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Random vibration of linear and nonlinear structural systems with singular matrices: A frequency domain approach

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ABSTRACT

A frequency domain methodology is developed for stochastic response determination of multi-degree-of-freedom (MDOF) linear and nonlinear structural systems with singular matrices. This system modeling can arise when a greater than the minimum number of coordinates/DOFs is utilized, and can be advantageous, for instance, in cases of complex multibody systems where the explicit formulation of the equations of motion can be a nontrivial task. In such cases, the introduction of additional/redundant DOFs can facilitate the formulation of the equations of motion in a less labor intensive manner. Specifically, relying on the generalized matrix inverse theory, a Moore-Penrose (M-P) based frequency response function (FRF) is determined for a linear structural system with singular matrices. Next, relying on the M-P FRF a spectral input-output (excitation-response) relationship is derived in the frequency domain for determining the linear system response power spectrum. Further, the above methodology is extended via statistical linearization to account for nonlinear systems. This leads to an iterative determination of the system response mean vector and covariance matrix. Furthermore, to account for singular matrices, the generalization of a widely utilized formula that facilitates the application of statistical linearization is proved as well. The formula relates to the expectation of the derivatives of the system nonlinear function and is based on a Gaussian response assumption. Several linear and nonlinear MDOF structural systems with singular matrices are considered as numerical examples for demonstrating the validity and applicability of the developed frequency domain methodology.

1. Introduction

The dynamic analysis of systems subjected to stochastic excitations has been extensively studied over the last decades; see for instance Refs. [1,2] and [3] for some indicative books, as well as Refs. [4] and [5] for some recent techniques related to the path integral concept. In general, in the field of random vibration of structural systems [6] modeling the system by utilizing the minimum number of coordinates (generalized coordinates) yields not only non-singular, but also positive

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definite matrices. Note, however, that an alternative modeling of the system equations of motion that employs additional/redundant degrees-of-freedom (DOFs)/coordinates may be preferable, particularly in the field of multi-body system dynamics, for a number of reasons. These may include decreased complexity and computational cost associated with the formulation of the equations of motion. Thus, although formulating the equations of motion by employing redundant DOFs yields singular system matrices, this alternative modeling scheme appears advantageous in many cases; see Refs. [7–15] for a detailed discussion on the topic.

Clearly, determining the dynamic response of structural systems with singular matrices poses significant challenges as standard solution techniques such as those based on a state-variable formulation cannot be utilized, at least in a straightforward manner. In this regard, relying on the concept of the Moore-Penrose (M-P) generalized inverse Udwadia and co-workers (e.g. [16]) determined the dynamic response of systems with singular matrices subject to deterministic excitation. Subsequently, the authors developed in Refs. [17,18] generalized random vibration time-domain techniques for determining the response of linear and nonlinear structural systems subject to stochastic excitations; see also Ref. [19] for an alternative treatment based on polynomial matrix theory.

In this paper, standard frequency domain random vibration solution methodologies (e.g. [6]) are generalized to account for systems with singular matrices. To this aim, based on the theory of generalized matrix inverses, an M-P based frequency response function (FRF) is derived for a structural system with singular matrices. Next, relying on the M-P FRF the celebrated standard input-output (excitation-response) relationship in the frequency domain is generalized for determining the system response power spectrum. Finally, the above derived frequency domain relationship is utilized in conjunction with a recently developed statistical linearization technique [18] for determining the response statistics of nonlinear systems with singular matrices. The validity of the herein developed frequency domain random vibration techniques is demonstrated by pertinent numerical examples including several linear and nonlinear systems with singular matrices.

2. Moore-Penrose theory elements

In this section, some elements of the generalized matrix inverse theory pertaining to the Moore-Penrose (M-P) matrix inverse, are provided for completeness.

Definition 1. If $\mathbf{A} \in \mathbb{C}^{m\times n}$ then $\mathbf{A}^{+}$ is the unique matrix in $\mathbb{C}^{n\times m}$ so that

$$\begin{aligned} &\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}, \quad &&\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}, \\ &(\mathbf{A}\mathbf{A}^{+})^{*} = \mathbf{A}\mathbf{A}^{+}, \quad &&(\mathbf{A}^{+}\mathbf{A})^{*} = \mathbf{A}^{+}\mathbf{A}. \end{aligned} \tag{1}$$

The matrix $\mathbf{A}^{+}$ is known as the M-P inverse of $\mathbf{A}$ and Eq. (1) represents the so-called M-P equations. In general, the M-P inverse of a square matrix exists for any arbitrary $\mathbf{A} \in \mathbb{C}^{n\times n}$, and if $\mathbf{A}$ is non-singular, $\mathbf{A}^{+}$ coincides with $\mathbf{A}^{-1}$. Further, the M-P inverse of any $m \times n$ matrix $\mathbf{A}$ can be determined, for instance, via a number of recursive formulae (e.g., [20,21]), and provides a tool for solving equations of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2}$$

where $\mathbf{A}$ is a rectangular $m \times n$ matrix, $\mathbf{x}$ is an n vector and $\mathbf{b}$ is an m vector. For a singular square matrix $\mathbf{A}$, i.e. $det\mathbf{A} = 0$, utilizing the M-P inverse, Eq. (2) yields

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{y}, \tag{3}$$

where $\mathbf{y}$ is an arbitrary n vector and $\mathbf{I}$ is the identity matrix. A more detailed presentation of the topic can be found in Refs. [20] and [22].

3. Frequency domain stochastic response analysis of linear systems with singular matrices

In this section, the response of linear systems with singular matrices subject to stochastic excitation is determined via a frequency domain approach. Note that the herein developed frequency domain response analysis methodology can be construed as an alternative to a recently developed time domain technique [17].

3.1. Linear systems with standard non-singular matrices

Some elements of the frequency domain stochastic response analysis of systems with standard non-singular matrices are provided in the following for completeness. In this regard, the statistics of the system response, $\mathbf{q}(t)$, to an external excitation, $\mathbf{Q}(t)$, are determined in the frequency domain by utilizing input-output relationships, involving the FRF matrix $\boldsymbol{\alpha}(\omega)$ [6]. Specifically, consider the equations of motion of an n-DOF linear system given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t), \tag{4}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ denote the $n \times n$ mass, damping and stiffness matrices of the system, respectively, and $\mathbf{q}$ corresponds to the n (generalized) coordinates vector. The n vector $\mathbf{Q}$ denotes the excitation vector that is applied to the system. Note that utilizing generalized coordinates for formulating the system equations of motion yields matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ that are not only non-singular, but also symmetric and positive definite. Next, to determine the system FRF matrix $\boldsymbol{\alpha}(\omega)$, consider an excitation of the form

$$\mathbf{Q}(t) = \mathbf{Q}_0 \exp(i\omega t), \tag{5}$$

where ω denotes the frequency and $\mathbf{Q}_0$ is an amplitude vector. Considering next the response displacement vector to be of the form

$$\mathbf{q}(t) = \boldsymbol{\alpha}(\omega)\mathbf{Q}(t), \tag{6}$$

where $\boldsymbol{\alpha}(\omega)$ is the $n \times n$ FRF matrix and substituting Eqs. (5)–(6) into Eq. (4) yields

$$\boldsymbol{\alpha}(\omega) = \mathbf{R}^{-1}, \tag{7}$$

where

$$\mathbf{R} = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}. \tag{8}$$

Further, a spectral excitation-response (input-output) relationship can be determined by utilizing the FRF matrix of Eq. (7) in the form

$$\mathbf{S}_{\mathbf{q}}(\omega) = \boldsymbol{\alpha}(\omega)\mathbf{S}_{\mathbf{Q}}(\omega)\boldsymbol{\alpha}^{\mathrm{T}*}(\omega), \tag{9}$$

where $\mathbf{S}_{\mathbf{q}}(\omega)$ and $\mathbf{S}_{\mathbf{Q}}(\omega)$ are the system response and excitation power spectrum matrices, respectively, $\mathbf{Q}(t)$ represents an arbitrary stationary stochastic vector process, and $\boldsymbol{\alpha}^{\mathrm{T}*}(\omega)$ denotes the conjugate transpose of $\boldsymbol{\alpha}(\omega)$; see Ref. [6] for a more detailed presentation. Furthermore, system response second-order statistics can be readily determined based on Eq. (9). For instance, utilizing Eq. (9) the response displacement and velocity moments $\mathrm{E}\left[q_i^2(t)\right]$ and $\mathrm{E}\left[\dot{q}_i^2(t)\right]$ are given, respectively, by

$$\mathrm{E}\left[q_i^2(t)\right] = \int_{-\infty}^{\infty} S_{q_i q_i}(\omega)\mathrm{d}\omega \tag{10}$$

and

$$\mathrm{E}\left[\dot{q}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 S_{q_i q_i}(\omega)\mathrm{d}\omega. \tag{11}$$

3.2. Linear systems with singular matrices

It can be argued that there are cases where utilizing more than the minimum number (redundant) degrees-of-freedom (DOFs) for formulating the equations of motion of a complex dynamical system can be advantageous, especially from a computational efficiency perspective; see Refs. [17,13] for a detailed discussion. In this regard the $n-$ DOF system of Eq. (4) can be alternatively modeled as an $l-$ DOF system ($l \geq n$) of the form

$$\mathbf{M}_{\mathbf{x}}\ddot{\mathbf{x}} + \mathbf{C}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{K}_{\mathbf{x}}\mathbf{x} = \mathbf{Q}_{\mathbf{x}}(t). \tag{12}$$

In Eq. (12), $\mathbf{M}_{\mathbf{x}}$, $\mathbf{C}_{\mathbf{x}}$ and $\mathbf{K}_{\mathbf{x}}$ are the $l \times l$ mass, damping and stiffness matrices, respectively, $\mathbf{x}$ is the l coordinates vector and $\mathbf{Q}_{\mathbf{x}}$ is the l vector of external forces. Note that due to the utilization of additional/redundant DOFs, $\mathbf{M}_{\mathbf{x}}$, $\mathbf{C}_{\mathbf{x}}$, and $\mathbf{K}_{\mathbf{x}}$ are singular matrices. Moreover, constraint equations

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{13}$$

where $\mathbf{A}$ is an $m \times l$ matrix, need to be included as well [16]. Next, considering (for convenience and without loss of generality) the vector $\mathbf{b}$ to be of the form

$$\mathbf{b} = \mathbf{F} - \mathbf{E}\dot{\mathbf{x}} - \mathbf{L}\mathbf{x}, \tag{14}$$

the original system of Eq. (4) can be alternatively modeled, via employing the redundant coordinates vector $\mathbf{x}$, as

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{15}$$

where the $(m + l) \times l$ matrices $\bar{\mathbf{M}}_{\mathbf{x}}$, $\bar{\mathbf{C}}_{\mathbf{x}}$ and $\bar{\mathbf{K}}_{\mathbf{x}}$ denote the augmented mass, damping and stiffness matrices for the system, defined as

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{M}_{\mathbf{x}} \\ \mathbf{A} \end{bmatrix}, \tag{16}$$

$$\bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{C}_{\mathbf{x}} \\ \mathbf{E} \end{bmatrix} \tag{17}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{K}_{\mathbf{x}} \\ \mathbf{L} \end{bmatrix}, \tag{18}$$

respectively. Also, the $(m + l)$ augmented excitation vector is given by

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{Q}_{\mathbf{x}} \\ \mathbf{F} \end{bmatrix}. \tag{19}$$

A detailed presentation of the derivation of Eq. (15) can be found in Refs. [17,18], and is also outlined, for convenience, in Appendix A.

Next, focusing on the frequency domain, the problem of determining the FRF matrix of a system with singular mass, damping and stiffness matrices is considered. In this regard, the system of Eq. (15) is excited by a harmonic force of the form defined in Eq. (5). The system response is given by

$$\mathbf{x}(t) = \boldsymbol{\alpha}_{\mathbf{x}}(\omega)\bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{20}$$

where $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$ is the $l \times (m + l)$ FRF matrix. Next, Eq. (20) is differentiated twice with respect to time and the obtained expressions, along with Eq. (20), are substituted in Eq. (15) yielding

$$\mathbf{R}_{\mathbf{x}}\boldsymbol{\alpha}_{\mathbf{x}}(\omega) = \mathbf{I}. \tag{21}$$

In Eq. (21) the $(m + l) \times l$ matrix $\mathbf{R}_{\mathbf{x}}$ is given by

$$\mathbf{R}_{\mathbf{x}} = -\omega^2\bar{\mathbf{M}}_{\mathbf{x}} + i\omega\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}. \tag{22}$$

Further, the M-P inverse of the matrix $\mathbf{R}_{\mathbf{x}}$ is employed for solving Eq. (21). Specifically, utilizing Eq. (3), the FRF matrix takes the form

$$\boldsymbol{\alpha}_{\mathbf{x}}(\omega) = \mathbf{R}_{\mathbf{x}}^{+} + (\mathbf{I} - \mathbf{R}_{\mathbf{x}}^{+}\mathbf{R}_{\mathbf{x}})\mathbf{Y}, \tag{23}$$

where $\mathbf{R}_{\mathbf{x}}^{+}$ is the $l \times (m + l)$ M-P inverse of $\mathbf{R}_{\mathbf{x}}$ and $\mathbf{Y}$ is an arbitrary $l \times (m + l)$ matrix.

It is noted that the presence of the arbitrary matrix $\mathbf{Y}$ on the right hand side of Eq. (23) yields a non-unique solution for the FRF matrix. Nevertheless, depending on the rank of $\mathbf{R}_{\mathbf{x}}$, a uniquely defined FRF matrix can be derived. Specifically, any $m \times n$ matrix $\mathbf{E}$ can be written as

$$\mathbf{E} = \mathbf{F}\mathbf{G}, \tag{24}$$

where the $m \times r$ matrix $\mathbf{F}$ has full column rank, i.e. $rank\mathbf{F} = r$, and the $r \times n$ matrix $\mathbf{G}$ has full row rank, i.e. $rank\mathbf{G} = r$; the expression given by Eq. (24) corresponds to the so-called full rank factorization of an arbitrary matrix [22,23]. Then, it can be proved that the M-P inverse of $\mathbf{E}$ is given by

$$\mathbf{E}^{+} = \mathbf{G}^{*}(\mathbf{F}^{*}\mathbf{E}\mathbf{G}^{*})^{-1}\mathbf{F}^{*}, \tag{25}$$

where the symbol '*' denotes the conjugation operator; a detailed proof of Eq. (25) can be found in Ref. [22]. Therefore, it is readily seen that if $\mathbf{R}_{\mathbf{x}}$ has full rank, its M-P inverse takes the form

$$\mathbf{R}_{\mathbf{x}}^{+} = (\mathbf{R}_{\mathbf{x}}^{*}\mathbf{R}_{\mathbf{x}})^{-1}\mathbf{R}_{\mathbf{x}}^{*} \tag{26}$$

and taking into account Eq. (26), the expression

$$\mathbf{I} - \mathbf{R}_{\mathbf{x}}^{+}\mathbf{R}_{\mathbf{x}} = \mathbf{0}, \tag{27}$$

holds true. Combining Eq. (27) with Eq. (23), the FRF matrix is uniquely defined as

$$\boldsymbol{\alpha}_{\mathbf{x}}(\omega) = \mathbf{R}_{\mathbf{x}}^{+}. \tag{28}$$

Next, following Ref. [6] the standard spectral excitation-response relationship of Eq. (9) is generalized and given in the form

$$\mathbf{S}_{\mathbf{x}}(\omega) = \boldsymbol{\alpha}_{\mathbf{x}}(\omega)\mathbf{S}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(\omega)\boldsymbol{\alpha}_{\mathbf{x}}^{T*}(\omega), \tag{29}$$

where $\mathbf{S_x}(\omega)$ and $\mathbf{S}_{\bar{\mathbf{Q}}_\mathbf{x}}(\omega)$ are the system response and excitation power spectrum matrices, respectively. Further, system response second-order statistics can be readily determined based on Eq. (29). For instance, utilizing Eq. (29) the response displacement and velocity moments $\mathrm{E}\left[x_i^2(t)\right]$ and $\mathrm{E}\left[\dot{x}_i^2(t)\right]$ are given, respectively, as

$$\mathrm{E}\left[x_i^2(t)\right] = \int_{-\infty}^{\infty} S_{x_i x_i}(\omega)\mathrm{d}\omega \tag{30}$$

and

$$\mathrm{E}\left[\dot{x}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 S_{x_i x_i}(\omega)\mathrm{d}\omega. \tag{31}$$

It is deemed appropriate to note at this point that the evaluation of the FRF matrix $\boldsymbol{\alpha}_\mathbf{x}(\omega)$ of Eq. (28) can be simplified in many cases by circumventing the computation of the M-P inverse of $\mathbf{R_x}$ of Eq. (22). Specifically, in the context of generalizing the classical modal analysis treatment to account for systems with singular matrices, it was shown recently in Ref. [24] that the problem of determining the natural frequencies of the augmented system given in Eq. (15) is related to solving an eigenvalue problem for the $l \times l$ matrix $\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{K}}_\mathbf{x}$ and determining the $l \times l$ modal matrix, $\bar{\boldsymbol{\Psi}}$. In this regard, considering the transformation

$$\mathbf{x} = \bar{\boldsymbol{\Psi}}\mathbf{p}, \tag{32}$$

the system governing equation of motion Eq. (15) becomes

$$\mathbf{L}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{N}\mathbf{p} = \mathbf{P}. \tag{33}$$

In Eq. (33), $\mathbf{L}$, $\mathbf{N}$ denote the $l \times l$ diagonal matrices given by

$$\mathbf{L} = \bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{M}}_\mathbf{x}\bar{\boldsymbol{\Psi}} \tag{34}$$

and

$$\mathbf{N} = \bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{K}}_\mathbf{x}\bar{\boldsymbol{\Psi}}, \tag{35}$$

respectively, whereas the l vector $\mathbf{P}$ has the form

$$\mathbf{P} = \bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{Q}}_\mathbf{x}. \tag{36}$$

Further, the $l \times l$ matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{C}}_\mathbf{x}\bar{\boldsymbol{\Psi}} \tag{37}$$

and, in general, is not a diagonal matrix; see Ref. [24] for a more detailed presentation. Nevertheless, in many cases, and based on a reasonable assumption of light damping (e.g. [6,25]), a satisfactory approximation can be obtained by neglecting the off-diagonal elements of $\mathbf{D}$; thus, yielding a diagonal $\mathbf{D}$ matrix. In this regard, clearly, the FRF matrix of the system of Eq. (33) is given by

$$\boldsymbol{\Lambda}(\omega) = \mathbf{R}_\Lambda^{-1}, \tag{38}$$

where

$$\mathbf{R}_\Lambda = -\omega^2\mathbf{L} + i\omega\mathbf{D} + \mathbf{N}. \tag{39}$$

Further, considering Eqs. (32) and (36), as well as the relation $\mathbf{P} = \boldsymbol{\Lambda}(\omega)\mathbf{P}$, leads to

$$\mathbf{x} = \bar{\boldsymbol{\Psi}}\boldsymbol{\Lambda}(\omega)\bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+\bar{\mathbf{Q}}, \tag{40}$$

which, combined with Eq. (20), yields

$$\boldsymbol{\alpha}_\mathbf{x}(\omega) = \bar{\boldsymbol{\Psi}}\boldsymbol{\Lambda}(\omega)\bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{M}}_\mathbf{x}^+. \tag{41}$$

Finally, the FRF matrix obtained in Eq. (41) can be further simplified if taken into account that the FRF matrix $\boldsymbol{\Lambda}(\omega)$ is diagonal. In this regard, Eq. (41) yields

$$\boldsymbol{\alpha}_\mathbf{x}(\omega) = \left\{\sum_{k=1}^{l} \mathbf{x}^{(k)}\mathbf{y}^{(k)}\alpha'_k(\omega)\right\}\bar{\mathbf{M}}_\mathbf{x}^+, \tag{42}$$

where $\mathbf{x}^{(k)}$, $\mathbf{y}^{(k)}$, $k = 1, 2, \ldots, l$ correspond to the $k-th$ column of the modal matrix $\bar{\boldsymbol{\Psi}}$ and to the $k-th$ row of $\bar{\boldsymbol{\Psi}}^{-1}$, respectively. Finally, $\alpha'_k(\omega)$, $k = 1, 2, \ldots, l$ is the $k-th$ diagonal element of the matrix $\boldsymbol{\Lambda}(\omega)$.

Note that Eq. (42) is a rather useful series expression for $\boldsymbol{\alpha}_\mathbf{x}(\omega)$, which circumvents the potentially cumbersome numerical evaluation of the M-P inverse indicated in Eq. (28). Also, in many applications, the series may be truncated to only the first few terms, with little loss of accuracy.

4. Frequency domain stochastic response analysis of nonlinear systems with singular matrices

Consider next a nonlinear version of the system of Eq. (4) given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{Q}(t), \tag{43}$$

where $\mathbf{\Phi}$ is a nonlinear n vector depending on the coordinates vector $\mathbf{q}$ and its derivatives up to order two.

Further, taking into account Eqs. (12)–(15), the general form of the equations of motion for the augmented l– DOF nonlinear system ($l \geq n$) becomes

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} + \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{44}$$

where the $m + l$ augmented nonlinear vector of the system takes the form

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{\Phi}_{\mathbf{x}} \\ \mathbf{0} \end{bmatrix}. \tag{45}$$

A more detailed presentation on the construction of the equations of motion for a nonlinear system with singular matrices can be found in Ref. [18].

4.1. Generalized statistical linearization of nonlinear systems – a frequency domain approach

The statistical linearization approximate methodology has been one of the most efficient and versatile approaches for determining the stochastic response of nonlinear structural and mechanical systems [6,26]. The main objective of the methodology relates to the replacement of the original nonlinear system with an equivalent linear one by appropriately minimizing the error vector corresponding to the difference between the two systems. Thus, closed form analytical expressions available for the response statistics of linear systems can be readily used. One of the reasons for the wide utilization of the methodology in diverse engineering applications relates to the typically used Gaussian response assumption in conjunction with the mean square error minimization criterion. The above elements facilitate the derivation of closed form expressions for the equivalent linear elements (e.g., stiffness, damping coefficients, etc.) as functions of the response statistics.

Next, an equivalent to Eq. (44) linear system is sought in the form

$$(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_{\mathbf{e}})\ddot{\mathbf{x}} + (\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{e}})\dot{\mathbf{x}} + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{e}})\mathbf{x} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{46}$$

where $\bar{\mathbf{M}}_{\mathbf{e}}$, $\bar{\mathbf{C}}_{\mathbf{e}}$ and $\bar{\mathbf{K}}_{\mathbf{e}}$ denote the $(m + l) \times l$ equivalent linear mass, damping and stiffness matrices, respectively, to account for the nonlinearity of the original system.

Comparing Eqs. (15) and (46), clearly, the FRF matrix of the equivalent linear system of Eq. (46) is given by

$$\boldsymbol{\alpha}_{\mathbf{e}}(\omega) = \mathbf{R}_{\mathbf{e}}^{+}, \tag{47}$$

where

$$\mathbf{R}_{\mathbf{e}} = -\omega^2(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_{\mathbf{e}}) + i\omega(\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{e}}) + (\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{e}}). \tag{48}$$

Without loss of generality, it has been assumed in Eq. (47) that the $\mathbf{R}_{\mathbf{e}}$ matrix has full rank. In a different case, Eq. (23) should be considered. Further, the response statistics are determined via applying Eqs. (30)–(31).

Following Ref. [18], the basic steps for determining the equivalent linear matrices are concisely reviewed next for completeness. Further, to account for singular matrices, a generalization of a formula [6,27] based on a Gaussian response assumption and related to the expectation of the derivatives of the nonlinear function $\bar{\mathbf{\Phi}}_{\mathbf{x}}$ is proved for the first time in the literature. Specifically, minimizing the mean square error, $\mathrm{E}\left[\boldsymbol{\varepsilon}^2\right]$, where the error vector, $\boldsymbol{\varepsilon}$, is defined as

$$\boldsymbol{\varepsilon} = \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) - \bar{\mathbf{M}}_{\mathbf{e}}\ddot{\mathbf{x}} - \bar{\mathbf{C}}_{\mathbf{e}}\dot{\mathbf{x}} - \bar{\mathbf{K}}_{\mathbf{e}}\mathbf{x} \tag{49}$$

yields

$$\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right] = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\begin{bmatrix} \mathbf{K}_{i*}^{e\mathrm{T}} \\ \mathbf{C}_{i*}^{e\mathrm{T}} \\ \mathbf{M}_{i*}^{e\mathrm{T}} \end{bmatrix}, \; i = 1, 2, \ldots, (m + l). \tag{50}$$

To simplify further Eq. (50) the following proposition is introduced, which can be construed as a generalization of the theorem proved in [27].

Proposition 1. *Let the* $3l$ *vector* $\hat{\mathbf{x}}$ *be a zero mean jointly Gaussian random vector and* $\bar{\boldsymbol{\Phi}}_{\mathbf{x}}: \mathbb{R}^{3l} \to \mathbb{R}^{3l}$ *be a smooth multivariate function. Then, the expression*

$$\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right] = \mathrm{E}\left[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right], \tag{51}$$

holds true. A proof of Eq. (51) is provided in the Appendix B.

Moreover, it is noticed that for the singular matrix $\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]$, the M-P inverse matrix on the left hand side of Eq. (51) is also singular, and thus, taking into account Eqs. (1), (3) and (51) yields

$$\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right] = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\left[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right] + \left\{\mathbf{I} - \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\right\}\mathbf{w}, \tag{52}$$

where $\mathbf{w}$ is an arbitrary $3l$ vector. Clearly, for $\mathbf{w} = \mathbf{0}$ a particular solution for $\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right]$ is obtained in the form

$$\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right] = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\left[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right]. \tag{53}$$

Note that Eq. (53) was utilized in Ref. [18], and can be construed as a direct generalization of the standard relationship for non-singular matrices [6,27]. Nevertheless, the step of arbitrarily choosing the solution of Eq. (53) for $\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right]$ corresponding to $\mathbf{w} = \mathbf{0}$ can be circumvented by directly treating Eq. (51). In this regard, Eqs. (50) and (51) are pre-multiplied by $\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}$ and $\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]$, respectively, yielding

$$\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\begin{bmatrix}\mathbf{k}_{i*}^{e\mathrm{T}}\\ \mathbf{c}_{i*}^{e\mathrm{T}}\\ \mathbf{m}_{i*}^{e\mathrm{T}}\end{bmatrix} = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\begin{bmatrix}\dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\mathbf{x}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\dot{\mathbf{x}}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\ddot{\mathbf{x}}}\end{bmatrix}, i = 1, 2, \ldots, (m+l), \tag{54}$$

where $\mathbf{m}_{i*}^{e\mathrm{T}}$, $\mathbf{c}_{i*}^{e\mathrm{T}}$ and $\mathbf{k}_{i*}^{e\mathrm{T}}$ correspond to the ith row of $\bar{\mathbf{M}}_{\mathbf{e}}$, $\bar{\mathbf{C}}_{\mathbf{e}}$ and $\bar{\mathbf{K}}_{\mathbf{e}}$, respectively. Also, $\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}$ is the ith component of the nonlinear vector $\bar{\boldsymbol{\Phi}}_{\mathbf{x}}$ and the $3l$ vector $\hat{\mathbf{x}}$ is defined as $\hat{\mathbf{x}} = (\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})^{\mathrm{T}}$.

Apparently, the equivalent linear mass, damping and stiffness matrices can be determined by solving Eq. (54). However, the $3l \times 3l$ matrix $\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]$ is a priori assumed to be singular as a result of the redundant coordinates modeling scheme [18]. Therefore, by employing its M-P inverse, $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}$, and taking into account Eqs. (3) and (54) yields

$$\begin{bmatrix}\mathbf{k}_{i*}^{e\mathrm{T}}\\ \mathbf{c}_{i*}^{e\mathrm{T}}\\ \mathbf{m}_{i*}^{e\mathrm{T}}\end{bmatrix} = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\begin{bmatrix}\dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\mathbf{x}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\dot{\mathbf{x}}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\ddot{\mathbf{x}}}\end{bmatrix} + \mathbf{g}(\mathbf{y}), i = 1, 2, \ldots, (m+l), \tag{55}$$

where m_{ij}^{e}, c_{ij}^{e} and k_{ij}^{e} denote the elements of the equivalent linear augmented matrices; and $\mathbf{g}(\mathbf{y}) = \left(\mathbf{I} - \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\right)\mathbf{y}$, denotes the arbitrary part of the solution. Although the existence of $\mathbf{g}(\mathbf{y})$ implies a non-unique solution for the equivalent linear matrices, it was recently proved in Ref. [18] that the solution obtained by setting the arbitrary vector equal to zero is at least as good, in terms of minimizing the mean square error, as any other possible solution corresponding to a non-zero arbitrary vector. In this regard, setting $\mathbf{y} = \mathbf{0}$ in the arbitrary part of the solution and substituting in Eq. (55) yields

$$\begin{bmatrix}\mathbf{k}_{i*}^{e\mathrm{T}}\\ \mathbf{c}_{i*}^{e\mathrm{T}}\\ \mathbf{m}_{i*}^{e\mathrm{T}}\end{bmatrix} = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathrm{E}\begin{bmatrix}\dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\mathbf{x}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\dot{\mathbf{x}}}\\ \dfrac{\partial\bar{\boldsymbol{\Phi}}_{i,\mathbf{x}}}{\partial\ddot{\mathbf{x}}}\end{bmatrix}, i = 1, 2, \ldots, (m+l). \tag{56}$$

Clearly, in the case where the minimum number of coordinates is utilized, $\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+} = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{-1}$, and thus, Eq. (56) takes

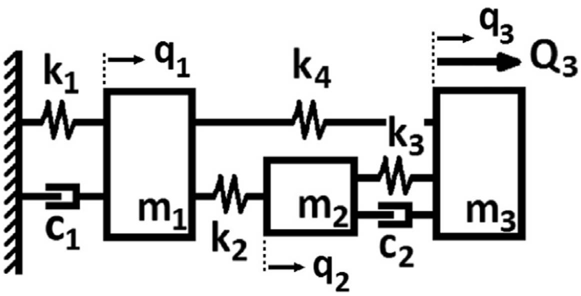

Fig. 1. A three degree-of-freedom linear system under stochastic excitation.

the well-established form used in the standard implementation of statistical linearization [6,18].

Further, determining the equivalent linear matrices in Eq. (54) requires knowledge of the response covariance matrix $E\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^T\right]$. Thus, an additional set of equations relating the covariance matrix and the equivalent linear matrices is required. In this regard, the herein derived frequency domain input-output Eq. (29) is utilized. Overall, the developed generalized statistical linearization methodology can be construed as the frequency domain alternative to a recently proposed time-domain methodology [18].

4.2. Mechanization of the generalized statistical linearization methodology

Regarding the numerical implementation of the method, Eqs. (30)–(31) and Eq. (56) comprise a coupled nonlinear system of equations yielding the equivalent linear matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$, and $\bar{\mathbf{K}}_e$ as well as the system response covariance matrix. For the solution of the coupled nonlinear system, any standard numerical optimization scheme can be applied [28]. Nevertheless, the following iterative procedure can be utilized as an alternative straightforward approach.

The first step consists of selecting initial values for the equivalent linear matrices. In this regard, $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$, and $\bar{\mathbf{K}}_e$ are set equal to null matrices. Next, following the selection of an appropriate convergence criterion, the following two steps are repeated successively:

- A value for the system response covariance matrix is computed via Eqs. (30) and (31).
- Combining Eq. (56) with the system response covariance matrix obtained in the previous step, updated values for the equivalent linear matrices are calculated.

The iterative method stops when convergence is attained.

5. Numerical examples

5.1. Linear systems with singular matrices

As a numerical example the 3-DOF linear system of rigid masses shown in Fig. 1, is considered. The first mass m_1 is attached to the foundation by a linear spring and a linear damper with coefficients k_1 and c_1, respectively. It is also connected to the other two masses m_2 and m_3 by two linear springs with coefficients k_2 and k_4. Finally, the mass m_2 is connected to the third mass by a linear spring with coefficient k_3 and a linear damper with damping coefficient c_2. Further, the system is excited by a stochastic force $Q_3(t)$ applied on mass m_3 and modeled as a white-noise process with a correlation function $w_{Q_3}(t) = 2\pi S_0 \delta(t)$. The value S_0 stands for the (constant) power spectrum value of $Q_3(t)$. The generalized displacements of the masses m_1, m_2 and m_3 due to the applied force, are denoted by q_1, q_2 and q_3, respectively.

Following a standard Newtonian, or Lagrangian approach [29], the linear system equations of motion have the form given in Eq. (4), where the 3×3 mass, damping and stiffness matrices are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \tag{57}$$

and

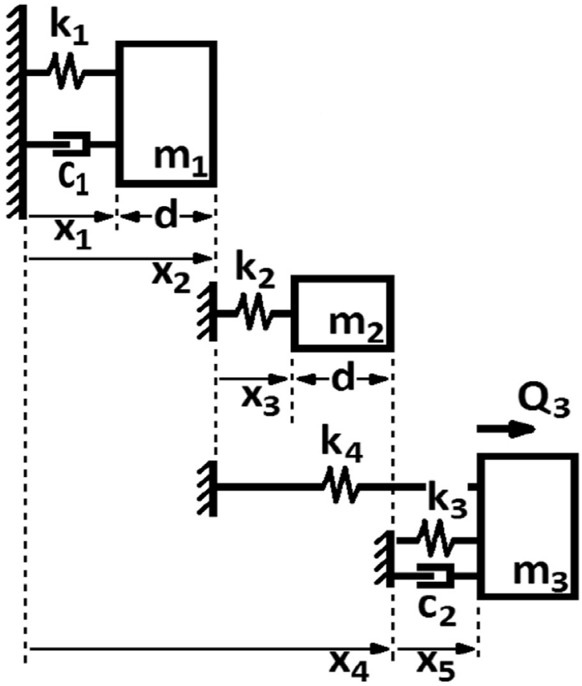

Fig. 2. A three degree-of-freedom linear system under stochastic excitation utilizing redundant coordinates.

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}, \tag{58}$$

respectively. Further, the displacement vector is given as

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \tag{59}$$

whereas the excitation vector $\mathbf{q}(t)$ is given by

$$\mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ Q_3 \end{bmatrix}. \tag{60}$$

The parameters values in this example are $m_1 = m_3 = 2$, $m_2 = 1$, $c_1 = c_2 = 0.1$ and $k_1 = k_2 = k_3 = k_4 = 1$, and $S_0 = 10^{-3}$. Considering next Eqs. (9)–(10), the stationary covariance matrix of the system response displacement is given by

$$\mathbf{V}_{\mathbf{q}} = \begin{bmatrix} 0.0493 & 0.0623 & 0.0644 \\ 0.0623 & 0.0805 & 0.0846 \\ 0.0644 & 0.0846 & 0.0916 \end{bmatrix}, \tag{61}$$

whereas the stationary covariance matrix of the system response velocity is

$$\mathbf{V}_{\dot{\mathbf{q}}} = \begin{bmatrix} 0.0106 & 0.0110 & 0.0086 \\ 0.0110 & 0.0142 & 0.0131 \\ 0.0086 & 0.0131 & 0.0170 \end{bmatrix}. \tag{62}$$

Next, to demonstrate the herein developed frequency domain based methodology for systems with singular matrices, the system shown in Fig. 1 is decomposed into several separate systems, which are treated independently. In particular, as it is seen in Fig. 2, the number of modeling coordinates used for deriving the system equations of motion is increased by two. In this regard, the coordinates vector of the redundant DOFs system becomes

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} \bar{x}_1 & x_2 & \bar{x}_3 & x_4 & \bar{x}_5 \end{bmatrix}, \tag{63}$$

where $\bar{x}_1$, $\bar{x}_3$ and $\bar{x}_5$ correspond to the displacements of the masses m_1, m_2 and m_3 and the coordinates x_2, x_4 correspond to the additional DOFs. Note, however, that the sub-systems are related via two constraint equations, namely

$$x_1 + d = x_2 \tag{64}$$

and

$$x_2 + x_3 + d = x_4, \tag{65}$$

where d is the physical length of the masses (same for m_1, m_2 and m_3). The constraint equations can also be written as

$$\bar{x}_1 + l_{1,0} + d = x_2 \tag{66}$$

and

$$x_2 + \bar{x}_3 + l_{3,0} + d = x_4, \tag{67}$$

where $l_{1,0}$ is the unstretched length of the mass m_1, and $l_{3,0}$ is the unstretched length of m_3.

To derive the system equations of motion, the total kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{\bar{x}}_1^2 + \frac{1}{2}m_2(\dot{x}_2 + \dot{\bar{x}}_3)^2 + \frac{1}{2}m_3(\dot{x}_4 + \dot{\bar{x}}_5)^2 \tag{68}$$

and the total potential energy by

$$V = \frac{1}{2}k_1\bar{x}_1^2 + \frac{1}{2}k_2\bar{x}_3^2 + \frac{1}{2}k_3\bar{x}_5^2 + \frac{1}{2}k_4(-x_2 + x_4 + \bar{x}_5)^2. \tag{69}$$

Next, the standard variational formulation [29] involving the Lagrangian function $L(\mathbf{x}, \dot{\mathbf{x}}) = T - V$ leads to the Euler-Lagrange equations, and thus, to the system equations of motion of the form of Eq. (15). In particular, the mass, damping and stiffness matrices become

$$\mathbf{M_x} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & 0 & 0 & m_3 & m_3 \\ 0 & 0 & 0 & m_3 & m_3 \end{bmatrix}, \mathbf{C_x} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 \end{bmatrix} \tag{70}$$

and

$$\mathbf{K_x} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_4 & 0 & -k_4 & -k_4 \\ 0 & 0 & k_2 & 0 & 0 \\ 0 & -k_4 & 0 & k_4 & k_4 \\ 0 & -k_4 & 0 & k_4 & k_3 + k_4 \end{bmatrix}, \tag{71}$$

respectively. Also, differentiating twice with respect to time Eqs. (66)–(67), the 2×5 matrix $\mathbf{A}$ defined in Eq. (13) takes the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \tag{72}$$

whereas the 2×1 vector $\mathbf{b}$ becomes

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{73}$$

Next, taking into account Eqs. (13)–(14) and (16)–(18), and substituting the parameters values, the 7×5 augmented mass, damping and stiffness matrices of the system become

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.8 & 0.2 & 0.2 & 0.4 & 0.4 \\ 0.8 & 0.2 & 0.2 & 0.4 & 0.4 \\ -0.4 & 0.4 & 0.4 & 0.8 & 0.8 \\ 0.4 & 0.6 & 0.6 & 1.2 & 1.2 \\ 0 & 0 & 0 & 2 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.04 & 0 & 0 & 0 & 0 \\ 0.04 & 0 & 0 & 0 & 0 \\ -0.02 & 0 & 0 & 0 & 0 \\ 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{74}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.4 & 0.2 & -0.2 & -0.2 & -0.2 \\ 0.4 & 0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.6 & 0.6 & 0.6 & 0.6 \\ 0.2 & -0.4 & 0.4 & 0.4 & 0.4 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{75}$$

Further, employing Eq. (19), the augmented excitation vector is given by

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} 0.2Q_3 \\ 0.2Q_3 \\ 0.4Q_3 \\ 0.6Q_3 \\ Q_3 \\ 0 \\ 0 \end{bmatrix}. \tag{76}$$

Next, to determine the system response statistics via the herein developed frequency domain methodology, the 7×5 matrix $\mathbf{R}_{\mathbf{x}}$ is obtained via Eq. (22). Furthermore, utilizing Eq. (28) the FRF matrix $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$ is determined. It is noted that Eq. (28) is utilized instead of Eq. (23) as the 7×5 matrix $\mathbf{R}_{\mathbf{x}}$ has full rank, i.e. *rank* $\mathbf{R}_{\mathbf{x}} = 5$, and thus, the FRF matrix is uniquely defined. Next, combining Eq. (29) with Eq. (30), the covariance matrix of the system response displacement is given by

$$\mathbf{V}_{\bar{\mathbf{x}}} = \begin{bmatrix} 0.0493 & 0.0493 & 0.0130 & 0.0623 & 0.0021 \\ 0.0493 & 0.0493 & 0.0130 & 0.0623 & 0.0021 \\ 0.0130 & 0.0130 & 0.0052 & 0.0182 & 0.0019 \\ 0.0623 & 0.0623 & 0.0182 & 0.0805 & 0.0040 \\ 0.0021 & 0.0021 & 0.0019 & 0.0040 & 0.0030 \end{bmatrix} \tag{77}$$

and combining Eq. (29) with Eq. (31), the covariance matrix of the system response velocity is determined to be

$$\mathbf{V}_{\dot{\bar{\mathbf{x}}}} = \begin{bmatrix} 0.0106 & 0.0106 & 0.0004 & 0.0110 & -0.0024 \\ 0.0106 & 0.0106 & 0.0004 & 0.0110 & -0.0024 \\ 0.0004 & 0.0004 & 0.0028 & 0.0032 & 0.0013 \\ 0.0110 & 0.0110 & 0.0032 & 0.0142 & -0.0011 \\ -0.0024 & -0.0024 & 0.0013 & -0.0011 & 0.0050 \end{bmatrix}. \tag{78}$$

For the comparison of the results obtained by the standard and the herein proposed methodology, the matrices given by Eqs. (77)–(78) are compared to those given by Eqs. (61)–(62). Indicatively, it is seen that the variances $\mathrm{E}\left[q_1^2\right]$ and $\mathrm{E}\left[\dot{q}_1^2\right]$ coincide with their counterparts, i.e. $\mathrm{E}\left[\bar{x}_1^2\right]$ and $\mathrm{E}\left[\dot{\bar{x}}_1^2\right]$. Further, considering the equations that connect the reference systems depicted in Fig. 2, i.e. $\bar{x}_3 = q_2 - q_1$ and $\bar{x}_5 = q_3 - q_2$, yields

$$\mathrm{E}\left[\bar{x}_3^2\right] = \mathrm{E}\left[q_1^2\right] + \mathrm{E}\left[q_2^2\right] - 2\mathrm{E}\left[q_1 q_2\right] = 0.0052 \tag{79}$$

and

$$\mathrm{E}\left[\bar{x}_5^2\right] = \mathrm{E}\left[q_2^2\right] + \mathrm{E}\left[q_3^2\right] - 2\mathrm{E}[q_2 q_3] = 0.0030. \tag{80}$$

Therefore, the variances computed in Eqs. (79)–(80) are equal to the corresponding ones in positions (3, 3) and (5, 5) of matrix $\mathbf{V}_{\bar{\mathbf{x}}}$. The same agreement for the response velocity variances can be readily verified by comparing Eq. (62) with Eq. (78).

Further, as noted in Section 3.2, the FRF matrix $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$ can be alternatively determined without computing the M-P inverse of the matrix $\mathbf{R}_{\mathbf{x}}$ in Eq. (22). Instead, a generalized modal analysis approach can be employed. In this regard, following closely Ref. [24], the modal matrix for the system in Fig. 2 is computed as

$$\bar{\Psi} = \begin{bmatrix} 0.1740 & 0.4880 & 0.5151 & 0.0000 & -0.0000 \\ 0.1740 & 0.4880 & 0.5151 & -0.1305 & 0.8116 \\ -0.6401 & -0.3004 & 0.1517 & 0.3601 & 0.1281 \\ -0.4661 & 0.1877 & 0.6668 & -0.8507 & 0.5554 \\ 0.5590 & -0.6310 & 0.0413 & 0.3601 & 0.1281 \end{bmatrix}. \quad (81)$$

Next, utilizing the transformation of Eq. (32) and taking into account Eqs. (34)–(35) and (37), the system equation of motion of Eq. (33) arises. Also, the 5×5 diagonal FRF matrix of Eq. (38) becomes

$$\Lambda(\omega) = \begin{bmatrix} \frac{1}{-\omega^2 + 0.1162i\omega + 2.5726} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{-\omega^2 + 0.0703i\omega + 1.7620} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{-\omega^2 + 0.0135i\omega + 0.1655} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-\omega^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{-\omega^2} \end{bmatrix}. \quad (82)$$

Furthermore, combining Eq. (41), or Eq. (42), with Eqs. (81)–(82), the FRF matrix is determined, and thus, the covariance matrix of the system response displacement is given by

$$\mathbf{V_x} = \begin{bmatrix} 0.0492 & 0.0492 & 0.0132 & 0.0624 & 0.0020 \\ 0.0492 & 0.0492 & 0.0132 & 0.0624 & 0.0020 \\ 0.0132 & 0.0132 & 0.0050 & 0.0182 & 0.0019 \\ 0.0624 & 0.0624 & 0.0182 & 0.0806 & 0.0039 \\ 0.0020 & 0.0020 & 0.0019 & 0.0039 & 0.0032 \end{bmatrix}, \quad (83)$$

which is in agreement with Eq. (77) obtained via utilizing the MP inverse of $\mathbf{R_x}$.

5.2. Nonlinear systems with singular matrices

5.2.1. 2-DOF nonlinear system with singular matrices

A $2-DOF$ nonlinear system of rigid masses m_1 and m_2 shown in Fig. 3 is considered as the first numerical example. The mass m_1 is connected to the foundation by a nonlinear spring of the linear-plus-cubic type and by a linear damper with coefficient c_1. Further, the mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. The system is excited by a random force $Q_2(t)$ which is modeled as a white-noise process with a correlation function $w_{Q_2}(t) = 2\pi S_0 \delta(t)$, where S_0 is the (constant) power spectrum value of $Q_2(t)$. Finally, the generalized displacements are given by q_1 and q_2.

The system equations of motion are written in the matrix form of Eq. (4), where the matrices **M**, **C** and **K** are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}; \quad (84)$$

whereas the coordinate vector **q** and the excitation vector **Q** are defined as

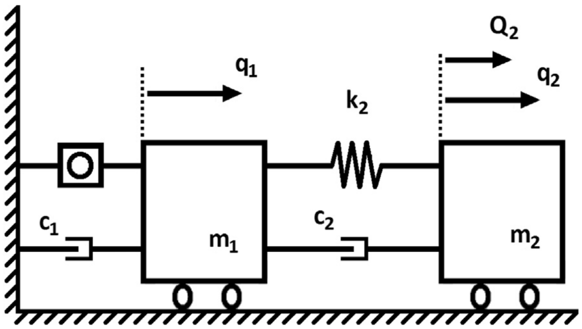

Fig. 3. A two degree-of-freedom nonlinear structural system under stochastic excitation.

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{85}$$

and

$$\mathbf{Q} = \begin{bmatrix} 0 \\ Q_2 \end{bmatrix}, \tag{86}$$

respectively. Further, the nonlinear function $\mathbf{\Phi}$ takes the form

$$\mathbf{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \begin{bmatrix} \varepsilon_1 k_1 q_1^3 \\ 0 \end{bmatrix}. \tag{87}$$

Next, taking into account Eq. (87) and the fact that the minimum number of DOFs are used in modeling the system equations of motion, i.e $\mathrm{E}\left[\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathrm{T}}\right]^{+} = \mathrm{E}\left[\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathrm{T}}\right]^{-1}$, Eq. (56) yields

$$\mathbf{K_e} = \begin{bmatrix} 3\varepsilon_1 k_1 \sigma_{q_1}^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{88}$$

Further, the standard statistical linearization procedure is applied. The parameters values used are $m_1 = m_2 = m = 1$, $c_1 = c_2 = c = 0.1$, $k_1 = k_2 = k = 1$, and $S_0 = 10^{-3}$. Also, the value of the power spectrum for the excitation is $S_0 = 10^{-3}$. Regarding the numerical implementation, convergence is attained after eight iterations, subject to the criterion $\left|\frac{\mathbf{K_e^{j+1}} - \mathbf{K_e^{j}}}{\mathbf{K_e^{j}}}\right| > 10^{-5}$, where the j index denotes the $j-th$ iteration and the initial value $\mathbf{K_e^0}$ is set equal to zero. At the end of the iterative solution procedure, the covariance matrix of the system response displacement is determined as

$$\mathbf{V_q} = \begin{bmatrix} 0.0386 & 0.0639 \\ 0.0639 & 0.1102 \end{bmatrix}, \tag{89}$$

whereas the covariance matrix of the system response velocity is

$$\mathbf{V_{\dot{q}}} = \begin{bmatrix} 0.0178 & 0.0252 \\ 0.0252 & 0.0458 \end{bmatrix}. \tag{90}$$

Next, utilizing a redundant coordinates modeling scheme, the three coordinates $\bar{x}_1$, x_2 and $\bar{x}_3$ shown in Fig. 4 are considered, whereas the constraint equation

$$x_2 = x_1 + d, \tag{91}$$

with d being the length of mass m_1, serves to connect the two sub-systems of mass m_1 and mass m_2. Differentiating Eq. (91) twice with respect to time, the constraint equation is written in the matrix form given by Eq. (13), where

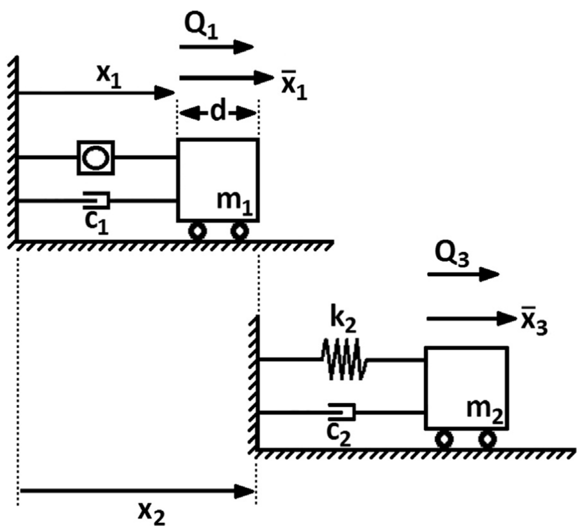

Fig. 4. A two degree-of-freedom nonlinear structural system under stochastic excitation utilizing redundant coordinates.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \tag{92}$$

and

$$b = 0. \tag{93}$$

Thus, the augmented mass, damping and stiffness matrices, defined in Eqs. (16)–(18), become

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \tag{94}$$

while the augmented excitation vector and the nonlinear vector of the system which are defined in Eqs. (19) and (45), respectively, are given by

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} 0.5w(t) \\ 0.5w(t) \\ w(t) \\ 0 \end{bmatrix} \tag{95}$$

and

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0 \\ 0 \end{bmatrix}. \tag{96}$$

Note that the variable $\bar{x}_1$ in Eq. (96) corresponds to the displacement of the first mass and is defined as $\bar{x}_1 = x_1 - l_{1,0}$, where $l_{1,0}$ is the unstretched length of the spring k_1.

Applying next the generalized statistical linearization methodology, Eq. (56) is utilized for determining the equivalent linear stiffness matrix, $\bar{\mathbf{K}}_{\mathbf{E}}$, yielding

$$\bar{\mathbf{K}}_{\mathbf{e}} = \frac{3}{2}\varepsilon_1 k_1 \sigma_{\bar{x}_1}^2 \begin{bmatrix} r(1,1) & r(2,1) & r(3,1) \\ r(1,1) & r(2,1) & r(3,1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{97}$$

In Eq. (97), $r(i,j)$ denotes the (i,j) element of the matrix $\mathbf{R} = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]$; see Ref. [18] for more details. Note also that, due to the presence of the non-unitary matrix r, the equivalent stiffness matrix in Eq. (97) has more non-zero elements than the corresponding one in Eq. (88). Further, the same convergence criterion as the one employed in deriving Eqs. (89)–(90), is used, whereas convergence is reached after eight iterations.

In particular, noticing that in this case the 4×3 matrix $\mathbf{R}_{\mathbf{e}}$, has full rank, and thus, Eq. (28) is used for determining the FRF matrix $\boldsymbol{\alpha}_{\mathbf{x}}(\omega)$, the covariance matrix of the system response displacement is determined to be

$$\mathbf{V}_{\bar{\mathbf{x}}} = \begin{bmatrix} 0.0386 & 0.0386 & 0.0253 \\ 0.0386 & 0.0386 & 0.0253 \\ 0.0253 & 0.0253 & 0.0210 \end{bmatrix}, \tag{98}$$

whereas the system response velocity covariance matrix is computed as

$$\mathbf{V}_{\dot{\bar{\mathbf{x}}}} = \begin{bmatrix} 0.0178 & 0.0178 & 0.0074 \\ 0.0178 & 0.0178 & 0.0074 \\ 0.0074 & 0.0074 & 0.0132 \end{bmatrix}. \tag{99}$$

Comparing the results, it is seen that the variance $\mathrm{E}\left[q_1^2\right]$ in Eq. (89) coincide with the variance $\mathrm{E}\left[\bar{x}_1^2\right]$ in Eq. (98). Similarly, the variances $\mathrm{E}\left[\dot{q}_1^2\right]$ and $\mathrm{E}\left[\dot{\bar{x}}_1^2\right]$ in Eqs. (90) and (99), coincide with each other. Further, taking into account the expression $\bar{x}_3 = q_2 - q_1$ that relates the two reference systems yields

$$\mathrm{E}\left[\bar{x}_3^2\right] = \mathrm{E}\left[q_2^2\right] + \mathrm{E}\left[q_1^2\right] - 2\mathrm{E}\left[q_1 q_2\right] = 0.0210 \tag{100}$$

and

$$\mathrm{E}\left[\dot{x}_3^2\right] = \mathrm{E}\left[\dot{q}_2^2\right] + \mathrm{E}\left[\dot{q}_1^2\right] - 2\mathrm{E}\left[\dot{q}_1\dot{q}_2\right] = 0.0132, \tag{101}$$

which agree with the corresponding values in Eqs. (98)–(99).

At this point, it should be noted that the herein obtained results are in total agreement with the ones obtained when the problem is solved by following an alternative time-domain methodology recently developed by the authors [18].

5.2.2. 3-DOF nonlinear system with singular matrices

In this example, nonlinearities are considered in the system studied in Section 5.1. Specifically, it is assumed that the damping force connecting mass m_1 with the foundation is given by $c_1\dot{x}_1\left(1 + \epsilon\left|\dot{x}_1\right|\right)$. In this regard, the system mass, damping and stiffness matrices, as well as the system coordinates and the vector of the excitation force are given by Eqs. (57) and (59)–(60), respectively. Finally, the nonlinear vector $\mathbf{\Phi}$ of Eq. (43) takes the form

$$\mathbf{\Phi} = \begin{bmatrix} \epsilon_1 c_1 \dot{q}_1\left|\dot{q}_1\right| \\ 0 \\ 0 \end{bmatrix}. \tag{102}$$

Following next the standard statistical linearization approach [6], the equivalent linear damping matrix of the system becomes

$$\mathbf{C_e} = \frac{4\epsilon_1 c_1}{\sqrt{2\pi}}\sqrt{\mathrm{E}\left[\dot{q}_1^2\right]}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{103}$$

Regarding the numerical implementation of the iterative solution scheme, the convergence criterion is given by $\left|\frac{\mathbf{c}_\mathbf{e}^{j+1} - \mathbf{c}_\mathbf{e}^{j}}{\mathbf{c}_\mathbf{e}^{j}}\right| > 10^{-5}$, where j denotes the $j-th$ iteration and $\mathbf{C}_\mathbf{e}^0$ is set equal to zero. After eight iterations, the covariance matrices of the system response displacement and velocity are given by

$$\mathbf{V_q} = \begin{bmatrix} 0.0379 & 0.0477 & 0.0491 \\ 0.0477 & 0.0616 & 0.0646 \\ 0.0491 & 0.0646 & 0.0702 \end{bmatrix} \tag{104}$$

and

$$\mathbf{V_{\dot{q}}} = \begin{bmatrix} 0.0084 & 0.0085 & 0.0063 \\ 0.0085 & 0.0110 & 0.0099 \\ 0.0063 & 0.0099 & 0.0133 \end{bmatrix}, \tag{105}$$

respectively.

Next, utilizing the redundant coordinates modeling, the augmented nonlinear vector of Eq. (45) becomes

$$\bar{\mathbf{\Phi}}_\mathbf{x} = \begin{bmatrix} 0.4\epsilon_1 c_1\dot{x}_1\left|\dot{x}_1\right| \\ 0.4\epsilon_1 c_1\dot{x}_1\left|\dot{x}_1\right| \\ -0.2\epsilon_1 c_1\dot{x}_1\left|\dot{x}_1\right| \\ 0.2\epsilon_1 c_1\dot{x}_1\left|\dot{x}_1\right| \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{106}$$

Then, the equivalent damping matrix $\mathbf{C_e}$ is obtained by applying the generalized statistical linearization methodology; that is, Eq. (56) yields

$$\bar{\mathbf{C}}_\mathbf{e} = \frac{0.8\epsilon_1 c_1}{\sqrt{2\pi}}\sqrt{\mathrm{E}\left[\dot{x}_1^2\right]}\begin{bmatrix} 2r(6,6) & 2r(7,6) & 2r(8,6) & 2r(9,6) & 2r(10,6) \\ 2r(6,6) & 2r(7,6) & 2r(8,6) & 2r(9,6) & 2r(10,6) \\ -r(6,6) & -r(7,6) & -r(8,6) & -r(9,6) & -r(10,6) \\ r(6,6) & r(7,6) & r(8,6) & r(9,6) & r(10,6) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{107}$$

whereas the iterative solution procedure using the same convergence criterion as in Eqs. (89)–(90) yields the response covariance matrices

$$\mathbf{V}_{\bar{\mathbf{x}}} = \begin{bmatrix} 0.0379 & 0.0379 & 0.0098 & 0.0477 & 0.0014 \\ 0.0379 & 0.0379 & 0.0098 & 0.0477 & 0.0014 \\ 0.0098 & 0.0098 & 0.0041 & 0.0139 & 0.0016 \\ 0.0477 & 0.0477 & 0.0139 & 0.0616 & 0.0029 \\ 0.0014 & 0.0014 & 0.0016 & 0.0029 & 0.0027 \end{bmatrix} \tag{108}$$

and

$$\mathbf{V}_{\dot{\bar{\mathbf{x}}}} = \begin{bmatrix} 0.0084 & 0.0084 & 0.0001 & 0.0085 & -0.0022 \\ 0.0084 & 0.0084 & 0.0001 & 0.0085 & -0.0022 \\ 0.0001 & 0.0001 & 0.0025 & 0.0026 & 0.0010 \\ 0.0085 & 0.0085 & 0.0026 & 0.0110 & -0.0012 \\ -0.0022 & -0.0022 & 0.0010 & -0.0012 & 0.0046 \end{bmatrix}. \tag{109}$$

Taking into account the equations that connect the reference systems, i.e. $\bar{x}_3 = q_2 - q_1$ and $\bar{x}_5 = q_3 - q_2$, it can be readily verified that the covariance matrices in Eqs. (104)–(105) are in total agreement with the respective ones in Eqs. (108)–(109).

6. Conclusions

In this paper, a frequency domain methodology has been developed for stochastic response determination of MDOF linear and nonlinear structural systems with singular matrices. Specifically, relying on the generalized matrix inverse theory, a M-P FRF has been determined for a linear structural system with singular matrices. In this regard, a rather useful series expansion for the M-P FRF has been presented as well, which circumvents the potentially cumbersome numerical evaluation of the M-P inverse. Next, relying on the M-P FRF a spectral input-output (excitation-response) relationship has been derived in the frequency domain for determining the linear system response power spectrum. Further, the above methodology has been extended via statistical linearization to account for nonlinear systems. This has led to an iterative determination of the system response mean vector and covariance matrix. Furthermore, to account for singular matrices, the generalization of a widely utilized formula that facilitates the application of statistical linearization has been proved as well. The formula relates to the expectation of the derivatives of the system nonlinear function and is based on a Gaussian response assumption. It is noted that the herein developed frequency domain response analysis methodology can be construed as an alternative to a recently developed time domain technique [17,18]. Several linear and nonlinear MDOF structural systems with singular matrices have been considered as numerical examples demonstrating the validity and applicability of the developed frequency domain methodology.

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Appendix A

In this Appendix, a concise presentation of the formulation of Eq. (15), is provided for completeness. Further details can be found in Refs. [30,17,18]. In this regard, following a redundant DOFs modeling scheme, the $n-$ DOF system of Eq. (4) is construed as a collection of sub-systems modeled separately, yielding an overall $l-$ DOF system ($l \geq n$) with governing equations of motion given by Eq. (12). Further, additional constraint equations given by Eq. (13) arise that connect the aforementioned subsystems [30,17]. Subsequently, the constraint equations imply a number of additional forces, $\mathbf{Q}_{\mathbf{x}}^{c}(t)$, and thus, Eq. (12) is transformed into

$$\mathbf{M}_{\mathbf{x}}\ddot{\mathbf{x}} + \mathbf{C}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{K}_{\mathbf{x}}\mathbf{x} = \mathbf{Q}_{\mathbf{x}}(t) + \mathbf{Q}_{\mathbf{x}}^{c}(t). \tag{A.1}$$

Furthermore, virtual displacements that are denoted by the non-zero l vector $\mathbf{w}_c$, appear due to the additional forces $\mathbf{Q}_{\mathbf{x}}^{c}(t)$; these displacements satisfy the condition

$$\mathbf{A}\mathbf{w}_c = \mathbf{0}, \tag{A.2}$$

and at any instant of time t can be expressed as

$$\mathbf{w}_c^{\mathrm{T}}\mathbf{Q}_{\mathbf{x}}^{c} = \mathbf{w}_c^{c}\mathbf{N}. \tag{A.3}$$

The l vector $\mathbf{N}$ in Eq. (A.3) describes the nature of the non-ideal constraints and can be obtained by experimentation and/or observation [16]. Exploiting at this point the concept of the M-P generalized matrix inverses, and particularly Eq. (3), the solution to Eq. (A.2) becomes

$$\mathbf{w}_c = \tilde{\mathbf{A}}\mathbf{y}, \tag{A.4}$$

where

$$\tilde{\mathbf{A}} = \mathbf{I} - \mathbf{A}^{+}\mathbf{A}, \tag{A.5}$$

and $\mathbf{y}$ is an arbitrary l vector. Substituting then Eq. (A.4) into Eq. (A.3) and manipulating, the expression

$$\tilde{\mathbf{A}}\mathbf{Q}_{\mathbf{x}}^{c} = \tilde{\mathbf{A}}\mathbf{N}, \tag{A.6}$$

arises. Next, pre-multiplying Eq. (A.1) by Eq. (A.5) and taking into account Eq. (A.6) yields

$$\tilde{\mathbf{A}}\left\{\mathbf{M}_x\ddot{\mathbf{x}} + \mathbf{C}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{K}_{\mathbf{x}}\mathbf{x}\right\} = \tilde{\mathbf{A}}(\mathbf{Q}_{\mathbf{x}} + \mathbf{N}). \tag{A.7}$$

Further, assuming for simplicity that the m vector $\mathbf{b}$ in Eq. (13) is of the form given by Eq. (14), and considering Eq. (A.7) yields

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{A}}(\mathbf{Q}_{\mathbf{x}} + \mathbf{N}) \\ \mathbf{F} \end{bmatrix}; \tag{A.8}$$

$\bar{\mathbf{M}}_{\mathbf{x}}$, $\bar{\mathbf{C}}_{\mathbf{x}}$ and $\bar{\mathbf{K}}_{\mathbf{x}}$ denote the augmented mass, damping and stiffness matrices defined by Eqs. (16)–(18), respectively. Finally, considering ideal constraints, i.e. $\mathbf{N} = \mathbf{0}$, Eq. (A.8) takes the form given by Eq. (15), where the augmented excitation vector $\bar{\mathbf{Q}}_{\mathbf{x}}$ is defined by Eq. (19). A more detailed presentation of the topic can be found in Refs. [30,17,18].

Appendix B

In this Appendix, Eq. (51) is proved. In this regard, taking into account the definition of the expected value and assuming that the joint Gaussian pdf of $\hat{\mathbf{x}}$ is denoted by $p(\hat{\mathbf{x}})$, the expression

$$\mathrm{E}\left[\nabla\bar{\mathbf{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right] = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\nabla\bar{\mathbf{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})p(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}}^{\mathrm{T}}, \tag{B.1}$$

holds true. However, as it is noted in Section 4.1, the matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ is singular. Therefore, considering a multivariate Gaussian distribution with a singular covariance matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ [31–33], the pdf of $\hat{\mathbf{x}}$ is given by

$$\mathbf{P}\left(\hat{\mathbf{x}}\right) = \left((2\pi)^k\left|\mathbf{B}^{\mathrm{T}}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathbf{B}\right|\right)^{-\frac{1}{2}} exp\left\{-\frac{1}{2}\hat{\mathbf{x}}^{\mathrm{T}}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\hat{\mathbf{x}}\right\}. \tag{B.2}$$

In Eq. (B.2), the M-P inverse of the matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ has the form

$$\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+} = \mathbf{B}\left(\mathbf{B}^{\mathrm{T}}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathbf{B}\right)^{-1}\mathbf{B}^{\mathrm{T}}, \tag{B.3}$$

and $\mathbf{B}$ satisfies the relationship

$$\left|\mathbf{B}^{\mathrm{T}}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]\mathbf{B}\right| = \lambda_1\lambda_2\dots\lambda_\rho, \tag{B.4}$$

where λ_i, $i = 1, 2, \dots, \rho$ denote the non-zero eigenvalues of the singular matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ [31].

Next, following closely Ref. [27], the right hand side of Eq. (B.1) is integrated by parts yielding

$$\mathrm{E}[\nabla\bar{\mathbf{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})] = \boldsymbol{r} - \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\mathbf{x}}\nabla p(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}}^{\mathrm{T}}, \tag{B.5}$$

where $\boldsymbol{r}$ is a $3l$ vector with

$$r_i = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\left\{\bar{\Phi}_{i,\mathbf{x}}p(\hat{\mathbf{x}})\Big|_{x_i=-\infty}^{x_i=+\infty}\right\}\prod_{\substack{j=1\\ i\neq j}}^{3l}\mathrm{d}x_j,\ i = 1, 2, \dots, 3l. \tag{B.6}$$

Without loss of generality, the quantity in the brackets in Eq. (B.6) is assumed next to be zero at $x_i = \pm\infty$. This is further substantiated by the form of $\bar{\boldsymbol{\Phi}}_{\mathbf{x}}$ ordinarily met in practice; see also Ref. [27]. Thus, Eq. (B.5) becomes

$$\mathrm{E}[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})] = -\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\mathbf{x}}\nabla p(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}}^{\mathrm{T}}. \tag{B.7}$$

Furthermore, note that applying the nabla operator to Eq. (B.2) yields

$$\nabla p(\hat{\mathbf{x}}) = -\frac{1}{2}p(\hat{\mathbf{x}})\nabla\left(\hat{\mathbf{x}}^{\mathrm{T}}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\hat{\mathbf{x}}\right) \tag{B.8}$$

and noticing that $\nabla\left(\hat{\mathbf{x}}^{\mathrm{T}}\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\hat{\mathbf{x}}\right) = 2\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\hat{\mathbf{x}}$, Eq. (B.8) becomes, equivalently,

$$\nabla p(\hat{\mathbf{x}}) = -p(\hat{\mathbf{x}})\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\hat{\mathbf{x}}. \tag{B.9}$$

Finally, considering Eqs. (B.9) and (B.7) takes the form

$$\mathrm{E}\left[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right] = \mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}p(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}}^{\mathrm{T}} \tag{B.10}$$

or, equivalently,

$$\mathrm{E}\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}\right]^{+}\mathrm{E}\left[\bar{\Phi}_{i,\mathbf{x}}\hat{\mathbf{x}}\right] = \mathrm{E}\left[\nabla\bar{\boldsymbol{\Phi}}_{\mathbf{x}}(\hat{\mathbf{x}})\right], \tag{B.11}$$

which proves the Proposition in Section 4.1, and thus, Eq. (51) holds true.

References

[1] Y.K. Lin, Probabilistic Theory of Structural Dynamics, McGraw-Hill, New York, 1967.
[2] M. Grigoriu, Stochastic Calculus, Applications in Science and Engineering, Birkhauser, Boston, 2002.
[3] J. Li, J. Chen, Stochastic Dynamics of Structures, J. Wiley and Sons, Singapore, 2009.
[4] A. Naess, J.M. Johnsen, Response statistics of nonlinear, compliant offshore structures by the path integral solution method, Probab. Eng. Mech. 8 (2) (1993) 91–106.
[5] I.A. Kougioumtzoglou, A. Di Matteo, P.D. Spanos, A. Pirrotta, M. Di Paola, An efficient wiener path integral technique formulation for stochastic response determination of nonlinear MDOF systems, ASME J. Appl. Mech. 82 (10) (2015) http://dx.doi.org/10.1115/1.4030890. (101005-101005-7).
[6] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, Dover Publications, New York, 2003.
[7] W.O. Schiehlen, Dynamics of complex multibody systems, SM Arch. 9 (1984) 159–195.
[8] P.E. Nikravesh, R.A. Wehage, O.K. Kwon, Euler parameters in computational kinematics and dynamics. Part 1, ASME J. Mech. Trans. Autom. 107 (3) (1985) 358–365, http://dx.doi.org/10.1115/1.3260722.
[9] R. Featherstone, Robot Dynamics Algorithms, Kluwer, New York, 1987.
[10] D.S. Bae, E.J. Haug, A recursive formulation for constraint mechanical system dynamics: Part II. Closed loop systems, Mech. Struct. Mach. 15 (4) (1987) 481–506.
[11] J.H. Critchley, K.S. Anderson, A generalized recursive coordinate reduction method for multi-body dynamic systems, Int. J. Multiscale Comp. Eng. 1 (2–3) (2003) 181–200.
[12] A. Laulusa, O.A. Bauchau, Review of classical approaches for constraint enforcement in multibody systems, ASME J. Comput. Nonlinear Dyn. 3 (1) (2007) 011004–011008, http://dx.doi.org/10.1115/1.2803257.
[13] A.D. Schutte, F.E. Udwadia, New approach to the modeling of complex multi-body dynamical systems, J. Appl. Mech. 78 (2) . (021018-021018-11).
[14] L. Mariti, N.P. Belfiore, E. Pennestrì, P.P. Valentini, Comparison of solution strategies for multibody dynamics equations, Int. J. Numer. Methods Eng. 88 (7) (2011) 637–656.
[15] F.E. Udwadia, T. Wanichanon, On general nonlinear constrained mechanical systems, Numer. Algebra Control Optim. 3 (3) (2013) 425–443.
[16] F.E. Udwadia, P. Phohomsiri, Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics, Proc. R. Soc. A 462 (2006) 2097–2117.
[17] V.C. Fragkoulis, I.A. Kougioumtzoglou, A.A. Pantelous, Linear random vibration of structural systems with singular matrices, ASCE J. Eng. Mech. 142 (2) . (04015081-11).
[18] V.C. Fragkoulis, I.A. Kougioumtzoglou, A.A. Pantelous, Statistical linearization of nonlinear structural systems with singular matrices, ASCE J. Eng. Mech. 142 (9) . (04016063-11).
[19] E.N. Antoniou, A.A. Pantelous, I.A. Kougioumtzoglou, A. Pirrotta, Response determination of linear dynamical systems with singular matrices: a polynomial matrix theory approach, Appl. Math. Model. 42 (2017) 423–440.
[20] S.L. Campbell Jr., C.D. Meyer, Generalized Inverses of Linear Transformations, Dover Publications, New York, 1979.
[21] T.N.E. Greville, Some applications of the pseudo inverse of a matrix, SIAM Rev. 2 (1960) 15–22.
[22] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, 2nd ed. Springer-Verlag, New York, 2003.
[23] R. Piziak, P.L. Odell, Full rank factorization of matrices, Math. Mag. 72 (3) (1999) 193–201.
[24] A.A. Pantelous, A. Pirrotta, Modal analysis of multi-degree-or-freedom systems with singular matrices: analytical dynamics approach, ASCE J. Eng. Mech. http://dx.doi.org/10.1061/(ASCE)EM.1943–7889.0001232.
[25] R.W. Clough, J. Penzien, Dynamics of Structures, 2nd ed. McGraw-Hill, New York, 1993.
[26] L. Socha, Linearization methods for stochastic dynamic systems, Lect. Notes Phys. 730, Springer-Verlag, Berlin Heidelberg, 2008.
[27] T.S. Atalik, S. Utku, Stochastic linearization of multi-degree-of-freedom non-linear systems, Earthq. Eng. Struct. Dyn. 4 (4) (1976) 411–420.
[28] J. Nocedal, S. Wright, Numerical Optimization, 2nd ed. Springer, New York, 2006.
[29] C. Lanczos, The Variational Principles of Mechanics, 4th ed. Dover, New York, 1986.
[30] F.E. Udwadia, A.D. Schutte, Equations of motion for general constrained systems in Lagrangian mechanics, Acta Mech. 213 (1) (2010) 111–129.
[31] C.G. Khatri, Some results for the singular normal multivariate regression models, Sankyā Ser. A 30 (3) (1968) 267–280.
[32] C.R. Rao, Linear Statistical Inference and Its Applications, 2nd ed. John Wiley and Sons, New York, USA, 1973.
[33] M.S. Srivastava, C.G. Khatri, An Introduction to Multivariate Statistics, North-Holland, New York, 1979.