Università degli Studi di Pisa

Facoltà di Scienze Matematiche Fisiche e Naturali
Corso di Laurea Specialistica in Scienze Fisiche e Astrofisiche
A. A. 2004-05

Tesi di Laurea Specialistica

Cosmological Perturbation Theory in a Matter Dominated Universe: the Gradient Expansion

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Introduction

The idea underlying the theory of spacetime perturbations is the same that we have in any perturbative formalism: we try to find approximate solutions of some field equations (Einstein Equations), considering them as "small" deviations from a known exact solution (the background: usually the Friedmann-Robertson-Walker (FRW) metric).
The complications in General Relativity, as in any other spacetime theory, arise from the fact that we have to perturbe not only the fields in a given geometry -fields describing the matter content in literal sense or scalar fields as the inflaton for the Inflation or the quintessence for the Dark Energy-, but the geometry itself, that is the metric.
The necessity for the development of such a formalism resides in the difficulty of Einstein Equations resolution, and in the fact that relatively few physically interesting exact solutions of the Einstein Equations are known. From the point of view of Cosmology, the ultimate aim of perturbation theory is to provide an appropriate tool for understanding the large-scale clustering of matter in galaxies and clusters of galaxies, its properties and its origin.
In this thesis we limit ourselves to the study of universes dominated by a perfect pressureless fluid, called dust or simply matter, that we assume to be irrotational as well. In the synchronous and comoving gauge, we present the calculation at first and second order of the perturbative functions of the so-called gradient expansion technique, and compare such a technique with the standard perturbation approach: our approach is analytical and the analysis fully relativistic.
The standard theory is based on the perturbations of a homogenous and isotropic FRW background metric considering the (small) fluctuations of that metric, deviations including a priori all the three perturbation modes: scalar, vector and tensor modes. In other words, we assume FRW as a good zeroth order approximation for describing our universe. Observations tell us that the universe is far from being homogenous and isotropic at small scales. To take into account of these inhomogeneities, the perturbative expansion is needed, and it is implemented through space and time functions, whose form in terms of the so-called peculiar gravitational potential is determined at different orders solving iteratively Einstein Equations (the linear or first order approach is the most common but in the last decade some cosmologists have begun stopping at second order).
In the thesis the starting point is exactly the standard one: two physical variables are introduced, the "volume expansion" and the "shear", and the Einstein Equations are written in the ADM formalism. The perturbation procedure, on the other hand, is different. We start with a spatial metric containing the perturbative functions Ψ and χ_{ij} of the standard theory, containing in turn all the orders of this expansion: at the initial time we deal with a "seed" metric con-

formally related to FRW by an exponential space-dependent factor. Then we consider as perturbation parameter not the magnitude of the deviation from the background, but the spatial gradients content, so that the zeroth order metric (or the zeroth order of any other field) is the one not containing spatial derivatives.

Counting the gradients content at different orders means considering the typical scale lengths on which the metric (and other fields) varies spatially being larger, in different approximation, than the characteristic times on which the same quantities vary in time: the result is a non-linear approximation method which allows us to study how cosmological inhomogeneities grow from initial perturbations, our "seed" (generated by inflationary fluctuations).

Therefore, in this thesis, after describing irrotational dust dynamics (Chapter 1), commenting our gauge choice (Chapter 2) and summarizing basic ideas of cosmological perturbations theory (Chapter 3), we get Ψ and χ_{ij} up to the second order (the order with four spatial gradients) solving respectively expansion and shear evolution equations. We check energy and momentum constraints (Chapter 4), we carry on comparing our result with the standard ones by a suitable procedure, and finally we show the form that the magnetic part of the Weyl Tensor assumes within this approach (Chapter 5).

Chapter 1

Describing our Universe

This thesis deals with departures from an ideal homogenous and isotropic FRW (Friedmann-Robertson-Walker) cosmological model. Before going into the technicalities of the cosmological perturbations, we want in this chapter to outline the state of the art of the present cosmology, pointing out the ideas and techinques underlying the standard description of the universe in different contexts and phases of its history.
In particular, from a qualitative point of view, we present the cosmological model that is able to give the best fit to the complete set of high-quality data available at present, that is the standard "ΛCDM Hot Big Bang" model; we briefly show the problems left unsolved by this standard model and the reasons which lead us to invoke alternative scenarios for the early universe, such as Inflation. Finally, as matter today is clustered in galaxies and clusters of galaxies, a complete description of the universe should include a description of deviations from homogeneity: we then resort to Inflation as the simplest viable mechanism for generating the observed perturbations, and briefly overview the possible approaches used at present to study the evolution of such perturbations and hence the observable large-scale mass distribution.
The treatment of this Chapter is not meant to be exhaustive and precise as it could be [4], [3], [1],...: some subjects and the overall formalism are gone on in much more detail in following chapters.

1.1 The standard cosmological model

General Relativity, together with symmetry assumptions of the metric and assumptions about the matter content of the universe, is one of the foundamental tools for the study of cosmology: it indeed has produced in the last decades a quite remarkably successfull picture of the history of our universe.
While General Relativity is in principle capable of describing the cosmology of any given distribution of matter, it is extremely fortunate that our universe appears to be homogenous and isotropic on the largest scales. Together, homogeneity and isotropy allow us to extend the Copernican Principle to the Cosmological Principle, stating that all spatial positions in the universe are essentially equivalent.
In the past the Cosmological Principle served as a useful tool in keeping the dis-

cussion focused on some well-defined and useful problems (homogenous models, their relative merits and possible tests). Nowadays, precise tests have emerged and the results do agree with the idea of the Cosmological Principle at least as a zeroth order guidelines. If on scales $\gtrsim$ tens of Mpc we see galaxies and galaxies clusters in one-dimensional and bidimensional structures (filaments and sheets) and vacuum regions without galaxies even up to 50-100 Mpc, three sets of observations -galaxy counting, extragalactic radio sources, CMB temperature smoothness- give some evidence that matter distribution and motion are quite accurately isotropic on scales $\gg 10^2$ Mpc and comparable to our Hubble length, at least within our visible patch [9]. Fluctuations from homogeneity and isotropy are thought to be of the order of $\frac{\delta\rho}{\rho} \sim 10^{-5}$ [10], thus they can be neglected at a first approach to the subject.

FRW cosmological models

A purely kinematic consequence of requiring homogeneity and isotropy of our spatial sections [1] is the Friedman-Robertson-Walker (FRW) metric, which enables us to describe the overall geometry and evolution of the universe in terms of two cosmological parameters accounting for the spatial curvature and the overall expansion or contraction of the universe:

$$dS^2_{FRW} = a^2(\tau)\ [-d\tau^2 + \frac{dr^2}{1-\kappa\ r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\ d\phi^2]. \tag{1.1}$$

τ is the *conformal time* related to the cosmic proper time t by the relation $dt = a(t)d\tau$. By rescaling the radial coordinate, we can choose the curvature constant κ to take only discrete values +1, -1 or 0 corresponding to closed, open, or flat spatial geometries. These are local statements, which should be expected from a local theory such as General Relativity: the global topology of the spatial sections may be that of the covering spaces but it need not be.
A combination of high redshift supernova and Large Scale Structure (LSS) data and measurements of the cosmic microwave background (CMB) anisotropies strongly favors for a spatially flat model, then we will almost always assume such a constraint.

We next turn to cosmological dynamics, in the form of differential equations governing the evolution of the scale factor $a(t)$; these come from applying Einstein Equations (E.E.):

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.2}$$

where it is common to assume that the matter content of the universe is a perfect fluid, for which

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \tag{1.3}$$

The pressure p is necessarily isotropic, for consistency with the FRW metric; ρ is the energy density in the rest frame of the fluid, and u^μ is the 4-velocity in

[1] In this Chapter we are supposing a (1+3)-dimensional spacetime and spatial sections have to be intended as slices at constant time: see later Section 2.1.

comoving coordinate (see later Section 2.2).

The *cosmological constant* Λ term can be interpreted as particle physics processes yielding an effective stress-energy tensor for the vacuum of $\Lambda g_{\mu\nu}/8\pi G$, and we have introduced it in E.E. because recent observations (luminosity-redshift of SNIA and the CMB anisotropies measurements) suggest an acceleration of the universe expansion and thus the requirement of a non standard fluid, called Dark Energy. With Λ we mean the simplest form of Dark Energy, that is an energy component indipedent of time, spatially homogenous and with an equation of state:

$$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}. \tag{1.4}$$

Thus, for brevity, from now on we will not explicit it in the equations but treat it as any other (even if particular) energy component.

With this simplified description for matter, equations (1.2) can be rewritten as follows

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{\kappa}{a^2} \tag{1.5a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i), \tag{1.5b}$$

where $H(t)$ is the *Hubble parameter*, overdots denote derivatives with respect to time t and the index i labels all different possible types of energy components in the universe. The first equation is often called *Friedmann equation* and is a constraint equation, the second one is sometime referred to as *acceleration equation* and is an evolution equation. A third useful equation -not independent of these last two- is the continuity equation $T^{\mu\nu}{}_{;\mu}$. With our assumptions it reads

$$\dot{\rho} = -3H(\rho + p) \tag{1.6}$$

which implies that the expansion of the universe (as specified by H) can lead to local changes in the energy density. Let us note that there is no notion of conservation of "total energy", as energy can be interchanged between matter and the spacetime geometry.
The FRW equations can be solved quite easily supposing that one single energy component dominates. Within a fluid approximation, defining an equation of state parameter w which relates the pressure p to the energy density ρ by $p = w\rho$, the ordinary energy contributions of our universe such as dust and radiation are distinguished by, respectively, $w = 0$ and $w = 1/3$. On the contrary, a cosmological constant is characterized by $w = -1$ (equation (1.4)).
Equation (1.6) is easily integrated to yield

$$\rho \propto a^{-3(1+w)}. \tag{1.7}$$

Then Friedmann equation (1.5a) with $\kappa = 0$ and $w \neq -1$ is solved by

$$a(t) \propto t^{2/[3(1+w)]}. \tag{1.8}$$

General qualitative features of the future evolution of FRW universe can now be seen. If $\kappa = 0$ or -1, Friedmann equation (1.5a) shows that $\dot{a}$ can never become

zero (apart from $t = 0$): thus, if the universe is presently expanding, it must continue to expand forever. Indeed, for any energy content with $p \geq 0$, ρ must decrease as a increases at least as rapidly as a^{-3}, the value for dust. Thus, $\rho a^2 \to 0$ as $a \to \infty$. Hence for $\kappa = 0$ the expansion velocity $\dot{a}$ asymptotically approaches zero as $t \to \infty$, while if $\kappa = -1$ we have $\dot{a} \to 1$ as $t \to \infty$. Otherwise, if $\kappa = +1$, the universe cannot expand forever but there is a critical value a_c such that $a \leq a_c$: at a finite time after $t = 0$ the universe achieves a maximum size a_c and then begins to recontract.
The presence of a vacuum energy alters the fate of the universe and the above simple conclusions: if $\Lambda < 0$, the universe will eventually recollapse independent of the sign of κ. For large values of Λ even a closed universe will expand forever. Table 1.1 summarizes the behaviour of the most important sources of energy density in cosmology in the case of a flat universe.

Type of Energy	w	$\rho(a)$	$a(t)$	H(t)
Dust	0	a^{-3}	$t^{2/3}$	$\frac{2}{3t}$
Radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	$\frac{1}{2t}$
Cosmological Constant	-1	$const$	e^{Ht}	$\sqrt{\frac{\Lambda}{3}}$

Table 1.1: The behaviour of the scale factor and Hubble constant applie to the case of a flat universe; behaviours of energy density are perfectly general.

There are three foundamental features of FRW spacetimes which we are going to discuss:

- expansion (or contraction) $\Longrightarrow$ gravitational redshift (or blueshift);
- existence of an initial singularity, the Big Bang;
- existence of particle horizons.

Expansion and Redshift The first striking result of FRW models is that universe cannot be static but must be expanding or contracting. This conclusion follows immediately from equation (1.5b) written in the simple form

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a. \tag{1.9}$$

(1.9) tells us that $\ddot{a} < 0$ if $\rho + 3p > 0$ and $\ddot{a} > 0$ if $\rho + 3p < 0$: in any case, the universe must always either be expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$) (with the possible exception of an instant of time when expansion changes over to contraction, as in the case $\kappa = +1$). Let us comment the nature of this expansion or contraction: the distance scale between all isotropic observers changes with time, but there is no preferred center of expansion or contraction. Indeed, if the distance (measured on the homogenous slice) between two isotropic observer at time t is r, the rate of change of r is

$$v \equiv \frac{dr}{dt} = \frac{r}{a}\frac{da}{dt} = Hr \tag{1.10}$$

where $H(t)$ is the well-known Hubble parameter and (1.10) is known as *Hubble Law*. Let us still note that the expansion speed can be greater than the speed of light without any harmful thought .
The expansion of the universe is confirmed in accordance with equation (1.10): the most direct observational evidence for that comes from the *redshift* of the spectral lines of distant galaxies. The idea is that a local observer detecting light from a distant emitter sees a redshift in frequency or, in other words, the wavelength λ of each photon increases in proportion to the amount of expansion, as any other physical scale is stretched by expansion. The solution of all redshift problems (as illustrated in Figure 1.1) in Special and General Relativity is governed by the following two facts: first, light travels on null geodesics; secondly, the frequency of a light signal of wave vector k^μ measured by an observer with 4-velocity u^μ is $\nu = -k_\mu u^\mu$. Thus we can always find the observed frequency by calculating the null geodesic determined by the initial value of k^μ at the emission point and then calculating the right hand side of the former expression at the observation point [1]. The redshift factor is then given by

$$z \equiv \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\nu_1}{\nu_2} - 1 = \frac{a(t_2)}{a(t_1)} - 1. \qquad (1.11)$$

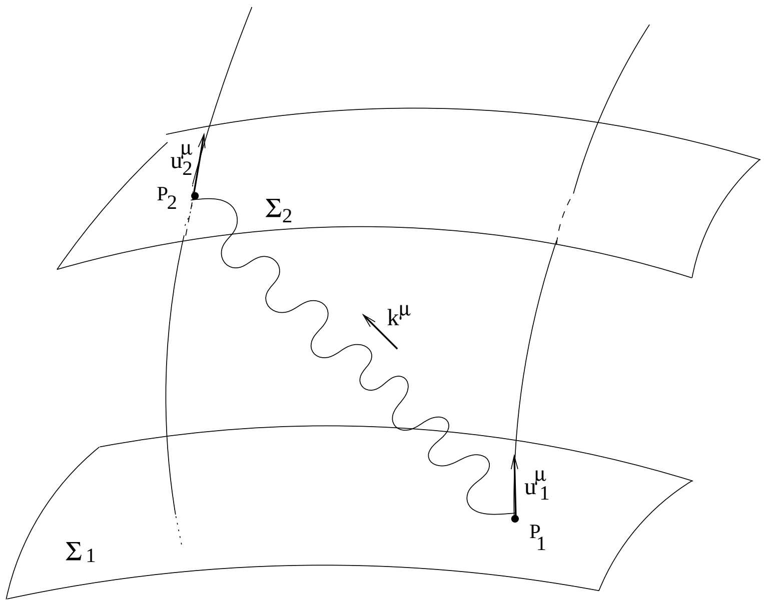

Figure 1.1: A spacetime diagram showing the emission of a light signal at event P_1 and its reception at event P_2

It is possible to relate the redshift to the relative velocity of the two observers in the case of small scales (i.e. less than cosmological scales) such that the expansion velocity is non-relativistic. In this case, for light emitted say by nearby galaxies, we have $t_2 - t_1 \approx r$, where r is the present proper distance to the galaxy; furthermore, $a(t_2) \approx a(t_1) + (t_2 - t_1)\dot{a}$. Thus we find

$$z_{non\ rel} \approx \frac{\dot{a}}{a}\, r = Hr \qquad (1.12)$$

which is the linear redshift-distance relationship discovered by Hubble. The redshifts of distant galaxies will deviate from this linear law depending on exactly how $a(t)$ varies with t.

The redshift z is often used in place of the scale factor: to be complete, $z, t, a(t), \rho(t)$ and the temperature T are all used as variables to refer to different phases of the universe history (Tables 1.1).

Big Bang singularity Both matter and radiation dominated flat universes present a singularity at $t = 0$ in which $a = 0$. Thus, under the assumption of homogeneity and isotropy, General Relativity makes the striking prediction that at a time $t = \int_0^1 \frac{da}{a\ H(a)} = \frac{2}{3(1+w)H_0} \sim H_0^{-1}$ ago the universe was in a singular state: the distance between all "points of space" was zero, the density of matter and the curvature of spacetime infinite. This singularity state of the universe is referred to as *Big Bang*, and the quantity H_0^{-1}, known as the *Hubble time*, provides a useful estimate of the time scale for which the universe has been around. [2]
The nature of this singularity is that resulting from a homogenous contraction of space down to "zero size". The Big Bang does not represent an explosion of matter concentrated at a preexisting point: it does not make sense to ask about the state of the universe "before" the Big Bang because spacetime structure itself is singular at $t = 0$; thus General Relativity leads to the viewpoint that universe began at the Big Bang. For many years it was generally believed that the prediction of a singular origin was due merely to the assumptions of exact homogeneity and isotropy, that if these assumptions were relaxed one would get a non-singular "bounce" at small a rather than a singularity. The Singularity Theorem of General Relativity [1] shows that singularities are generic features of cosmological solutions. Of course, at the extreme conditions very near the Big Bang one expects that quantum effects will become important, and predictions of classical General Relativity are expected to break down.

Particle horizons We shall demonstrate now the third crucial point of FRW spacetimes: FRW cosmological models presuppose the existence of non-trivial *particle horizons*, where, by this expression, we mean in general the boundary of the observable region at a generic time t, or the boundary between the worldlines that can be seen by an observer at a certain point of spacetime and those one that cannot be seen (see Figure (1.2)). In General Relativity the question about how much of our universe can be observed at a given point is due, and indeed, in spite of the fact that the universe was vanishingly small at early times, the expansion precluded causal contact from being established throughout the universe.

The photons travel on null paths characterized by $dr = \frac{dt}{a(t)} = d\tau$: the physical distance that a photon could have travelled since tha Bang until time t, the distance to the particle horizon, is

$$R_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \tag{1.13}$$

[2] The subscript "0" means that the quantity is evaluated at $t = t_{NOW}$.

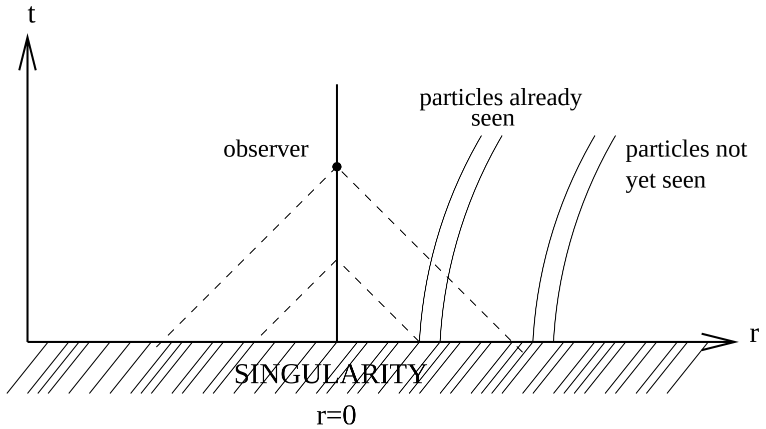

Figure 1.2: The causal structure of FRW spacetime near the Big Bang singularity: particle horizons arise when the past light cone of an observer terminates at a finite time t or conformal time τ.

An observer at a time t is able to receive a signal from all other isotropic observers if and only if the integral of (1.13) diverges : in this case the flat FRW metric is conformally related to Minkowski spacetime and there is no particle horizon. On the other hand, if the integral converges, FRW model is conformally related only to a portion of Minkowski spacetime (the one above a $t = const$ surface) and particle horizon does occur. It is not difficult to see that the integral converges in all FRW models with equation of state parameter $w \in (0,1)$:

$$R_H(t) = \begin{cases} 2t = H^{-1}(t) \propto a^2 & \text{(radiation)} \\ 3t = 2H^{-1}(t) \propto a^{3/2} & \text{(dust).} \end{cases} \tag{1.14}$$

As $H(t)^{-1}$ is the age of the universe, $H(t)^{-1}$ is called the *Hubble Radius*, as it is the distance that light can travel in a Hubble time $H(t)$. If the particle horizon exists then it would coincide, up to numerical factor, with the Hubble radius: for this reason, in the context of standard cosmology (when $\omega > -1/3$) horizon and Hubble radius are used interchangeably.
These conclusions are not true anymore in the case of non standard matter, that is $w \notin (0,1)$: in the case of a cosmological constant (for example, during Inflation or in the later time of universe history), particle horizon and Hubble radius are not equal as the horizon distance grows exponentially in time relative to the Hubble radius.
A physical length scale λ is within the horizon if $\lambda < R_H \sim H^{-1}$; in terms of the corresponding comoving wavenumber k, $\lambda = 2\pi a/k$, we will have the following rule:

$$\frac{k}{a} \ll H^{-1} \Longrightarrow \qquad \text{scale } \lambda \text{ outside the horizon and no causality}$$
$$\frac{k}{a} \gg H^{-1} \Longrightarrow \qquad \text{scale } \lambda \text{ within the horizon and causality.}$$

Therefore, in a universe described by FRW with standard matter content such as dust or radiation, there will always exist regions not causally connected:

any comoving length scale evolves in time with a power law t^α with $\alpha < 1$ ($\kappa = 0$), thus its rate of increase is always smaller than the rate of increase in the Hubble horizon size, which is linear in time. Thus, for example, the size of a comoving region corresponding at present to a supercluster (say $\sim 30Mpc$ at $t \approx 10^9 years$) was comparable to the horizon at epoch shortly before the recombination ($t \approx 10^5 years$) and was much greater than the horizon at some earlier epoch.

These considerations about the existence of particle horizons and of causally disconnected regions in FRW models lead to very interesting issues. We begin presenting one of them (known as Horizon problem), postponing a brief discussions of the shortcomings of the standard cosmological model as described until here to a next paragraph.
As mentioned earlier, we have good reasons to believe that the present universe is homogenous and isotropic to a very high degree of precision. Now, many ordinary systems, such as gas confined in a box, often are found in extremely homogenous and isotropic states: the usual explanation of that state is that they have had an opportunity to self-interact and thermalize, exactly as in a box filled with gas initially in an inhomogenous state, these inhomogeneities quickly "wash out" on a time scale of the order of the transit time across the box. However this type of explanation cannot possibly apply to a universe with particle horizons, since different portions cannot even send signals to each other, far less interact sufficiently to thermalize each other. Thus, in order to explain the homogeneity and isotropy of the present universe, one must postulate that either (a) the universe was born in an extremely homogenous and isotropic state, or (b) the very early universe differed significantly from the FRW models so that no horizons were present; the inhomogeneities and anisotropy then "damped out" by some mechanisms and the universe approached the FRW models that fit present observations. Unfortunately, if the first point of view may appear rather unnatural and a profession of faith, the second one suffers not only from the absence of a plausible picture of evolution from a chaotic to a FRW state, but for the fact that gravity promotes inhomogeneity, not homogeneity. Later we will see how a third way is now accepted, the one of an *inflationary phase* of the very early universe.

Brief outline of universe evolution

The above considerations should be almost sufficient to understand and justify the basic aspects of the evolution of our universe from the Big Bang to the present in the standard picture. Two points should be still clarified for completeness:

- the various particles inhabiting the universe can be usefully characterized according to three criteria: in equilibrium vs. out of equilibrium (decoupled), bosonic vs. fermionic, and relativistic (velocities near to c) vs. non relativistic (dust);

- much of the history of the standard Big Bang model can be easily described by assuming that one of the components dominates the total energy density.

As mentioned earlier, the cosmological energy conservation (equation (1.6)) tells us that the decrease of the scale factor a as one goes back towards the past has the same local effect on the matter as if the matter were placed in a box whose walls contract at the same rate. Thus (in agreement with Table 1.1) the contribution of radiation compared with ordinary matter increases in the past, and there must be a period in the early times of universe evolution in which this radiation should have been the dominant contribution to the energy. The present radiation energy contribution to the universe energy density is represented by the CMB energy density, wich is about 1000 times smaller than the present mass density contribution of matter. One would expect the radiation-filled model of the universe to be a good approximation for the dynamics of the universe before a stage in which the scale factor a was more than few 1000 times smaller than its present value, while the dust filled model should be a good approximation afterwards. In the context of this separation, another important issue is whether the interactions of matter or radiation proceed on a rapid enough time scale for thermalization to occur locally (within the particle horizon). A given species remains in *thermal equilibrium* with the surrounding thermal plasma as long as its interaction rate is larger than the expansion rate of the universe. A particle species for which the interaction rates have fallen below the expansion rate is said to have *frozen out* or *decoupled.* As good rule of thumb, the expansion rate in the early universe is "slow", and particles tend to be in thermal equilibrium (unless they are very weakly coupled); in our current universe, no species are in equilibrium with the background plasma (represented by the CMB photons).

The basic picture of the evolution of our universe can then be told as follows: the universe began with a singularity state as a hot ($T \to \infty$), dense ($\rho \to \infty$) soup of matter and radiation in thermal equilibrium. The energy content of early universe was dominated by radiation: at these early times thermal equilibrium held and other specific phenomena took place such as primordial nucleosynthesis. However, as the universe evolved, thermal equilibrium was not maintained and the ordinary matter contribution began to dominate the energy content of the universe (about 4×10^4 years after the Bang): the dynamics of the universe became that of a dust filled FRW model characterized by the CMB photons background, matter-antimatter asymmetry and cosmological structure formation.

There is no room in this thesis to fill the details of this schematic and full of gaps evolutionary history, and to discuss for example the very complex first few minutes of universe life characterized by symmetry breakings and phase transitions, and other [4]: more interesting, even in relation to the following developments, is to underline the good predictions of the *Hot Big Bang* model and to understand how it faces recent observations and some theorethical questions.

Parametrizing the universe: shortcomings of the standard model

Earlier we introduced global parameters such as *expansion factor* $a(t)$, *spatial curvature* κ and *Hubble parameter* $H(t)$, the latter defined by

$$H(t) \doteqdot \frac{\dot{a}}{a} = \frac{a'}{a^2} \text{ or } \mathcal{H}(\tau) \doteqdot \frac{a'}{a} \tag{1.15}$$

where the dot denotes differentiation with respect to t and the prime differentiation with respect to τ. In addition, it is useful to define several other measurable cosmological parameters.
The Friedmann equation (1.5a) suggests to define a *critical density* ρ_c and a cosmological *density parameter* Ω_{tot}

$$\rho_c \doteq \frac{3H^2}{8\pi G} \text{ and } \Omega_{tot} \doteq \frac{\rho}{\rho_c} \tag{1.16}$$

such that it can be rewritten as follows

$$\frac{\kappa}{a^2} = H^2(\Omega_{tot} - 1) \tag{1.17}$$

From equation (1.17), one can distinguish the different cases

$$\begin{array}{lcccccc} \rho < \rho_c & \leftrightarrow & \Omega_{tot} < 1 & \leftrightarrow & \kappa = -1 & \leftrightarrow & open \\ \rho = \rho_c & \leftrightarrow & \Omega_{tot} = 1 & \leftrightarrow & \kappa = 0 & \leftrightarrow & flat \\ \rho > \rho_c & \leftrightarrow & \Omega_{tot} > 1 & \leftrightarrow & \kappa = +1 & \leftrightarrow & closed. \end{array} \tag{1.18}$$

It is often necessary to distinguish different contributions to the density, and therefore convenient to define present-day density parameters for pressureless matter Ω_m, relativistic particles Ω_r, and for the vacuum Ω_v. This last one is equal to $\Omega_\Lambda = \Lambda/3H^2$ in models with cosmological constant, i.e. constant vacuum energy density. Then the Friedmann equation becomes

$$\frac{\kappa}{a_0^2} = H_0^2(\Omega_m + \Omega_r + \Omega_v - 1) \tag{1.19}$$

where the subscript 0 indicates present-day values.
One way to quantify the deceleration (or acceleration) of the universe expansion of equation (1.5b) is the *deceleration parameter* q_0 defined as

$$q_0 \doteq -\left(\frac{a\ddot{a}}{\dot{a}^2}\right)_0 = \frac{1}{2}\Omega_m + \Omega_r + \frac{1+3w}{2}\Omega_v. \tag{1.20}$$

The expansion accelerates if $q_0 < 0$ and this equation shows that $w < -1/3$ for the vacuum may lead to an accelerating expansion.
It is usual to express the Hubble parameter and hence all the previous parameters in terms of the *scaled Hubble parameter* h for which

$$H \equiv 100h \; km \; s^{-1} \; Mpc^{-1}. \tag{1.21}$$

The term "cosmological parameters" is increasing its scope because of the rapid advances in observational cosmology of the last ten years which are leading to the establishment of the first high precision cosmological model. The most accurate model of the universe requires consideration of a wide range of different types of observations, with complementary probes providing consistency checks, lifting parameter degeneracies, and enabling the strongest constraints to be placed. Hence, nowadays, the term "cosmological parameters" not only refers to the original usage of simple numbers as the above ones describing the global dynamics and properties of the universe, but also includes the parametrization of some functions describing the nature of perturbations in the universe, and physical parameters of the state of the universe. Typical comparison of cosmological models with observational data now feature about ten parameters, shown in Table 1.2 (see [36] and [11]).

Parameter	Symbol	Value
Hubble Parameter	h	0.73 ± 0.03
Total matter density	Ω_m	$\Omega_m h^2 = 0.134 \pm 0.006$
Baryon Density	Ω_b	$\Omega_b h^2 = 0.023 \pm 0.001$
Cosmological Constant	Ω_Λ	$\Omega_v = 0.72 \pm 0.05$
Radiation Density	Ω_r	$\Omega_r h^2 = 2.47 \times 10^{-5}$
Density perturbation amplitude	$\Delta^2_{\mathcal{R}}(k_*)$	see later P(k)
Density perturbation spectral index	n	$n = 0.97 \pm 0.03$
Tensor to scalar ratio	r	$r < 0.53$ $(95\% conf)$
Ionization optical lenght	τ	$\tau = 0.15 \pm 0.07$

Table 1.2: The basic set of cosmological parameters: uncertainities are one-sigma/68% confidence unless otherwise stated.

We have by now most of the ingredients needed to understand the first half of the shown parameters; the second one will be in part justified in the continuation, while the ionization optical depth will not be commented at all in this thesis. The spatial curvature does not appear in the list because it can be determined from the other parameters using (1.17) or (1.19), and the total present matter density is indicated as usual as a sum of baryonic matter and dark matter densities, namely $\Omega_m = \Omega_{dm} + \Omega_b$. With appropriate arguments, the parameter set listed above can be reduced to seven parameters as the smallest set that can usefully be compared to the present cosmological data set. Of course this is not the unique possible choice: one could instead use parameters derived from those basic ones such as the age of the universe, the present horizon distance, the present CMB and neutrino background temperatures, the epoch of matter-radiation equality, the epoch of transition to an accelerating universe, the baryon to photon ratio, ... Furthermore, different types of observations are sensitive to different subsets of the full cosmological parameter set.

Having in mind the above parametrization and Table 1.2 as mirror of the disposable observational data, we can proceed in evaluating the standard cosmological model. Among the most notable achievements of Hot Big Bang FRW standard model are

- the prediction of cosmological expansion;
- the prediction and explanation of the presence of a relic background radiation with temperature of order of few K, the CMB;
- the explanations of the cosmic abundance of light elements;
- the possibility to insert in this picture the structure formation phenomenon.

On the contrary, the most severe problems that it has to face can be summarized in the following interesting issues.

- Horizon problem.
  Under the term "horizon problem" a wide range of facts is included, all related to the existence of particle horizons in FRW models. We have already discussed the main point of the question: we want now to delineate some more quantitative aspects of it.

According to the standard model, photons and the other components such as electrons and baryons decoupled at a temperature of 0.3 eV. Recalling the preceding discussions, this happened when the rate of interaction of photons with, say, electrons and protons became of the order of the Hubble size (that is, of the horizon size), and the expansion made not possible the reverse reaction of $p+e^+ \rightarrow H+\gamma$. The temperature of 0.3 eV corresponds to the so-called ***surface of last-scattering***, posed at a redshift $z_{LS} \approx 1100$, after the matter-radiation equivalence and hence in matter era. From the epoch of last-scattering onwards, photons free-stream and now are measurable in the well known CMB, whose spectrum is consistent with that of a black-body at a temperature of 2.726 ± 0.01K. Then let us look at two photons from different parts of the sky: the lengh corresponding to our present Hubble radius at the time of last-scattering was (remembering that $T \propto a^{-1}$)

$$\lambda_{H_0}(t_{LS}) = R_H(t_0)\left(\frac{a(t_{LS})}{a(t_0)}\right) = R_H(t_0)\left(\frac{T_0}{T_{LS}}\right)$$

During the matter domination $H^2 \propto a^{-3} \propto T^3$, and at last-scattering

$$H_{LS}^{-1} = R_H(t_0)\left(\frac{T_0}{T_{LS}}\right)^{3/2} \ll R_H(t_0)$$

Being $T_0 \sim 2.7K \sim 10^{-4}$ eV $\ll T_{LS}$, the length corresponding to our present Hubble radius was much much larger that the horizon at that time. Because CMB experiments like COBE and WMAP tells us that our two photons have nearly the same temperature to a precision of 10^{-5}, we are forced to say that those two photons were very similar even if they could not talk to each other, and that the universe at last-scattering was homogenous and isotropic in a physical region about some order greater than the causally connected one!
Not only the homogeneity of the CMB is able to tell us important things, but nowadays the measured temperature fluctuations (consequences of density inhomogeneities) are a mine of information too, and another striking feature of the CMB is that photons at the last-scattering surface which were causally disconnected have the same small anisotropies ([10]). The standard model cannot say anything with reference to this.

- Flatness problem and the peculiarity of initial conditions.
  The Friedmann equation tells us that

$$(\Omega_{tot} - 1) = \kappa / \, H^2 a^2$$

therefore (we implicitly consider from now on $\Omega \equiv \Omega_{tot}$) $(\Omega - 1) \rightarrow 0$ for $t \rightarrow 0$ in both cases of radiation and matter domination: in other words, given $(\Omega(t) - 1)$ at a given time t, Ω has to depart from 1 both in open and closed cases. Present observations tell us that $(\Omega_0 - 1)$ is of order unity (i.e. $\in (0, \sim 1)$). Let us calculate the same value at some early time of universe, say at *Planck time* (at $t \approx 10^{-43}$ s or T$\sim 10^{19}$ GeV):

$$\frac{|\Omega - 1|_{T=T_{Pl}}}{|\Omega - 1|_{T=T_0}} \approx \left(\frac{a^2(t_{Pl})}{a^2(t_0)}\right) \approx \left(\frac{T_0^2}{T_{Pl}^2}\right) \approx \mathcal{O}(10^{-64})$$

A very problematic question arises, because how can it be possible that Ω had been so near the critical value able to lead to the universe observed today? Even small deviations of Ω from 1 at early time would have led to the collapse or the cooling of the universe in *few* 10^{-43}s, respectively in the case of $\kappa = +1$ or $\kappa = -1$. In order to get the correct value $(\Omega_0 - 1)$ at present, the value $(\Omega - 1)$ at early times had to be fined-tuned to values amazingly close to zero, but without being exactly zero. This is the reason why the flatness problem is also dubbed the "fine-tuning problem".

- Existence of Dark Matter.
  We have a remarkable convergence on the value of the density parameter in matter ($w = 0$): $\Omega_m = 0.28 \pm 0.05$. We call *baryonic matter* or simply ordinary matter anything made of atoms and their constituents, and this would include all of stars, planets, gas and dust in the universe. Ordinary baryonic matter, it turns out, is not enough to account for the observed matter density:
  $$\Omega_b \sim 0.043 \pm 0.002 \ll \Omega_m$$
  This determination comes from a variety of methods: direct evaluation of baryons, consistency with the CMB power spectrum, and agreement with the predictions of primordial nucleosynthetis, which places the constraint $\Omega_b \leq 0.12$. Most of the matter density must therefore be in the form of non-baryonic matter, or *dark matter*. Candidates for dark matter include the lightest supersymmetric particle, the axion, but in the past essentially every known particle of the Standard Model of particle physics and predicted particles of Supersymmetry theories have been ruled out as a candidate for it. The things we know are that it has no significant interactions with other matter, so as to have escaped detection thus far, and that its particles have negligible velocity, i.e. they are "cold".

- Evidence of accelerated expansion.
  Astonishignly, in recent years, it appears that an effect of accelerating expansion ($q_0 < 0$) has been observed in the Supernova Hubble diagram: the common position in the last years is to invoke the existence of another energy component (different from matter and radiation), and comparison with the prediction of FRW models leads of course to favor a vacuum-dominated universe. In this picture, current data indicate that the vacuum energy is indeed the largest contributor to the cosmological density budget, with $\Omega_v = 0.72 \pm 0.05$, [11]. The nature of this dominant term is presently uncertain, but much effort is being invested in dynamical models, under the catch-all heading of *quintessence*, or Dark Energy.

- The problem of perturbations unknown origin.

The first issues arise from a combination of observational facts and theoretical principles, and together with the last one they find the best model solution in the *Inflationary paradigm*. The Dark Matter and the Dark Energy problems force us to take into account an ampler cosmological model referred to by various names, including "ΛCDM Hot Big Bang" model, the concordance cosmology, or the standard cosmological model. But the sense of accomplishment at having measured all the numbers above is somewhat tempered by the realization that we do not understand very well any of them. For instance, there are many proposals for the nature of Dark Matter, but no consensus as to which is correct. Even the baryon density, now measured to an accuracy of a few percent, lacks an underlying theory able to predict it even within orders of magnitude. Finally the nature of the Dark Energy remains a mystery, even if very recent works have suggested viable mechanisms able to explain the acceleration without invoking an extra energy component [37].

1.2 Inflation

The horizon problem is a relevant problem of the standard cosmology because at its heart there is simply causality. From the considerations made so far, it appears that solving the shortcomings of the standard model requires at least an important modification to how the information can propagate in the early universe, and hence that the universe has to go through a primordial period during which the physical scale λ evolves faster than the horizon scale H^{-1}. Cosmological Inflation is such a mechanism.
The foundamental idea of Inflation is that the universe undergoes a period of accelerated expansion, defined as a period when $\ddot{a} > 0$, at early times. The effect of this acceleration is to quickly expand a small region of space to a huge size, reducing the spatial curvature in the process, making the universe extremely close to flat. In addition, the horizon size is greatly increased, so that distant points on the CMB actually are in causal contact.

An *inflationary stage* is defined as a period of the universe during which the latter accelerates. From previous sections we have learned that

$$\ddot{a} > 0 \Longleftrightarrow (\rho + 3p) < 0 \tag{1.22}$$

and that such a condition is not satisfied neither during a radiation-dominated phase nor in a matter-dominated phase. Even if it is sufficient that $p < -\rho/3$, in order to study the properties of the period of inflation, we assume the extreme condition $p = -\rho$ which considerably simplifies the analysis and that we have already met in terms of a cosmological constant. We recall briefly that in the case of such an energy component

$$\rho \propto const \tag{1.23}$$
$$H_I \propto const \tag{1.24}$$
$$a(t) = a_i \; e^{H_I(t-t_i)} \propto e^{H_I t} \tag{1.25}$$
$$R_H^I(t) \propto H_I^{-1} \; e^{H_I t} \tag{1.26}$$

where the subscript (or superscript) I indicates that we refer to an inflation quantity and t_i denotes the time at which inflation starts. Contrary to what happens in FRW dust or radiation filled universes, a comoving length scale increases faster than the particle horizon and much faster than the Hubble size. By the way, Inflation is a phase of the history of the universe occurring before the era of nucleosynthetis ($t \approx 1s$, T $\approx$ 1 MeV) during which the light elements abundances were formed: this is because nucleosynthetis is the earliest epoch we have experimental data from, and as already seen they are in agreement with

the predictions of the Hot Big Bang model. However, the thermal hystory of the universe before that stage is almost unknown and many models of Inflation are set to be around the Plank time ($t_{Pl} \approx 10^{-43}s$). It is common, even in reponse to other tasks, to think of a period of *reheating* at the end of Inflation during which thermal equilibrium is established and radiation era begins.
It is useful to have a general expression to describe how much Inflation occurs once it has begun. This is typically quantified by the number of *e-folds*, defined by

$$N(t) \doteqdot \ln\left(\frac{a(t_f)}{a(t)}\right) \text{ and } N_{tot} = \ln\left(\frac{a(t_f)}{a(t_i)}\right) \tag{1.27}$$

Resolution of the horizon problem Thanks to Inflation any comoving length scale observable at present has been causally connected at some primordial stage of the evolution of the universe, removing the horizon problem. This can be easily seen with the help of Figure 1.3. Let us consider length scales λ which are within the horizon today ($\lambda < H^{-1}(t_0) \equiv H_0^{-1}$) but were outside the horizon for some previous period ($\lambda > H^{-1}(t_{past})$) during the matter or radiation era. If there is a period (inflation) during which physical length scales grow faster than H^{-1}, such today observable scales had a chance to be within the horizon in that early period again ($\lambda < H_I^{-1}$): in fact, during the inflationary epoch the Hubble radius is constant and the condition satisfied.

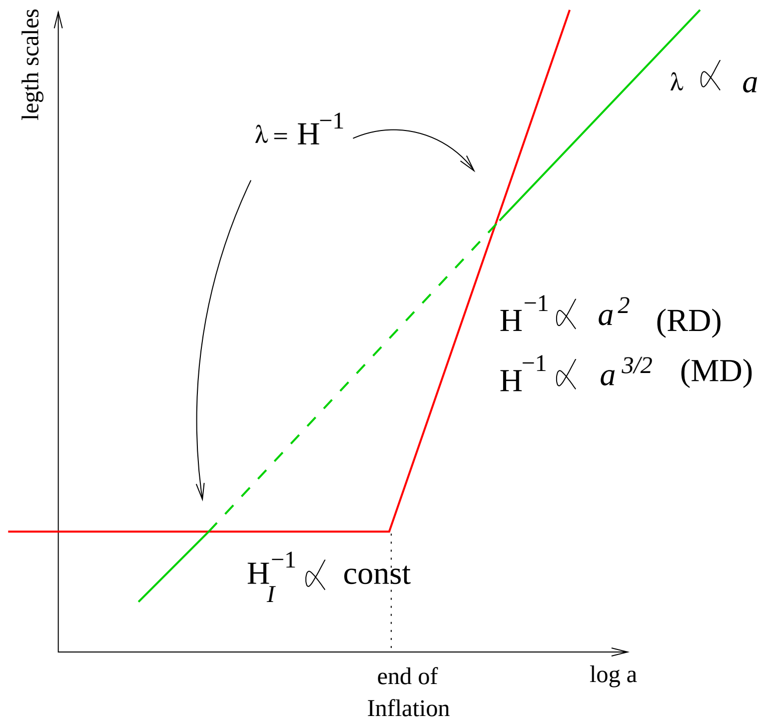

Figure 1.3: Hubble scale and a physical scale as a function of the scale factor a [10].

Let us see how long Inflation must be sustained in order to solve the horizon problem and let the present day largest observable scale re-enter the horizon during Inflation. The largest observable scale is of course the present Hubble radius H_0 and we want it to be reduced during Inflation to a value $\lambda_{H_0}(t_i)$ smaller than the value of the Hubble size H_I^{-1} during Inflation. This gives

$$\lambda_{H_0}(t_i) = H_0^{-1}\left(\frac{a(t_f)}{a(t_0)}\right)\left(\frac{a(t_i)}{a(t_f)}\right) = H_0^{-1}\left(\frac{T_0}{T_f)}\right) e^{-N_{tot}} \lesssim H_I^{-1}$$

(where we have neglected for simplicity the short period of matter-domination). Then the condition for solving the horizon problem is

$$N_{tot} \gtrsim \ln(\frac{T_0}{H_0}) - \ln(\frac{T_f}{H_I}) \approx 67 + \ln(\frac{T_f}{H_I}). \tag{1.28}$$

More precise valutations give $N_{tot} \gtrsim 60$.

Inflation and flatness problem Inflation solves elegantly the flatness problem, thanks to the fact that the Hubble scale is constant and

$$\Omega - 1 = \frac{k}{a^2 H_I^2} \propto 1/a^2.$$

We have seen that to reproduce a value of $(\Omega_0 - 1)$ of order unity today the initial value of $(\Omega - 1)$ at Planck time must be $|\Omega - 1| \sim 10^{-60}$. Since we identify the beginning of the radiation era with the end of Inflation, and the time scale of Inflation is Planck time, we require $|\Omega - 1|_{t=t_f} \sim 10^{-60}$.
During Inflation

$$\frac{|\Omega - 1|_{t=t_f}}{|\Omega - 1|_{t=t_i}} = \left(\frac{a_i}{a_f}\right)^2 = e^{-2N_{tot}}$$

Taking $|\Omega - 1|_{t=t_i}$ of order unity, it is enough to require that $N_{tot} \approx 60$ to solve the flatness problem. From the point of view of the fine-tuning, Inflation avoids the hindrance of an enormous fine-tuning, because the density parameter Ω is driven to 1 with exponential precision. Let us note that if the period of Inflation lasts longer than 60 e-folding the present-day value of Ω_0 will be equal to unity with a great precision. Thus we could say that a generic prediction of Inflation is $\Omega_0 = 1$, and current data on CMB anisotropies confirm this prediction.

Inflation as driven by a slowly-rolling scalar field

Knowing the various advantages of having a period of accelarated expansion phase, the next task consists in finding a model that satisfies the conditions mentioned above. There are many models of Inflation. Today most of them are based on a new scalar field, the *inflaton* ϕ.
We consider modelling matter in the early universe by the inflaton, a real scalar field which moves with a potential $V(\phi)$. Its Lagrangian then reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + V(\phi) \tag{1.29}$$

and the stress-energy tensor is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}\phi_{,\mu}\phi_{,\nu} + V(\phi)\right) \tag{1.30}$$

The corresponding energy density ρ_ϕ and pressure p_ϕ are

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2} \tag{1.31a}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{(\nabla\phi)^2}{6a^2} \tag{1.31b}$$

where it is evident that if the gradient term were dominant, we would obtain $p_\phi = -\frac{\rho_\phi}{3}$, not enough to drive Inflation.
In the case of an homogenous field $\phi(t, \vec{x}) = \phi(t)$, the inflaton behaves with a perfect fluid and expression (1.31) become

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{1.32a}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \tag{1.32b}$$

The equation of motion for the homogenous inflaton is

$$\Box\phi = \frac{dV}{d\phi} \text{ i.e. } \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{1.33}$$

which can be thought of as the usual Klein-Gordon equation of motion for a scalar field in Minkowski space, but with a friction term $3H\dot{\phi}$ due to the expansion of the universe. The Friedmann equation with such a scalar field as the sole energy source is

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1.34}$$

Let us now quantify under which circumstances a scalar field may give rise to a period of Inflation. First of all, let us note that requiring $V(\phi) \gg \dot{\phi}^2$ implies from expressions (1.32) that the potential energy of the scalar field is the dominant contribution to both the energy density and the pressure, and hence $p_\phi \simeq -\rho_\phi$: from this simple calculation, we realize that a scalar field whose energy dominates the universe and whose potential energy dominates over the kinetic term can mimic a cosmological constant dominated universe, and then gives Inflation. Inflation is driven by the vacuum energy of the inflaton field.
If $\dot{\phi}^2 \ll V(\phi)$, the scalar field is slowly rolling down its potential and this is the reason why such a period is called *slow-roll*. The so-called *slow-roll approximation* consists in two conditions:

- neglecting the kinetic term of ϕ compared to the potential energy;
- assuming a flat potential so that $\ddot{\phi}$ is negligible as well in (1.33).

In this approximation, the Friedmann equation (1.34) and the field equation (1.33) are written

$$H^2 \simeq \frac{8\pi G}{3}V(\phi) \tag{1.35}$$

$$3H\dot{\phi} \simeq -V'(\phi) \tag{1.36}$$

where in this context $V'(\phi) = \frac{dV}{d\phi}$. That is, the friction due to the expansion is balanced by the acceleration due to the slope of the potential. The slow-roll conditions can be rewritten as follows

- $\dot{\phi}^2 \ll V(\phi) \Longrightarrow \frac{(V')^2}{V} \ll H^2$;
- $\ddot{\phi} \ll 3H\dot{\phi} \Longrightarrow V'' \ll H^2$.

If we define the following *slow-roll parameters*

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{1.37a}$$

$$\eta \equiv \frac{1}{8\pi G}\left(\frac{V''}{V}\right) \tag{1.37b}$$

the slow-roll conditions hold if $|\epsilon| \ll 1$ and $|\eta| \ll 1$.
It is now easy to see in another sense how the slow-roll approximation yields inflation. Let us recall that Inflation is defined by $\ddot{a} > 0$, or in other terms

$$\frac{\ddot{a}}{a^2} = \dot{H} + H^2 > 0$$

$\dot{H} > 0$ cannot be for a scalar potential (as p cannot be $< -\rho$): the acceleration condition can be translated to

$$-\frac{\dot{H}}{H^2} = \epsilon < 1$$

As soon as this condition fails, Inflation ends: in general, slow-roll inflation is attained if $\epsilon \ll 1$ and $|\eta| \ll 1$, where the latter condition helps to ensure that inflation will continue for a sufficient period.
Within this approximation, the total number of e-folds between the beginning and the end of Inflation is

$$N_{tot} \equiv \ln\left(\frac{a(t_f)}{a(t_i)}\right) = \int_{t_i}^{t_f} H dt \simeq -8\pi G \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi. \tag{1.38}$$

Concluding, Inflation is cosmologically attractive but serious problems are left unsolved with it: on the one hand, we cannot say if the universe in its earliest stages satisfied the conditions for Inflation to light up (i.e. for inflaton to undergo slow rollover); on the other hand, there are no experimental evidences even for the existence of a neutral spin zero boson far less for the existence of the inflaton in particular.

1.3 Foundamental ideas of Structure Formation

As already mentioned, the Cosmological Principle and hence the inhomogeneity of the universe have played a curious role in the history of modern cosmology: if the overall properties of the universe are very close to being homogenous and hence much of universe dynamics as a whole can be said thanks to the assumption of homogeneity and isotropy on the largest scales, on the other hand telescopes reveal a wealth of details on scales varying from single galaxies to large structures of size far exceeding 10^2 Mpc. Understanding the existence of these structure is one of the principal task of modern cosmology, and this study is usually performed with different techniques and approximation schemes, depending on the specific range of scales under analysis.

The interest in the large-scale mass distribution traces back to the Thirthies with Lemaitre, who pointed out that if the evolving homogenous and isotropic world model is a reasonable first appoximation (we now say zeroth order approximation), then the next step is to account for the departures from homogeneities in the observed structures. As the Cosmological Principle cannot be expected from general arguments and physical principles, nor the existence of galaxies can be deduced from general principles because we do not know how to specify initial conditions: we have been left with Lemaitre' s program consisting in trying to find the character of density fluctuations in the early universe and modelling the physical processes that have operated subsequently to develop such fluctuations into the irregularities we observe today.
Much work has been done in the last decades and now we can follow a great part of the evolution of initial perturbations to present structures thanks to a long list of cosmological schemes and methods. But before going into some more detailed description of the idea of structure formation we want still to stress on the nature of the Cosmological Principle. If it were really a principle, as initially suggested by Milne, the Cosmological Principle should be compared to a law of nature: on the contrary, now it is common sense to intend it as a philosophical assumption which allows us to circumevent our inability to obtain information about the universe outside our past light-cone by assuming that a symmetry principle exists everywhere. By assuming the Cosmological Principle, we assume that we are able to determine conditions many Hubble radii away from us by using observational data within our past light-cone, whose region of influence is, by definition, limited to one Hubble radius. It is exactly this point that should lead us to treat the Cosmological Principle as a subtle approach. Moreover, homogeneity could only apply on the average over many galaxies: we should then keep in mind that when we refer to homogeneity and isotropy of the universe we tacitly assume that spatial smoothing over some suitably large filtering scale has been applied exactly with the purpose of letting the fine-grained details to be ignored.

A great deal of structure formation theory is based on the study of just one scalar field, namely the *density perturbation field* defined as

$$\delta(t,\vec{x}) \equiv \frac{\rho(t,\vec{x}) - \rho_b(t)}{\rho_b(t)} \tag{1.39}$$

where ρ_b represents the unperturbed mean value of the background universe density, in the FRW model. In specific cases, this field is related to the *Newtonian peculiar gravitational potential* $\varphi(\vec{x})$ through the Poisson equation which in an expanding universe reads

$$\nabla^2\varphi(t,\vec{x}) = 4\pi G\; a^2(t)\rho_b(t)\;\delta(t,\vec{x}). \tag{1.40}$$

There are many different notations used to describe the density perturbations and their evolution, both in terms of the quantities used to describe the perturbations as metric deviations and of the definition of an appropriate statistical treatment. The former approach will be clearer only in the following chapters and it is the heart of the thesis; for now, we want to give a sketch of the latter. A critical feature of the quantity δ is that it inhabits a universe that is isotropic and homogenous in its large-scale properties: this suggest a statistical reformulation of Cosmological Principle, that is that the statistical properties of δ

should also be statistically homogenous. In other words, δ reflects a stationary random process: every spatial position $\vec{x}_i$ is associated to a stochastic variable $\delta(\vec{x}_i)$, with $i = 1, 2, ...N$ and $N \to \infty$, and all the probability densities on a finite number of points $P_{\vec{x}_1,\vec{x}_N,...,\vec{x}_N}(\delta_1, \delta_2, ...\delta_N)$ are invariant under translations, rotations and reflection of the points set $\vec{x}_1, \vec{x}_N, ..., \vec{x}_N$. The universe we observe is the statistical realization of $\delta(\vec{x})$ thought as a stochastic field, and in this language the unperturbed density of FRW background universe corresponds to the average over the statistical ensemble, $\rho_b \equiv \langle\rho(\vec{x})\rangle$.
Cosmological density fields are an example of *ergodic process*, in which the average over a large volume tends to the same answer as the average over a statistical ensemble.
It is usual to describe δ as a Fourier superposition:

$$\delta(\vec{x}) = \sum \hat{\delta}(\vec{k})\; e^{-i\vec{k}\vec{x}} \tag{1.41}$$

The cross-terms vanish when we compute the variance in the field, which is just a sum over modes of the *power spectrum*

$$\langle\delta^2\rangle = \sum |\hat{\delta}(\vec{k})|^2 \equiv \sum P(k) \tag{1.42}$$

where the statistical isotropic nature of the fluctuations allows us to write $P(k)$ rather than $P(\vec{k})$. Another quantity which describes the statistical properties of δ is the *autocorrelation function*, which is related to the power spectrum through Fourier transformation and hence gives the same description of the density field: for this reason, we skip for brevity the introduction of this further concept.
The physical meaning of the power spectrum is the following: $P(k) \propto |\hat{\delta}(\vec{k})|^2$, the latter being the amplitude of plane waves with wavelength $\lambda = 2\pi/k$; then the value of the spectrum at every k tells us how much the contribution of k-scale fluctuations is important in the Fourier sum in order to form the generic perturbation $\delta(\vec{x})$ in configurations space. In other words, $P(k)$ is a measure of the power of the fluctuations of wavenumber k.
A stochastic field is said to be *Gaussian* if the phases of the Fourier modes describing fluctuations at different scales λ are uncorrelated, that is if the amplitudes of waves of different wavenumbers are randomly drawn from a Rayleigh distribution of width given by the power spectrum. The density perturbation field is Gaussian (see later): this means that if we could do a very big number of statistical realizations of the universe, in any point $\vec{x}$ the distribution of the observed value of $\delta(\vec{x})$ in all those universes would be a Gaussian centered in zero. In momentum space, because the Fourier transformation of a Gaussian is still a Gaussian, the same description applies.
A Gaussian distribution is univocally described by its average and its variance: thus, in our case, what we need for describing the density fluctuation field $\delta(\vec{x})$ is just its power spectrum.
Assuming for $P(k)$ a simple functional form allows us doing simple and useful considerations. The most convenient power spectra are the so-called *power-law* power spectra

$$P(k) \propto k^{n-1} \tag{1.43}$$

where the exponential index n is called *spectral index*; these are often called *scale-free* power spectra because their logaritmic slopes are the same at every scale, and hence they are characterized by no particular physical scale. Among the others, a case of particular interest is the Harrison-Zel'dovich spectrum, which corresponds to a power spectrum with $n = 1$.

Inflation and cosmological perturbations

In order for structure formation to occur, there must have been small preexisting fluctuations on physical length scales when they crossed the Hubble radius in the radiation-dominated or matter-dominated eras. In the standard Big Bang model these small perturbations have to be put by hand, because it is impossible to produce fluctuations on any length scale while it is larger than the horizon. Since the goal of cosmology is to understand the universe on the basis of physical laws, this appeal to initial condition is unsatisfactory. The challenge is therefore to give an explanation to the small "seed" perturbations which allow the gravitational growth of the matter perturbations.
The simplest mechanism for generating the observed perturbations is the inflationary cosmology, as mentioned in previous sections. Although originally introduced as a possible solutions of already seen problems such as the horizon and flatness problems, as an unexpected bonus, Inflation has the useful property to generate spectra of both density perturbations and gravitational waves, through the amplification of quantum fluctuations: these perturbations extend from extremely short scales to scales considerably in excess of the size of the observable universe.
In the simplest inflationary model introduced earlier, Inflation is driven by a slowly-rolling scalar field, the inflaton: this latter can be split in

$$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x}), \tag{1.44}$$

where ϕ_0 is the classical (infinite wavelength) field, that is the expectation value of the inflaton field on the initial isotropic and homogenous state, whose stress-energy tensor and equation of motion have been already expressed in (1.32) and (1.33); $\delta\phi(t, \vec{x})$ represents the quantum fluctuations around ϕ_0. This separation is justified by the fact that quantum fluctuations are much smaller than the classical value and therefore negligibile when looking at the classical evolution, as done in previous pages. Nevertheless, exactly those quantum fluctuations are responsible for the creation of initial perturbations whose evolution can now be seen in the large-scale structure of the universe.
It is not possible to describe the generation of perturbations of a scalar field in this context: the machinery needed fot such a task is almost the same formalism developed throughout the thesis, at least a linear theory of cosmological perturbations would be needed. Anyway, we can give a heuristic explanation of why we expect that during Inflation such fluctuations are indeed present and how these inflaton fluctuations will induce in turn pertubations of the metric [10].
If we take equation (1.33) adding the non-homogenous term $-\nabla^2\phi/a^2$, and split the inflaton field as in (1.44), the quantum perturbation $\delta\phi$ satisfies the equation of motion

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} + V''\delta\phi = 0. \tag{1.45}$$

Differentiating (1.33) with respect to time t and taking H constant (we are during inflationary phase!) we find

$$(\phi_0)^{\cdots} + 3H\ddot{\phi}_0 + V''\dot{\phi}_0 = 0. \tag{1.46}$$

Let us consider for simplicity the limit $k^2/a^2 \ll 1$ and let us disregard the gradient term. Under this condition we see that $\dot{\phi}_0$ and $\delta\phi$ solve the same equation. The solutions have therefore to be related to each other by a constant of proportionality which depends upon time, that is

$$\delta\phi = -\dot{\phi}_0 \; \delta t(\vec{x}).$$

This tell us that

$$\phi(t, \vec{x}) = \phi_0(t - \delta t(\vec{x}), \vec{x}),$$

that is the inflaton field does not acquire the same value at a given time t in all the space. On the contrary, when the inflaton is rolling down its potential, it acquires different values from one spatial point $\vec{x}$ to the other. Then inflaton field is not homogenous and fluctuations are present.

These fluctuations will induce fluctuations of the metric: any perturbation in the inflaton field means a perturbation of the stress-energy tensor; a perturbation in the stress-energy tensor implies, through E.E., a perturbation of the metric. On the other hand, a perturbation of the metric induces a backreaction on the evolution of the inflaton through the perturbed Klein-Gordon (K.G.) equation of the inflaton field: hence,

$$\delta\phi \Longrightarrow \delta T_{\mu\nu} \stackrel{E.E}{\Longrightarrow} \delta g_{\mu\nu} \stackrel{K.G.}{\Longrightarrow} \delta\phi \tag{1.47}$$

During Inflation the scale factor grows exponentially, while the Hubble radius remains almost constant. Consequently the wavelength of a quantum fluctuation soon exceeds the Hubble radius, stretched by the inflationary expansion. The amplitude of the fluctuations therefore become "frozen in". Once Inflation has ended, however, the Hubble radius increases faster than the scale factor, so -in the way we have already seen- the fluctuations eventually reenter the Hubble radius and hence the horizon during the radiation- or matter- dominated eras. The number of e-folds which are needed to let our present horizon scale of about 10^4 Mpc to reenter the horizon during Inflation is about 60, as we have seen in previous Section: all the fluctuations which exited the horizon in a very narrow interval of about 10 e-folds around 60 e-folds of Inflation length have reentered with physical wavelengths in the range accessible to cosmological observations and of interest for structure formation today, that is the range scale between 1 and 10^4 Mpc. These spectra provide a distinctive signature of Inflation.

The simplest models generate two types of perturbations: density perturbations which come from fluctuations in the inflaton scalar field and the corresponding scalar metric perturbations (which we will define better in Chapter 3), and gravitational waves which are tensor metric fluctuations. The former experience gravitational instability and lead to structure formation, while the latter can influence the cosmic microwave background anisotropies.

In terms of the power spectra of these perturbations, with the working assumption of initial power-law spectrum for both density perturbations and gravitational waves,

$$\begin{aligned} P(k) &\propto k^{n-1} && \text{scalar or density perturbations} \\ P_{grav}(k) &\propto k^{n_{grav}} && \text{gravitational waves,} \end{aligned}$$

the spectral indices are in some way related to the slow-roll parameters [9]:

$$n \simeq 1 - 6\epsilon + 2\eta \qquad n_{grav} \simeq -2\epsilon. \tag{1.48}$$

The simplest Inflation models predict adiabatic fluctuations and a level of non-Gaussianity which is too small to be detected by any experiment so far conceived. *Adiabaticity* means that all types of material in the universe share a common perturbation, so that if the spacetime is foliated by constant-density hypersurfaces, then all fluids and fields are homogenous on those slices, with the perturbations completely described by the variation of the spatial curvature of the slices. The second part of Table 1.2 can now be understood and used for getting the values of the perturbations creation that give the best agreement between models and observations.

Standard scenario of structure formation

After the perturbations are created in the early universe, they undergo a complex evolution up until the time they are observed in the present universe. In summary, the key ingredients for understanding the observed structures in the universe within the standard inflationary scenario are summarized as follows.

- The universe is composed mainly by non-baryonic dark matter. The evidence for this matter being dark (i.e. interacting only with gravity) come from the dynamics of clusters of galaxies and of galaxy haloes.

- Baryons are present in the amount predicted by the Big Bang Nucleosynthesis, some percent of the density required to close the universe.

- At recombination (redshift $z \sim 1000$, in the matter era) the universe is well described by a FRW metric. Small deviations from homogeneity and isotropy do exist: $\delta\rho/\rho \sim 10^{-5}$. These deviations are created during an inflationary period in the early universe: quantum fluctuations of the inflaton field are excited during Inflation and stretched to cosmological scales. At the same time, the inflaton fluctuations being connected to the metric perturbations through E.E., ripples on the metric are also excited and stretched to cosmological scales.

- Gravity acts as a messanger since it communicates to baryons and photons the small seed perturbations once a given wavelength becomes smaller than the horizon scale after Inflation.

- Cosmic structures form by *gravitational instability* (which we will see in some aspects later): this process is driven by the gravity of the dark matter component of the universe, up to the formation of the first non-linear systems, the *dark matter haloes*.

- Galaxies and luminous systems form later by the *dissipative collapse* of gas (baryonic matter) in the potential wells of dark matter haloes.

- Within this scenario, the most successful model coherent with observations is *hierarchical clustering*, with the dominant dark matter being cold, that is non relativistic, and where the initial density power spectrum is such that larger systems form later by the assembly of pre-existing smaller units.

The details of this complex process are determined by the values of cosmological parameters. On the other hand, the comparison between observations and structure formation models is developed on different fronts: CMB, large-scale clustering properties, peculiar motions of galaxies, gravitational lensing, properties of large-scale structure, dark matter haloes structure, galaxy counting,....
The techniques developed for modelling the details of the above described scenario are various and can be divided in three groups: analytical techniques, numerical simulations, and semi-analytical methods. If we want to set the approach of our thesis against such a distinction of methods, we should of course underline its analytical nature.
Density fluctuations δ are called *linear* until they are much smaller than 1, $\delta \ll 1$: within this limit, as we will see, it will be sufficient to study their evolution using a perturbative theory up to first order. When gravitational growth leads to $\delta \to 1$, we talk about *non-linear regime* and a first order perturbation expansion is no more applicable, forcing us to go at the following orders. In our thesis the calculations will be performed up to second order in our perturbative technique.
Finally, as structure formation study involves a wide range of scales under analysis, let us recall that General Relativity is of course the more complete and appropriate tool to handle gravitational interactions. However when the scales under analysis do not exceed the Hubble radius, the Newtonian approximation can be applied as a limiting case of the full relativistic theory, consisting in perturbing only the time-time component of the FRW metric tensor by an amount $2\varphi/c^2$, in contrast with a general metric perturbation as the one that we will see in Chapter 3. Wanting to be able to deal with cosmological perturbations of any length scale (from super-horizon to small scales), in the thesis our analysis will be fully relativistic.

Gravitational Instability As last task of this Chapter we want briefly to delineate the simplest model for the generation of cosmological structure, that is gravitational instability. The fact that a fluid of self-gravitating particles is unstable to the growth of small inhomogeneities was first pointed out by Jeans in the late Twenties and is known as the Jeans instability.
Expanding the perturbation matter density ρ in plane waves as already mentioned earlier, the growth of small matter inhomogeneities of wavelength smaller than the Hubble scale is governed by a Newtonian equation:

$$\ddot{\hat{\delta}}(\vec{k}) + 2H\dot{\hat{\delta}}(\vec{k}) + \hat{\delta}(\vec{k})\left(\frac{v_s^2\; k^2}{a^2} - 4\pi G\rho_b\right) = 0 \tag{1.49}$$

where $v_s^2 = \partial p/\partial\rho$ is the square of the sound speed. Competition between the pressure term and the gravity term in the last term of equations (1.49) determines whether or not pressure can counteract gravity. The *Jeans scale* or the *Jeans wavenumber* are scale values which arise naturally from the physical content of the process and which distinguishes two different regimes. Defining them as

$$k_J^2 \equiv \frac{1}{v_s^2}\, 4\pi G\rho_b \text{ and } \lambda_J^2 \equiv v_s^2\, \frac{\pi}{G\rho_b}, \tag{1.50}$$

perturbations with wavenumber larger than the Jeans wavenumber are stable and oscillate: the density fluctuation $\delta(t,\vec{x})$ evolves in time and space as a sound waves; pertubations with smaller wavenumber are Jeans unstable and can grow, eventually undergoing in a gravitational collapse:

$k > k_J \Longrightarrow$ OSCILLATION: SOUND WAVE
$k < k_J \Longrightarrow$ GRAVITATIONAL INSTABILITY: STATIONARY WAVE.

The solutions of equation (1.49) or the relativistic equivalent equation depends on the circumstances: many cases can be studied according to the time period of universe under analysis (before or later than the matter-radiation equivalence), to the length scales involved (sub or super horizon), and to the type of energy component dominating (radiation, matter or dark matter) [3], [?]. In a matter dominated universe, because the expansion tends to pull particles away from one another, the growth of matter density perturbations is only a power law. In a radiation-dominated universe, the expansion is so rapid that the matter perturbations grow very slowly, as $\ln a$; if we consider radiation density perturbations in a radiation-dominated universe, then the situation is different, because perturbations grow as a^2. Considering δ as the baryonic matter density perturbation field, then

$$\delta(t) \propto \begin{cases} \ln a(t) & \text{(radiation domination)} \\ a(t) & \text{(dust domination).} \end{cases} \tag{1.51}$$

Therefore, perturbations of baryonic matter density which we can see in galaxies and stars may grow only in a matter dominated period. When Dark Energy begins to dominate, that is for $z \leq 1$, perturbations stop growing.

Chapter 2

Dust Cosmology: frame and formalism

In this thesis we deal with irrotational and pressureless fluid dominated universes, studying the perturbation theory in a synchronous and comoving system of coordinates.
In this Chapter we outline the formalism used throughout the work.
We give a precise characterization of the fluid, define the synchronous and comoving gauge choice and derive the equations governing the evolution of such a fluid. We note that the possibility of making these two gauge choices simultaneously is a peculiarity of irrotational dust, that spatial coordinates in this gauge are Lagrangian coordinates and that the so-called slicing and threading of spacetime are the same. In this simple frame, we see that E.E. can be divided in 4 constraints and 6 evolution equations, the so-called *energy* and *momentum constraints* and *evolution equations* of the ADM approach.

2.1 Space-time splittings, gauge choices and general hypotheses

When we talk about our spacetime we mean a $(1 + n)$-dimensional manifold $(\mathcal{M}, g_{\mu\nu})$ with Lorentzian metric of signature (-,+,...+) and $n = 3$, namely a curved spacetime described by metric components where the curvature is created by (and reacted back on) energy and momentum. Although General Relativity makes no fundamental distinction between time and space, actually we do, and in order to obtain field equations comparable with those of Newtonian gravity (and Electrodynamics) we need indeed a decomposition procedure of Einstein Equations (E.E.), conservation equations and other geometrical and physical quantities.
In what follows we will always assume $(\mathcal{M}, g_{\mu\nu})$ be a *globally hyperbolic spacetime*. A spacetime is *globally hyperbolic* if it possesses a Cauchy surface Σ: for us, it will be sufficient to think of a Cauchy surface as an embedded C^0 submanifold of $\mathcal{M}$, representing an "instant of time" throughout the universe. The fundamental feature of a globally hyperbolic spacetime is that the entire future and past history of the universe can be predicted (or retrodicted) from

conditions at the instant of time represented by Σ. In other world, the Cauchy Problem can be solved.
Actually we invoke such a feature of our universe not for predictability issues, but to decompose our spacetime [1]:

Theorem 1. *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic spacetime. Then a global time function t can be chosen such that each surface of constant t is a Cauchy surface: thus $\mathcal{M}$ can be foliated by Cauchy surfaces and the topology of $\mathcal{M}$ is $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy surface.*

It is thanks to this theorem that -from a very general point of view- we can slice our spacetime in hypersurfaces at constant t and then implement gauge choices, or view the spatial metric on a three-dimensional hypersurface as the dynamical variable in General Relativity. But let us procede step by step.

Let n^μ be the unit normal vector field to the hypersurface Σ_t: the spacetime metric $g_{\mu\nu}$ induces a spatial metric (i.e. a three-dimensional Riemannian metric) $h_{\mu\nu}$ on each Σ_t by the formula

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{2.1}$$

This is known as *orthogonal decomposition of the metric* and we will often refer to this slicing of spacetime as (3+1) splitting.
(3+1) splitting is complementary to the alternative and more general (1+3) split called "threading" (see [7]): there the fundamental geometrical objects used for charting spacetime are a series of timelike worldlines $x^\mu(\lambda, \mathbf{q})$, where λ is an affine parameter measuring proper time along the worldline and $\mathbf{q}$ gives a unique label (e.g., a spatial Lagrangian position vector) to each different "thread".
In principle we will be inclined to use the splitting in hypersurfaces and define our geometrical variables in such a context: anyway, it is worth bearing in mind from now on that in the particular frame which we will adopt the two descriptions are the same.

Gauge choices

Theorem 1 tells us that a splitting of our spacetime is possible but does not provide a precise procedure: the different splitting procedures deal with coordinates or gauge choices.
General Relativity is invariant under diffeomorphisms; diffeomorphisms are coordinate transformations in some sense and choosing the coordinate systems means fixing the chart between open subsets of $\mathcal{M}$ and open subsets of $\mathbb{R}^{n+1}$.
This invariance under diffeomorphisms reflects the redudancy in the description of spacetime geometry by metric components $g_{\mu\nu}$ and can be seen in the indetermination of E. E. system: it is also known as *gauge freedom*. In other words, the diffeomorphisms comprise the gauge freedom of any theory formulated in terms of tensor fields on a spacetime manifold: in particular, diffeomorphisms comprise the gauge freedom of General Relativity [1].
In what follows we will then refer to a gauge (or gauge choice) as a coordinates choice or more loosely to a family of coordinates choices, and a gauge transformation as equivalent to a coordinates transformation.
There are two different ways by which we can implement a gauge choice:

- we can impose a suitable number of relations among gauge-dependent variables: in terms of coordinates, $1+n$ are the coordinates transformations then $1+n$ are the gauge conditions;
- or given a $1+n$ spacetime, we can slice it in space-like hypersurfaces at t=const where we fix spatial coordinates, and thread it in time-like lines (orthogonal to hypersurfaces) along which we make the time coordinate flowing.

We will use these two recipes later to define our special gauge choice: there we will see in detail how the two approaches give the same result.
Concerning the gauge transformation as change of coordinates system, we can write it formally as an (infinitesimal) traditional coordinate transformation:

$$x^{\mu} \rightarrow \bar{x}^{\mu} = x^{\mu} + \epsilon\, \xi^{\mu} \tag{2.2}$$

where ϵ is a (small) parameter and ξ^{μ} a 4-dimensional vector. According to the decomposition of spatial vectors on Σ given in Appendix A and having separated time and space parts of $\xi^{\mu} = (\xi^{0}, \xi^{i})$, the latter can still be decomposed in a scalar (irrotational) and a solenoidal components:

$$\xi^{0} \doteqdot \alpha \qquad \xi^{i} \doteqdot \partial^{i}\beta + d^{i} \text{ (with } \partial_{i} d^{i} = 0) \tag{2.3}$$

In terms of components then a gauge transformation is implemented with 2 scalars and 1 transverse vector:

$$x^{0} \rightarrow \bar{x}^{0} = x^{0} + \epsilon\, \alpha \tag{2.4}$$

$$x^{i} \rightarrow \bar{x}^{i} = x^{i} + \epsilon\, (\partial^{i}\beta + d^{i}) \tag{2.5}$$

Exstrinsic curvature

As already mentioned, we may view a globally hyperbolic spacetime as representing the time development of a Riemannian metric on a fixed 3-dimensional manifold. A quantity which expresses a well-defined notion of "time derivative" of the spatial metric on a hypersurface embedded in $\mathcal{M}$ is the *extrinsic curvature*. Having in mind the general orthogonal decomposition of the metric given in equation (2.1) and adding the unit time-like condition for vectors $n^{\mu}n_{\mu} = -1$, then extrinsic curvature is defined as follows

$$K_{\mu\nu} \doteqdot \frac{1}{2}\mathfrak{L}_{n} h_{\mu\nu} \tag{2.6}$$

where $\mathfrak{L}_{n}$ is the Lie derivative along n. [1]

As $h_{\mu\nu}$ is purely spatial, extrinsic curvature is purely spatial too: then it would have been preferable writing

$$K_{ij} \doteqdot \frac{1}{2}\mathfrak{L}_{n} h_{ij} \tag{2.8}$$

[1] Expressions of Lie derivative along ξ are:

$$\mathfrak{L}_{\xi} f = f_{,\mu}\xi^{\mu} \tag{2.7a}$$

$$\mathfrak{L}_{\xi} Z^{\mu} = Z^{\mu}_{,\nu}\xi^{\nu} - \xi^{\mu}_{,\nu}Z^{\nu} \tag{2.7b}$$

$$\mathfrak{L}_{\xi} T^{\mu\nu} = T_{\mu\nu,\sigma}\xi^{\sigma} + \xi^{\sigma}_{,\mu}T_{\sigma\nu} + \xi^{\sigma}_{,\nu}T_{\mu\sigma} \tag{2.7c}$$

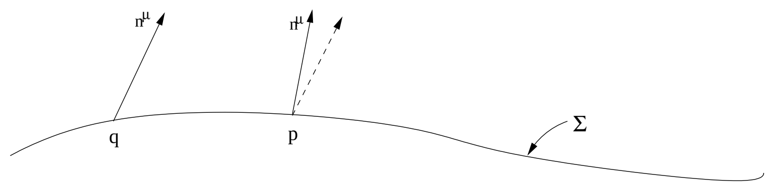

Figure 2.1: Notion of the extrinsic curvature of a hypersurface Σ. The failure of the parallel transported vector along a geodesic from q to p to coincide with n^μ at p corresponds intuitively to the bending of Σ in the spacetime in which it is embedded. The formula $K_{\mu\nu} = \frac{1}{2}\mathfrak{L}_n h_{\mu\nu} = h^\alpha_\mu \, n_{\nu;\alpha}$ shows that $K_{\mu\nu}$ directly measures this failure.

Furthermore, extrinsic curvature is symmetric, $K_{ij} = K_{ji}$, and its trace is often denoted by K:

$$K \doteqdot K^a_{\ a} = h^{ab} K_{ab} \tag{2.9}$$

We will note later that extrinsic curvature assumes interesting physical meanings according to the gauge choice.

2.2 Characterization of the matter content

The geometry of spacetime is determined by its energy content through the stress-energy tensor. The matter (or radiation) content of the universe may be described in two convenient ways, related to the two eulerian and lagrangian approaches of hydrodynamics, and strictly connected to the (3+1) and (1+3) splittings of spacetime.
The **eulerian approach** consists in a fluid approximation: a fluid is a dense set of particles treated as a continuum. This continuum is described by a vector field (that we assume to be unique) representing the average velocity of matter in the neighborhood of each point of spacetime.
The **lagrangian approach** uses a particle distribution function in order to follow each matter element along its worldline and labeling it with a unique spatial position vector **q**.
In any case, the matter *4-velocity* of a particle is defined to be the unit tangent (as measured by $g_{\mu\nu}$) to its worldline:

$$u^\mu = \frac{dx^\mu}{d\lambda} \text{ with } d\lambda^2 \doteqdot -dS^2 \text{ and such that } u^\mu u_\mu = -1 \tag{2.10}$$

In the (3+1) split, spacetime is naturally described by Eulerian observers sitting in the space-like hypersurfaces with constant spatial coordinates; in the (1+3) split, spacetime is described by Lagrangian observers moving along the worldline which define the threading.

Although we prefer a (3+1) splitting, we will have in mind the latter point of view when defining the other kinematic quantities of matter content, even if definitions are coeherent in any of the two approaches.

Stress-energy tensor

The stress-energy tensor in E.E. provides the source for the metric variables: as the FRW metric is our zeroth order solution of the universe, the stress-energy tensor of the background matter is forced to take a perfect fluid form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{2.11}$$

where with perfect fluid we generally mean a patch of matter isotropic in its rest frame and characterized only by pressure and energy density. We then a priori exclude any extra terms corresponding to bulk and shear viscosity (respectively, the isotropic stress generated when an imperfect fluid is rapidily compressed or expanded, and the stress due to the shear -see below), thermal conduction and other physical processes.
To these restrictions we add our requirement of matter content being pressureless and hence collisionless: such a pressureless fluid is often called *dust* or *cold dust* and is described by a very simple stress-energy tensor, namely

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{2.12}$$

Other kinematic quantities

Let V^μ be a time-like unit vector field, tangent vector to a congruence of time-like curves; the following quantities are defined:

$$\text{PROJECTION TENSOR } h_{\mu\nu} = g_{\mu\nu} + V_\mu V_\nu \tag{2.13a}$$

$$\text{VECTOR-GRADIENT TENSOR } \Theta_{\mu\nu} \equiv \frac{1}{2} h_\mu^{\ \alpha}\, h_\nu^{\ \beta}(V_{\alpha;\beta} + V_{\beta;\alpha}) \tag{2.13b}$$

$$\text{EXPANSION } \Theta \equiv V^\mu_{\ ;\mu} \tag{2.13c}$$

$$\text{SHEAR } \sigma_{\mu\nu} \equiv \Theta_{\mu\nu} - \frac{1}{3}\, h_{\mu\nu}\, \Theta \tag{2.13d}$$

$$\text{VORTICITY OR TWIST } \omega_{\mu\nu} \equiv h_\mu^{\ \alpha}\, h_\nu^{\ \beta}(V_{\alpha;\beta} - V_{\beta;\alpha}) \tag{2.13e}$$

$$\text{ACCELERATION } a_\mu \equiv V_{\mu;\nu}\, V^\nu = \dot{V}_\mu \tag{2.13f}$$

These time-like curves could represent the histories of small test particles, in which case they would be geodesics, or they might represent the flow lines of a generic fluid: hence, quantities of (2.13) assume specific physical meanings depending whether the time-like unit vector is the normal vector field to a family of space-like hypersurfaces n^μ, the 4-matter velocity u^μ or geodesics tangents ξ^μ of free particles.

$V^\mu = n^\mu$) If $V^\mu = n^\mu$ then the projection tensor is the well known spatial metric and Θ represents the volume expansion rate of the hypersurfaces along the normal vector.

$V^\mu = u^\mu$) If $V^\mu = u^\mu$, $h_{\mu\nu}$ is at each point a projection tensor into the rest space of an observer moving with 4-velocity u^μ; the velocity-gradient tensor determines the rate of change of distance of neighbouring particles in the fluid and Θ its isotropic volume expansion. The shear tensor $\sigma_{\mu\nu}$ (the trace part of $\Theta_{\mu\nu}$) determines the distorsion arising in the fluid flow leaving the volume constant: the direction of the principal axes of shear (its eigenvectors) are unchanged by the distorsion, but all other directions are changed. Finally, the vorticity tensor $\omega_{\mu\nu}$ determines a rigid rotation of patch of fluid with respect to a local inertial rest frame leaving one direction (the axis of rotation) fixed (see Figure 2.2).

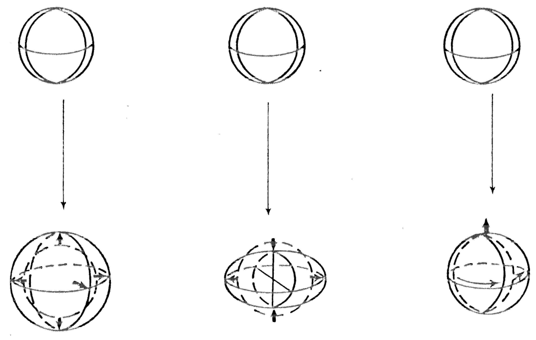

Figure 2.2: It is probably easiest to understand the meaning of some of the defined quantities by considering how a sphere of fluid particles changes during the elapse of a small increment in proper time, choosing 0 at the centre of the sphere: *(a)* action of expansion Θ alone; *(b)* action of shear $\sigma_{\mu\nu}$ alone; *(c)* action of vorticity $w_{\mu\nu}$.

As one moves along one of such families curve, expansion, shear and vorticity change with precise evolution equations, knowing the Riemann tensor. Among the others, we concentrate our attention on the *Raychaudhuri Equation*, the equation for the rate of change of the expansion Θ which plays a central role throughout the thesis:

$$\frac{d\Theta}{dS} = -\mathcal{R}_{\mu\nu}V^{\mu}V^{\nu} + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\Theta^2 + \dot{V}^{\mu}{}_{;\mu} \tag{2.14}$$

$$\text{(where } \omega^2 = \tfrac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} \geq 0 \text{ and } \sigma^2 = \tfrac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0\text{)}$$

From it one sees that vorticity induces expansion (+ sign) as might be expected by analogy with centrifugal forces, while shear induces contraction (- sign).
We do not derive here equation (2.14) in fully generality but we postpone the task to a next section, where we will adopt a precise gauge choice and hypothesis on matter in order to express the Ricci tensor through E.E. Anyway, let us remark that the Raychaudhury equation is valid apart from E.E..

We recall that another hypothesis that our matter content will have to satisfy is to be not only pressureless but also irrotational, that is with $\omega_{\mu\nu} \equiv 0$: the reason of such a requirement will be manifest in next section.

2.3 The syncrhonous and comoving system of coordinates

Defining the synchrounous gauge

We begin following the first approach outlined in the previous sections.
Let $(\mathcal{M}, g_{\mu\nu})$ be a manifold with metric of signature (-,+...+): the **synchronous gauge** is defined by the conditions

$$g_{00} = -1, \;\; g_{0i} = 0$$

In terms of coordinates, if $dim\mathcal{M} = 1+n$ then we must specify $1+n$ conditions, because $1+n$ are the coordinates tranformations: g_{00} carries with itself one degree of freedom and defines the temporal coordinate (the slicing), while the n-vector g_{0i} fixes the spatial coordinates.
In terms of components under spatial transformations (scalars, vectors and tensors -see later Appendix A-), a gauge choice is implemented with 2 scalars and 1 transverse vector: 1 scalar comes from g_{00}, 1 scalar and 1 vector from g_{0i}.
Let then see the properties of such a coordinate system.

Fact 1. $g_{00} = -1 \implies$ *temporal coordinate* $x^0 \equiv$ *proper time* η.
Indeed, between two events at the same spatial coordinates, we have
$dS^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2 d\eta^2 = g_{00}dx^0dx^0 \Rightarrow d\eta = \frac{1}{c}\sqrt{-g_{00}}dx^0$
In other words, $g_{00} = -1$ implies that the proper-time distance between two neighboring hypersurfaces along the normal vector coincides with the coordinate-time distance defining these hypersurfaces.
For this reason, we will even refer to this condition with the expression *proper-time slicing.*

Fact 2. $g_{0i} = 0 \implies \neq$ *space-coordinates clocks synchronization.*
Indeed, the rate of deviation from simultaneity between two clocks at different spatial coordinates measuring the same events is $\Delta x^0_{SIM} = -\frac{g_{0i}dx^i}{g_{00}}$ (see the usual radar-rangin experiment). In this case, the time coordinate of an event marked by two clocks at different spatial coordinates coincide.

Another way of defining the synchronous gauge refers to the second approach seen earlier. Let $(\mathcal{M}, g_{\mu\nu})$be a $1+n$-dimensional spacetime.
Synchronous gauge: foliation of $\mathcal{M}$ in n-hypersurfaces at $t = const$ on which we put spatial coordinates such that clocks are synchronized, and identification of normal geodesics as time-lines along which we let the time-coordinate flowing.

$$\Sigma_t \perp geodesics$$

Such a geometrical construction is possible thanks to the next general features.

Lemma 1. *Let Σ be a n-dimensional submanifold of $\mathcal{M}$ with Riemannian metric; let n^μ the vector normal to Σ in a generic point $p \in \Sigma$. Then n^μ has the direction of time (it is inside the light-cone of p).*

Lemma 2 (Existence and unicity of geodesics)**.** *Given $p \in \mathcal{M}$ and V_p the tangent space at p of $\mathcal{M}$, then for any $T^\mu \in V_p$ there always exists a unique geodesic through p with tangent T^μ.*

Applied to our situation, these two lemmas allow us to define a sensible **prescription for the coordinates choice**. The n-dimensional embedded submanifolds of $\mathcal{M}$ are our space-like hypersurfaces at constant time, whose tangent spaces can be naturally viewed as n-dimensional subspaces of the tangent space of $\mathcal{M}$. We begin referring to a single hypersurface at constant time, which we could call Σ_{IN}: for brevity, we will avoid this specification remembering that the possibility of extending the construction to all Σ_t is not obvious but feasible and viable. So, let p be a generic point of Σ and n^μ the unique

vector $\in V_p$ orthogonal to all vectors in $V_p(\Sigma)$: for the lemma 1, this vector does not lie in $V_p(\Sigma)$. Then we can construct the unique geodesic through p with tangent n^μ and fix the coordinates as follows. We choose arbitrary coordinates $(x^1, ..., x^n)$ on a portion of Σ: then we label each point q in a neighborhood of that portion of Σ with the parameter t along the geodesic on which it lies and with the coordinates $x^1, ..., x^n$ of the point $p \in \Sigma$ from which the geodesic emanated.
In a sufficiently small neighborhood of each $p \in \Sigma$, the map $q \rightarrow (t, x^1, ..., x^n)$ defines the chart we wished to construct.[2]
Moreover, one could demonstrate that the geodesics remain orthogonal to all the hypersurfaces Σ_t [1], showing that the prescription for the coordinate choice can be extended to all the spacetime.

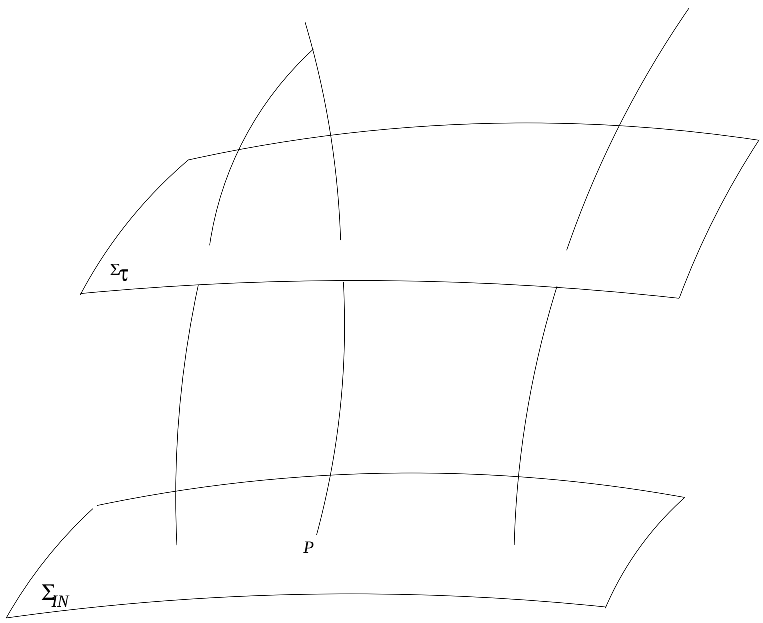

Figure 2.3: Construction of Gaussian Normal coordinates or synchronous gauge.

This geometrical construction, otherwise the first one, shows much more directly the connections between the physical concept of system of coordinates and the mathematical one of chart of a manifold.
There is more. The geodesics emanating from Σ may eventually cross or run into a singularity. This occurence is harmful in the (3+1) frame because the hypersurfaces (exactly by the definition of embedded submanifold) shoul not cross themselves or the others in order to preserve the chart being one to one and onto: in that case, on the contrary, two different sets of x^μ label the same spacetime event. This is the reason why the threading (1+3) description is more general than the slicing one [7] and in some cases preferable.

[2]We use here the time label t consistently with global iperbolicity theorem: anyway, t is still just a time coordinate or parametrization, that one we called x^0: when later we will assume a synchronous gauge then we will be allowed to use t as proper time.

Equivalence of the two definitions

The synchronous frame in the first approach presents the following properties.

Fact 3. $g_{00} = -1,\ g_{0i} = 0 \implies$ *time-lines (with $x^1 = ... = x^n = const$) are orthogonal to hypersurfaces at $t = const$.*
In other words, the rate of deviation of a constant space-coordinate line from a line normal to a constant time hypersurface is null.

Indeed, let us write the n-vector tangent to the time-like lines:

$$\xi^\mu = \frac{dx^\mu}{d\lambda} \text{ with } d\lambda = (-dS^2)^{\frac{1}{2}} :\ \xi^0 = -1,\ \xi^i = 0$$

Let us write the n-vector $\perp \Sigma_t$: $n_\mu = \frac{\partial t}{\partial x^\mu}$ with $n_0 = 1,\ n_i = 0$.
Then $n^0 = g^{0\rho}\ n_\rho = -1, n^i = n_i \ \Rightarrow \ n^0 \equiv \xi^0$ and $n^i \equiv \xi^i$.$\Box$

Fact 4. $g_{00} = -1,\ g_{0i} = 0 \implies$ *time-like lines are geodesics of all spacetime.*

Indeed, let ξ^μ be the tangent n-vector to lines defined by the equation $x^1 = ... = x^n = const : \xi^0 = -1,\ \xi^i = 0$. Let us remember the geodesic equation:

$$\frac{dx^{\mu 2}}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0$$

We can easily see that ξ^μ is solution of the equation. In fact
$\frac{d\xi^0}{dS} + \Gamma^0_{\nu\rho}\xi^\nu\xi^\rho = 0 + \Gamma^0_{00}\xi^0\xi^0 + \Gamma^0_{0j}\xi^0\xi^j + \Gamma^0_{ij}\xi^i\xi^j = 0$ and $\frac{d\xi^i}{dS} + \Gamma^i_{\nu\rho}\xi^\nu\xi^\rho = 0$, being $\Gamma^0_{00} = \Gamma^i_{00} = 0$.$\Box$

In other words, if $g_{00} = -1,\ g_{0i} = 0$, vectors orthogonal to the hypersurfaces are (tangent to) time-like lines of constant space-coordinates and time-like lines are geodesics. These features of synchronous conditions allow to implement the geometrical construction of the second way demostrating the equivalence of the two approaches. Yet, they are less general than the two lemmas seen earlier, that is why we preferred to show the two definitions separately.

Other characterizations of synchronous gauge

- A synchronous gauge choice is in principle always possible for a spacetime like our own, 1+3-dimensional with Lorentzian metric.
- The synchronous gauge choice is not unique: gauge-fixing conditions or geometrical construction do not eliminate the gauge freedom, neither in time slicing nor in space-coordinates setting. They leave a so called residual gauge freedom. Infact:
  - 1^{st} approach: a metric such as

    $$dS^2 = -dt^2 + h_{ij}\ dx^i dx^j$$

    admits any time-coordinate transformation and any space-coordinates tranformation.
  - 2^{nd} approach: although the chart is well defined, a residual gauge freedom arises from the freedom to adjust the initial settings of the clocks (to choose the Σ_{IN}) and to choose the initial spatial coordinate labels (the origin of space-coordinates).

- In the synchronous gauge, there exists a natural choice of reference system, that one of "fundamental observers" who fall freely along the normal geodesics carrying clocks reading time t. Because the spatial coordinates x^i of each fundamental observer are held fixed with time, the x^i in synchronous gauge are Lagrangian coordinates.

- In the synchronous gauge, it is not possible to put at rest ($\vec{v} = 0$) all the matter filling the space: it is to say that a synchronous system is not necessarily a comoving gauge.

- $K_{\mu\nu} = \frac{1}{2}\mathfrak{L}_\xi h_{\mu\nu} = \frac{1}{2}\,\frac{\partial h_{\mu\nu}}{\partial t}$.

The comoving gauge

The comoving gauge, unlike the synchronous one, deals with the content of matter in our spacetime.
Let $(\mathcal{M}, g_{\mu\nu})$ be a manifold with metric of signature (-,+...+): the **comoving gauge** is defined as the frame in which all filling space matter is at rest:

$$u^i = 0$$

This condition fixes spatial coordinates only: u^i carries with itself only 3 degrees of freedom in terms of coordinates and 1 scalar and 1 solenoidal vector in terms of components. We then should call this condition space-coordinates choice rather than gauge. What about the time slicing?
Following [6], we stress that there are several possibilities in associating this space-coordinates choice to a time-slicing: for example, one could take $g_{00} = u^i = 0$, in what can be called *comoving proper-time gauge*, or $g_{0i} = u^i = 0$ that is a *comoving time-orthogonal gauge.*
Consistently with [16] and [17], we will think of the latter alternative as our comoving gauge and we specify the definitions as follows: Let $(\mathcal{M}, g_{\mu\nu})$ be a manifold with metric of signature (-,+...+): the **comoving gauge** is defined by the conditions:

$$u^i = g^0_i = 0$$

The quantity $(u^i - g^i_0)$ can be shown to be a scalar under gauge transformations ([6]: it transforms under gauge change only with the α of law (2.4)): the condition $u^i - g^i_0 = 0$ fixes a slicing such that the matter $(1+n)$-velocity is orthogonal to the constant time hypersurfaces (*velocity-orthogonal slicing*). The conditions $u^i = g_{0i} = 0$ impose a space-coordinates choice such that the fluid is at rest and clocks are synchronized.

In terms of the geometrical approach: Let $(\mathcal{M}, g_{\mu\nu})$be a $1+n$-dimensional spacetime. **Comoving gauge**: foliation of $\mathcal{M}$ in n-hypersurfaces at $t = const$ on which we put spatial coordinates such that clocks are synchronized and fluid at rest, and identification of normal matter worldlines as time-lines along which we let the time-coordinate flowing.

$$\Sigma_t \perp \textit{matter worldlines}$$

For this coordinate system we have the following properties:

- The "scalar condition" completely eliminates the gauge freedom associated with initial hypersurface choice, while on hypersurfaces there remains a residual gauge freedom related to the origin of spatial coordinates.

- In the comoving gauge, the stress-energy tensor satisfies $T_i^0 = 0$.

- In the comoving gauge, there exists a natural choice of reference system: the one of observers comoving with the matter flow, that is observers seated on particles and then moving along their worldlines.

- $K_{\mu\nu} = \frac{1}{2}\mathfrak{L}_u h_{\mu\nu} = \frac{1}{2}\,\frac{\partial h_{\mu\nu}}{\partial t}$

Pressureless and irrotational fluid: synchronous and comoving gauge

As pointed out earlier, a synchronous system is not necessarily comoving with the matter. Is there a particular situation in which the two gauge choices can be taken simultaneously?

Fact 5. $p = 0 \Longrightarrow$ *a synchronous gauge can be comoving.*
Let us remember that

- trajectories of particles subjected only to gravitational forces are geodesic lines

- trajectories of particles subjected to pressure forces (i.e. non-gravitational forces) are not geodesic lines.

By pressureless fluid ($p = 0$) we mean non-collisional fluid, that is a fluid with no pressure forces. Then, this fluid trajectories are geodesics: worldlines $\equiv$ geodesics. If $p = 0$ there' s no contradiction in choosing a synchronous gauge which is comoving as well. □

Actually, the condition $p = 0$ is not the only necessary condition for having a synchronous and comoving gauge.
Let us write the fluid $(1+n)$-velocity in comoving coordinates: $u^\mu = (1, 0)$. If we are in a synchronous gauge as well, $u_\mu = (-1, 0)$.
Let us then see the vorticity: as shown in previous sections,

$$\omega_{\mu\nu} = u_{\mu;\nu} - u_{\nu;\mu} \text{ with } u_{\mu;\nu} = u_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} u_\alpha$$

Then

$$\omega_{\mu\nu} = u_{\mu,\nu} - u_{\nu,\mu} = 0 \text{ for the particular chosen frame.} \tag{2.15}$$

But (2.15) is a tensor equality which must be verified in any coordinates system. We can then deduce that in a synchronous but non comoving gauge *curl* $\vec{v} = 0$ and that

$p = 0$ AND $\omega = 0 \Longrightarrow$
SYNCHRONOUS AND COMOVING GAUGES CAN BE TAKEN SIMULTANEOUSLY

Throughout the thesis we will then work in this special frame, assuming all the good properties of each two gauges. In particular, our protagonist variable (stressing on its purely spatial nature) will be the *velocity-gradient tensor or extrinsic curvature*:

$$\Theta^i{}_j = u^i{}_{;j} = \frac{1}{2} h^{ia}\ \dot{h_{aj}} = K^i_j. \tag{2.16}$$

Our spacetime will be described by fundamental observers moving along particle geodesics≡worldlines and we will naturally be led to follow a lagrangian approach. Actually, because with this choice we are taking the threads to correspond to the worldlines of comoving observers in the slicing framework (lines of fixed $\vec{x}$), then the two (3+1) and (1+3) descriptions of Bertschinger paper [7] are the same and it will be possible to switch from the eulerian approach to the lagrangian one without problems.

2.4 Einstein Equations in ADM formalism

The next goal is to rewrite E.E. taking advantage of the frame fixed earlier and separating the operation of spatial derivatives and time derivative: we are going to present the (3+1) spacetime decomposition of E.E. into constraints and evolution equations developed in detail by Arnowitt, Deser & Misner in 1962 [12].
Einstein Equations read

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\ \mathcal{R} = k^2\ T_{\mu\nu} \tag{2.17}$$

(with $k^2 = \frac{8\pi G}{c^4}$ and $c = 1$)

In our frame, the line-element is $dS^2 = -dt^2 + h_{ij}(t, \vec{x})\ dx^i dx^j$, extrinsic curvature and velocity-gradient tensor concide (2.16) and geometrical quantities are expressed as reported in Appendix B. Let's then write down E.E component by component:

0-0) $\mathcal{R}^0{}_0 - \frac{1}{2}\delta^0_0\ \mathcal{R} = k^2\ T^0_0$ and substituting from Appendix B,

$$-\dot{\Theta} + \Theta^a{}_b\ \Theta^b{}_a - \frac{1}{2}({}^{(3)}\mathcal{R} + 2\dot{\Theta} + \Theta^2 + \Theta^a{}_b\ \Theta^b{}_a) = k^2\ T^0_0 \text{ i.e.}$$

$$\Theta^2 - \Theta^a{}_b\ \Theta^b{}_a + {}^{(3)}\mathcal{R} = -2\ k^2\ T^0_0$$

0-j) $\mathcal{R}^0_j - \frac{1}{2}\delta^0_j\ \mathcal{R} = k^2\ T^0_j$ i.e. $\Theta^a{}_{j|a} - \Theta_{,j} = -k^2\ T^0_j$

i-j) $\mathcal{R}^i_j - \frac{1}{2}\delta^i_j\ \mathcal{R} = k^2\ T^i{}_j$ with $\mathcal{R} = -k^2\ T$: then

$\mathcal{R}^i_j = k^2(T^i{}_j - \frac{1}{2}\delta^i_j\ T)$ and from Appendix B

$${}^{(3)}\mathcal{R}^i{}_j + \dot{\Theta}^i{}_j + \Theta\ \Theta^i{}_j = k^2(T^i{}_j - \frac{1}{2}\delta^i_j\ T)$$

Until now we just used the hypotesis of synchronous gauge. Equations obtained are clearly separated in 1+3 constraints and 6 evolution equations: in fact, equations arising form ${G^0}_\mu$ involve only a single time derivative of spatial metric, while those arising from ${G^i}_\mu$ have one time derivative of extrinsic curvature and hence two time derivatives of spatial metric. Equations (2.18a) and (2.18b) are known respectively as *ADM Energy Constraint* and *ADM Momentum Constraint*; equations (2.18c) are simply called *ADM Evolution Equations*:

$$\Theta^2 - {\Theta^a}_b\ {\Theta^b}_a + {}^{(3)}\mathcal{R} = -2\ k^2\ T^0_0 \tag{2.18a}$$

$${\Theta^a}_{j|a} - \Theta_{,j} = -k^2\ T^0_j \tag{2.18b}$$

$$\dot{\Theta}^i{}_j + \Theta\ {\Theta^i}_j + {}^{(3)}{\mathcal{R}^i}_j = k^2({T^i}_j - \frac{1}{2}\delta^i_j\ T) \tag{2.18c}$$

One could desire to specify those equations accordingly to the matter content of the universe which he is drawing. In our case, $T_{\mu\nu} = \rho\ u_\mu u_\nu$ with $u^\mu = (1,0,0,0)$ because of comoving coordinates and $u_\mu = (-1,0,0,0)$ because of synchronous coordinates, then it is straighforward to obtain the *ADM Einstein Equations in dust universes*:

$$\Theta^2 - {\Theta^a}_b\ {\Theta^b}_a + {}^{(3)}\mathcal{R} = +16\pi G\rho \tag{2.19a}$$

$${\Theta^a}_{j|a} = \Theta_{,j} \tag{2.19b}$$

$$\dot{\Theta}^i{}_j + \Theta\ {\Theta^i}_j + {}^{(3)}{\mathcal{R}^i}_j = 4\pi G\rho\ \delta^i_j \tag{2.19c}$$

The main advantage of this formalism is that there is only one dimensionless (tensor) variable in the evolution equations, namely the spatial metric tensor h_{ij}, which is present with its partial time derivatives through ${\Theta^i}_j$ and with its spatial gradients through the spatial Ricci curvature ${}^{(3)}\mathcal{R}^i_j$. The only remaining variable is the density ρ, that one could replace from the energy constraint or indeed rewrite in terms of h_{ij} by solving the *continuity equation*

$$\dot{\rho} = -\Theta\ \rho \tag{2.20}$$

The redundancy of disposable equations is again manifest: which equations to take? One possibility is to discard equations (2.19a) - (2.19b) and to be left with exactly as many second-order in time equations as unknown fields: ADM constraint equations would then be regarded as providing initial-value constraints on geometrical and matter variables. If these constraints are satisfied initially (this is required for a consistent metric), if equations (2.19c) are used to evolve the metric while matter variables are evolved so as to locally conserve the net energy-momentum, then ADM constraints will be in principle fulfilled at all later times, and may eventually be used to check the qualities of subsequent calculations. (In effect, E.E. have built into themeselves the requirement of energy-momentum conservation for the matter via Bianchi Identities.)
We will follow exactly this road, after having manipulated a little eqs. (2.19c).

Raychaudhuri equation

In (2.14), we reported Raychaudhuri Equation, the evolution equation along time-like curves of the expansion rate Θ. Now, ADM evolution equations govern indeed the evolution of the extrinsic curvature tensor ${\Theta^i}_j$: being Θ the trace part of extrinsic curvature, then the trace of (2.18c) or (2.19c) should give exactly Raychaudhuri equation. This is what happens, even if we could rewrite it in several ways. One should take infact the trace directly of (2.19c) or of (2.18c) (remebering that in our case $tr\ T^i_j = 0$) to obtain

$$\dot{\Theta} + \Theta^2 + {}^{(3)}\mathcal{R} = k^2({}^{(3)}T - \frac{3}{2}\ T) = 12\pi G\rho,$$

and then could use the Energy Constraint (2.19a) in order to substitute ${}^{(3)}\mathcal{R}$ or ρ. We report here both of possibilities, but we will be inclined to use the second one to avoid calculating later the perturbed expression of energy density:

$$\dot{\Theta} + \Theta^{ab}\ \Theta_{ab} + 4\pi G\ \rho = 0 \tag{2.21a}$$

$$\dot{\Theta} + \frac{1}{4}\Theta^2 + \frac{3}{4}\Theta^{ab}\ \Theta_{ab} + \frac{1}{4}{}^{(3)}\mathcal{R} = 0 \tag{2.21b}$$

Note that if one takes equation (2.14), expresses Ricci tensor through E.E. applying the hypotheses of sinchronous and comoving gauge and pressureless and irrotational perfect fluid, he will find the Raychaudhuri Equation in the form given in (2.21a).

In fact, $\dfrac{d\Theta}{dS} = -\mathcal{R}_{\mu\nu}V^\mu V^\nu + 2w^2 - 2\sigma^2 - \dfrac{1}{3}\Theta^2 + \dot{V^\mu}_{;\mu}$;

we are in the case $V^\mu = u^\mu$ and we are following particles along their worldlines, then

$\dfrac{d\Theta}{dt} = -\mathcal{R}_{\mu\nu}u^\mu u^\nu + 2w^2 - 2\sigma^2 - \dfrac{1}{3}\Theta^2 + \dot{u^\mu}_{;\mu}$.

Now, $u^\mu = (1,0,0,0)$ and $w = 0$ so $\dfrac{d\Theta}{dt} = -\mathcal{R}_{\mu\nu}u^\mu u^\nu - 2\sigma^2 - \dfrac{1}{3}\Theta^2$.

$\mathcal{R}_{\mu\nu}u^\mu u^\nu = k^2(T_{\mu\nu} - \dfrac{1}{2}g_{\mu\nu}\ T)\ u^\mu u^\nu = \dfrac{1}{2}k^2\ \rho$ and $\sigma^2 = \Theta_{\mu\nu}\ \Theta^{\mu\nu} - \dfrac{1}{3}\Theta^2$

$\Longrightarrow \dot{\Theta} + \Theta_{\mu\nu}\ \Theta^{\mu\nu} + 4\pi G\rho = 0 \qquad \square$

This should demostrate in a very specific case the evolution equation of the expansion.

Conformal rescaling and FRW background subtraction

With the purpose of making the metric pertubations of the Einstein-de Sitter background, it is convenient (as suggested in [19]) to factor out the homogenous and isotropic solution of the above evolution equations: to this aim we also perform a conformal rescaling of the metric with *conformal factor* $a(t)$, the scale-factor of FRW models, and change the time variable to the conformal time τ, defined by $d\tau = \frac{dt}{a(t)}$. The line-element is then written in the form

$$dS^2 = a^2(\tau)\ [-d\tau^2 + \gamma_{ij}(\tau,\vec{x})dx^i dx^j] \tag{2.22}$$

where $a^2(\tau)\gamma_{ij}(\tau,\vec{x}) \equiv h_{ij}(t(\tau),\vec{x})$.

We recall here briefly the properties and solutions of the FRW universe filled with a perfect fluid of dust ($n = 3$), that is the properties of the Einstein- de Sitter background:

$$dS^2_{FRW} = a^2(\tau)\ [-d\tau^2 + \frac{dr^2}{1-\kappa\ r^2} + r^2 d\theta 2 + r^2 \sin^2\theta\ d\phi^2] \tag{2.23a}$$

$$^{(3)}\mathcal{R}_{ijkl} = \kappa(\gamma_{ik}\ \gamma_{jl} - \gamma_{il}\ \gamma_{jk}) \tag{2.23b}$$

$$^{(3)}\mathcal{R}_{ij} = 2\kappa\ \gamma_{ij} \tag{2.23c}$$

$$^{(3)}\mathcal{R} = 6\kappa \tag{2.23d}$$

$$(\frac{a'}{a})^2 = \frac{8\pi\ G}{3}\ \rho_b\ a^2 - \kappa \tag{2.23e}$$

$$2(\frac{a''}{a}) - (\frac{a'}{a})^2 + \kappa = 0 \tag{2.23f}$$

$$\dot{\rho}_b = -3\frac{a'}{a}\ \rho_b \tag{2.23g}$$

where primes denote differentiation with respect to the conformal time τ, κ represents the curvature parameter of FRW models and ρ_b the energy density of the background.

By subtracting the isotropic Hubble-flow, we introduce a *peculiar velocity-gradient tensor* or *conformal extrinsic curvature*:

$$\theta^i{}_j = a\ \tilde{u}^i{}_{;j} - \frac{a'}{a}\delta^i_j = \frac{1}{2}\gamma^{ia}\ \gamma'_{aj} \tag{2.24}$$

$$(\text{ with } \tilde{u}^\mu = (1/a, 0, 0, 0))$$

such that

$$\Theta^i{}_j = \frac{1}{a}(\theta^i{}_j + \frac{a'}{a}\delta^i_j) \text{ and } \Theta = \frac{1}{a}(\theta + 3\frac{a'}{a}) \tag{2.25}$$

We are ready to rewrite our equations 2.19 in the new formalism: in detail, we want

- to express everything in terms of conformal time τ: $dt = a(t)d\tau$
- to replace the unknown h_{ij} with the conformal spatial metric γ_{ij} (see 2.25)
- to subtract from the above equations the background FRW Einstein-de Sitter zeroth order solution (see 2.23).

We report the results, having introduced the *density contrast* $\delta = (\rho - \rho_b)/\rho_b$ and renamed the conformal Ricci curvature of the three-space $\mathcal{R}^i_j = {}^{(3)}\mathcal{R}^i_j(\gamma) = a^2\ {}^{(3)}\mathcal{R}^i{}_j(h)$: in what follows we will sometimes refer to these equations as *ADM rescaled perturbed Einstein Equations.*

$$\theta^2 - \theta^a{}_b\ \theta^b{}_a + 4\frac{a'}{a}\theta + (\mathcal{R} - 6\kappa) = +16\pi G\ a^2\delta\rho_b \tag{2.26a}$$

$$\theta^a{}_{j|a} = \theta_{,j} \tag{2.26b}$$

$$\theta^i{}_j{}' + 2\frac{a'}{a}\theta^i{}_j + \theta\theta^i{}_j + \frac{a'}{a}\theta\ \delta^i{}_j + (\mathcal{R}^i{}_j - 2\kappa\delta^i{}_j) = (4\pi G\ a^2\ \rho_b\delta)\delta^i_j \tag{2.26c}$$

From now on the bar denotes covariant derivatives in the three-space with metric γ_{ij}. The calculation of (2.26) requires some attention: it's worth remembering that

$$\text{if } \theta_{ij} = \frac{1}{2}\gamma'_{ij} \text{ then } \theta^{ij} = -\frac{1}{2}\gamma'^{ij}$$

and that the presence of time derivative must always be handle with care.□

Equation (2.26c) will be the equations through which we will calculate perturbed metric at first and second order. As shown in Chapter 4, our perturbed spatial metric will be written down as function of two perturbative functions, one with trace, the other one traceless: as last step of the chapter, we want to split the evolution equations in their trace and traceless part, so that the Raychaudhuri equation governs the evolution of the trace of spatial metric, while the traceless part of (2.26c) has the traceless perturbative function as each order solution of spatial metric.
The former is obtained taking the trace of (2.26c) (as already done some page ago), using the Energy Constraint (2.26a) in order to express the matter content and remembering expansion and shear (peculiar) definitions (see (2.13)); the latter substituting expression for (peculiar) expansion as function of (peculiar) shear, $\theta^i{}_j = \sigma^i{}_j + \frac{1}{3}\theta\delta^i_j$; we suppose to deal with spatially flat universes, namely $\kappa = 0$:

$$\theta' + 2\frac{a'}{a}\theta + \frac{1}{2}\theta^2 + \frac{3}{2}\sigma^2 = -\frac{1}{4}\mathcal{R} \tag{2.27a}$$

$$\sigma^i{}_j{}' + 2\frac{a'}{a}\sigma^i{}_j + \theta\,\sigma^i{}_j = -(\mathcal{R}^i{}_j - \frac{1}{3}\mathcal{R}\delta^i_j) \tag{2.27b}$$

Equations (2.27) are still a system of six indipendent equations: one degree of freedom comes from the Raychaudhuri equation, 5 from the evolution equation of shear.

The following Table resumes the formalism introduced in this Chapter and adopted throughout this thesis:

FRAME AND FORMALISM
matter content: IRROTATIONAL ($\omega = 0$) DUST ($p = 0$)
metric background: EINSTEIN-DE SITTER UNIVERSE $dS^2_{FRW} = a^2(\tau)\,[-d\tau^2 + \frac{dr^2}{1-k\,r^2} + r^2 d\theta 2 + r^2\sin^2\theta\, d\phi^2]$
matter background: $T^{\mu\nu} = \rho u^\mu u^\nu$
gauge choice: SYNCHRONOUS+COMOVING
evolution equations: trace part) $\theta' + 2\frac{a'}{a}\theta + \frac{1}{2}\theta^2 + \frac{3}{4}\sigma^2 = -\frac{1}{4}\mathcal{R}$ traceless part) $\sigma^i{}_j{}' + 2\frac{a}{a}\sigma^i{}_j + \theta\,\sigma^i{}_j = -(\mathcal{R}^i{}_j - \frac{1}{3}\mathcal{R}\delta^i_j)$

Chapter 3

Standard Perturbation Theory at First and Second Order

As emphasized in the Introduction and in the Chapter 1, the study of the large-scale structure of the universe and its origin is usually performed with different techniques and approximations, depending on the specific range of scale under analysis. The full relativistic theory rather than the Newtonian approximation is needed when one of the following three situations occurs: strong gravitational fields, relativistic motion ($v \sim c$) for both sources and test particles, scales larger than the Hubble radius. In terms of density irregularities or more generally of cosmological perturbations, these situations are expressed as pronounced amplitudes of irregularities, high local density and perturbation wavelengths larger than the Hubble horizon size.
In this Chapter we lay the essential ideas of full relativistic cosmological perturbations theory as developed by Lifchitz, Peebles, Bardeen, Kodama & Sasaki, and others, since the Sixties ([24], [25], [18], [6],...). We present the usual classification of metric perturbations, define the notions of gauge choice and gauge transformations in the perturbative context trying to make it clear why such a terminology has been adopted in connection with the standard concepts of Chapter 2, and briefly discuss the consequences of gauge invariance. Nevertheless we do not dwell upon elegant gauge-invariant formalisms such those of Bardeen and Kodama & Sasaki, but we prefer to summarize the standard results in the sychronous gauge at first and second order, having in mind a further comparison with the alternative technique worked out in Chapter 4. In what follows we will refer to the formalism of this Chapter as Standard Perturbation Theory.

3.1 Ideas of the Standard Perturbation Theory

From a very general point of view, the idea underlying the theory of cosmological perturbations is to find approximate solutions of some field equations regarding them as small deviations from a known exact background solution. In our case,

we restrict the background spacetime (or zeroth order solution) to belong to a certain class, namely FRW spatially homogenous and isotropic spacetimes; the equations we have to try to solve are of course E.E..
In General Relativity, like in any other spacetime theory, the difficulties arise from the fact that not only fields in a given geometry have to be perturbed, but the geometry itself; besides, coordinate invariance complicates General Relativity compared with other gauge theories (like Electrodynamics in Minkowski spacetime) in which the spacetime coordinates are fixed while other variables change under the appropriate gauge transformations.

There are two practical methods for getting the equations of a perturbed system:

- One could derive the Euler-Lagrange perturbed equations from an Action Principle: the $(r+1)^{th}$ order perturbation of the action $\mathcal{S}$ of a system produces r^{th} order Euler-Lagrange equations;
- or one could directly write equations of the system and perturb them around the background solution.

We will follow exactly the second approach, as suggested at the beginning.
The perturbed spacetime is often called the *physical spacetime* $(\mathcal{M}, g_{\mu\nu})$, while we refer to the unperturbed spacetime with known solution as the *background* $(\mathcal{M}_0, g_{\mu\nu}^{FRW})$. Being as general as possible, let T be any relevant tensor field representing a physical or geometrical quantity in the spacetime of interest and satisfying some field equations, and let $T_{(0)}$ be the known value that the same quantity has in the given unperturbed background. If the deviation from the known exact solution $T_{(0)}$ is small, it makes sense to look for an approximate solution by expanding T in Taylor series in a suitable parameter ϵ.
Consider the equation

$$\mathcal{E}(T) = 0 \tag{3.1}$$

for the unknown function or, more generally, for a collection of functions or tensor fields T. In the case of interest, T is the spacetime metric $g_{\mu\nu}$ (possibly together with variables describing the matter content like the stress-energy tensor $T_{\mu\nu}$), and $\mathcal{E}$ are the E.E.
The basic assumption in perturbation theory is the existence of a parametric family of solutions of the field equations, to which the unperturbed background spacetime belongs [1]:

$$\mathcal{E}(T_\epsilon) = 0 \qquad \text{such that} \tag{3.2}$$

- ϵ is real;
- T_ϵ is a differentiable function of ϵ (and T_ϵ can be written as $T(\epsilon)$);
- $\epsilon = 0$ identifies the background: $T_\epsilon|_{\epsilon=0} = T_{(0)}$.

In cosmology and in many other cases in general relativity, one deals with a one-parameter family of models $(\mathcal{M}_\epsilon, T_\epsilon)$. In some applications, ϵ is a dimensionless parameter arising naturally from the physical problem one is dealing with: in that case one expects the perturbative solution to accurately approximate the exact one for reasonably small ϵ. In other problems, ϵ can be introduced as a purely formal parameter, and in the end, for convenience, one can choose $\epsilon = 1$.

This is exactly what we will do: the physical spacetime $\mathcal{M}_\epsilon$ will eventually be identified by $\epsilon = 1$.
In any case, the parameter ϵ is used for Taylor expanding these T_ϵ: as in elementary analysis, the idea is to evaluate the deviation from the zeroth order term by differentiation of the function of interest. In particular ([15]), the procedure consists in differentiating at different orders the equations and at each step solving them.
For example, as first step one can derive a simpler equation from equation (3.2) by differentiating it once with respect to ϵ and setting ϵ equal to zero: the equation thus obtained is a linear equation for the first derivative of T with respect to ϵ, namely $\delta T_{(1)} = \left(\frac{dT}{d\epsilon}\right)_{\epsilon=0}$. Since linear equations are generally much easier to solve than nonlinear ones, it may be feasible to solve the former even if (3.2) is intractable: if this is the case, an expression as $T_{(0)} + \epsilon\delta T_{(1)}$ should yeld a good approximation to T_ϵ, and the quality of the approximation can be improved repeating the procedure at the following orders. Then at second order, the second derivative with respect to ϵ at $\epsilon = 0$ gives an equation which is linear in the second order perturbation $\delta T_{(2)}$, and where the first order perturbation now appears as known source terms. This can obviously be extended to higher orders, giving an iterative procedure to calculate $\Delta T_\epsilon = T_\epsilon - T_{(0)}$ to the required accuracy.
The result can be written as follows

$$T_\epsilon = T_{(0)} + \epsilon \left(\frac{\partial T}{\partial \epsilon}\right)_{\epsilon=0} + \frac{1}{2}\epsilon^2 \left(\frac{\partial^2 T}{\partial \epsilon^2}\right)_{\epsilon=0} + ... \quad \text{or} \tag{3.3a}$$

$$T_\epsilon = T_{(0)} + \epsilon \;\delta T_{(1)} + \frac{1}{2}\epsilon^2 \;\delta T_{(2)} + ... \tag{3.3b}$$

where T_ϵ lives in the perturbed world, $T_{(0)}$ in the background, $\delta T_{(r)} = \left(\frac{\partial^r T}{\partial \epsilon^r}\right)_{\epsilon=0}$ represents the r^{th} order correction to T with respect to the background value (the r^{th} order perturbation) and ϵ gives a weight of such a correction.

3.2 Implementing the perturbations

Having delineated the general ideas underlying the making of perturbations, we want now to specify the procedure to the case under study. As discussed before, we set $\epsilon = 1$ to describe our physical spacetime. We will expand the quantities of interest up to second order: this is recent and due choice, for the increasing of calculations complexity as one goes at higher orders.
In order to take into account the geometry of spacetime and the matter content, two are the relevant quantities to be perturbed: obviously, the spacetime metric (and hence all the useful geometrical quantities $\Gamma^\mu_{\nu\rho}, R_{\mu\nu}, R$) and the stress-energy tensor.

Classification of metric perturbations

Expression (3.3b) for small perturbations of the metric is rewritten as follows:

$$g_{\mu\nu}(t,\vec{x}) = g^{FRW}_{\mu\nu}(t,\vec{x}) + \delta g^{(1)}_{\mu\nu}(t,\vec{x}) + \frac{1}{2}\delta g^{(2)}_{\mu\nu}(t,\vec{x}) \tag{3.4}$$

A widely common use (especially when the expansion was stopped at first order) is to generically expand the perturbations in Fourier coefficients or in any other

basis eigenfunctions, so that any (Fourier) component or mode is naturally associated to a wavenumeber and wavelength. We will not adopt directly this point of view, but prefer a more common approach consisting in splitting of perturbations in different spatial symmetry components, called *modes* as well.
The components of a perturbed spatially flat FRW metric can be written as [13]

$$g_{00} = -\,a^2(\tau)(1 + 2\phi_{(1)} + \phi_{(2)}) \tag{3.5a}$$

$$g_{0i} = a^2(\tau)(\omega_i^{(1)} + \frac{1}{2}\omega_i^{(2)}) \tag{3.5b}$$

$$g_{ij} = a^2[(1 - 2\Psi_{(1)} - \Psi_{(2)})\delta_{ij} + \chi_{ij}^{(1)} + \frac{1}{2}\chi_{ij}^{(2)}] \tag{3.5c}$$

where τ is the conformal time and the i-j components have been split in a trace part and a traceless one: $\chi_i^{(r)i} = 0$.
The perturbation variables or perturbative functions ($\phi,\ \psi,\ \omega_i,\ \chi_{ij}$) are treated exclusively as 3-tensors of rank 0, 1, or 2 according to the number of indices: they all live on the 3-dimensional hypersurfaces Σ of the unperturbed world and their components are raised and lowered using δ_{ij} and δ^{ij} by definition. The standard decomposition of spatial vectors and tensors into scalar and transverse parts of Appendix A then applies:

- ϕ, ψ are scalars by their own;
- $\omega_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)\perp}$, with $\omega^{(r)}$ a scalar and $\omega_i^{(r)\perp}$ a solenoidal vector, $\partial^i \omega_i^{(r)\perp} = 0$;
- $\chi_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)\top}$, with $\chi^{(r)}$ a suitable function, $\chi_i^{(r)\perp}$ a solenoidal vector, $\partial^i \chi_{ij}^{(r)\top} = 0$; hereafter, $D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$.

Equations (3.5) are completely general: $g_{\mu\nu}$ has 10 independent components and we have introduced 10 independent fields, 1+1+3+5 for $\phi + \psi + \vec{\omega} + \chi$. Moreover, as the most general perturbations of the metric, they contain all the possible *scalar, vector and tensor modes*: four scalar parts each having 1 degree of freedom ($\phi,\ \psi,\ \omega,\ \chi$), two vector parts each having 2 degrees of freedom ($\omega^\perp,\ \chi^\perp$), and one tensor part having 2 degrees of freedom ($\chi^\top$, which is symmetric, traceless and transverse). The total number of degrees of freedom is again 10 as it must be.
There are several reasons for having entered in this mathematical classification of perturbations. First of all, let us still note that, being the components of the perturbed metric $g_{00},\ g_{0i},\ g_{ij}$ respectively a scalar, a vector and a tensor under spatial coordinate transformations, then a scalar perturbation only would affect all the three components, a vector perturbation only would affect $g_{oi},\ g_{ij}$ leaving g_{00} unperturbed, and a tensor perturbation would affect excusively the space-space components g_{ij}. Furthermore, different perturbations have distinct physical meanings and represent distinct physical phenomena. In the language of the (3+1)-formalism, ϕ is interpreted as the amplitude of perturbation in the *lapse function*, which represents the ratio of the proper-time distance to the coordinate-time distance between two neighboring constant-time hypersurfaces; ω is interpreted as the amplitude of perturbation in the *shift vector*, which represent the rate of deviation of a constant space-coordinate line from a line

normal to a constant-time hypersurface; ψ can be seen as the amplitude of the perturbation of a unit spatial volume, and finally χ represents the anisotropic distorsion of each constant-time hypersurface [6]. The other vector and tensor perturbative functions have no such an easy interpretation. From a wider point of view, ordinary Newtonian gravity is a scalar phenomenon, i.e. corresponds to the scalar mode, being the Newtonian potential a 3-scalar; the vector and tensor modes, on the contrary, represent the relativistic effects of gravitomagnetism and gravitational radiation, which have no counterpart in Newtonian gravity although they are similar to electromagnetic phenomena. Scalar metric perturbations are associated to density perturbations, which experience gravitational instability and lead to structure formation; tensor metric fluctuations produce gravitational waves, which are not foundamental at all in structure formation but can reveal themselves in other phenomena, for example in the cosmic microwave background anisotropies.
The spatial decomposition can also be applied to the Einstein and stress-energy tensors (see below), allowing us to clearly see (at least in some coordinate system) the physical sources for each type of phenomenon. Finally, the classification will help us to eliminate unphysical gauge degree of freedom, remembering that a gauge choice needs two scalars and one transverse vector conditions.

Perturbing the matter content

Our background is the Einstein-de Sitter universe, a FRW matter-dominated spacetime. As extensively discussed in the previous Chapter, the matter we consider is irrotational dust and the corresponding stress-energy tensor is that of equation (2.12). Let us recall that this is a very special and appropriate case, but even other types of stress-energy tensors are largely considered, as those, for example, of scalar fields. Anyway, we limit the treatment of the perturbations of the matter content to the stress-energy tensor of our interest, because if the general idea is always the same the practical notations are rather different.
Equation (3.3b) is of course rewritten as follows

$$T_{\mu\nu} = T^{DUST}_{\mu\nu} + \delta T^{(1)}_{\mu\nu} + \frac{1}{2}\delta T^{(2)}_{\mu\nu} \tag{3.6}$$

being $T^{DUST}_{\mu\nu} = \rho u^\mu u^\nu$. Therefore we must digress to discuss the perturbations of energy density and 4-velocity. Energy density is a scalar, then it can be affected by scalar perturbations only; the 4-velocity, on the contrary, can be affected by both scalar and vector perturbations:

$$\rho = \rho_{(0)}(t) + \delta^{(1)}\rho + \frac{1}{2}\delta^{(2)}\rho \tag{3.7}$$

$$u^\mu = \frac{1}{a}(\delta^\mu_0 + v^\mu_{(1)} + \frac{1}{2}v^\mu_{(2)}) \tag{3.8}$$

Here, we have already assumed comoving coordinates in the background; the velocity perturbation $v^\mu_{(r)}$ can as usual be split into a scalar and a vector part, while the time component $v^0_{(r)}$ is related at any order to the lapse perturbation $\phi_{(r)}$ (see [13]).
We do not linger over writing down the explicit form of the second-order perturbed stress-energy tensor even because we will not need it in the continuation:

anyway, it is interesting to note that, even if the background $T^{(0)}_{\mu\nu}$ is that of a perfect fluid, a general perturbation leads to the appearance of extra terms such as isotropic stress perturbations (with scalar perturbations only) and shear stress perturbations, that is anisotropic stress perturbations.

3.3 Gauge choice and gauge dependence in perturbation theory

In the previous Sections, some problems dealing with the comparison of quantities between the real world and the unperturbed one have been neglected and brought forward. To be honest, it is worthwile to remind that in order to make the comparison of tensors meaningful at all, one has to consider them at the same point : but T and $T_{(0)}$ of Section 3.1 were defined on different manifolds, respectively $\mathcal{M}$ and $\mathcal{M}_0$, thus we would be allowed to compare them only after a prescription for identifying points of those different spacetimes is given.
Likewise and for the same reason, perturbations such as those of the metric and of the stress-energy tensor,

$$\Delta g_{\mu\nu} = \delta g^{(1)}_{\mu\nu} + \frac{1}{2}\delta g^{(2)}_{\mu\nu} \tag{3.9}$$

$$\Delta T_{\mu\nu} = \delta T^{(1)}_{\mu\nu} + \frac{1}{2}\delta T^{(2)}_{\mu\nu} \tag{3.10}$$

of equations (3.4)-(3.6), are well defined (univocally) only when a coordinate choice has been made.

Roughly speaking, a *gauge choice* in cosmological perturbations theory is a one-to-one correspondence (a map) between points in the background $\mathcal{M}_0$ and points in the physical spacetime $\mathcal{M}$. A change in this correspondence, keeping the background coordinates fixed, is then called a *gauge transformation*, and it can be formally expressed in terms of a coordinates transformation in the perturbed world à la manière of equations (2.2) or (2.4).
The essence of the "gauge problem", that has created a great deal of confusion in the past, consists in two strictly related points:

- arbitrariness in choosing the map between $\mathcal{M}_0$ and $\mathcal{M}$;
- gauge dependence of the value of perturbations.

The second point is probably the most problematic: the perturbation in some quantity is the difference between the value it has at a point in the physical spacetime and the value at the *corresponding point* in the background. A gauge transformation induces a coordinate transformation in the physical spacetime, but it also changes the point in the background corresponding to a given point in the physical world. Thus, the value of the perturbation in the quantity will not be invariant under gauge transformations if the quantity is nonzero and position dependent in the background.
Two essentially different ways of handling the perturbations have been then developed in the literature:

- the usual one works with coordinates: the gauge is fixed, perturbations of the metric components are considered, solutions are written in that gauge and appropriate relations are used to pass to other gauges and verify the consistency of the results;
- the other approach consists in formulating the problem in terms of gauge-invariant variables and trying to understand the physical meaning of such variables.

As already anticipated, we will adopt the gauge-fixing way.

Let us formalize the idea of gauge choice as map between the two spacetimes (see Figure 3.1,[15]). First of all, let us suppose having fixed a coordinate system $\{x^\mu\}$ in the background: any $p \in \mathcal{M}_0$ is labeled by $x^\mu(p)$. Apart from the way of constructing the correspondence, the map a priori depends from the parameter ϵ: we will later greatly simplify the treatment by taking $\epsilon = 1$ as usual.
A first way of defining the point identification map consists in carrying the background coordinate over $\mathcal{M}_\epsilon$:

$$\begin{aligned} \psi_\epsilon &: \mathcal{M}_0 \to \mathcal{M}_\epsilon \\ &p \mapsto O = \psi_\epsilon(p) \qquad \text{with} \qquad x^\mu(p) \equiv x^\mu(O) \end{aligned}$$

O is the point on the physical spacetime corresponding to p through the diffeomorphism ψ_ϵ; ψ_ϵ assigns the same coordinate labels between related points, and defines in every respect a gauge choice in the perturbed world: this is the reason why we call such a map choice a gauge choice as well. A change in the map ψ_ϵ, keeping the background coordinates fixed, is a gauge transformation.
We could as well use a different gauge φ_ϵ and think of O as the point of $\mathcal{M}$ corresponding to a different point q in the background, with coordinates $x^\mu(q)$:

$$\begin{aligned} \varphi_\epsilon &: \mathcal{M}_0 \to \mathcal{M}_\epsilon \\ &q \mapsto O = \varphi_\epsilon(q) = \psi_\epsilon(p) \qquad \text{with} \qquad x^\mu(q) \neq x^\mu(O) \end{aligned}$$

There is then another reason for calling those correspondences between the different spacetimes with the same terminology of standard gauge facts: the two different ways of mapping $\mathcal{M}_\epsilon$ through the coordinate system of $\mathcal{M}_0$ suggest a one-to-one correspondence between different points in the background, that is an *active coordinate transformation* on the unperturbed world. Otherwise a standard gauge transformation (or passive transformation) which changes coordinate labels to each point keeping the manifold fixed, the composition of maps

$$\begin{aligned} \Phi_\epsilon &: \mathcal{M}_0 \to \mathcal{M}_\epsilon \to \mathcal{M}_0 \\ &p \mapsto q = \Phi_\epsilon(p) = \varphi_\epsilon^{-1}(\psi_\epsilon(p)) \end{aligned}$$

is a gauge transformation which does not change the coordinate label system but moves the points on the manifold, and then evaluate the coordinates of the new points: $\bar{x}^\mu(\epsilon, q) = \Phi^\mu_\epsilon(x^\alpha(p))$.

With the same approach of Section 3.1, in order to compute at the desired order of accuracy the effects of a gauge transformation, we need a Taylor expansion. The latter up to 2^{nd} order of the transformation $\bar{x}^\mu(\epsilon) = \Phi^\mu_\epsilon(x^\alpha)$ between

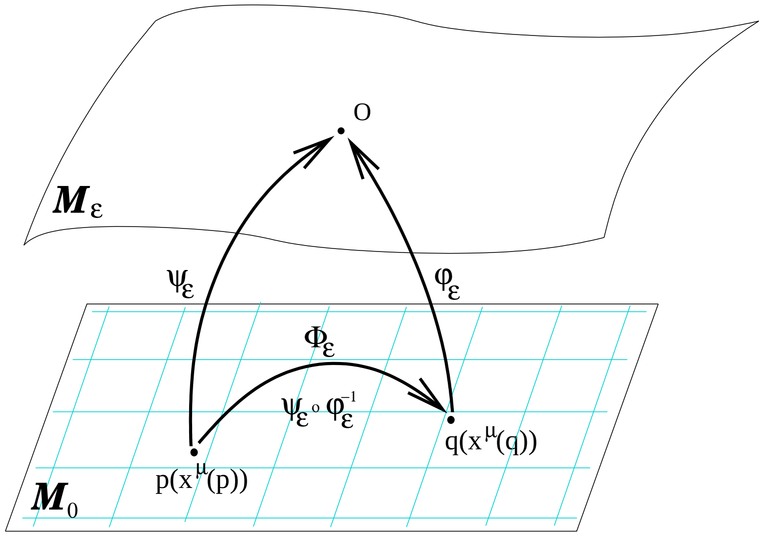

Figure 3.1: Active coordinates choice on the background as composition of two gauges between $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$

the coordinates of any pair of points of the background can be written as follows ([15],[13], $\epsilon = 1$):

$$\bar{x}^\mu = x^\mu + \xi^\mu_{(1)} + \frac{1}{2}(\xi^\mu_{(1),\nu}\ \xi^\nu_{(1)} + \xi^\mu_{(2)}) \tag{3.11}$$

where $\xi_{(1)}$ and $\xi_{(2)}$ are two indipendent vector fields and closely related to that one of equation (2.2). The gauge transformation under (3.11) up to 2^{nd} order of a generical tensor is

$$\bar{T} = T + \mathfrak{L}_{\xi_{(1)}} T + \frac{1}{2}(\mathfrak{L}^2_{\xi_{(1)}} + \mathfrak{L}_{\xi_{(2)}})T \tag{3.12}$$

In the light of these new formalism, the generic perturbation is rewritten more carefully as

$$\Delta T_\epsilon = \psi^*_\epsilon T - T_{(0)} \text{ and } \delta T_{(r)} = \left(\frac{\partial^r \psi^*_\epsilon T}{\partial \epsilon^r}\right)_{\epsilon=0} \tag{3.13}$$

and the first and second order perturbations of T transform under a gauge transformation up to second order as

$$\delta\bar{T}_{(1)} = \delta T_{(1)} + \mathfrak{L}_{\xi_{(1)}} T_{(0)} \tag{3.14a}$$

$$\delta\bar{T}_{(2)} = \delta T_{(2)} + 2\mathfrak{L}_{\xi_{(1)}}\delta T_{(1)} + \mathfrak{L}^2_{\xi_{(1)}} T_{(0)} + \mathfrak{L}_{\xi_{(2)}} T_{(0)} \tag{3.14b}$$

First order gauge transformations

As a practical application of all the theory developed in these last few pages, we write down at least the first order gauge transformations of the perturbative functions presented earlier; we have in mind the usual decomposition of gauge vector ξ as indicated in (2.3):

$$\bar{\phi}_{(1)} = \phi_{(1)} + \alpha'_{(1)} + \frac{a'}{a}\alpha_{(1)} \tag{3.15a}$$

$$\bar{\omega}_i^{(1)} = \omega_i^{(1)} - \alpha_{,i}^{(1)} + \beta_{,i}^{(1)\prime} + d_i^{(1)\prime} \tag{3.15b}$$

$$\bar{\Psi}_{(1)} = \Psi_{(1)} - \frac{1}{3}\nabla^2\beta_{(1)} - \frac{a'}{a}\alpha_{(1)} \tag{3.15c}$$

$$\bar{\chi}_{ij}^{(1)} = \chi_{ij}^{(1)} + 2D_{ij}\beta_{(1)} + d_{i,j}^{(1)\prime} + d_{j,i}^{(1)\prime} \tag{3.15d}$$

$$\bar{\delta\rho}^{(1)} = \delta\rho^{(1)} + \rho'_{(0)}\ \alpha_{(1)} \tag{3.15e}$$

$$\bar{v}_{(1)}^0 = v_{(1)}^0 - \frac{a'}{a}\alpha_{(1)} - \alpha'_{(1)} \tag{3.15f}$$

$$\bar{v}_{(1)}^i = v_{(1)}^i - \beta_{(1)}^{\prime,i} - d_{(1)}^{\prime i} \tag{3.15g}$$

Gauge transformations of second order perturbations are much more complicated than these and far exceed the necessity of this thesis [13]. The only thing that is important to point out is the form of such transformation rules: for example, the gauge transformation of the lapse perturbation (3.15a) or of the velocity perturbation time-component (3.15f) are expressed only in terms of α; (3.15d) shows that the tensor modes of $\chi_{ij}^{\top}$ are gauge invariant at the linear level, as tensor type gauge transformations cannot exist. In any case, they suggest practical methods for gauge fixing.

Implementing gauge choices

Having demonstrated the meaning of the gauge choice in perturbation theory, as last task of this section we want to give some practical prescriptions for fixing it. The procedure we follow is that of the first approach outlined in Chapter 2: analogously to what done earlier, we must impose two relations among the gauge-dependent variables, one for fixing the slicing and one for the space-coordinates. The simplest way to specify the time slicing is to require one of those quantities transforming only with α to vanish; for each time slicing the standard way to eliminate the spatial coordinate gauge freedom is to require a quantity whose gauge transformation involves β and d_i to vanish. Consistently with Section 2.3, we thus have the following definitions:

The **synchronous gauge** in perturbation theory is defined by the conditions

$$\phi = \omega_i = 0$$

The **comoving gauge** in perturbation theory is defined by the conditions

$$v_i = \omega_i = 0$$

Other possibilities are indicated in Table 3.1.

Proper-time slicing	$\phi = 0$
Synchronous gauge	$\phi = \omega_i = 0$
Comoving proper-time gauge	$\phi = v_i = 0$
Velocity-orthogonal silcing	$v_i = \omega_i$
Comoving time-orthogonal gauge	$v_i = \omega_i = 0$
Velocity-orthogonal isotropic gauge	$v_i = \omega_i,\, \chi_{ij} = 0$
Longitudinal gauge	$\omega_i = \chi_{ij} = 0$
Poisson gauge	${\omega_i}^{,i} = {\chi_{ij}}^{,j} = 0$

Table 3.1: Examples of possible gauge choices in perturbation theory ([6], [13])

3.4 Standard perturbations at 1^{st} and 2^{nd} order of Einstein-de Sitter universe in the synchronous-comoving gauge

We finally present the calculation of the metric and matter perturbations up to second order of the Einstein-de Sitter universe in the standard perturbation theory. The final aim is to compare the results of this chapter to those ones obtained with the Gradient Expansion Technique presented in the next two Chapters.
From now on we will always work in synchronous and comoving coordinates, essentially for a reason of convenience in performing calculations: as a matter of fact already second order calculations are almost invariably a computational tour de force. The simpler form of the gauge-invariant variables often makes it easy to find analytical solutions and avoids misunderstandings around incidental unphysical modes; but a gauge-invariant second order treatment is not completely at hand, and in the case under study there are no particular problems in solving equations. In general, it is not necessary to use gauge-invariant variables during a calculation, and indeed many cosmologists continue successfully to use the synchronous gauge: in the end, when the results are converted to measurable quantities -spacetime scalars- the gauge modes automatically get cancelled. Of course, some more attention must be paid in numerical solutions, where the gauge modes can swamp the physical ones and the consequent round-off can produce significant numerical errors. But this is not our case: yes, we are going to get approximate metric solutions, but at every order E.E. are analytically solved. Unfortunately, the computationally more convenient gauge does not necessary coincide with the most interesting one; for example, the Poisson gauge, otherwise the synchronous one, would allow a more direct comparison with the standard Newtonian and Eulerian approaches adopted in Large Scale Structure studies. In any case, one is always free to switch to other gauges making good use of the gauge transformation rules mentioned in the previous Section and references therein.

Let us then specify the formalism outlined in Chapters 2 and 3 to our task. The components of a perturbed spatially flat FRW metric in the synchronous and comoving gauge are written as follows (see equations (3.5) and the gauge conditions of the previous page):

$$g_{00} = -1 \tag{3.16a}$$

$$g_{0i} = 0 \tag{3.16b}$$

$$g_{ij} = a^2[(1 - 2\Psi_{(1)} - \Psi_{(2)})\delta_{ij} + \chi^{(1)}_{ij} + \frac{1}{2}\chi^{(2)}_{ij}] \tag{3.16c}$$

Then the rescaled spatial metric tensor -the only variable in our equations- reads

$$\gamma_{ij} = (1 - 2\Psi_{(1)} - \Psi_{(2)})\delta_{ij} + \chi^{(1)}_{ij} + \frac{1}{2}\chi^{(2)}_{ij} \tag{3.17}$$

The stress-energy tensor is the usual $T_{\mu\nu} = \rho u_\mu u_\nu$ and the Einstein-de Sitter background is described by a scale factor $a(\tau) \propto \tau^2$ (as mentioned in Chapter 1). The spatial curvature is set to zero; the density contrast already introduced in Section 2.4 reads in the new formalism $\delta = \Delta\rho/\rho$, so that the density contrast expansion corresponding to equation (3.7) is

$$\delta(\tau, \vec{x}) = \delta^{(1)}(\tau, \vec{x}) + \frac{1}{2}\delta^{(2)}(\tau, \vec{x}) \tag{3.18}$$

The background mass density is $\rho_b \equiv \rho_{(0)}$: we can take its mean value as $\rho_{(0)} = 3/\ 2\pi G a^2(\tau)\tau^2$. With these notations and hypotheses we can rewrite E.E.(2.26) as follows:

$$\theta^2 - \theta^a{}_b\ \theta^b{}_a + \frac{8}{\tau}\theta + \mathcal{R} = +\frac{24}{\tau^2}\delta \tag{3.19a}$$

$$\theta^a{}_{j|a} = \theta_{,j} \tag{3.19b}$$

$$\theta^i{}_j{}' + \frac{4}{\tau}\theta^i{}_j + \theta\theta^i{}_j + \frac{2}{\tau}\theta\ \delta^i{}_j + \mathcal{R}^i{}_j = \left(\frac{6}{\tau^2}\delta\right)\delta^i_j \tag{3.19c}$$

Using the energy constraint (3.19a) and taking the trace of the evolution equations (3.19c), the Raychaudhuri equation takes the final form:

$$\theta' + \frac{2}{\tau}\theta + \theta^a{}_b\ \theta^b{}_a + \frac{6}{\tau^2}\delta = 0 \tag{3.20}$$

We say that in these equations the really indipendent degree of freedom is γ_{ij} because, through the continuity equation $T^{\mu\nu}_{;\nu}$ written in the form (2.20) of Section 2.4, the exact solution for the density contrast is known and can be written as

$$\delta(\tau, \vec{x}) = (1 + \delta_{IN}(\vec{x}))[\gamma(\tau, \vec{x})/\gamma_{IN}(\vec{x})]^{-1/2} - 1. \tag{3.21}$$

Here $\gamma = det\gamma_{ij}$ and the subscript "IN" denotes the value of quantities at some initial time [14].

Calculation scheme and initial conditions The calculation scheme consists in an iterative procedure: the unknown spatial metric (with its 6 degrees of freedom) is known at the zeroth order and, according to the desired accuracy, if r is the expansion order of quantities then r is the number of steps of this scheme. We stop our Taylor series at second order, therefore two are the steps we have to fulfill. At every order, E.E. in the form given in 3.19 are written in

terms of γ_{ij} expanded up to the corresponding r^{th} order, and they are solved in the r^{th} order perturbations $\Psi_{(r)}$ and $\chi_{ij}^{(r)}$: the $(r-1)^{th}$ order metric perturbations (calculated at the previous step) appear as known source terms. The same procedure has to be applied to the expression (3.21) to obtain the density contrast.
The idea underlying the calculations should now be almost clear: actually, the practical procedure presents many passages and difficulties which we are not going to cover and explain, being outside the purpose of the thesis. We will just report the main results referring to the literature for more details [13].

Let us now briefly discuss the key issue of the initial conditions and other well-founded hypotheses. The situation is simplified with the following considerations:

- we neglect linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations as Inflation: then $\omega_i^{(1)\perp} = \chi_i^{(1)\perp} = 0$;
- we neglect linear tensor modes since they play a negligibile role for large scale structure formation: then $\chi_{ij}^{(1)\top} = 0$.

We decide to fix the initial conditions at the end of Inflation, that is at the time when the cosmological perturbations relevant today for the large scale structure formation are well outside the Hubble radius, i.e. when the comoving wavelength aL of a given perturbation mode is such that $aL \gg H^{-1}$, $H = \frac{a'}{a^2}$ being the horizon size, as extensively seen in Chapter 1. Information for such a valutation come from the study of curvature perturbation ζ evolution -a gauge-invariant variable expressing the curvature perturbation on uniform density hypersurfaces (see [14]). In conclusion, our constraints about the initial conditions are summarized by

- $\delta_{IN} = 0$;
- $\chi_{IN}^{(1)} = 0$ (for residual gauge fixing).

Linear order solutions

At 1^{st} order the growing-mode [1] solutions for a dust filled universe in the synchronous-comoving gauge read

$$\psi^{(1)}(\tau, \vec{x}) = \frac{5}{3}\varphi(\vec{x}) + \frac{\tau^2}{18}\nabla^2\varphi(\vec{x}) \tag{3.22a}$$

$$\chi_{ij}^{(1)} = -\frac{\tau^2}{3}\left(\varphi_{,ij} - \frac{1}{3}\,\delta_{ij}\nabla^2\varphi\right) \tag{3.22b}$$

$$\delta^{(1)} = \frac{\tau^2}{6}\nabla^2\varphi \tag{3.22c}$$

where $\varphi(\vec{x})$ is the so-called *peculiar gravitational potential* related to δ_{IN} through the cosmological Poisson equation (1.40) or (3.22c) itself. A foundamental result of the standard linear perturbation theory is that at first order scalar, vector and tensor modes are decoupled and evolve indipendently [6]:

[1] We only consider modes not decaying with time

Theorem 2. *In a FRW spacetime, scalar, vector and tensor equations, if they are covariant with respect to the coordinate transformation in Σ, linear in the unknown geometrical quantities and second order at most in the sense of differential equations, are decomposed into groups of equations each of which contains only components of one type. Therefore the three types of perturbations completely decouple from each other dynamically.*

This is of course true even in the case of Fourier or harmonic functions expansions: there, the temporal evolution of expansion coefficients is determined by a linear system of differential equations, thus there is no coupling among different wavelenght modes .

Let us still note that if we had not neglected tensor modes we would have obtained, by linearizing the traceless part of $\theta^i{}_j$ evolution equation, the equation of the free propagation of *gravitational waves* in the Einstein-de Sitter universe:

$$\chi_{ij}^{(1)\top\prime\prime} + \frac{4}{\tau}\chi_{ij}^{(1)\top\prime} - \nabla^2\chi_{ij}^{(1)\top} = 0 \tag{3.23}$$

This is why tensor modes are associated to gravitational radiation and people often refer to them as gravitational waves.

Second order solutions

At 2^{nd} order the corresponding growing-mode solutions for a dust filled universe in the synchronous-comoving gauge are written, in terms of the gravitational potential as well, as follows:

$$\psi^{(2)} = -\frac{50}{9}\varphi^2 - \frac{5}{54}\,\tau^2\,\varphi^{,a}\varphi_{,a} - \frac{\tau^4}{252}\left(\frac{10}{3}\varphi^{,ab}\varphi_{,ab} - \left(\nabla^2\varphi\right)^2\right) \tag{3.24a}$$

$$\begin{aligned}\chi_{ij}^{(2)} =& -\frac{10}{9}\tau^2\varphi_{,i}\varphi_{,j} + \frac{10}{27}\tau^2\varphi^{,a}\varphi_{,a}\delta_{ij} \\ &+ \frac{\tau^4}{126}\big(19\,\varphi^{a}{}_{,i}\varphi_{,aj} - \frac{19}{3}\varphi^{,ab}\varphi_{,ab}\,\delta_{ij} \\ &- 12\,\varphi_{,ij}\,\nabla^2\varphi + 4\,(\nabla^2\varphi)^2\delta_{ij}\big) + \triangle_{ij}^{(2)}\end{aligned} \tag{3.24b}$$

$$\delta^{(2)} = \frac{10}{9}\tau^2\left(-\frac{1}{4}\,\varphi^{,a}\varphi_{,a} + \varphi\,\nabla^2\varphi\right) + \frac{\tau^4}{126}\left(5(\nabla^2\varphi)^2 + 2\,\varphi^{,ab}\varphi_{,ab}\right) \tag{3.24c}$$

Here $\triangle_{ij}^{(2)}$ describes second-order tensor modes generated by linear scalar perturbations and possible time-independent terms arising from the initial conditions but is not necessary for our purposes.

The principal and general phenomenon of second-order perturbation theory is *mode mixing*. Interesting consequences of this fact are ([13]):

- tensor modes $\chi_{ij}^{(2)\top}$ are no more gauge invariant;
- primordial density fluctuations act as seeds for second-order gravitational waves and second-order vector modes;
- density fluctuations can be generated from primordial tensor modes.

Chapter 4

Gradient Expansion Technique

In this Chapter the core of the thesis is presented, that is the calculation up to four spatial gradients of the perturbed spatial metric in synchronous and comoving gauge of a matter-dominated universe, within the context of the Gradient Expansion Technique. The latter is a method for expanding and solving E.E. in a series of terms containing the perturbative functions Ψ and χ_{ij}, according to the number of spatial gradients they contain. This is alternative to the standard technique introduced in Chapter 3.
The idea of the Gradient Expansion Approximation traces back to the Sixties with Lifchitz & Khalatnikov [25]; later, different approaches to this approximation method have been followed according to the field of application and final goal [26], [27], [30], [31], [32].
The formalism worked out in Chapter 2 is assumed: all the work of the following two chapters has been performed in full relativistic approach, fixing the gauge, assuming conformal time τ and hence with all quantities rescaled by the isotropic FRW background with an expansion factor $a(t)$. The description applies to a matter-dominated universe, a universe filled with pressureless fluid assumed to be irrotational, and E.E. are written in the ADM formalism in the way shown in Section 2.4.
Thus, after having introduced our "seed" spatial metric, explained the nature of our expansion, and presented the iteration scheme used for getting the solutions, we proceed in the calculation of our purely spatial physical quantities and spatial hypersurfaces geometrical quantities in terms of the perturbative functions subsequently up to two derivatives terms (called first order) and four derivatives terms (called second order).
The calculations and the resolution of the equations have been carried out with analytical methods: nevertheless, the correctness of the results has been verified with internal consistency checks (such as Energy and Momentum Constraint of ADM formalism), and at the end controlled with the help of *MATHEMATICA* codes for symbolic computations using EinS package [23].

4.1 The starting spatial metric and background comparison

In the synchronous and comoving gauge, the line-element is written as in equation (2.22), that is

$$dS^2 = a^2(\tau)\ [-d\tau^2 + \gamma_{ij}(\tau,\vec{x})dx^i dx^j], \tag{4.1}$$

thus we can focus only on the quantities lying on constant time hypersurfaces Σ of Chapter 2, starting with the rescaled spatial metric tensor γ_{ij}. Let us write it in a very general form as follows

$$\gamma_{ij} = e^{-2\Psi}(\delta_{ij} + \chi_{ij}). \tag{4.2}$$

Ψ and χ_{ij} are the well-known functions of time and space of the Standard Perturbation Theory of Chapter 3, with χ_{ij} being a traceless tensor containing the three modes: scalar χ, solenoidal vector $\chi_i^{\perp}$ and symmetric tensor $\chi_{ij}^{\top}$.
Ψ and χ contain all the perturbative orders of this technique:

$$\Psi = \Psi_{(0)} + \Psi_{(1)} + \frac{1}{2}\Psi_{(2)} + ... \tag{4.3}$$

$$\chi_{ij} = \chi_{ij}^{(0)} + \chi_{ij}^{(1)} + \frac{1}{2}\chi_{ij}^{(2)} + ... \tag{4.4}$$

Broadly speaking, if in the Standard Perturbation Theory the expansion parameter of Taylor series is the magnitude of deviations from the background, in the Gradient Expansion Technique the expansion parameter is the *number of spatial derivatives*: in other words, all physical and geometrical quantities of interest are expanded in a series on the basis of their spatial gradients content. Let T be a generic field, then

$$T = T_{(0)} + T_{(1)} + \frac{1}{2}T_{(2)} + ... \tag{4.5}$$

where

- $T_{(0)}$ contains zero spatial derivatives
- $T_{(1)}$ contains two spatial derivatives

  ⋮
- $T_{(r)}$ contains $2r$ spatial derivatives.

Our choice to associate the first order to a content of *two* spatial derivatives rather than one, and to consider the second order terms as containing *four* spatial derivatives, and so on with the r^{th} order corresponding to $2r$ spatial gradients lies in the form of our equations. In what follows, similarly to what done in Section 3.4, the calculation procedure will consist in an iterative resolution of E.E. suitable to give the perturbation functions at increasing orders: γ_{ij} will be the only variable of our equations and will be obtained through an evolution equation like (2.26c). Now, the spatial gradient content of equation (2.26c) is two and these spatial gradients appear in the spatial curvature tensors $\mathcal{R}_{ij}$ and scalar $\mathcal{R}$: therefore the solutions of the metric containing zero and one spatial derivatives can be found in just one iteration neglecting those terms, which means that they solve the same equation. We have to wait for a two-gradient metric for having a non trivial source term of the same gradient content in $\mathcal{R}_{ij}$ and $\mathcal{R}$, that is $\mathcal{R}_{ij}$ and $\mathcal{R}$ written as functions of a metric containing zero gradients. The same considerations apply at following steps, with jumps of two gradients among subsequent solutions for the spatial metric γ_{ij}.

The form of the spatial metric (4.2) is not accidental: supposing for a moment that we are allowed to expand the exponential, we write

$$\gamma_{ij} = (1 - 2\Psi + ...)\delta_{ij} + \chi_{ij} - 2\Psi\ \chi_{ij} + ... \tag{4.6}$$

Then, disregarding mixed terms of type $\Psi\ \chi_{ij}$, one could see that our spatial metric is formally similar to the one given in the standard theory (3.17) at least in the aspect it assumes at its standard first order. Nevertheless, analogies apply only at a formal level and only want to suggest that a higher order comparison between standard results of Chapter 3 and gradient expansion results can be engaged in, but with an appropriate procedure (see Chapter 5). Many differences arise: let us stress that our metric as written in (4.2) is not approximated: it contains all the perturbative orders of this technique. Furthermore, if in the standard technique the zeroth order terms express properties of the FRW background, here the comparison with the background is less obvious.
The flat FRW metric contains no spatial derivatives (nor temporal derivatives) so we should recover it in the zeroth order terms. But cutting off Higher than 0 Derivatives Terms (H0DT), the metric reads

$$\gamma_{ij}^{(0)} = e^{-2\Psi_{(0)}}(\delta_{ij} + \chi_{ij}^{(0)}). \tag{4.7}$$

with $\Psi_{(0)}$ and $\chi_{ij}^{(0)}$ a priori functions of time and space.
From standard linear perturbation results (3.22), admitting here the same initial conditions set in Section 3.4 at $\tau = \tau_{IN} = 0$, we know that Ψ contains at least a zero derivative term, while the traceless tensor χ_{ij} has at least two spatial gradients:

$$\begin{aligned}
\Psi_{ST}^{(1)}(\tau, \vec{x}) &= \frac{5}{3}\varphi(\vec{x}) + \frac{\tau^2}{18}\nabla^2\varphi(\vec{x}) \\
\chi_{ij\ ST}^{(1)}(\tau, \vec{x}) &= -\frac{\tau^2}{3}\left(\varphi_{,ij}(\vec{x}) - \frac{1}{3}\ \delta_{ij}\nabla^2\varphi(\vec{x})\right)
\end{aligned}$$

where the subscript "ST" stands for standard. The Ψ zero derivatives term is the term not depending on time. In χ_{ij} there are no time-indipendent nor zero derivatives terms for χ_{ij}, neither at standard first order nor at the second one (see equation (3.24)). Thus in our formalism we will assume from now on

$$\Psi_{(0)}(\tau, \vec{x}) = \frac{5}{3}\varphi(\vec{x}) \equiv \Psi(\tau = 0, \vec{x}) = \Psi_{IN} \tag{4.8a}$$

$$\chi_{ij}^{(0)}(\tau, \vec{x}) = \chi_{ij}^{IN} = 0 \tag{4.8b}$$

With this initial assumptions, equation (4.7) can be rewritten as

$$\gamma_{ij}^{(0)} = e^{-2\Psi_{(0)}(\vec{x})}\ \delta_{ij} = e^{-\frac{10}{3}\varphi(\vec{x})}\delta_{ij} \tag{4.9}$$

where φ is the so-called gravitational potential and the zeroth order metric is conformally related to the flat space metric by a space-dependent factor $e^{-\frac{10}{3}\varphi(\vec{x})}$.
Therefore, rather than having a FRW background coinciding with the zeroth order approximation, here the idea is to let a *seed spatial metric* $\gamma_{ij}^{IN} \equiv \gamma_{ij}^{(0)}$ evolve in time with the perturbative functions Ψ and χ from the end of Inflation until present time, producing the necessary ingredient for gravitational instability to develop.

Initial conditions from Inflation In order to compare the two techniques at least at the lowest orders, we have earlier assumed the same initial conditions of Section 3.4 to compute the first two orders of Ψ_{ST} and χ_{ij}^{ST}: thus we have specialized our quantities on the basis of those hypotheses. Let us briefly linger over this choice.
Since the cosmological perturbations are generated during Inflation as widely discussed in Chapter 1, it is physically natural to set initial conditions for the gravitational perturbations Ψ and χ_{ij} at the end of Inflation, effectively coinciding with $\tau = \tau_{IN} = 0$. This way, a gauge-invariant formulation of inflationary perturbations theory [14] tells us that the spatial perturbation of the metric is related to ζ, the gauge-invariant comoving curvature perturbation, and hence to the gravitational potential through an expression as $h_{ij} = a^2 \, e^{-2\zeta}\delta_{ij} = a^2 e^{-\frac{10}{3}\varphi} \, \delta_{ij}$. Therefore, even without making any parallelism with the standard gauge-dependent theory of Chapter 3 but only assuming Inflation as the simplest mechanism for generating perturbations, we have that the initial conditions at $\tau = 0$ are $\Psi_{IN} \equiv 5\varphi/3$ and $\chi_{ij}^{IN} = 0$. The initial condition $\delta_{IN} = 0$ is also assumed. Since cosmological perturbations generated during single-field models of Inflation are very nearly Gaussian with a nearly flat power spectrum ($n \simeq 1$)[14], [11], we notice by the way that φ should be regarded as a nearly scale-invariant, quasi-Gaussian random field.
Thanks to these points, in what follows we will be allowed to write χ rather than χ_{ij} regarding to the contribution of $D_{ij}\chi$ of the traceless part of the spatial metric.

4.2 The expansion scheme

In this perturbative technique the expansion parameter is the number of spatial derivatives. We now want to comment this rule and understand the physical meaning behind it.
A first rough idea can be obtained by a dimensional point of view. The perturbative functions Ψ and χ are dimensionless: in the natural units system, the dimension of a spatial derivative is the inverse of a length (L^{-1}) or, in other terms, a wavenumber k. The two gradients contained -say- in the first order of Ψ, $\Psi_{(1)}$, give a contribution $\sim (L^{-2} = k^2)$ in the dimensions, the four gradients in $\psi^{(2)}$ give a contribution of $\sim (L^{-4} = k^4)$, and so on. In order to have at every order $[\psi^{(r)}] = 1$, we need a factor $L^{2r} \sim t^{2r}$, which can come from a suitable power of conformal time: for every spatial derivative a power of the conformal time appears.
Therefore the Gradient Expansion consists in a perturbative expansion in even powers of (τk): the lowest (zeroth) order solution corresponds to the so-called

long wavelength approximation (or *separate universe*, with $(\tau k) \ll 1$ [32], [27], [28]); adding the higher order gradients leads to a more accurate solution, which hopefully converges toward the exact one.
The long wavelength approximation consists in neglecting spatial gradients of the variables describing the cosmological models: these spatial gradients have to be considered negligible in comparison with the time derivatives of the above variables, and this should now be clear having in mind the expansion parameter (τk):

$$\tau k \ll 1 \Longleftrightarrow k \ll \tau^{-1} \Longleftrightarrow \frac{\partial}{\partial x} \ll \frac{\partial}{\partial \tau}$$

Since the time-scale of variation in cosmology is given by the local Hubble expansion rate, the zeroth order approximation consists in neglecting inhomogeneities varying over a scale smaller that the Hubble horizon, or conversely in studying inhomogeneities larger than the Hubble radius: adding the following orders is equivalent to getting information about perturbation scales as they become smaller than the Hubble horizon [32].
For completeness, we translate what explained until now in terms of our spatial metric γ_{ij}, following [30]. The conditions $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial \tau}$ is rewritten as

$$\gamma_{ij,k} \ll \gamma'_{ij}.$$

The characteristic comoving length on which the metric varies is L: $\gamma_{ij,k} \sim L^{-1}\gamma_{ij}$. As said, the Hubble time is the characteristic proper time on which the metric evolves at a point x^k: in conformal time, $\gamma'_{ij} \sim aH\gamma_{ij}$.
Thus we can conclude that

$$(\tau k) \ll 1 \Longleftrightarrow aL \gg H^{-1}, \tag{4.10}$$

which precisely means that the characteristic scale of spatial variation is bigger than the Hubble radius.
Nevertheless, the actual range of validity of the Gradient Expansion Technique is not only restricted to the description of inhomogeneities on super-Hubble scales: as we will see later in Chapter 5, it can be applied also to sub-horizon wavelength perturbations [37].

The overall computation procedure to obtain Ψ and χ_{ij} at different orders is similar to the one described in Section 3.4. In what follows, we write down all the useful geometrical quantities as functions of the spatial metric defined earlier (equation (4.2)) up to first and second order in the gradient expansion; we introduce the two physical variables, the expansion rate θ and the shear σ^i_j as defined in Chapter 2, and iteratively solve the E.E.. These are written order by order in their space-space components as the evolution equations for θ (the Raychaudhuri equation) and σ^i_j, namely the equations (2.27). Knowing the zeroth order solution of Ψ and χ of the equations (4.8), we have in mind an iteration scheme suitable for getting explicit expressions of Ψ and χ in terms of φ. In other words:

- Solving the Raychaudhuri equation up to 1^{st} order (2DT) $\Longrightarrow \Psi_{(1)}$
  Solving the shear evolution equation up to 1^{st} order (2DT) $\Longrightarrow \chi^{(1)}_{ij}$
- Solving the Raychaudhuri equation up to 2^{nd} order (4DT) $\Longrightarrow \Psi_{(2)}$
  Solving the shear evolution equation up to 2^{nd} order (4DT) $\Longrightarrow \chi^{(2)}_{ij}$.

4.3 Gradient expansion technique at 1^{st} order

Definitions and quantities up to 1^{st}order

Spatial metric and inverse spatial metric

Let us begin calculating the inverse spatial metric of the general metric of equation (4.2): the expansion procedure and the cutting off of the Higher than 2 Derivative Terms (H2DTs) will be similar in all the following computations up to first order.
First of all, let us notice that the exponential in the spatial metric cannot be expanded in power series of Ψ, because a priori $\Psi_{(0)} = \frac{5}{3}\varphi$ can be large. The gravitational potential $\varphi(\vec{x})$ can in general be splitted in two parts: $\varphi = \varphi_L + \varphi_S$, where φ_L is the long-wavelength mode and φ_S short wavelength modes such that $\varphi_S/\varphi \sim 10^{-5}$ from CMB constraints. There are no known upper limits on φ_L: therefore, we will factor out $e^{-10/3\varphi}$ in almost all our following expressions. By the way, let us note that the spatial differentiation of φ_L is neglectable, as by definition spatial gradients see spatial variations on small scales and on small scales φ_L is almost constant.
The inverse spatial metric is given solving the following equation in terms of the unknown γ^{aj}:

$$\gamma_{ia}\ \gamma^{aj} = \delta_i^j. \tag{4.11}$$

This can be written as

$$e^{-2\Psi}(\delta_{ia} + \chi_{ia})\ [A(\delta^{aj} + \delta\gamma^{aj})] = \delta_i^j \text{ where } \Psi = \Psi_{(0)} + \Psi_{(1)} \text{ and } \chi_{ij} = \chi_{ij}^{(1)}.$$

The factor A is straightforward given by $A = e^{+2\Psi}$, with $\Psi = \Psi_{(0)} + \Psi_{(1)}$. For the tensor coefficient $\delta\gamma^{aj}$ we write:

$$\begin{aligned}
&(\delta_{ia} + \chi_{ia})\ (\delta^{aj} + \delta\gamma^{aj}) = \delta_i^j;\\
&\delta_{ia}\ \delta^{aj} + \chi_{ia}\ \delta^{aj} + \delta_{ia}\delta\gamma^{aj} + \chi_{ia}\delta\gamma^{aj} = \delta_i^j, \text{ that is}\\
&\chi^j_{i(1)} + \delta\gamma_i^j + \chi_{ia}^{(1)}(\delta\gamma^{aj}_{(0)} + \delta\gamma^{aj}_{(1)}) = 0.
\end{aligned}$$

The term $\chi_{ia}^{(1)}\ \delta\gamma^{aj}_{(1)}$ is certainly a Higher than 2 Derivative Term (H2DT) so can be neglected: the result is

$$\delta\gamma_i^j = -\chi_i^j \text{ that is } \delta\gamma^{ij} = -\chi^{ij} \text{ with } \chi^{ij} \equiv \delta^{im}\delta^{jn}\chi_{mn}.$$

Then we can write the *inverse spatial metric* as follows

$$\gamma^{ij} = e^{2\Psi}(\delta^{ij} - \chi^{ij}) \tag{4.12}$$

$$(\text{with } \Psi = \Psi_{(0)} + \Psi_{(1)} \text{ and } \chi^{ij} = \chi^{ij}_{(1)})$$

Velocity-gradient tensor and expansion rate

An analogous computation can be applied to express the expansion rate in terms of the perturbative functions Ψ and χ_{ij}. The definition of the velocity-gradient tensor is given in (2.24):

$$\theta^i_j = \frac{1}{2}\gamma^{ia}\gamma'_{aj}.$$

Now, $\gamma^{ia} = e^{2\Psi}(\delta^{ia} - \chi^{ia})$ and $\gamma_{aj} = e^{-2\Psi}(\delta_{aj} + \chi_{aj})$.

$$\gamma'_{aj} = (-2\Psi')e^{-2\Psi}(\delta_{aj} + \chi_{aj}) + e^{-2\Psi}\chi'_{aj}.$$

Then, $\frac{1}{2}\gamma^{ia}\gamma'_{aj} = -\Psi'\delta^i_j + \frac{1}{2}\chi^i_j{}' + H2DTs,$

where the H2DTs are terms like $\Psi'\ \chi^{ia}\chi_{aj}$ or $\frac{1}{2}\chi^{ia}\chi'_{aj}$. The resulting *velocity-gradient tensor* and *expansion rate* up to two spatial gradients are written as follows

$$\theta^i_j = -\Psi'\delta^i_j + \frac{1}{2}\chi^i_j{}' \tag{4.13}$$

$$\theta = -3\Psi' \tag{4.14}$$

(with $\Psi = \Psi_{(0)} + \Psi_{(1)}$ and $\chi_{ij} = \chi^{(1)}_{ij}$).

Shear

The shear can be computed thanks to the definition given in Chapter 2:

$$\sigma^i_j = \theta^i_j - \frac{1}{3}\delta^i_j\ \theta.$$

Up to first order we obtain

$$\sigma^i_j = \frac{1}{2}\chi^i_j{}' \tag{4.15}$$

(with $\chi^i_j = \chi^{i\,(1)}_j$).

Christoffel Symbols

On 3-dimensional hypersurfaces Σ, the Chrsitoffel Symbols are defined

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{ia}(\gamma_{aj,k} + \gamma_{ak,j} - \gamma_{jk,a}).$$

Using equations (4.2) and (4.12) for the metric and its inverse, and neglecting all terms like $\chi_{ij,k}$ and $\Psi_{,k}\chi^{ia}$ because they contain at least three spatial gradients, we get

$$\Gamma^i_{jk} = -\Psi_{,k}\delta^i_j - \Psi_{,j}\delta^i_k + \Psi^{,i}\delta_{jk} \tag{4.16}$$

(with $\Psi = \Psi_{(0)}$).

Let us note that Γ^i_{jk} contains only one spatial derivative up to our first order.

Ricci Tensor

The Ricci tensor is defined as the contraction of the Riemann tensor, which we will not explicit, and reads

$$\mathcal{R}_{jm} = -\Gamma^a_{ja,m} + \Gamma^a_{jm,a} + \Gamma^a_{ba}\Gamma^b_{jm} - \Gamma^a_{mb}\Gamma^b_{ja}.$$

Using (4.16), up to two derivative terms, we get

$$\mathcal{R}_{jm} = \Psi_{,jm} + (\nabla^2\Psi)\delta_{jm} + \Psi_{,j}\ \Psi_{,m} - (\nabla\Psi)^2\delta_{jm} \tag{4.17}$$

(with $\Psi = \Psi_{(0)}$).

Because the zeroth order term of Ψ coincide with its initial value (4.8a), we can write $\mathcal{R}^{(1)}_{jm} = \mathcal{R}_{jm}(\Psi_{(0)}) = \mathcal{R}_{jm}(\Psi_{IN}) = \mathcal{R}^{IN}_{jm}$, as extensively done in Appendix C.

Scalar Curvature

Taking the trace of Ricci tensor (4.17), the Scalar Curvature is

$$\mathcal{R} = e^{2\Psi}[4(\nabla^2\Psi) - 2(\nabla\Psi)^2] \tag{4.18}$$

(with $\Psi = \Psi_{(0)}$ and $\mathcal{R} = \mathcal{R}^{(1)} = \mathcal{R}_{IN}$).

Evolution equations for θ and σ^i_j at 1^{st} order

The evolution equations for θ and σ^i_j have been deduced in Chapter 2 and read as in (2.27). Considering the background scale factor being $a(\tau) \propto \tau^2$, they can be rewritten as follows:

$$\theta' + \frac{4}{\tau}\theta + \frac{1}{2}\theta^2 + \frac{3}{2}\sigma^2 = -\frac{1}{4}\mathcal{R} \tag{4.19a}$$

$$\sigma^{i\,\prime}_j + \frac{4}{\tau}\sigma^i_j + \theta\ \sigma^i_j = -(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j), \tag{4.19b}$$

where $\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\ \sigma^{ab}$.

Raychaudhuri equation UP2DT

As cleraly shown in Appendix C, θ and σ^i_j contain at least two spatial gradients: therefore, terms like θ^2 and σ^2 contain more than two derivatives terms. Dropping from equation (4.19a) the H2DTs we obtain

$$\theta' + \frac{4}{\tau}\theta = -\frac{1}{4}R_{IN}, \tag{4.20}$$

and hence, using the expression at first order for θ (4.14), the equation we have to solve in the unknown $\Psi_{(1)}$ is

$$\Psi''_{(1)} + \frac{4}{\tau}\Psi'_{(1)} = \frac{1}{12}\mathcal{R}_{IN}. \tag{4.21}$$

Writing

$$s_\Psi(\vec{x}) = \frac{1}{12}\mathcal{R}_{IN} = \frac{1}{12}e^{2\Psi_{(0)}}[4(\nabla^2\Psi_{(0)}) - 2(\nabla\Psi_{(0)})^2], \tag{4.22}$$

the solution is

$$\Psi_{(1)} = \frac{1}{10}\tau^2 s_\Psi(\vec{x}) - \frac{1}{3\tau^3}c_1 + c_2, \tag{4.23}$$

where c_1 and c_2 are integration constants. As function of φ the source $s_\Psi(\vec{x})$ reads

$$s_\Psi(\vec{x}) = \frac{5}{9}e^{\frac{10}{3}\varphi(\boldsymbol{x})}[(\nabla^2\varphi(\vec{x})) - \frac{5}{6}(\nabla\varphi(\vec{x}))^2]. \tag{4.24}$$

Then, considering only the growing mode, we get

$$\Psi_{(1)} = \frac{1}{18}\tau^2 e^{\frac{10}{3}\varphi}[(\nabla^2\varphi) - \frac{5}{6}(\nabla\varphi)^2] \tag{4.25}$$

Evolution equation of shear UP2DT

Dropping from equation (4.19b) the H2DT $\theta\ \sigma^i_j$, we obtain the following equation:

$$\sigma^{i\,\prime}_j + \frac{4}{\tau}\sigma^i_j = -(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j)_{IN} \tag{4.26}$$

Substituting in the equation above the expression for σ^i_j as function of $\chi^{(1)}_{ij}$ (see (4.15)), we get

$$\chi^{i\,\prime\prime}_{j(1)} + \frac{4}{\tau}\chi^{i\,\prime}_{j(1)} = -2(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j)_{IN}. \tag{4.27}$$

Isolating the source term as

$$s_\chi(\vec{x}) = -2(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j)_{IN}, \tag{4.28}$$

the solution is

$$\chi^i_{j(1)} = \frac{1}{10}\tau^2 s_\chi(\vec{x}) - \frac{1}{3\tau^3}c_1 + c_2. \tag{4.29}$$

Expliciting the source as function of φ we have

$$s_\chi(\vec{x}) = -\frac{10}{3}e^{\frac{10}{3}\varphi}[D^i_j\varphi + \frac{5}{3}(\varphi^{,i}\varphi_{,j} - \frac{1}{3}(\nabla\varphi)^2\delta^i_j)]. \tag{4.30}$$

Then, considering only the growing mode, the first order result for the traceless coefficient $\chi^{(1)}_{ij}$ reads

$$\chi^i_{j(1)} = -\frac{1}{3}\tau^2 e^{\frac{10}{3}\varphi}[D^i_j\varphi + \frac{5}{3}(\varphi^{,i}\varphi_{,j} - \frac{1}{3}(\nabla\varphi)^2\delta^i_j)] \tag{4.31}$$

4.4 Gradient expansion technique at 2^{nd} order

Definitions and quantities up to 2^{nd} order

By second order in this technique we mean keeping only quantities which contain at most four spatial derivatives.

Spatial metric and inverse spatial metric

The spatial metric and its inverse read respectively up to our second order

$$\gamma_{ij} = e^{-2\Psi}(\delta_{ij} + \chi_{ij}) \tag{4.32}$$

$$\gamma^{ij} = e^{2\Psi}(\delta^{ij} - \chi^{ij} + \chi^{i\,(1)}_a\chi^{aj}_{(1)}) \tag{4.33}$$

(with $\Psi = \Psi_{(0)} + \Psi_{(1)} + \frac{1}{2}\Psi_{(2)}$ and $\chi_{ij} = \chi^{(1)}_{ij} + \frac{1}{2}\chi^{(2)}_{ij}$).

In fact, $\gamma_{ia}\ \gamma^{aj} = \delta_i^j$,

$$e^{-2\Psi}(\delta_{ia} + \chi_{ia})\ [A(\delta^{aj} + \delta\gamma^{aj})] = \delta_i^j \Longrightarrow A = e^{2\Psi}.$$

$$(\delta_{ia} + \chi_{ia})\ (\delta^{aj} + \delta\gamma^{aj}) = \delta_i^j.$$

Now, $\chi_{ia} = \chi_{ia}^{(1)} + \frac{1}{2}\chi_{ia}^{(2)}$

$$\delta\gamma^{aj} = \delta\gamma^{aj}_{(0)} + \delta\gamma^{aj}_{(1)} + \frac{1}{2}\delta\gamma^{aj}_{(2)}.$$

From the first order we know that, $\delta\gamma^{aj}_{(0)} = 0$ and $\delta\gamma^{aj}_{(1)} = -\chi^{aj}_{(1)}$.

Then, $(\chi_{ia}^{(1)} + \frac{1}{2}\chi_{ia}^{(2)})\ \delta^{aj} + (-\chi^{aj}_{(1)} + \frac{1}{2}\delta\gamma^{aj}_{(2)})\ \delta_{ia} + (\chi_{ia}^{(1)} + \frac{1}{2}\chi_{ia}^{(2)})(-\chi^{aj}_{(1)} + \frac{1}{2}\delta\gamma^{aj}_{(2)}) = 0;$

$$\chi_i^{j(1)} + \frac{1}{2}\chi_i^{j(2)} - \chi^j_{i(1)} + \frac{1}{2}\delta\gamma^j_{i(2)} - \chi_{ia}^{(1)}\chi^{aj}_{(1)} + H4DT = 0$$

$$\Longrightarrow \delta\gamma^j_{i(2)} = -\chi_i^{j(2)} + 2\ \chi_{ia}^{(1)}\ \chi^{aj}_{(1)}\Box$$

Velocity-gradient tensor and expansion rate

Performing the calculation similarly to what indicated earlier for (4.13) and (4.14), and using expressions above (4.32) and (4.33), the velocity-gradient tensor and the expansion rate read respectively

$$\theta^i_j = -\Psi'\delta^i_j + \frac{1}{2}\chi^{i\,\prime}_j - \frac{1}{2}\chi^{ia}_{(1)}\ \chi^{(1)\prime}_{aj} \tag{4.34}$$

$$\theta = -3\Psi' - \frac{1}{2}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab} \tag{4.35}$$

(with $\Psi = \Psi_{(0)} + \Psi_{(1)} + \frac{1}{2}\Psi_{(2)}$ and $\chi^i_j = \chi^{i\,(1)}_j + \frac{1}{2}\chi^{i\,(2)}_j$).

Shear

The shear is obtained taking the traceless part of the gradient-velocity tensor, thus using (4.34) and (4.35) one obtains

$$\sigma^i_j = \frac{1}{2}\chi^{i\,\prime}_j - \frac{1}{2}\chi^{ia}_{(1)}\chi^{(1)\prime}_{aj} + \frac{1}{6}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab}\delta^i_j \tag{4.36}$$

(with $\chi^i_j = \chi^{i\,(1)}_j + \frac{1}{2}\chi^{i\,(2)}_j$).

Christoffel Symbols

Likewise at the first order, the Christoffel Symbols cannot fill up the number of gradients content set by the order, and at the second order they contain only three spatial derivatives:

$$\begin{aligned}\Gamma^i_{jk} =&(-\Psi_{,k}\delta^i_j - \Psi_{,j}\delta^i_k + \Psi^{,i}\delta_{jk})+\\ &+\frac{1}{2}(\chi^i_{j,k} + \chi^i_{k,j} - \chi^{,i}_{jk})+\\ &+(\Psi^{,i}_{(0)}\ \chi^{(1)}_{jk} - \Psi^{(0)}_{,a}\ \chi^{ia}_{(1)}\ \delta_{jk})\end{aligned} \tag{4.37}$$

(with $\Psi = \Psi_{(0)} + \Psi_{(1)}$ and $\chi^i_j = \chi^{i\,(1)}_j$), where we have highlighted that the Christoffel Symbols at second order are composed of three parts:

$$\Gamma^i_{jk} = \Gamma^i_{jk}(\Psi) + \Gamma^i_{jk}(\chi) + \Gamma^i_{jk}(\Psi \cdot \chi)$$

Ricci Tensor

With a straightforward but long calculation, the other geometrical quantities follow. The Ricci tensor with four spatial gradient is written as function of Ψ and χ_{ij} containing at most two spatial derivatives:

$$\begin{aligned} R_{jm} =& \Psi_{,jm} + (\nabla^2\Psi)\delta_{jm} + \Psi_{,j}\Psi_{,m} - (\nabla\Psi)^2\delta_{jm} + \\ &+ \frac{1}{2}(\chi^a_{j,ma} + \chi^a_{m,ja} - \nabla^2\chi_{jm}) + \\ &+ [(\nabla^2\Psi_{(0)})\chi_{jm} - \Psi^{(0)}_{,ab}\ \chi^{ab}\ \delta_{jm} - \Psi^{(0)}_{,a}\ \chi^{ab}_{,b}\ \delta_{jm} + \\ &+ \frac{1}{2}\Psi^{;a}_{(0)}(-\chi_{am,j} - \chi_{aj,m} + \chi_{mj,a}) - (\nabla\Psi_{(0)})^2\chi_{jm} + \Psi^{(0)}_{,a}\ \Psi^{(0)}_{,b}\ \chi^{ab}\ \delta_{jm}] \end{aligned} \tag{4.38}$$

$$\text{(with } \Psi = \Psi_{(0)} + \Psi_{(1)} \text{ and } \chi^i_j = \chi^{i\,(1)}_j),$$

where we note again that

$$R_{jm} = R_{jm}(\Psi) + R_{jm}(\chi) + R_{jm}(\Psi \cdot \chi).$$

Scalar Curvature

The Scalar Curvature reads

$$\begin{aligned} \mathcal{R} =& e^{2\Psi}[4(\nabla^2\Psi) - 2(\nabla\Psi)^2] + \\ &+ e^{2\Psi_{(0)}}[\chi^{ab}_{(1),ab}] + \\ &+ e^{2\Psi_{(0)}}[-4\chi^{ab}_{(1)}\ \Psi^{(0)}_{,ab} - 4\chi^{ab}_{(1),b}\ \Psi^{(0)}_{,a} + 2\chi^{ab}_{(1)}\ \Psi^{(0)}_{,a}\ \Psi^{(0)}_{,b}] \end{aligned} \tag{4.39}$$

$$\text{(with } \Psi = \Psi_{(0)} + \Psi_{(1)}).$$

Also the Scalar Curvature can be divided into three parts according with the argument, and it contains several mixed terms of the kind $\Psi\ \chi$:

$$R = R(\Psi) + R(\chi) + R(\Psi \cdot \chi).$$

In Appendix C the explicit expressions of every contribution are presented.

Ricci Tensor and Scalar curvature in terms of φ

As we can see from expressions (4.38) and (4.39) of the Ricci tensor and the Scalar Curvature up to four spatial gradients, they are functions of Ψ and χ at most at first order, that is up to two gradient terms. Then, having solved the first step of our iteration scheme and obtained the results (4.25) and (4.31), $\mathcal{R}_{ij}$ and $\mathcal{R}$ up to 4DTs are completely known. In the following we write down the result of a straightforward calculation that eventually makes use of Appendix C.

$$
\begin{aligned}
\mathcal{R}^i_j \;&= e^{\frac{10}{3}\varphi}\;[\frac{5}{3}\varphi^{,i}_{,j} + \frac{5}{3}(\nabla^2\varphi)\;\delta^i_j + \frac{25}{9}\varphi^{,i}\varphi_{,j} - \frac{25}{9}(\nabla\varphi)^2\delta^i_j] + \\
&+\tau^2 e^{\frac{20}{3}\varphi}[+\frac{5}{18}\;\varphi_{,ab}\;\varphi^{,ab}\;\delta^i_j - \frac{5}{18}\varphi^{,ai}\;\varphi_{,aj} + \frac{5}{9}\;(\nabla^2\varphi)\;\varphi^{,i}_{,j} + \\
&\quad + \frac{25}{27}\;\varphi_{,a}\;\varphi_{,b}\;\varphi^{,ab}\;\delta^i_j - \frac{25}{54}(\;\varphi^{,a}\;\varphi_{,aj}\;\varphi^{,i} + \;\varphi_{,a}\;\varphi^{,ai}\;\varphi_{,j}) + \\
&\quad + \frac{25}{27}(\nabla^2\varphi)\;\varphi^{,i}\;\varphi_{,j} - \frac{25}{27}(\nabla^2\varphi)\;(\nabla\varphi)^2\delta^i_j + \\
&\quad + \frac{125}{162}(\nabla\varphi)^2(\nabla\varphi)^2\;\delta^i_j - \frac{125}{162}(\nabla\varphi)^2\;\varphi^{,i}\;\varphi_{,j}]
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
\mathcal{R} \;&= e^{\frac{10}{3}\varphi}\;[\frac{20}{3}(\nabla^2\varphi) - \frac{50}{9}(\nabla\varphi)^2] + \\
&+\tau^2 e^{\frac{20}{3}\varphi}[+\frac{5}{9}\;\varphi_{,ab}\;\varphi^{,ab} + \frac{5}{9}\;(\nabla^2\varphi)^2 + \\
&\quad + \frac{50}{27}\;\varphi_{,a}\;\varphi_{,b}\;\varphi^{,ab} - \frac{50}{27}(\nabla^2\varphi)\;(\nabla\varphi)^2 + \\
&\quad + \frac{125}{81}(\nabla\varphi)^2(\nabla\varphi)^2]
\end{aligned}
\tag{4.41}
$$

Evolution equations for θ and σ^i_j at 2^{nd} order

We want to solve E.E. in order to get the complete expressions up to four derivatives for the metric coefficients. For this task we use the evolution equations for θ and σ^i_j already met several times:

$$\theta' + \frac{4}{\tau}\theta + \frac{1}{2}\theta^2 + \frac{3}{2}\sigma^2 = -\frac{1}{4}\mathcal{R} \tag{4.42}$$

$$\sigma^{i\,\prime}_j + \frac{4}{\tau}\sigma^i_j + \theta\;\sigma^i_j = -(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j) \tag{4.43}$$

In what follows we use all the results at second order given earlier and we have in mind an expansion for θ and σ^i_j as

$$\theta = \theta_{(0)} + \theta_{(1)} + \frac{1}{2}\theta_{(2)}$$

$$\sigma^i_j = \sigma^{i\,(0)}_j + \sigma^{i\,(1)}_j + \frac{1}{2}\sigma^{i\,(2)}_j,$$

where $\theta_{(0)} = \sigma^{i\,(0)}_j = 0$. We aim to obtain the expressions for $\Psi_{(2)}$ and $\chi^i_{j\,(2)}$ in terms of φ and its derivatives. As we will see, the procedure is the same as that at the previous order, but is much more complicated for the presence of a greater number of terms to express first of all in terms of $\Psi_{(1)}$ and $\chi^i_{j\,(1)}$, and then in terms of φ. The result will be two expressions of 4DTs, in which the four gradients will distributed in one φ, or in two φ, or in three φ, and so on. Among those types of terms a precise hierarchy exists:

$$\xrightarrow[\text{(subdominance)}]{4grad(\varphi),\quad 4grad(\varphi^2),\quad 4grad(\varphi^3),\ ...}$$

With the symbol $4grad(\varphi)$ we mean terms like $\varphi_{,abcd}$, $\nabla^2\varphi^{;i}_{,j}$, or $\nabla^2(\nabla^2\varphi)$; the symbol $4grad(\varphi^2)$ means terms like $\varphi_{,ab}\ \varphi^{,ab}$, $(\nabla^2\varphi)^2$ or $(\nabla^2\varphi)\ \varphi^{;i}_{,j}$; $4grad(\varphi^3)$ indicates terms like $\varphi_{,a}\ \varphi_{,b}\ \varphi^{,ab}$ or $(\nabla^2\varphi)\ (\nabla\varphi)^2$, and so on.
We have already mentioned how the peculiar gravitational potential can be thought as a sum of a longwavelength mode φ_L and a collection of short wavelength modes φ_S: the spatial derivative can affect only the latter, whose magnitude with respect to φ is of the order of 10^{-5}. The idea is to compare terms like

$$\nabla^2\varphi \longleftrightarrow (\nabla\varphi)^2$$

Recalling earlier notations, $\nabla^2\varphi \propto (k\tau)^2\varphi_S$ while $(\nabla\varphi)^2 \propto (k\tau)^2\varphi_S\varphi_S$. Gradients being the same number, the number of φ_S determines the order of magnitude: hence

$$(\nabla^2\varphi \propto (k\tau)^2\varphi_S)\ \gg\ ((\nabla\varphi)^2 \propto (k\tau)^2\varphi_S\varphi_S)$$

Up to four spatial gradients, we will procede step by step producing at the beginning only the leading terms $4grad(\varphi)$, and then turning to the complete expressions in terms of $4grad(\varphi^2)$, $4grad(\varphi^3)$, and so on.

Raychaudhuri equation UP4DT

Dropping H4DTs like $\theta_{(1)} \times \theta_{(2)}$ or $\sigma^{(1)}_{ab} \times \sigma^{ab}_{(2)}$ from (4.42), we obtain :

$$\theta' + \frac{4}{\tau}\theta + \frac{1}{2}\theta^2_{(1)} + \frac{3}{2}\sigma^2_{(1)} = -\frac{1}{4}\mathcal{R}. \tag{4.44}$$

Subtracting the corresponding equation at first order (4.20) and taking the known terms at the right hand side of the equation, it becomes

$$\frac{1}{2}\theta'_{(2)} + \frac{4}{\tau}\frac{1}{2}\theta_{(2)} = -\frac{1}{4}(\mathcal{R} - \mathcal{R}_{IN}) - \frac{1}{2}\theta^2_{(1)} - \frac{3}{2}\sigma^2_{(1)}. \tag{4.45}$$

Using the expression (4.35) for the second order terms of the expansion rate, and isolating again the known solutions at the previous order for $\chi^{(1)}_{ij}$, the equation to solve reads

$$\Psi''_{(2)} + \frac{4}{\tau}\Psi'_{(2)} = \tau^2\mathcal{S}_\Psi(\vec{x}), \tag{4.46}$$

where the source function $\boldsymbol{\mathcal{S}}_\Psi(\tau,\vec{x}) \equiv \tau^2\mathcal{S}_\Psi(\vec{x})$ is

$$\boldsymbol{\mathcal{S}}_\Psi(\tau,\vec{x}) = \frac{1}{6}(\mathcal{R}-\mathcal{R}_{IN}) + \frac{1}{3}\theta^2_{(1)} + \sigma^2_{(1)} - \frac{1}{3}({\chi^{ia}}_{(1)}\chi^{(1)\prime}_{ai})' - \frac{4}{3\tau}({\chi^{ia}}_{(1)}\chi^{(1)\prime}_{ai}). \tag{4.47}$$

Thus the solution expressed in terms of the source is

$$\Psi_{(2)} = \frac{1}{28}\tau^4\mathcal{S}_\Psi(\vec{x}) - \frac{1}{3\tau^3}c_1 + c_2. \tag{4.48}$$

Following the leading term $\nabla^2(\nabla^2\varphi)$
Now we go on and look for contributions in $\mathcal{S}_\Psi(\boldsymbol{x})$ to terms $\sim \nabla^2(\nabla^2\varphi)$, remembering that (up to terms functions of a single φ)

- $\Psi_{(0)} = \frac{5}{3}\varphi$
- $\Psi_{(1)} = \frac{1}{18}\tau^2 e^{\frac{10}{3}\varphi}(\nabla^2\varphi)$
- $\chi^i_{j\,(1)} = -\frac{1}{3}\tau^2 e^{\frac{10}{3}\varphi}[D^i_j\varphi(\boldsymbol{x})]$
- $(\mathcal{R} - \mathcal{R}_{IN}) = \frac{1}{2}\mathcal{R}^{(2)}(\Psi) + \frac{1}{2}\mathcal{R}^{(2)}(\chi) + \frac{1}{2}\mathcal{R}^{(2)}(\Psi\cdot\chi)$.

Then we see with the help of Appendix C that

- $\frac{1}{2}\mathcal{R}^{(2)}(\Psi) \Rightarrow e^{2\Psi_{(0)}}\ 4\nabla^2\Psi_{(1)}$
- $\frac{1}{2}\mathcal{R}^{(2)}(\chi) \Rightarrow e^{2\Psi_{(0)}}\ \chi^{ab}_{(1),ab}$
- $\frac{1}{2}\mathcal{R}^{(2)}(\Psi\cdot\chi) \Rightarrow$ terms $\varphi\cdot\varphi$, like every other addendum like $\Psi\cdot\Psi$, $\chi\cdot\chi$ and $\Psi\cdot\chi$.

Making the calculation, we obtain that there's no leading contribution to $\Psi_{(2)}$ like $\nabla^2(\nabla^2\varphi)$.

Complete expression for $\mathcal{S}_\Psi(\vec{x})$
Then we write down the complete expression for the source of $\Psi_{(2)}$, stressing that the effective leading terms are those with four gradients distributed in two φ (that is $4grad(\varphi^2)$):

$$\begin{aligned}\mathcal{S}_\Psi = e^{\frac{20}{3}\varphi}\ \frac{1}{9}\ [&-\frac{10}{3}\ \varphi_{,ab}\ \varphi^{,ab} + \frac{23}{9}\ (\nabla^2\varphi)^2 \\ &-\frac{100}{9}\ \varphi_{,a}\ \varphi_{,b}\ \varphi^{,ab} + \frac{35}{27}(\nabla^2\varphi)\ (\nabla\varphi)^2 + \\ &-\frac{1675}{324}(\nabla\varphi)^2(\nabla\varphi)^2].\end{aligned} \tag{4.49}$$

We conclude that, considering only the growing mode, $\Psi_{(2)}$ reads

$$\begin{aligned}\Psi_{(2)} = \tau^4\ e^{\frac{20}{3}\varphi}\ \frac{1}{252}[&-\frac{10}{3}\ \varphi_{,ab}\ \varphi^{,ab} + \frac{23}{9}\ (\nabla^2\varphi)^2 \\ &-\frac{100}{9}\ \varphi_{,a}\ \varphi_{,b}\ \varphi^{,ab} + \frac{35}{27}(\nabla^2\varphi)\ (\nabla\varphi)^2 + \\ &-\frac{1675}{324}(\nabla\varphi)^2(\nabla\varphi)^2].\end{aligned} \tag{4.50}$$

Evolution equation of shear UP4DT

Dropping H4DTs such as $\theta_{(1)} \times \sigma^{(2)}_{ij}$ and $\theta_{(2)} \times \sigma^{(2)}_{ij}$ from (4.43), we write

$$\sigma^{i\,\prime}_j + \frac{4}{\tau}\sigma^i_j + \theta_{(1)}\ \sigma^{i(1)}_j = -(\mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j). \tag{4.51}$$

Subtracting the corrisponding equation at first order (4.26) and isolating the known terms in the right hand side of the equation, we obtain the following equation

$$\frac{1}{2}\sigma^{i\,\prime}_{j\,(2)} + \frac{4}{\tau}\frac{1}{2}\sigma^{i}_{j\,(2)} = -[(\mathcal{R}^i_j - \mathcal{R}^i_{jIN}) - \frac{1}{3}(\mathcal{R} - \mathcal{R}_{IN})\delta^i_j] - \theta_{(1)}\ \sigma^{i(1)}_j. \qquad (4.52)$$

Substituting the expression of $\sigma^i_{j\,(2)}$ using (4.36), the equation we have to solve in the unknown $\chi^{(2)}_{ij}$ is

$$\chi^{i\,\prime\prime}_{j\,(2)} + \frac{4}{\tau}\chi^{i\,\prime}_{j\,(2)} = \tau^2\mathcal{S}_\chi(\vec{x}), \qquad (4.53)$$

where the source function $\boldsymbol{\mathcal{S}}_\chi(\tau,\vec{x}) = \tau^2\mathcal{S}_\chi(\vec{x})$ reads

$$\begin{aligned}\boldsymbol{\mathcal{S}}_\chi(\tau,\vec{x}) = & -4[(\mathcal{R}^i_j - \mathcal{R}^i_{jIN}) - \frac{1}{3}(\mathcal{R} - \mathcal{R}_{IN})\delta^i_j] - 4\theta_{(1)}\ \sigma^{i(1)}_j + \\ & + 2({\chi^{ia}}_{(1)}\chi^{(1)\,\prime}_{aj})^\prime - \frac{2}{3}({\chi^{ab}}_{(1)}\chi^{(1)\,\prime}_{ab})^\prime\delta^i_j + \\ & + \frac{8}{\tau}({\chi^{ia}}_{(1)}\chi^{(1)\,\prime}_{aj}) - \frac{8}{3\tau}({\chi^{ab}}_{(1)}\chi^{(1)\,\prime}_{ab})\delta^i_j.\end{aligned} \qquad (4.54)$$

Then the solution is

$$\chi^i_{j\,(2)} = \frac{1}{28}\tau^4\mathcal{S}_\chi(\vec{x}) - \frac{1}{3\tau^3}c_1 + c_2. \qquad (4.55)$$

Following the leading terms $(\nabla^2\varphi)^{;i}_{,j}$ and $\nabla^2(\nabla^2\varphi)\delta^i_j$

Let us go on and look for contributions in $\mathcal{S}_\chi(\vec{x})$ to terms like $(\nabla^2\varphi)^{;i}_{,j}$ and like $\nabla^2(\nabla^2\varphi)\delta^i_j$. For this task we remember that

- $\chi^i_{j\,(2)}$ has to be traceless;
- $(\mathcal{R}^i_j - \mathcal{R}^i_{jIN}) = \frac{1}{2}\mathcal{R}^{i(2)}_j(\Psi) + \frac{1}{2}\mathcal{R}^{i(2)}_j(\chi) + \frac{1}{2}\mathcal{R}^{i(2)}_j(\Psi\cdot\chi)$;
- $(\mathcal{R} - \mathcal{R}_{IN})$ does not contribute (see equation (4.41)).

Then we see with the help of Appendix C that

- $\frac{1}{2}\mathcal{R}^{i(2)}_j(\Psi) \Rightarrow e^{2\Psi_{(0)}}[\Psi^{;i}_{(1),j} + \nabla^2\Psi_{(1)}\delta^i_j]$;
- $\frac{1}{2}\mathcal{R}^{i(2)}_j(\chi) \Rightarrow e^{2\Psi_{(0)}}\frac{1}{2}[\chi^{ia}_{(1),aj} + \chi^{a,i}_{(1)j,a} - \nabla^2\chi^i_{(1)j}]$;
- $\frac{1}{2}\mathcal{R}^{i(2)}_j(\Psi\cdot\chi) \Rightarrow$ terms $\varphi\cdot\varphi$ (like every other addenda like $\Psi\cdot\Psi$, $\chi\cdot\chi$ and $\Psi\cdot\chi$).

Making the calculation, we obtain that there's no dominant contribution to $\chi^i_{j\,(2)}$ like $(\nabla^2\varphi)^{;i}_{,j}$ and $\nabla^2(\nabla^2\varphi)\delta^i_j$.

Complete expression for $\mathcal{S}_\chi(x)$
Consequence of the previous paragraph is that the effective leading terms of the source of $\chi_{(2)}$ are made of four gradient on two φ, namely $4grad(\varphi^2)$. $\mathcal{S}_\chi(\vec{x})$ reads

$$\begin{aligned}\mathcal{S}_\chi = +\frac{1}{9}\, e^{\frac{20}{3}\varphi}\, [&+ 38\, (\varphi_{,aj}\ \varphi^{,ai} - \frac{1}{3}\ \varphi_{,ab}\ \varphi^{,ab}\ \delta^i_j) + \\ &- \frac{128}{3}\ ((\nabla^2\varphi)\ \varphi^{,i}_{,j} - \frac{1}{3}\ (\nabla^2\varphi)^2\ \delta^i_j) + \\ &+ \frac{890}{27}(\nabla^2\varphi)\ (\nabla\varphi)^2\ \delta^i_j - \frac{250}{9}(\nabla\varphi)^2\ \varphi^{,i}_{,j} - \frac{640}{9}(\nabla^2\varphi)\ \varphi^{,i}\ \varphi_{,j} + \\ &- \frac{380}{9}\ \varphi_{,a}\ \varphi_{,b}\ \varphi^{,ab}\ \delta^i_j + \frac{190}{3}(\varphi_{,a}\ \varphi_{,j}\ \varphi^{,ai} + \varphi_{,aj}\ \varphi^{,a}\ \varphi^{,i}) + \\ &+ \frac{1600}{27}\ ((\nabla\varphi)^2\ \varphi^{,i}\ \varphi_{,j} - \frac{1}{3}(\nabla\varphi)^2\ (\nabla\varphi)^2\ \delta^i_j)]\end{aligned} \tag{4.56}$$

We conclude that, considering only the growing mode, $\chi^{(2)}_{ij}$ reads

$$\begin{aligned}\chi^i_{j(2)} = \tau^4\ e^{\frac{20}{3}\varphi}\ \frac{1}{252}\ [&+ 38\, (\varphi_{,aj}\ \varphi^{,ai} - \frac{1}{3}\ \varphi_{,ab}\ \varphi^{,ab}\ \delta^i_j) + \\ &- \frac{128}{3}\ ((\nabla^2\varphi)\ \varphi^{,i}_{,j} - \frac{1}{3}\ (\nabla^2\varphi)^2\ \delta^i_j) + \\ &+ \frac{890}{27}(\nabla^2\varphi)\ (\nabla\varphi)^2\ \delta^i_j - \frac{250}{9}(\nabla\varphi)^2\ \varphi^{,i}_{,j} - \frac{640}{9}(\nabla^2\varphi)\ \varphi^{,i}\ \varphi_{,j} + \\ &- \frac{380}{9}\ \varphi_{,a}\ \varphi_{,b}\ \varphi^{,ab}\ \delta^i_j + \frac{190}{3}(\varphi_{,a}\ \varphi_{,j}\ \varphi^{,ai} + \varphi_{,aj}\ \varphi^{,a}\ \varphi^{,i}) + \\ &+ \frac{1600}{27}\ ((\nabla\varphi)^2\ \varphi^{,i}\ \varphi_{,j} - \frac{1}{3}(\nabla\varphi)^2\ (\nabla\varphi)^2\ \delta^i_j)]\end{aligned} \tag{4.57}$$

4.5 Check of constraints

Expressions (4.25) and (4.31) up to two spatial gradients, and expressions (4.50) and (4.57) up to four spatial gradients are the solutions we aimed at. A possible procedure to check the coherence of these results consists in taking advantage of the ADM Constraint Equations of Chapter 2.

Momentum Constraint

We begin for semplicity testing the Momentum Constraint (2.26b):

$$\theta^a{}_{j|a} = \theta_{,j}.$$

If we check the Momentum Constraint for a gradient-velocity tensor and an expansion rate up to two derivatives terms, then we will verify an equality with three spatial gradients in every addendum because of the simple and covariant differentiation. To check the Momentum Constraint for a gradient-velocity tensor and an expansion rate expressed up to four derivatives terms, then we have to verify an equality with five spatial gradients in every addendum.

The procedure is straightforward and can be performed calculating the right and left hand sides of the equality in terms of Ψ and χ_{ij}, and then expressing everything in terms of the gravitational potential. We do not write all the passages: the Momentum Constraint is verified to both first and second order.

Energy Constraint

Verifying the ADM Energy Constraint (2.26a) is less straightforward because the density contrast δ is involved. Indeed, until now we have always tried to avoid the necessity to calculate the perturbation of the matter density expressing the equations of interest in terms of the geometrical quantities with the help of the energy constraint equation itself (see Section 2.4).
Let us write the Energy Constraint in the following form, referring to equation (3.19a):

$$\frac{2}{3}\theta^2 - 2\sigma^2 + \frac{8}{\tau}\theta + \mathcal{R} = +\frac{24}{\tau^2}\delta.$$

With the exception of δ, all the quantities in the above equation can be expressed in terms of the gravitational potential up to two or four gradients without problems. Let us then stop a little to obtain a useful expression for the density contrast.
The temporal evolution of the density contrast is governed by the following equation, which is the analogous of the continuity equation (2.20) presented in Chapter 2:

$$\delta' = -\theta\ \delta. \tag{4.58}$$

Given that $\theta = \frac{1}{2}\gamma^{ia}\gamma'_{aj} = \frac{\partial}{\partial\tau}\gamma^{1/2}$ [5], where $\gamma \equiv det\ \gamma_{ij}$, we can write the solution of (4.58) in the form

$$1+\delta = (1+\delta_{IN})\left(\frac{\gamma}{\gamma_{IN}}\right)^{-1/2}. \tag{4.59}$$

The determinant of our metric can be calculated, and at least up to 2DT reads

$$\gamma = e^{-6\Psi}. \tag{4.60}$$

Therefore, $\gamma_{IN} = e^{-6\Psi_{(0)}} = e^{-6\Psi_{IN}}$ and we can express the density contrast up to two spatial gradient.

In fact, $1+\delta = (1+\delta_{IN})\left(\frac{\gamma}{\gamma_{IN}}\right)^{-1/2} =$

$$= (1+\delta_{IN})\left(e^{-6(\Psi-\Psi_{IN})}\right)^{-1/2} = (1+\delta_{IN})\ e^{3(\Psi-\Psi_{IN})}.$$

Assuming $\delta_{IN} = 0$, the first order expression for the density contrast is

$$\delta_{(1)} = \frac{\tau^2}{6}\ e^{\frac{10}{3}\varphi}[\nabla^2\varphi - \frac{5}{6}(\nabla\varphi)^2]. \tag{4.61}$$

Similarly one can proceed to obtain the second order term for the density contrast, getting all the helpful tools for verifying the constraint. Thus, the calculation is straightforward and the outcome turned out to be successful.

Chapter 5

Comparing Perturbative Techniques. Other Results.

Having obtained the expressions for the metric coefficients Ψ and χ_{ij} in the previous Chapter, we want now to comment them briefly and show some secondary results. First of all, we see how the Gradient Expansion results are related to those of the Standard Perturbation Theory, giving the complete expression of the metric up to four spatial gradients; then we introduce the Weyl tensor and see the form that its magnetic part assumes within this expansion method.

5.1 Comparison between standard theory and gradient expansion

In order to perform a comparison among the results of the two perturbative techniques presented in this thesis, we have to write down the complete expression that the spatial metric assumes up to the second order in the respective approaches. In what follows we will label the quantities of the Standard Perturbation Theory with the superscript "ST", trying to avoid any confusion.
In the standard theory, the perturbed spatial metric in terms of Ψ and χ_{ij} up to second order reads as in (3.17), that is

$$\gamma_{ij}^{ST} = \delta_{ij} - 2\Psi_{(1)}^{ST}\delta_{ij} - \Psi_{(2)}^{ST}\delta_{ij} + \chi_{ij}^{(1)ST} + \frac{1}{2}\chi_{ij}^{(2)ST}. \tag{5.1}$$

In order to write the spatial metric in the gradient method some more attention must be paid. As done in Chapter 4, we factor out the term $e^{-2\Psi_{(0)}}$, and we expand the exponential in the function $\tilde{\Psi}$, which here formally comprise only the first and the second order terms: $\tilde{\Psi} = \Psi_{(1)} + 1/2\ \Psi_{(2)}$. Developing equation

(4.2) up to the right number of spatial gradients (four), we write

$$\begin{aligned}\gamma_{ij} &= e^{-2\Psi}(\delta_{ij}+\chi_{ij}) = \\ &= e^{-2\Psi_{(0)}}(1-2\tilde{\Psi}+2\tilde{\Psi}^2+H4DT)\,(\delta_{ij}+\chi_{ij}) = \\ &= e^{-2\Psi_{(0)}}(1-2\Psi_{(1)}-\Psi_{(2)}+2\Psi^2_{(1)})\delta_{ij}+e^{-2\Psi_{(0)}}(1-2\Psi_{(1)})(\chi^{(1)}_{ij}+\frac{1}{2}\chi^{(2)}_{ij}) = \\ &= e^{-2\Psi_{(0)}}(1-2\Psi_{(1)}-\Psi_{(2)}+2\Psi^2_{(1)})\delta_{ij}+ \\ &+ e^{-2\Psi_{(0)}}(1-2\Psi_{(1)})\chi^{(1)}_{ij}+e^{-2\Psi_{(0)}}\frac{1}{2}\chi^{(2)}_{ij}+H4DT.\end{aligned}$$

Then the spatial metric in the gradient approach up to four spatial derivatives reads

$$\begin{aligned}\gamma_{ij} &= e^{-2\Psi_{(0)}}(1-2\Psi_{(1)}-\Psi_{(2)}+2\Psi^2_{(1)})\delta_{ij}+ \\ &+ e^{-2\Psi_{(0)}}(\chi^{(1)}_{ij}+\frac{1}{2}\chi^{(2)}_{ij})-e^{-2\Psi_{(0)}}2\;\Psi_{(1)}\;\chi^{(1)}_{ij}.\end{aligned} \tag{5.2}$$

The following step consists in using expressions got in Chapters 3 and 4 in order to write the spatial metrics γ^{ST}_{ij} and γ_{ij} in terms of the peculiar gravitational potential φ. We proceed for this task and the following calculations treating separately the trace and the traceless part of the metric.

Trace part of the metric

In the Standard Theory the trace part of the spatial metric as function of the gravitational potential can be obtained substituting the equations (3.22a) and (3.24a) of $\Psi^{ST}_{(1)}$ and $\Psi^{ST}_{(2)}$ in

$$\gamma^{ST}_{ij\;(trace)} = \delta_{ij}-2\Psi^{ST}_{(1)}\delta_{ij}-\Psi^{ST}_{(2)}\delta_{ij}.$$

The resulting expression is

$$\begin{aligned}\gamma^{ST}_{ij\;trace} &= \delta_{ij}-\frac{10}{3}\varphi\;\delta_{ij}+\frac{50}{9}\varphi^2\delta_{ij}+ \\ &+\frac{\tau^2}{9}\left(-\nabla^2\varphi+\frac{5}{6}\;(\nabla\varphi)^2\right)\delta_{ij}+ \\ &+\frac{\tau^4}{252}\left(\frac{10}{3}\varphi^{,ab}\varphi_{,ab}-\left(\nabla^2\varphi\right)^2\right)\delta_{ij},\end{aligned} \tag{5.3}$$

where we have separated the different contributions according to the number of gradients (zero the first line, two the second one, four the third) and according to the powers of the gravitational potential φ ($2grad(\varphi)$ or $2grad(\varphi^2)$, and $4grad(\varphi^2)$).
The trace part of the Gradient Technique spatial metric is written using equations (4.25) and (4.50) in

$$\gamma^{(trace)}_{ij} = e^{-2\Psi_{(0)}}(1-2\Psi_{(1)}-\Psi_{(2)}+2\Psi^2_{(1)})\delta_{ij}.$$

Now, let us note that the four spatial gradients contributions to the standard metric (5.3) are of one type only, namely $4grad(\varphi^2)$: with the aim to rewrite $\gamma^{(trace)}_{ij}$ in terms of φ, we can limit ourselves to the leading terms of type

$4grad(\varphi^2)$ in the expression (4.50): hence the comparison up to four spatial derivatives will be able to control the coherence of the two approaches only up to those leading terms in gradient expansion.
Neglecting contributions of the kind $4grad(\varphi^3)$ or $4grad(\varphi^4)$, the trace of the metric is

$$\begin{aligned}\gamma_{ij}^{(trace)} &= e^{-\frac{10}{3}\varphi}\delta_{ij}+ \\ &+ \frac{\tau^2}{9}\left(-(\nabla^2\varphi)+\frac{5}{6}(\nabla\varphi)^2\right)\delta_{ij}+ \\ &+ e^{+\frac{10}{3}\varphi}\tau^4\frac{1}{252}\left(\frac{10}{3}\,\varphi_{,ab}\;\varphi^{,ab}-(\nabla^2\varphi)^2\right)\delta_{ij}.\end{aligned} \tag{5.4}$$

To see the formal equivalence of the two expressions (5.3) and (5.4) it is sufficient now to expand the exponential: this procedure adds powers of φ to the already existing terms, but not spatial gradients.

Traceless part of the metric

In the Standard Theory we obtain the traceless part of the spatial metric substituting the expression for $\chi_{ij}^{(1)ST}$ and $\chi_{ij}^{(2)ST}$ with the help of equations (3.22b) and (3.24b) in

$$\gamma_{ij\;(traceless)}^{ST} = \chi_{ij}^{(1)ST} + \frac{1}{2}\chi_{ij}^{(2)ST}.$$

The result is

$$\begin{aligned}\gamma_{ij\;(traceless)}^{ST} &= +\frac{\tau^2}{3}\left(-\varphi_{,ij}+\frac{1}{3}\nabla^2\varphi\;\delta_{ij}\right)+ \\ &+ \frac{\tau^2}{9}\left(-5\varphi_{,i}\varphi_{,j}+\frac{5}{3}\varphi^{,a}\varphi_{,a}\;\delta_{ij}\right)+ \\ &+ \frac{\tau^4}{252}\left(19\;\varphi^{ai}\varphi_{,aj}-\frac{19}{3}\varphi^{,ab}\varphi_{,ab}\;\delta_{ij}\right) \\ &+ \frac{\tau^4}{252}\left(-12\;\varphi_{,ij}\;\nabla^2\varphi+4\;(\nabla^2\varphi)^2\delta_{ij}\right),\end{aligned} \tag{5.5}$$

where the expression is manifestly traceless, and we can note the different contributions of type $2grad(\varphi)$, $2grad(\varphi^2)$ and $4grad(\varphi^2)$.
For writing the analogous formula in the Gradient Technique, we proceed as done earlier factoring out $e^{-2\Psi_{(0)}}$ and formally expanding it only at the end of the calculation. We use (4.31) and (4.57) for $\chi_{ij}^{(1)}$ and $\chi_{ij}^{(2)}$ respectively, and (4.25) for $\Psi_{(1)}$ in

$$\gamma_{ij}^{(traceless)} = e^{-2\Psi_{(0)}}(\chi_{ij}^{(1)}+\frac{1}{2}\chi_{ij}^{(2)}) - e^{-2\Psi_{(0)}}2\;\Psi_{(1)}\;\chi_{ij}^{(1)}.$$

Neglecting $4grad(\varphi^3)$ and $4grad(\varphi^4)$ terms, we get

$$\begin{aligned}\gamma_{ij}^{(traceless)} &= \frac{\tau^2}{3}\left(-\varphi_{,ij}+\frac{1}{3}\nabla^2\varphi\;\delta_{ij}\right)+\\&+\frac{\tau^2}{9}\left(-5\varphi_{,i}\varphi_{,j}+\frac{5}{3}\varphi^{,a}\varphi_{,a}\;\delta_{ij}\right)+\\&+e^{+\frac{10}{3}\varphi}\tau^4\frac{1}{252}\left(+19\;\varphi_{,aj}\;\varphi^{,ai}-\frac{19}{3}\;\varphi_{,ab}\;\varphi^{,ab}\;\delta^i_j\right)\\&+e^{+\frac{10}{3}\varphi}\tau^4\frac{1}{252}\left(-12(\nabla^2\varphi)\;\varphi^{,i}_{,j}+4\;(\nabla^2\varphi)^2\;\delta^i_j\right).\end{aligned} \tag{5.6}$$

Again the expansion of the exponential $e^{+\frac{10}{3}\varphi}$ up to its constant term shows the equivalence of the results between the two perturbative techniques [1].
Some observations can be proposed: we have seen that the comparison can be carried into effect only with an appropriate procedure consisting principally in cutting off many terms of the Gradient Expansion spatial metric. This fact reflects the property of this technique and the form of the general metric: even if Ψ is obtained up to a finite number of spatial gradients, γ_{ij} will necessary contain gradient terms of any order; in other terms, solving for the coefficients Ψ and χ_{ij} up to $2r$ spatial gradients one obtains terms of any order in the conventional perturbative expansion containing up to $2r$ gradients.
Furthermore, having in mind the complete results for Ψ and χ_{ij} up to four spatial gradients and the distinction in different terms like $4grad(\varphi^m)$, we can check that terms of order r in the expansion contain the peculiar gravitational potential φ to power m, with $2r > m > r$. We have already seen that a precise hierarchy exists among those terms according with the number of φ, that is φ_S: the dominant contribution comes from terms of the type $(\partial^2\varphi)^r$, followed by those proportional to $(\partial^2\varphi)^{r-1}(\partial\varphi)^2$. We can deduce that the actual limit of validity of our expansion at order r is set by $(\tau k)^{2r}\varphi^r \lesssim 1$: being $\varphi_S \sim 10^{-5}$, this allows us to consider also perturbations with wavelength comparable or smaller than the Hubble radius.

5.2 Weyl tensor and its magnetic part

Einstein Equations are second-order partial differential equations for $g_{\mu\nu}$ which relate the spacetime curvature expressed in terms of the Ricci tensor and the Scalar Curvature to the energy local sources described in the stress-energy tensor. The Scalar Curvature is the contraction of the Ricci tensor, which in turn is the trace over the second and fourth (or equivalently, the first and third) indices of the Riemann tensor $\mathcal{R}^{\alpha}_{\beta\mu\nu}$:

$$\mathcal{R}_{\beta\nu} = \mathcal{R}^{\rho}_{\beta\rho\nu} \text{ and } \mathcal{R} = \mathcal{R}^{\rho}{}_{\rho}$$

The trace free part of the Riemann tensor is called the *Weyl tensor*, $C_{\alpha\beta\mu\nu}$;: it has many characterizations and we introduce it for its cosmological implications.

[1] A priori the exponential $e^{+\frac{10}{3}\varphi}$ could not be expanded because φ can be as large as it wants, for the presence of contributes of the kind φ_L. Two are the possibilities to arrange this situation: one could assume $\varphi_L \equiv 0$, or the long-wavelength part of the factor $e^{+\frac{5}{3}\varphi}$, associated with each spatial gradient, can be re-absorbed by a redefinition of the spatial coordinates [37].

The Riemann tensor satisfies a series of symmetry properties:

$$\mathcal{R}_{\alpha\beta\mu\nu} = \mathcal{R}_{\mu\nu\alpha\beta} \tag{5.7a}$$

$$\mathcal{R}_{\alpha\beta\mu\nu} = -\mathcal{R}_{\beta\alpha\mu\nu} = -\mathcal{R}_{\alpha\beta\nu\mu} \tag{5.7b}$$

$$\mathcal{R}_{\alpha\beta\mu\nu} + \mathcal{R}_{\alpha\nu\beta\mu} + \mathcal{R}_{\alpha\mu\nu\beta} = 0, \tag{5.7c}$$

to which the Bianchi Identites (5.9) can be added. The set of symmetries (5.7) are such that there are $\frac{1}{12}(1+n)^2((1+n)^2-1)$ algebraically independent components of $\mathcal{R}_{\alpha\beta\mu\nu}$ [2], where $1+n$ as usual denotes the total dimension of our spacetime. $\frac{1}{2}(1+n)(n+2)$ is the number of indipendent components of the Riemann tensor that can be represented by the components of the Ricci tensor. If $n=0$, $\mathcal{R}_{\alpha\beta\mu\nu}=0$; if $n=1$, there is one independent component of $\mathcal{R}_{\alpha\beta\mu\nu}$, which is essentially the function $\mathcal{R}$. If $n=2$, the Ricci tensor (which is given algebraically by the local stress-energy tensor through E.E.) completely determines the curvature tensor. If $n \geq 3$, the remaining components of the Riemann tensor are represented by the Weyl tensor or, in other words, the Weyl tensor is that part of the Riemann tensor that cannot be obtained from the Ricci tensor: it is defined by [1]

$$\begin{aligned} C_{\alpha\beta\mu\nu} \equiv \mathcal{R}_{\alpha\beta\mu\nu} &+ \frac{2}{n-1}\left(g_{\alpha\mu}\mathcal{R}_{\nu\beta} - g_{\alpha\nu}\mathcal{R}_{\mu\beta} - g_{\alpha\mu}\mathcal{R}_{\nu\alpha} + g_{\beta\nu}\mathcal{R}_{\mu\alpha}\right) \\ &- \frac{2}{n(n-1)}\mathcal{R}\left(g_{\alpha\mu}g_{\nu\beta} - g_{\alpha\nu}g_{\mu\beta}\right). \end{aligned} \tag{5.8}$$

As the last two terms on the right hand side have the Riemann symmetries (5.7), it follows that $C_{\alpha\beta\mu\nu}$ has also these symmetries as well as it is trace free on all its indices.

An alternative characterization of the Weyl tensor is given by the fact that it behaves in a very simple manner under conformal transformations of the metric ($\hat{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu}$), and for this reason is sometimes called the *conformal tensor*, being $\hat{C}_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}$.

As the Ricci tensor is given by the E.E. and hence, physically, it gives the contribution to the spacetime curvature from local sources, then the Weyl tensor is that part of the curvature which is not determined locally by the energy distribution. For example, Newtonian tidal forces are represented in the Weyl tensor. However, the Weyl tensor cannot be entirely arbitrary: the Riemann tensor must satisfy the already mentioned *Bianchi Identities*:

$$\mathcal{R}_{\alpha\beta\mu\nu;\rho} + \mathcal{R}_{\alpha\beta\rho\mu;\nu} + \mathcal{R}_{\alpha\beta\nu\rho;\mu} = 0. \tag{5.9}$$

Using the definition (5.8), these can be rewritten as equations of motion of the Weyl tensor as follows ([7] or[2]):

$$C_{\alpha\beta\mu\nu}{}^{;\nu} = J_{\alpha\beta\mu}, \tag{5.10}$$

where (with from now on $n=3$)

$$J_{\alpha\beta\mu} \equiv \mathcal{R}_{\mu\alpha;\beta} - \mathcal{R}_{\mu\beta;\alpha} + \frac{1}{6}g_{\mu\beta}\mathcal{R}_{;\alpha} - \frac{1}{6}g_{\mu\alpha}\mathcal{R}_{;\beta}. \tag{5.11}$$

These equations are rather similar to Maxwell's equations of Electrodynamics $F_{\mu\nu}{}^{;\nu} = J_\mu$, where $F_{\mu\nu}$ is the electromagnetic field tensor and J_μ is the source current. Thus, in some sense, the Bianchi Identities of the Weyl tensor can be regarded as its field equations giving that part of the curvature at a point that depends on the matter distribution at other points.

The magnetic part of the Weyl tensor in the Gradient Technique

One can proceed with the analogy of the Electrodynamics splitting the Weyl tensor into two second-rank tensors known as the *electric* and *magnetic parts* of the Weyl tensor. Likewise in Electrodynamics $F_{\mu\nu}$ is composed of two contributions, the electric and magnetic fields E_μ and H_μ whose values and forms depend on the coordinate system, the decomposition of the Weyl tensor depends on the gauge choice, or more generally on the assumed spacetime splitting.
Adopting the usual synchronous and comoving gauge choice with the geodesic lines coinciding with the worldlines of the particles fluid, and with the normal vector field n^μ to the hypersurfaces Σ coinciding with the geodesics tangents ξ^μ and the matter 4-velocity field u^μ, the electric and magnetic parts of the Weyl tensor read, respectively,[7]

$$E_{\mu\nu} \equiv u^\alpha u^\beta C_{\mu\alpha\nu\beta} \tag{5.12a}$$

$$H_{\mu\nu} \equiv \frac{1}{2}\eta_{\alpha\beta\rho\mu}\, u^\rho u^\delta\, C^{\alpha\beta}{}_{\nu\delta} - \frac{1}{2}\eta_{\alpha\beta\rho\nu}\, u^\rho u^\delta\, C^{\alpha\beta}{}_{\mu\delta}, \tag{5.12b}$$

where $\eta_{\alpha\beta\mu\nu} \equiv (-g)^{-1/2}\epsilon_{\alpha\beta\mu\nu}$, with g being the determinant of the metric $g_{\mu\nu}$ and $\epsilon_{\alpha\beta\mu\nu}$ being the four dimensional completely antisymmetric Levi-Civita symbol. It can be shown that $E_{\mu\nu}$ and $H_{\mu\nu}$ are both symmetric, traceless, and flow-orthogonal. Therefore they have each 5 independent components, half as many as the Weyl tensor, and thanks to our gauge choice they live in the purely spatial 3-dimensional hypersurfaces at constant time Σ.
$E_{\mu\nu}$ is also called the *tidal force field*, since it contains the part of the gravitational field which describes tidal interactions: tidal forces act on the fluid flow by inducing shear distortions, and indeed the evolution equation of the shear contains as its source the electric part of the Weyl tensor [20]. The tensor $H_{\mu\nu}$ is related to that part of the gravitational field which describes gravitational waves, which have no Newtonian counterpart [22].
The magnetic part of the Weyl tensor plays an interesting role in the nonlinear dynamics of cosmological perturbations of an irrotational collisionless fluid. In fact, the dynamics of self-gravitating perfect fluid is greatly simplified under three assumptions: the fluid is collisionless ($p = 0$), it has zero vorticity, and $H_{\mu\nu} = 0$. If the former two conditions have been used throughout and are wide enough to allow for many cosmological cases, the third assumption is more problematic. If the magnetic component is switched off, all the equations for the dynamics take a strictly local form: the matter and spacetime curvature variables evolve independently along different fluid worldlines [20]. If such hypotheses were satisfied, no information could be exchanged among different fluid elements: signal exchange can occur via gravitational radiation and also via sound waves, but none of these wave modes is allowed when $p = H_{\mu\nu} = 0$. Furthermore, the condition $H_{\mu\nu} = 0$ cannot be taken as an exact constraint for the general cosmological case, not being suitable to study cosmological structure formation.
Let us then investigate the form that the magnetic part of the Weyl tensor assumes in the context of the gradient expansion. For this task, we rewrite the equation (5.12b) as follows, in line with the formalism adopted until here:

$$\mathcal{H}^i{}_j = \frac{1}{2}\gamma_{jm}\,[\eta_\gamma^{mab}\,\theta^i{}_{a|b} + \eta_\gamma^{iab}\,\theta^m{}_{a|b}], \tag{5.13}$$

where $\eta_\gamma^{abc} \equiv \gamma^{-1/2}\epsilon^{abc}$, the bar denotes covariant derivatives in the 3-space with metric γ_{ij}, and θ_{ij} is the conformal rescaled velocity-gradient tensor.
If the geometrical and physical quantities in the definition are written up to $2r$ spatial derivatives terms, then the magnetic tensor contains $2r+1$ spatial gradients, for the covariant differentiation. We have in mind the usual expansion

$$\mathcal{H}^i{}_j = \mathcal{H}^{i(0)}_{\ j} + \mathcal{H}^{i(1)}_{\ j} + \frac{1}{2}\mathcal{H}^{i(2)}_{\ j}.$$

From equation (5.13), we can already stand that, in our conventions, $\mathcal{H}^{i(0)}_{\ j} = \mathcal{H}^{i(1)}_{\ j} = 0$. In fact, the lowest spatial derivative contribution to $\mathcal{H}_{ij}$ is of the form

$$\mathcal{H}^i_j \propto \frac{1}{2}\gamma^{(0)}_{jm}\ e^{3\Psi}\ \epsilon^{mab}\ \theta^{i(1)}_{\ a\ \ |b},$$

where we have used that the determinant of the spatial metric is $\gamma = e^{-6\Psi}$. But $\theta^{i(1)}_{\ a\ \ |b}$ is at least a 3DT, thus up to two spatial gradients there is no contribution to the magnetic part of the Weyl tensor.
The second order term can then be calculated as usual using the results obtained in Chapter 4. Up to second order, the term $\theta^{i(1)}_{\ a\ \ |b}$ can contain at most 3 spatial gradients, and reads

$$\begin{aligned}\theta^{i(1)}_{\ a|b} = \tau e^{\frac{10}{3}\varphi}[& -\frac{1}{3}\varphi^{,i}_{,ab} + \\ & + \frac{5}{9}\varphi^{,n}\varphi^{,i}_{,n}\delta_{ab} - \frac{5}{9}\varphi^{,i}\varphi_{,ab} + \\ & - \frac{5}{9}\varphi_{,a}\varphi^{,i}_{,b} + \frac{5}{9}\varphi^{,n}\varphi_{,na}\delta^i_b].\end{aligned} \tag{5.14}$$

Now, the first three terms of equation (5.14) are symmetric for exchange of indices a and b: therefore, because of the presence of the Levi-Civita symbol in the definition (5.13) , they do not contribute to the magnetic tensor $\mathcal{H}^i{}_j$. The latter, up to our second order, is different from being null and reads

$$\mathcal{H}^i{}_j = \frac{\tau}{2}e^{\frac{15}{3}\varphi}\delta_{jm}[\epsilon^{mab}(-\frac{5}{9}\varphi_{,a}\varphi^{,i}_{,b} + \frac{5}{9}\varphi^{,n}\varphi_{,na}\delta^i_b) + \epsilon^{iab}(-\frac{5}{9}\varphi_{,a}\varphi^{,m}_{,b} + \frac{5}{9}\varphi^{,n}\varphi_{,na}\delta^m_b)]. \tag{5.15}$$

The magnetic part of the Weyl tensor does not contain terms with a single φ, that is $\mathcal{H}^i{}_j(3grad(\varphi)) = 0$.

Conclusions

Approximation methods have been and are very important in General Relativity and its applications to Cosmology and Relativistic Astrophysics. In this thesis we have presented the so-called Gradient Expansion Technique, computing the expressions up to four spatial gradients of the perturbative functions Ψ and χ_{ij} in an irrotational matter-dominated universe.

Our gradient expansion approach is slightly different from the ones already existing in literature: we have perturbed Einstein Equations in a given precise gauge rather than beginning with a relativistic action principle; we have written the spatial metric γ_{ij} with the scalar perturbative function Ψ appearing in the argument of an exponential and allowing the FRW background solution to have a spatial dependence; finally we have solved Einstein Equations in the form of evolution equations of the ADM formalism, and we have set the initial conditions as provided by standard Inflation.

The Gradient Expansion Technique has shown itself to be much more handy than the standard one, for the simplicity and relative brevity of the computations. Furthermore, this approximation methods has shown itself to be non-perturbative in the sense that by solving for the metric coefficients Ψ and χ_{ij} up to $2r$ spatial gradients one obtains terms of any order in the standard perturbative expansion containing up to $2r$ spatial gradients.

Our particular approach allowed us to compare quite directly the results obtained in the Gradient Expansion with those of the Standard Theory: the comparison has shown the coherence of the two sets of results, and hence the consistency of the method.

Thanks to the wide wavelength-range of validity of the Gradient Expansion, this scheme is suitable to study the large-scale structure formation and issues related with it, from the study of perturbations generation during Inflation, to the problem of the backreaction, and the derivation of the Zel'dovich approximation for General Relativity describing the formation of pancake structure in matter-dominated universes [27], [28], [29], [37].

A possible further development of the work presented in this thesis could be the exstension of the computations in our approach in the case of a universe dominated by the cosmological constant Λ, in line with the standard cosmological model of the present universe, or in the case of a scalar field dominated universe.

Appendix A

Decomposition of spatial vectors and tensors

In order to study perturbations on the invariant $n-$space Σ, we first classify them into three groups on the basis of their behaviour under the transformation of space-coordinate x^k: the *scalar type*, *vector type* and *tensor type*.

A vector quantity v^i on Σ can be decomposed as

$$v^i = \partial^i v + v_\perp^i \text{ such that } \partial_i \, v_\perp^i = 0. \tag{A.1}$$

v represents the scalar (or longitudinal or irrotational) component of the space-vector v^i, while $v_\perp^i$ represents the transverse (divergence-free or solenoidal) proper vector part of it.

Similarly, a symmetric traceless second-rank tensor T_{ij} on Σ can be decomposed into a sum of parts, called longitudinal, solenoidal, and transverse:

$$T_{ij} = D_{ij} \; T + (\partial_i T_j^\perp + \partial_j T_i^\perp) + T_{ij}^\top \tag{A.2}$$

with (in the case $n = 3$)

$$D_{ij} \equiv \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \tag{A.3a}$$

$$\partial^i T_i^\perp = 0 \tag{A.3b}$$

$$\partial^i T_{ij}^\top = 0. \tag{A.3c}$$

The longitudinal tensor T is also called the scalar part of T_{ij}, the solenoidal part $T_j^\perp$ is also called the vector part, and the transverse-traceless part $T_{ij}^\top$ is also called the tensor part of the spatial-tensor on Σ.

For a more general decomposition of non-traceless tensors see [6]. Let us note that the decomposition in scalar, vector and tensor parts of a spatial tensor is not unique: T and $T_i^\perp$ are defined only up to a constant, and additional freedom may appear [7].

Appendix B

Synchronous gauge: geometrical quantities

FRAME:

$$dS^2 = -dt^2 + h_{ij}(t, \vec{x})\ dx^i dx^j$$

$$\Theta^i{}_j = \frac{1}{2}\ h^{ia}\ \dot{h}_{aj}$$

$$(t - coordinates)$$

Christoffel Symbols

$$\Gamma^0_{00} = \Gamma^0_{0j} = \Gamma^j_{00} = 0$$

$$\Gamma^0_{ij} = \Theta_{ij}; \quad \Gamma^i_{0j} = \Theta^i{}_j; \quad \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk}$$

Riemann Tensor

$$\mathcal{R}^0_{000} = \mathcal{R}^j_{000} = \mathcal{R}^0_{00j} = 0$$

$$\mathcal{R}^0_{i0j} = \dot{\Theta}_{ij} - \Theta_{aj}\ \Theta^a{}_i; \quad \mathcal{R}^i_{00j} = \dot{\Theta}^i{}_j + \Theta^i{}_a\ \Theta^a{}_j; \quad \mathcal{R}^0_{0ij} = \Theta_{ai}\ \Theta^a{}_j - \Theta_{aj}\ \Theta^a{}_i$$

$$\mathcal{R}^0_{ijk} = -\Theta_{ij,k} + \Theta_{ik,j} + \Theta_{aj}\ {}^{(3)}\Gamma^a_{ik} - \Theta_{ak}\ {}^{(3)}\Gamma^a_{ij}$$

$$\mathcal{R}^i_{0jk} = -\Theta^i{}_{j,k} + \Theta^i{}_{k,j} + \Theta^a{}_k\ {}^{(3)}\Gamma^i_{aj} - \Theta^a{}_j\ {}^{(3)}\Gamma^i_{ak}$$

$$\mathcal{R}^i_{j0k} = -\Theta^i{}_{j,k} + {}^{(3)}\Gamma^i_{jk,0} + \Theta^i{}_a\ {}^{(3)}\Gamma^a_{jk} - \Theta^a{}_j\ {}^{(3)}\Gamma^i_{ak}$$

$$\mathcal{R}^i_{jkl} = {}^{(3)}\mathcal{R}^i_{jkl} + \Theta^i{}_k\ \Theta_{jl} - \Theta^i{}_l\ \Theta_{jk}$$

Ricci Tensor

$$\mathcal{R}_{00} = -\dot{\Theta} - \Theta^a{}_b\ \Theta^b{}_a; \quad \mathcal{R}^0{}_0 = -\dot{\Theta} + \Theta^a{}_b\ \Theta^b{}_a$$

$$\mathcal{R}_{0i} = \Theta^a{}_{i|a} - \Theta_{|i}$$

$$\mathcal{R}_{ij} = {}^{(3)}\mathcal{R}_{ij} + \dot{\Theta}_{ij} - 2\Theta_{aj}\ \Theta^a{}_i + \Theta\ \Theta_{ij}$$

$$\mathcal{R}^i{}_j = {}^{(3)}\mathcal{R}^i{}_j + \dot{\Theta}^i{}_j + \Theta\ \Theta^i{}_j$$

Scalar Curvature

$$\mathcal{R} = {}^{(3)}\mathcal{R} + 2\dot{\Theta} + \Theta^2 + \Theta^a{}_b\ \Theta^b{}_a$$

Appendix C

Different orders contributions to the calculated quantities

In the text, the geometrical and physical quantities of interests have been expressed in terms of the pertubative functions Ψ and χ and their derivatives. We want in this Appendix to work on them in order to distinguish the different contributions to different orders in gradient content.
The results of this procedure will be really useful for performing the calculations.

Velocity-gradient tensor and expansion rate

Having in mind an expansion for θ^i_j and θ like

$$\theta = \theta_{(0)} + \theta_{(1)} + \frac{1}{2}\theta_{(2)} \text{ and } \theta^i_j = \theta^{i\,(0)}_j + \theta^{i\,(1)}_j + \frac{1}{2}\theta^{i\,(2)}_j,$$

we can expand equations (4.13) and (4.14) as follow

$$\theta^i_j = -(\Psi_{(0)} + \Psi_{(1)})'\delta^i_j + \frac{1}{2}\chi^{i(1)\prime}_j = \theta^{i\,(0)}_j + \theta^{i\,(1)}_j,$$
$$\theta = -(\Psi_{(0)} + \Psi_{(1)})' = -3\Psi'_{(1)} = \theta_{(0)} + \theta_{(1)},$$

where we note that $\Psi_{(0)}$ does not dipend on time and hence $\theta^{i\,(0)}_j$ and $\theta_{(0)}$ are null.
Up to 2^{nd} order, θ^i_j and θ are given by (4.34) and (4.35). Analogously, we procede and separate the different order contributions:

$$\theta^i_j = -(\Psi_{(0)} + \Psi_{(1)} + \frac{1}{2}\Psi_{(2)})'\delta^i_j + \frac{1}{2}(\chi^i_{j(1)} + \frac{1}{2}\chi^i_{j(2)})' - \frac{1}{2}\chi^{ia}_{(1)}\ \chi^{(1)\prime}_{aj} =$$
$$= -\Psi'_{(1)}\delta^i_j - \frac{1}{2}\Psi'_{(2)}\delta^i_j + \frac{1}{2}{\chi^i_{j(1)}}' + \frac{1}{2}\frac{1}{2}{\chi^i_{j(2)}}' - \frac{1}{2}\chi^{ia}_{(1)}\ \chi^{(1)\prime}_{aj} =$$
$$= (-\Psi'_{(1)}\delta^i_j + \frac{1}{2}{\chi^i_{j(1)}}') + (-\frac{1}{2}\Psi'_{(2)}\delta^i_j + \frac{1}{2}\frac{1}{2}{\chi^i_{j(2)}}' - \frac{1}{2}\chi^{ia}_{(1)}\ \chi^{(1)\prime}_{aj}) =$$
$$= \theta^{i\,(1)}_j + \frac{1}{2}\theta^{i\,(2)}_j.$$

$$\theta = -3\Psi' - \frac{1}{2}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab} = -3(\Psi_{(1)} + \frac{1}{2}\Psi_{(2)})' - \frac{1}{2}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab} =$$
$$= -3\Psi'_{(1)} - 3\frac{1}{2}\Psi'_{(2)} - \frac{1}{2}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab} = \theta_{(1)} + \frac{1}{2}\theta_{(2)}.$$

So we can conclude that the *velocity-gradient tensor* can be written as follows

$$\theta^i_j = \theta^{i\,(1)}_j + \frac{1}{2}\theta^{i\,(2)}_j \text{ with} \tag{C.1a}$$
$$\theta^i_{j\,(1)} = -\Psi'_{(1)}\delta^i_j + \frac{1}{2}\chi^i_{j(1)}{}' \tag{C.1b}$$
$$\frac{1}{2}\theta^{i\,(2)}_j = \frac{1}{2}[-\Psi'_{(2)}\delta^i_j + \frac{1}{2}\chi^i_{j(2)}{}' - \chi^{ia}_{(1)}\ \chi^{(1)\prime}_{aj}]. \tag{C.1c}$$

And the *expansion rate* reads

$$\theta = \theta_{(1)} + \frac{1}{2}\theta_{(2)} \text{ where} \tag{C.2a}$$
$$\theta_{(1)} = -3\Psi'_{(1)} \tag{C.2b}$$
$$\frac{1}{2}\theta_{(2)} = \frac{1}{2}[-3\Psi'_{(2)} - \chi^{ab}_{(1)}\chi^{(1)\prime}_{ab}]. \tag{C.2c}$$

Shear

Frim 4.15) we see that $\sigma^i_j = \sigma^i_{j(1)}$. Up to 2^{nd} order, from (4.36) we write

$$\sigma^i_j = \frac{1}{2}(\chi^i_{j\,(1)} + \frac{1}{2}\chi^i_{j\,(2)})' - \frac{1}{2}\chi^{ia}_{(1)}\chi^{(1)\prime}_{aj} + \frac{1}{6}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab}\delta^i_j =$$
$$= \frac{1}{2}\chi^{i\,\prime}_{j\,(1)} + \frac{1}{2}[\frac{1}{2}\chi^{i\,\prime}_{j\,(2)} - \chi^{ia}_{(1)}\chi^{(1)\prime}_{aj} + \frac{1}{3}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab}\delta^i_j].$$

Therefore for the *shear* we conclude

$$\sigma^i_j = \sigma^i_{j(1)} + \frac{1}{2}\sigma^i_{j(2)} \text{ with} \tag{C.3a}$$
$$\sigma^i_{j(1)} = \frac{1}{2}\chi^{i\,\prime}_{j\,(1)} \tag{C.3b}$$
$$\frac{1}{2}\sigma^i_{j(2)} = \frac{1}{2}[\frac{1}{2}\chi^{i\,\prime}_{j\,(2)} - \chi^{ia}_{(1)}\chi^{(1)\prime}_{aj} + \frac{1}{3}\chi^{ab}_{(1)}\chi^{(1)\prime}_{ab}\delta^i_j]. \tag{C.3c}$$

Ricci Tensor

At 1^{st} order, the Ricci tensor is given by (4.17) with $\Psi = \Psi_{(0)}$. Because $\Psi_{(0)} = \Psi(\tau = 0)$, then we could even call $\Psi_{(0)} = \Psi_{IN}$ and write $\mathcal{R}^{(1)}_{jm} = \mathcal{R}^{IN}_{jm}$.
Up to 2^{nd} order we can split $\mathcal{R}_{jm}$ in two ways, according with the order or according with the argument:

$$\mathcal{R}_{jm} = \mathcal{R}^{(0)}_{jm} + \mathcal{R}^{(1)}_{jm} + \frac{1}{2}\mathcal{R}^{(2)}_{jm}$$

$$\mathcal{R}_{jm} = \mathcal{R}_{jm}(\Psi) + \mathcal{R}_{jm}(\chi) + \mathcal{R}_{jm}(\Psi \cdot \chi),$$

where the zeroth order term $\mathcal{R}^{(0)}_{jm}$ is null. But (4.38) suggests the gradient content of single addenda:

- $\mathcal{R}_{jm}(\Psi)$ contains 2DTs and 4DTs
- $\mathcal{R}_{jm}(\chi)$ contains 4DTs
- $\mathcal{R}_{jm}(\Psi\cdot\chi)$ contains 4DTs.

Therefore, a making sense expansion for the Ricci Tensor is

$$\mathcal{R}_{jm} = \mathcal{R}^{(1)}_{jm}(\Psi) + \frac{1}{2}\mathcal{R}^{(2)}_{jm}(\Psi) + \frac{1}{2}\mathcal{R}^{(2)}_{jm}(\chi) + \frac{1}{2}\mathcal{R}^{(2)}_{jm}(\Psi\cdot\chi), \tag{C.4}$$

where

$$\mathcal{R}^{(1)}_{jm}(\Psi) = \Psi^{(0)}_{,jm} + (\nabla^2\Psi_{(0)})\delta_{jm} + \Psi^{(0)}_{,j}\Psi^{(0)}_{,m} - (\nabla\Psi_{(0)})^2\delta_{jm} \tag{C.5a}$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{jm}(\Psi) = \Psi^{(1)}_{,jm} + (\nabla^2\Psi_{(1)})\delta_{jm} + \Psi^{(0)}_{,j}\Psi^{(1)}_{,m} + \Psi^{(1)}_{,j}\Psi^{(0)}_{,m} + -2(\partial_a\Psi_{(0)})(\partial^a\Psi_{(1)})\delta_{jm} \tag{C.5b}$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{jm}(\chi) = +\frac{1}{2}(\chi^a_{j,ma} + \chi^a_{m,ja} - \nabla^2\chi_{jm}) \text{ with} \chi_{ij} = \chi^{(1)}_{ij} \tag{C.5c}$$

$$\begin{aligned}\frac{1}{2}\mathcal{R}^{(2)}_{jm}(\Psi\chi) =& (\nabla^2\Psi_{(0)})\chi_{jm} - \Psi^{(0)}_{,ab}\,\chi^{ab}\,\delta_{jm} - \Psi^{(0)}_{,a}\,\chi^{ab}_{,b}\,\delta_{jm}\\ &+\frac{1}{2}\Psi^{;a}_{(0)}(-\chi_{am,j} - \chi_{aj,m} + \chi_{mj,a}) + \\ &-(\nabla\Psi_{(0)})^2\chi_{jm} + \Psi^{(0)}_{,a}\,\Psi^{(0)}_{,b}\,\chi^{ab}\,\delta_{jm}.\end{aligned} \tag{C.5d}$$

Scalar Curvature

We can apply the same procedure to the Scalar Curvature $\mathcal{R}$. At first order (two derivatives), it reads as in (4.18) with $\Psi = \Psi_{(0)} = \Psi(\tau = 0)$: then we can write $\mathcal{R}_{(1)} = \mathcal{R}_{\mathcal{IN}}$. At second order two different splittings can be made:

$$\mathcal{R} = \mathcal{R}^{(0)} + \mathcal{R}^{(1)} + \frac{1}{2}\mathcal{R}^{(2)}$$

$$\mathcal{R} = \mathcal{R}(\Psi) + \mathcal{R}(\chi) + \mathcal{R}(\Psi\cdot\chi),$$

where the zeroth order term $\mathcal{R}^{(0)}$ is null. Similarly to the Ricci tensor case, the complete expression for the scalar curvature (4.39) suggests that

- $\mathcal{R}(\Psi)$ contains 2DTs and 4DTs
- $\mathcal{R}(\chi)$ contains 4DTs
- $\mathcal{R}(\Psi\cdot\chi)$ contains 4DTs.

Thus we can write

$$\mathcal{R} = \mathcal{R}^{(1)}(\Psi) + \frac{1}{2}\mathcal{R}^{(2)}(\Psi) + \frac{1}{2}\mathcal{R}^{(2)}(\chi) + \frac{1}{2}\mathcal{R}^{(2)}(\Psi\cdot\chi), \tag{C.6}$$

where

$$\mathcal{R}^{(1)}(\Psi) \quad = e^{2\Psi_{(0)}}[4\nabla^2\Psi_{(0)} - 2(\nabla\Psi_{(0)})^2] \tag{C.7a}$$

$$\begin{aligned}\frac{1}{2}\mathcal{R}^{(2)}(\Psi) \quad =& e^{2\Psi_{(0)}}[4\nabla^2\Psi_{(1)} - 4(\partial^a\Psi_{(0)}\ \partial_a\Psi_{(1)})]+ \\ & e^{2\Psi_{(0)}}(2\Psi_{(1)})[4(\nabla^2\Psi_{(0)}) - 2(\nabla\Psi_{(0)})^2]\end{aligned} \tag{C.7b}$$

$$\frac{1}{2}\mathcal{R}^{(2)}(\chi) \quad = + e^{2\Psi_{(0)}}[\chi^{ab}_{(1),ab}] \tag{C.7c}$$

$$\frac{1}{2}\mathcal{R}^{(2)}(\Psi\chi) = e^{2\Psi_{(0)}}[-4\chi^{ab}_{(1)}\ \Psi^{(0)}_{,ab} - 4\chi^{ab}_{(1),b}\ \Psi^{(0)}_{,a} + 2\chi^{ab}_{(1)}\ \Psi^{(0)}_{,a}\ \Psi^{(0)}_{,b}]. \tag{C.7d}$$

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