A Survey of Classical and Recent Results in RLC Circuit Synthesis

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Abstract— **The motivation provided by mechanical network synthesis to make a fresh attack on certain questions in circuit synthesis will be briefly recalled. The classical early work on RLC synthesis, beginning with the works of Foster and Cauer and culminating in the Bott-Duffin construction, will be explained in a tutorial manner. Recent work on RLC synthesis by Jiang and Smith and Hughes and Smith will be introduced. The proof in T.H. Hughes and M.C. Smith, 2014, "On the minimality and uniqueness of the Bott-Duffin procedure", *IEEE Trans. Aut. Contr.*, (to appear), showing the surprising result that the Bott-Duffin construction for a biquadratic minimum function is the simplest possible among series-parallel circuits, will be explained.**

Key words. Circuit synthesis, passivity, realisation, inerter

AMS subject classification. 70Q05, 93C05, 94C99

I. Background

The question of the fundamental performance limits achievable in a *passive* suspension system led to a realisation question in mechanical networks [1]. To allow an exact analogy between electrical and mechanical networks for the purpose of synthesising an arbitrary positive-real function a new ideal mechanical modelling element was proposed, the *inerter*. The talk will begin by recalling the reasoning and justification for such a definition, the approximations involved in dealing with practical devices, and the motivation for reopening the study of RLC (resistor-inductor-capacitor) realisations.

The talk will recall some of the main results of classical circuit synthesis: Foster's Reactance theorem, the constructions of Brune and Darlington, reactance extraction, and the Bott-Duffin procedure. An introduction will also be provided to the enumerative method for RLC synthesis and the recent results of [2].

The publication in 1949 of the procedure of Bott and Duffin [3] was, and still is, a landmark in circuit synthesis. It provided the first proof that any positive-real function may be realised as the impedance of an RLC circuit. The apparent extravagance in the number of elements used by the procedure has persisted for more than 60 years as a major puzzle. In the forthcoming paper [4], it is shown, surprisingly, that the circuits produced by this procedure contain the least possible number of reactive elements and the least possible number of resistors for the realisation
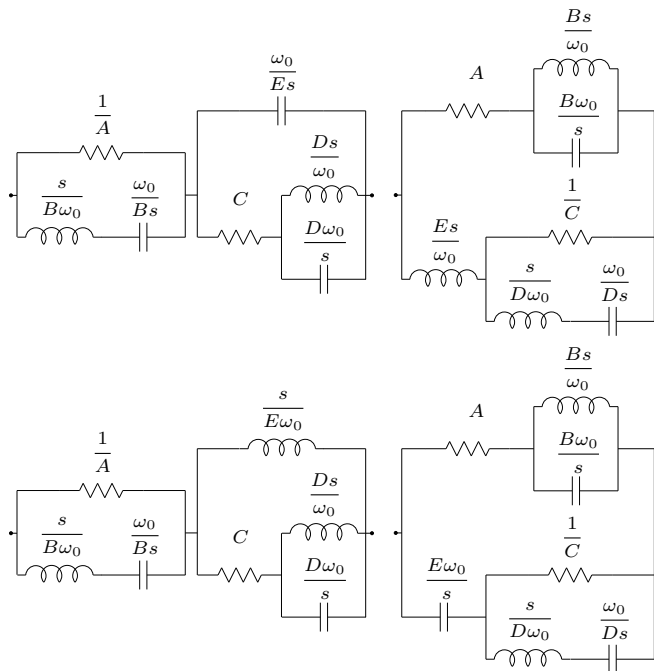
Fig. 1.   Series-parallel circuit realisations of a minimum function which contain five reactive elements. $A, B, C, D, E > 0$.

of certain positive-real functions among all series-parallel circuits. The talk will explain this proof. In the next section we describe some of the key results from that paper.

II. Minimality and Uniqueness of the Bott-Duffin Realisation Procedure

In [4], series-parallel realisations of a particular class of positive-real functions known as *minimum functions* are examined, with a particular emphasis on biquadratic minimum functions. A minimum function is a positive-real function $H(s)$ which has no poles or zeros on the imaginary axis, and which possesses a minimum frequency $\omega_0 > 0$ at which $\Re(H(j\omega_0)) = 0$. A series of lemmas in [4] characterised those series-parallel circuits whose impedance (or admittance) satisfies some of the conditions of a minimum function, in particular the condition for the real part to be zero at some point on the imaginary axis. In this manner, it was shown that there are no series parallel circuits which contain fewer than five reactive elements and which realise a minimum function. In addition, those minimum functions which are realised by series-parallel circuits containing exactly five reactive elements were found. The following theorem, from [4], summarises these results.
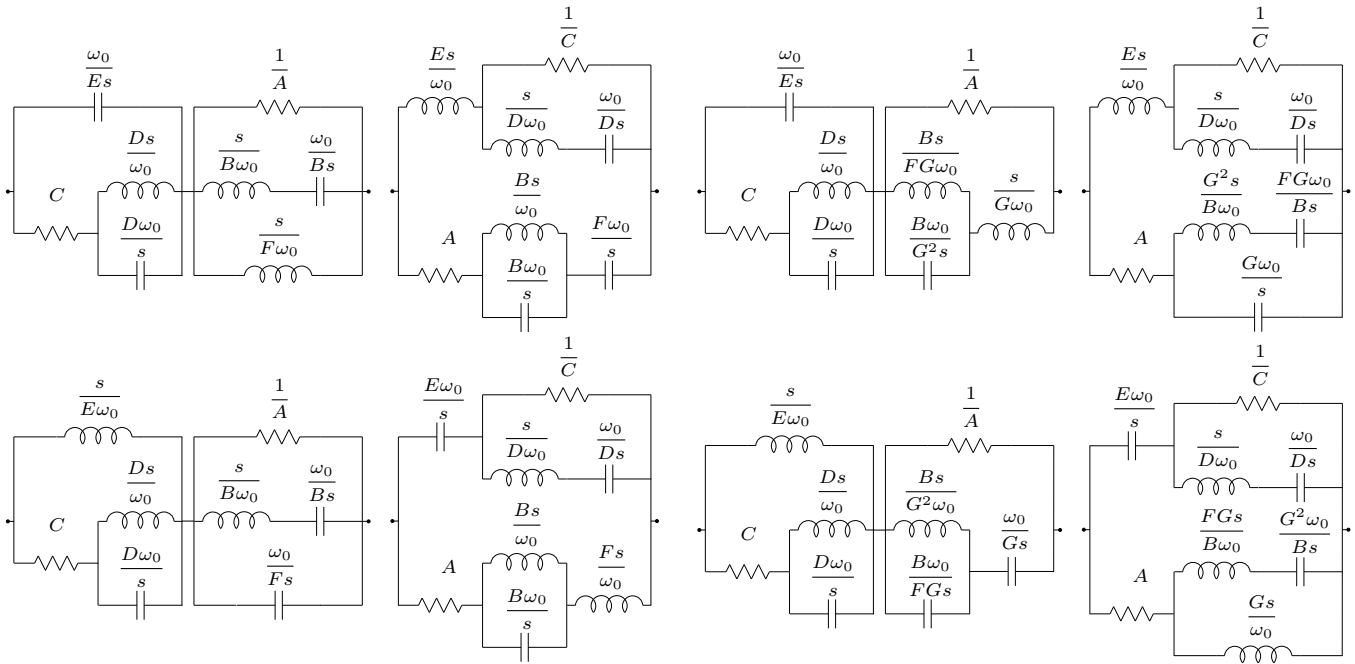
Fig. 2. Series-parallel circuit realisations of a biquadratic minimum function which contain six reactive elements. $G = B+F$, $B = AD/C$, $F = A/(CE)$, $D = (\sqrt{AC} - 1)(A + CE^2)/(AE)$, $A, E > 0$, $AC > 1$.

Theorem 1 ([4], Theorem 5): Let $H(s)$ be the impedance of a series-parallel circuit N where $H(s)$ is a minimum function with ω_0 a minimum frequency. Then N contains at least five reactive elements and at least two resistors. Furthermore, if N contains exactly five reactive elements, then either $H(s)$, $H(\omega_0^2/s)$, $1/H(s)$, or $1/H(\omega_0^2/s)$ takes the form

$$\frac{s^2+\omega_0^2}{As^2+B\omega_0 s+A\omega_0^2} + \left(\frac{s^2+\omega_0^2}{Cs^2+D\omega_0 s+C\omega_0^2} + \frac{Es}{\omega_0}\right)^{-1} \quad (1)$$

for some $A, B, C, D, E > 0$. If in addition N contains exactly two resistors, then N is one of the circuits in Fig. 1.

By showing that the function in equation (1) cannot be biquadratic, the following theorem was obtained:

Theorem 2 ([4], Theorem 6): Let $H(s)$ be the impedance of a series-parallel circuit N where $H(s)$ is a biquadratic minimum function. Then N contains at least six reactive elements and at least two resistors.

The circuits produced by the Bott-Duffin procedure for the realisation of a biquadratic minimum function contain exactly six reactive elements and two resistors. From Theorem 2, it follows that these circuits contain the least possible number of reactive elements, and the least possible number of resistors, among all series-parallel circuits which realise a biquadratic minimum function.

By considering those series-parallel circuits which contain exactly six reactive elements and exactly two resistors and which realise a minimum function, the following theorem was shown:

Theorem 3 ([4], Theorem 10): Let N be a series-parallel circuit which contains at most six reactive elements and at most two resistors. Further let N have impedance $H(s)$

which is a biquadratic minimum function with minimum frequency ω_0. Then N is one of the circuits in Fig. 2.

The circuit on the top left hand corner of Fig. 2 is the circuit obtained from the Bott-Duffin procedure for the realisation of a biquadratic minimum function whose imaginary part is positive at the minimum frequency. Its immediate neighbour is the circuit obtained by Bott and Duffin in the case where the imaginary part is negative at the minimum frequency. The remaining circuits can be obtained from these two by the application of simple circuit transformations. As a consequence of Theorem 3, these are the only series-parallel circuits which contain six reactive elements and two resistors and which realise a biquadratic minimum function.

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