $\tilde{\varphi}(\xi) = \varphi(\xi)|\xi|^s$.

We also consider $P_{<k} := \sum_{l<k} P_l$, which essentially restricts the Fourier transform to frequencies $\lesssim 2^k$.

Define also the function $\psi(\xi) = \chi(\xi/4) - \chi(4\xi)$. Note that ψ has similar support properties as φ and $\psi(\xi)\varphi(\xi) = \varphi(\xi)$. Thus, we may also define the operators Z_k by $\widehat{Z_k f}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi)$. By the construction, $Z_k P_k = P_k$ and $Z_k = P_{k-2} + \ldots + P_{k+1}$.
Recall a version of the Calderón commutator estimate (see for example Lemma 2.1 in the work of Rodnianski and Tao, [**161**]), which reads

$$\|[P_k, f]g\|_{L^r} \leq C2^{-k} \|\nabla f\|_{L^q} \|g\|_{L^p},$$

whenever $1 \leq r, p, q \leq \infty$ and $1/r = 1/q + 1/p$.
Also of interest will be the properties of products under the action of P_k. Starting with the relations

$$P_k(fg) = \sum_{\ell,m} P_k(f_\ell g_m),$$

$$P_k(f_\ell g_m) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} P_k(\xi)P_\ell(\xi-\eta)\hat{f}(\xi-\eta)P_m(\eta)\hat{g}(\eta)e^{2\pi i x\cdot\xi}d\xi d\eta,$$

we exploit the property $\operatorname{supp} P_k(\xi) \subseteq \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ and see that the sum can be restricted to the set

$$\{|\ell - m| \geq 2 + N_0,\ |\max(\ell, m) - k| \leq 3\} \cup \{|\ell - m| \leq 1 + N_0,\ k \leq \max(\ell, m) + 3\},$$

where $N_0 \geq 1$ is an arbitrary number. This domain can be enlarged slightly using the inequality $\max(\ell, m) \leq \ell + 1 + N_0$ provided $|\ell - m| \leq 1 + N_0$. So we can restrict the sum over the union of the following sets (the first two are disjoint for $N_0 \geq 5$, while the third one can overlap with them)

$$\{m \leq k - N_0 + 1,\ |\ell - k| \leq 3\}, \{\ell \leq k - N_0 + 1,\ |m - k| \leq 3\}$$

and

$$\{|\ell - m| \leq 1 + N_0,\ \ell \geq k - N_0 - 4\}.$$

In conclusion, for any two (Schwartz) functions f, g we have the pointwise estimate

$$\begin{aligned}|P_k(fg)(x)| \quad &\leq \quad \sum_{l \geq k-N_0-4}\ \sum_{|m-\ell|\leq 1+N_0} |P_k(f_l g_m)(x)| + \\ &+ \quad |P_k(f_{\leq k-N_0+1}\ g_{k-3\leq\cdot\leq k+3})(x)| + \\ &\quad +|P_k(f_{k-3\leq\cdot\leq k+3}\ g_{\leq k-N_0+1})(x)|.\end{aligned}$$

Taking for determinacy $N_0 = 7$, we get

$$\begin{aligned}&|P_k(fg)(x)| \leq |f_{\leq k-6}(x)g_k(x)| + |[P_k, f_{\leq k-6}]g_{k-3\leq\cdot\leq k+3}(x)| + \\ &+|P_k(f_{k-3\leq\cdot\leq k+3}\ g_{\leq k-6})(x)| + \sum_{l\geq k-11}\ \sum_{|m-\ell|\leq 8} |P_k(f_l g_m)(x)|.\end{aligned}$$

In particular, we need an appropriate (product like!) expression for
$P_k(A\nabla u)$. The main term is clearly when ∇u is in high frequency mode, while $\vec{A}$ is low frequency. More precisely, according to our considerations above,

$$P_k(A\nabla u) = A_{\leq k-6}\nabla u_k + E^k,$$

where $E^k(x)$ satisfies the pointwise estimate

$$|E^k(x)| \leq |[P_k, A_{\leq k-6}]\nabla u_{k-3\leq \cdot k+3}(x)| + \tag{6.2.17}$$
$$\sum_{l\geq k-11}\ \sum_{|m-\ell|\leq 8} |P_k(A_l \cdot \nabla u_m)(x)| + |P_k(A_{k-3\leq \cdot \leq k+3} \cdot \nabla u_{\leq k-6})(x)|$$

Note that in terms of L^p behavior and Littlewood-Paley theory, one treats these error terms as if they would appear in the form $(\partial_x A)u$.

2.2. Besov space versions of the "local smoothing space". The space Y was introduced as the closure of $S(\mathbb{R}\times\mathbb{R}^n)$ with respect to the norm in (6.1.9), where

$$\|\phi\|_{Y_k} = 2^{k/2} \sum_m 2^{-m/2} \|P_k\phi\|_{L^2_t L^2(|x|\sim 2^m)} . \tag{6.2.18}$$

We can replace $\|F\|_{L^2(|x|\sim 2^m)}$ by the comparable expression $\|\varphi(2^{-m}\cdot)F\|_{L^2}$. This will be done frequently (and without much discussions) in the sequel in order to make use of the Plancherel's theorem, which is of course valid only in the global L^2 space. We mention also that the norm $\|\phi\|_Y$ is scale invariant for rescale factors any diadic number.

We show that the "local smoothing space" defined as a closure of Schwartz functions ϕ with respect to "local smoothing norms"

$$\sum_m 2^{m/2} \left\| D_x^{-1/2}\phi(t,x)\right\|_{L^2_t L^2(|x|\sim 2^m)}$$

can be embedded in Y.

LEMMA 6.1. *There is a constant $C = C(n)$, so that for every Schwartz function ϕ we have*

$$\|\phi\|_Y \leq C \sum_m 2^{m/2} \left\| D_x^{-1/2}\phi(t,x)\right\|_{L^2_t L^2(|x|\sim 2^m)} . \tag{6.2.19}$$

Proof. Taking into account the definition of the space Y, it is sufficient to establish the estimate:

$$\|\phi\|_{Y_k} \leq C \sum_m 2^{m/2} \left\| D_x^{-1/2}\phi_k(t,x)\right\|_{L^2_t L^2(|x|\sim 2^m)}$$

for any integer k. Using the scale invariance of the estimate we see that we lose no generality taking $k=0$. Thus, we have to verify the estimate

$$\sum_m 2^{m/2} \left\|\varphi(2^{-m}\cdot)P_0\phi\right\|_{L^2_t L^2_x} \leq C \sum_m 2^{m/2} \left\| \varphi(2^{-m}\cdot) D_x^{-1/2}\phi_0(t,x)\right\|_{L^2_t L^2_x} .$$

Since

$$P_0\phi = \sum_{|k|\leq 2} P_0 D_x^{1/2} D_x^{-1/2} P_k\phi = \sum_{|k|\leq 2}\sum_{\ell\in\mathbb{Z}} \widetilde{P}_0 \varphi(2^{-\ell}\cdot) D_x^{-1/2} P_k \phi,$$

we can apply the triangle inequality, and reduce the proof to the following estimate

$$\sum_m 2^{m/2} \sum_{\ell\in\mathbb{Z}} \left\|\varphi(2^{-m}\cdot)\widetilde{P}_0 \varphi(2^{-\ell}\cdot) D_x^{-1/2} P_k\phi\right\|_{L^2_t L^2_x} \leq$$
$$\leq C \sum_\ell 2^{\ell/2} \left\|\varphi(2^{-\ell}\cdot) D_x^{-1/2}\phi_k(t,x)\right\|_{L^2_t L^2_x} ,$$

where $k \in \mathbb{Z}, |k| \leq 2$. This estimate follows easily from

$$\left\| \varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f \right\|_{L^2_x} \leq C \|f\|_{L^2_x}, \ m \leq \ell+1 \tag{6.2.20}$$

$$\left\| \varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f \right\|_{L^2_x} \leq C2^{-m} \|f\|_{L^2_x}, \ m \geq \ell+2 \tag{6.2.21}$$

and the obvious observation that

$$\sum_{m \leq \ell+1} 2^{m/2} + \sum_{m \geq \ell+2} 2^{m/2}2^{-m} \lesssim 2^{\ell/2}.$$

The estimate (6.2.20) is obvious, while the proof of (6.2.21) follows from

$$\varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f = [\varphi(2^{-m}\cdot), \widetilde{P}_0]\varphi(2^{-\ell}\cdot)f, \ m \geq \ell+2$$

and the Calderón estimate

$$\left\| [\varphi(2^{-m}\cdot), \widetilde{P}_0]g \right\|_{L^2_x} \leq C2^{-m} \|g\|_{L^2_x}.$$

This completes the proof of the lemma. □

REMARK 6.1. *Note that the argument in the proof of this lemma implies also the estimates*

$$\|P(D)f_k\|_Y \lesssim \|f_k\|_Y = \|f\|_{Y_k}, \ \forall k \in \mathbb{Z} \tag{6.2.22}$$

for any pseudodiferential operator with symbol $P(\xi) \in C_0^\infty(\mathbb{R}^n)$.

We have also the following estimate (dual to (6.2.19))

LEMMA 6.2. *There is a constant* $C = C(n)$*, so that for every Schwartz function* $\phi \in Y'$*, we have*

$$\sup_m 2^{-m/2} \left\| D_x^{1/2}\phi(t,x) \right\|_{L^2_t L^2(|x|\sim 2^m)} \leq C_n \|\phi\|_{Y'}. \tag{6.2.23}$$

REMARK 6.2. *Some generalizations of the previous two lemmas can be seen in Theorem 1.6 and Theorem 1.7 in* [**69**].

3. Estimates for the bilinear form $Q(F,G)$

The sesquilinear form

$$Q(F,G) = \int \int_{t>s} \langle e^{i(t-s)\Delta}F(s), G(t)\rangle_{L^2(\mathbb{R}^n)} ds dt$$

with Schwartz functions F, G, was used in [**115**] to derive Strichartz estimates (with endpoint) and this estimates can be expressed in terms of Q

$$|Q(F,G)| \leq C \|F\|_{L_t^{q_1'} L_x^{r_1'}} \|G\|_{L_t^{q_2'} L_x^{r_2'}}, \tag{6.3.24}$$

for all Strichartz pairs (q_1, r_1), (q_2, r_2).

We have the following estimate that can be obtained by applying Lemma 3 from the work of Ionescu-Kenig [**95**].

THEOREM 6.2. *There exists a constant $C = C(n)$ so that for any integer k, any $F(t, x) \in S(\mathbb{R}\times\mathbb{R}^n)$ and $G(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$*

$$|Q(F_k, G_k)| \leq C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} 2^{-k/2} \|\varphi(2^{-m}\cdot)F_k\|_{L^2_t L^2_x} \right) \|G_k\|_{L^2_t L^{2n/(n+2)}_x}. \tag{6.3.25}$$

We have also the following energy-smoothing estimate.

THEOREM 6.3. *There exists a constant $C = C(n)$ so that for any integer k, any $F(t, x) \in S(\mathbb{R}\times\mathbb{R}^n)$ and $G(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$*

$$|Q(F_k, G_k)| \leq C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} 2^{-k/2} \|\varphi(2^{-m}\cdot)F_k\|_{L^2_t L^2_x} \right) \|G_k\|_{L^1_t L^2_x}. \tag{6.3.26}$$

Before proving these theorems, we verify some of the smoothing estimates used in this chapter.

3.1. Estimates in the local smoothing space. For $n = 1$ we have the following smoothing estimates (see Kenig, Ponce, Vega [**116, 117**])

$$2^{k/2}\|e^{-it\Delta}f_k\|_{L^\infty_x L^2(\gamma)} \leq C\|f_k\|_{L^2}, \tag{6.3.27}$$

$$2^{k/2}\left\|\int_{s<t} e^{-i(t-s)\Delta}F_k(s)ds\right\|_{L^\infty_x L^2_t} \leq C\|F_k\|_{L^1_x L^2_t}, \tag{6.3.28}$$

as well as

$$2^{k/2}\left\|\int_\gamma e^{-it\Delta}F_k(t)dt\right\|_{L^2} \leq C\|F_k\|_{L^1_x L^2(\gamma)} \tag{6.3.29}$$

for any interval $\gamma \subseteq \mathbb{R}_t$. Here $C > 0$ is a constant independent of f, F, γ.

For $n > 1$ we may assume

$$\text{supp } \widehat{f}(\xi) \subseteq \{|\xi'| \leq \xi_1/10, \ \ \xi' = (\xi_2, \cdots, \xi_n)\}. \tag{6.3.30}$$

Then we have the representation

$$\left(e^{-it\Delta}f\right)(x_1, x') = \tag{6.3.31}$$

$$= c\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{it|\xi'|^2+i(x'-y')\xi'}\left(e^{-it\Delta_1}f\right)(x_1, y')\, d\xi' dy',$$

where $\Delta_1 = \partial^2_{x_1}$.

This representation and the one–dimensional estimates (6.3.27), (6.3.28) and (6.3.29) lead to the following statement:

LEMMA 6.3. *There exists a constant C depending only on the dimension, so that for any $f \in S(\mathbb{R}^n), F \in S(\mathbb{R} \times \mathbb{R}^n)$, satisfying* (6.3.30) *and*

$$\text{supp}_\xi \ \widehat{F}(t, \xi) \subseteq \{|\xi'| \leq \xi_1/10, \ \ \xi' = (\xi_2, \cdots, \xi_n)\} \tag{6.3.32}$$

we have

$$2^{k/2}\|e^{-it\Delta}f_k\|_{L^\infty_{x_1} L^2_{x'}, L^2_t(\gamma)} \leq C\|f_k\|_{L^2_{x'}, L^2_{x_1}} \tag{6.3.33}$$

$$2^{k/2}\left\|\int_{s<t} e^{-i(t-s)\Delta}F_k(s)ds\right\|_{L^\infty_{x_1} L^2_{x',t}} \leq C\|F_k\|_{L^1_{x_1} L^2_{x',t}}, \tag{6.3.34}$$

and

$$2^{k/2}\left\|\int_\gamma e^{-it\Delta}F_k(t)dt\right\|_{L^2_{x_1}L^2_{x'}} \leq C\|F_k\|_{L^1_{x_1}L^2_{x'}L^2_t(\gamma)} \tag{6.3.35}$$

for any interval $\gamma \subseteq \mathbb{R}_t$.

Proof. To prove (6.3.33) we use (6.3.31) and find

$$\left(\widehat{e^{-it\Delta}f}\right)(x_1,\xi') = \tag{6.3.36}$$

$$= ce^{it|\xi'|^2}\left(e^{-it\Delta_1}\widehat{f}\right)(x_1,\xi') = ce^{-it\Delta_1}\left(e^{it|\xi'|^2}\widehat{f}(\cdot,\xi')\right)(x_1).$$

Note that

$$P_k(\xi_1) \sim P_k(\xi) \tag{6.3.37}$$

for $\xi \in \text{supp}_\xi\ \widehat{f}$ due to (6.3.30). From this observation, the one dimensional estimate (6.3.27) and the Plancherel identity imply (6.3.33), since

$$\|\hat{f}(x_1,\xi')\|_{L^2_{\xi'}L^\infty_{x_1}} \geq \|\hat{f}(x_1,\xi')\|_{L^\infty_{x_1}L^2_{\xi'}} = \|f(x_1,x')\|_{L^\infty_{x_1}L^2_{x'}}.$$

In a similar way we prove (6.3.34) and (6.3.35). This completes the proof of the lemma. □

Applying Hölder inequality

$$\|g\|_{L^1_{x_1}} \lesssim \sum_{m\in\mathbb{Z}} 2^{m/2}\|g\|_{L^2_{x_1}(|x|\sim 2^m)},\ \sup_{m\in\mathbb{Z}} 2^{-m/2}\|g\|_{L^2_{x_1}(|x|\sim 2^m)} \leq \|g\|_{L^\infty_{x_1}},$$

we obtain

COROLLARY 6.1. *The smoothing estimates* (6.1.6)*,* (6.1.7)*,* (6.1.8) *are satisfied.*

By Corollary 6.1 one gets

$$\begin{aligned}|Q(F_k,G_k)| \leq C_n(\sum_m 2^{-k/2}2^{m/2}\,\|F_k\|_{L^2_tL^2(|x|\sim 2^m)})\times \\ \times(\sum_m 2^{-k/2}2^{m/2}\,\|G_k\|_{L^2_tL^2(|x|\sim 2^m)}).\end{aligned} \tag{6.3.38}$$

After this prepartion, we turn to

3.2. Proof of Theorem 6.2: Bilinear smoothing-Strichartz estimate. The estimate (6.3.25) is scale invariant and for this we can take $k=0$. We have the relation

$$Q(F,G) = \int\int_{\mathbb{R}^2}\langle e^{i(t-s)\Delta}F(s),G(t)\rangle_{L^2(\mathbb{R}^n)}dsdt-$$

$$-\int\int_{t<s}\langle e^{i(t-s)\Delta}F(s),G(t)\rangle_{L^2(\mathbb{R}^n)}dsdt$$

For the form

$$Q_0(F,G) = \left\langle \int_{\mathbb{R}} ds\ e^{-is\Delta}F(s), \int_{\mathbb{R}} dt\ e^{-it\Delta}G(t)\right\rangle_{L^2(\mathbb{R}^n)}$$

we can apply the Cauchy inequality and via (6.1.3) and (6.1.7) we get

$$|Q_0(F_0,G_0)| \leq C\left(\sum_{m\in\mathbb{Z}} 2^{m/2}\|\varphi(2^{-m}\cdot)F_0\|_{L^2_tL^2_x}\right)\|G_0\|_{L^2_tL^{2n/(n+2)}_x}.$$

Hence it remains to evaluate the form

$$Q^*(F,G) = \int \int_{t<s} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)}$$

and verify the inequality

$$|Q^*(F_0,G_0)| \le C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} \|\varphi(2^{-m}\cdot)F_0\|_{L^2_t L^2_x} \right) \|G_0\|_{L^2_t L^{2n/(n+2)}_x}. \tag{6.3.39}$$

To prove (6.3.39) it is sufficient to consider F with

$$\operatorname{supp}_\xi \ \widehat{F}(t,\xi) \subseteq \{|\xi'| \le \xi_1/10, \ \ \xi' = (\xi_2, \cdots, \xi_n)\}. \tag{6.3.40}$$

Also, note that

$$Q^*(F,G) = \int_{\mathbb{R}\times\mathbb{R}^n} F(s,y)\overline{u(s,y)}\, dsdy,$$

where u is a solution to the free Schrödinger equation $i\partial_t u + \Delta u = G$ having initial data identically 0.

With (6.3.40) in mind, applying Lemma 3 from Ionescu-Kenig [**95**], we get

$$\|D^{1/2}_{x_1} u\|_{L^\infty_{x_1} L^2_{x',t}} \lesssim \|G\|_{L^2_t L^{2n/(n+2)}_x}. \tag{6.3.41}$$

Here and below we use the notations $x = (x_1, x')$, $x' = (x_2, \cdots, x_n)$. So we have

$$|Q^*(F,G)| \le C \left(\|D^{-1/2}_{x_1} F\|_{L^1_{x_1} L^2_{x',t}} \right) \|G\|_{L^2_t L^{2n/(n+2)}_x}. \tag{6.3.42}$$

Thus, we need to establish the inequality

$$\|D^{-1/2}_{x_1} F_0\|_{L^1_{x_1} L^2_{x',t}} \lesssim \|F_0\|_{Y_0} = \sum_{m\in\mathbb{Z}} 2^{m/2} \|\varphi(2^{-m}\cdot)F_0\|_{L^2_t L^2_x}.$$

For the purpose it is sufficient to apply (6.2.22), the Hölder inequality

$$\|g\|_{L^1_{x_1}} \lesssim \sum_{m\in\mathbb{Z}} 2^{m/2} \|g\|_{L^2_{x_1}},$$

and note that

$$D^{-1/2}_{x_1} F_0 = P(D) F_0,$$

for some $P(\xi) \in C^\infty_0(\mathbb{R}^n)$ due to our assumption (6.3.40). This completes the proof of the theorem.

3.3. Proof of Theorem 6.3: bilinear energy – smoothing estimate. The proof follows the same line of the proof of Theorem 6.2 with the following changement: in the place of Ionescu-Kenig inequality (6.3.41) we use

$$\sup_t \left\| \int_0^t e^{i(t-s)\Delta} F_k(s,\cdot) ds \right\|_{L^2_x} \le C \left(\sum_{m\in\mathbb{Z}} 2^{m/2} 2^{-k/2} \|F_k\|_{L^2_t L^2(|x|\sim 2^m)} \right). \tag{6.3.43}$$

This estimate is trivial, since by the L^2 energy conservation, the left-hand side of this inequality is equal to

$$\sup_t \left\| \int_0^t e^{-is\Delta} F_k(s,\cdot) ds \right\|_{L^2_x}$$

and applying the estimate (6.1.7), we can finish the proof as before.

4. Proof of Theorem 6.1

We start by some reductions of the problem. First, note that (4.1.1) is in the form

$$\begin{cases} \partial_t u - i\Delta u + 2A\nabla u = \tilde{F}(t,x) \\ u(0,x) = f(x), \end{cases} \tag{6.4.44}$$

where $\tilde{F} = F - div(A)u - i(\sum_j A_j^2)u$. We claim that is suffices to prove

$$\|u\|_{X'} \leq C_n(\|f\|_{L^2} + \|\tilde{F}\|_X), \tag{6.4.45}$$

for the solutions of (6.4.44). Indeed, assuming the validity of (6.4.45) and since by our assumptions and Sobolev embedding $\|\nabla A\|_{L^\infty_t L^{n/2}_x} + \|A\|_{L^\infty_t L^n_x} \leq C \|\nabla A\|_{L^\infty_t L^{n/2}_x} \leq C\varepsilon$, we have

$$\begin{aligned} &\|u\|_{X'} \leq C\|f\|_{L^2} + C\|\tilde{F}\|_X \leq \\ &\leq C\|f\|_{L^2} + C\|F\|_X + C(\|\nabla A\|_{L^\infty_t L^{n/2}_x} + \|A\|^2_{L^\infty_t L^n_x})\|u\|_{L^2 L^{2n/(n-2)}} \\ &\leq C_n\|f\|_{L^2} + C_n\|F\|_X + C_n\varepsilon\|u\|_{L^2 L^{2n/(n-2)}} \leq \\ &\leq C_n\|f\|_{L^2} + C_n\|F\|_X + C_n\varepsilon\|u\|_{X'}. \end{aligned}$$

It follows that

$$\|u\|_{X'} \leq C\|f\|_{L^2} + C\|F\|_X,$$

as claimed, as long as $\varepsilon : C_n\varepsilon < 1/2$.

Thus, we concentrate on showing (6.4.45) for the solutions of (6.4.44), where we denote the right hand side by F again.

Next, we take a Littlewood-Paley projection of (6.4.44). We get

$$\partial_t u_k - i\Delta u_k = F_k - 2A_{<k-6}\nabla u_k - 2E^k := H_k,$$

where E^k is the error term $E^k = P_k(A\nabla u) - A_{\leq k-6}\nabla u_k$ given by (6.2.17).

We will show that the solution to $\partial_t u_k - i\Delta u_k = H_k$ with initial data $u_k(0,x) = f_k$, satisfies the estimate

$$\|u_k\|_{X'} \leq C\|f_k\|_{L^2} + C\|H_k\|_X. \tag{6.4.46}$$

First we will show how (6.4.46) implies Theorem 6.1 and then we proceed to show (6.4.46).

4.1. (6.4.46) implies Theorem 6.1.

Apply (6.4.46) to u_k. We have

$$\begin{aligned} &\|u_k\|_{X'} \leq C\|f_k\|_{L^2} + C(\|F_k\|_X + \left\|E^k\right\|_{L^2 L^{2n/(n+2)}} + \\ &+C\sum_m 2^{m/2} 2^{-k/2}\|A_{<k-6}\nabla u_k\|_{L^2 L^2(|x|\sim 2^m)}). \end{aligned} \tag{6.4.47}$$

We will need the following estimates:

$$(\sum_k \|E^k\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \leq C_n\varepsilon(\sum_k \|u\|^2_{L^2 L^{2n/(n-2)}})^{1/2} \leq C_n\varepsilon\|u\|_{X'} \tag{6.4.48}$$

$$\sum_m 2^{m/2} 2^{-k/2}\|A_{<k-6}\nabla u_k\|_{L^2 L^2(|x|\sim 2^m)} \leq C_n\varepsilon\|u_k\|_{X'_k}. \tag{6.4.49}$$

Let us show how based on (6.4.48) and (6.4.49), we finish the proof of Theorem 6.1. Plugging in these estimates in (6.4.47), using the definition (6.1.12) for X' and square summing in k yields

$$\|u\|_{X'} = \left(\sum_k \|u_k\|^2_{X'_k}\right)^{1/2} \leq C_n(\|f\|_{L^2} + \|F\|_X) + C_n\varepsilon \|u\|_{X'},$$

whence with the choice of $\varepsilon : C_n\varepsilon < 1/2$,

$$\|u\|_{X'} \leq C_n(\|f\|_{L^2} + \|F\|_X).$$

Thus, in this section, remains to show (6.4.48) and (6.4.49).

4.1.1. *Proof of* (6.4.49). Let $\widetilde{k}$ be integer with $|k - \widetilde{k}| \leq 3$. We have

$$\begin{aligned}
&\sum_m 2^{m/2}2^{-k/2} \left\|A_{<k-6}\nabla u_{\widetilde{k}}\right\|_{L^2L^2(|x|\sim 2^m)} \lesssim \\
&\lesssim \left(\sum_m 2^m \|A_{<k-5}\|_{L^\infty L^\infty(|x|\sim 2^m)}\right) \sup_m 2^{-m/2}2^{-k/2} \left\|\nabla u_{\widetilde{k}}\right\|_{L^2L^2(|x|\sim 2^m)} \leq \\
&\leq C_n\varepsilon \sup_m 2^{-m/2}2^{-k/2} \left\|\nabla u_{\widetilde{k}}\right\|_{L^2L^2(|x|\sim 2^m)}.
\end{aligned}$$

This last expression is very similar to $\left\|u_{\widetilde{k}}\right\|_{X'}$. We will show that it is controlled by it, which of course is enough to establish (6.4.49).

Fix an m. Then

$$2^{-m/2}2^{-k/2} \left\|\nabla u_{\widetilde{k}}\right\|_{L^2L^2(|x|\sim 2^m)} \lesssim 2^{-m/2}2^{k/2} \left\|\varphi(2^{-m}\cdot)Q_{\widetilde{k}}u_{\widetilde{k}}\right\|_{L^2L^2},$$

where Q_k acts as a (vector) multiplier $\psi(2^{-k}\xi)2^{-k}\xi$. We have by the Calderón commutator estimate[1] and the Bernstein inequality

$$\begin{aligned}
&2^{-m/2}2^{k/2} \|\varphi(2^{-m}\cdot)Q_k u_k\|_{L^2L^2} \leq 2^{-m/2}2^{k/2} \|Q_k(\varphi(2^{-m}\cdot)u_k)\|_{L^2L^2} + \\
&+2^{-m/2}2^{k/2} \|[Q_k, \varphi(2^{-m}\cdot)]u_k\|_{L^2L^2} \lesssim 2^{-m/2}2^{k/2} \|\varphi(2^{-m}\cdot)u_k\|_{L^2L^2} + \\
&+2^{-k/2} \|u_k\|_{L^2_tL^{2n/(n-3)}} \lesssim 2^{-m/2}2^{k/2} \|\varphi(2^{-m}\cdot)u_k\|_{L^2L^2} + \|u_k\|_{L^2_tL^{2n/(n-2)}} \\
&\leq C_n \|u_k\|_{X'_k}.
\end{aligned}$$

4.1.2. *Proof of* (6.4.48). We treat E^k on a term-by-term basis in (6.2.17). For the first term, by Calderón commutators,

$$\begin{aligned}
&(\sum_k \|[P_k, A_{<k-6}]\nabla u_k\|^2_{L^2L^{2n/(n+2)}})^{1/2} \lesssim \\
&\lesssim (\sum_k \|\nabla A_{<k-6}\|^2_{L^\infty L^{n/2}} \|u_{k-3\leq\cdot\leq k+3}\|^2_{L^2L^{2n/(n-2)}})^{1/2} \lesssim \\
&\lesssim \sup_k \|\nabla A_{<k-6}\|_{L^\infty L^{n/2}} (\sum_k \|u_{k-3\leq\cdot\leq k+3}\|^2_{L^2L^{2n/(n-2)}})^{1/2} \lesssim \\
&\lesssim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{X'}.
\end{aligned}$$

[1]We are using the particular form $\left\|[Q_k, \varphi(2^{-m}\cdot)]u_k\right\|_{L^2L^2} \lesssim 2^{-k}2^{-m} \left\|(\nabla\varphi)(2^{-m}\cdot)\right\|_{L^{2n/3}} \|u_k\|_{L^{2n/(n-3)}} = 2^{-k}2^{m/2} \|u_k\|_{L^{2n/(n-3)}}$

For the second term, we have by standard Littlewood-Paley theory

$$(\sum_k \|P_k G\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim \|G\|_{L^2 L^{2n/(n+2)}} ,$$
$$\|(\sum_l |g_l|^2)^{1/2}\|_{L^p} \sim \|g\|_{L^p} \qquad \text{for all} \quad 1 < p < \infty,$$

whence with $m, \ell \in \mathbb{Z}$ with $|m - \ell| \leq 8$ we have

$$(\sum_k \|P_k(\sum_{|\ell - m| \leq 8} A_\ell \cdot \nabla u_m)\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \sim$$
$$\sim \| \sum_{|\ell - m|\leq 8} A_\ell \cdot \nabla u_m \|_{L^2 L^{2n/(n+2)}} \lesssim$$
$$\lesssim \|(\sum_\ell 2^{2l} |A_\ell|^2)^{1/2}\|_{L^\infty L^{n/2}} \|(\sum_m |\tilde{P}_m u|^2)^{1/2}\|_{L^2 L^{2n/(n-2)}} \sim$$
$$\sim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon \|u\|_{X'} .$$

For the third term in (6.2.17), observe that since for all $1 \leq p \leq 2$,

$$(\sum_k \|G^k\|^2_{L^p})^{1/2} \leq C_n \|(\sum_k |G^k|^2)^{1/2}\|_{L^p}$$

we can estimate as follows:

$$(\sum_k \|P_k (A_{k-3 \leq \cdot \leq k+3} \cdot \nabla u_{<k-5})\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim$$
$$\lesssim (\sum_k 2^{2k} \|A_{k-3\leq \cdot \leq k+3} \tilde{P}_{<k-5} u\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim$$
$$\lesssim \|(\sum_k 2^{2k} |A_{k-3 \leq \cdot \leq k+3}|^2 |\tilde{P}_{<k-5} u|^2)^{1/2}\|_{L^2 L^{2n/(n+2)}} \lesssim$$
$$\lesssim \|(\sum_k 2^{2k} |A_{k-3\leq \cdot \leq k+3}|^2)^{1/2}\|_{L^\infty L^{n/2}} \| \sup_k |\tilde{P}_{<k-5} u| \|_{L^2 L^{2n/(n-2)}} \lesssim$$
$$\lesssim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon \|u\|_{X'} .$$

Here, we have used the pointwise estimate(see section 6.1, Chapter I, **[186]**) $\sup_k |\tilde{P}_{<k-5} u|(x) \leq CM(u)(x)$, where $M(u)$ is the Hardy-Littlewood maximal function and therefore

$$\| \sup_k |\tilde{P}_{<k-5} u| \|_{L^p} \leq C \|u\|_{L^p}$$

for all $1 < p < \infty$.

4.2. Proof of (6.4.46)**.** The nontrivial part of (6.4.46) is the case when u_k is the solution to $\partial_t u_k - i\Delta u_k = H_k$ with zero initial data $u_k(0, x) = 0$. Then the fact that the norm of X_k has three components implies that the inequality

$$\|u_k\|_{X'} \leq C \|H_k\|_X$$

is equivalent to the following nine inequalities

$$\|u_k\|_{L^2_t L^{2n/(n-2)}} \leq C \|H_k\|_{L^2_t L^{2n/(n+2)}} , \tag{6.4.50}$$

$$\|u_k\|_{L^2L^{2n/(n-2)}} \leq C\,\|H_k\|_{L^1_tL^2}\,, \tag{6.4.51}$$

$$\|u_k\|_{L^2L^{2n/(n-2)}} \leq C\sum_m 2^{m/2}2^{-k/2}\,\|H_k\|_{L^2_tL^2(|x|\sim 2^m)}\,, \tag{6.4.52}$$

$$\|u_k\|_{L^\infty L^2} \leq C\,\|H_k\|_{L^2_tL^{2n/(n+2)}}\,, \tag{6.4.53}$$

$$\|u_k\|_{L^\infty L^2} \leq C\,\|H_k\|_{L^1_tL^2}\,, \tag{6.4.54}$$

$$\|u_k\|_{L^\infty L^2} \leq C\sum_m 2^{m/2}2^{-k/2}\,\|H_k\|_{L^2_tL^2(|x|\sim 2^m)}\,, \tag{6.4.55}$$

$$2^{k/2}\sup_m 2^{-m/2}\,\|u_k\|_{L^2_tL^2(|x|\sim 2^m)} \leq C\,\|H_k\|_{L^2_tL^{2n/(n+2)}}\,, \tag{6.4.56}$$

$$2^{k/2}\sup_m 2^{-m/2}\,\|u_k\|_{L^2_tL^2(|x|\sim 2^m)} \leq C\,\|H_k\|_{L^1_tL^2}\,, \tag{6.4.57}$$

and

$$\begin{aligned} &2^{k/2}\sup_m 2^{-m/2}\,\|u_k\|_{L^2_tL^2(|x|\sim 2^m)} \leq \\ &\leq C\sum_m 2^{m/2}2^{-k/2}\,\|H_k\|_{L^2_tL^2(|x|\sim 2^m)}\,. \end{aligned} \tag{6.4.58}$$

The estimates (6.4.50), (6.4.51), (6.4.53) and (6.4.54) are Strichartz inequalities (see (6.1.4) for the general case).

The estimate (6.4.58) is smoothing - smoothing estimate established in Corollary 6.1 (actually they follow from the bilinear estimate (6.3.38)).

The estimates (6.4.52), (6.4.56) are smoothing - endpoint Strichartz inequalities following from the bilinear estimate of Theorem 6.2.

Finally, the estimates (6.4.55), (6.4.57) are smoothing - energy inequalities following from bilinear estimate of Theorem 6.3.

The first inequality is the usual Strichartz estimate, while the second one is equivalent to (6.3.25).

This completes the proof of the inequality (6.4.46) and of Theorem 6.1.

5. On the Spectrum of Δ_A

We conclude this chapter with an interesting theorem, regarding the spectrum of Δ_A.

THEOREM 6.4. *Let $n \geq 3$ and $A = A(x) : \mathbf{R}^n \to \mathbf{R}^n$ be a real-valued vector potential, such that the smallness conditions of Theorem 6.1 are satisfied. Let also $V = V(x) : \mathbf{R}^n \to \mathbf{R}^1$, with $\|V\|_{L^{n/2}} << 1$. Then the spectrum of $-\Delta_A + V$ does not contain eigenvalues.*

This is a standard corollary of the Strichartz estimates in higher dimensions. Note that the requirement $n \geq 3$ is necessary, and in fact such result fails in dimensions one and two.

Proof. Assume that there is an eigenvalue λ with eigenvector f for $-\Delta_A + V$. Then $u(t,\cdot) = e^{i\lambda t}f$ is a solution to the Schrödinger equation

$$u_t = i(-\Delta_A + V)u.$$

It follows from the Strichartz estimates of Theorem 6.1

$$\begin{aligned} &\|u\|_{L^2(0,T)L^{2n/(n-2)}} \leq C(\|f\|_{L^2} + \|Vu\|_{L^2(0,T)L^{2n/(n+2)}}) \leq \\ &\leq C(\|f\|_{L^2} + \|V\|_{L^{n/2}}\,\|u\|_{L^2L^{2n/(n-2)}}) \leq \\ &C\,\|f\|_{L^2} + C\varepsilon\,\|u\|_{L^2(0,T)L^{2n/(n-2)}}\,. \end{aligned}$$

Clearly, if $C\varepsilon < 1/2$, we have that

$$\|u\|_{L^2(0,T)L^{2n/(n-2)}} \lesssim \|f\|\, L^2,$$

for every $T > 0$, which is a impossible, since $\|u\|_{L^2(0,T)L^{2n/(n-2)}} \geq CT^{1/2}\,\|f\|_{L^{2n/(n-2)}}$. □

In dimension two, one may consider the Aharonov-Bohm type vector potentials (i.e. of the form $A(r,\theta) = g(r)\psi(\theta)(\sin(\theta), -\cos(\theta)))$, for which $-\Delta_A + V$ is unitarily equivalent to $-\Delta + V$, [**12**]. For the Schrödinger operators $-\Delta + V$ however, it is well-known that eigenvalues may exists for arbitrarily small (and smooth compactly) potentials V. This is due to B. Simon [**176**], see also [**158**, p.274, Theorem XIII.80]. Therefore, such result must fail in dimensions two.

Similar examples must be easier to construct in dimension one.

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