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Tilings and other combinatorial results

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ABSTRACT

In this dissertation we treat three tiling problems and three problems in combinatorial geometry, extremal graph theory and sparse Ramsey theory.

We first consider tilings of $\mathbb{Z}^n$. In this setting a tile is just a finite subset of $\mathbb{Z}^n$. We say that a given tile T tiles $\mathbb{Z}^n$ if we can use isometric copies of T to form a partition of $\mathbb{Z}^n$. Obviously, some tiles tile $\mathbb{Z}^n$ and some do not. Chalcraft observed that some tiles of the latter kind do tile $\mathbb{Z}^{n+1}$ or at least $\mathbb{Z}^d$ for some $d > n$. He conjectured that, in fact, such d exists for any given tile. We prove this conjecture in Chapter 2. We begin this chapter by analysing a key first case (an interval in the line, with one point removed) which serves as an introduction to some of the important ideas that recur in all three of our tiling results.

Next, we present a related problem concerning the Boolean lattice $2^{[n]}$. We prove a conjecture of Lonc, which states that for any poset P of size a power of 2, if P has a greatest and a least element, then there is a positive integer k such that $2^{[k]}$ can be partitioned into isomorphic copies of P. We present this result in Chapter 3. In the same chapter we prove a more general theorem that can be useful in other scenarios where a product space is being partitioned into copies of a given set. This is the most technically complicated chapter of this dissertation.

The third tiling problem is about vertex-partitions of the hypercube graph Q_n. Offner asked the following question: if G is a subgraph of Q_n such that the order of G is a power of 2, must it be possible to partition the vertex set of Q_d, for some d, into isomorphic copies of G? In Chapter 4 we answer this question in the affirmative. Our proof makes use of the machinery set up for the previous result and also includes some new ideas.

We follow up with a question in combinatorial geometry. For a set $P \subset \mathbb{R}^2$, a line in P is a maximal collinear subset of P. Pór and Wood considered what happens if a finite set $P \subset \mathbb{R}^2$ with no large lines is coloured with a fixed number of colours. In particular, they wanted to know whether monochromatic lines can always be found in such colourings, provided that $|P|$ is large. They conjectured that for all $k, l \geq 2$ there exists an $n \geq 2$ with the following property: if $|P| \geq n$ and if P does not contain a line of cardinality larger than l, then every colouring of P with k colours produces a monochromatic line. Their conjecture is obviously true for $l = 2$ and in the case $k = 2$ it is an immediate corollary of the Motzkin–Rabin theorem. We construct arbitrarily large counterexamples for the case $k = l = 3$, disproving the conjecture for all $k, l \geq 3$. Our construction is short, and it is based on simple properties of cubic curves. It is presented in Chapter 5.

We move on to a problem in extremal graph theory. For any graph, we say that a given edge is triangular if it forms a triangle with two other edges. A natural question

arises: how few triangular edges can there be in a graph with a given number of vertices and edges? For graphs of sufficiently large order we prove a conjecture of Füredi and Maleki that gives an exact formula for this minimum. Our proof, which is given in Chapter 6, is fairly long and it consists of a repeated application of two main ideas.

Finally, Chapter 7 is concerned with a question about degrees of vertices in directed hypergraphs. One natural way to prescribe an orientation to an r-uniform graph H is to assign for each of its edges one of the $r!$ possible orderings of its elements. Then, similarly to the in-degree and out-degree in graphs, for any vertex $v \in V(H)$ and any index $i \in [r]$, we define the i-degree of v to be the number of edges that have v in the i-th position of their ordering. More generally, for any set of p vertices A and any set of p indices $I \subset [r]$, we define the I-degree of A to be the number of edges that contain vertices A in precisely the positions labelled by I. Motivated by an old theorem of Hakimi, Caro and Hansberg were interested in determining whether a given r-uniform hypergraph admits an orientation where every set of p vertices has some I-degree equal to 0. They conjectured that a certain obvious Hall-type necessary condition is sufficient. We show that this is true for r large (for given p), but false in general. Our counterexample is based on a new technique in sparse Ramsey theory that may be of independent interest.

DECLARATION

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared here. The results in Chapter 2 were obtained in collaboration with Imre Leader and Ta Sheng Tan, and my contribution was about 33%. Chapter 3 is based on joint work with Imre Leader and István Tomon, and my contribution was about 50%. Chapter 6 contains results obtained together with Shoham Letzter, and my contribution was about 50%.

This dissertation is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution.

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Contents

CHAPTER 1

Introduction

This dissertation focuses on several topics in combinatorics. About half of it is devoted to tiling problems and the rest covers three questions in combinatorial geometry, extremal graph theory and sparse Ramsey theory. In this chapter we introduce these topics and give an overview of our results.

1 Tilings

We discuss three tiling problems coming from different areas of combinatorics. All three problems concern partitions of some product structure S^d of arbitrarily large dimension d into copies of a fixed set.

The first problem is motivated by the following fundamental question: how can we determine if a given polyomino P *tessellates* the plane, meaning that the plane can be covered by isometric copies of P overlapping only at the boundaries? Despite its appeal as a recreational mathematics problem, this question might be extremely hard or even impossible to solve. Indeed, following Berger's work on Wang tiles [2], Golomb [19] proved in 1970 that the problem of determining whether a finite set of polyominoes $P_1, \dots, P_n$ tessellates the plane is undecidable. It is not known whether this problem for single polyominoes is decidable.

We consider a variation of this question due to Chalcraft. Any polyomino can be seen as a finite connected subset of $\mathbb{Z}^2$. We generalise this definition to any number of dimensions by saying that a *tile* in $\mathbb{Z}^n$ is any non-empty finite subset of $\mathbb{Z}^n$; note that a tile does not have to be connected. Just as any polyomino can be made into a three-dimensional figure by giving it unit depth, any tile in $\mathbb{Z}^n$ can be naturally embedded into $\mathbb{Z}^m$ for any $m \geq n$. Now, even if a tile in $\mathbb{Z}^n$ does not tile $\mathbb{Z}^n$, it may still tile $\mathbb{Z}^m$ for some $m > n$ (we say that T *tiles* $\mathbb{Z}^m$ if $\mathbb{Z}^m$ can be partitioned into isometric copies of T); in fact, it is easy to construct one-dimensional tiles that tile $\mathbb{Z}^2$ but not $\mathbb{Z}$. Quite remarkably, Chalcraft conjectured that any tile tiles $\mathbb{Z}^m$

for sufficiently large m. According to its author, this conjecture dates back to at least 1992, but the earliest surviving record of it seems to be from 2008 [46, 47]. We prove this conjecture in Chapter 2. To make the presentation easier to follow, we first examine a key first case where the tile T is one-dimensional and has very simple structure: it is an interval with one element missing. Chapter 2 is based on a joint paper with Leader and Tan [28].

In Chapter 3, which contains joint work with Leader and Tomon [29], we discuss a tiling problem in the context of partially ordered sets (posets). Given a finite poset P, we wish to determine whether there exists some m for which the Boolean lattice $2^{[m]}$ can be partitioned into isomorphic copies of P. Of course, for such an m to exist, P must have a minimal and a maximal element, and its order must be a power of 2. We prove that these conditions are in fact sufficient, thus resolving a conjecture of Lonc [42] from 1991.

The third tiling problem is about vertex-partitions of the hypercube graph Q_n. There is a rich background of problems concerning vertex-partitions of dense graphs (see, for example, [5, 11, 12, 32, 40]), but our result is closer in flavour to the work of Hamming [35] from the late 1940s on perfect error correcting codes. Indeed, a perfect k-error correcting code is a partition of the vertices of Q_n into Hamming balls of radius k. We prove the following theorem, which resolves a conjecture of Offner [53] from 2014. For any graph H, if H is a subgraph of Q_d for some d and if the order of H is a power of 2, then, for all sufficiently large n, the hypercube Q_n admits an H-factor. This proof is presented in Chapter 4 and it also appears in [27].

In the three tiling problems that we have presented our aim is to partition a product space. As a result, our proofs follow a similar structure. In particular, one idea is essential for all three proofs: we show that, for some r and m, the relevant product space S^m admits coverings of two following types by copies of the tile:

- A covering of the first type covers every element of S^m exactly r times.
- A covering of the second type covers every element $x \in S^m$ exactly $1 + a_x r$ times, where $a_x \in \mathbb{N} \cup \{0\}$ is allowed to vary with x.

Crucially, we prove that the existence of such coverings implies the existence of the desired partition of $S^{m'}$ for some $m' \geq m$. Therefore, our task reduces to constructing coverings of these two types. In the three proofs we obtain them by using different methods:

- It is easy to construct the two coverings of $\mathbb{Z}^m$, but $\mathbb{Z}^m$ being infinite makes it slightly harder to use them to obtain the desired partition.

- We use a somewhat technical averaging argument to construct a covering of the Boolean lattice $2^{[m]}$ of the first type. A covering of the second type is more difficult to obtain. The key idea here is that two coverings can 'cancel each other out' when working modulo r.

- Obtaining a covering of Q_m of the first type is trivial. We construct a covering of the second type by induction on the dimension m. We combine the 'cancellation' idea from above with the observation that, if we split a subgraph $H \subset Q_d$ into two halves $H^-, H^+ \subset Q_{d-1}$, then certain coverings of Q_{m-1} by copies of H^- extend to coverings of Q_m by copies of H.

2 Multicoloured lines in the plane

In Chapter 5 we consider a problem about finite sets of points in the plane. We define a *line* in $P \subset \mathbb{R}^2$ to be a maximal set of collinear points in P and we try to find a line that is 'small' in some sense. A very natural notion is that of *ordinary lines*, which are lines containing exactly two points of P. The Sylvester-Gallai theorem [17, 59] from 1944 asserts that if P is finite and if not all of its points lie on one line then P has at least one ordinary line. In 2013, Green and Tao [21] proved a much stronger result: they showed that, in fact, P must have at least $|P|/2$ ordinary lines, provided that $|P|$ is sufficiently large, thus resolving a longstanding conjecture of Dirac and Motzkin [6].

We examine a different notion of a 'small' line. Instead of thinking about the cardinality of lines, we seek monochromatic lines in finite colourings of P. The Motzkin–Rabin theorem [49] from the 1960s says that, if P is finite and not contained in a single line, then every colouring of P with two colours produces a monochromatic line. In an attempt to generalise this result, Pór and Wood [54] conjectured in 2010 that for any fixed integers $k, l \geq 1$, every sufficiently large finite set $P \subset \mathbb{R}^2$ either contains a line on at least $l+1$ points or is such that every colouring of P with k colours produces a monochromatic line. We disprove this conjecture by constructing a counterexample, which makes use of simple properties of cubic curves. The results of this chapter appear in [25].

3 Minimising the number of triangular edges

Chapter 6, which is based on a joint paper with Letzter [30], contains a result in extremal graph theory. We study the following question: what is the smallest possible number of edges contained in triangles in a graph of fixed order n and size e? For brevity, we call edges contained in triangles *triangular*.

If instead of minimising the number of triangular edges we were trying to minimise the number of triangles, we would arrive at another question which is fairly well understood by now. Here the story begins in 1941 with an observation of Rademacher [55] that an n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges must have at least $\lfloor n^2/4 \rfloor$ triangles. This bound is realised by a balanced bipartite graph with an edge added to one of the vertex classes. In 1955, Erdös [9] conjectured that a similar bound holds for any n-vertex graph with $\lfloor n^2/4 \rfloor + l$ edges, where $1 \leq l \leq \lfloor n/2 \rfloor$, that is, any such graph should have at least $l\lfloor n^2/4 \rfloor$ triangles. This conjecture was proved by Lovász and Simonovits [43] in 1975. More recently, in 2008, Razborov [57] asymptotically determined the minimal possible number of triangles in an n-vertex graph with $\lfloor n^2/4 \rfloor + l$ edges where $l = \Omega(n^2)$.

Returning to the number of triangular edges, Erdös, Faudree and Rousseau [13] proved in 1992 that any n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges has at least $2\lfloor n/2 \rfloor + 1$ triangular edges. This bound is best possible. What if the graph has more edges? A natural example comes to mind: we take suitably chosen integers $a, b, c \geq 0$ and define the graph $G(a, b, c)$, which consists of a clique A of order a and two independent sets B, C of sizes b, c respectively, such that all edges between B and $A \cup C$ are present, while all possible between A and C are missing (see Figure 1.1). In 2014 Füredi and Maleki [16] conjectured that, for any fixed n and e, the smallest

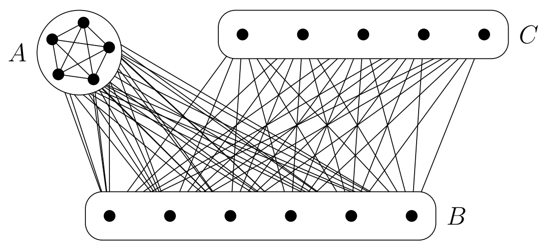

Figure 1.1: The graph $G(a, b, c)$ (here $a = 5, b = 6, c = 5$).

number of triangular edges is achieved by (a subgraph of) $G(a, b, c)$ for some a, b, c. They proved their conjecture approximately, with an additive error term of order $O(n)$. We resolve the conjecture of Füredi and Maleki for sufficiently large n. Our bound on n does not depend on e, that is, we establish a constant n_0 such that the conjecture holds for all $n \geq n_0$ and all e such that $\lfloor n^2/4 \rfloor + 1 \leq e \leq \binom{n}{2}$.

Our proof does not directly build on the approximate result of Füredi and Maleki, but it incorporates some of their key ideas. In particular, we treat the given graph as being weighted and we keep shifting the weights of its vertices in a manner that does not decrease the total weight of the triangular edges, until we obtain a weighted graph of a relatively simple structure. However, in contrast to the approximate result, we need our final graph to correspond to a blow-up of a non-weighted graph, and so all weights have to be integral. One of our main new ideas is a trick that replaces a large independent set of vertices by a much smaller set of vertices of larger, but still integral, weights. However, this trick only somewhat simplifies the structure of the given graph, and we keep modifying it in various ways until it takes the desired form.

4 Directed hypergraphs and sparse Ramsey theory

In the final chapter of this dissertation, Chapter 7, we study a question of Caro and Hansberg [3] from 2012 about degrees of sets of vertices in directed hypergraphs. There is more than one natural way to define an orientation of an edge of a hypergraph; Caro and Hansberg considers the notion where an *orientation* of $e = \{v_1, \dots, v_r\}$ is a choice of one of the possible $r!$ orderings of $v_1, \dots, v_r$. We say that a hypergraph is *directed* if its edges have orientations. The notions of in-degree and out-degree extend to this context: given a directed r-uniform hypergraph H, for any vertex $v \in V(H)$ and any index $i \in [r]$ we define the i-degree of v to be the number of edges that have vertex v in the i-th position of their orientation. Moreover, a similar definition can be made for sets of multiple vertices: for distinct $v_1, \dots, v_p \in V(H)$ and distinct $i_1, \dots, i_p \in [r]$ we define the $\{i_1, \dots, i_p\}$-degree of $\{v_1, \dots, v_p\}$ to be the number of edges whose orientation contains precisely the vertices $v_1, \dots, v_p$, in some order, in positions $i_1, \dots, i_p$.

Caro and Hansberg considered r-uniform hypergraphs H that admit an orientation such that for every set $A \subset V(H)$ of p vertices there exists a set $I \subset [r]$ of p indices such that the I-degree of A is 0. They derived a condition on the density of certain subhypergraphs of H, which is necessary for such an orientation to exist, and asked whether it is sufficient. We answer their question in the negative, showing that the condition is not sufficient for $r = 4, p = 2$. However, we prove that it is sufficient for any fixed p and sufficiently large r.

We attack this problem by relating it to a question about set mappings. We make a general conjecture, of which we are able to prove enough to show that the question of Caro and Hansberg has a positive answer if p is fixed and r is large. Conversely, the case $r = 4, p = 2$ is a sparse-Ramsey-type problem for graphs: we seek a graph G

that admits a 6-colouring avoiding a monochromatic 4-clique but whose every such colouring produces a non-monochromatic 4-clique whose colouring follows a certain pattern. Our construction of such G is inspired by the amalgamation method, which was introduced by Nešetřil and Rödl [50–52] in the 1970s.

The results of this chapter appear in [26].

CHAPTER 2

Tilings of $\mathbb{Z}^n$

1 Introduction

Let T be a *tile*, by which we mean a finite non-empty subset of $\mathbb{Z}^n$ for some n. It is natural to ask if $\mathbb{Z}^n$ can be partitioned into copies of T, that is, into subsets each of which is isometric to T. If such a partition exists, we say that T *tiles* $\mathbb{Z}^n$.

For instance, consider the following tiling of $\mathbb{Z}^2$ by copies of the C-shaped pentomino.

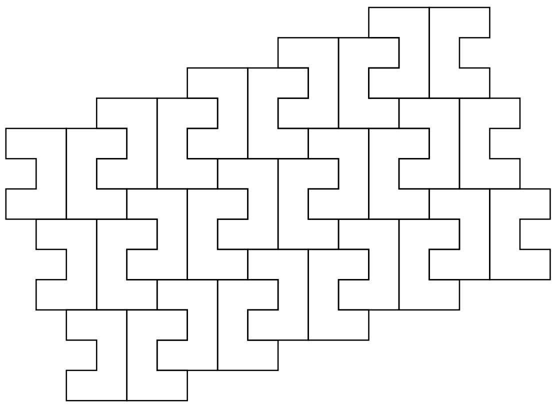

Figure 2.1: The C-shaped pentomino tiles $\mathbb{Z}^2$.

As another example, the one-dimensional tile `X.X` (to be understood as $\{1, 3\}$) tiles $\mathbb{Z}$, and so does `XX.X`. On the other hand, `XX.XX` is a one-dimensional tile that does not tile $\mathbb{Z}$. Does it tile some space of higher dimension? The following diagram shows that `XX.XX` does tile $\mathbb{Z}^2$.

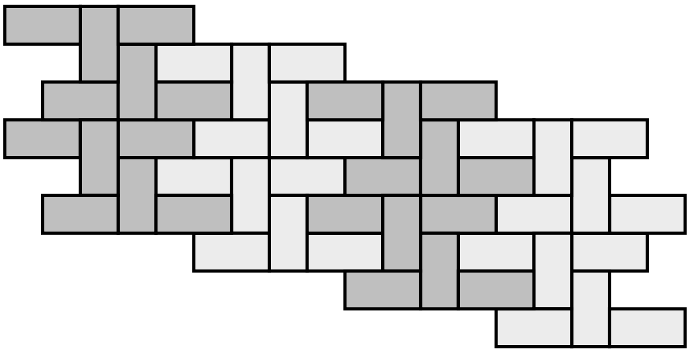

Figure 2.2: This pattern is formed from disjoint copies of `XX.XX`; copies of the pattern may be stacked vertically to tile $\mathbb{Z}^2$.

A similar pattern works for `XXX.XX` in $\mathbb{Z}^2$. However, one can check by hand that `XXX.XXX` does not tile $\mathbb{Z}^2$. Does it tile $\mathbb{Z}^3$, or $\mathbb{Z}^d$ for some d? What about more complicated one-dimensional tiles?

Let us now consider a couple of two-dimensional examples. Let T denote the 3×3 square with the central point removed. Clearly T does not tile $\mathbb{Z}^2$, since the hole in a copy of T cannot be filled. However, in $\mathbb{Z}^3$ there is enough space for one copy of T to fill the hole of another. (Of course, this in no way implies that T does tile $\mathbb{Z}^3$.)

For a 'worse' example, consider the 5×5 square with the central point removed. Two copies of such tile cannot be interlinked in $\mathbb{Z}^3$. However, there is, of course, enough space in $\mathbb{Z}^4$ to fill the hole, as demonstrated in the following diagram.

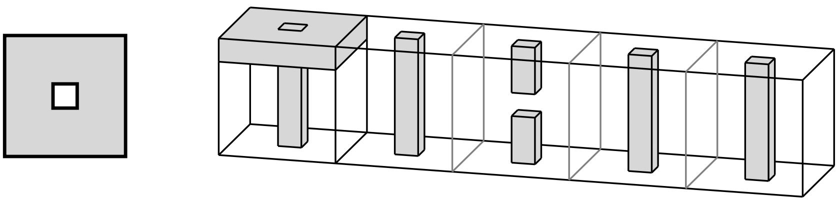

Figure 2.3: The diagram on the right is four-dimensional and shows a $5 \times 5 \times 5 \times 5$ region of $\mathbb{Z}^4$. Let x_1, x_2, x_3, x_4 be the directions of $\mathbb{Z}^4$. Each of the five $5 \times 5 \times 5$ cubes corresponds to a fixed value of x_1. Increasing the value of x_1 by 1 means jumping from a cube to the cube on its right. This four-dimensional diagram contains two copies of the two-dimensional tile depicted on the left side. One copy is horizontal and can be found in the top left part of the diagram. The second copy is formed by the vertical columns.

Chalcraft [46, 47] made a rather daring conjecture that every tile $T \subset \mathbb{Z}$, or even $T \subset \mathbb{Z}^n$, does tile $\mathbb{Z}^d$ for some d.

Conjecture 2.1 (Chalcraft [46, 47]). *Let $T \subset \mathbb{Z}^n$ be a tile. Then T tiles $\mathbb{Z}^d$ for some d.*

It is not important if reflections are allowed when forming copies of a tile. Indeed, any reflection of an n-dimensional tile can be obtained by rotating it in $n+1$ dimensions. It is also not important if only connected tiles are considered, as it is an easy exercise to show that any disconnected tile in $\mathbb{Z}^n$ tiles a connected tile in $\mathbb{Z}^{2n}$.

In this chapter we prove Chalcraft's conjecture.

Theorem 2.2. *Let $T \subset \mathbb{Z}^n$ be a tile. Then T tiles $\mathbb{Z}^d$ for some d.*

Interestingly, the problem is not any easier for tiles $T \subset \mathbb{Z}$. Indeed, the proof for one-dimensional tiles seems to us to be as hard as the general problem.

The plan of the chapter is as follows. In Section 2 we prove a special case of the theorem, namely when T is an interval in $\mathbb{Z}$ with one point removed. The aim of this section is to demonstrate some of the key ideas in a simple setting. The proof of the general case builds on these ideas and on several additional ingredients. We give a proof of Theorem 2.2 in Section 3. Finally, in Section 4 we give some open problems.

We end the section with some general background. A lot of work has been done about tiling $\mathbb{Z}^2$ by polyominoes (a polyomino being a connected tile in $\mathbb{Z}^2$). Golomb [18] proved that every polyomino of size at most 6 tiles $\mathbb{Z}^2$. In [19] he also proved that there is no algorithm which decides, given a finite set of polyominoes, if $\mathbb{Z}^2$ can be tiled with their copies – this is based on the work of Berger [2], who showed a similar undecidability result for Wang tiles (which are certain coloured squares). However, it is not known if such an algorithm exists for single polyominoes. A related unsolved problem is to determine whether there is a polyomino which tiles $\mathbb{Z}^2$ but such that every tiling is non-periodic. On the other hand, Wijshoff and van Leeuwen [63] found an algorithm which determines if disjoint translates (rather than translates, rotations and reflections) of a single given polyomino tile $\mathbb{Z}^2$. A vast number of results and questions regarding tilings of $\mathbb{Z}^2$ by polyominoes and other shapes are compiled in Grünbaum and Shephard [24].

One may also wish to know if a given polyomino tiles some finite region of $\mathbb{Z}^2$, say a rectangle. This class of questions has also received significant attention, producing many beautiful techniques and invariants – see, for example, [4, 20, 39]. In the context of this chapter, we observe that there are tiles which cannot tile any (finite) cuboid of any dimension. For example, consider the plus-shaped tile of size 5 in $\mathbb{Z}^2$: this tile cannot cover the corners of any cuboid. In fact, there are one-dimensional such tiles. For example, let $T \subset \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$, be a symmetric tile (meaning that $-T$ is a translate of T) whose associated polynomial $p(x) = \sum_{t \in T} x^t$ does not have all of its non-zero roots on the unit circle – it turns out that such T cannot tile a cuboid (see [47]). On the other hand, the situation turns out to be different if we switch from Euclidean to ℓ_1 metric. We will discuss this variant of the problem in Chapter 4.

2 Tiling $\mathbb{Z}^d$ by an interval minus a single point

2.1 Overview

Before starting the proof of Theorem 2.2, we demonstrate some of the key ideas in a simple setting, where the tile is a one-dimensional interval with one point removed. We give a self-contained proof of the general case in Section 3, but it will build on the ideas in this section.

We write $[k] = \{1, \dots, k\}$.

Theorem 2.3. *Fix integers $k \geq 3$ and $i \in \{2, \dots, k-1\}$ and let T be the tile $[k] \setminus \{i\}$. Then T tiles $\mathbb{Z}^d$ for some d.*

The tile $T = [k] \setminus \{i\}$ will remain fixed throughout this section.

The proof is driven by two key ideas. A first natural idea is to use *strings*, where a string is a one-dimensional infinite line in $\mathbb{Z}^d$ with every k-th point removed. Note that any string is a union of disjoint copies of T. An obvious way to use strings would be to partition $\mathbb{Z}^d$ into them. Although this is an attractive idea, it is not possible, for the following simple reason: if we consider just the fixed cuboid $[k]^d \subset \mathbb{Z}^d$, then every string intersects it in exactly 0 or $k-1$ points, but the order of $[k]^d$ is not divisible by $k-1$.

This suggests a refinement of the idea. We will try to use strings parallel to $d-1$ of the d directions, while the remaining direction will be special and copies of T parallel to it will be used even without forming strings. In other words, we will view $\mathbb{Z}^d$ as $\mathbb{Z} \times \mathbb{Z}^{d-1}$, that is, as being partitioned into $(d-1)$-dimensional *slices* according to the value of the first coordinate. We will first put down some tiles parallel to the first direction (each such tile intersects multiple slices), and then complete the tilings in each slice separately by strings.

To do this we need another idea. What subsets of $\mathbb{Z}^{d-1}$ can be tiled by strings? Note that a partial tiling of $\mathbb{Z}^{d-1}$ by strings can be identified with a partial tiling of the discrete torus $\mathbb{Z}_k^{d-1}$ (where $\mathbb{Z}_k$ denotes the integers modulo k), where a tile in $\mathbb{Z}_k^{d-1}$ means any line with one point removed. The size of $\mathbb{Z}_k^{d-1}$ is $k^{d-1} \equiv 1 \pmod{k-1}$, so any such partial tiling of $\mathbb{Z}_k^{d-1}$ must leave out $1 \pmod{k-1}$ points. Of course, it is far from true that any subset of $\mathbb{Z}_k^{d-1}$ of size a multiple of $k-1$ may be partitioned into tiles. However, our plan is to find a large supply of sets that do have this property. In particular, it turns out that a key idea will be to find a large set $C \subset \mathbb{Z}_k^{d-1}$ such that for any choice of distinct elements $x_1, \ldots, x_m \in C$ with $m \equiv 1 \pmod{k-1}$, T does tile $\mathbb{Z}_k^{d-1} \setminus \{x_1, \ldots, x_m\}$.

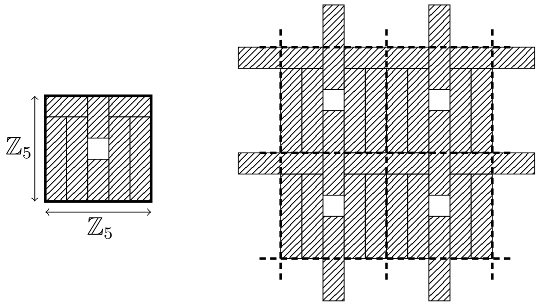

Figure 2.4: A partial tiling of $\mathbb{Z}_k^2$ (here $k = 5$) corresponds to a partial tiling of $\mathbb{Z}^2$ by strings.

These ideas work together as follows (see Figure 2.5). First, in $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$ (for large d) we find a subset X which is a disjoint union of translates of $T \times \{0\}^{d-1}$ and has the property that for any $n \in \mathbb{Z}$ the set $\{x \in \mathbb{Z}_k^{d-1} : (n, x) \in X\}$ is a subset of C of size congruent to 1 modulo $k-1$. Then T tiles $(\{n\} \times \mathbb{Z}_k^{d-1}) \setminus X$. This holds for all $n \in \mathbb{Z}$, so in fact T tiles $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$, and hence it tiles $\mathbb{Z}^d$, establishing Theorem 2.3.

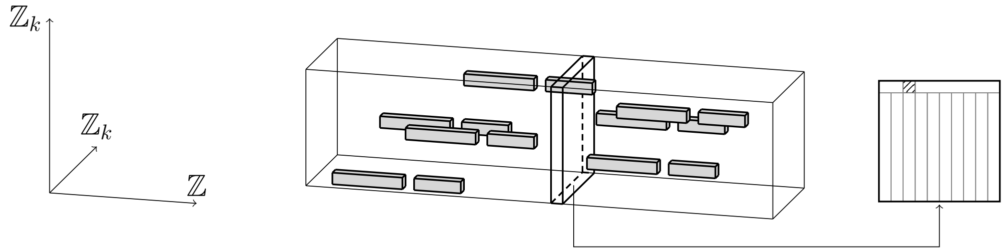

Figure 2.5: The aim is to put down tiles parallel to one of the directions so that the remainder of each slice could be tiled by strings. This diagram only symbolically visualises this principle. In particular, the slices here are two-dimensional, while in the proof they can have much higher dimension.

The rest of this section is organised as follows. In Section 2.2 we consider partial tilings by strings. In Section 2.3 we consider the special direction. Both ideas are combined in Section 2.4, where a full proof of Theorem 2.3 is given.

2.2 Tiling $\mathbb{Z}_k^d$ with some elements removed

For any $1 \le j \le d$, define the *j-th corner* of $\mathbb{Z}_k^d$ to be $c_{j,d}$ where

$$c_{j,d} = (\underbrace{0, \dots, 0, \overset{j\text{-th coordinate}}{\overset{\downarrow}{k-1}}, 0, \dots, 0}_{d \text{ coordinates}}) \in \mathbb{Z}_k^d.$$

Write $C_d = \{c_{j,d} : j = 1, \dots, d\}$ for the set of corners.

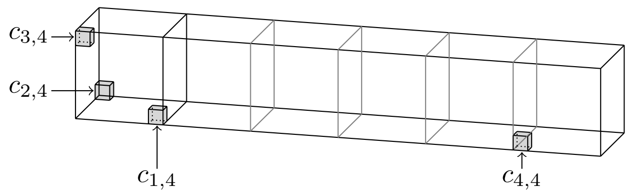

Figure 2.6: The set of corners C_4 when $k = 6$. In this diagram the space $\mathbb{Z}_6^4 = \{(x_1, x_2, x_3, x_4) : x_i \in \{0, \dots, 5\}\}$ is split from left to right, according to the value of x_4, into 6 three-dimensional slices.

Looking ahead, our aim later will be to provide some copies of T in the x_1-direction in $\mathbb{Z} \times \mathbb{Z}_k^d$, at heights corresponding to points of C_d, and in such a way that what remains in each $\mathbb{Z}_k^d$ can be partitioned into lines with one point removed. But first we need to create a useful supply of such subsets of $\mathbb{Z}_k^d$.

Recall that $|\mathbb{Z}_k^d| \equiv 1 \pmod{k-1}$, so if T tiles some set $X \subset \mathbb{Z}_k^d$ (here and in the remainder of this section T is identified with its image under the projection $\mathbb{Z} \to \mathbb{Z}_k$, so its copies in $\mathbb{Z}_k^d$ are lines with one point removed), then $|\mathbb{Z}_k^d \setminus X| \equiv 1 \pmod{k-1}$. In this section we will prove Lemma 2.4, which is an approximate converse of this statement.

Lemma 2.4. *Let $d \ge 1$ and suppose that $S \subset C_d$ is such that $|S| \equiv 1 \pmod{k-1}$ and $|S| \le d - \log_k d$. Then T tiles $\mathbb{Z}_k^d \setminus S$.*

In fact, this lemma holds even without the assumption that $|S| \le d - \log_k d$, but we keep it for the sake of simpler presentation.

We will prove Lemma 2.4 at the end of this section. Meanwhile, we collect the tools needed for the proof. In fact, there are several ways to prove Lemma 2.4. The method outlined here is quite general, and we will build on it in Section 3.

We start with a simple proposition.

Proposition 2.5. *Let $d \geq 1$ and $x \in \mathbb{Z}_k^d$. Then T tiles $\mathbb{Z}_k^d \setminus \{x\}$.*

Proof (see Figure 2.7). Use induction on d. If $d = 1$, then $\mathbb{Z}_k \setminus \{x\}$ is itself a translate of T. Now suppose that $d \geq 2$ and write $x = (x_1, \ldots, x_d)$, $\hat{x} = (x_1, \ldots, x_{d-1})$. By the induction hypothesis, for each $j \in \mathbb{Z}_k$, $(\mathbb{Z}_k^{d-1} \setminus \{\hat{x}\}) \times \{j\}$ can be tiled with copies of T. It remains to tile $\{\hat{x}\} \times (\mathbb{Z}_k \setminus \{x_d\})$, but this is itself a copy of T. $\square$

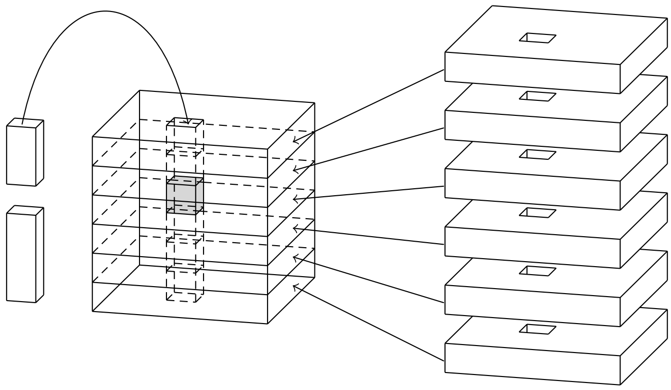

Figure 2.7: The induction step in the proof of Proposition 2.5. The grey cube represents x. The vertical column in which x lies, without x itself, is a copy of T. Each horizontal slice minus the point in this column can be tiled by the induction hypothesis.

Let $X \subset \mathbb{Z}_k^d$ (for any $d \geq 1$) be such that T tiles $\mathbb{Z}_k^d \setminus X$. We will say that such X is a *hole* in $\mathbb{Z}_k^d$. The intuition for X is that it is a set that remains uncovered after an attempt to tile $\mathbb{Z}_k^d$ by copies of T.

We can identify X with a higher-dimensional set $X' = X \times \{0\} \subset \mathbb{Z}_k^{d+1}$. One can easily verify that X' is a hole in $\mathbb{Z}_k^{d+1}$. More importantly, we will show in the following proposition that a single additional point of X' can be covered in exchange for leaving the $(d+1)$-st corner of $\mathbb{Z}_k^{d+1}$ uncovered (see Figure 2.9). This is why, for any $S \subset \mathbb{Z}_k^d$, we define

$$S^\dagger = (S \times \{0\}) \cup \{c_{d+1,d+1}\} \subset \mathbb{Z}_k^{d+1}.$$

Note that the definition of $S^\dagger$ and the definition of S being a hole depend not only on S, but also on the dimension of the underlying discrete torus $\mathbb{Z}_k^d$. For $m \geq 1$, we will use the shorthand $S^{\dagger(m)}$ to denote the result of m consecutive applications of the † operation to S, that is,

$$S^{\dagger(m)} = S^{\overbrace{\dagger\ldots\dagger}}\!\!\underbrace{}_{m} \subset \mathbb{Z}_k^{d+m}.$$

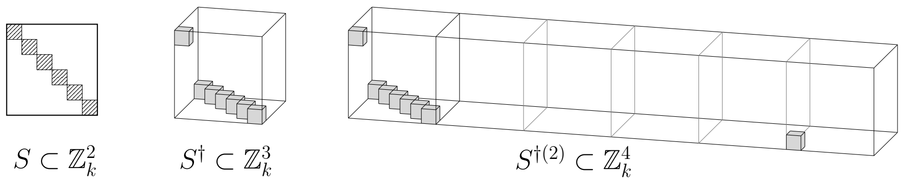

Figure 2.8: Suppose S is the subset of $\mathbb{Z}_k^2$ given in the diagram on the left (here $k = 6$). The diagram in the middle depicts $S^\dagger$, and the diagram on the right depicts $S^{\dagger(2)}$. Observe that S is a hole in $\mathbb{Z}_k^2$, but $S^\dagger$ and $S^{\dagger(2)}$ are not holes in $\mathbb{Z}_k^3$ and $\mathbb{Z}_k^4$, respectively.

Proposition 2.6. *Let $d \geq 1$ and let $X \subset \mathbb{Z}_k^d$ be a hole. Then for each $x \in X$ the set $(X \setminus \{x\})^\dagger$ is a hole in $\mathbb{Z}_k^{d+1}$.*

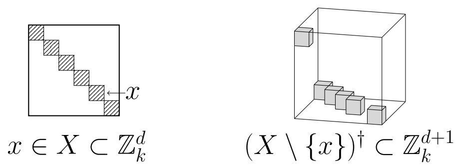

Figure 2.9: An illustration of the statement of Proposition 2.6. The aim is to show that T tiles $\mathbb{Z}_k^{d+1} \setminus (X \setminus \{x\})^\dagger$.

Proof (see Figure 2.10). Use (i) a tiling of $\mathbb{Z}_k^d \setminus X$ for $(\mathbb{Z}_k^d \setminus X) \times \{0\}$, and (ii) one copy of T to cover $\{x\} \times (\mathbb{Z}_k \setminus \{k-1\})$. By Proposition 2.5, (iii) $(\mathbb{Z}_k^d \setminus \{(0, \ldots, 0)\}) \times \{k-1\}$ and (iv) $(\mathbb{Z}_k^d \setminus \{x\}) \times \{i\}$, $i \in \{1, \ldots, k-2\}$, can each be tiled by copies of T. $\square$

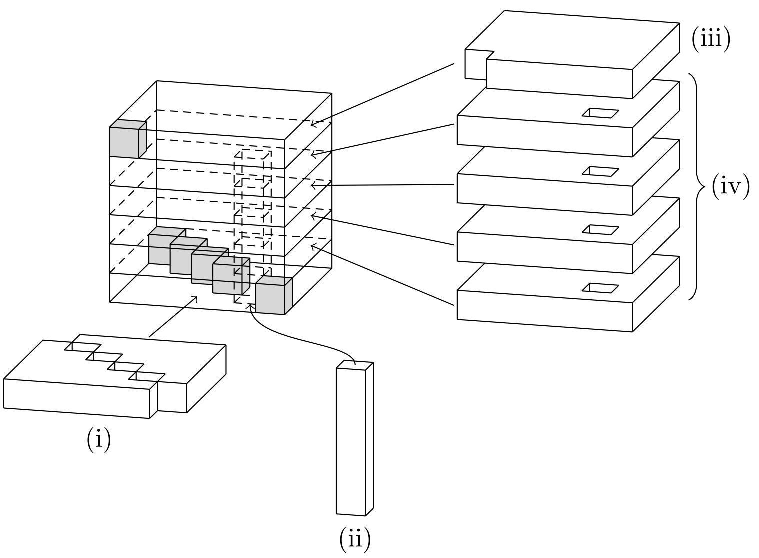

Figure 2.10: The bottom horizontal piece (i) is tilable because X is a hole, and the other horizontal pieces (iii) and (iv) are tilable by Proposition 2.5. The remaining vertical column (ii) is a copy of T.

We will apply Proposition 2.6 inductively, that is, in the form of the following corollary.

Corollary 2.7. *Let $d \geq 1$ and let $X \subset \mathbb{Z}_k^d$ be a hole. Then for any distinct elements $x_1, \ldots x_m \in X$, the set $(X \setminus \{x_1, \ldots, x_m\})^{\dagger(m)}$ is a hole in $\mathbb{Z}_k^{d+m}$.* □

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4. Write $|S| = m$ and $r = d - m$. By symmetry, we can assume that

$$S = \{c_{j,d} : j = r+1, \ldots, d\}.$$

Our aim is to prove that S is a hole in $\mathbb{Z}_k^d$. Note that $S = \emptyset^{\dagger(m)}$, where the empty set $\emptyset$ is considered as a subset of $\mathbb{Z}_k^r$. Therefore by Corollary 2.7 it suffices to find a hole $X \subset \mathbb{Z}_k^r$ with $|X| = m$.

This can be done by partitioning $\mathbb{Z}_k^r$ into a singleton $\{x\}$ and copies of T (this can be done by Proposition 2.5), and letting X be the union of $\{x\}$ and the appropriate number of copies of T. By assumption, $m \equiv 1 \pmod{k-1}$ so the only potential problem with this construction of X is if $|\mathbb{Z}_k^r| < m$. However, this is ruled out by the assumption that $m \leq d - \log_k d$. □

2.3 Using one special direction to get T-tilable slices

The purpose of this section is to demonstrate that tiles in the first direction in $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$ (that is, translates of $T \times \{0\}^{d-1}$) can be combined in such a way that the uncovered part of each slice can be tiled by copies of T using Lemma 2.4. The exact claim is as follows.

Lemma 2.8. *There exists a number $\ell \geq 1$ such that for any $d \geq 1$ and any set $C \subset \mathbb{Z}_k^{d-1}$ of order $|C| \geq \ell$ there is a set $X \subset \mathbb{Z} \times C$, satisfying:*

(a) X is a union of disjoint sets of the form $(T+n) \times \{c\}$ with $n \in \mathbb{Z}$ and $c \in C$;

(b) $|(\{n\} \times C) \cap X| \equiv 1 \pmod{k-1}$ for every $n \in \mathbb{Z}$;

(c) $|(\{n\} \times C) \cap X| \leq \ell$ for every $n \in \mathbb{Z}$.

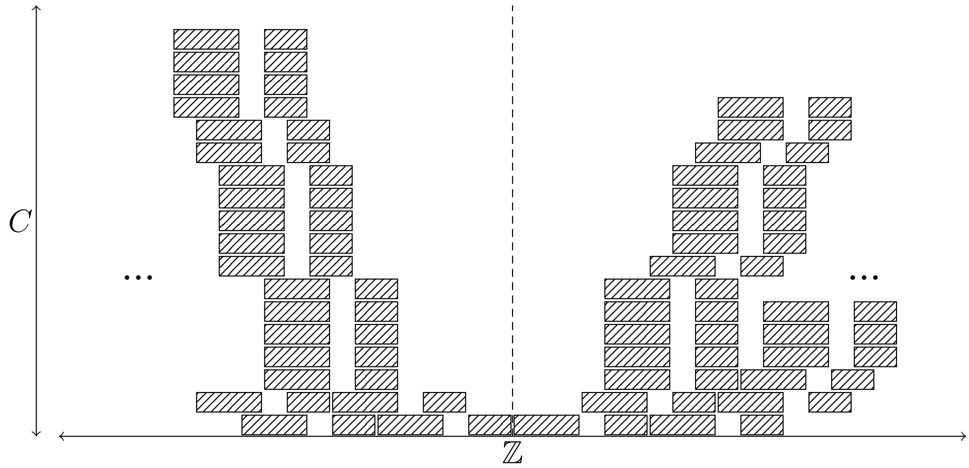

Figure 2.11: A possible construction of X. In this example the aim is to have 1 modulo 6 elements covered in each column.

We start with the following trivial proposition.

Proposition 2.9. *There is a function $f : \mathbb{Z} \to \{0, \dots, k-2\}$ such that for each $x \in \mathbb{Z}$*

$$\sum_{y \in T} f(x-y) \equiv 1 \pmod{k-1}.$$

Proof. Start by defining $f(n) = 0$ for $-k+1 \leq n \leq -1$. Now define $f(n)$ for $n \geq 0$ as follows. Suppose that for some $n \geq 0$ the values of $f(j)$ are already defined for all j such that $-k+1 \leq j \leq n-1$. Then the value of $f(n)$ is uniquely defined by

$$f(n) \equiv 1 - \sum_{y \in T \setminus \{1\}} f(n+1-y) \pmod{k-1}.$$

Define $f(n)$ for all $n \leq -k$ in a similar way. □

Now Lemma 2.8 can be proved quickly.

Proof of Lemma 2.8. Write $\ell = 2k(k-2)$ and suppose that $|C| = \ell$. Let $f : \mathbb{Z} \to \{0, \dots, k-2\}$ be as given by Proposition 2.9. The aim is to choose subsets $S_n \subset C$ for every $n \in \mathbb{Z}$, with orders satisfying $|S_n| = f(n)$, and such that $S_m \cap S_n = \emptyset$ whenever $m \neq n$ and $(T+m) \cap (T+n) \neq \emptyset$. Then X can be taken to be $\bigcup_{n \in \mathbb{Z}} (T+n) \times S_n$.

Fix any enumeration of $\mathbb{Z}$, and define the sets S_n one by one in that order. When defining S_n, there can be at most $2k-1$ choices of m with S_m already defined and $m - n \in T - T$. Moreover, $|S_m| \leq k-2$ for each m. Therefore to be able to find S_n it is enough to have $|C| - (2k-1)(k-2) \geq f(n)$. Finally, this condition is ensured by the choice of ℓ, completing the proof in the case when $|C| = \ell$. If $|C| > \ell$, we are done by restricting to a subset of C of size exactly ℓ. □

2.4 Completing the proof of Theorem 2.3

It was noted in Section 2.1 that Lemmas 2.4 and 2.8 together imply that for some $d \geq 1$

$$T \text{ tiles } \mathbb{Z} \times \mathbb{Z}_k^{d-1}, \tag{2.1}$$

and therefore

$$T \text{ tiles } \mathbb{Z}^d, \tag{2.2}$$

implying Theorem 2.3. However, some abuse of notation is already present in the statement of (2.1). In this section we will carefully explain what is meant by (2.1), why it follows from the two lemmas and how it implies (2.2). In doing so, we will complete the proof of Theorem 2.3.

To avoid confusion, within this section we use quite precise language. Although this might seem pedantic here, for later it will be very important to have precise notation available. We denote the elements of $\mathbb{Z}_k$ by $\overline{x}$ for $x \in \mathbb{Z}$ (instead of identifying them with x, which was our preferred notation in the rest of the section), and we will denote the image of T under the natural projection $\pi : \mathbb{Z} \to \mathbb{Z}_k$ by $\pi(T)$ rather than simply by T.

Theorem 2.3. *Fix integers $k \geq 3$ and $i \in \{2, \ldots, k-1\}$ and let T be the tile $[k] \setminus \{i\}$. Then T tiles $\mathbb{Z}^d$ for some d.*

Proof. Fix a large d (more precisely, first let ℓ be as given by Lemma 2.8 and then fix d such that $d - 1 - \log_k(d-1) \geq \ell$).

Denote the projection map $\mathbb{Z} \to \mathbb{Z}_k$ by π, and consider the following subsets of $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$:

$$\begin{aligned}
\mathsf{T}_1 &= \; T \;\times \left\{\,\overline{0}\,\right\} \times \left\{\,\overline{0}\,\right\} \times \cdots \times \left\{\,\overline{0}\,\right\},\\
\mathsf{T}_2 &= \left\{\,0\,\right\} \times \pi(T) \times \left\{\,\overline{0}\,\right\} \times \cdots \times \left\{\,\overline{0}\,\right\},\\
&\;\;\vdots\\
\mathsf{T}_d &= \left\{\,0\,\right\} \times \left\{\,\overline{0}\,\right\} \times \left\{\,\overline{0}\,\right\} \times \cdots \times \pi(T).
\end{aligned}$$

Recall from Section 2.2 the definition of

$$C_{d-1} = \left\{ (\underbrace{\overline{0}, \ldots, \overline{0}, \overset{j\text{-th coordinate}}{\overset{\downarrow}{\overline{k-1}}}, \overline{0}, \ldots, \overline{0}}_{d-1 \text{ coordinates}}) : j = 1, \ldots, d-1 \right\} \subset \mathbb{Z}_k^{d-1}.$$

By Lemma 2.8, there is a set $X \subset \mathbb{Z} \times C_{d-1}$, which is a union of disjoint translates of T_1 and for each $n \in \mathbb{Z}$ satisfies $|(\{n\} \times C_{d-1}) \cap X| \leq d-1-\log_k(d-1)$ and $|(\{n\} \times C_{d-1}) \cap X| \equiv 1 \pmod{k-1}$. Hence, by Lemma 2.4, $(\{n\} \times \mathbb{Z}_k^{d-1}) \setminus X$ is a union of disjoint translates of $\mathsf{T}_2, \ldots, \mathsf{T}_d$ for each $n \in \mathbb{Z}$. Therefore $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$ is a union of disjoint translates of $\mathsf{T}_1, \ldots, \mathsf{T}_d$ (this is exactly what is meant by (2.1)).

More explicitly, there are integers $1 \leq t(\alpha) \leq d$ and $x_1(\alpha), \ldots, x_d(\alpha) \in \mathbb{Z}$, indexed by $\alpha \in A$, such that $\mathbb{Z} \times \mathbb{Z}_k^{d-1}$ is the disjoint union

$$\mathbb{Z} \times \mathbb{Z}_k^{d-1} = \bigsqcup_{\alpha \in A} \Big[\mathsf{T}_{t(\alpha)} + \big(\, x_1(\alpha),\, \overline{x_2(\alpha)}\,,\, \ldots\,,\, \overline{x_d(\alpha)}\,\big)\Big].$$

From this it follows that, in fact, $\mathbb{Z}^d$ is T-tilable. Indeed, consider the following subsets of $\mathbb{Z}^d$:

$$\begin{aligned}
\mathsf{T}'_1 &= \;T\; \times \{0\} \times \{0\} \times \cdots \times \{0\},\\
\mathsf{T}'_2 &= \{0\} \times \;T\; \times \{0\} \times \cdots \times \{0\},\\
&\;\;\vdots\\
\mathsf{T}'_d &= \{0\} \times \{0\} \times \{0\} \times \cdots \times \;T.
\end{aligned}$$

Then we can express $\mathbb{Z}^d$ as the disjoint union

$$\mathbb{Z}^d = \bigsqcup_{\substack{\alpha \in A\\ c_2, \ldots, c_d \in \mathbb{Z}}} \Big[\mathsf{T}'_{t(\alpha)} + \big(x_1(\alpha),\, x_2(\alpha) + kc_2\,,\, \ldots\,,\, x_d(\alpha) + kc_d\,\big)\Big]. \qquad \square$$

3 The general case

Recall the statement of the main theorem.

Theorem 2.2. *Let $T \subset \mathbb{Z}^n$ be a tile. Then T tiles $\mathbb{Z}^d$ for some d.*

In this section we prove the main theorem by generalising the approach demonstrated in Section 2. We have to account for two ways in which Theorem 2.3 is a special case: firstly, the tile can be multidimensional; secondly, even in the one-dimensional case the tile can have more complicated structure than in Section 2.

It turns out that dealing with the first issue does not add significant extra difficulty to the proof, provided that the right setting is chosen. Namely, most of the intermediate results will be stated in terms of abelian groups rather than integer lattices. This way a multidimensional tile $T \subset \mathbb{Z}^b$ can be considered as being one-dimensional, if $\mathbb{Z}^b$ (rather than $\mathbb{Z}$) is chosen as the underlying abelian group. Moreover, this point of view is vital for comparing periodic tilings of an integer lattice with tilings of a discrete torus, already an important idea in the proof of the special case.

On the other hand, dealing with the second issue requires significant effort. It involves finding the right way to generalise the two key ideas from Section 2, as well as introducing a new ingredient that allows the argument to be applied iteratively.

We now introduce some definitions. Given an abelian group G, we call any non-empty subset $T \subset G$ a *tile* in G. Given abelian groups $G_1, \ldots, G_d$ and corresponding tiles $T_i \subset G_i$, consider the following subsets of $G_1 \times \cdots \times G_d$:

$$\begin{aligned}
\mathsf{T}_1 &= \; T_1 \; \times \{0\} \times \cdots \times \{0\}, \\
\mathsf{T}_2 &= \{0\} \times \; T_2 \; \times \cdots \times \{0\}, \\
&\;\;\vdots \\
\mathsf{T}_d &= \{0\} \times \{0\} \times \cdots \times \; T_d.
\end{aligned}$$

Any translate of such T_i (that is, a set of the form $\mathsf{T}_i + x$ for $x \in G_1 \times \cdots \times G_d$) is called a *copy of* T_i. We say that a subset $X \subset G_1 \times \cdots \times G_d$ is $(T_1, \ldots, T_d)$*-tilable* if X is a disjoint union of copies of $T_1, \ldots, T_d$.

It will often be the case that $(G_1, T_1) = \cdots = (G_d, T_d) = (G, T)$. Then we will use the term T*-tilable* as a shorthand for $(T, \ldots, T)$-tilable.

More generally, we may consider subsets of $G_1^{d_1} \times \cdots \times G_m^{d_m}$ where $G_1, \ldots, G_m$ are abelian groups with tiles $T_i \subset G_i$. In this setting we would say that a subset is $(d_1 \cdot T_1, \ldots, d_m \cdot T_m)$-tilable. In other words, each $d_i \cdot T_i$ replaces

$$\underbrace{T_i, \ldots, T_i}_{d_i}.$$

However, we suppress "1·" in the notation. So, for example, we could say that a subset of $G_1^7 \times G_2 \times G_3^{10}$ is $(7 \cdot T_1, T_2, 10 \cdot T_3)$-tilable.

3.1 A summary of the proof

Let T be a fixed finite tile in $\mathbb{Z}^b$. Without loss of generality assume that $T \subset [k]^b$ for some $k \geq 1$. Then, writing $\pi : \mathbb{Z}^b \to \mathbb{Z}_k^b$ for the projection map, $\pi(T)$ is a tile in $G = \mathbb{Z}_k^b$.

In the light of the argument from Section 2, one might hope to find a positive integer d and a large family $\mathcal{F}$ of disjoint subsets of G^d with the property that whenever a subfamily $\mathcal{S} \subset \mathcal{F}$ with $|\mathcal{S}| \equiv 1 \pmod{|T|}$ is chosen, the set $G^d \setminus (\bigcup_{S \in \mathcal{S}} S)$ is $\pi(T)$-tilable. However, this seems to be achievable only in the case when $\pi(T)$ is in a certain sense a 'dense' subset of G.

If $\pi(T)$ is sparse, we achieve a weaker aim. Namely, we find a certain set $X \subset G^d$ which has sufficiently nice structure and is a denser subset of G^d than $\pi(T)$ is of G. Also, we find a large family $\mathcal{F}$ of disjoint subsets of X such that for any $\mathcal{S} \subset \mathcal{F}$ of appropriate size $X \setminus (\bigcup_{S \in \mathcal{S}} S)$ is $\pi(T)$-tilable. Taking copies of T in the special direction, we can now tile $\mathbb{Z}^b \times X$.

Repeating this process, we can use copies of T and $\mathbb{Z}^b \times X$ to tile $\mathbb{Z}^p \times Y$ for an even denser subset $Y \subset G^l$. After finitely many iterations of this procedure we tile the whole of $\mathbb{Z}^q \times G^m$ for some possibly large q and m. From this it follows that $\mathbb{Z}^{q+bm}$ is T-tilable.

The rest of this section is organised as follows. In Section 3.2 we show how any tile in a (finite) abelian group H can be used to *almost* tile a sufficiently nice denser subset of H^d for some d. This is the most complicated part of the proof, but it shares a similar structure with the simpler argument in Section 2.2.

In Section 3.3 we show how one special dimension can be used to cover the gaps in every slice. The argument is almost identical to the one in Section 2.3.

In Section 3.4 we observe some simple transitivity properties of tilings. They enable the iterative application of the process. The ideas in this section are fairly straightforward.

Finally, in Section 3.5 we compile the tools together and complete the proof of Theorem 2.2.

3.2 Almost tiling denser multidimensional sets

Our goal is to prove the following lemma.

Lemma 2.10. *Let $T \subsetneq G$ be a tile in a finite abelian group G. Then there is a set $A \subset G$, with $T \subsetneq A$, having the following property. Given any $d_0 \geq 1$, there is some $d \geq d_0$ and a family $\mathcal{F}$ consisting of at least d_0 pairwise disjoint subsets of A^d such that*

$$G \times \left(A^d \setminus \bigcup_{S \in \mathcal{S}} S \right) \subset G^{d+1}$$

is T-tilable whenever $\mathcal{S} \subset \mathcal{F}$ satisfies $|\mathcal{S}| \equiv 1 \pmod{|T|}$.

Before presenting the proof, we make a few definitions that will hold throughout this section. First, let G and T be fixed as in the statement of Lemma 2.10. Since $T \neq G$, we can fix an $x \in G$ such that $T + x \neq T$. Define

$$
\begin{aligned}
T^{\mathrm{up}} &= T + x, \\
C^{\mathrm{up}} &= T^{\mathrm{up}} \setminus T, \\
C_{\mathrm{down}} &= T \setminus T^{\mathrm{up}}, \\
A &= T \cup T^{\mathrm{up}}
\end{aligned}
$$

(see Figures 2.12 and 2.13).

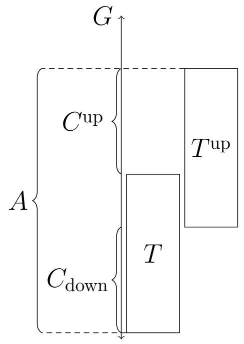

Figure 2.12: An illustration of the definitions.

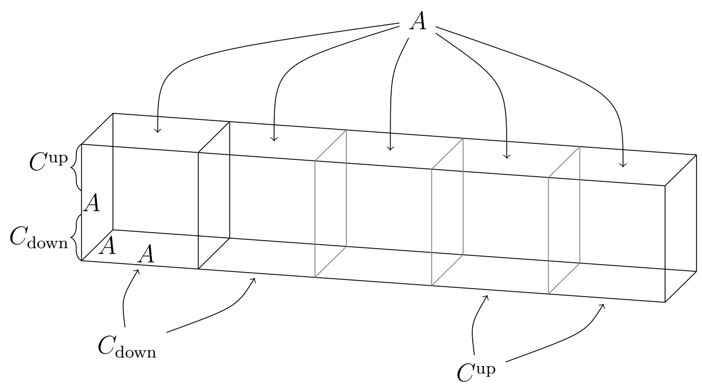

Figure 2.13: A four-dimensional diagram of A^4. The sets C_{down} and C^{up} are marked on two of the axes. In this example $|A| = 5$ and $|C_{\mathrm{down}}| = |C^{\mathrm{up}}| = 2$. This and the following four-dimensional diagrams in this section should be understood more generally as depicting A^d for any d, the three-dimensional slices representing copies of A^{d-1}.

We will use A from this definition in the proof of Lemma 2.10. For the family

$\mathcal{F}$ we will take all sets of the following form. For any integers $1 \le i \le d$, write

$$C_{i,d} = \underbrace{C_{\text{down}} \times \cdots \times C_{\text{down}} \times \overset{\substack{i\text{-th component}\\ \downarrow}}{C^{\text{up}}} \times C_{\text{down}} \times \cdots \times C_{\text{down}}}_{d \text{ components}} \subset A^d$$

(see Figure 2.14). Also write $C_{0,d} = (C_{\text{down}})^d$. Note that if $i \neq j$, then $C_{i,d} \cap C_{j,d} = \emptyset$.

Finally, as T is fixed, we can simply say *tilable* instead of T-tilable.

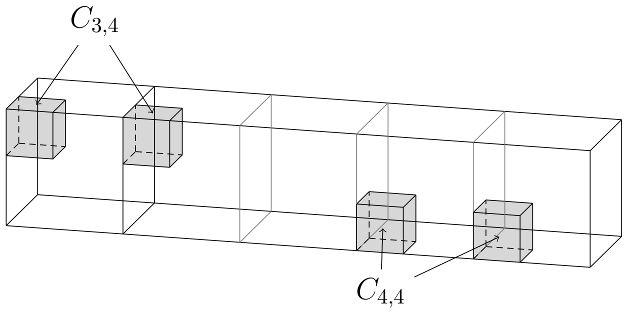

Figure 2.14: A four-dimensional diagram, which extends the previous diagram. Note that $C_{3,4}$ and $C_{4,4}$ both intersect two three-dimensional slices, because in this example $|C_{\text{down}}| = |C^{\text{up}}| = 2$.

One of the reasons why these definitions are useful is that they allow the following analogue of Proposition 2.5.

Proposition 2.11. *For any integers $d \ge 1$ and $0 \le i \le d$, the set $A^d \setminus C_{i,d}$ is tilable.*

Proof (see Figure 2.15). Use induction on d. If $d = 1$, observe that $A = C^{\text{up}} \sqcup T = C_{\text{down}} \sqcup T^{\text{up}}$, and so $A \setminus C_{i,1}$ ($= A \setminus C^{\text{up}}$ or $A \setminus C_{\text{down}}$) is a translate of T.

Now suppose that $d \ge 2$ and without loss of generality assume that $i \neq d$. By the induction hypothesis, for each $g \in A$, the slice $(A^{d-1} \setminus C_{i,d-1}) \times \{g\}$ can be T-tiled. It remains to tile the set $C_{i,d-1} \times (A \setminus C_{\text{down}}) = C_{i,d-1} \times T^{\text{up}}$, but this is obviously a union of disjoint copies of T. □

We now make a series of definitions that are useful for lifting subsets of lower-dimensional spaces to higher-dimensional spaces.

A *basic set* is a set of the form A^d, $G \times A^d$ or $\{g\} \times A^d$ for some $g \in G$, with d any positive integer. Let X be a subset of a basic set Ω and write $\Omega = W \times A^d$ (so $W = A^0$, G or $\{g\}$ for some $g \in G$). We define

$$X^\dagger = (X \times C_{\text{down}}) \cup (W \times C_{d+1,d+1}) \subset W \times A^{d+1}$$

(see Figure 2.16).

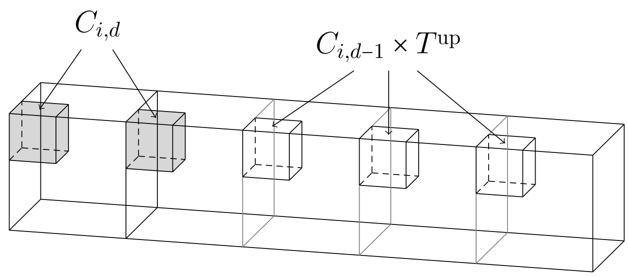

Figure 2.15: The induction step in the proof of Proposition 2.11. The set $C_{i,d-1} \times T^{\mathrm{up}}$ is a union of copies of T^{up}. In each slice it remains to tile a copy of $A^{d-1} \setminus C_{i,d-1}$. This can be done by the induction hypothesis.

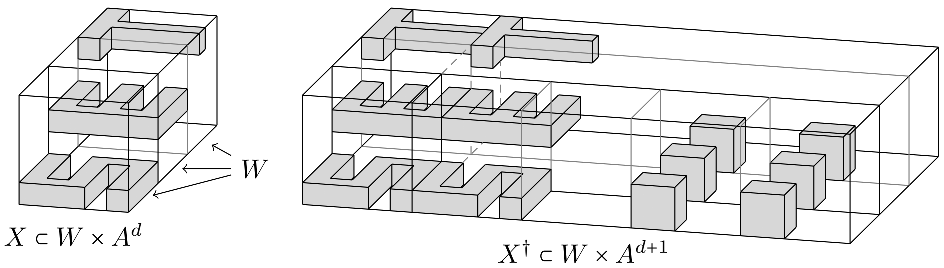

Figure 2.16: An illustration of the definition of $X^\dagger$, building on Figure 2.13. The diagram on the left is four-dimensional and represents a generic set $X \subset W \times A^d$. The diagram on the right is five-dimensional and represents the corresponding $X^\dagger$. We stress that this is an abstract illustration. In particular, here $|A| = 5$ and $|W| = 3$, while in fact we always have either $|W| = 1$ or $|W| = |G| \geq |A|$.

Moreover, for any $m \geq 1$ we use the shorthand $X^{\dagger(m)}$ to denote the result of m consecutive applications of the † operation to X, that is,

$$\begin{aligned} X^{\dagger(m)} &= X\underbrace{^{\dagger\cdots\dagger}}_{m} \\ &= \big(X \times C_{0,m}\big) \cup (W \times C_{d+1,d+m}) \cup \cdots \cup (W \times C_{d+m,d+m}) \\ &\subset W \times A^{d+m}. \end{aligned}$$

For the final definition, we say that X is a *hole* in Ω if $\Omega \setminus X$ is tilable. Note that these definitions depend not only on X, but also on the underlying basic set Ω. Therefore we will only use them when the underlying set is explicitly stated or clear from the context.

Proposition 2.12. *Let $d \geq 1$ and let X be a hole in A^d. Suppose that $C_{i,d} \subset X$ for* *some* $0 \leq i \leq d$. *Then* $(X \setminus C_{i,d})^\dagger$ *is a hole in* A^{d+1}.

Proof (see Figure 2.17). Partition $A^{d+1} \setminus (X \setminus C_{i,d})^\dagger$ into four sets

(i) $C_{i,d} \times (A \setminus C^{\text{up}})$ — tilable, because $A \setminus C^{\text{up}} = T$;

(ii) $(A^d \setminus X) \times C_{\text{down}}$ — tilable, because $A^d \setminus X$ is tilable;

(iii) $(A^d \setminus C_{i,d}) \times (A \setminus (C^{\text{up}} \cup C_{\text{down}}))$ — tilable by Proposition 2.11;

(iv) $(A^d \setminus C_{0,d}) \times C^{\text{up}}$ — tilable by Proposition 2.11. □

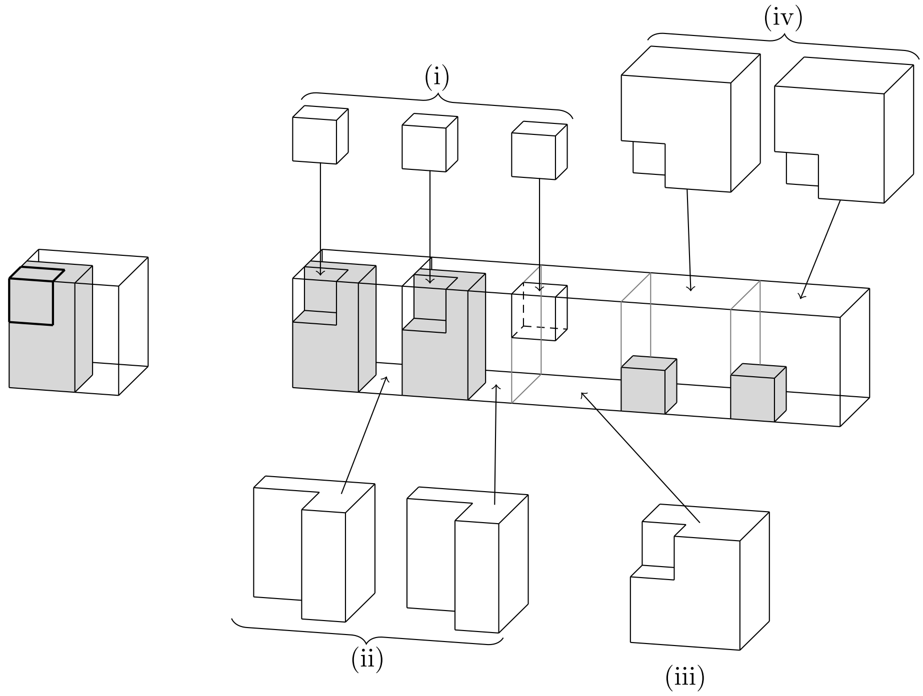

Figure 2.17: An illustration of the proof of Proposition 2.12. The three-dimensional diagram on the left represents a hole $X \subset A^d$ which contains $C_{i,d}$. The four-dimensional diagram on the right represents $(X \setminus C_{i,d})^\dagger$ and demonstrates why it is a hole in A^{d+1}.

This proposition is the most useful for us in the form of the following corollary.

Corollary 2.13. *Let* $d \geq 1$ *and suppose that* $0 \leq i_1, \dots, i_m \leq d$ *are distinct integers. Then*

$$\left(A^d \setminus (C_{i_1,d} \cup \dots \cup C_{i_m,d})\right)^{\dagger(m)}$$

is a hole in A^{d+m}.

Proof. Use induction on m. The base case $m = 1$ is a special case of Proposition 2.12, so suppose that $m \geq 2$. Note that

$$\begin{aligned}&\big(A^d \setminus (C_{i_1,d} \cup \cdots \cup C_{i_m,d})\big)^{\dagger(m)}\\=&\Big(\big(A^d \setminus (C_{i_1,d} \cup \cdots \cup C_{i_{m-1},d})\big)^{\dagger(m-1)} \setminus C_{i_m,d+m-1}\Big)^{\dagger}\end{aligned}$$

so it is a hole in A^{d+m-1} by the induction hypothesis and Proposition 2.12. $\square$

Now we have the tools needed for the proof of Lemma 2.10.

Proof of Lemma 2.10. Fix any $d \geq (1 + |G|/|T|)d_0$ and write $\mathcal{F} = \{C_{i,d} : i = 1, \ldots, d\}$. By symmetry, it is enough to find a tiling for the set

$$M_m = G \times \big(A^d \setminus (C_{d-m+1,d} \cup \cdots \cup C_{d,d})\big)$$

for every choice of $m \leq d_0$ with $m \equiv 1 \pmod{|T|}$. Fix one such value of m, and let $M = M_m$ be the corresponding set that we have to tile.

Define $r = d - m$ and $\Omega = G \times A^r$. We will construct a partition $\mathcal{B}$ of the set Ω, satisfying:

- $\mathcal{B}$ consists of the set $Y_0 = G \times C_{0,r}$ and copies of the tile T;
- for each $1 \leq i \leq r$, there is some $y_i \in G$ such that the set $Y_i = (T + y_i) \times C_{i,r}$ is exactly the union of some copies of T in $\mathcal{B}$;
- each $y \in G$ appears at least $t = (m-1)/|T|$ times in the list $y_1, \ldots, y_r$.

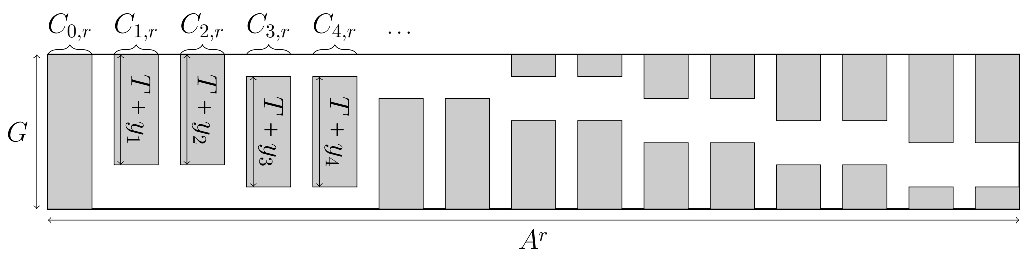

Figure 2.18: By constructing the partition $\mathcal{B}$ we show that the set $\bigcup_{i=0}^{r} Y_i$ (grey in this diagram) is a hole in $\Omega = G \times A^r$. In fact, $\bigcup_{i \in I \cup \{0\}} Y_i$ is a hole for any $I \subset [r]$.

We start the construction by fixing any list $y_1, \ldots, y_r$ such that each member of G appears exactly t times in $y_1, \ldots, y_{t|G|}$ (in particular, this list satisfies the final condition displayed above). Note that such a list exists since $r \geq t|G|$.

Now we use induction to construct, for each $0 \le j \le r$, a partition $\mathcal{B}_j$ of $G \times A^j$ such that the first two conditions are satisfied when $\mathcal{B}$ and r are replaced by $\mathcal{B}_j$ and j.

Let $\mathcal{B}_0 = \{G\}$. Having defined $\mathcal{B}_{j-1}$, let $\mathcal{B}_j$ consist of the following sets (see Figure 2.19):

(i) $G \times C_{0,j}$,

(ii) $X \times \{b\}$ for each $X \in \mathcal{B}_{j-1}$ that is a copy of T and each $b \in C_{\text{down}}$,

(iii) $\{g\} \times \{a\} \times T^{\text{up}}$ for each $g \in G \setminus (T + y_j)$ and each $a \in A^{j-1}$,

(iv) $(T + y_j) \times \{a\} \times \{b\}$ for each $a \in A^{j-1}$ and each $b \in T^{\text{up}} = A \setminus C_{\text{down}}$.

One can easily check that $\mathcal{B}_j$ is a partition of $G \times A^j$ with the required properties. In particular, the sets of the first two types cover $G \times A^{j-1} \times C_{\text{down}}$, and the remaining sets cover $G \times A^{j-1} \times (A \setminus C_{\text{down}})$.

This concludes the construction of $\mathcal{B}$.

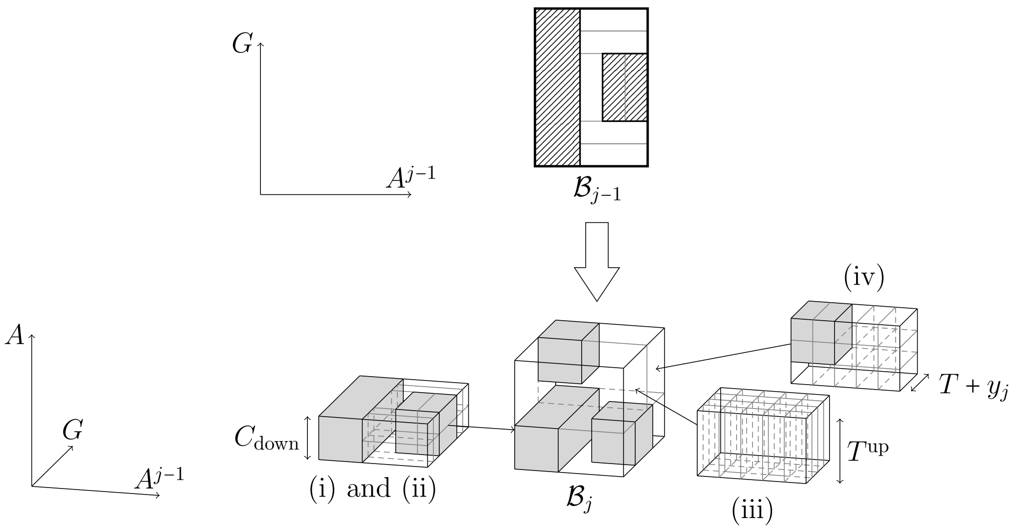

Figure 2.19: The induction step in the construction of the partition $\mathcal{B}$.

Define (recalling that $Y_0 = G \times C_{0,r}$ and $Y_i = (T + y_i) \times C_{i,r}$ for $1 \le i \le t|G|$)

$$S = \Omega \setminus \left(\bigcup_{i=0}^{t|G|} Y_i \right).$$

The point is that S is tilable by the restriction of $\mathcal{B}$, and hence $S \times C_{0,m}$ is also tilable. Therefore it only remains to prove that $M \setminus (S \times C_{0,m})$ is tilable, because this would imply that M is tilable. Observe that $M \setminus (S \times C_{0,m}) = (G \times A^d) \setminus S^{\dagger(m)}$, so it remains to prove that $S^{\dagger(m)}$ is a hole.

To prove this, fix any $g \in G$ and write $\Omega_g = \{g\} \times A^r$. Then Ω_g intersects Y_0 and exactly $t|T| = m - 1$ of the $Y_1, \ldots, Y_{t|G|}$. In other words,

$$\Omega_g \cap S = \{g\} \times \left(A^r \setminus \bigcup_{k=1}^{m} C_{j_k,r} \right)$$

for some $0 = j_1 < j_2 < \cdots < j_m \leq r$. By Corollary 2.13, $(\Omega_g \cap S)^{\dagger(m)}$ is a hole in $\{g\} \times A^d$. This holds for any $g \in G$, so in fact $S^{\dagger(m)} = \bigcup_{g \in G} (\Omega_g \cap S)^{\dagger(m)}$ is a hole in $G \times A^d$, completing the proof. □

3.3 Using one special dimension to cover certain subsets in slices

In this section we show how one special dimension can be used to lay foundations for a tiling so that the tiling can be completed in each slice separately using Lemma 2.10.

Here is the main result of this section. Its statement and proof are very similar to Lemma 2.8 from Section 2.

Lemma 2.14. *Let $t, b \geq 1$ be integers and T a finite tile in $\mathbb{Z}^b$. Further, let S be a set and let $\mathcal{F}$ be a family consisting of at least $(t-1)|T|^2$ pairwise disjoint subsets of S. Then there is a set $X \subset \mathbb{Z}^b \times S$, satisfying:*

- *X is a union of disjoint sets of the form $(T + x) \times A$ with $x \in \mathbb{Z}^b$ and $A \in \mathcal{F}$, and*
- *for each $x \in \mathbb{Z}^b$ there is some $m \equiv 1 \pmod t$ such that $\{y \in S : (x, y) \in X\}$ is a union of m distinct members of $\mathcal{F}$.*

We will deduce Lemma 2.14 from the following simple deconvolution type statement.

Proposition 2.15. *Let $t \geq 1$ and $b \geq 0$ be integers, T a finite tile in $\mathbb{Z}^b$, and $f : \mathbb{Z}^b \to \mathbb{Z}$ a function. Then there is a function $g : \mathbb{Z}^b \to \{0, \ldots, t-1\}$ such that for each $x \in \mathbb{Z}^b$*

$$\sum_{y \in T} g(x - y) \equiv f(x) \pmod t.$$

Proof. Use induction on b. The base case $b = 0$ is trivial ($\mathbb{Z}^0$ being the trivial group), so suppose that $b \geq 1$. For any $n \in \mathbb{Z}$, write

$$T_n = \{x \in \mathbb{Z}^{b-1} : (x, n) \in T\}.$$

Without loss of generality, assume that $T_0 \neq \emptyset$ and $T_n = \emptyset$ for all $n < 0$. Write k for the greatest integer such that $T_k \neq \emptyset$. In other words, $[0, k]$ is the minimal interval containing the projection of T in the last coordinate.

Set $g(x, n) = 0$ for all $x \in \mathbb{Z}^{b-1}$ and all $n \in \mathbb{Z}$ such that $-k \leq n \leq -1$. The next step is to define $g(x, n)$ for all $n \geq 0$ and $x \in \mathbb{Z}^{b-1}$. Consider $N = 0, 1, \ldots$ in turn, at each step having defined $g(x, n)$ whenever $-k \leq n \leq N - 1$ and $x \in \mathbb{Z}^{b-1}$. By the induction hypothesis, we can define $g(x, N)$ so that for all $x \in \mathbb{Z}^{b-1}$

$$\sum_{y \in T_0} g(x - y, N) \equiv f(x, N) - \sum_{\substack{1 \leq j \leq k \\ z \in T_j}} g(x - z, N - j) \pmod{t}.$$

The final step is to define $g(x, n)$ when $n \leq -k - 1$. The argument is similar. Consider $N = -k - 1, -k - 2, \ldots$ in turn, at each step having defined $f(x, n)$ whenever $n \geq N + 1$. By the induction hypothesis, we can define $g(x, N)$ so that for each $x \in \mathbb{Z}^{b-1}$

$$\sum_{y \in T_k} g(x - y, N) \equiv f(x, N + k) - \sum_{\substack{0 \leq j \leq k-1 \\ z \in T_j}} g(x - z, N + k - j) \pmod{t}.$$

These three steps together define g completely, and it is easy to see that g satisfies the required condition. $\square$

We get Lemma 2.14 as a quick corollary. In its proof we write $\mathbb{N}$ for $\{1, 2, \ldots\}$.

Proof of Lemma 2.14. Let $g : \mathbb{Z}^b \to \{0, \ldots, t - 1\}$ be such that for each $x \in \mathbb{Z}^b$

$$\sum_{y \in T} g(x - y) \equiv 1 \pmod{t}.$$

Let $z_1, z_2, \ldots$ be any enumeration of the elements of $\mathbb{Z}^b$. We will define sets $F_1, F_2, \ldots \subset \mathcal{F}$ such that $|F_n| = g(z_n)$ for any $n \in \mathbb{N}$, and $F_m \cap F_n = \emptyset$ for any distinct $m, n \in \mathbb{N}$ with $(T + z_m) \cap (T + z_n) \neq \emptyset$. Then we will be done by taking

$$X = \bigcup_{\substack{n \in \mathbb{N} \\ A \in F_n}} \big[(T + z_n) \times A\big].$$

The F_n can be defined inductively. Indeed, suppose that for some $N \in \mathbb{N}$ the sets $F_1, \ldots, F_{N-1}$ are already defined. Then we can define F_N to consist of exactly $g(z_N)$ elements of $\mathcal{F}$ that are not contained in F_n for any $n \le N-1$ with $(T+z_n) \cap (T+z_N) \neq \emptyset$. This is possible, because we have at most $|T|^2 - 1$ choices for such n, and

$$g(z_N) + (|T|^2 - 1) \max_{n \le N-1} |F_n| \le (t-1)|T|^2 \le |\mathcal{F}|. \qquad \square$$

3.4 General properties of tilings

In this section we prove some transitivity results for tilings. The underlying theme is, expressed very roughly, 'if B is A-tilable with the help of k extra dimensions, and C is B-tilable with the help of ℓ extra dimensions, then C is A-tilable with the help of $k + \ell$ extra dimensions'.

To avoid making the notation, which is already somewhat cumbersome, even more complicated we allow ourselves to abuse it in places where this is unlikely to create ambiguity. For example, given a tiling $X = \bigsqcup X_\alpha$ we may refer to the sets X_α as tiles (technically, they are not tiles, but copies of tiles). Otherwise, the proofs in this section are fairly straightforward.

Proposition 2.16. *Let G, $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$ be abelian groups with tiles $A \subset B \subset C \subset G$, $T_i \subset B_i \subset G_i$ and $U_i \subset C_i \subset H_i$. Suppose that*

$$B_1 \times \cdots \times B_m \times B \ \text{ is } \ (T_1, \ldots, T_m, A)\text{-tilable} \tag{2.3}$$

and that

$$C_1 \times \cdots \times C_n \times C^d \ \text{ is } \ (U_1, \ldots, U_n, d \cdot B)\text{-tilable.} \tag{2.4}$$

Then

$$B_1 \times \cdots \times B_m \times C_1 \times \cdots \times C_n \times C^d \ \text{ is } \ (T_1, \ldots, T_m, U_1, \ldots, U_n, d \cdot A)\text{-tilable.}$$

Let us unravel the statement of this proposition. Intuitively, condition (2.3) asserts that 'B is almost A-tilable' – the extra dimensions $B_1, \ldots, B_m$ are used to fill the gaps. Similarly, condition (2.4) asserts that 'C^d is almost B-tilable' – here we use the extra dimensions $C_1, \ldots, C_n$. Finally, the conclusion states that 'C^d is almost A-tilable' – we use all the extra dimensions, $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$, to complete this tiling.

Proof. For each tile X in the $(U_1, \ldots, U_n, d \cdot B)$-tiling of $C_1 \times \cdots \times C_n \times C^d$, partition the set $B_1 \times \cdots \times B_m \times X$ in one of the two following ways:

- if X is a copy of B, then partition $B_1 \times \ldots \times B_m \times X$ into its $(T_1, \ldots, T_m, A)$-tiling;
- otherwise (that is, if X is a copy of one of the $U_1, \ldots, U_n$), partition the set into copies of X, namely $\{b\} \times X$ for each $b \in B_1 \times \ldots \times B_n$.

This produces a $(T_1, \ldots, T_m, U_1, \ldots, U_n, d \cdot A)$-tiling of $B_1 \times \cdots \times B_m \times C_1 \times \cdots \times C_n \times C^d$. □

In the proof of the main theorem, we will apply this result in the following more compact form.

Corollary 2.17. *Let G and H be abelian groups with tiles $T \subset G$ and $A \subset B \subset H$. Suppose that*

$$G^k \times H^\ell \times B^d \quad \textit{is} \quad (k \cdot T, (\ell + d) \cdot A)\textit{-tilable} \tag{2.5}$$

and that

$$G^u \times H^v \quad \textit{is} \quad (u \cdot T, v \cdot B)\textit{-tilable.} \tag{2.6}$$

Then

$$G^{du+k} \times H^{dv+\ell} \quad \textit{is} \quad ((du + k) \cdot T, (dv + l) \cdot A)\textit{-tilable.}$$

Proof. Use induction on d. The base case $d = 0$ is trivial, so suppose that $d \geq 1$.

Rewrite (2.5) to state that

$$G^k \times H^l \times B^{d-1} \times B \text{ is } (k \cdot T, (\ell + d - 1) \cdot A, A)\text{-tilable.}$$

Now Proposition 2.16 applied to this and (2.6) implies that

$$G^k \times H^l \times B^{d-1} \times G^u \times H^v \text{ is } (k \cdot T, (\ell + d - 1) \cdot A, u \cdot T, v \cdot A)\text{-tilable,}$$

which after reordering and combining terms becomes the statement that

$$G^{u+k} \times H^{v+l} \times B^{d-1} \text{ is } ((u + k) \cdot T, (v + l + d - 1) \cdot A)\text{-tilable.}$$

Finally, apply the induction hypothesis to this and (2.6) to conclude the proof. □

The following straightforward proposition allows tilings to be lifted via surjective homomorphisms.

Proposition 2.18. *Let G, H and $G_1, \ldots, G_n$ be abelian groups with tiles $T \subset G$ and $U_i \subset G_i$, and let $\rho : G \to H$ be a surjective homomorphism that is injective on*

T. If $G_1 \times \cdots \times G_n \times H$ is $(U_1, \ldots, U_n, \rho(T))$-tilable, then $G_1 \times \cdots \times G_n \times G$ is $(U_1, \ldots, U_n, T)$-tilable.

Proof. For any tile X in the $(U_1, \ldots, U_n, \rho(T))$-tiling of $G_1 \times \cdots \times G_n \times H$, let $\hat{X}$ denote the set

$$\hat{X} = \{(x_1, \ldots, x_n, x) \in G_1 \times \cdots \times G_n \times G : (x_1, \ldots, x_n, \rho(x)) \in X\}.$$

For every X, partition $\hat{X}$ in one of the two following ways:

- if X is a copy of U_i for some $1 \leq i \leq n$, then partition $\hat{X}$ into copies of U_i in the obvious way;
- if X is a copy of $\rho(T)$, then $X = \{(x_1, \ldots, x_n)\} \times \rho(T + x)$ for some $x_i \in G_i$ and $x \in G$. Hence $\hat{X} = \{(x_1, \ldots, x_n)\} \times (T + x + \ker(\rho))$, and as ρ is injective on T, this can be partitioned into copies of T.

Since the sets $\hat{X}$ partition $G_1 \times \cdots \times G_n \times G$, this produces a $(U_1, \ldots, U_n, T)$-tiling for it. □

An inductive application of this proposition gives the following result, which we will use in the proof of the main theorem.

Corollary 2.19. *Let G and H be abelian groups, and let $T \subset G$ be a tile. Moreover, suppose that a surjective homomorphism $\rho : G \to H$ is injective on T. If $G^k \times H^\ell$ is $(k \cdot T, \ell \cdot \rho(T))$-tilable, then $G^{k+\ell}$ is T-tilable.* □

3.5 Proof of the main theorem

The tools needed for the proof Theorem 2.2 are now available.

Theorem 2.2. *Let $T \subset \mathbb{Z}^n$ be a tile. Then T tiles $\mathbb{Z}^d$ for some d.*

Proof. Without loss of generality assume that $T \subset [k]^b$, where $k \in \mathbb{N}$. Write $G = \mathbb{Z}_k^b$ and let $\pi : \mathbb{Z}^b \to G$ be the projection map. In particular, $\pi(T)$ is a tile in G.

Claim 2.20. *Suppose that $A \subset G$ is a tile. Then there exist integers $p \geq 0$ and $q \geq 1$ such that $(\mathbb{Z}^b)^p \times G^q$ is $(p \cdot T, q \cdot A)$-tilable.*

Proof of claim. Use reverse induction on $|A|$. If $|A| = |G|$ then in fact $A = G$, and the claim holds with $p = 0$, $q = 1$. So suppose that $|A| \leq |G| - 1$.

Applying Lemma 2.10 to the tile A with fixed large d_0 produces a number $d_1 \geq d_0$, a set B such that $A \subsetneq B \subset G$ and a family $\mathcal{F}$ ($|\mathcal{F}| \geq d_0$) of pairwise disjoint subsets of B^{d_1} with the property that for any subfamily $\mathcal{S} \subset \mathcal{F}$ of size satisfying $|\mathcal{S}| \equiv 1 \pmod{|A|}$, the set

$$G \times \left(B^{d_1} \setminus \bigcup_{S \in \mathcal{S}} S \right)$$

is A-tilable.

Since d_0 is large, Lemma 2.14 gives a set $X \subset \mathbb{Z}^b \times G \times B^{d_1}$ that is a disjoint union of copies of T, and such that for every $x \in \mathbb{Z}^b$ the slice $\{y \in G \times B^{d_1} : (x, y) \in X\}$ is a hole in $G \times B^{d_1}$. Therefore $\mathbb{Z}^b \times G \times B^{d_1}$ is $(T, (d_1 + 1) \cdot A)$-tilable.

By the induction hypothesis, there exist $u \geq 0$ and $v \geq 1$ such that $(\mathbb{Z}^b)^u \times G^v$ is $(u \cdot T, v \cdot B)$-tilable. Now apply Corollary 2.17 to conclude that the claim holds with $p = d_1 u + 1$ and $q = d_1 v + 1$. This proves the claim. $\square$

To finish the proof of the theorem, apply the claim to the tile $\pi(T)$. This gives $p \geq 0$ and $q \geq 1$ such that $(\mathbb{Z}^b)^p \times G^q$ is $(p \cdot T, q \cdot \pi(T))$-tilable. Hence, by Corollary 2.19, $(\mathbb{Z}^b)^{p+q}$ is T-tilable. $\square$

4 Concluding remarks and open problems

We mention in passing that all our tilings are (or can be made to be) periodic. Also, our copies of T arise only from translations and permutations of the coordinates – in particular, ‘positive directions stay positive’.

We have made no attempt to optimise the dimension d in Theorem 2.2. What can be read out of the proof is the following.

Theorem 2.2’. *Let $T \subset \mathbb{Z}^n$ be a tile and suppose that $T \subset [k]^n$. Then T tiles $\mathbb{Z}^d$, where $d = \lceil \exp(100(n \log k)^2) \rceil$.*

Thus our upper bound on d is superpolynomial in the variable k^n. We believe that there should be an upper bound on d in terms only of the size and dimension of T. Even in the case $n = 1$ this seems to be a highly non-trivial question.

Conjecture 2.21. *For any positive integer t there is a number d such that any tile $T \subset \mathbb{Z}$ with $|T| \leq t$ tiles $\mathbb{Z}^d$.*

On the other hand, it is easy to see that there cannot be a bound just in terms of the dimension of the tile. Indeed, given any d it is possible to find a one-dimensional tile that does not tile $\mathbb{Z}^d$. Such a tile T can be constructed by fixing an integer k and taking two intervals of length k, distance $k^2 - 1$ apart, where in between the intervals only every k-th point is present in the tile. For example, if $k = 4$ then the resulting tile would be

$$\texttt{XXXX...X...X...X...XXXX}$$

Suppose that T tiles $\mathbb{Z}^d$. Choose a large integer N and consider the cuboid $[N]^d$. Fix one of the d directions and only consider the copies of T in this direction that intersect $[N]^d$. Since tiles do not overlap, there are at most $O(N^d/k^2)$ such tiles and they cover at most $O(N^d/k)$ elements of $[N]^d$. Since there are d directions, at most $O(dN^d/k)$ elements of the cuboid can be covered with tiles, but this number is less than N^d for large k. Therefore, if k is large enough, then T does not tile $\mathbb{Z}^d$.

Finally, apart from examples where tiles are sparse and do not stack, we do not have any tools for establishing reasonable lower bounds on the dimension d. It would be interesting to find a family of dense one-dimensional tiles which require arbitrarily large dimension. We expected that intervals with the central point removed would have this property. However, shockingly to us, Metrebian showed that this is not the case. In fact, he proved that for any k the tile $\underbrace{\texttt{XXXXX}}_{k}\texttt{.}\underbrace{\texttt{XXXXX}}_{k}$ tiles $\mathbb{Z}^4$. His work is very recent and has not yet been published. It is not known whether the same holds for intervals with an arbitrary point removed.

CHAPTER 3

Decompositions of the Boolean lattice

1 Introduction

Let $2^{[n]}$ denote the *Boolean lattice* of dimension n, that is, the poset (partially ordered set) whose elements are the subsets of $[n] = \{1, \ldots, n\}$, ordered by inclusion.

An important property of the Boolean lattice is that any finite poset P can be embedded into $2^{[n]}$ for sufficiently large n. Here by an *embedding* of a poset P into a poset Q we mean an injection $f : P \to Q$ such that $f(x) \leq_Q f(y)$ if and only if $x \leq_P y$. For any embedding $f : P \to Q$, we call the image $f(P)$ a *copy* of P in Q.

Now, if P is fixed and n is large, then $2^{[n]}$ contains many copies of P. So a natural question arises: can $2^{[n]}$ be partitioned into copies of P? Of course, for such a partition to exist, the size of P must divide the size of $2^{[n]}$, that is, $|P|$ must be a power of 2 (we would like to emphasise that we denote by $|P|$ the number of elements of P and not the number of relations). Moreover, P must have a greatest and a least element. Lonc [42] conjectured that these obvious necessary conditions are in fact sufficient.

Conjecture 3.1 (Lonc [42])**.** *Let P be a poset of size 2^k with a greatest and a least element. Then, for sufficiently large n, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

The case where P is a chain of size 2^k was originally conjecture by Sands [58]. Griggs [22] proposed a slightly stronger conjecture that, for any positive integer c and for sufficiently large n, it is possible to partition $2^{[n]}$ into chains of length c and at most one other chain. Both conjectures were proved by Lonc [42]. The question of minimising the dimension n in Griggs' conjecture in terms of the length of the chain c has received attention from several authors, including Elzobi and Lonc [7] and Griggs, Yeh and Grinstead [23]. Recently, Tomon [61] proved that the smallest sufficient n is of order $\Theta(c^2)$. Related questions on partitioning $2^{[n]}$ into chains of almost equal lengths have also been examined, by Füredi [14], Hsu, Logan, Shahriari and Towse [36, 37] and Tomon [60].

As we mentioned in the previous paragraph, Lonc himself verified Conjecture 3.1 in the case where P is a chain. Furthermore, it is easy to extend this result to products of chains. In fact, for any two posets P, Q, if $2^{[n]}$ can be partitioned into copies of P and $2^{[m]}$ can be partitioned into copies of Q, then $2^{[n+m]}$ can be partitioned into copies of $P \times Q$. However, apart from some small cases that can be checked by hand, chains and their products were the only two cases for which Lonc's conjecture had been confirmed.

In this chapter we resolve the conjecture in full generality.

Theorem 3.2. *Let P be a poset of size 2^k with a greatest and a least element. Then, for sufficiently large n, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

The plan of the chapter is as follows. In Section 2 we give the most important definitions and outline the structure of the proof of Theorem 3.2. We give the actual proof in Sections 3 and 4: Section 3 contains a general argument, which works in various settings where a partition of a product set into smaller sets is sought, and might be of independent interest; Section 4 contains ideas that are particular to partitioning $2^{[n]}$ into copies of a fixed poset. Finally, in Section 5 we give some open problems.

2 Overview of the proof

2.1 Weak partitions

A key idea in the proof will be the interplay between partitions and two weaker notions, called r-partitions and $(1 \bmod r)$-partitions, which we now describe.

Let P be a poset. Recall that a set $A \subset 2^{[n]}$ is a *copy* of P if the poset induced on A by $2^{[n]}$ is isomorphic to P. We define $\mathcal{F}_n(P)$ to be the family of all copies of P in $2^{[n]}$.

Let X be a set, and let $\mathcal{F}$ be any family of subsets of X. A $\mathbb{Z}^+$*-valued weight function* (or simply a *weight function*) on $\mathcal{F}$ is an assignment of non-negative integer weights to the members of $\mathcal{F}$. For an element $x \in X$, the *multiplicity* of x for a weight function is the total weight of those members of $\mathcal{F}$ that contain x. So, for example, X can be partitioned into members of $\mathcal{F}$ if and only if there exists a $\mathbb{Z}^+$-valued weight function on $\mathcal{F}$ for which every element of X has multiplicity 1. For a positive integer r, we say that

- $\mathcal{F}$ *contains an r-partition of X* if there is a weight function on $\mathcal{F}$ for which every element of X has multiplicity r ;
- $\mathcal{F}$ *contains a* $(1 \text{ } mod \text{ } r)$*-partition of* X if there is a weight function on $\mathcal{F}$ for which every $x \in X$ has multiplicity $1+rk_x$, where $k_x \in \{0, 1, \dots\}$ may depend on x.

Our strategy revolves around establishing a close relation between r-partitions, $(1 \bmod r)$-partitions and actual partitions of sets. Obviously, if $\mathcal{F}$ contains a partition of X, then $\mathcal{F}$ contains an r-partition and a $(1 \bmod r)$-partition of X for every r. Our aim is to go in the opposite direction. Namely, our strategy consists of two steps: firstly, we will show that if there exists an r such that $\mathcal{F}$ contains an r-partition and a $(1 \bmod r)$-partition of X, then we can use these weak partitions to get an actual partition of X^m for some m; secondly, we will show that, for some n and r, $\mathcal{F}_n(P)$ does contain an r-partition and a $(1 \bmod r)$-partition of $2^{[n]}$.

It is not immediately obvious that this strategy should work. For instance, it is not clear that finding weak partitions of $2^{[n]}$ is easier than finding an actual partition. However, this will turn out to be the case in Section 4, where we prove the following lemmas.

Lemma 3.3. *Let P be a finite poset with a greatest and a least element. Then there exist positive integers n and r such that the family of copies of P in $2^{[n]}$ contains an r-partition of $2^{[n]}$.*

Lemma 3.4. *Let P be a finite poset of size 2^k that has a greatest and a least element, and let r be a positive integer. Then there exists a positive integer n such that the family of copies of P in $2^{[n]}$ contains a* $(1 \text{ } mod \text{ } r)$*-partition of $2^{[n]}$.*

A key part of the argument will be to see how to use these seemingly much weaker results can be used to find an actual partition of $2^{[n]}$. We will discuss this in the following subsection.

2.2 Product systems

We will prove a very general theorem, which, applied to Lemmas 3.3 and 3.4, will imply our main result.

Let S be a set. For two sets $A \subset S^m, B \subset S^n$ with $m \leq n$, we say that B is a *copy* of A if B can be obtained by taking a product of A with a singleton set in S^{n-m} and permuting the coordinates. More precisely, for a permutation π of $\{1, \dots, n\}$ and

$x = (x_1, \ldots, x_n) \in S^n$, we define $\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Moreover, for any $X \subset S^n$, we define $\pi(X) = \{\pi(x) : x \in X\}$. Finally, for any $X \subset S^m$ and $Y \subset S^{n-m}$, we define $X \times Y = \{(x_1, \ldots, x_m, y_1, \ldots, y_{n-m}) : (x_1, \ldots, x_m) \in X, (y_1, \ldots, y_{n-m}) \in Y\}$. Note that we abuse the notation slightly and identify $S^m \times S^{n-m}$ with S^n, which allows us to consider $X \times Y$ as a subset of S^n. With these definitions, B is a copy of A if $B = \pi(A \times \{y\})$ for some permutation π of $\{1, \ldots, n\}$ and some $y \in S^{n-m}$.

Note that this definition does not exactly agree with the definition of a copy of a poset, which we made in Section 1. Indeed, there may exist two sets $A, B \subset 2^{[n]}$ such that $2^{[n]}$ induces the same poset on A and B, but such that B cannot be obtained from A by permuting the coordinates. However, we think that this abuse of notation is not harmful, because it will always be clear from the context which definition of a copy should be used. Moreover, if sets $A \subset 2^{[n]}$ and $B \subset 2^{[m]}$ are copies in the new sense, then they are also copies when considered as posets. Therefore, the two definitions are in fact closely related.

The following theorem is vital for our strategy.

Theorem 3.5. *Let S be a finite set and let $\mathcal{F}$ be a family of subsets of S. Suppose that there exists a positive integer r such that $\mathcal{F}$ contains an r-partition and a $(1 \bmod r)$-partition of S. Then there exists a positive integer n such that S^n can be partitioned into copies of members of $\mathcal{F}$.*

It is straightforward to deduce our main theorem from Lemmas 3.3 and 3.4 and Theorem 3.5. Indeed, let P be a poset of size 2^k with a greatest and a least element. Lemma 3.3 implies that there are positive integers r and u such that $\mathcal{F}_u(P)$ contains an r-partition of $2^{[u]}$. Now Lemma 3.4 implies that there is a positive integer v such that $\mathcal{F}_v(P)$ contains a $(1 \bmod r)$-partition of $2^{[v]}$. Setting $m = \max\{u, v\}$, $\mathcal{F}_m(P)$ contains both an r-partition and a $(1 \bmod r)$-partition of $2^{[m]}$. We can now apply Theorem 3.5 with $\mathcal{F} = \mathcal{F}_m(P)$ and $S = 2^{[m]}$ to finish the proof. (Note that if $B \subset 2^{[mn]}$ is a copy of some $A \in \mathcal{F}_m(P)$, then the poset that $2^{[mn]}$ induces on B is isomorphic to P, and hence $B \in \mathcal{F}_{mn}(P)$.)

3 Partitions in product systems

Our aim in this section is to prove Theorem 3.5.

As in the statement of the theorem, we let $\mathcal{F}$ be a family of subsets of a finite set S and we suppose that r is a natural number such that $\mathcal{F}$ contains an r-partition and $(1 \bmod r)$-partition of S. The set S, family $\mathcal{F}$ and number r will remain fixed throughout this section.

Lemma 3.6. *For any sets $A, B \subset S$, there exists a positive integer n such that $S^2 \times (A \cup B)^n$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.*

The proof of Lemma 3.6 is by far the most complicated part of this chapter. We will prove Lemma 3.6 in the next subsection. Now, with Lemma 3.6 at our disposal, we will prove Theorem 3.5.

Proposition 3.7. *Let $A, B \subset S$ and suppose that there exist positive integers p, q such that*

- *S^p can be partitioned into copies of members of $\mathcal{F} \cup \{A\}$, and*
- *$S^2 \times A^q$ can be partitioned into copies of members of $\mathcal{F} \cup \{B\}$.*

Then S^{pq+2} can be partitioned into copies of members of $\mathcal{F} \cup \{B\}$.

Proof. Partition S^p into sets $X_1, \dots, X_u, Y_1, \dots, Y_v$, where every X_i is a copy of A and every Y_j is a copy of a member of $\mathcal{F}$. We denote $\mathcal{X} = \{X_1, \dots, X_u\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_v\}$. Then $S^{pq+2} = S^2 \times (S^p)^q$ is the disjoint union of sets $S^2 \times Z_1 \times \cdots \times Z_q$ with $Z_i \in \mathcal{X} \cup \mathcal{Y}$ for all i. We separate these sets into two families, namely,

$$\mathcal{A} = \left\{S^2 \times Z_1 \times \cdots \times Z_q : Z_i \in \mathcal{X} \text{ for all } i\right\},$$
$$\mathcal{B} = \left\{S^2 \times Z_1 \times \cdots \times Z_q : Z_i \in \mathcal{X} \cup \mathcal{Y} \text{ for all } i \text{ and } Z_j \in \mathcal{Y} \text{ for some } j\right\}.$$

Each member of $\mathcal{A}$ is a copy of $S^2 \times A^q$, so it can be partitioned into copies of members of $\mathcal{F} \cup \{B\}$. Moreover, each member of $\mathcal{B}$ can be partitioned into copies of some member of $\mathcal{F}$ in an obvious way. Since together these sets form a partition of S^{pq+2}, we are done. $\square$

Proof of Theorem 3.5 (assuming Lemma 3.6). Since $\mathcal{F}$ contains an r-partition of S with $r \geq 1$, and since S is finite, we can find finitely many sets $B_1, \dots, B_k \in \mathcal{F}$ that cover S. We define $A_i = B_1 \cup \cdots \cup B_i$ for every $1 \leq i \leq k$. So, in particular, $A_k = S$.

We will use reverse induction on i to prove that there exist positive integers $p_1, \dots, p_k$ such that, for every $1 \leq i \leq k$, S^{p_i} can be partitioned into copies of members of $\mathcal{F} \cup \{A_i\}$. If $i = k$, then $A_k = S$, and the statement is trivially true with, say, $p_k = 1$. So we may assume that $1 \leq i \leq k-1$. Since $A_{i+1} = A_i \cup B_{i+1}$, it follows from Lemma 3.6 that there exists a positive integer q such that $S \times (A_{i+1})^q$ can be partitioned into copies of members of $\mathcal{F} \cup \{A_i, B_{i+1}\}$. However, B_{i+1} is a member of $\mathcal{F}$, so $\mathcal{F} \cup \{A_i, B_{i+1}\} = \mathcal{F} \cup \{A_i\}$. Combining this with the induction hypothesis for $i+1$ and Proposition 3.7, we see that S^{p_i}, where $p_i = p_{i+1}q + 2$, can be partitioned into copies of members of $\mathcal{F} \cup \{A_i\}$.

In particular, the statement holds for $i = 1$. Since $A_1 = B_1 \in \mathcal{F}$, it says that S^{p_1} can be partitioned into copies of members of $\mathcal{F}$, as required. $\square$

3.1 Proof of Lemma 3.6

Here we will prove Lemma 3.6.

Lemma 3.6. *For any sets $A, B \subset S$, there exists a positive integer n such that $S^2 \times (A \cup B)^n$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.*

We start by picking two sets $A, B \subset S$; these sets will be fixed throughout the subsection. We define $U = A \cup B$, $\bar{A} = U \setminus A$ and $\bar{B} = U \setminus B$. Moreover, for any integers $1 \le i \le d$, we define

$$C_{i,d} = \underbrace{\bar{A} \times \cdots \times \bar{A} \times \overset{\substack{i\text{-th component}\\ \downarrow}}{\bar{B}} \times \bar{A} \times \cdots \times \bar{A}}_{d \text{ components}}.$$

We also define $C_{0,d} = \bar{A}^d$. Our aim is to prove that there exists a positive integer n such that $S \times U^n$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

At certain points in the proof we will be conjuring up extra elbow space by 'blowing up' S^k, for some k, into S^{k+1}. It turns out that sometimes a set $X \subset S^k$ can be usefully identified with a larger set $X \times \bar{A} \subset S^{k+1}$. The following simple proposition is an example of this idea.

Proposition 3.8. *Let $k \ge 1$ and let $X \subset U^k$ be such that $U^k \setminus X$ can be partitioned into copies of A and B. Then $U^{k+1} \setminus \big(X \times \bar{A}\big)$ can be partitioned into copies of A and B.*

Proof. Partition $U^{k+1} \setminus \big(X \times \bar{A}\big)$ into sets $\big(U^k \setminus X\big) \times \bar{A}$ and $U^k \times A$; the first of these sets can be partitioned into copies of $U^k \setminus X$, and the second – into copies of A. $\square$

If we could prove that U^k, for some k, can be partitioned into copies of A and B (that is, without using $\mathcal{F}$), then we would be done. Of course, this is not possible in general. However, we can partition U^k with one $C_{i,k}$ removed.

Proposition 3.9. *For any integers $k \ge 1$ and $0 \le i \le k$, the set $U^k \setminus C_{i,k}$ can be partitioned into copies of A and B.*

Proof. We use induction on k. If $k = 1$, then, depending on the value of i, $U \setminus C_{i,1}$ is either A or B. If $k \ge 2$, we may assume that $i \ne k$ (in fact, there are only two distinct cases: $i = 0$ and $i \neq 0$). By the induction hypothesis, $U^{k-1} \setminus C_{i,k-1}$ can be partitioned into copies of A and B. However, $C_{i,k} = C_{i,k-1} \times \bar{A}$, so we are done by Proposition 3.8. $\square$

Proposition 3.8 says that if we can partition a subset of U^k, then we can also partition an 'equivalent' subset of U^{k+1}. The following proposition allows us to use the extra space in U^{k+1} to slightly modify this subset.

Proposition 3.10. *Let $X \subset U^k$ be such that $U^k \setminus X$ can be partitioned into copies of A and B. Suppose that X contains the set $C_{i,k}$ for some $0 \leq i \leq k$. Then the set $U^{k+1} \setminus Y$, where*

$$Y = \big(X \times \bar{A}\big) \cup C_{k+1,k+1} \setminus C_{i,k+1},$$

can also be partitioned into copies of A and B.

Proof. Partition $U^{k+1} \setminus Y$ into four sets Z_1, Z_2, Z_3, Z_4, where

$$\begin{aligned}
Z_1 &= (U^k \setminus C_{0,k}) \times \bar{B}, \\
Z_2 &= (U^k \setminus C_{i,k}) \times (A \cap B), \\
Z_3 &= C_{i,k} \times B, \\
Z_4 &= (U^k \setminus X) \times \bar{A}.
\end{aligned}$$

It is evident from Figure 3.1 that these four sets do partition $U^{k+1} \setminus Y$. The sets Z_1

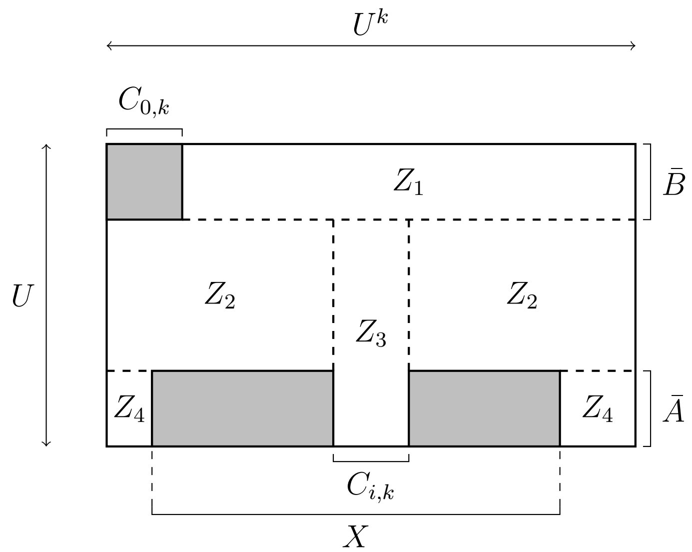

Figure 3.1: The set $Y \subset U^{k+1}$ is shaded. The four sets Z_1, Z_2, Z_3, Z_4 partition $U^{k+1} \setminus Y$.

and Z_2 can be partitioned into copies of A and B by Proposition 3.9. The set Z_3

is obviously a union of disjoint copies of B. Finally, Z_4 is a union of disjoint copies of $U^k \setminus X$, so it can be partitioned into copies of A and B by the assumption on X. $\square$

The previous proposition enables us to make one change to the set X when we go one dimension up, that is, from U^k to U^{k+1}. To make multiple changes, we apply this proposition multiple times. This is exactly the content of Corollary 3.11.

Corollary 3.11. *Let k, l be non-negative integers and let $I \subset \{0, \dots, k\}$, $J \subset \{k+1, \dots, k+l\}$ be sets such that $|J| = |I|$. Then the set $U^{k+l} \setminus Y$, where*

$$Y = \left(U^k \times \bar{A}^l\right) \cup \left(\bigcup_{j \in J} C_{j,k+l}\right) \setminus \left(\bigcup_{i \in I} C_{i,k+l}\right),$$

can be partitioned into copies of A and B.

Proof. We shall apply induction on l. If $l = 0$, then $|J| = |I| = 0$, so $U^k \setminus Y = \emptyset$, and hence the conclusion trivially holds.

Now suppose that $l \geq 1$. We will split the argument into two cases, depending on whether or not $k + l \in J$. If $k + l \in J$, then we write $j^* = k + l$ and we pick any $i^* \in I$. We define $I^* = I \setminus \{i^*\}$ and $J^* = J \setminus \{j^*\}$. Finally, we define

$$Y^* = \left(U^k \times \bar{A}^{l-1}\right) \cup \left(\bigcup_{j \in J^*} C_{j,k+l-1}\right) \setminus \left(\bigcup_{i \in I^*} C_{i,k+l-1}\right).$$

By the induction hypothesis, U^{k+l-1} can be partitioned into copies of A and B. Moreover, $Y = \left(Y^* \times \bar{A}\right) \cup C_{k+l,k+l} \setminus C_{i^*,k+l}$, so we can apply Proposition 3.10 to finish the proof in this case.

On the other hand, if $k + l \notin J$, then we define

$$Y' = \left(U^k \times \bar{A}^{l-1}\right) \cup \left(\bigcup_{j \in J} C_{j,k+l-1}\right) \setminus \left(\bigcup_{i \in I} C_{i,k+l-1}\right)$$

and observe that $Y = Y' \times \bar{A}$. Moreover, $U^{k+l-1} \setminus Y'$ can be partitioned into copies of A and B by the induction hypothesis, and hence it follows from Proposition 3.8 that the same holds for $U^{k+l} \setminus Y$. $\square$

Recall that our ultimate goal in this subsection is to partition $S^2 \times U^n$, for some $n \geq 1$, into copies of members of $\mathcal{F} \cup \{A, B\}$. We cannot achieve this goal just yet, but we have already provided ourselves with tools, in the form of Propositions 3.8

to 3.10 and Corollary 3.11, that allow us to partition $U^k \backslash X$, for various k and various sets X, into copies of A and B. Our strategy now can be roughly described as follows. We will take a large n and we will slice $S^2 \times U^n$ up into copies of $S \times U^n$. We will partition big parts of these slices into copies of members of $\mathcal{F} \cup \{A, B\}$, leaving out gaps that we can control. Then we will combine the gaps across all slices, and we will fill them in with copies of members of $\mathcal{F}$. The following proposition will tell us what gaps we should leave in the slices so that their union could be filled in later on.

Proposition 3.12. *Let t be a positive integer and take not necessarily distinct sets $P_1, \ldots, P_t \in \mathcal{F}$. Define $Q_0, \ldots, Q_t \subset S \times U^t$ by setting*

$$Q_i = \begin{cases} P_i \times C_{i,t} & \text{if } 1 \leq i \leq t, \\ S \times C_{0,t} & \text{if } i = 0. \end{cases}$$

Then the set $(S \times U^t) \setminus (Q_0 \cup \cdots \cup Q_t)$ can be partitioned into copies of members of $\mathcal{F} \cup \{A \cup B\}$.

Proof. We use induction on t. We take $t = 0$ to be the base case. Although the set $C_{0,0}$ had not been defined, we may interpret $S \times C_{0,0}$ and $S \times U^0$ as both being the set S, in which case the conclusion says that the empty set can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$, which is trivially true.

Now suppose that $t \geq 1$. We write $X = Q_0 \cup \cdots \cup Q_t$ and $X^* = (S \times C_{0,t-1}) \cup (P_1 \times C_{1,t-1}) \cup \cdots \cup (P_{t-1} \times C_{t-1,t-1})$. By the induction hypothesis, $(S \times U^{t-1}) \setminus X^*$ can be partitioned into copies of $\mathcal{F} \cup \{A, B\}$. Moreover, using the fact that $X = (X^* \times \bar{A}) \cup Q_t$, we can partition $(S \times U^t) \setminus X$ into three sets Y_1, Y_2, Y_3, where

$$\begin{aligned} Y_1 &= \big((S \times U^{t-1}) \setminus X^*\big) \times \bar{A}, \\ Y_2 &= \big((S \times U^{t-1}) \setminus (P_t \times C_{0,t-1})\big) \times A), \\ Y_3 &= P_t \times U^{t-1} \times (A \cap B). \end{aligned}$$

It is clear from Figure 3.2 that these sets do partition $(S \times U^t) \backslash X$. Moreover, Y_1 can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$ (by the induction hypothesis); Y_2 is trivially a disjoint union of copies of A; Y_3 is a disjoint union of copies of P_t, which is a member of $\mathcal{F}$. $\square$

We recall that, for some positive integer r, $\mathcal{F}$ contains an r-partition of S. In other words, there exist not necessarily distinct sets $P_1, \ldots, P_m \in \mathcal{F}$ such that every

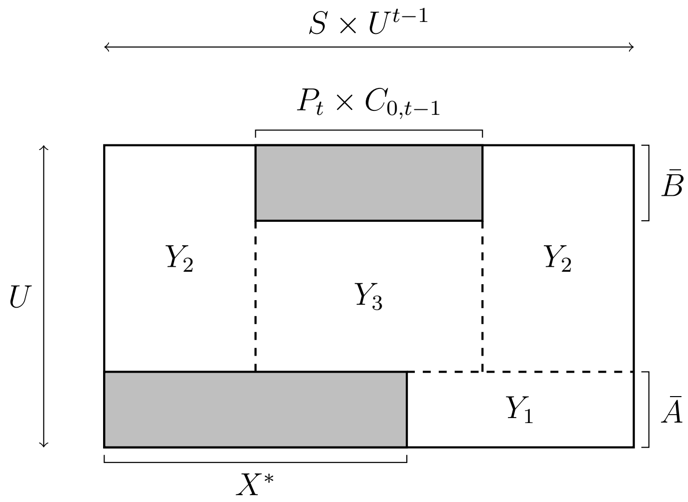

Figure 3.2: The set X is shaded; Y_1, Y_2, Y_3 partition $(S \times U^{t-1}) \setminus X$.

element of S is contained in precisely r of them. We will use the sets $P_1, \ldots, P_m$ to prove the following proposition.

Proposition 3.13. *Let r be as above. For any positive integer k there exists an integer $l \geq k$ with the following property. For any distinct numbers $j_1, \ldots, j_t \in \{1, \ldots, l\}$, if $t \leq k$ and $t \equiv 1 \pmod{r}$, then the set $S \times \left(U^l \setminus \bigcup_{u=1}^{t} C_{j_u,l}\right)$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.*

Proof (see Figure 3.3). Given k, fix any $l \geq k+(k-1)m/r$. Given distinct $j_1, \ldots, j_t \in \{1, \ldots, l\}$, we may assume (after a permutation of coordinates, if necessary), that $\{j_1, \ldots, j_t\} = \{l-t+1, \ldots, l\}$. We denote this set by J. Since $t \leq k$ and $t \equiv 1 \pmod{r}$ by assumption, we may write $t = ar+1$ for some integer $0 \leq a \leq (k-1)/r$. We will prove that the set

$$Y = S \times \left(U^l \setminus \bigcup_{j \in J} C_{j,l}\right)$$

can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

Extend $P_1, \ldots, P_m$ to a longer list $P_1, \ldots, P_{am}$ by setting $P_{i+m} = P_i$ for every $m+1 \leq i \leq am$. The only important property of this new list is that every member of the original list is repeated exactly a times. Moreover, set $P_0 = S$. Then every element of S is contained in exactly $ar+1 = t$ members of the list $P_0, \ldots, P_{am}$. We define

$$X = (S \times U^{am} \times \bar{A}^{l-am}) \setminus \left(\bigcup_{i=0}^{am} P_i \times C_{i,l}\right).$$

Since $X = ((S \times U^{am}) \setminus (\bigcup_{i=0}^{am} P_i \times C_{i,am})) \times \bar{A}^{l-am}$, it follows from Proposition 3.12 that X can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$. Since $\min J > l - k \geq am$, the set X is disjoint from $S \times C_{j,l}$ for any $j \in I$, and hence $X \subset Y$. Therefore, it only remains to prove that $Y \setminus X$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

For any $z \in S$, we denote by S_z the cross-section of $Y \setminus X$ at z, that is,

$$Y_z = \{y \in U^l : (z, y) \in Y \setminus X\}.$$

For the moment, let us focus on one fixed $z \in S$. By construction of $P_0, \dots, P_{am}$, there are exactly t values of i for which $z \in P_i$. Let I be the set of these values. Then

$$Y_z = U^l \setminus \left((U^{am} \times \bar{A}^{l-am}) \cup \left(\bigcup_{j \in J} C_{j,l} \right) \setminus \left(\bigcup_{i \in I} C_{i,l} \right) \right).$$

Since $|I| = |J| = t$, $I \subset \{0, \dots, am\}$ and $J \subset \{am+1, \dots, l\}$, Corollary 3.11 implies that Y_z can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

Now we are done: $Y = X \cup \left(\bigcup_{z \in S} \{z\} \times Y_z\right)$, and we have proved that X and every Y_z can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$. $\square$

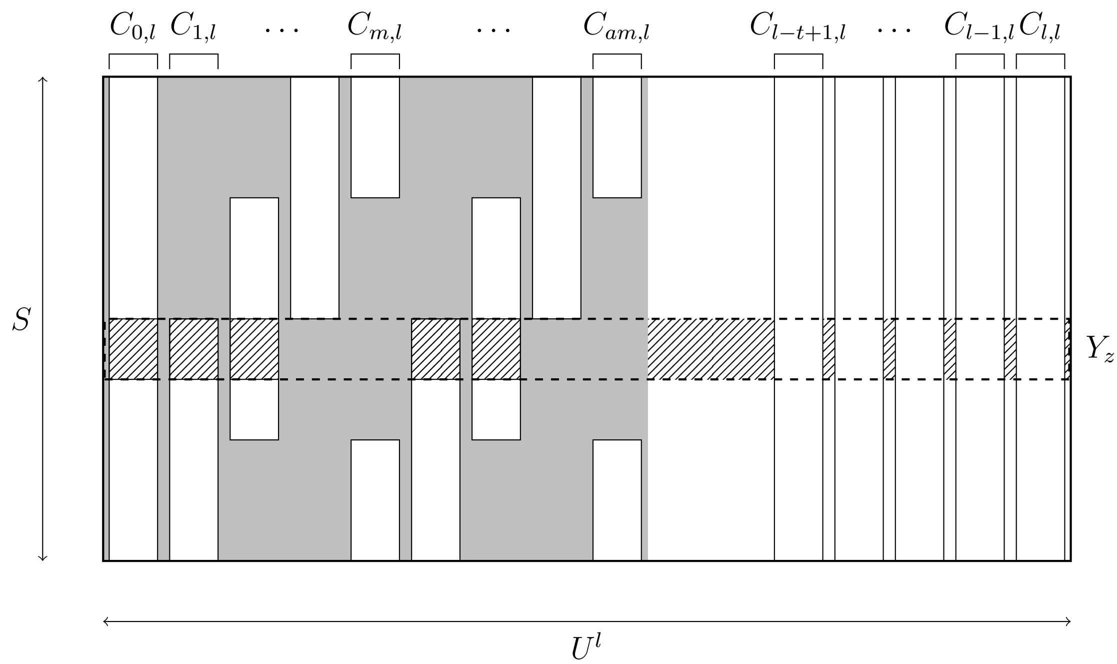

Figure 3.3: The set X is shaded, a slice Y_z is hatched diagonally. Proposition 3.12 and Corollary 3.11, respectively, imply that X and Y_z can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

We are now ready to prove Lemma 3.6.

Proof of Lemma 3.6. We begin by recalling that $\mathcal{F}$ contains a (1 mod r)-partition of S. In other words, there exists a family of not necessarily distinct sets $R_1, \dots, R_k \in \mathcal{F}$ such that every $x \in S$ is contained in exactly $1 + ra_x$ members of this family, where a_x is an integer. Furthermore, Proposition 3.13 provides us with a positive integer $n \geq k$ such that, for any set $I \subset \{1, \dots n\}$ that satisfies $|I| \equiv 1 \pmod r$ and $|I| \leq k$, the set $S \times \left(U^n \setminus \bigcup_{i \in I} C_{i,n}\right)$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$. We will show that $S \times U^n$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$.

We define

$$X = (S \times U^n) \setminus \left(\bigcup_{i=1}^{k} R_k \times C_{i,n}\right)$$

and, for any $y \in S$, we let X_y denote the cross-section of X at y, that is, $X_y = \{x \in U^n : (y, x) \in X\}$. Any $y \in S$ is contained in $1 + ra_y$ members of the family $R_1, \dots, R_k$. Therefore, if we write $J_y = \{j \in [k] : y \in R_j\}$, then $|J_y| \equiv 1 \pmod r$ and $|J_y| \leq k$. Moreover, it is easy to see that

$$X_y = U^n \setminus \left(\bigcup_{j \in J_y} C_{j,n}\right).$$

By Proposition 3.13, $S \times X_y$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$. Therefore, so can be $S \times X$, which is the disjoint union of sets $S \times \{y\} \times X_y$, $y \in S$.

Finally, observe that $S^2 \times U^n$ is the disjoint union of $S \times X$ and sets $S \times R_i \times C_{i,n}$, $1 \leq i \leq k$. Each set $S \times R_i \times C_{i,n}$ is trivially a union of disjoint copies of R_i, which is a member of $\mathcal{F}$. Therefore, $S^2 \times U^n$ can be partitioned into copies of members of $\mathcal{F} \cup \{A, B\}$, as required. □

4 Weak partitions

4.1 Constructing an r-partition of $2^{[n]}$

Our aim in this subsection is to prove Lemma 3.3, which asserts the existence of an r-partition of $2^{[n]}$ into copies of P for some n, r. Our proof is somewhat technical, but not very difficult.

Recall that by our earlier definition a weight function is an assignment of non-negative integer weights to sets from some selected family. We now extend this definition to allow more general weights. Namely, given a set $V \subset \mathbb{R}$ and a set family $\mathcal{F}$, a *V-valued weight function on $\mathcal{F}$* is a function $w : \mathcal{F} \to V$. Usually, we will take V to be $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Q}^+$, where S^+ is defined to be $S \cap [0, \infty)$ for any $S \subset \mathbb{R}$. We note that a weight function in the old sense is precisely a $\mathbb{Z}^+$-valued weight function in the new sense.

Moreover, if $\mathcal{F}$ is a family of subsets of some set X, for any $x \in X$ we define the *multiplicity of x for w*, denoted $N_w(x)$, to be the total weight assigned to the members of $\mathcal{F}$ that contain x. That is,

$$N_w(x) = \sum_{\substack{A \in \mathcal{F} \\ x \in A}} w(A).$$

Moreover, for any $Y \subset X$, we set $N_w(Y) = \sum_{y \in Y} N_w(y)$. With these definition at hand, we can restate Lemma 3.3 in a form that is slightly more convenient for the proof.

Lemma 3.3'. *Let P be a finite poset with a greatest and a least element. Then there exist a positive integer n and a $\mathbb{Q}^+$-valued weight function w on the copies of P in $2^{[n]}$ such that $N_w(x) = 1$ for all $x \in 2^{[n]}$.*

To see why Lemma 3.3' is equivalent to Lemma 3.3, observe that a $\mathbb{Q}^+$-valued weight function w on a finite set family $\mathcal{F}$ can be made into a $\mathbb{Z}^+$-valued weight function by multiplying it by the least common multiple of the denominators of the $w(A)$ for $A \in \mathcal{F}$. Moreover, if $N_w(x) = 1$ for all x, then the resulting $\mathbb{Z}^+$-valued weight function rw satisfies $N_{rw}(x) = r$ for all x.

The main idea in the proof is to look for a weight function that is symmetric with respect to all permutations of the ground set $\{1, \ldots, n\}$. Such a weight function can be obtained by averaging any another weight function over all permutations of $\{1, \ldots, n\}$. This idea essentially removes the need to consider the structure of the poset P, and converts Lemma 3.3' into a question about finding a certain weight function on the power set of $\{0, \ldots, n\}$. This is reflected in the following definition.

Let P be a poset and n a positive integer. Moreover, let w be a $\mathbb{Q}^+$-valued weight function on the copies of P in $2^{[n]}$. We define a new $\mathbb{Q}^+$-valued weight function w^{sym}, also on the copies of P in $2^{[n]}$, by setting

$$w^{\mathrm{sym}}(A) = \frac{1}{n!} \sum_{\pi \in \mathrm{Perm}(n)} w\big(\pi(A)\big)$$

for all A that are copies of P in $2^{[n]}$. Here $\mathrm{Perm}(n)$ denotes the set of permutations of $\{1, \dots, n\}$ and we recall that $\pi(A)$ denotes the image of A after permuting the coordinates of $2^{[n]}$ according to π.

Since elements of $2^{[n]}$ are subsets of $\{1, \dots, n\}$, it makes sense to write $|x|$ for $x \in 2^{[n]}$ to denote the size of x. We partition $2^{[n]}$ into levels $L_0, \dots, L_n$, where $L_k = \{x \in 2^{[n]} : |x| = k\}$. Then, for any $x \in L_k$,

$$N_{w^{\mathrm{sym}}}(x) = \frac{1}{\binom{n}{k}} N_w(L_k).$$

Therefore, our task is reduced to finding w such that $N_w(L_k) = \binom{n}{k}$ for all k. To this aim, we would like to have a tool for embedding P into $2^{[n]}$ while keeping control on levels into which we map the elements of P. The following proposition provides us with such a tool.

We say that a set $A \subset \mathbb{Z}$ is *d-scattered* if, for any distinct $i, j \in A$, we have $|i - j| \geq d$.

Proposition 3.14. *Let P be a finite poset with a greatest and a least element. Then there exists a positive integer d such that, for any integer $n \geq (|P| - 1)d$ and any d-scattered set $A \subset \{0, \dots, n\}$ of size $|P|$, there exists an embedding $\phi : P \to 2^{[n]}$ satisfying*

$$\{|\phi(x)| : x \in P\} = A.$$

In other words, for any $0 \leq k \leq n$,

$$|L_k \cap \phi(P)| = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We start by recalling that, since P is finite, it can be embedded into $2^{[k]}$ for some k. Let $\psi : P \to 2^{[k]}$ be an embedding which maps the greatest element of P to the greatest element of $2^{[k]}$ and the least element of P to the least element of $2^{[k]}$. We write $s = |P|$ and list the elements of P as $p_1, \dots, p_s$ in the order where $0 = |\psi(p_1)| \leq \dots \leq |\psi(p_s)| = k$.

We will prove that $d = k$ works. Indeed, take any integer $n \geq (s-1)k$ and let $A \subset \{0, \dots, n\}$ be a k-scattered set of size s. Then $A = \{a_1, \dots, a_s\}$, where $0 \leq a_1 < \dots < a_s \leq n$ and $a_{i+1} \geq a_i + k$ for all $0 \leq i \leq s-1$. For every $1 \leq i \leq s$, we set

$$\phi(p_i) = \psi(p_i) \cup \{k+1, \dots, k + a_i - |\psi(p_i)|\}.$$

To prove that $\phi : P \to 2^{[n]}$ is a well-defined embedding, we have to check that $0 \leq a_1 - |\psi(p_1)| \leq \dots \leq a_s - |\psi(p_s)| \leq n - k$. However, if we prove this, then it is trivial to see that $|\phi(p_i)| = a_i$ for all i, as required.

First, we observe that $a_1 - |\psi(p_1)| = a_1 \geq 0$ and $a_s - |\psi(p_s)| = a_s - k \leq n - k$. Furthermore, for any $1 \leq i \leq s-1$, we have $a_{i+1} - |\psi(p_{i+1})| \geq a_i + k - k = a_i \geq a_i - |\psi(p_i)|$, and so we are done. $\square$

Proposition 3.15. *Let X be a finite set and t a positive integer. If $f : X \to \mathbb{Q}^+$ is a function such that*

$$t \max_{x\in X} f(x) \leq \sum_{x\in X} f(x),$$

then there exists a $\mathbb{Q}^+$-valued weight function w on the family of t-element subsets of X, such that $N_w(x) = f(x)$ for all $x \in X$.

Proof. Let r be the least common multiple of the denominators of the $f(x)$ over all $x \in X$. After multiplying f by tr, we may assume that f takes values in $\mathbb{Z}^+$ and that $\sum_{x\in X} f(x)$ is divisible by t. We denote $\sum_{x\in X} f(x) = Nt$ and we will use induction on N.

If $f(x) = 0$ for all $x \in X$, then the result is trivial. Therefore, we may assume that $N \geq 1$. Let $S = \{x \in X : f(x) > 0\}$ and $T = \{x \in X : f(x) = N\}$. Since

$$t \max_{x\in X} f(x) \;\leq\; \sum_{x\in X} f(x) \;\leq\; |S| \max_{x\in X} f(x),$$

it follows that $|S| \geq t$. Moreover, $N|T| \leq \sum_{x\in X} f(x) = Nt$, and hence $|T| \leq t$. Therefore, there exists a set A such that $T \subset A \subset S$ and $|A| = t$.

We define $g : X \to \mathbb{Z}^+$ by setting

$$g(x) = \begin{cases} f(x) - 1 & \text{if } x \in A, \\ f(x) & \text{otherwise.} \end{cases} .$$

Then $\sum_{x\in X} g(x) = (N-1)t$ is non-negative and divisible by t. Moreover, since $T \subset A$, we have $g(x) \leq N - 1$ for all $x \in X$. Therefore, by the induction hypothesis, there exists a $\mathbb{Q}^+$-valued weight function w' on the t-element subsets of X, such that $N_{w'}(x) = g(x)$ for all $x \in X$. We define

$$w(B) = \begin{cases} w'(A) + 1 & \text{if } B = A, \\ w'(B) & \text{if } B \subset X,\ |B| = t \text{ and } B \neq A. \end{cases}$$

This w satisfies the required conditions. $\square$

It is easy to deduce Lemma 3.3' from Propositions 3.14 and 3.15.

Proof of Lemma 3.3'. Let P be a finite poset with a greatest and a least element. Recall that our aim is to find, for some positive integer n, a $\mathbb{Q}^+$-valued weight function w on the copies of P in $2^{[n]}$, such that $N_w(L_i) = \binom{n}{i}$ for all $0 \le i \le n$. Indeed, then $N_{w^{\mathrm{sym}}}(x) = 1$ for all $x \in 2^{[n]}$.

Let d be such that, for any $n \ge (|P|-1)d$ and any d-scattered set $A \subset \{0, \dots, n\}$ of size $|P|$, there exists a copy of P in $2^{[n]}$, say C, such that $\{|x| : x \in C\} = A$. The existence of such a number d is guaranteed by Proposition 3.14. Set $k = |P|d$.

Choose n large enough to satisfy the inequality $k\binom{n}{\lceil n/2\rceil} \le 2^n$. Then Proposition 3.15 gives a $\mathbb{Q}^+$-valued weight function w' on the k-element subsets of $\{0, \dots, n\}$ that satisfies $N_{w'}(i) = \binom{n}{i}$ for all $0 \le i \le n$.

Let B be a k-element subset of $\{0, \dots, n\}$. If we consider the elements of B in increasing order and take every dth element, we obtain a d-scattered set. In this way we can partition B into d-scattered sets $B_1, \dots, B_d$, each of size $k/d = |P|$. We say that B *splits* into sets $B_1, \dots, B_d$.

By splitting k-element sets we obtain a $\mathbb{Q}^+$-valued weight function w'' on d-scattered $|P|$-element subsets of $\{0, \dots, n\}$. More precisely, we define $w''(A) = \sum w'(B)$, summing over all k-element sets $B \subset \{0, \dots, n\}$ with the property that A is one of the sets into which B splits. Note that we have $N_{w''}(i) = N_{w'}(i) = \binom{n}{i}$ for all $0 \le i \le n$.

Finally, for any d-scattered $|P|$-element set $A \subset \{0, \dots, n\}$ we choose one copy of P in $2^{[n]}$, denoted C_A, such that $\{|x| : x \in C_A\} = A$. We define a $\mathbb{Q}^+$-valued weight function w on the copies of P in $2^{[n]}$ by setting

$$w(C) = \begin{cases} w''(A) & \text{if } C = C_A \text{ for some } d\text{-scattered } |P|\text{-element set } A \subset \{0, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that every d-scattered $|P|$-element set $A \subset \{0, \dots, n\}$ contributes $w''(A)$ towards both $N_{w''}(i)$ and $N_w(L_i)$ for every $i \in A$, and 0 towards both $N_{w''}(j)$ and $N_w(L_j)$ for every $j \notin A$. Therefore, $N_w(L_i) = N_{w''}(i) = \binom{n}{i}$ for all $0 \le i \le n$, as required. □

4.2 Constructing a (1 mod r)-partition of $2^{[n]}$

Here we prove Lemma 3.4, which asserts the existence of an (1 mod r)-partition of $2^{[n]}$ into copies of P for some n. This proof is shorter, but slightly trickier than that of Lemma 3.3. We begin by recasting Lemma 3.4 in a form which is stronger, but more convenient to work with.

Lemma 3.4'. *Let P be a poset of size 2^k with a greatest and a least element. Then there exist a positive integer n and a $\mathbb{Z}$-valued weight function w on the copies of P in $2^{[n]}$ satisfying $N_w(x) = 1$ for all $x \in 2^{[n]}$.*

We remark that Lemma 3.4' does imply Lemma 3.4, because the $\mathbb{Z}$-valued weight function w can be converted into a suitable $\mathbb{Z}^+$-valued weight function w' by choosing $w'(A) \in \{0, \dots, r-1\}$ such that $w'(A) \equiv w(A) \pmod{r}$, for all A.

Proof of Lemma 3.4'. Since P is finite, it can be embedded into $2^{[d]}$, for some d, by an embedding which maps the greatest and the least elements of P to the corresponding elements of $2^{[d]}$. We will show that $n = 2d-1$ works.

We say that a function $f : 2^{[n]} \to \mathbb{Z}$ is *realisable* if there exists a $\mathbb{Z}$-valued weight function w on the copies of P in $2^{[n]}$, such that $N_w(x) = f(x)$ for all $x \in 2^{[n]}$. We note that if f, g are realisable functions, then so are $f+g$ and $f-g$. Our aim is to show that the constant 1 function on $2^{[n]}$ is realisable.

For any $A \subset 2^{[n]}$, we define $1_A : 2^{[n]} \to \{0,1\}$ to be the indicator function of A. Clearly, if A is a copy of P, then 1_A is realisable.

We denote the greatest and the least elements of $2^{[n]}$ by x_+, x_-. Let $x \in 2^{[n]}$. If $|x| \geq d$, then there exists an embedding $2^{[d]} \to 2^{[n]}$ which maps the greatest element of $2^{[d]}$ to x. Therefore, in $2^{[n]}$, we can find a copy of P whose greatest element is x. We denote this copy by A. Moreover, if we denote $B = A \setminus \{x\}$, then $B \cup \{x\}$ and $B \cup \{x_+\}$ are copies of P. Therefore, the function $1_{\{x\}} - 1_{\{x_+\}} = 1_{B\cup\{x\}} - 1_{B\cup\{x_+\}}$ is realisable.

Similarly, if $|x| \leq d$, then there exists an embedding $2^{[d]} \to 2^{[n]}$ which maps the least element of $2^{[d]}$ to x. Then we can find a copy of P in $2^{[n]}$, which we denote by A, with the property that x is the least element of A. We write $B = A \setminus \{x\}$ and observe that $A \cup \{x\}$ and $A \cup \{x_-\}$ are copies of P. Therefore, the function $1_{\{x\}} - 1_{\{x_-\}} = 1_{B\cup\{x\}} - 1_{B\cup\{x_-\}}$ is realisable.

In particular, for any $x \in 2^{[n]}$, at least one of the functions $1_{\{x\}} - 1_{\{x_+\}}$ and $1_{\{x\}} - 1_{\{x_-\}}$ is realisable. Moreover, if $|x| = d$, then both of them are. Therefore, by choosing any $x_0 \in 2^{[n]}$ with $|x_0| = d$, we can see that $1_{\{x_+\}} - 1_{\{x_-\}} = (1_{\{x_0\}} - 1_{\{x_-\}}) - (1_{\{x_0\}} - 1_{\{x_+\}})$ is realisable. We conclude that, in fact, for any $x, y \in 2^{[n]}$, the function $1_{\{x\}} - 1_{\{y\}}$ is realisable.

Let $f, g : 2^{[n]} \to \mathbb{Z}$ be two functions that satisfy $\sum_{x\in 2^{[n]}} f(x) = \sum_{x\in 2^{[n]}} g(x)$. Then the difference $f-g$ can be expressed as a sum of functions of the form $1_{\{x\}} - 1_{\{y\}}$ with $x, y \in 2^{[n]}$, so $f-g$ is realisable. Hence, f is realisable if and only if g is realisable. Therefore, to prove that the constant 1 function is realisable, it is enough to find one realisable function f such that $\sum_{x\in 2^{[n]}} f(x) = 2^n$. However, we know that $|P| = 2^k$ and, trivially, $k \leq n$, so we can take $f = 2^{n-k} \cdot 1_A$ for any $A \subset 2^{[n]}$ which is a copy of P. $\square$

5 Concluding remarks and open problems

In the proof of Theorem 3.2 we do not explicitly keep track of a value of n that would be sufficient. This is to make the proof more readable. Moreover, we did not put any serious effort into finding a good bound. The following bound can be extracted from the proof.

Theorem 3.2'. *There exists an absolute constant $C > 0$ with the following property. Let P be a poset of size 2^k with a greatest and a least element. Then, for any integer $n \geq 2^{|P|^C}$, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

It is interesting to ask what happens if P does not satisfy the conditions required by Theorem 3.2. Of course, then it is impossible to partition $2^{[n]}$ into copies of P. However, what if we are allowed to leave a small number of elements of $2^{[n]}$ uncovered? For example, if P does not have a greatest and/or a least element, then the greatest and/or the least element of $2^{[n]}$ are the only ones that obviously cannot be covered by copies of P. Lonc [42] conjectured that, if n is large and if an obvious divisibility condition is satisfied, then $2^{[n]}$ with its greatest and least element removed can be partitioned into copies of P.

Conjecture 3.16 (Lonc [42])**.** *Let P be a finite poset. If n is sufficiently large and if $|P|$ divides $2^n - 2$, then it is possible to partition $2^{[n]}$, with its greatest and least element removed, into copies of P.*

In the spirit of Griggs' conjecture it is reasonable to hope that, even if we do not impose any divisibility conditions for $|P|$, for sufficiently large n, $2^{[n]}$ can be partitioned into copies of P and a set of size c, where $c < |P|$. Or perhaps one can bound c by a weaker constant which depends on P.

Question 3.17. *Let P be a finite poset. Must there exist a constant $c = c(P)$ such that, for any n, it is possible to cover all but at most c elements of $2^{[n]}$ by disjoint copies of P?*

We remark that Conjecture 3.16 would give a positive answer to Question 3.17 in the case where $|P|$ is not a multiple of 4.

CHAPTER 4

Partitions of the hypercube

1 Introduction

A famous theorem of Wilson [64] states that, for any finite graph H and for any sufficiently large integer n which satisfies certain divisibility conditions, the edges of the complete graph K_n can be covered by disjoint copies of H. Such a cover is called an *H-decomposition* of K_n. The divisibility conditions required by Wilson's theorem are obviously necessary for an H-decomposition of K_n to exist: $\binom{n}{2}$ must be divisible by $e(H)$ and $n-1$ must be divisible by the highest common factor of the degrees of the vertices of H. Therefore, as long as we are only interested in large n, Wilson's theorem tells us exactly when K_n admits an H-decomposition. On the other hand, the general question of determining whether an arbitrary graph G has a H-decomposition is very difficult, and various special cases of this question have attracted significant attention.

In this chapter we examine a related question: we are concerned with partitioning the vertices – not edges – of a given graph G into copies of H. More precisely, for finite graphs G, H, we say that a set $A \subset V(G)$ is an *H-set* if the induced subgraph $G[A]$ is isomorphic to H. We consider the following question: can $V(G)$ be partitioned into H-sets?

In contrast to Wilson's theorem, this question is not interesting in the case where G is a complete graph: obviously, $V(K_n)$ can be partitioned into H-sets if and only if $H = K_m$ where m divides n. Instead, we focus on the case where G is the hypercube Q_n, that is, the graph with vertex set $\{0,1\}^n$ where two n-tuples are adjacent if and only if they differ in precisely one entry.

Let H be a finite graph and let n be large. Can we quickly determine whether $V(Q_n)$ can be partitioned into H-sets? Of course, there is an obvious necessary divisibility condition: $|H|$ must be a power of 2. Moreover, this condition alone is not sufficient because H may not be isomorphic to any induced subgraph of any hypercube Q_n. For example, H could be a non-bipartite graph or, say, it could be a bipartite graph, of size a power of 2, that contains $K_{3,2}$ as a subgraph. Note $K_{3,2}$ is not a subgraph of any Q_n since any two vertices that are distance 2 apart in Q_n are joined by precisely two paths of length 2. Therefore, we should require H to be an induced subgraph of some hypercube. Offner [53] considered this problem in connection with coding theory. He asked if this condition together with the divisibility condition is sufficient.

Question 4.1 (Offner [53]). *Let H be an induced subgraph of Q_k for some k and suppose that $|H|$ is a power of 2. Must it be true that, for any sufficiently large n, $V(Q_n)$ can be partitioned into H-sets?*

This question bears resemblance to the celebrated work of Hamming [35] on error-correcting codes. Indeed, a perfect single-error-correcting code is a partition of $V(Q_n)$ into $K_{1,n}$-sets. Hamming showed that such a partition exists if and only if a natural divisibility condition is satisfied, namely, if $n = 2^r - 1$ for some r. Much later, Rogers (see [56]) asked if it is possible to partition the vertices of Q_n into antipodal paths, subject to the same divisibility condition. Here an *antipodal path* is a path of length n which starts and ends at two diagonally opposite vertices of Q_n. Rogers' question was answered by Ramras [56], who proved the following more general result: if $n = 2^r - 1$ and if T is a tree on $n + 1$ vertices which is an induced subgraph of Q_n, then $V(Q_n)$ can be partitioned into isometric copies of T.

Moreover, there is a clear connection between Offner's question and conjectures of Chalcraft [46, 47] and of Lonc [42], which were presented in Chapters 2 and 3. In this chapter we combine new ideas with tools developed in Chapter 3 to answer Offner's question.

Theorem 4.2. *Let H be an induced subgraph of Q_k for some k. If $|H|$ is a power of 2, then there exists a positive integer n such that the vertices of Q_n can be partitioned into H-sets.*

Of course, if the result holds for n, then it holds for all $n' \geq n$. Therefore, Theorem 4.2 answers Question 4.1.

2 Overview of the proof

It turns out that, in order to prove Theorem 4.2, it is convenient to view the hypercube Q_n as the metric space $\{0, 1\}^n$ where the distance between any two points

$x, y \in \{0,1\}^n$, denoted $d(x, y)$, is equal to the number of entries where x and y are different. With this definition, $d(x, y)$ equals 1 if and only if x and y are adjacent vertices of Q_n. If H is an induced subgraph of Q_k, then we can identify H with a subset of $\{0,1\}^k$. For any $n \geq k$, we say that a set $X \subset \{0,1\}^n$ is an *isometric copy of* H if there exists an isometry $\phi : \{0,1\}^k \to \{0,1\}^n$ which maps H to X. Clearly, any isometric copy of H in $\{0,1\}^n$ is an H-set, but an H-set need not be an isometric copy of H.

We deduce Theorem 4.2 from the following slightly stronger result.

Theorem 4.3. *Let X be a subset of $\{0,1\}^k$ for some k. If $|X|$ is a power of 2, then there exists a positive integer n such that $\{0,1\}^m$ can be partitioned into isometric copies of X.*

Our proof will rely on Theorem 3.5 from the previous chapter. We recall this result for reader's convenience.

Theorem 3.5. *Let S be a finite set and let $\mathcal{F}$ be a family of subsets of S. Suppose that there exists a positive integer r such that $\mathcal{F}$ contains an r-partition and a $(1 \bmod r)$-partition of S. Then there exists a positive integer n such that S^n can be partitioned into copies of members of $\mathcal{F}$.*

Recall that $\mathcal{F}$ contains an r-partition of S if there exists a collection of members of $\mathcal{F}$ (allowing repetitions) that covers every element of S exactly r times. Similarly, $\mathcal{F}$ contains a $(1 \bmod r)$-partition of S if it contains a collection covering every element of S $1 \pmod r$ times. For any subset $A \in \mathcal{F}$, a copy of A in S^n is the image of $A \times \{b\}$, for any $b \in S^{n-1}$, under any permutation of the coordinates.

To apply Theorem 3.5, we let $S = \{0,1\}^m$ for some large m and we take $\mathcal{F}$ to be the family of isometric copies of X in S. It turns out that the right choice for r is $r = |X|$.

Observation 4.4. *Let X be a non-empty subset of $\{0,1\}^k$ for some positive integer k. Then, for any $m \geq k$, the family of isometric copies of X in $\{0,1\}^m$ contains a $|X|$-partition of $\{0,1\}^m$.*

Proof. Let $m \geq k$ be given. We fix one isometric copy of X in $\{0,1\}^m$, which we denote by Y. Under addition modulo 2, for any $p \in \{0,1\}^m$, the set $Y + p = \{y + p : y \in Y\}$ is a subset of $\{0,1\}^m$. Moreover, it is an isometric copy of X.

By symmetry, all elements of $\{0,1\}^m$ are contained in $Y + p$ for the same number of choices of p. By double counting, this number must equal $2^m|Y|/2^m = |X|$. Therefore, the sets $Y + p$, where $p \in \{0,1\}^m$, form a $|X|$-partition of $\{0,1\}^m$. $\square$

Constructing a (1 mod $|X|$)-partition is rather more difficult, but also possible.

Lemma 4.5. *Let X be a non-empty subset of $\{0,1\}^k$ for some positive integer k, and let r be a power of 2. Then there exists an integer $m \geq k$ such that the family of all isometric copies of X in $\{0,1\}^m$ contains a (1 mod r)-partition of $\{0,1\}^m$.*

Although we are only going to use this lemma with $r = |X|$, we state it with r being any power of 2. This small detail allows us to prove this lemma by induction, which we do in Section 3.

We will now explain how Observation 4.4, Lemma 4.5, and Theorem 3.5 imply Theorem 4.3.

Proof of Theorem 4.3. Let X be a subset of $\{0,1\}^k$ such that $|X|$ is a power of 2. It follows from Lemma 4.5 that there exists a positive integer $m \geq k$ such that the family of isometric copies of X in $\{0,1\}^m$ contains a (1 mod $|X|$)-partition of $\{0,1\}^m$. By Observation 4.4, the family of isometric copies of X in $\{0,1\}^m$ also contains a $|X|$-partition of $\{0,1\}^m$. Therefore, it follows from Theorem 3.5 with $S = \{0,1\}^m$ that there exists a positive integer n such that $\{0,1\}^{mn}$ can be partitioned into copies of sets which are isometric copies of X in $\{0,1\}^m$. However, a copy of an isometric copy of X is itself an isometric copy of X, so we are done. $\square$

3 Constructing a $(1 \bmod r)$-partition of $\{0,1\}^n$

Here we prove Lemma 4.5. This section is is the heart of the chapter: it is the key new ingredient beyond the ideas presented in Chapter 3. First, we introduce some convenient notation. For any set $A \subset \{0,1\}^n$, we define

$$\begin{aligned} A_+ &= \left\{a \in \{0,1\}^{n-1} : (a,1) \in A\right\}, \\ A_- &= \left\{a \in \{0,1\}^{n-1} : (a,0) \in A\right\}. \end{aligned}$$

Proof of Lemma 4.5. Fix $r = 2^d$. We will use induction on k. If $k = 1$, then X is either a single point or the whole $\{0,1\}$, and so the conclusion holds with $n = 1$.

We now suppose that $k \geq 2$. At least one of the sets X_+ and X_- is not empty, so we may assume without loss of generality that $X_- \neq \emptyset$. Since X_- is a subset of $\{0,1\}^{k-1}$, the induction hypothesis implies the existence of a positive integer m such that the family of isometric copies of X_- in $\{0,1\}^m$ contains a (1 mod r)-partition of $\{0,1\}^m$. Moreover, we note that, for every set $A \subset \{0,1\}^m$ which is an isometric copy of X_-, there exists a set $B \subset \{0,1\}^{m+1}$ which is an isometric copy of X and which satisfies $B_- = A$. Therefore, it is possible to define a weight function on the family of isometric copies of X in $\{0,1\}^{m+1}$ in such a way that the multiplicity of every element of $\{0,1\}^m \times \{0\}$ is congruent to 1 (mod r). We do not impose any conditions on the multiplicities of elements of $\{0,1\}^m \times \{1\}$. For convenience, we denote that the multiplicity of any $x \in \{0,1\}^m \times \{1\}$ is congruent to $f(x) \pmod r$.

We will prove that the conclusion of Lemma 4.5 holds with $n = m + d + 1$. Let $x, y \in \{0,1\}^{d+1}$ be two elements that differ in exactly two entries. There exists an element $z \in \{0,1\}^{d+1}$ that differs from both x and y in exactly one entry. Then $\{0,1\}^m \times \{x,z\}$ is an isometric copy of $\{0,1\}^{m+1}$, while $\{0,1\}^m \times \{x\}$ and $\{0,1\}^m \times \{z\}$ are isometric copies of $\{0,1\}^m$. Therefore, there exists an isometry $\phi : \{0,1\}^m \times \{x,z\} \to \{0,1\}^{m+1}$ which maps $\{0,1\}^m \times \{x\}$ to $\{0,1\}^m \times \{0\}$ and $\{0,1\}^m \times \{z\}$ to $\{0,1\}^m \times \{1\}$. Hence, it is possible to assign integer weights to the isometric copies of X in $\{0,1\}^m \times \{x,z\}$ so that the multiplicity of every element of $\{0,1\}^m \times \{x\}$ is congruent to $1 \pmod r$, and the multiplicity of any $p \in \{0,1\}^m \times \{z\}$ is congruent to $f(\phi(p)) \pmod r$. We denote the resulting weight function by w'.

The restriction of ϕ to $\{0,1\}^m \times \{z\}$ maps this set isometrically onto $\{0,1\}^m \times \{1\}$. This map extends to an isometry $\{0,1\}^m \times \{y,z\} \to \{0,1\}^{m+1}$. Therefore, we can assign integer weights to the isometric copies of X in $\{0,1\}^m \times \{y,z\}$ in such a way that every element of $\{0,1\}^m \times \{y\}$ has multiplicity congruent to $1 \pmod r$, and any $p \in \{0,1\}^k \times \{z\}$ has multiplicity congruent to $f(\phi(p)) \pmod r$. We denote the resulting weight function by w''.

Although, technically, the weight functions w', w'' are only defined on isometric copies of X in, respectively, $\{0,1\}^m \times \{x,z\}$ and $\{0,1\}^m \times \{y,z\}$, we may suppose that they are defined and equal to 0 on the other isometric copies of X in $\{0,1\}^n$. Then $w' + (r-1)w''$, which we denote by $w_{x,y}$, is a weight function on the family of all isometric copies of X in $\{0,1\}^n$. Moreover, for any $p \in \{0,1\}^n$, the multiplicity of p for $w_{x,y}$ is congruent to

$$\begin{cases} 1 \pmod r & \text{if } p \in \{0,1\}^m \times \{x\}, \\ -1 \pmod r & \text{if } p \in \{0,1\}^m \times \{y\}, \\ 0 \pmod r & \text{otherwise.} \end{cases}$$

The existence of the weight functions $w_{x,y}$ simplifies our problem in the following way. Let us view $\{0,1\}^n$ as the product set $\{0,1\}^m \times \{0,1\}^{d+1}$. Given two elements $x, y \in \{0,1\}^{d+1}$ with $d(x,y) = 2$, we identify the pair (x,y) with both the directed edge $\overrightarrow{xy}$ on $\{0,1\}^{d+1}$ and the weight function $w_{x,y}$. Now, our aim is to find a family (allowing repetitions) of directed edges on $\{0,1\}^{d+1}$, whose every member joins two elements of $\{0,1\}^{d+1}$ that are distance 2 apart, and such that for any $v \in \{0,1\}^{d+1}$ the difference between the in-degree and out-degree of v is congruent to $1 \pmod p$. Indeed, such a family of directed edges corresponds to a weight function for which every element of $\{0,1\}^n$ has multiplicity congruent to $1 \pmod r$.

We will now construct a family of directed edges with the desired properties. Fix vertices $x^* = (0,\dots,0) \in \{0,1\}^{d+1}$ and $y^* = (1,0,\dots,0) \in \{0,1\}^{d+1}$. Note that, for any vertex $v \in \{0,1\}^{d+1}$, there exists a directed path starting from x^* or y^* and ending at v with the property that any two consecutive vertices on this path differ in exactly two entries. Such a path increases the difference between the in-degree and the out-degree of v by 1, decreases this parameter of its starting point (x^* or y^*) by 1 and does not change the value of this parameter for any other vertex. Now, for any vertex $v \in \{0,1\}^{d+1} \setminus \{x^*\}$ with an even number of 1's, select one such path from x^* to v. Similarly, for any $v \in \{0,1\}^{d+1} \setminus \{y^*\}$ with an odd number of 1's, select one such path from y^* to v. Let us combine all of these paths together to obtain a family of directed edges. It is clear that for any $v \in \{0,1\}^{d+1} \setminus \{x^*, y^*\}$ the difference between the in-degree and the out-degree of v is equal to 1. Moreover, excluding x^*, there are $2^d - 1$ vertices in $\{0,1\}^{d+1}$ with an even number of 1's. Therefore, the difference between the in-degree and the out-degree of x^* is $-(2^d-1) \equiv 1 \pmod r$. Similarly, the difference between the in-degree and the out-degree of y^* is also congruent to 1 $\pmod r$. This finishes the proof. $\square$

4 Concluding remarks and open problems

The statement of Theorem 4.3 is very similar to that of Chalcraft's conjecture. Indeed, the only difference is that, instead of an infinite space $\mathbb{Z}^n$, here we are dealing with a finite hypercube $\{0,1\}^n$. However, the results are, in fact, significantly different.

To illustrate this claim, we note that not every sensible finite version of Chalcraft's conjecture is true. First, there is the issue of choosing which metric to use. In $\mathbb{Z}^n$ or in any hypercube $[\ell]^n$ there are at least two natural choices of a metric: the Euclidean metric $d((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt{\sum_{i=1}^n (x_i-y_i)^2}$ and the graph metric $\sum_{i=1}^n |x_i - y_i|$. Chalcraft's conjecture (for $\mathbb{Z}^n$) is true for both metrics. Theorem 4.3 (for $[2]^n$) is independent of the choice of the metric, since if $X, Y \subset \{0,1\}^n$ are isometric copies with respect to one of the metrics then they are also isometric copies with respect to the other. However, the situation is different in $[\ell]^n$ for $\ell \geq 3$:

the obvious version of Chalcraft's conjecture is false for $[\ell]^n$ with the Euclidean metric. For example, take $\ell = 5$ and let $T \subset [5]^2$ be a plus-shaped set of size 5, as shown in Figure 4.1. Then, no matter what n we choose, it is impossible to partition $[5]^n$ into isometric copies of T because the corners of $[5]^n$ cannot be covered. Similar counterexamples exist for all $\ell \geq 3$.

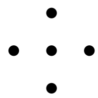

Figure 4.1: The plus-shaped set T.

Second, the situation does not become trivial even if we choose the graph metric. It turns out that, with this metric, the obvious version of Chalcraft's conjecture is true for $[\ell]^n$ where $\ell \geq 2$ is even. This fact can be verified in a similar way to Theorem 4.3; essentially, the only difference is that we have to partition $[\ell]^n$ into copies of $[2]^n$ before we can apply Observation 4.4 (it is also important to note that $[\ell]^n$ can be isometrically embedded into $[2]^m$ for sufficiently large m). However, the corresponding conjecture would be false for $[\ell]^n$ where $\ell \geq 3$ is odd. Indeed, we will demonstrate that even the corresponding version of the weaker Theorem 4.2 is false.

We define P_ℓ^n to be the graph with vertex set $[\ell]^n$ where two vertices $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ are adjacent if $\sum_{i=1}^n |x_i - y_i| = 1$. We say that a vertex is odd if the sum of its entries is odd; otherwise, that vertex is even.

Proposition 4.6. *Let $\ell \geq 3$ be an odd integer. Then there exists a graph H satisfying*

- *H is isomorphic to an induced subgraph of P_ℓ^m for some m*
- *$|H|$ is a power of ℓ*
- *for any n, it is impossible to partition the vertices of P_ℓ^n into induced copies of H.*

Proof. Fix an odd integer $\ell \geq 3$ and write A_n and B_n for the number of even and odd vertices in P_ℓ^n, respectively. For any n the graph P_ℓ^n contains a Hamiltonian path, which visits vertices of alternating parity, so we have $|A_n - B_n| \leq 1$. However, $A_n + B_n = |P_\ell^n| = \ell^n$ is odd, so in fact $|A_n - B_n| = 1$. In particular, $A_n \not\equiv 0 \pmod{\ell}$.

Now, choose m sufficiently large so that P_ℓ^m contains an induced connected subgraph on ℓ even and $\ell^2 - \ell$ odd vertices. Denote this subgraph by H. We claim that, for any n, it is impossible to partition the vertices of P_ℓ^n into induced copies of H. Indeed, each induced copy of H in P_ℓ^n contains ℓ or $\ell^2 - \ell$ even vertices.

Therefore, the total number of even vertices covered by such a partition would be divisible by ℓ. However, as we saw previously, the number of even vertices in P_ℓ^n is not. $\square$

It would be interesting to know if Theorem 4.2 is particular to the hypercubes Q_n or if it holds for powers of other graphs as well. More specifically, let G, H be finite graphs. For any n, we define G^n to be the graph with vertex set $V(G)^n$, where $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are adjacent if and only if there exists an index $i' \in [n]$ such that $u_i = v_i$ for all $i \neq i'$ and $u_{i'}, v_{i'}$ are adjacent vertices of G. We remark that, with this definition, Q_n is the nth power of the path P_2 consisting of a single edge. What are the natural conditions on H that would make it reasonable to believe that, for some n, G^n can be partitioned into H-sets? Obviously, $|H|$ has to divide $|G|^n$, so we should assume that every prime factor of $|G|$ also divides $|H|$. We should also require H to be isomorphic to an induced subgraph of G^k for some k; in fact, we may assume that H is isomorphic to an induced subgraph of G itself. However, this is not enough. First, it may still not be possible to cover G^n with copies of H. Moreover, Proposition 4.6 tells us that even the extra assumption that G can be covered by copies of H would not be enough. After examining why $G = Q_n$ works and $G = P_3^n$ does not, we see that Observation 4.4 breaks down because P_3^n is not vertex-transitive. We conjecture that Theorem 4.2 holds whenever we replace Q_n by another vertex-transitive graph.

Conjecture 4.7. *Let G be a finite vertex-transitive graph and let H be an induced subgraph of G. If every prime factor of $|H|$ divides $|G|$, then there exists a positive integer n such that G^n can be partitioned into induced copies of H.*

What happens if instead of partitioning the vertices of Q_n we attempt to partition the edges? If we want to partition the edge set of Q_n into copies of a fixed graph H, then the obvious necessary divisibility condition is $e(H)|2^{n-1}n$, which is satisfied whenever n is a multiple of $e(H)$. Therefore, as long as H is isomorphic to a subgraph of Q_k for some k, we may expect that such a partition exists for some n. Along with I. Leader and T.S. Tan we make the following conjecture.

Conjecture 4.8. *Let H be a non-empty subgraph of Q_k for some k. Then there exists a positive integer n such that the edges of Q_n can be covered by edge-disjoint copies of H (the copies of H are not required to be induced).*

It seems to be difficult to prove Conjecture 4.8 even in very special cases, when we choose H to be a fairly simple graph. For example, we do not know if the conjecture is true when H is Q_k with one edge removed.

On the other hand, the case when H is a path is well understood. Indeed, the edges of Q_n can be partitioned into antipodal paths of the form $(x_1, x_2, \ldots, x_n) \to (1-x_1, x_2, \ldots, x_n) \to (1-x_1, 1-x_2, \ldots, x_n) \to \cdots \to (1-x_1, 1-x_2, \ldots, 1-x_n)$ with $x_1+\cdots+x_n$ even. Therefore, $E(Q_n)$ can be partitioned into copies of P_{k+1} whenever n is a multiple of k. Moreover, for odd n, Erde [8] and Anick and Ramras [1] independently determined exactly when $E(Q_n)$ can be partitioned into copies of P_{k+1}: this can be done if and only if $k \leq n$ and $k|2^{n-1}n$. For even n not everything is known yet. Erde conjectured that in this case the obviously necessary conditions $k \leq 2^n$ and $k|2^{n-1}n$ are sufficient.

CHAPTER 5

Multicoloured lines in the plane

1 Introduction

In this chapter we consider colourings of finite sets in the plane. For a finite set $S \subset \mathbb{R}^2$, a *line* in S is a maximal set of collinear points of S. Pór and Wood posed the following conjecture about monochromatic lines.

Conjecture 5.1 (Pór and Wood [54]). *For all integers $k \geq 1$ and $l \geq 2$ there exists an integer n such that the following statement holds for all finite sets $S \subset \mathbb{R}^2$ of size at least n. If S does not contain a line on at least $l+1$ points, then every colouring of S with k colours produces a monochromatic line.*

The motivation for this conjecture comes from the Hales-Jewett theorem. By a *combinatorial line* in the grid $[l]^n$ (where $[l]$ stands for the set $\{1, 2, \ldots, l\}$) we mean a set of the form

$$\{(x_1, \ldots, x_n) \in [l]^n : x_i = x_j \text{ for all } i, j \in I \text{ and } x_i = a_i \text{ for all } i \notin I\}$$

for fixed $I \subset [n], I \neq \emptyset$ and fixed a_i for all $i \in [n] \setminus I$. Now the Hales-Jewett theorem can be stated as follows.

Theorem 5.2 (Hales and Jewett). *For all integers $k, l \geq 1$, there exists an integer n such that every colouring of $[l]^n$ with k colours produces a monochromatic combinatorial line.*

Conjecture 5.1 is a natural geometric version of this theorem, where the lines are not necessarily parallel to a fixed set of axes, and the ambient set can be any set without many collinear points.

For $l = 2$ the result is trivial: we may take $n = k + 1$ and by the pigeonhole principle there exists a line containing two points of the same colour. The case $k = 2$

is a special case of the Motzkin-Rabin theorem [49]. In this chapter we demonstrate by a counter-example that the conjecture is false in the next smallest case $k = l = 3$, and hence it is false whenever $k, l \geq 3$.

Theorem 5.3. *For any $n \geq 2$, there exists a set $S \subset \mathbb{R}^2$ of size n satisfying:*

- *no four points of S are collinear, and*
- *S can be coloured with three colours without creating a monochromatic line.*

2 Proof of Theorem 5.3

It is sufficient to find a set with the required properties in the projective plane $\mathbb{RP}^2$, because given a finite set $S \subset \mathbb{RP}^2$ one can choose a projective line $\ell \subset \mathbb{RP}^2$ that does not meet S and apply a projective transformation that sends ℓ to the line at infinity. The image of S under this transformation is contained in the affine plane $\mathbb{R}^2$ while the collinearity relations of the original set S are preserved.

Our counterexample is a finite subset of the irreducible cubic curve $y^2 = x^3 - x^2$. More precisely, we use a suitable subset of Γ where Γ is the set of non-singular points of this curve, that is, $\Gamma = \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 - x^2, x \neq 0\} \cup \{\mathcal{O}\} \subset \mathbb{RP}^2$ where $\mathcal{O}$ is the point at infinity contained in all lines parallel to the y-axis. It follows from Bézout theorem, or from thinking directly about equations of the form $(ax + b)^2 = x^3 - x^2$, that Γ does not contain a set of four collinear points. Moreover, it is a well-known fact in algebraic geometry that Γ forms an abelian group with the property that distinct points $P, Q, R \in \Gamma$ are collinear if and only if $P + Q + R = 0$, and that Γ is isomorphic to the circle group $\mathbb{R}/\mathbb{Z}$ (see [38], p. 19–20).

In fact, there is nothing very special about the curve $y^2 = x^3 - x^2$: any elliptic curve whose group is isomorphic to $\mathbb{R}/\mathbb{Z}$ would do. However, we choose this particular cubic curve (which is, in fact, not an elliptic curve as it contains a singular point $(0, 0)$) because it admits a simple explicit group isomorphism $\phi : \mathbb{R}/\mathbb{Z} \to \Gamma$, given by

$$\phi(x) = \begin{cases} (\cot(\pi x)^2 + 1,\ \cot(\pi x)(\cot(\pi x)^2 + 1)) & \text{if } x \neq 0, \\ \mathcal{O} & \text{if } x = 0. \end{cases}$$

This enables us to give a self-contained proof of the theorem without referring to any results from algebraic geometry. However, the reader familiar with elliptic curves can skip the proof of the following proposition.

Proposition 5.4. *Let x, y and z be distinct elements of $\mathbb{R}/\mathbb{Z}$. Then the points $\phi(x), \phi(y)$ and $\phi(z)$ are collinear if and only if $x+y+z=0$. Moreover, $\phi : \mathbb{R}/\mathbb{Z} \to \Gamma$ is a well-defined bijection.*

Proof. The fact that ϕ is a well-defined bijection follows from the basic properties of the cotangent function. To prove the equivalence of the geometric and algebraic relations, we will use the identity

$$\cot(x+y) = \frac{\cot(x)\cot(y) - 1}{\cot(x) + \cot(y)}, \tag{5.1}$$

which holds whenever $x+y, x, y$ are not multiples of π. For any $r \in \mathbb{R} \setminus \mathbb{Z}$ we define $c_r = \cot(\pi r)$.

Let us first examine what happens if one of $x, y, z \in \mathbb{R}/\mathbb{Z}$ is 0. Say, $x = 0$. In this case $\phi(z)$ is collinear with $\phi(x) = \mathcal{O}$ and $\phi(y)$ if and only if $\phi(z)$ is the reflection of $\phi(y)$ in the x-axis, that is, $z = -y$. Similarly, if two of the numbers (say, x and y) sum to 0, then the three points are collinear if and only if $\phi(z) = \mathcal{O}$, that is, $z = 0$. We may now assume that x, y, z are all non-zero and that no two of them sum to 0. In this case the points $\phi(x), \phi(y)$ and $\phi(z)$ are collinear if and only if

$$\frac{c_z(c_z^2+1) - c_x(c_x^2+1)}{(c_z^2+1) - (c_x^2+1)} = \frac{c_z(c_z^2+1) - c_y(c_y^2+1)}{(c_z^2+1) - (c_y^2+1)},$$

which after rearrangement becomes

$$c_z = -\frac{c_x c_y - 1}{c_x + c_y}.$$

It now suffices to observe that $z = -x-y$ is a solution by (5.1), and that it is unique in $\mathbb{R}/\mathbb{Z}$ since cot is injective on $(0, \pi)$. $\square$

Now we are ready to finish the proof of the theorem.

Proof of Theorem 5.3. As noted before, it is enough to construct a set $S' \subset \mathbb{RP}^2$ with the two required properties. We take $S' = \{\phi(i/n) : i = 0, \ldots, n-1\}$. If there were four collinear points $\phi(x), \phi(y), \phi(z), \phi(w)$ with $x, y, z, w \in \mathbb{R}/\mathbb{Z}$ distinct, then we would know from Proposition 5.4 that $z = w = -x-y$, giving a contradiction. Therefore there are no four collinear points in S'.

We colour the points of S' by the following rule:

$$\phi\left(\frac{i}{n}\right) \text{ is } \begin{cases} \text{red} & \text{if } 0 \le i < \frac{n}{3} \\ \text{green} & \text{if } \frac{n}{3} \le i < \frac{2n}{3} \\ \text{blue} & \text{if } \frac{2n}{3} \le i < n. \end{cases}$$

It remains to check that this colouring does not create a monochromatic line in S'. Suppose that $\ell \subset S'$ is a monochromatic line. Since S' contains at least two points, ℓ must also contain at two distinct points $\phi(i/n), \phi(j/n)$ with $0 \le i, j < n$. There exists an integer k satisfying $0 \le k < n$ and $k \equiv -i - j \pmod{n}$, possibly $k = i$ or $k = j$. Since $i/n + j/n + k/n = 0$ in $\mathbb{R}/\mathbb{Z}$, either $\phi(i/n), \phi(j/n), \phi(k/n)$ are distinct collinear points or $\phi(k/n)$ coincides with one of the other two points. In either case, ℓ passes through all of them and so they have the same colour. Therefore, we can write $i/n = x + \alpha$, $j/n = x + \beta$, $k/n = x + \gamma$ where $x \in \{0, 1/3, 2/3\}$ and $0 \le \alpha, \beta, \gamma < 1/3$. Now, $3x$ and $i/n + j/n + k/n = 3x + \alpha + \beta + \gamma$ are integers, and hence $\alpha + \beta + \gamma$ is also an integer. However, this is only possible if $\alpha = \beta = \gamma = 0$, which in particular implies that $i/n = j/n$, contradicting the assumption that $\phi(i/n) \neq \phi(j/n)$.

This finishes the proof. $\square$

3 Concluding remarks and open problems

It seems plausible that our counter-example to Conjecture 5.1 is essentially unique, by which we mean that, possibly, every counter-example is contained in a cubic curve except for at most a bounded number of points. Also, cubic curves do not contain lines on more than four points. Therefore, the following question seems interesting.

Question 5.5. *Let $k \ge 3$ and $l \ge 4$ be integers. Must there exist a number m such that for any finite set $S \subset \mathbb{R}^2$, if S contains at least m lines on exactly l points but no lines on $l+1$ or more points, then every k-colouring of S produces a monochromatic line?*

Note that, if $k \ge 4$, then it is not enough to ask for one long line (even if S has to be arbitrarily large): indeed, a counter-example to such a question could be obtained by taking our original cubic curve construction and extending one of the lines to length l by adding points of a fourth colour. However, for three colours there exists a cleaner version of the question.

Question 5.6. *Does there exist a number n such that for any finite set $S \subset \mathbb{R}^2$ of size at least n, if S contains a line on exactly four points and no lines on five or more points, then every colouring of S with three colours produces a monochromatic line?*

CHAPTER 6

Minimising the number of triangular edges

1 Introduction

Mantel [45] proved that a triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges. In other words, a graph on n vertices with at least $\lfloor n^2/4 \rfloor + 1$ edges contains a triangle. A natural question arises from this classical result: how many triangles must such a graph have? And, indeed, Rademacher (see [10]) extended Mantel's result by showing that any graph on n vertices with $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles, a bound that can readily be seen to be best possible (see Figure 6.1).

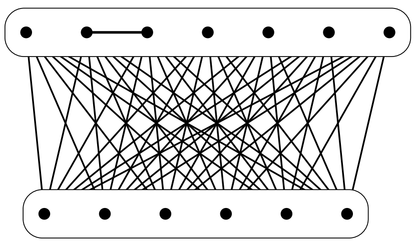

Figure 6.1: An n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges and $\lfloor n/2 \rfloor$ triangles.

Erdős [9] conjectured that a further generalisation holds: any graph on n vertices with at least $\lfloor n^2/4 \rfloor + l$ edges contains at least $l \lfloor n/2 \rfloor$ triangles, for every $1 \leq l < \lfloor n/2 \rfloor$. Erdős [9, 10] proved his conjecture for $l \leq cn$ for some constant $c > 0$. It is not hard to see that the bound on the number of triangles is best possible. Indeed, this bound can be achieved by adding l edges that do not span a triangle to the larger part of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ (see Figure 6.2). The bound on l can also be easily seen to be best possible.

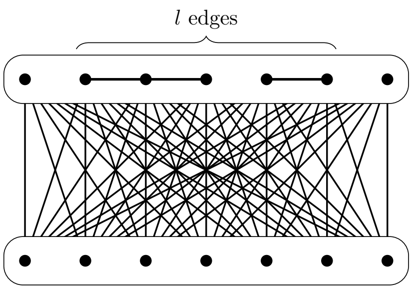

Figure 6.2: An n-vertex graph with $\lfloor n^2/4 \rfloor + l$ edges and $l\lfloor n/2 \rfloor$ triangles.

Erdős's conjecture was resolved by Lovász and Simonovits [43], who also characterised [44] the n-vertex graphs with $\lfloor n^2/4 \rfloor + l$ edges that minimise the number of triangles, for every $l \leq cn^2$ and some fixed $c > 0$. Razborov [57] asymptotically determined the minimal possible number of triangles in an n-vertex graph with $\lfloor n^2/4 \rfloor + l$ edges where $l = \Omega(n^2)$.

In this chapter we consider a similar problem, concerning the number of edges that are contained in a triangle (we shall call such edges *triangular edges*), rather than the number of triangles. The first result in this direction was obtained by Erdős, Faudree and Rousseau [13] who proved that any n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges has at least $2\lfloor n/2 \rfloor + 1$ triangular edges. This bound is best possible (see Figure 6.1).

It is very natural, similarly to the question about the number of triangles, to ask how many triangular edges an n-vertex graph with e edges must have, where e is an integer satisfying $\lfloor n^2/4 \rfloor < e \leq \binom{n}{2}$. After some thought, a natural example comes to mind. Given integers a, b, c, we denote by $G(a, b, c)$ the graph on $n = a + b + c$ vertices, which consists of a clique A of size a and two independent sets B and C of sizes b and c respectively, such that all edges between B and $A \cup C$ are present, and there are no edges between A and C (see Figure 6.3).

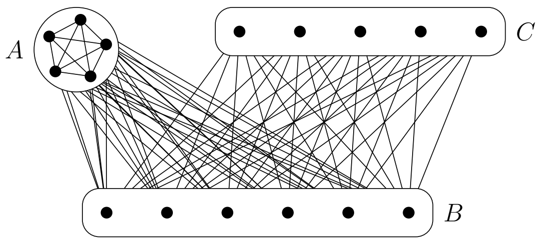

Figure 6.3: The graph $G(a, b, c)$ (here $a = 5, b = 6, c = 5$).

Note that the graph $G(a, b, c)$ has $\binom{a}{2} + b(a + c)$ edges and, as long as $a \geq 2$ and $b \geq 1$, precisely bc of them are non-triangular. We remark that the extremal example (depicted in Figure 6.1) for the aforementioned result by Erdős, Faudree and Rousseau [13] is isomorphic to $G(2, \lfloor n/2 \rfloor, \lceil n/2 \rceil - 2)$.

Füredi and Maleki [16] conjectured that the minimisers of the number of triangular edges are graphs of the form $G(a, b, c)$, or subgraphs of such graphs.

Conjecture 6.1 (Füredi and Maleki [16]). *Let n and $e > \lfloor n^2/4 \rfloor$ be integers and let G be an n-vertex graph with e edges that minimises the number of triangular edges. Then G is isomorphic to a subgraph of a graph $G(a, b, c)$ for some a, b, c.*

The condition that G is isomorphic to a subgraph of a graph $G(a, b, c)$ (rather than to $G(a, b, c)$ itself) is due to the fact that we specify the exact number of edges, so the minimiser may be isomorphic to $G(a, b, c)$ with a few edges removed from $A \cup B$.

The conjecture implies, in particular, that every n-vertex graph with e edges has at least $g(n, e)$ triangular edges, where $g(n, e)$ is defined by

$$g(n, e) = \min \left\{ e - bc : a + b + c = n, \binom{a}{2} + b(a + c) \geq e \right\}.$$

Füredi and Maleki [16] proved an approximate version of the latter statement, which reads as follows.

Theorem 6.2 (Füredi and Maleki [16]). *Every n-vertex graph with e edges has at least $g(n, e) - 3n/2$ triangular edges.*

It is worth noting that, if $e \geq (1/4 + \Omega(1))n^2$ and $e \leq (1/2 - \Omega(1))n^2$, then $g(n, e) = \Omega(n^2)$ and $\binom{n}{2} - g(n, e) = \Omega(n^2)$. Therefore, Theorem 6.2 is asymptotically sharp in the range where the edge density is bounded away from 1/2 and 1 by small positive constants.

Our main result is an exact version of Theorem 6.2: we shall prove that an n-vertex graph with e edges has at least $g(n, e)$ triangular edges, provided that n is large enough. However, the bound on n does not depend on e, that is, as long as $n \geq n_0$ for some n_0, our theorem holds for any e such that $\lfloor n^2/4 \rfloor \leq e \leq \binom{n}{2}$.

Before we precisely state our result, we make a few remarks. Firstly, it turns out to be more convenient to consider the clearly equivalent problem of maximising the number of non-triangular edges among n-vertex graphs with e edges. Thus, given a graph G, we denote by $t(G)$ the number of non-triangular edges in G. Secondly, given n and e, instead of restricting our attention to n-vertex graphs with exactly e edges, we consider n-vertex graphs with at least e edges. Since the removal of a triangular edge cannot decrease the number of non-triangular edges, this slight reformulation does not change the problem, and yet it allows us to concentrate on graphs $G(a, b, c)$ without having to consider their subgraphs.

We are now ready to state our main result.

Theorem 6.3. *There exists n_0 such that, for any graph G on at least n_0 vertices, there exists a graph $H = G(a, b, c)$ (for some integers a, b, c) such that $|H| = |G|$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$.*

We note that Theorem 6.3 comes close to proving Conjecture 6.1 (for sufficiently large n) as it shows that the minimum number of triangular edges is attained by a graph $G(a, b, c)$ or a subgraph of $G(a, b, c)$. However, we do not prove that such graphs are the only minimisers.

1.1 Structure of the chapter

The proof of our main result, Theorem 6.3, is divided into three parts, according to the number of edges in the graph G. We treat separately graphs that are close to being bipartite, that is, whose number of edges is close to $n^2/4$; graphs that are close to being complete, that is, whose number of edges is close to $\binom{n}{2}$; and the middle range, where the number of edges is bounded away from both $n^2/4$ and $\binom{n}{2}$ by a constant factor of n^2.

We state Theorems 6.4 to 6.6, which are the theorems corresponding to the aforementioned three ranges, in Section 2, and give an overview of their proofs. In Section 3 we introduce some notation and describe the tools that we shall use to prove these theorems. We prove the theorems in Sections 4 to 6. Theorem 6.4, which deals with graphs with about $n^2/4$ edges, is proved in Section 4; the proof of Theorem 6.5, for the middle range, which is the most difficult of the three and is the heart of this chapter, is given in Section 5; and Theorem 6.6 is proved in Section 6. We conclude the chapter with Section 7 where we make a few remarks and mention some open problems.

2 Overview

We split the proof of Theorem 6.3 into three parts, according to the number of edges in the graph. We state the theorems corresponding to these three parts here.

The following theorem deals with e that is close to $n^2/4$, that is, $e \leq (1/4 + \delta)n^2$, where δ is a sufficiently small constant.

Theorem 6.4. *There exist n_0 and $\delta > 0$ such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $n^2/4 \leq e \leq (1/4 + \delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

The next theorem considers the case where e is bounded away from $n^2/4$ and $\binom{n}{2}$, namely, where $(1/4 + \delta)n^2 \leq e \leq (1/2 - \delta)n^2$ for a constant $\delta > 0$.

Theorem 6.5. *For every $\delta > 0$ there exists n_0 such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $(1/4+\delta)n^2 \leq e \leq (1/2-\delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

Finally, we consider the remaining case, where e is close to $\binom{n}{2}$, that is, $e \geq (1/2 - \delta)n^2$ for a sufficiently small constant $\delta > 0$.

Theorem 6.6. *There exist n_0 and $\delta > 0$ such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $e \geq (1/2 - \delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

It is clear that Theorems 6.4 to 6.6 imply Theorem 6.3: we first take a small $\delta > 0$ that works for Theorems 6.4 and 6.6 and then we choose a sufficiently large n_0 that works for all three theorems.

We now give some insight into our proofs. The rough plan for the proof of each of the theorems is the same. Assuming that G is an n-vertex graph with at least e edges that maximises the number of non-triangular edges, we first obtain rough information about the structure of the graph. In each of the cases, we partition the vertices of G into parts A, B, C, which relate to the three parts in a graph $G(a, b, c)$, in a way that will be explained in the proofs. In the next stage we use lower bounds on the number of non-triangular edges (coming from examples $G(a, b, c)$) to estimate the sizes of the sets A, B, C. The final stage uses the estimates on the sizes and some case specific arguments to conclude that G has the required structure, namely, that it is isomorphic to the graph $G(|A|, |B|, |C|)$.

The proofs of the two extremal cases, where e is close to either $n^2/4$ or $\binom{n}{2}$, are considerably easier than that of the middle range. The main reason for this is that in the extremal cases it is fairly easy to show that the graph G should be close to a graph $G(a, b, c)$, whereas in the middle range getting any handle on the structure of the graph is hard, and the initial structural properties that we find are less restrictive than in the two extremal cases.

We introduce two tools, which will be helpful in the proof of the middle range. The first one is a process of 'compression' that allows us to 'simplify' a graph without decreasing the number of edges or non-triangular edges. The second is the 'exchange lemma', which allows us to 'exchange' edges to non-triangular edges and vice versa. In other words, it allows us to replace a graph by another graph with (somewhat) fewer edges, but more non-triangular edges and vice versa. Both of these tools will be presented and explained in greater detail in Section 3.

3 Tools

In this section we introduce the tools that we will use throughout the chapter. We start by describing some notation and simple definitions in Subsection 3.1. We introduce the notion of weighted graphs in Subsection 3.2 and list some results by Füredi and Maleki [16] that involve weighted graphs. An important tool in the proof of the middle range is the so-called Exchange Lemma, Lemma 6.13. We prove Lemma 6.13 and explain its importance in Subsection 3.3. Our last tool is the notion of *compressed graphs* which is a class of graphs with somewhat restrictive structure. In Subsection 3.4, we give our definition of a compressed graph and prove Lemma 6.16, which shows that, in order to prove Theorem 6.3, it suffices to prove it for compressed graphs.

3.1 Notation

The following notation is standard. Write $|G|$ for the order of a graph G and $e(G)$ for the number of edges in G. We denote the degree of a vertex u of G by $\deg_G(u)$, or $\deg(u)$ if G is clear from the context. Given a set U of vertices of G, we denote by $G[U]$ the graph induced by G on U.

We now turn to notation that is more specific to our context. An edge $e \in E(G)$ is called *triangular* if it is an edge of at least one triangle in G. Similarly, we say that e is *non-triangular* if it is not an edge of any triangle. We denote by $t(G)$ the number of non-triangular edges of G.

Given a vertex u, a vertex v is a *triangular neighbour* of u, if uv is a triangular edge. Similarly, the *triangular neighbourhood* of u is the set of triangular neighbours of u, and the *triangular degree* of u is the number of triangular edges adjacent to u. The notions of a *non-triangular neighbour*, *non-triangular neighbourhood* and *non-triangular degree* are defined similarly. We denote the non-triangular degree of u in G by $\deg_{\text{Non-}\Delta}(u)$. A vertex u is called *triangular* if $\deg_{\text{Non-}\Delta}(u) = 0$, that is, if all edges adjacent to u are triangular.

We say that a set of vertices $U \subseteq V(G)$ is a *set of clones* if any two vertices in U

have the same neighbourhood in G. In particular, a set of clones is an independent set. For example, in $G(a, b, c)$ the sets B and C are sets of clones. We remark that the notion of clones will play an important role in the definition of a compressed graph (which is given in Subsection 3.4).

We now introduce the natural notion of an optimal graph.

Definition 6.7. A graph G on n vertices is called *optimal* if there does not exist a graph H on n vertices such that either $t(H) > t(G)$ and $e(H) \geq e(G)$ or $e(H) > e(G)$ and $t(H) \geq t(G)$.

In other words, G is optimal if it maximises $t(G)$ among graphs with n vertices and at least $e(G)$ edges and, in addition, it maximises $e(G)$ among graphs with n vertices and at least $t(G)$ non-triangular edges.

It clearly suffices to prove the main result, Theorem 6.3, for optimal graphs. The following observation is a simple property of optimal graphs.

Observation 6.8. *Let G be an optimal graph and let u, v be vertices of G. Then at least one of* $\deg(u) \geq \deg(v) - 1$ *and* $\deg_{\textit{Non-}\Delta}(u) \geq \deg_{\textit{Non-}\Delta}(v) - 1$ *holds.*

Proof. Suppose that $\deg(u) \leq \deg(v) - 2$ and $\deg_{\text{Non-}\Delta}(u) \leq \deg_{\text{Non-}\Delta}(v) - 2$. Consider the graph G' obtained by removing the edges incident with u and adding the edges between u and the neighbours of v (do not add the loop uu if u, v are adjacent in G). Then $e(G') \geq e(G) - \deg(u) + \deg(v) - 1 > e(G)$ and, similarly, $t(G') > t(G)$, contradicting the assumption that G is optimal. $\square$

We shall use big-O notation extensively throughout this chapter, so, for the sake of clarity, we briefly explain how we interpret the symbols O, o and Ω. First of all, we always assume that n is large, so, whenever we write down a statement or an inequality, we only suppose it to hold for sufficiently large n. We write $f(n) = O(g(n))$ if there exists an absolute constant $C > 0$ such that $|f(n)| \leq Cg(n)$. In particular, the expression $f(n) = g(n) + O(h(n))$ consists of the following inequalities: $g(n) - Ch(n) \leq f(n) \leq g(n) + Ch(n)$. Similarly, $f(n) = o(g(n))$ means that $\lim_{n\to\infty} |f(n)|/g(n) = 0$. Finally, we write $f(n) = \Omega(g(n))$ if $f(n) \geq Cg(n)$ for an absolute constant $C > 0$. To ensure that this notation makes sense, we will only write $O(g(n)), o(g(n)), \Omega(g(n))$ for functions $g(n)$ which are positive for sufficiently large n. We remark that $\Omega(g(n))$ always denotes a positive quantity, while $O(g(n))$ and $o(g(n))$ may denote positive and negative quantities.

Throughout this chapter, we omit integer parts whenever they do not affect the argument.

3.2 Weighted graphs

Our most basic tool is the concept of a *weighted graph*, which is a graph whose vertices have been assigned non-negative real weights. The *total weight* of a weighted graph G is the sum of the weights of its vertices and is denoted by $|G|$. For technical reasons, throughout this chapter we require that the number of vertices of a weighted graph does not exceed its total weight. Equivalently, we require that the average weight of a vertex in a weighted graph is at least 1.

For containment purposes, we identify weighted graphs with their underlying graphs. For instance, given weighted graphs G and H, we say that H is a *weighted subgraph* of G if, as graphs, H is an induced subgraph of G. Note that this definition does not impose any conditions on the weight function of H. In particular, if H is a weighted subgraph of G then the weight in H of a vertex in H may be larger, or smaller, than its weight in G. Similarly, an edge of a weighted graph is *triangular* (*non-triangular*) if it is a triangular (non-triangular) edge of the underlying graph. We remark that, unless explicitly stated otherwise, vertices of zero weight are taken into account when switching to the underlying graph.

Given a weighted graph G with weight function $w : V(G) \to \mathbb{R}^{\geq 0}$ we define $e(G)$ to be the sum of $w(u)w(v)$ over all edges uv of G. Similarly, we define $t(G)$ to be the same sum over the non-triangular edges of G. Note that any graph G can be seen either as a graph or as a weighted graph whose every vertex has weight 1, and the definitions of $|G|, e(G)$ and $t(G)$ are independent of the point of view.

The notions of *degree* and *non-triangular* degree of a vertex may be similarly generalised to weighted graphs. For instance, the degree of a vertex u of a weighted graph is the sum of weights of the neighbours of u. Note that the degree and non-triangular degree of a vertex do not depend on the weight of that vertex itself. We use the notation $\deg(u)$ and $\deg_{\text{Non-}\Delta}(u)$ for the degree and the non-triangular degree of a vertex u in a weighted graph.

We now define *good weighted graphs* (see Figure 6.4), which are weighted equivalents of the graphs $G(a, b, c)$ (see Figure 6.3).

Definition 6.9. We call a weighted graph G *good* if its vertex set can be partitioned into a set K, which induces a clique, and a pair (u, v) of adjacent vertices such that uv is the only non-triangular edge in G.

Moreover, if there are no edges between v and K, and if u is adjacent to all vertices in K, then we say that G is a *very good weighted graph.* We remark that, according to this definition, if K consists of a single vertex, then G cannot be very good. Also, in this case G is good if and only if uv is the only edge in G, which

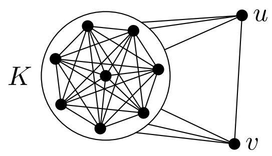

Figure 6.4: A good weighted graph.

gives $e(G) \leq |G|^2/4$.

Let G be a good graph with $e(G) > |G|^2/4$ and let $\{K, \{u, v\}\}$ be the partition of $V(G)$ as described above. We know that in this case K contains at least two vertices. Moreover, we may assume that the weight of v does not exceed the weight of u. Since uv is a non-triangular edge, u and v do not have common neighbours in K. Therefore, by removing the edges between v and K and adding all possible edges between u and K, we obtain a very good graph G' such that $|G'| = |G|$, $e(G') \geq e(G)$ and $t(G') = t(G) = w(u)w(v)$.

We observe that, provided that $a \geq 2$, a graph $G(a, b, c)$ can be represented by a very good weighted graph, by replacing the independent parts of sizes b and c by vertices of weight b and c respectively. This is an example of the correspondence between an independent set of clones I and a vertex of weight $|I|$ with the same neighbourhood, which we shall use on multiple occasions. We remark that, in general, good and very good weighted graphs may have non-integer weights.

Motzkin and Straus [48] used weighted graphs to give an alternative proof of Turán's theorem [62]. They pointed out that Turán's theorem for weighted graphs is very easy: given a weighted graph G, there exists a weighted graph H that satisfies $|H| = |G|$ and $e(H) \geq e(G)$, and, as a graph, is a complete subgraph of G. Therefore, among K_{r+1}-free weighted graphs with total weight $\alpha \geq r$, $e(G)$ is maximised when G is a complete graph with r vertices whose every vertex has weight α/r. If α/r is an integer, then this corresponds to a complete r-partite graph, implying Turán's theorem. However, if α/r is not an integer, then this argument gives only an approximate form of Turán's theorem, and Motzkin and Straus needed an additional argument to recover the full theorem.

Füredi and Maleki [16] modified the aforementioned observation of Motzkin and Straus to also give $t(H) \geq t(G)$ at the cost of making the structure of H more complicated.

Lemma 6.10 (Füredi and Maleki [16]). *Let G be a weighted graph with $t(G) > 0$. Then G contains a weighted subgraph H which is a good weighted graph and satisfies $|H| = |G|$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$.*

We will use both this result and the key observation that leads to its proof. We state and prove this observation next, but we do not present the careful analysis that Füredi and Maleki perform to complete the proof of Lemma 6.10.

Lemma 6.11 (Füredi and Maleki [16])**.** *Let G be a weighted graph and suppose that I is an independent set of three vertices. Then there exists a weighted graph H, which can be obtained from G by removing one of the vertices in I and, possibly, changing the weights of the other two vertices in I, such that $|H| = |G|$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$.*

Proof. Denote $I = \{u_1, u_2, u_3\}$, $d_i = \deg(u_i)$ and $t_i = \deg_{\text{Non-}\Delta}(u_i)$. It is not hard to see that there exist reals s_1, s_2, s_3, not all 0, such that $s_1 d_1 + s_2 d_2 + s_3 d_3 \geq 0$, $s_1 t_1 + s_2 t_2 + s_3 t_3 \geq 0$ and $s_1 + s_2 + s_3 = 0$. For real λ we denote by G_λ the weighted graph obtained by adding λs_i to the weight $w(u_i)$ of u_i for each $i \in [3]$; this definition is valid for the values of λ for which $w(u_i) + \lambda s_i \geq 0$ for all $i \in [3]$. Pick $\lambda > 0$ such that $w(u_i) + \lambda s_i \geq 0$ for $i \in [3]$ with equality for at least one value, say 1. Then $|G_\lambda| = |G|$, $e(G_\lambda) \geq e(G)$ and $t(G_\lambda) \geq e(G)$, so the weighted graph $H = G_\lambda \setminus \{u_1\}$ satisfies the requirements of the lemma. □

Füredi and Maleki deduce their main result from Lemma 6.10. We present their theorem with minor modifications, which make it more suitable for our application.

Corollary 6.12 (Füredi and Maleki [16])**.** *Let G be a weighted graph G with $|G| = n$. Then there exists a graph $H = G(a, b, c)$ satisfying $|H| = n$, $e(H) \geq e(G)$ and $t(H) \geq t(G) - 5n$.*

Proof. We begin by recalling that, according to our definition of a weighted graph, G has at most n vertices. The parameter $e(G)$ is maximised when the underlying graph of G is complete, in which case $2e(G) = \sum_{u \in V(G)} w(u)(n - w(u)) = n^2 - \sum_{u \in V(G)} w^2(u)$. By the arithmetic-quadratic mean inequality, the right hand side is maximised when $w(u) = 1$ for all u. Therefore, $e(G) \leq \binom{n}{2}$. As a result, we may assume that $t(G) > 5n$ because otherwise the complete graph K_n satisfies the requirements. We may also assume that $e(G) > n^2/4$ because otherwise $G(2, \lfloor n/2 \rfloor, \lceil n/2 \rceil - 2)$ works.

Let H be a good weighted graph that satisfies $|H| = n$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$, whose existence is ensured by Lemma 6.10. There exists a partition $\{K, \{u, v\}\}$ of $V(H)$ such that K induces a clique and uv is the only non-triangular edge in G. Denote the sum of weights (in H) of the vertices in K by α and the weights of u and v by β and γ; we may assume that $\beta \geq \gamma$. Trivially, we have

$t(G) = \beta\gamma$. Moreover, since no vertex in K is adjacent to both u and v, we also have $e(G) \leq \alpha^2/2 + \alpha\beta + \beta\gamma$.

We now show that for some integers $a, b, c \geq 0$ the graph $G' = G(a, b, c)$ has the desired properties. It is enough to choose a, b, c so that

$$a + b + c = n, \tag{6.1}$$

$$\binom{a}{2} + (n - b)b \geq \frac{\alpha^2}{2} + (n - \beta)\beta, \tag{6.2}$$

$$bc \geq \beta\gamma - 5n. \tag{6.3}$$

Of course, the plan is to set $a \approx \alpha, b \approx \beta, c \approx \gamma$, but there are some tedious details to check. We set $a = \lceil \alpha \rceil + 2$ and, depending on whether $\beta \geq n/2$ or $\beta < n/2$, either $b = \lfloor \beta \rfloor$ or $b = \lceil \beta \rceil$. Finally, we set $c = n - a - b$. Note that from the assumption that $t(G) \geq 5n$ it follows that $\beta, \gamma > 5$. In particular, since $c \geq \gamma - 4$, c is positive. Now, (6.1) is immediate from the definition; (6.3) is immediate from the fact that $b \geq \beta - 1 > 0$ and $c \geq \gamma - 4 > 0$; and the only case when (6.2) is not immediate is when $(n-1)/2 \leq \beta \leq (n+1)/2$. However, in this case the difference between $(n - \beta)\beta$ and $(n - b)b$ is at most 1, and it is compensated by the difference between $\binom{a}{2}$ and $\alpha^2/2$. □

3.3 Exchange lemma

The following lemma, Lemma 6.13, will be very useful in the proof of our main result in the middle range. Roughly speaking, it says that there exists a positive number ζ, which we informally call the 'exchange rate', with the following property. For any graph G, not too dense and not too sparse, and any number x, not too big and not too small, we can exchange x edges of G for at least ζx non-triangular edges. That is, there exists a graph H such that $|H| = |G|$, $e(H) \geq e(G) - x$ and $t(H) \geq t(G) + \zeta x$. Similarly, we can exchange x non-triangular edges for at least ζx edges.

This tool is very useful to us, because now we can arrive at a contradiction by finding a graph G whose either parameter $e(G)$ or $t(G)$ is too large, even if the other parameter is slightly smaller than what would normally be needed for a contradiction.

For any positive integer n and real $e \leq \binom{n}{2}$, we denote by $t(n, e)$ the maximum number of non-triangular edges among n-vertex graphs with at least e edges. Note that if $e \leq \lfloor n^2/4 \rfloor$, then $t(n, e) = \lfloor n^2/4 \rfloor$. Moreover, for any n, the function $t(n, e)$

is a non-increasing function of e.

Lemma 6.13. *For any $\delta > 0$ there exist $\zeta, \varepsilon, C > 0$ and n_0 such that the following holds for any weighted graph G on $n \geq n_0$ vertices and for any real x satisfying $Cn \leq x \leq \varepsilon n^2$.*

1. *If $e(G) \geq e + x$ for some real e satisfying $n^2/4 \leq e \leq (1/2 - \delta)n^2$, then $t(G) \leq t(n, e) - \zeta x$.*

2. *If $t(G) \geq t(n, e) + x$ for some real $e \geq (1/4 + \delta)n^2$, then $e(G) \leq e - \zeta x$.*

Here is a brief overview of the proof of Lemma 6.13. To prove the first statement, we note that by Lemma 6.10, we may assume that G is good. We shift the weights of the vertices in G so as to increase $t(G)$ while decreasing $e(G)$ only slightly. An upper bound on $t(G)$ then follows from Corollary 6.12. The second statement is proved in a similar way.

Proof of Lemma 6.13. Let $\delta \in (0, 1/10)$. To prove the first statement, suppose that n, e, x satisfy $e \leq (1/2 - \delta)n^2$ and $Cn \leq x \leq \varepsilon n^2$ for constants C and ε that will be determined later. Let G be a weighted graph such that $|G| = n$ and $e(G) \geq e + x$. We note that $t(n, e) \geq \frac{\delta^{3/2}}{2} n^2$. Indeed, the graph $G(a, b, c)$ where $c = \frac{\delta}{2} n$, $b = \sqrt{\delta}\, n$ and $a = n - b - c$ has at least $(1/2 - \delta)n^2$ edges and $\frac{\delta^{3/2}}{2} n^2$ non-triangular edges. By taking ε, ζ to satisfy $\varepsilon\zeta \leq \frac{\delta^{3/2}}{4}$, we may assume that

$$t(G) \geq \frac{\delta^{3/2}}{4} n^2, \tag{6.4}$$

because otherwise we get $t(G) \leq t(n, e) - \zeta x$ for free. By Lemma 6.10, we may assume that G is a good weighted graph, so $V(G)$ can be partitioned into a clique K and two adjacent vertices u and v such that uv is the only non-triangular edge. Denote by α the sum of weights of vertices in K and let β and γ be the weights of u and v respectively. By Inequality (6.4), we have $\beta, \gamma \geq \frac{\delta^{3/2}}{4} n$. Moreover, the removal of the edges spanned by K would make G bipartite, so we have $e(G) \leq n^2/4 + \alpha^2/2 \leq n^2/4 + \alpha n/2$. Recall that $e(G) \geq e + x \geq n^2/4 + x$, and hence $\alpha \geq 2x/n$.

Let G' be a weighted graph obtained by increasing the weight of u by x/n and decreasing the weights of the vertices in K so that the new sum of their weights is $\alpha - x/n$. Trivially, $e(G') \geq e(G) - x \geq e$ and $t(G') = (\beta + x/n)\gamma \geq t(G) + \frac{\delta^{3/2}}{4} x$. Furthermore, it follows from Corollary 6.12 that $t(G') \leq t(n, e) + 5n$, and hence $t(G) \leq t(n, e) + 5n - \frac{\delta^{3/2}}{4} x \leq t(n, e) - \left(\frac{\delta^{3/2}}{4} - \frac{5}{C}\right) x$. By taking C large and ζ small with respect to δ, we can ensure that $t(G) \leq t(n, e) - \zeta x$.

To prove the second statement, suppose that n, e, x satisfy $e \geq (1/4 + \delta)n^2$ and $10n < x \leq \varepsilon n^2$ for a sufficiently small constant $\varepsilon > 0$. Let G be a weighted graph such that $|G| = n$ and $t(G) \geq t(n, e)+x$. Note that by taking ε, ζ to satisfy $\varepsilon\zeta \leq \delta/2$, we may assume that

$$e(G) \geq \left(\frac{1}{4} + \frac{\delta}{2}\right) n^2, \tag{6.5}$$

because otherwise we can conclude immediately that $e(G) \leq e - \zeta x$. Furthermore, by Lemma 6.10, we may assume that G is a good weighted graph. In fact, since $e(G) > n^2/4$, we may assume that G is very good. Let $\{K, \{u, v\}\}$ be a partition of $V(G)$ into a clique K and two vertices u, v such that uv is the only non-triangular edge in G, u is adjacent to all vertices of K and there are no edges between v and K. Moreover, let α be the total weight of vertices in K and let β and γ be the weights of u and v. As before, it follows from Inequality (6.5) that $\alpha \geq \sqrt{\delta}n$. Moreover, K contains at least two vertices, so in particular a vertex $w \in K$ whose weight does not exceed $\alpha/2$. Let G' be the weighted graph obtained by reducing the weight of v by $x/2n$ (note that $\beta\gamma = t(G) \geq x$, so $\gamma \geq x/n$) and increasing the weight of w by the same amount. Then, since $x > 10n$,

$$t(G') = \beta(\gamma - x/2n) \geq t(G) - x/2 \geq t(n, e) + x/2 > t(n, e) + 5n. \tag{6.6}$$

Furthermore, since $\alpha \geq \sqrt{\delta}n$,

$$e(G') \geq e(G) + \frac{x}{2n} \cdot \frac{\alpha}{2} \geq e(G) + \frac{\sqrt{\delta}}{4}x.$$

By Corollary 6.12 and Inequality (6.6), $e(G') < e$, because otherwise there exists a graph H with n vertices, at least e edges and more than $t(n, e)$ non-triangular edges, which contradicts the definition of $t(n, e)$. Thus, $e(G) \leq e - \zeta x$ for any $\zeta \leq \frac{\sqrt{\delta}}{4}$. $\square$

3.4 Compressed graphs

We now present the notion of *compressed* graphs. Many proofs of Turán's theorem, including the one given by Motzkin and Straus [48], first show that, among K_{r+1}-free graphs on a given number of vertices, the greatest number of edges is achieved by a complete r-partite graph. As a result, it is enough to solve the problem for complete r-partite graphs. In this chapter the class of compressed graphs will play a role similar to that of complete r-partite graphs in the proof of Motzkin and Straus. Compressed graphs have fairly simple structure (though not quite as simple as complete r-partite graphs) and we shall see from Lemma 6.16 that it suffices to prove Theorem 6.3 for compressed graphs.

In the following definition, as well as the rest of the chapter, the logarithm is taken in base 2.

Definition 6.14. A graph G on n vertices is called *compressed* if the following assertions hold.

1. Any independent set $I \subset V(G)$ can be partitioned into at most $3 \log n$ sets of clones. Moreover, this can be done in such a way that at most four of the sets of clones into which I is partitioned have size larger than $3n^{1/3}$.

2. The set of triangular vertices in G, which we denote by U, induces a clique in G. Furthermore, the vertices of U all have the same neighbourhood outside of U.

To demonstrate how compressed graphs may be of use to us, we mention the following observation.

Observation 6.15. *Let G be a compressed graph on n vertices and let I be an independent set of size at least $45n^{1/3} \log n$. Then I contains a set of clones of size at least $|I|/5$.*

Indeed, let m be the size of the largest set of clones in I. Then Condition 1 of Definition 6.14 implies that $|I| \leq 4m + 9n^{1/3} \log n \leq 4m + |I|/5$, so $m \geq |I|/5$.

The following lemma shows that, for the purpose of proving Theorem 6.3, we may assume without loss of generality that the given graph is compressed.

Lemma 6.16. *Let G be a graph on n vertices. Then there exists a compressed graph H such that $|H| = n$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$.*

Proof. Given a graph G on n vertices, we let H be a weighted graph with the following properties.

- $|H| = n$, $e(H) \geq e(G)$ and $t(H) \geq t(G)$.

- All vertices of H have integer weights.

- The number of vertices of H is minimal under the first two conditions.

- The number of vertices of weight at least $3n^{1/3}$ is minimal under the first three conditions.

We shall show that the graph, obtained by replacing each vertex of H by a set of clones of size equal to the weight of the vertex, is compressed. To that end, we show that H has no independent set of size larger than $3 \log n$, and that the vertices with weight larger than $3n^{1/3}$ do not contain an independent set of size at least five.

We first show that every independent set of H contains at most $3 \log n$ vertices. Suppose to the contrary that H contains an independent set I of size $m \geq 3 \log n$. For any set $A \subseteq I$ we denote $S_A = \sum_{x \in A} \deg(x)$ and $T_A = \sum_{x \in A} \deg_{\text{Non-}\Delta}(x)$. Trivially, $S_A \leq n^2$ for every $A \subseteq I$. Since $\binom{m}{m/2} \geq 2^m/\sqrt{2m} \geq n^3/\sqrt{2n} > n^2$, it follows from considering sets of size $m/2$ that there exist distinct sets $A, B \subseteq I$ such that $|A| = |B|$ and $S_A = S_B$. By replacing A and B by $A \setminus B$ and $B \setminus A$, we may assume that $A \cap B = \emptyset$. Also, without loss of generality, $T_A \geq T_B$.

Let w be the minimum weight of a vertex in B. Consider the weighted graph H', obtained by increasing the weight of each vertex in A by w and decreasing the weight of each vertex in B by w (and removing vertices whose weight becomes 0). Then $|H'| = |H|$, $e(H') = e(H) + w(S_A - S_B) = e(H)$, $t(H') = t(H) + w(T_A - T_B) \geq t(H)$ and the number of vertices in H' is smaller than the number of vertices in H, contradicting the choice of H. It follows that every independent set of H contains at most $3 \log n$ vertices.

We now show that, given an independent set of five vertices $\{u_1, \ldots, u_5\}$ in H, at least one of the vertices u_i has weight at most $3n^{1/3}$. Indeed, suppose that the weight of each of the vertices exceeds $3n^{1/3}$. For any quintuple of non-negative integers $k = (k_1, \ldots, k_5)$, we denote $S_k = k_1 \deg(u_1) + \ldots + k_5 \deg(u_5)$ and $T_k = k_1 \deg_{\text{Non-}\Delta}(u_1) + \ldots + k_5 \deg_{\text{Non-}\Delta}(u_5)$. Consider only the quintuples k that satisfy $k_1 + \ldots + k_5 = 3n^{1/3}$: there are $\binom{3n^{1/3}+4}{4} \geq \frac{81}{24} n^{4/3}$ such quintuples and for each of them we have $S_k \leq 3n^{4/3}$. Thus, there exist distinct quintuples k and l, each of whose coordinates are non-negative integers summing to $3n^{1/3}$, such that $S_k = S_l$. Without loss of generality, we may assume that $T_k \geq T_l$.

Consider the weighted graph H', obtained by repeatedly adding $k_i - l_i$ to the weight of each vertex u_i, as long as all weights remain non-negative (note that this process will end because $k_i < l_i$ for some $i \in [5]$). The resulting weighted graph H' has the same number of vertices as H and satisfies $|H'| = |H|$, $e(H') = e(H)$ and $t(H') \geq t(H)$. Furthermore, since $|k_i - l_i| \leq 3n^{1/3}$, for some $i \in [5]$ the weight of u_i in H' is smaller than $3n^{1/3}$. In particular, H' has fewer vertices with weight at least $3n^{1/3}$ than H. This is, again, a contradiction to the choice of H. It follows that every independent set in H has at most four vertices with weight at least $3n^{1/3}$.

Recall that H has integer weights, so we may view it as a graph where a vertex of weight w represents a set of clones of size w. The graph H satisfies Condition 1 of Definition 6.14. Denote by U the set of triangular vertices in H. Adding all edges missing from $H[U]$ would not form triangles with edges that were previously non-triangular, so we may assume that U induces a clique in H. Let $u \in U$ be a vertex of maximum degree in H. For every $v \in U \setminus \{u\}$, we remove the edges between v and $V(H) \setminus U$ and add the edges between v and the neighbourhood of u in $V(H) \setminus U$. This process does not decrease the total number of edges and, moreover, all edges that were previously non-triangular remain non-triangular. We denote the resulting graph by H' and note that it satisfies Condition 2. Furthermore, if $I \subset V(H) \setminus U$ and $v \in U$ are such that $I \cup \{v\}$ is an independent set in H', then $I \cup \{u\}$ is an independent set in H. Therefore, H' retains Condition 1, and hence H' is compressed. $\square$

4 Almost bipartite

In this section we prove Theorem 6.4.

Theorem 6.4. *There exist n_0 and $\delta > 0$ such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $n^2/4 \leq e \leq (1/4 + \delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

Once again, some of the statements and inequalities that we write down only hold for sufficiently large n. Whenever this happens, we assume that n is, indeed, large enough to satisfy them.

Throughout this section we assume that G is a graph with n vertices and $e = (1/4 + \varepsilon)n^2$ edges, where $0 < \varepsilon \leq \delta$ for a small positive constant δ which we will (implicitly) determine later. Moreover, we assume that G is optimal (this means that increasing the number of edges reduces the number of non-triangular edges and vice versa, see Definition 6.7) and compressed (see Definition 6.14). In fact, for this proof we only need Condition 2 of Definition 6.14, which is much simpler than Condition 1.

However, to be able to make the assumptions described above, we have to deal with a small technicality regarding the condition $e(G) \leq (1/4 + \delta)n^2$. Indeed, it is true that given G it is always possible to find an optimal and compressed graph G' satisfying $|G'| = n, e(G') \geq e$ and $t(G') \geq t(G)$, but we cannot guarantee that $e(G') \leq (1/4 + \delta)n^2$ holds. To deal with this issue, we use the Exchange Lemma (Lemma 6.13). Indeed, if $e(G') \geq e + \Omega(n^2)$, then Lemma 6.13 implies that $t(G) \leq t(G') \leq t(n, e) - \Omega(n^2)$. If this happens, then we take a graph H with n vertices and at least e edges, satisfying $t(H) = t(n, e)$. By Corollary 6.12 there exists a graph

$H' = G(a,b,c)$ such that $|H'| = n, e(H') \geq e$ and $t(H') \geq t(n,e) - 5n \geq t(G)$, so we are done in this case. Therefore, we may assume that $e(G') \leq e + o(n^2)$, and so $e(G') \leq (1/4+\delta)n^2$ holds for a relaxed value of δ.

To get a rough idea about how large $t(G)$ is, we derive the following lower bound. Consider the graph $G(a,b,c)$ where $a = \lceil \sqrt{2\varepsilon}n \rceil + 1$, $b = \lceil n/2 \rceil$ and $c = n - a - b = \lfloor n/2 \rfloor - \lceil \sqrt{2\varepsilon}n \rceil - 1$. Then $e(G(a,b,c)) = \binom{a}{2} + (n-b)b \geq (1/4+\varepsilon)n^2$ and $t(G(a,b,c)) = bc \geq \left(1/4 - \sqrt{\varepsilon/2} - O(1/n)\right)n^2$. Since G is optimal, it follows that

$$t(G) \geq \left(\frac{1}{4} - \sqrt{\frac{\varepsilon}{2}} - O\left(\frac{1}{n}\right)\right)n^2. \tag{6.7}$$

Moreover, we have $e \geq \lfloor n^2/4 \rfloor + 1$, so in fact $\varepsilon n^2 \geq 1/2$ and therefore $1/n = O(\sqrt{\varepsilon})$. It follows that

$$t(G) \geq \left(\frac{1}{4} - O\left(\sqrt{\varepsilon}\right)\right)n^2. \tag{6.8}$$

We would like to make a brief comment regarding the use of big-O notation in Inequalities (6.7),(6.8) and other similar inequalities. According to our definition, $O(g(n))$ stands for a positive or negative quantity, so $-O(g(n))$ is exactly the same as $+O(g(n))$. We usually choose the sign before the big-O which looks more natural. However, we do not assume that $-O(g(n))$ is necessarily negative nor that $+O(g(n))$ is necessarily positive.

We divide the proof of Theorem 6.4 into four parts, represented by the following four propositions. In the first of these propositions we prove that G has the following structure (see Figure 6.5), which resembles a graph $G(a,b,c)$.

Proposition 6.17. *There exists a partition $\{A,B,C,D\}$ of $V(G)$ satisfying the following assertions.*

1. *All possible edges between B and C are present in G and are non-triangular. Moreover, $|B|,|C| \geq (1/2 - O(\sqrt{\varepsilon}))n$. In particular, B and C are independent sets.*

2. *There are no edges between A and C nor between B and D.*

3. *The induced subgraphs $G[A]$ and $G[D]$ do not have isolated vertices.*

4. *Every vertex in $A \cup D$ is incident with at most $O(\sqrt{\varepsilon}n)$ non-triangular edges of G. Moreover, the sets A and D do not span non-triangular edges (but there may be non-triangular edges between A and D).*

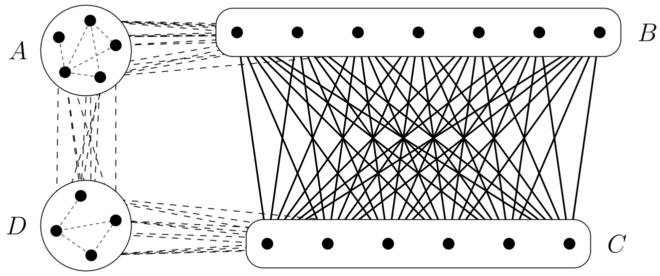

Figure 6.5: The partition $\{A, B, C, D\}$.

Here the proof of Theorem 6.4 splits into two cases: $\varepsilon \le \kappa/n$ and $\varepsilon \ge \kappa/n$, where κ is a small absolute positive constant that will be (implicitly) determined later. If ε is small, then we complete the proof directly.

Proposition 6.18. *There exists a constant $\kappa > 0$ with the property that if $\varepsilon \le \kappa/n$, then $G \cong G(a, b, c)$ for some a, b, c.*

If ε is large, then we first obtain sharp estimates for the sizes of the sets A, B, C, D.

Proposition 6.19. *Let G and A, B, C, D satisfy the conclusions of Proposition 6.17 and suppose that $|B| \ge |C|$ and that $\varepsilon \ge \kappa/n$ for some constant $\kappa > 0$. Then*

$$\begin{aligned} |A \cup D| &= \left(\sqrt{2\varepsilon} + O_\kappa(\varepsilon)\right) n, \\ |B| &= \left(\frac{1}{2} - O_\kappa(\varepsilon^{3/4})\right) n, \\ |C| &= \left(\frac{1}{2} - \sqrt{2\varepsilon} + O_\kappa(\varepsilon^{3/4})\right) n. \end{aligned}$$

Here $f(n) = O_\kappa(g(n))$ means that there exists a constant $C_\kappa > 0$, which depends on κ, such that $|f(n)| \le C_\kappa g(n)$.

We can now complete the proof of Theorem 6.4 in the case where ε is large.

Proposition 6.20. *Let $\kappa > 0$ be a constant. If $\varepsilon \ge \kappa/n$, then $G \cong G(a, b, c)$ for some a, b, c.*

Proposition 6.20 is a typical example of a statement that holds for sufficiently large n.

Proof of Theorem 6.4. Theorem 6.4 immediately follows from Propositions 6.18 and 6.20. □

The rest of this section is devoted to the proofs of Propositions 6.17 to 6.20, which are presented in separate subsections.

4.1 Structure of an optimal graph

In this subsection, we prove Proposition 6.17 (see also Figure 6.5). Recall that G is a fixed optimal graph with n vertices and $e = (1/4 + \varepsilon)n^2$ edges, where $0 < \varepsilon \leq \delta$ for a fixed small constant $\delta > 0$.

Proposition 6.17. *There exists a partition $\{A, B, C, D\}$ of $V(G)$ satisfying the following assertions.*

1. *All possible edges between B and C are present in G and are non-triangular. Moreover, $|B|, |C| \geq (1/2 - O(\sqrt{\varepsilon}))n$. In particular, B and C are independent sets.*

2. *There are no edges between A and C nor between B and D.*

3. *The induced subgraphs $G[A]$ and $G[D]$ do not have isolated vertices.*

4. *Every vertex in $A \cup D$ is incident with at most $O(\sqrt{\varepsilon}n)$ non-triangular edges of G. Moreover, the sets A and D do not span non-triangular edges (but there may be non-triangular edges between A and D).*

The first assertion follows fairly easily from the fact that the number of non-triangular edges is almost $n^2/4$. To complete the proof we use basic properties of optimal graphs.

Proof of Proposition 6.17. Let H be the spanning subgraph of G whose edges are the non-triangular edges of G. We note that H is a triangle-free graph with close to $n^2/4$ edges, which implies that H is close to being a complete bipartite graph. This enables us to find independent (with respect to G) sets U and W of size almost $n/2$ each, such that H contains almost all of the possible edges between them. This idea is rigorously implemented in the following claim.

Claim 6.21. *There exist disjoint non-empty independent sets $U, W \subseteq V(G)$ such that every vertex in U has at least $(1/2 - O(\varepsilon^{1/4}))n$ non-triangular neighbours in W and vice versa. In particular, $|U|, |W| \geq (1/2 - O(\varepsilon^{1/4}))n$.*

Proof. Inequality (6.8) states that $e(H) = t(G) \geq (1/4 - c\sqrt{\varepsilon})n^2$ for some absolute constant $c > 0$. From this we deduce that, writing $d = \sqrt{c}$, there are at most $2d\varepsilon^{1/4}$ vertices in H of degree smaller than $(1/2 - d\varepsilon^{1/4})n$. Indeed, suppose that we can find a set S consisting of exactly $2d\varepsilon^{1/4}n$ vertices of degree smaller than $(1/2 - d\varepsilon^{1/4})n$ in H. Since H is triangle-free, $e(H \setminus S) \leq (n - |S|)^2/4$. Hence,

$$\begin{aligned} e(H) &< \frac{(n-|S|)^2}{4} + |S|\left(\frac{1}{2} - d\varepsilon^{1/4}\right)n \\ &= \frac{n^2}{4} - |S|\left(d\varepsilon^{1/4}n - \frac{|S|}{4}\right) \\ &= \left(\frac{1}{4} - d^2\sqrt{\varepsilon}\right)n^2 \\ &= \left(\frac{1}{4} - c\sqrt{\varepsilon}\right)n^2, \end{aligned}$$

a contradiction.

Let $u \in V(G)$ be any vertex with $\deg_H(u) \geq (1/2 - d\varepsilon^{1/4})n$. Denote by U the set of vertices in $N_H(u)$ that have at least $(1/2 - d\varepsilon^{1/4})n$ neighbours in H. Since the edges of H are non-triangular in G, it follows that U is independent in G. Moreover, $|U| \geq \deg_H(u) - 2d\varepsilon^{1/4}n \geq (1/2 - O(\varepsilon^{1/4}))n$.

Now let $v \in U$ and denote by W the set of vertices in $N_H(v)$ whose degree in H is at least $(1/2 - d\varepsilon^{1/4})n$. As before, W is independent in G and has size at least $(1/2 - O(\varepsilon^{1/4}))n$. Finally, every vertex in U has at least $(1/2 - d\varepsilon^{1/4})n - (n - |U| - |W|) \geq (1/2 - O(\varepsilon^{1/4}))n$ non-triangular neighbours in W, and vice versa. $\square$

Let U and W be the disjoint independent sets given by Claim 6.21. The following similar claim allows us to enlarge U and W to obtain sets B and C which will be shown to satisfy the requirements of Proposition 6.17.

Claim 6.22. *There exist disjoint independent sets $B, C \subseteq V(G)$, satisfying $U \subseteq B$ and $W \subseteq C$ and $|B \cup C| \geq (1 - O(\sqrt{\varepsilon}))n$, such that every vertex in B has at least $2n/5$ non-triangular neighbours in C and vice versa.*

Proof. We first show that there are at most $O(\sqrt{\varepsilon}n)$ vertices of degree at most $21n/50$ in H (where H, the graph of non-triangular edges of G, was defined in the proof of the previous claim). To this end we recall Inequality (6.8), which states that $e(H) = t(G) \geq (1/4 - c\sqrt{\varepsilon})n$ for some absolute constant c. Importantly, this constant does not depend on δ, so we may choose δ to satisfy $c\sqrt{\delta} \leq 1/100$. Recall that δ is an upper bound for ε, so we have $c\sqrt{\varepsilon} \leq 1/100$.

Suppose that S is a set consisting of exactly $25c\sqrt{\varepsilon}n$ vertices of degree at most

$21n/50$ in H. Then, similarly to the previous claim,

$$\begin{aligned} e(H) &\le \frac{(n-|S|)^2}{4} + |S|\frac{21n}{50} \\ &= \frac{n^2}{4} - |S|\left(\frac{4n}{25} - \frac{|S|}{4}\right) \\ &< \left(\frac{1}{4} - 2c\sqrt{\varepsilon}\right)n^2, \end{aligned}$$

a contradiction to Inequality (6.8). Therefore, there are at most $O(\sqrt{\varepsilon}n)$ vertices with degree at most $21n/50$ in H.

Recall that every vertex in U has at least $(1/2-O(\varepsilon^{1/4}))n \ge 2n/5$ non-triangular neighbours in W and vice versa. Here we implicitly assume that δ is small enough to make this inequality true, and we shall do so throughout this proof.

Denote by X the set of vertices in $V(G)\setminus(U\cup W)$ whose degree in H is at least $21n/50$. We note that no vertex in X has neighbours in both U and W. Indeed, suppose that $v \in X$ is adjacent to $u \in U$ and $w \in W$. Since v is not adjacent to any non-triangular neighbour of either u or w, it has at most $O(\varepsilon^{1/4}n)$ neighbours in U and at most $O(\varepsilon^{1/4}n)$ neighbours in W, implying that $\deg_H(v) \le O(\varepsilon^{1/4}n)$, a contradiction to the assumption that $\deg_H(v) \ge 21n/50$.

Let Y be the set of vertices in X that are adjacent to vertices in U and, similarly, let Z be the set of vertices in X that have neighbours in W. Then every vertex in Y has at least $21/50n - O(\varepsilon^{1/4}n) \ge 2n/5$ non-triangular neighbours in U and no neighbours in W. In particular, since $|U| \le n-|W| < 4n/5$, any two vertices in Y share a non-triangular neighbour in U, and hence Y is an independent set in G. Denote $B = Y\cup W$ and $C = Z \cup U$. Then B and C are independent sets such that every vertex in B has at least $2n/5$ non-triangular neighbours in C, and vice versa. Furthermore, $V(G)\setminus(B\cup C)$ is the set of vertices with fewer than $21n/50$ neighbours in H, so $|V(G)\setminus(B\cup C)| = O(\sqrt{\varepsilon}n)$, finishing the proof of Claim 6.22. □

We can now finish the proof of Proposition 6.17. Let B and C be as in Claim 6.22. Since every vertex in $B\cup C$ has at least $2n/5$ non-triangular neighbours, it follows from Observation 6.8 and the assumption that G is optimal that every vertex in G has degree at least $2n/5-1$. We conclude (similarly to the proof of Claim 6.22) that no vertex in G has neighbours in both B and C. Indeed, suppose that some $v \in V(G)$ is adjacent to some $u \in B$ and $w \in C$. Since u has at least $2n/5$ non-triangular neighbours in C, v is adjacent to at most $|C|-2n/5$ vertices in C and, similarly, to at most $|B|-2n/5$ vertices in B. It follows that v has degree at most

$n/5$, a contradiction.

Since no vertex in G is adjacent to a vertex in B and a vertex in C, we may add all missing edges between B and C without creating new triangles. However, G is an optimal graph, so in fact all edges between B and C are present in G. Again, since vertices in B and C do not have common neighbours, all edges between B and C are non-triangular.

We may assume that $|B| \geq |C|$. Then $|B| \geq (1/2 - O(\sqrt{\varepsilon}))n$ and hence every vertex in C has non-triangular degree at least $(1/2 - O(\sqrt{\varepsilon}))n$. Again, by Observation 6.8, every vertex in G has degree at least $(1/2 - O(\sqrt{\varepsilon}))n$. Since B is an independent set, it follows that $n - |B| \geq (1/2 - O(\sqrt{\varepsilon}))n$. Therefore $|C| = |B \cup C| - |B| \geq (1/2 - O(\sqrt{\varepsilon}))n$.

We are now done with the first assertion of Proposition 6.17, and the remaining ones follow easily. Let A be the set of vertices outside of $B \cup C$ that are adjacent to a vertex in B and, similarly, let D be the set of vertices outside of $B \cup C$ that have a neighbour in C. Then $\{A, B, C, D\}$ forms a partition of G, because a vertex without neighbours in $B \cup C$ would have too small a degree. This establishes the second assertion.

To prove the third assertion, we may assume that every vertex in A has a neighbour in A: if some $u \in A$ has no neighbours in A, then we may add all edges between u and the vertices in B without creating new triangles and then reassign u to C. Similarly, we may assume that every vertex in D has a neighbour in D.

By inspecting the degrees, any two vertices in A have a common neighbour in B. Therefore, there cannot be any non-triangular edges with both ends in A or, similarly, with both ends in D. It remains to check that every vertex in $A \cup D$ is incident with at most $O(\sqrt{\varepsilon}n)$ non-triangular edges. Let $u \in A$ and let $v \in A$ by a neighbour of u. Since u and v have neighbours only in $A \cup D \cup B$ and the degree of v is at least $(1/2 - O(\sqrt{\varepsilon}))n$, it follows that u has at most $|A \cup D \cup B| - (1/2 - O(\sqrt{\varepsilon}))n = O(\sqrt{\varepsilon}n)$ non-triangular neighbours. The same holds for any vertex in D. This establishes the fourth assertion and completes the proof of Proposition 6.17. $\square$

4.2 Completing the proof if ε is small

We now prove Proposition 6.18, which completes the proof of Theorem 6.4 in the case where ε is small.

Proposition 6.18. *There exists a constant $\kappa > 0$ with the property that if $\varepsilon \leq \kappa/n$, then $G \cong G(a, b, c)$ for some a, b, c.*

Proof. It follows from the assumptions on the sets A, B, C, D that $|A \cup D| = O(\sqrt{\varepsilon}n)$ and that each vertex in $A \cup D$ is incident with at most $O(\sqrt{\varepsilon}n)$ non-triangular edges. Therefore the number of non-triangular edges with an end in $A \cup D$ is $O(\varepsilon n^2) = O(\kappa n)$. We show that, in fact, there are no such edges.

Suppose that uv is a non-triangular edge with $u \in A \cup D$. Without loss of generality, we may assume that $u \in A$ and $v \in B \cup D$. Observe that the neighbours of u are not adjacent to v. Let G' be the graph obtained by adding the edges between v and the neighbours of u in A, removing the edges between u and $A \setminus \{u\}$ and also adding all missing edges between u and B. Then $e(G') \geq e(G)$ and $t(G') \geq t(G) + |B| - O(\kappa n) > t(G)$, where the last inequality holds provided that we choose κ small enough. However, this contradicts the optimality of G, so there cannot be such an edge uv.

It is now easy to finish the proof. By what we have just proved, all the missing edges with both ends in $A \cup D$ may be added without causing a non-triangular edge to become triangular, and hence, since G is optimal, $G[A \cup D]$ is a clique. Similarly, all possible edges between A and B and between D and C are present in G. We may assume that $|B| \geq |C|$. Remove the edges between D and C and add all possible edges between D and B. The result graph G' is isomorphic to $G(|A \cup D|, |B|, |C|)$ and satisfies $|G'| = n$, $e(G') \geq e(G)$ and $t(G') \geq t(G)$. However, G is optimal, so we must have $e(G') = e(G)$ and $t(G') = t(G)$. Therefore, it must be the case that $D = \emptyset$ or $|B| = |C|$. If D is empty, then $G = G'$ and we are done. Let us suppose that $|B| = |C|$. Since $e(G') > n^2/4$, G' is not bipartite, and hence $|A \cup D| \geq 2$. Take any vertex $w \in A \cup D$. If we remove all edges between w and B, but add all possible edges between w and C, then we obtain a new graph which has the same number of edges, but more non-triangular edges than G. However, this contradicts the assumption that G is optimal. Therefore, it must be the case that $|B| > |C|$, and so we are done. $\square$

4.3 Sizes of A, B, C, D

In this subsection we prepare for the proof of Theorem 6.4 in the case where ε is large. In particular, we obtain good bounds for the sizes of the sets $A \cup D$, B and C.

Proposition 6.19. *Let G and A, B, C, D satisfy the conclusions of Proposition 6.17* *and suppose that* $|B| \geq |C|$ *and that* $\varepsilon \geq \kappa/n$ *for some constant* $\kappa > 0$. *Then*

$$\begin{aligned} |A \cup D| &= \left(\sqrt{2\varepsilon} + O_\kappa(\varepsilon)\right) n, \\ |B| &= \left(\frac{1}{2} - O_\kappa(\varepsilon^{3/4})\right) n, \\ |C| &= \left(\frac{1}{2} - \sqrt{2\varepsilon} + O_\kappa(\varepsilon^{3/4})\right) n. \end{aligned}$$

The proof is just a technical calculation, in which the main tool is the lower bound on $t(G)$ given by Inequality (6.7).

Proof. Denote $a = |A \cup D|$, $b = |B|$ and $c = |C|$ and write

$$\begin{aligned} a &= \left(\sqrt{2\varepsilon} + \alpha\right) n, \\ b &= \left(\frac{1}{2} - \beta\right) n, \\ c &= \left(\frac{1}{2} - \sqrt{2\varepsilon} + \beta - \alpha\right) n, \end{aligned}$$

where the quantities α and β are defined by these identities. We cannot assume that α and β are positive, but we have $-\sqrt{2\varepsilon} \leq \alpha \leq O(\sqrt{\varepsilon})$, where the second inequality comes from Proposition 6.17. Since there are at most $O(\varepsilon n^2)$ non-triangular edges with an end in $A \cup D$, we have

$$\begin{aligned} t(G) &\leq bc + O(\varepsilon n^2) \\ &\leq \frac{(n-a)^2}{4} + O(\varepsilon n^2) \\ &\leq \frac{n^2}{4} - \frac{an}{2} + O(\varepsilon n^2). \end{aligned}$$

Combining this with Inequality (6.7), which states that $t(G) \geq \left(1/4 - \sqrt{\varepsilon/2} - O(1/n)\right) n^2$
, we get

$$\frac{1}{4} - \sqrt{\frac{\varepsilon}{2}} - O_\kappa(\varepsilon) \;\leq\; \frac{t(G)}{n^2} \;\leq\; \frac{1}{4} - \frac{(\sqrt{2\varepsilon} + \alpha)}{2} + O(\varepsilon).$$

Therefore, $\alpha \leq O_\kappa(\varepsilon)$. Using the fact that $b \geq c$ and that any vertex in $A \cup D$ sends edges to only one of B and C, we obtain the following upper bound on the number of edges in G:

$$e(G) \leq b(n-b) + \frac{a^2}{2}.$$

Combining this with the definition $e(G) = (1/4 + \varepsilon)n^2$, we get

$$\begin{aligned}\frac{1}{4} + \varepsilon &\le \left(\frac{1}{2} - \beta\right)\left(\frac{1}{2} + \beta\right) + \frac{\left(\sqrt{2\varepsilon} + \alpha\right)^2}{2} \\ &= \frac{1}{4} - \beta^2 + \varepsilon + \alpha\left(\sqrt{2\varepsilon} + \frac{\alpha}{2}\right).\end{aligned}$$

It follows that $\beta^2 \le \alpha(\sqrt{2\varepsilon} + \alpha/2)$. In particular, $\alpha \ge 0$ and $\beta = O_\kappa(\varepsilon^{3/4})$, implying the assertions of Proposition 6.19. □

4.4 Completing the proof if ε is large

We are now able to complete the proof of Theorem 6.4 under the assumption that $\varepsilon \ge \kappa/n$ for some constant $\kappa > 0$. Here we will use the assumption that G is a compressed graph.

Proposition 6.20. *Let $\kappa > 0$ be a constant. If $\varepsilon \ge \kappa/n$, then $G \cong G(a, b, c)$ for some a, b, c.*

The proof consists of two stages. In the first stage we use the bounds from Proposition 6.19 to conclude that D is very small and that very few vertices in A are incident with non-triangular edges. In the second stage we show that if D is non-empty or if there exists a vertex in A with a non-triangular neighbour, then G can be manipulated to obtain a graph with more edges and more non-triangular edges, contradicting the assumption that G is optimal. It follows that G is isomorphic to a graph $G(a, b, c)$.

Proof of Proposition 6.20. We start by showing that the edges between $B \cup D$ and $A \cup C$ form an almost complete bipartite subgraph. We shall be using the estimates on the size of the sets $A \cup D$, B and C from Proposition 6.19. To be able to use Proposition 6.19, we assume, without loss of generality, that $|B| \ge |C|$. Note that κ is an absolute constant (implicitly determined in Proposition 6.18). Thus, we may remove the dependence on κ in the estimates of these sizes.

Claim 6.23. *Every vertex in $B \cup D$ is adjacent to all but $O(\varepsilon^{3/4}n)$ vertices in $A \cup C$. Furthermore, $|D| = O(\varepsilon^{3/4}n)$.*

Proof. The non-triangular degree of any vertex in C is at least $|B|$. Hence, by Observation 6.8, every vertex in G has degree at least $|B| - 1$. The vertices in $B \cup D$ are not adjacent to any vertex in B. Since $|B| \ge (1/2 - O(\varepsilon^{3/4}))n$, it follows that every vertex in $B \cup D$ is adjacent to all but $O(\varepsilon^{3/4}n)$ vertices in $V(G) \setminus B = A \cup C \cup D$. Since there are no edges between B and D, $|D| = O(\varepsilon^{3/4}n)$. $\square$

Denote by T the set of triangular vertices in A (recall that a triangular vertex is incident only with triangular edges) and let $S = A \setminus T$. We show that the vertices in S have few neighbours in A.

Claim 6.24. *Every vertex in S has $O(\varepsilon^{3/4}n)$ neighbours in A.*

Proof. Let $u \in S$ and let v be a non-triangular neighbour of u. Then $v \in B \cup D$, because there are no edges between A and C, and there are no non-triangular edges with both ends in A. Recall that, by Claim 6.23, v is adjacent to all but $O(\varepsilon^{3/4}n)$ vertices in A. Since uv is non-triangular, u and v have no common neighbours, implying that u has $O(\varepsilon^{3/4}n)$ neighbours in A. $\square$

We conclude that almost all of the vertices in A are triangular.

Claim 6.25. $|T| \geq (\sqrt{2\varepsilon} - O(\varepsilon^{3/4}))n^2$.

Proof. By removing the edges with both ends in A or both ends in D from G, we remain with a bipartite graph, so $(1/4 + \varepsilon)n^2 = e(G) \leq n^2/4 + e(G[A]) + e(G[D])$. Since $|D| = O(\varepsilon^{3/4}n)$, we have $e(G[D]) = O(\varepsilon^{3/2}n^2)$, and hence $e(G[A]) \geq (\varepsilon - O(\varepsilon^{3/2}))n^2$.

Claim 6.24 implies that $e(G[A]) - e(G[T]) \leq O(|S|\varepsilon^{3/4}n) \leq O(|A|\varepsilon^{3/4}n) \leq O(\varepsilon^{5/4}n^2)$, where the rightmost inequality is a consequence of Proposition 6.19. Therefore, $e(G[T]) \geq (\varepsilon - O(\varepsilon^{5/4}))n^2$, and so $|T| \geq (\sqrt{2\varepsilon} - O(\varepsilon^{3/4}))n$, as required. $\square$

Since G is compressed, T induces a clique and any two vertices in T have the same neighbourhood outside of T. In particular, if a vertex $v \in S$ is adjacent to a vertex in T, then v is adjacent to all vertices in T. However, this cannot happen since, by Claim 6.24, v has at most $O(\varepsilon^{3/4}n)$ neighbours in A, while, by Claim 6.25, there are at least $\Omega(\sqrt{\varepsilon}n)$ vertices in T. Therefore, there are no edges between T and S.

In the following claim we deduce that, in fact, all vertices in A are triangular. The key observation is that a pair of adjacent vertices in S can be replaced by one vertex in C and one in T, increasing both the number of edges and the number of non-triangular edges.

Claim 6.26. *The set S is empty.*

Proof. Suppose that S contains a vertex u. By Proposition 6.17, u has a neighbour $v \in A$. Since there are no edges between T and S, we conclude that $v \in S$. In particular, u and v have no neighbours in T. Now let H be the graph obtained from G by removing the vertices u and v and adding new vertices x and y, where x is joined by edges to all of B and y is joined to all of $B \cup T$. It follows from Claim 6.23 to 6.25 that $e(H) \geq e(G) - O(\varepsilon^{3/4}n) + (\sqrt{2\varepsilon} - O(\varepsilon^{3/4}))n > e(G)$. Recall that, by Proposition 6.17, the non-triangular degree of any vertex in A is at most $O(\sqrt{\varepsilon}n)$, implying that $t(H) \geq t(G) - O(\sqrt{\varepsilon}n) + |B| > t(G)$. Therefore, H has more edges and more non-triangular edges than G, contradicting the optimality of G. Thus, S is empty. $\square$

Similarly, we prove that D is empty. The trick here is to replace two adjacent vertices in D by one vertex in C and one in A.

Claim 6.27. *The set D is empty.*

Proof. Suppose that D is non-empty, so we may pick adjacent vertices $u, v \in D$. Consider the graph H, obtained by removing the vertices u and v and adding new vertices x and y with x joined to all of B and y joined to all of $A \cup B$. It follows from the bounds given by Proposition 6.19 and Claim 6.23 that $e(H) \geq e(G) + (\sqrt{2\varepsilon} - O(\varepsilon^{3/4}))n > e(G)$. Moreover, since $A = T$ is a clique of triangular vertices, the addition of x and y does not destroy any non-triangular edges in $G \setminus \{u, v\}$. Since u and v have at most $O(\sqrt{\varepsilon}n)$ non-triangular neighbours, we have $t(H) \geq t(G) + (1/2 - O(\sqrt{\varepsilon}))n > t(G)$, contradicting the assumption that G is optimal. $\square$

Now the proof of Proposition 6.20 is complete. Indeed, we know from Claim 6.26 that $A = T$. This means that A induces a clique and that every vertex in A is adjacent to every vertex in B. Therefore, $G = G(|A|, |B|, |C|)$. $\square$

5 Middle range

In this section we prove Theorem 6.5, in which we consider the case where the graph is neither close to being complete nor close to being complete bipartite. Out of the three ranges, the middle range turns out to be the hardest to prove. One of the main difficulties that arises here is that, unlike in the other two ranges, we cannot directly conclude that the graph G has structure similar to that of $G(a, b, c)$.

Theorem 6.5. *For every $\delta > 0$ there exists n_0 such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $(1/4+\delta)n^2 \leq e \leq (1/2-\delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

Fix $\delta > 0$. Throughout this section we assume that G is a compressed and optimal graph with n vertices and e edges, where $(1/4+\delta)n^2 \le e \le (1/2-\delta)n^2$. As in the rest of the chapter, the statements that we write down hold for sufficiently large n. Moreover, since δ is fixed, the constants implied by big-O notation may depend on δ.

We split the proof of Theorem 6.5 into four stages, as described by the four following propositions. In the first stage we show that G has many triangular vertices (that is, vertices that are incident only with triangular edges).

Proposition 6.28. *G has $\Omega(n)$ triangular vertices.*

In the second stage we conclude that G admits the following structure (see Figure 6.6). Although Proposition 6.29 gives much less information than Proposition 6.17 from Section 4, it still shows that G vaguely resembles a graph $G(a,b,c)$.

Proposition 6.29. *There exists a partition $\{A,B,C\}$ of $V(G)$ such that all parts have size $\Omega(n)$ and the following properties are satisfied.*

1. *A is the set of triangular vertices in G, it spans a clique and its vertices are adjacent to all of B and none of C.*

2. *B may be partitioned into $O(1)$ sets of clones and a remainder consisting of at most $O(\sqrt{n}\log n)$ vertices.*

3. *C may be partitioned into $O(1)$ sets of clones, each having $\Omega(n)$ non-triangular neighbours in B, and a remainder of size $O(n^{1/3}\log n)$.*

In the third stage we show that the number of edges (and non-triangular edges) in G is close to the number of edges (and non-triangular edges) in $G(|A|,|B|,|C|)$.

Proposition 6.30. *Let A,B,C be as in Proposition 6.29 and denote $a=|A|, b=|B|, c=|C|$. Then $e(G) = a^2/2 + ab + bc + O(n^{7/4}\sqrt{\log n})$ and $t(G) = bc + O(n^{7/4}\sqrt{\log n})$.*

In the final fourth stage we complete the proof of Theorem 6.5.

Proposition 6.31. *$G \cong G(a,b,c)$ for some a,b,c.*

Proof of Theorem 6.5. The proof is immediate from Propositions 6.28 to 6.31. The only slight technicality is that when we replace a graph with at most $(1/2-\delta)n^2$ edges by an optimal and compressed graph, the number of edges may increase and exceed this bound. However, Lemma 6.13 implies that this condition is still satisfied for a relaxed value of δ. $\square$

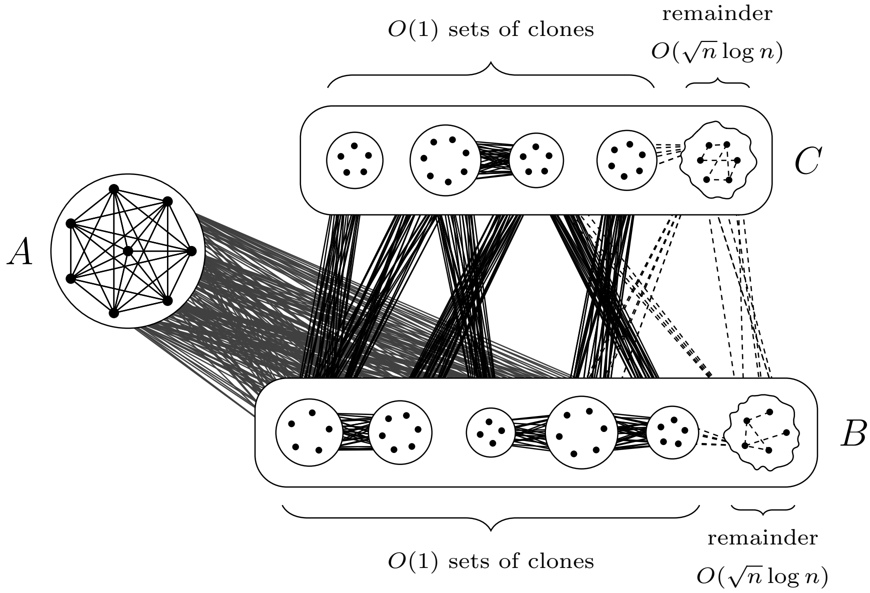

Figure 6.6: The partition $\{A, B, C\}$

We now turn to the proofs of Propositions 6.28 to 6.31. We present them in separate subsections.

5.1 Many triangular vertices

In this subsection we prove Proposition 6.28.

Proposition 6.28. *G has $\Omega(n)$ triangular vertices.*

The main ingredients of this proof are a somewhat unexpected application of Lemma 6.13 and the assumption that G is compressed. First, we conclude from Lemma 6.10 that G has a large clique. Then, we partition the graph into fairly large independent sets of clones and a very dense part, using the fact that G is compressed. It is then possible to conclude that only few of the vertices of the clique are incident with non-triangular edges.

Proof of Proposition 6.28. Our first aim is to show that G has a clique of size at least $\Omega(n)$. This can be done fairly easily, as shown in the proof of the following claim.

Claim 6.32. *G has a clique of size $\Omega(n)$.*

Proof. By Lemma 6.10, there exists a good weighted subgraph H of G satisfying $|H| = |G| = n$, $e(H) \geq e(G)$, $t(H) \geq t(G)$ (see Definition 6.9 for the definition of a good weighted graph). Let $\{K, \{u, v\}\}$ be a partition of $V(H)$ into a clique K and an edge uv, which is the only non-triangular edge of H.

Let α be the sum of the weights of vertices in K and let m be the number of vertices in K. Let β and γ be the weights of u and v and suppose that $\beta \geq \gamma$. Note that $\alpha + \beta + \gamma = n$. By the Cauchy-Schwarz inequality, the contribution of the vertices in K towards $e(H)$ is maximised if all of these vertices have weight α/m. Therefore this contribution does not exceed $(\alpha/m)^2\binom{m}{2} = (1-1/m)\alpha^2/2$. Moreover, since no vertex is adjacent to both u and v, the contribution of the edges between K and $\{u, v\}$ towards $e(H)$ is maximised when every vertex in K is adjacent to u, but not v. Hence,

$$e(G) \leq e(H) \leq \left(1 - \frac{1}{m}\right)\frac{\alpha^2}{2} + \alpha\beta + \beta\gamma. \tag{6.9}$$

In particular, since $\beta\gamma \leq n^2/4$, we have $e(G) \leq n^2/4 + \alpha n$. Recall that $e(G) \geq (1/4 + \delta)n^2$. It follows that $\alpha \geq \delta n$.

Denote $b = \lceil \beta \rceil$, $c = \lceil \gamma \rceil$ and $a = n - b - c$ and consider the graph $F = G(a, b, c)$. Note that $t(G) \leq t(H) = \beta\gamma \leq bc = t(F)$. Since G is optimal, it follows that $e(G) \geq e(F)$. Therefore,

$$\begin{aligned}
e(G) \geq e(F) &= \binom{a}{2} + ab + bc \\
&\geq \frac{(\alpha - 2)(\alpha - 3)}{2} + (\alpha - 2)\beta + \beta\gamma \\
&\geq \frac{\alpha^2}{2} + \alpha\beta + \beta\gamma - 2.5n \\
&= \left(1 - \frac{5n}{\alpha^2}\right)\frac{\alpha^2}{2} + \alpha\beta + \beta\gamma.
\end{aligned}$$

Comparing this with (6.9), we have $m \geq \alpha^2/(5n) \geq \delta^2 n/5$. It follows that G has a clique of size at least $\delta^2 n/5$. □

Recall that G is compressed. Hence, by Observation 6.15, every independent set of size $5\sqrt{n}$ in G contains a set of clones of size $\sqrt{n}$.

We construct a set $U \subseteq V(G)$ as follows. We start with $U = \emptyset$. At each stage, if the complement $U^{\mathsf{c}} = V(G) \setminus U$ contains an independent set I of size $5\sqrt{n}$, then I contains a set of clones of size at least $\sqrt{n}$. We add this set of clones to U and continue until U^{c} has no independent set of size $5\sqrt{n}$. Observe that the resulting set U is a disjoint union of sets of clones each of size at least $\sqrt{n}$, while the complement U^{c} has no independent set of size $5\sqrt{n}$ (see Figure 6.7).

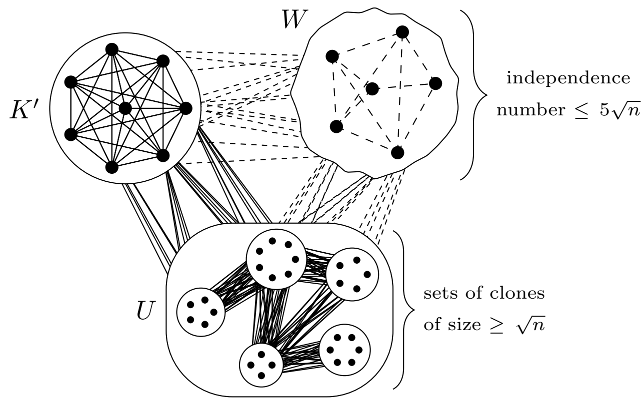

Figure 6.7: The sets U, W and K'.

In the following claim we deduce from Lemma 6.13 that $G[U^{\mathsf{c}}]$ is very dense.

Claim 6.33. *$G[U^{\mathsf{c}}]$ has $O(n^{3/2})$ non-edges.*

Proof. Since $G[U^{\mathsf{c}}]$ has no independent set of size at least $5\sqrt{n}$, every vertex in G has at most $5\sqrt{n}$ non-triangular neighbours in U^{c}. It follows that there are at most $5n^{3/2}$ non-triangular edges with at least one end in U^{c}.

Let m denote the number of non-edges in $G[U^{\mathsf{c}}]$. By adding these edges to G we obtain a graph G' with n vertices and $e(G)+m$ edges such that $t(G') \geq t(G) - 5n^{3/2}$. It follows from Lemma 6.13 that $m = O(n^{3/2})$. □

Let K be a largest clique in G, so $|K| = \Omega(n)$ by Claim 6.32. Let $K' = K \setminus U$ and denote $W = U^{\mathsf{c}} \setminus K'$ (see Figure 6.7). Note that, since U contains no clique of size greater than $\sqrt{n}$, we have $|K'| \geq |K| - \sqrt{n} = \Omega(n)$. In the following claim we use the structure of U and Claim 6.33 to deduce that almost all vertices in K' are triangular.

Claim 6.34. *All but $O(\sqrt{n})$ vertices in K' are triangular.*

Proof. Since K' is a clique, any vertex in the complement $V(G) \setminus K'$ sends at most one non-triangular edge to K'. In fact, if $u \in V(G) \setminus K'$ has a non-triangular neighbour in K', then u has no other neighbours in K'.

Denote by m the number of vertices in W that have a non-triangular neighbour in K'. Then the number of missing edges in $G[U^{\mathsf{c}}]$ is at least $m(|K'| - 1) = \Omega(mn)$. From Claim 6.33 we conclude that $m = O(\sqrt{n})$. Therefore, there are $O(\sqrt{n})$ vertices in K' with a non-triangular neighbour in U^{c}.

Finally, U is a union of at most $\sqrt{n}$ sets of clones, and any one set of clones can send non-triangular edges to at most one vertex in K'. Therefore, there are at most $\sqrt{n}$ vertices in K' that have a non-triangular neighbour in U. □

The proof of Proposition 6.28 is complete. Indeed, K' consists of $\Omega(n)$ vertices and all but $O(\sqrt{n})$ of them are triangular. $\square$

5.2 Structure

In this subsection we build on the fact that G has $\Omega(n)$ triangular vertices and prove that, in terms of structure, G has some similarities with a graph $G(a, b, c)$. In particular, we prove that the vertices of G can be partitioned into three linearly sized sets A, B, C such that A is a clique and all edges between A and B are present in G, while all edges between A and C are missing. We do not yet prove that the sets B, C are independent, but we show that both of them can be partitioned into a small number of independent sets (see Figure 6.6). Our main tool in this subsection is the assumption that G is compressed, and we also use Lemma 6.13.

Proposition 6.29. *There exists a partition $\{A, B, C\}$ of $V(G)$ such that all parts have size $\Omega(n)$ and the following properties are satisfied.*

1. *A is the set of triangular vertices in G, it spans a clique and its vertices are adjacent to all of B and none of C.*

2. *B may be partitioned into $O(1)$ sets of clones and a remainder consisting of at most $O(\sqrt{n}\log n)$ vertices.*

3. *C may be partitioned into $O(1)$ sets of clones, each having $\Omega(n)$ non-triangular neighbours in B, and a remainder of size $O(n^{1/3}\log n)$.*

Proof. Denote by A the set of triangular vertices in G. Since G is compressed, A induces a clique and the vertices of A have the same neighbourhood outside of A. Denote this neighbourhood by B and let $C = V(G) \setminus (A \cup B)$. Property 1 follows.

Note that the graph $G(a, b, c)$, where $c = \delta n/2$, $b = \sqrt{\delta}n$ and $a = n - b - c$, has at least $(1/2 - \delta)n^2$ edges and $\delta^{3/2}n^2/2$ non-triangular edges. Hence, since G is optimal and $e(G) \leq (1/2 - \delta)n^2$, it follows that $t(G) = \Omega(n^2)$.

By Proposition 6.28 we have $|A| = \Omega(n)$. Note that there are no non-triangular edges with both ends in $A \cup B$, and so the number of non-triangular edges in G is at most $|C|n$. Since $t(G) = \Omega(n^2)$, it follows that $|C| = \Omega(n)$. We will deduce that $|B| = \Omega(n)$ from a stronger statement that almost all vertices in C have $\Omega(n)$ non-triangular neighbours in B.

Claim 6.35. *All but $O(1)$ vertices of C have $\Omega(n)$ non-triangular neighbours in B.*

Proof. Let $c > 0$ and $k \in \mathbb{N}$ be constants. Suppose that there is a set $Z \subseteq C$ of size k whose every vertex has at most cn non-triangular neighbours in B. Our aim is to show that if c is sufficiently small and k is sufficiently large, then the existence of such a set Z would lead to a contradiction.

Consider the graph G', obtained from G by adding the edges between Z and A. Then $e(G') = e(G) + k|A|$ and $t(G') \geq t(G) - ckn - \binom{k}{2} \geq t(G) - 2ckn$. Provided that k is sufficiently large, Lemma 6.13 implies that $t(G') \leq t(G) - \zeta k|A|$ for some constant $\zeta > 0$ that does not depend on c or k. Therefore, $\zeta|A| \leq 2cn$ must hold. However, we may choose c small enough to make this false, thus obtaining a contradiction. □

The previous claim provides us with a set $C' \subseteq C$ such that $|C \setminus C'| = O(1)$ and every vertex in C' has $\Omega(n)$ non-triangular neighbours in B. The following claim implies that C' may be partitioned into $O(1)$ independent sets.

Claim 6.36. *There exists a set $S \subseteq B$ of size $O(1)$ such that every vertex in C' has a non-triangular neighbour in S.*

Proof. We construct $S = \{u_1, \ldots, u_k\}$ by choosing the elements $u_1, \ldots, u_k \in B$ and certain corresponding subsets $I_1, \ldots, I_k \subseteq B$ in the following way. Suppose that $u_1, \ldots, u_j$ and $I_1, \ldots, I_j$ have been chosen, where $j \geq 0$. Let U_j be the set of vertices in C' that have a non-triangular neighbour in $\{u_1, \ldots, u_j\}$ (so, in particular, $U_0 = \emptyset$). If $U_j = C'$, we stop the process. Otherwise, pick a vertex $v \in C' \setminus U_j$ and consider the set N consisting of the non-triangular neighbours of v in B. By the definition of C', we have $|N| = \Omega(n)$. Moreover, since N is independent and G is compressed, N contains a set of clones of size at least $|N|/5$. Denote this set of clones by I_{j+1} and pick $u_{j+1} \in I_{j+1}$ arbitrarily.

It is clear that when the process terminates, every vertex in C' has a non-triangular neighbour in the resulting set S. It remains to check that the process stops after $O(1)$ steps. Indeed, suppose that it ran for k steps. The sets $I_1, \ldots, I_k$ are pairwise disjoint and have size at least $\Omega(n)$ each, whence $k = O(1)$. □

The non-triangular neighbourhoods of the vertices in S cover C'. Therefore, C' can be partitioned into $O(1)$ independent sets. Since G is compressed, each independent set can be partitioned into $O(\log n)$ sets of clones, all but at most four of which have size $O(n^{1/3})$. By combining the sets of clones of size $O(n^{1/3})$ into one set, we get a partition of C' into $O(1)$ sets of clones and a remainder of size $O(n^{1/3} \log n)$. Note that, by definition, every vertex in C' has $\Omega(n)$ non-triangular neighbours in B. Now, throw all of the $O(1)$ vertices of $C \setminus C'$ into the remainder to get a partition of C that satisfies Property 3.

It remains to prove Property 2. Partition C into sets Z, C'' where $|Z| = O(n^{1/3} \log n)$ and C'' is a union of $O(1)$ sets of clones. Let Y be the set of vertices in B that do not have non-triangular neighbours in C'' and denote $B' = B \setminus Y$. First, we will show that B' can be partitioned into $O(1)$ independent sets. Indeed, B' is covered by the non-triangular neighbourhoods of vertices in C'', and each of them is an independent set. Moreover, C'' is a union of $O(1)$ sets of clones, and so there are $O(1)$ distinct such neighbourhoods. Second, we will prove that $|Y| = O(\sqrt{n} \log n)$.

Claim 6.37. $|Y| = O(\sqrt{n} \log n)$.

Proof. Recall that A is the set of triangular vertices in G. Since Y is disjoint from A, every vertex in Y has a non-triangular neighbour, and that neighbour must be in Z. That is, the non-triangular neighbourhoods of vertices in Z cover Y. Since Z is a union of $O(\log n)$ sets of clones, Y can be partitioned into $O(\log n)$ independent sets. In particular, Y contains an independent set I of size $\Omega(|Y|/\log n)$.

Let G' be the graph obtained from G by adding all possible edges spanned by $|I|$. Then $e(G') = e(G) + \binom{|I|}{2}$ and $t(G') \geq t(G) - |I||Z| \geq t(G) - O(|I| n^{1/3} \log n)$. This is a contradiction to Lemma 6.13 unless $\binom{|I|}{2} = O(n)$ or $\binom{|I|}{2} = O(|I| n^{1/3} \log n)$. In either case $|I| = O(\sqrt{n})$, and so $|Y| = O(\sqrt{n} \log n)$, as required. $\square$

The proof of Proposition 6.29 is now complete. We have already proved Properties 1 and 3 and that A, B, C are all of size $\Omega(n)$. To prove Property 2, recall that B is partitioned into a set Y of size $O(\sqrt{n} \log n)$ and a set B' which is a union of $O(1)$ independent sets. It follows from a similar argument as earlier that B' can be partitioned into $O(1)$ sets of clones and a remainder of size $O(n^{1/3} \log n)$. Assigning Y to this remainder gives the desired partition of B. $\square$

5.3 Sizes

In the previous subsection we proved that $V(G)$ can be partitioned into sets A, B, C that correspond to the three parts of the graph $G(|A|, |B|, |C|)$. In this subsection we consider the sizes of the sets A, B, C. We show that the number of edges (and non-triangular edges) of G is very close to the number of edges (and non-triangular edges) of $G(|A|, |B|, |C|)$.

Proposition 6.30. *Let A, B, C be as in Proposition 6.29 and denote $a = |A|, b = |B|, c = |C|$. Then $e(G) = a^2/2 + ab + bc + O(n^{7/4}\sqrt{\log n})$ and $t(G) = bc + O(n^{7/4}\sqrt{\log n})$.*

In the proof of this proposition we revisit Füredi and Maleki's [16] proof of Theorem 6.2 which is an approximate version of our main theorem. In their proof, Füredi and Maleki repeatedly apply Lemma 6.11, which eliminates one vertex at a time from any independent set of size 3. Here, we will do the same thing, but we will keep tight control on the independent sets to which we apply this lemma.

Proof of Proposition 6.30. Recall that by Proposition 6.29 both sets B and C can be partitioned into $O(1)$ sets of clones and a remainder of size $O(\sqrt{n}\log n)$. Let G' be the graph obtained by removing the edges incident with vertices in this remainder. Then $e(G') \geq e(G) - O(n^{3/2}\log n)$ and $t(G') \geq t(G) - O(n^{3/2}\log n)$.

The following claim is a variation of Lemma 6.10. It allows us to approximate G' by a weighted subgraph whose intersection with C induces a clique.

Claim 6.38. *There is a weighted subgraph H of G' such that $|H| = n$, $e(H) \geq e(G')$ and $t(H) \geq t(G')$, which has the following properties.*

- *At least two vertices in A are present in H. Moreover, with at most one exception, the vertices in A that are present in H have weight* 1.
- *All vertices in B are present in H and have weight* 1.
- *The vertices in C that are present in H induce a clique.*

Proof. We perform the following process to obtain the weighted graph H. Initially, we set H to be G' with every vertex having weight 1. Then we perform multiple steps, during which we modify the weights of the vertices in $A \cup C$ (and remove some of these vertices) so that, at any given time, A has at most one vertex with weight not equal to 1. At each step we select vertices $u \in A$ and $v, w \in C$. We take u to be the unique vertex in A of weight not equal to 1, and if there is no such vertex, then we take it to be an arbitrary vertex remaining in A. We take v and w to be any pair of non-adjacent (in G') vertices in C. If choosing u, v, w according to these rules is impossible, then we terminate the process.

Suppose that we successfully selected the vertices u, v, w. They form an independent set, and so by Lemma 6.11 it is possible to remove one or two of these vertices and redistribute their weight on the remaining ones so that the new weights are positive, the total weight does not change and $e(H), t(H)$ do not decrease.

It is clear that this process terminates, because each step decreases the number of vertices remaining in H. Let us consider the resulting weighted graph H. Since the process terminated, either no vertices of A are present in H, or the remaining vertices of C induce a clique. We show that, in fact, at least two vertices remain in A, and so the latter condition must hold.

Suppose that fewer than two vertices in A remain in H. Denote by m the size of the largest clique that can be formed from vertices remaining in H. Since the vertex set of G' can be partitioned into A and $O(1)$ independent sets, we have $m = O(1)$. Apply Lemma 6.10 to obtain a good weighted subgraph F of H, with xy being its only non-triangular edge, such that $|F| = n$, $e(F) \geq e(G')$ and $t(F) \geq t(G')$. Let β and γ be the weights of x and y in F and suppose that $\beta \geq \gamma$. Then $\alpha = n - \beta - \gamma$ is the sum of the weights of the other vertices in F. We have $t(F) = \beta\gamma$ and, as in Inequality (6.9) from Claim 6.32, $e(F) \leq (1 - 1/m)\alpha^2/2 + \alpha\beta + \beta\gamma$. It follows that $t(G) \leq \beta\gamma + O(n^{3/2} \log n)$ and $e(G) \leq \alpha^2/2 + \alpha\beta + \beta\gamma - \Omega(n^2)$. Consider the graph $G'' = G(n - \lceil\beta\rceil - \lceil\gamma\rceil, \lceil\beta\rceil, \lceil\gamma\rceil)$. It is easy to check that $t(G'') \geq t(G) - O(n^{3/2} \log n)$ and $e(G'') \geq e(G) + \Omega(n^2)$. This is a contradiction to Lemma 6.13, since G is optimal. Therefore, at least two vertices in A are present in H.

It follows that the set of vertices in C that are present in H induces a clique. Hence, the weighted graph H satisfies the requirements of Claim 6.38. $\square$

Let H be a weighted graph as given by Claim 6.38, so in particular, $e(H) \geq e(G) - O(n^{3/2} \log n)$ and $t(H) \geq t(G) - O(n^{3/2} \log n)$. By Lemma 6.13, since G is optimal,

$$\begin{aligned} e(H) &= e(G) + O\big(n^{3/2} \log n\big), \\ t(H) &= t(G) + O\big(n^{3/2} \log n\big). \end{aligned} \tag{6.10}$$

We remark that these two lines express both upper and lower bounds for the quantities $e(H)$ and $t(H)$. In the following claim we prove that, in fact, only one vertex of C is present in H.

Claim 6.39. *Exactly one vertex of C is present in H. Moreover, all but at most $O(n^{3/4}\sqrt{\log n})$ vertices in B are non-triangular neighbours of that vertex.*

Proof. Write $u_1, \ldots, u_m$ for the vertices of C that appear in H, and let $N_1, \ldots, N_m$ be their non-triangular neighbourhoods in B. Since the set $\{u_1, \ldots, u_m\}$ forms a clique, there are no edges between u_i and N_j for $i \neq j$. In particular, the sets $N_1, \ldots, N_m$ are pairwise disjoint.

Let $Z = B \setminus (N_1 \cup \cdots \cup N_m)$. Since the intersection of H with A induces a clique on at least two vertices and since all edges between B and the intersection of H with A are present in H, the vertices in Z are not incident with any non-triangular edges in H. We will show that $|Z| = O(n^{3/4}\sqrt{\log n})$. Indeed, recall that B is the union of $O(1)$ independent sets and a remainder of size at most $O(\sqrt{n}\log n)$. Thus, provided that $|Z| \geq C\sqrt{n}\log n$ for a sufficiently large constant C, there exists an independent set $I \subseteq Z$ of size $\Omega(|Z|)$. Consider the weighted graph H' obtained from H by adding the edges spanned by I. Then $e(H') = e(H) + \Omega(|Z|^2) \geq e(G) - O(n^{3/2}\log n) + \Omega(|Z|^2)$ and $t(H') = t(H) \geq t(G) - O(n^{3/2}\log n)$. It follows from Lemma 6.13 that $|Z| = O(n^{3/4}\sqrt{\log n})$.

Our aim is to prove that $m = 1$. We assume for contradiction that $m \geq 2$. In particular, since by Proposition 6.29 and the definition of G', in G' every vertex in C is either isolated or has $\Omega(n)$ non-triangular neighbours in B, the vertices $u_1, \ldots, u_m$ have the latter property. In other words, $|N_i| = \Omega(n)$ for every i.

For each i, let γ_i denote the weight of u_i in H. We will show that $\gamma_i = \Omega(n)$ for every i. Indeed, fix any i. Denote by H_i the weighted graph obtained from H by adding all edges spanned by N_i. Since N_i is an independent set in H, we have $e(H_i) \geq e(G) + \Omega(n^2)$ and $t(H_i) \geq t(G) - |N_i|\gamma_i - O(n^{3/2}\log n)$. By Lemma 6.13, $\gamma_i = \Omega(n)$.

Write $\beta_i = |N_i|$. Construct a weighted graph F, starting from H and carrying out the following steps. Firstly, remove all edges with an end in Z. Secondly, replace each set N_i by a vertex v_i of weight β_i. Finally, connect each vertex v_i to all of the vertices in A (that are present in H) as well as to u_i and v_j for every $j \neq i$ (see Figure 6.8). We have $e(F) \geq e(H) - |Z|n \geq e(G) - O(n^{7/4}\sqrt{\log n})$ and $t(F) \geq t(H) \geq t(G) - O(n^{3/2}\log n)$.

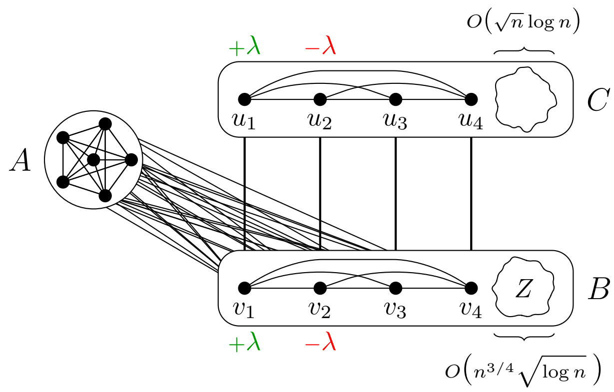

Figure 6.8: The graph F.

Pick any real λ such that $|\lambda| \leq \min\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$. Let F_λ be the weighted graph obtained from F by adding λ to the weights of u_1 and v_1 and subtracting λ from the weights of u_2 and v_2. Clearly, $|F_\lambda| = |F| = n$ and it is easy to check that $e(F_\lambda) = e(F)$.

If $m \geq 3$, then the only non-triangular edges in F are $u_i v_i$. Hence, in this case,

$$\begin{aligned} t(F_\lambda) &= t(F) - (\beta_1\gamma_1 + \beta_2\gamma_2) + (\beta_1 + \lambda)(\gamma_1 + \lambda) + (\beta_2 - \lambda)(\gamma_2 - \lambda) \\ &= t(F) + (\beta_1 + \gamma_1 - \beta_2 - \gamma_2)\lambda + 2\lambda^2. \end{aligned}$$

If $\beta_1 + \gamma_1 \geq \beta_2 + \gamma_2$, then take $\lambda = \min\{\beta_1, \gamma_1, \beta_2, \gamma_2\}$. Otherwise, take $\lambda = -\min\{\beta_1, \gamma_1, \beta_2, \gamma_2\}$. In either case, $|\lambda| = \Omega(n)$ and $t(F_\lambda) \geq t(F) + \Omega(n^2) \geq t(G) + \Omega(n^2)$, contradicting Lemma 6.13.

This calculation is slightly different in the case when $m = 2$, because then we have to account for the edge u_1u_2, which is also non-triangular. In this case

$$\begin{aligned} t(F_\lambda) = t(F) - (\beta_1\gamma_1 + \beta_2\gamma_2 + \gamma_1\gamma_2) + (\beta_1 + \lambda)(\gamma_1 + \lambda) + (\beta_2 - \lambda)(\gamma_2 - \lambda) \\ + (\gamma_1 + \lambda)(\gamma_2 - \lambda) \\ = t(F) + (\beta_1 - \beta_2)\lambda + \lambda^2. \end{aligned}$$

We may reach a contradiction to Lemma 6.13 by choosing λ of the same sign as $\beta_1 - \beta_2$ and with $|\lambda| = \min\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$. We conclude that $m = 1$, completing the proof of the claim. $\square$

Recall that, among the vertices in A that are present in H, at most one has weight not equal to 1. In the following claim we show that this weight cannot be very large.

Claim 6.40. *The weight in H of any vertex in A is $O(n^{3/4}\sqrt{\log n})$.*

Proof. Let u be a vertex of A of maximal weight in H, and let ω be its weight. Suppose that $\omega > 1$, in which case all other vertices in A have weight 1 in H.

Replace the vertex u by a clique of size $\lfloor \omega \rfloor$ whose vertices have weight $\omega / \lfloor \omega \rfloor$ and are adjacent to all of $(A \setminus \{u\}) \cup B$ and denote the resulting weighted graph by H'. We have to check the technical condition that the average weight of a vertex in H' is at least 1. However, this can be easily verified, since the total weight of H' is an integer and H' has at most one vertex whose weight is smaller than 1 (namely, the only vertex of C that remains in H').

By replacing u with a clique, we create new edges inside the clique, and these edges contribute $\binom{\lfloor \omega \rfloor}{2}(\omega/\lfloor \omega \rfloor)^2 = \Omega(\omega^2)$ towards $e(H')$. Therefore, we have $t(H') = t(H) \geq t(G) - O(n^{3/2}\log n)$ and $e(H') = e(H) + \Omega(\omega^2) \geq e(G) - O(n^{3/2}\log n) + \Omega(\omega^2)$. It follows from Lemma 6.13 that $\omega = O(n^{3/4}\sqrt{\log n})$. $\square$

Recall that a, b, c are the sizes of the sets A, B, C in the original graph G. Let α, β, γ be the sums of weights (in H) of the vertices in these sets, summing over vertices present in H. So, for example, $\beta = b$ and γ is the weight of the single vertex in C that is present in H. Clearly, $\alpha + \gamma = a + c$, because both sides are equal to $n - b$. Now, we use the properties of H that we have proved to get good bounds on $e(H)$ and $t(H)$ in terms of α, β, γ.

Recall that the set A induces a clique in G, so its remainder induces a clique in H. Combined with Claim 6.40, this implies that the contribution of the edges within A to $e(H)$ is $\alpha^2/2 - O(n^{3/2} \log n)$. By Claim 6.39, the set B contains an independent set of size at least $|B| - O(n^{3/4}\sqrt{\log n})$. Therefore, the contribution of the edges within B to $e(H)$ (and in particular to $t(H)$) is $O(n^{7/4}\sqrt{\log n})$. Moreover, Claim 6.39 implies that the edges between B and C contribute $\beta\gamma - O(n^{7/4}\sqrt{\log n})$ to both $e(H)$ and $t(H)$. Putting this together, we get

$$\begin{aligned} e(H) &= \alpha^2/2 + \alpha\beta + \beta\gamma + O\big(n^{7/4}\sqrt{\log n}\big) \\ t(H) &= \beta\gamma + O\big(n^{7/4}\sqrt{\log n}\big). \end{aligned} \tag{6.11}$$

Again, we remark that these are both upper and lower bounds for the quantities $e(H)$ and $t(H)$. We deduce that α almost equals a and γ almost equals c.

Claim 6.41. *$\alpha = a + O(n^{3/4}\sqrt{\log n})$ and $\gamma = c + O(n^{3/4}\sqrt{\log n})$.*

Proof. We can read the inequality $\alpha \leq a + O(n^{3/4}\sqrt{\log n})$ off Claim 6.40. To get the corresponding lower bound on α, we consider the quantity $e(H) - t(H)$. On one hand, the inequalities in (6.11) give

$$e(H) - t(H) = \alpha^2/2 + \alpha\beta + O\big(n^{7/4}\sqrt{\log n}\big).$$

On the other hand, we can use the inequalities in (6.10) to get

$$\begin{aligned} e(H) - t(H) &= e(G) - t(G) + O(n^{3/2} \log n) \\ &\geq a^2/2 + ab + O(n^{3/2} \log n), \end{aligned}$$

where the latter inequality comes from the fact that the quantity $e(G) - t(G)$ counts the triangular edges in G, and all vertices in A are triangular. Recall that $b = \beta$. Combining the two inequalities for $e(H) - t(H)$ we get $\alpha \geq a - O(n^{3/4}\sqrt{\log n})$, so $\alpha = a + O(n^{3/4}\sqrt{\log n})$. To complete the proof of the claim, note that $a + c = \alpha + \gamma$. $\square$

Proposition 6.30 follows from the last claim and the bounds given by (6.10) and (6.11). □

5.4 End of the proof

We are now ready to complete the proof of Theorem 6.5.

Proposition 6.31. *$G \cong G(a, b, c)$ for some a, b, c.*

We gradually get closer to proving that $G \cong G(a, b, c)$. We start by showing that b is much bigger than c, which leads to the conclusion that C spans no non-triangular edges. This implies that almost all possible edges between B and C are present in G and are non-triangular, by Proposition 6.30. In fact, using the fact that G is compressed, we deduce that there are large subsets of B and of C that span a complete bipartite graph consisting of non-triangular edges. With some more effort, using the optimality of G, we conclude that B and C themselves induce a complete bipartite graph, thus completing the proof.

Proof of Proposition 6.31. Let $\{A, B, C\}$ be the partition of $V(G)$ given by Proposition 6.29. As in the statement of Proposition 6.30, write $a = |A|, b = |B|, c = |C|$. We start by showing that b is significantly larger than c.

Claim 6.42. *We have $b \geq c + \Omega(n)$.*

Proof. Suppose to the contrary that $b \leq c + o(n)$. Then we have $n = a + b + c \geq 2b + a - o(n)$, and hence $b \leq (n - a + o(n))/2$. Since $a = \Omega(n)$, we can conclude that $b \leq n/2 - \Omega(n)$.

Consider the graph $H = G(a, b, c)$. Proposition 6.30 implies that $e(H) = e(G) + O(n^{7/4}\sqrt{\log n})$ and $t(H) = t(G) + O(n^{7/4}\sqrt{\log n})$. Consider also the graph $H' = G(a, b+d, c-d)$, where $d = n^{1.999}$. Note that $b+d \leq n/2 - \Omega(n)$. Therefore, from the expression $e(H) = \binom{a}{2} + (n - b)b$ and the corresponding expression for $e(H')$, we can see that $e(H') \geq e(H) + \Omega(dn) = e(G) + \Omega(dn)$. Similarly, $t(H') = (b + d)(c - d) \geq bc - o(dn)$, and so $t(H') \geq t(H) - o(dn) = t(G) - o(dn)$. However, this contradicts Lemma 6.13. Therefore, $b \geq c + \Omega(n)$. □

In the following claim we conclude that C spans no non-triangular edges.

Claim 6.43. *There are no non-triangular edges with both ends in C.*

Proof. By Proposition 6.30 there are $bc + o(n^2)$ non-triangular edges in G, and each one of them is incident with a vertex in C. Therefore, some vertex in C has at least

$b - o(n)$ non-triangular neighbours. Thus, by Observation 6.8, every vertex in C has degree at least $b - o(n)$, and so the sum of the degrees of any two vertices in C is at least $2b - o(n) > b + c = |B \cup C|$, where the latter inequality comes from the previous claim. Since the vertices in C have neighbours only in $B \cup C$, it follows that any pair of vertices in C have a common neighbour, and hence they cannot be joined by a non-triangular edge. $\square$

Recall that by Proposition 6.29 both sets B and C can be partitioned into $O(1)$ sets of clones and a remainder of size $O(\sqrt{n}\log n)$. In such a partition of B consider the sets of clones of size at least $n^{9/10}$ and let B' be their union. Similarly, let C' be the union of the sets of clones in the partition of C that have size at least $n^{9/10}$ and denote $Z = C \setminus C'$ and $Y = B \setminus B'$. Then $|Y| = O(n^{9/10})$ and $|Z| = O(n^{9/10})$. We show that all possible edges between B' and C' are present in G and are non-triangular.

Claim 6.44. *All possible edges between B' and C' are present in G and are non-triangular. In particular, B' and C' are independent sets.*

Proof. From the previous claim we know that every non-triangular edge in G has one end in B and one end in C. Suppose that there exists a pair of vertices, one in B' and one in C', that are not joined by a non-triangular edge. Then there are two sets of clones of size at least $n^{9/10}$, one contained in B' and the other in C', between which there are no non-triangular edges. But then $t(G) \le bc - n^{9/5}$, contradicting Proposition 6.30. $\square$

In the following claim we obtain additional information about Y and Z, which brings us closer to showing that B and C induce a complete bipartite graph.

Claim 6.45. *There are no edges between B' and Y and between C' and Z. Moreover, every vertex in B has at least $c - o(n)$ neighbours in C, and every vertex in C has at least $b - o(n)$ neighbours in B.*

Proof. Any vertex in C' has at least $|B'| = b - o(n)$ non-triangular neighbours, so Observation 6.8 implies that every vertex in G has degree at least $b - o(n)$. Claim 6.42 implies that $|C| + |B \setminus B'|$, which does not exceed $c + o(n)$, is smaller than this quantity, and hence every vertex in C has a neighbour in B'. Therefore, because all possible edges between B' and C' are present in G and are non-triangular (by Claim 6.44), there are no edges between C' and Z. In particular, every vertex in C has at most $o(n)$ neighbours in C, so it has at least $b - o(n)$ neighbours in B.

Pick any vertex $u \in Y$. Since u is not in A, u has a non-triangular neighbour $v \in C$. We have just proved that v has at least $b - o(n)$ neighbours in B'. Therefore, u has at most $o(n)$ neighbours in B. Now suppose that u is adjacent to a vertex in B'. Then u has no neighbours in C'. Hence, u has at most $a + o(n)$ neighbours, of which at most $o(n)$ are non-triangular. However, any vertex of B' has at least $a + c - o(n)$ neighbours and at least $c - o(n)$ of them are non-triangular. This contradicts Observation 6.8. Therefore, u has no neighbours in B'.

It remains to verify that every vertex in Y has at least $c - o(n)$ neighbours in C. Let us again consider $u \in Y$ and denote by d the number of its neighbours in C. Then the degree of u is at most $a + d + o(n)$ and its non-triangular degree is at most d. By Observation 6.8, applied to u and any vertex in B', we know that $a + d \geq a + c - o(n)$ or $d \geq c - o(n)$. In either case $d \geq c - o(n)$. □

In the following claim we prove that no edges are spanned by Z. We use a trick that we have used several times before, replacing a pair of adjacent vertices in Z by copies of vertices in A and B', increasing the number of both edges and non-triangular edges.

Claim 6.46. *The set Z is independent.*

Proof. Suppose to the contrary that Z contains a pair of adjacent vertices u, v. Then the non-triangular neighbours of u are all in B and they are not adjacent to v. By Claim 6.45, v has $b - o(n)$ neighbours in B, and hence u has at most $o(n)$ non-triangular neighbours. Likewise, v has at most $o(n)$ non-triangular neighbours.

Consider the graph G' obtained from G by removing u, v and adding new vertices x and y, where x is joined by edges to all vertices in $A \cup B$, and y is joined to all vertices in B'. We have $e(G') \geq e(G) + a - o(n)$ and $t(G') \geq t(G) + b - o(n)$. This contradicts the optimality of G, because $a = \Omega(n)$ and $b = \Omega(n)$. □

A similar trick enables us to conclude that Y spans no edges. Here we replace two adjacent vertices in Y by copies of vertices in A and B'.

Claim 6.47. *The set Y is independent.*

Proof. Suppose that there exists a pair of adjacent vertices $u, v \in Y$. Let G' be the graph obtained from G by removing u, v and adding new vertices x, y with x joined to all of $A \cup B'$ and y joined to all of $A \cup C'$.

Let us compare $e(G')$ and $t(G')$ with $e(G)$ and $t(G)$. By Claim 6.44, there are no edges in G between $\{u, v\}$ and B'. Therefore, the removal of u, v removes at most $2(a+c+o(n))$ edges. On the other hand, the addition of x, y creates $2a+b+c-o(n)$

new edges. Therefore, $e(G') \geq e(G) + b - c - o(n) > e(G)$. Furthermore, since u, v have at least $c - o(n)$ neighbours in C each, there are at most $o(n)$ vertices in C that are adjacent to precisely one of u, v. As a result, u, v are incident with at most $o(n)$ non-triangular edges in G. Since the addition of x, y creates $c - o(n)$ new non-triangular edges, we have $t(G') \geq t(G) + c - o(n) > t(G)$. This contradicts the optimality of G, because $c = \Omega(n)$. □

Proposition 6.31 easily follows from Claim 6.44 to 6.47. Indeed, these claims together imply that B and C are independent sets in G. Therefore, if there were any missing edges between B and C, we could add them to G without creating new triangles. Since G is an optimal graph, all possible edges between B and C are present. It follows that $G \cong G(|A|, |B|, |C|)$. □

6 Almost complete

In this section we prove Theorem 6.6.

Theorem 6.6. *There exist n_0 and $\delta > 0$ such that the following holds. Let G be a graph with $n \geq n_0$ vertices and e edges, where $e \geq (1/2 - \delta)n^2$. Then there exists a graph $H = G(a, b, c)$ such that $|H| = n$, $e(H) \geq e$ and $t(H) \geq t(G)$.*

The proof in this range is easier than in the other two ranges, though far from immediate. We start by making the usual assumption that G is an optimal and compressed graph, even though we do not use the full strength of the latter assumption: we only need Condition 2 from Definition 6.14.

If very few (namely, $2n - 8$ or fewer) edges are missing from G, then we directly prove that $G \cong G(a, b, c)$ for some a, b, c. For the remaining range, we partition the vertices of G, according to their degrees, into sets A, B, C with the aim of showing that $G \cong G(|A|, |B|, |C|)$. We first prove that the sets have the correct orders of magnitude using a rough lower bound on $t(G)$. We are then able to prove better estimates for the sizes of the sets, and, finally, we deduce that G has the required structure.

Proof of Theorem 6.6. Fix a sufficiently small constant $\delta > 0$ (whose value can be determined from the proof) and let G be an optimal and compressed graph with n vertices and $\binom{n}{2} - \varepsilon n^2$ edges, where $0 \leq \varepsilon \leq \delta$. We first consider the case $e(G) \geq \binom{n}{2} - (2n - 9)$.

Claim 6.48. *If $e(G) \geq \binom{n}{2} - (2n - 9)$, then $G \cong G(a, b, c)$ for some a, b, c.*

Proof. If G has no non-triangular edges, then it is a clique by optimality, so we are done. We claim that G does not have two independent non-triangular edges. Indeed, if uv and xy are such edges, then for any other vertex w one of the two possible edges uw and vw is missing, as well as one of xw and yw. Therefore, G has at least $2n-8$ missing edges, contradicting our assumption. Therefore, since the triangle-free edges cannot form a triangle, they form a star. Let $uv_1, \dots, uv_k$ be the non-triangular edges. Then the set $A = V(G) \setminus \{u, v_1, \dots, v_k\}$ is the set of triangular vertices in G, so A induces a clique and all of the vertices in A have the same neighbourhood in $V(G) \setminus A$. Now, there are two possibilities: either u is adjacent to all of A, or u is not adjacent to any vertex in A. In the former case, there are no edges between A and $\{u_1, \dots, u_k\}$, and so $G \cong G(|A|, 1, k)$. In the latter case, optimality of G implies that all possible edges between A and $\{u_1, \dots, u_k\}$ are present in G, and so $G \cong G(|A|, k, 1)$. $\square$

From this point onwards we assume that $e(G) \leq \binom{n}{2} - (2n-8)$. In particular, $\varepsilon \geq (2 - o(1))/n$. We wish to prove that G is isomorphic to the graph $G(a, b, c)$ for some parameters a, b, c. To get some idea on how large these parameters should be, we observe that $a \approx n$, because the number of missing edges is small. Now, the number of missing edges, $ac + \binom{b}{2} + \binom{c}{2}$, can be reasonably approximated by $cn + b^2/2$. Subject to b, c being non-negative reals such that $cn + b^2/2 \geq \varepsilon n$, the quantity bc is maximised when $b = \sqrt{2\varepsilon/3}\,n$, $c = (2\varepsilon/3)n$. Therefore, we expect G to be isomorphic to $G(a, b, c)$ with $b \approx \sqrt{2\varepsilon/3}\,n$ and $c \approx (2\varepsilon/3)n$. We can use this conclusion to get a lower bound on $t(G)$.

Claim 6.49. $t(G) = \Omega(\varepsilon^{3/2} n^2)$.

Proof. Let $G' = G(a, b, c)$, where $b = \lfloor \sqrt{2\varepsilon/3}\,n \rfloor$, $c = \lfloor (2\varepsilon/3)n \rfloor$ and $a = n - b - c$. There at most $cn + b^2/2 \leq \varepsilon n$ edges missing from G', so $t(G') \geq t(G)$. We now find a lower bound for $t(G')$ by a simple computation, but we have to be careful with rounding errors.

We have $\varepsilon n^2 \geq 2n - 9$, implying that $(2\varepsilon/3)n > 1$, and hence $c = \lfloor (2\varepsilon/3)n \rfloor = \Omega(\varepsilon n)$. Similarly, $b = \Omega(\sqrt{\varepsilon}\,n)$. It follows that $t(G') = bc = \Omega(\varepsilon^{3/2} n^2)$. Since G is optimal, we have $t(G) \geq t(G') = \Omega(\varepsilon^{3/2} n^2)$. $\square$

We now define three sets $A, B, C \subseteq V(G)$ that correspond to the three parts of a graph $G(a, b, c)$. Let C be the set of vertices of degree at most $3n/4$, let B be the set of vertices in $V(G) \setminus C$ that have a non-triangular neighbour in C, and let $A = V(G) \setminus (B \cup C)$. Since any two vertices in $A \cup B$ have at least $n/2$ common neighbours, there are no non-triangular edges with both ends in $A \cup B$. Therefore, all vertices in A are triangular, so A induces a clique and its vertices have the same neighbourhood in $V(G) \setminus A$.

The next step is to obtain tight bounds for the sizes of A, B, C. First, we determine the order of magnitude of $|B|$ and $|C|$.

Claim 6.50. *$|B| = \Theta(\sqrt{\varepsilon}n)$ and $|C| = \Theta(\varepsilon n)$. Moreover, every vertex of B is an end of $\Omega(\sqrt{\varepsilon}n)$ missing edges.*

Proof. By definition, every vertex in C is an end of at least $n/4$ non-edges. Since there are εn^2 non-edges in total, we have $|C| = O(\varepsilon n)$. We know from the previous claim that there are at least $\Omega(\varepsilon^{3/2}n^2)$ non-triangular edges. All of these edges have at least one end in C, and so some vertex in C has at least $\Omega(\sqrt{\varepsilon}n)$ non-triangular neighbours. Therefore, by Observation 6.8, every vertex in G has at least $\Omega(\sqrt{\varepsilon}n)$ neighbours.

Pick any $v \in B$. By the definition of B, v has a non-triangular neighbour $u \in C$. This means that v is not adjacent to any neighbours of u, and so v is an end of at least $\Omega(\sqrt{\varepsilon}n)$ non-edges. Therefore, $|B| = O(\sqrt{\varepsilon}n)$. Moreover, since every non-triangular edge has both ends in C, or one in B and one in C, we have $|B||C|+|C|^2/2 \geq \Omega(\varepsilon^{3/2}n^2)$, which implies that $|B| = \Omega(\sqrt{\varepsilon}n)$ and $|C| = \Omega(\varepsilon n)$. $\square$

An immediate consequence of the previous claim is that $|A| = (1 - O(\sqrt{\varepsilon}))n$. Recall that all vertices in A have the same neighbourhood in $V(G)\setminus A$. In particular, each vertex in $B \cup C$ is adjacent either to all vertices in A or to none of them. Since the vertices in B have degree at least $3n/4$, they are all adjacent to all of A, and, similarly, there are no edges between A and C. We can use this fact to give a better upper bound on $|C|$.

Claim 6.51. *There exists a constant $\xi > 0$ such that $|C| \leq (1 - \xi)\varepsilon n$.*

Proof. Every vertex in B is an end of $\Omega(\sqrt{\varepsilon}n)$ missing edges and $|B| = \Theta(\sqrt{\varepsilon}n)$, so there are $\Omega(\varepsilon n^2)$ missing edges with an end in B. Since all edges between A and C are missing, we have $(1 - O(\sqrt{\varepsilon}))n|C| + \Omega(\varepsilon n^2) \leq \varepsilon n^2$. Therefore, $|C| \leq (1 - \Omega(1))\varepsilon n/(1 - O(\sqrt{\varepsilon}))$, and the claim follows provided that ε is sufficiently small. $\square$

It is now possible to accurately relate the sizes of B and C. Write $|C| = \gamma\varepsilon n$, where $\Omega(1) = \gamma \leq 1 - \xi$. Define $\beta = \sqrt{2(1-\gamma)}$ and note that $\beta = \Theta(1)$.

Claim 6.52. *$|B| = \beta\sqrt{\varepsilon}n + O(\varepsilon n)$. Moreover, there are at least $|B||C| - O(\varepsilon^2 n^2)$ non-triangular edges between B and C.*

Proof. Let G' be the graph $G(a,b,c)$, where $c = |C|$, $b = \lfloor \beta\sqrt{\varepsilon} n \rfloor$ and $a = n-b-c$. It is easy to see that $b^2/2+cn \le \varepsilon n^2$. In particular, we have $e(G') \ge \binom{n}{2} - \varepsilon n^2 = e(G)$. Therefore, since G is optimal, $t(G) \ge t(G') = bc$.

Let us come back to the graph G. Since every non-triangular edge has an end in C, some vertex in C has at least b non-triangular neighbours. It follows from Observation 6.8 that every vertex in G has degree at least $b-1$. Moreover, since vertices in C are adjacent only to vertices in $B \cup C$, we have $|B| \ge b - c - 1 = b - O(\varepsilon n)$.

Every vertex in B has a non-triangular neighbour, and therefore is an end of at least $b-1$ missing edges. Hence, there are at least $|B|(b-1)/2$ missing edges with an end in B. Since there are no edges between A and C, we have

$$\frac{1}{2}|B|(b-1) + \big(1 - O(\sqrt{\varepsilon})\big)\, cn \;\le\; \varepsilon n^2 \;\le\; \frac{1}{2}b^2 + cn + O(\sqrt{\varepsilon} n),$$

where the latter inequality follows from the definition of b. It follows that $|B|b \le b^2 + O(\sqrt{\varepsilon} cn)$, and hence $|B| \le b + O(\varepsilon n)$. To finish the proof, observe that $t(G) \ge bc = |B||C| - O(\varepsilon^2 n^2)$ and recall that the non-triangular edges of G are either spanned by C (there are $O(\varepsilon^2 n^2)$ such edges) or they have one end in B and the other in C. $\square$

A standard trick of replacing two vertices by copies of other vertices, which we have been using throughout the chapter, allows us to conclude that C is an independent set.

Claim 6.53. *The set C is independent. Moreover, every vertex in C is adjacent to all but at most $O(\varepsilon n)$ vertices in B.*

Proof. The second conclusion of Claim 6.52 implies that some vertex in C has at least $|B| - O(\varepsilon n)$ non-triangular neighbours in B. As a consequence, B contains an independent set I of size $|B| - O(\varepsilon n)$. Moreover, Observation 6.8 implies that every vertex in C is adjacent to all but at most $O(\varepsilon n)$ vertices in $B \cup C$.

Suppose that C contains a pair of adjacent vertices u, v. Let G' be the graph obtained from G by removing the vertices u and v and adding new vertices x and y where x is adjacent to all of $A \cup B$, and y is adjacent to all of I. The removal of u and v decreases the total number of edges by at most $2(|B| + |C|) = O(\sqrt{\varepsilon} n)$, while the addition of x and y increases this number by at least $|A| = (1 - O(\sqrt{\varepsilon}))n$. Therefore, $e(G') > e(G)$. Moreover, since u and v are adjacent, they do not form non-triangular edges with their common neighbours. Hence, u and v have at most $O(\varepsilon n)$ non-triangular neighbours in total. On the other hand, the addition of x and

y adds $|I| = \Omega(\sqrt{\varepsilon}n)$ non-triangular edges. Therefore, $t(G') > t(G)$, a contradiction to the optimality of G. $\square$

Finally, we prove that B is an independent set.

Claim 6.54. *The set B is independent.*

Proof. Suppose that $u, v \in B$ are adjacent. There are at most $|C|$ non-triangular edges with an end in $\{u, v\}$, because every vertex can only be a non-triangular neighbour of at most one of u and v. Moreover, by definition, every vertex in B has a non-triangular neighbour. Let $w \in C$ be a non-triangular neighbour of u. Since the edge uw is non-triangular, it follows that u is not adjacent to any of the neighbours of w. By Claim 6.53, w is adjacent to all but at most $O(\varepsilon n)$ vertices in B. Therefore, u has at most $O(\varepsilon n)$ neighbours in B and, likewise, so does v.

Let G' be the graph obtained by replacing u and v with new vertices x and y where x is adjacent to all of $A \cup C$ and y is adjacent to all of $(A \cup B) \setminus \{u, v\}$. We have $t(G') \geq t(G)$ and $e(G') \geq e(G) + |B| - 2 - 2|C| - O(\varepsilon n) > e(G)$, contradicting the optimality of G. Therefore, $\square$

We have proved that B and C are independent, A is complete, and its vertices are adjacent to all of B and none of C. We may add any missing edges between B and C without creating new triangles, so by the optimality of G, there are in fact no missing edges between B and C. Therefore, G is isomorphic to $G(|A|, |B|, |C|)$, completing the proof of Theorem 6.6. $\square$

7 Concluding remarks

We note that we have not fully resolved Conjecture 6.1.

Conjecture 6.1 (Füredi and Maleki [16])**.** *Let n and $e > \lfloor n^2/4 \rfloor$ be integers and let G be an n-vertex graph with e edges that minimises the number of triangular edges. Then G is isomorphic to a subgraph of a graph $G(a, b, c)$ for some a, b, c.*

Theorem 6.3 shows that the minimum number of triangular edges among n-vertex graphs with e is attained by (a subgraph of) a graph $G(a, b, c)$. However, we have not shown that such graphs are the only minimisers. Nevertheless, we believe that this fact can be proved (for sufficiently large n) by retracing our proofs. In any case, we are only able to prove the conjecture for sufficiently large n, and it would be interesting to extend our result to work for all n.

We have not specified explicitly how large n should be in order for our proof to work, mainly because, due to the complexity of the proof, it is quite hard to find such an explicit bound. Nevertheless, we expect this bound to be 'reasonably small': say, much smaller than a bound that may arise from the use of the regularity lemma, because the inequalities we need to hold are polynomial in n.

The following question arises from Conjecture 6.1, by considering edges on K_r for $r \geq 4$. To simplify the notation, for any graph H and an edge e of some other graph G, we say that e is an *H-edge* if it is contained in a subgraph of G isomorphic to H.

Question 6.55. *What is the smallest number of K_r-edges that a graph with n vertices and e edges may have? Which graphs with n vertices and e edges minimise this quantity?*

It seems reasonable to believe that the extremal examples are analogues of graphs $G(a, b, c)$, namely, they may be formed by adding a clique to one of the parts of a complete $(r-1)$-partite graph with n vertices.

There is another natural generalisation, where we consider odd cycles instead of cliques.

Question 6.56. *What is the smallest number of C_{2k+1}-edges that a graph with n vertices and e edges may have? Which graphs minimise this quantity?*

It turns out that the case $k \geq 2$ is quite different from $k = 1$ (that is, where the odd cycle is a triangle). Erdős, Faudree and Rousseau [13] proved that, for any fixed $k \geq 2$, any graph with n vertices and $\lfloor n^2/4 \rfloor + 1$ edges has at least $11n^2/144 + O(n)$ C_{2k+1}-edges. In contrast, the number of triangular edges can be as small as $2\lfloor n/2 \rfloor + 1$, as mentioned in the introduction. So, the jump in the number of C_{2k+1}-edges (for $k \geq 2$) is very sharp, while the jump in the number of triangular edges is much smoother.

In the same paper, Erdős, Faudree and Rousseau conjectured a stronger statement: they conjectured that, for any fixed $k = 2$, any graph with n vertices and $\lfloor n^2/4 \rfloor + 1$ edges has at least $2n^2/9 + O(n)$ C_{2k+1}-edges. This bound can be attained by (a subgraph of) the union of a complete graph on roughly $2n/3$ vertices and a balanced complete bipartite graph on the remaining vertices. However, an example by Füredi and Maleki [16] shows that the conjecture is false: they constructed n-vertex graphs with $\lfloor n^2/4 \rfloor + 1$ edges and $(0.213 \ldots + o(1))n^2$ C_5-edges. The example is somewhat similar to a graph $G(a, b, c)$: here we have four sets A, B, C, D such that A induces a clique and all possible $A - B$, $B - C$ and $C - D$ edges are present. The

$C - D$ edges are not C_5-edges, but all other edges are. The aforementioned bound is obtained by optimising the sizes of A, B, C, D. Füredi and Maleki also calculated, asymptotically, the minimum possible number of C_{2k+1}-edges (for $k \geq 2$) in n-vertex graphs with e edges, where $e = \gamma n^2$ for any fixed constant $1/4 < \gamma < 1/2$. Their findings provided supporting evidence that the conjecture of Erdős, Faudree and Rousseau should be true for $k \geq 3$.

Very recently, more progress on Question 6.56 was made by Grzesik, Hu and Volec [31]. For any fixed $k \geq 2$, they obtained asymptotically sharp bounds for the smallest possible number of C_{2k+1}-edges in a graph with n vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges, using the method of flag algebras. In particular, they almost confirmed the conjecture of Erdős, Faudree and Rousseau for $k \geq 3$ (with an error term of $o(n^2)$ instead of $O(n)$) and proved that the construction of Füredi and Maleki is asymptotically best for $k = 2$.

We believe that the method of Grzesik, Hu and Volec should be sufficient to give the exact smallest number of C_{2k+1}-edges in a graph with n vertices and e edges, for any fixed $k \geq 2$, provided that n is sufficiently large. Furthermore, their stability result should be sufficient to establish that, for sufficiently large n, the construction described earlier is the unique extremal construction. However, Grzesik, Hu and Volec do not claim these results in their paper and many technical details would have to be checked to make sure that these results could indeed be proved. Answering these questions without the assumption that n is large is an interesting problem, which is still open.

Finally, all aforementioned problems are special cases of the following very general question.

Question 6.57. *Fix any graph F. What is the smallest possible number of F-edges in a graph with n vertices and e edges? What are the extremal examples?*

Füredi and Maleki [15] calculated this minimum, asymptotically, for 3-chromatic graphs F and for $e = \gamma n^2$ where γ is fixed and satisfies $1/4 < \gamma < 1/2$. For any other F, this problems is wide open. Finally, we note that it is possible to go even further and generalise the problem to the context of hypergraphs.

CHAPTER 7

Directed hypergraphs and sparse Ramsey theory

1 Introduction

When does a graph G admit an orientation such that every out-degree is at most k? An obvious necessary condition is that $|E(H)| \leq k|V(H)|$ for every subgraph H. Indeed, suppose that G has such an orientation and let H be a subgraph of G. In H the sum of out-degrees of the vertices equals the number of edges. Moreover, each vertex contributes at most k to this sum, giving the condition. Hakimi [33] observed that this condition is in fact sufficient.

Proposition 7.1 (Hakimi [33]). *Let G be a graph and $k \geq 0$ an integer. Then G admits an orientation such that every vertex has out-degree at most k if and only if all subgraphs $H \subset G$ satisfy $|E(H)| \leq k|V(H)|$.*

In fact, Hakimi proved a stronger result that determines, for any fixed graph G, all possible out-degree sequences that can be obtained by giving G an orientation. Proposition 7.1, which is a special case of Hakimi's result, is a simple consequence of Hall's marriage theorem.

What about hypergraphs? Suppose that an r-uniform hypergraph G is given an *orientation*, by which we mean that for each edge e one of the possible $r!$ orderings of the vertices in e is chosen. The ordering chosen for a particular edge e is called the *orientation of* e. Note that if $r = 2$ then this coincides with the usual definition of graph orientation. We will often denote an orientation of G by $D(G)$ and the corresponding orientation of an edge e by $D(e)$.

Given an orientation $D(G)$, a vertex v and an index $i \in [r] = \{1, 2, \ldots, r\}$ we define the *i-degree* of v, written $d_i(v)$, to be the number of edges e such that v is in the i-th position of $D(e)$. For example, if $r = 2$ then $d_1(v)$ is the out-degree of v.

When does an r-uniform hypergraph G admit an orientation such that the 1-degree of each vertex is at most k? Again, an obvious necessary condition is that $|E(H)| \le k|V(H)|$ for every subgraph $H \subset G$. Caro and Hansberg showed that this condition is sufficient.

Theorem 7.2 (Caro and Hansberg [3]). *Let G be an r-uniform hypergraph and $k \ge 0$ an integer. Then G admits an orientation such that $d_1(v) \le k$ for all vertices v if and only if all subgraphs $H \subset G$ satisfy $|E(H)| \le k|V(H)|$.*

Caro and Hansberg proved this result by constructing a suitable maximal flow on H, and a simple proof via Hall's marriage theorem is also possible.

Now, in contrast to the situation for graphs, for hypergraphs there is a sensible notion of degree for sets of multiple vertices. For example, given an orientation $D(G)$ and a pair of vertices u, v, we can define $d_{12}(u, v)$ to be the number of edges e such that u and v (in some order) are in the first two positions of $D(e)$. So, if $V = [5]$ and $E = \{(4, 5, 1), (4, 1, 3), (1, 4, 2)\}$, then $d_{12}(1, 4) = 2$.

More generally, for a p-set of vertices $A = \{v_1, \dots, v_p\} \subset V$ and a p-set of indices $I \subset [r]$, the *I-degree* of A, denoted by $d_I(A)$, is the number of edges e such that the elements of $D(e)$ in positions labeled by I are $v_1, \dots, v_p$ in some order. More formally, $d_I(A)$ is the number of edges e such that if we write $D(e) = (x_1, \dots, x_r)$ then $\{x_i \colon i \in I\}$ is exactly the set A.

There is also an equally natural variant of this notion where the mutual order of $v_1, \dots, v_p$ is important. However, the types of questions examined by Caro and Hansberg and by us turn out to be not very interesting with this alternative definition of degree. Therefore, in this chapter we mainly consider 'unordered' degrees, but we give a brief analysis of the notion of 'ordered' degrees in Section 5.

Caro and Hansberg asked if a similar result to their Theorem 7.2 can be found for degrees of multiple vertices.

Question 7.3 (Caro and Hansberg [3]). *Fix integers $k \ge 0$ and $1 \le p \le r$. Which r-uniform hypergraphs G admit an orientation such that $d_{[p]}(A) \le k$ holds for every p-set of vertices A?*

Again there is an obvious necessary condition: if G has such an orientation then for each family of p-sets $U \subset V(G)^{(p)}$ at most $k|U|$ edges e satisfy $e^{(p)} \subset U$ (here we write $X^{(p)}$ for the family of all p-subsets of X). Indeed, given U, every e with $e^{(p)} \subset U$ contributes 1 to the sum $\sum_{A \in U} d_{[p]}(A)$, and this sum cannot exceed $k|U|$.

It turns out that Question 7.3 can be answered in a fairly simple way: just like in the earlier similar scenarios, an application of Hall's marriage theorem shows that the aforementioned necessary condition is sufficient. We give this simple proof in Section 6. However, the story does not end here. Caro and Hansberg's interest in Theorem 7.2 and in Question 7.3 was mainly to answer questions of the following type. When does an r-uniform hypergraph G admit an orientation such that for every vertex v there exists some $i \in [r]$ such that $d_i(v) \leq k$?

It is once again easy to obtain a necessary condition: such an orientation would partition the vertices of G into sets $V_1, \dots, V_r$ where each v is assigned to some V_i with i satisfying $d_i(v) \leq k$. Theorem 7.2 applied to the induced subgraphs $G[V_i]$ gives a necessary condition: $|E(H)| \leq k|V(H)|$ for all subgraphs $H \subset G[V_i]$ and all i. Caro and Hansberg proved that this condition is sufficient.

Theorem 7.4 (Caro and Hansberg [3]). *Let G be an r-uniform hypergraph and $k \geq 0$ an integer. Then the following statements are equivalent:*

- *H admits an orientation such that for each vertex v some $i \in [r]$ satisfies $d_i(v) \leq k$*
- *$V(G)$ can be partitioned into r sets $V_1, \dots, V_r$ such that for each j and each $U \subset V_j$ there are at most $k|U|$ edges contained in U.*

They also examined a similar natural question for degrees of multiple vertices. When can an r-uniform hypergraph G be given an orientation such that for any p-set of vertices A there is some p-set $I \subset [r]$ such that $d_I(A) = 0$? Such an orientation would partition $V^{(p)}$ into $\binom{r}{p}$ sets W_I, $I \in [r]^{(p)}$, where each $A \in V^{(p)}$ is thrown into some W_I with $d_I(A) = 0$. For any I and any edge e the p-set of vertices that are in positions labeled by I in $D(e)$ must not belong to W_I. So there are no edges whose p-sets all belong to a single W_I, giving a necessary condition. Caro and Hansberg asked if, similarly to the case $p = 1$, this condition is sufficient.

Question 7.5 (Caro and Hansberg [3]). *Let $1 \leq p \leq r$ be integers and G an r-uniform hypergraph. Suppose that $V(G)^{(p)}$ can be coloured with $R = \binom{r}{p}$ colours in such a way that there does not exist an edge whose p-subsets all have the same colour. Must G admit an orientation such that for any p-set of vertices A there is some p-set $I \subset [r]$ such that $d_I(A) = 0$?*

The main aim of this chapter is to show that the answer to Question 7.5 is positive for r much larger than p, but negative in general.

Theorem 7.6. *For every integer $p \geq 1$ there exists a constant $r_0 = r_0(p)$ such that the answer to Question 7.5 is yes whenever $r \geq r_0$. However, the answer is no for $(r, p) = (4, 2)$.*

Here is a very brief overview of our proof. When r is large we prove that every function from $[r]^{(p)}$ to $\mathbb{N}^{(p)}$ satisfies a certain property, which we call the fixed intersection property. We then use a simple counting argument to deduce that in this case the answer to Question 7.5 is positive. On the other hand, when $(r, p) = (4, 2)$, we reduce Question 7.5 to a question about the existence of a graph which satisfies a certain Ramsey property but does not satisfy another Ramsey property. Finally, we use a new amalgamation-type technique to construct such a graph.

The structure of this chapter is as follows. In Section 2 we make a few important definitions and give a more detailed overview of the proof of Theorem 7.6. In Section 3 we prove our main theorem in the case where r is large. In Section 4 we construct a graph whose existence implies a negative answer to Question 7.5 when $(r, p) = (4, 2)$. Finally, in Section 6 we suggest some open problems.

2 Overview of the proof

It turns out that the following notion is crucial to understanding Question 7.5.

Definition 7.7. Let $1 \leq p \leq n$ be integers and $f\colon [n]^{(p)} \to [n]^{(p)}$ a function. We say that f *fixes an intersection* if there exist distinct $x, y \in [n]^{(p)}$ such that $|f(x) \cap f(y)| = |x \cap y|$. Moreover, if every nonconstant function $f\colon [n]^{(p)} \to [n]^{(p)}$ fixes an intersection then we say that $[n]^{(p)}$ has the *fixed intersection property.*

The connection between the fixed intersection property and our problem is as follows. Let G be an r-uniform hypergraph with an $\binom{r}{p}$-colouring of $V(G)^{(p)}$ such that there does not exist and edge e with $e^{(p)}$ monochromatic. Let us label the colours by elements of $[r]^{(p)}$. Our aim is to give an orientation to every edge e such that for all $A \in e^{(p)}$ of colour $c(A) \in [r]^{(p)}$, the set of vertices in positions indexed by $c(A)$ in the orientation of e does not equal A.

Let us focus our attention on a single edge e. We may label the vertices of e by $1, \ldots, r$. The restriction of the colouring to $e^{(p)}$ gives a non-constant function $c : [r]^{(p)} \to [r]^{(p)}$. Therefore, if $[r]^{(p)}$ has the fixed intersection property, then there exist distinct $A, B \in [r]^{(p)}$ such that $|c(A) \cap c(B)| = |A \cap B|$. Let π be a random orientation of e, where each one of the possible $r!$ orientations is chosen with equal probability. We have

$$\begin{aligned}\mathbb{P}\left[\pi(A) = c(A) \text{ for some } A \in [r]^{(p)}\right] &\leq \sum_{A \in [r]^{(p)}} \mathbb{P}\left[\pi(A) = c(A)\right] \\ &= 1\end{aligned}$$

and the inequality is strict unless the events '$\pi(A) = c(A)$', $A \in [r]^{(p)}$, are disjoint. However, if there exist distinct $A, B \in [r]^{(p)}$ with $|c(A) \cap c(B)| = |A \cap B|$, then it is possible to have $\pi(A \cap B) = c(A) \cap c(B)$, $\pi(A \setminus B) = c(A) \setminus c(B)$ and $\pi(B \setminus A) = c(B) \setminus c(A)$, in which case $\pi(A) = c(A)$ and $\pi(B) = c(B)$ happen at the same time. Therefore, if p, r are such that $[r]^{(p)}$ has the fixed intersection property, then with positive probability we have $\pi(A) \neq c(A)$ for all $A \in [r]^{(p)}$. Any such π gives the desired orientation for e.

We have proved the following statement.

Proposition 7.8. *The answer to Question 7.5 is yes for any choice of integers $r \geq p \geq 1$ such that $[r]^{(p)}$ has the fixed intersection property.* □

Our aim now is to understand when $[n]^{(p)}$ has the fixed intersection property. It is not difficult to see that $[2p]^{(p)}$ does not have it for any $p \geq 2$. This can be demonstrated by choosing $y = [p], \bar{y} = \{p+1, \ldots, 2p\}$ and defining $f\colon [2p]^{(p)} \to [2p]^{(p)}$ by

$$f(x) = \begin{cases} y & \text{if } x = y \text{ or } x = \bar{y} \\ \bar{y} & \text{otherwise.} \end{cases}$$

This f is non-constant and does not fix an intersection. On the other hand, in Section 3 we prove that, for any fixed p, $[n]^{(p)}$ has the fixed intersection property for sufficiently large n.

Theorem 7.9. *For every integer $p \geq 1$ there exists a constant $n_0 = n_0(p)$ such that if $n \geq n_0$ then $[n]^{(p)}$ has the fixed intersection property.*

We prove Theorem 7.9 by a repeated application of Ramsey's theorem. As an immediate corollary we get one half of our main theorem.

Corollary 7.10. *The answer to Question 7.5 is positive if r is sufficiently large, given p.* □

We conjecture that $n = 2p \geq 4$ is actually the only case where $[n]^{(p)}$ does not have the fixed intersection property. A positive answer to this conjecture would give a more precise version of our main result. However, we state this conjecture mainly because we find it interesting on its own.

Conjecture 7.11. *Let n, p be positive integers. If $n > 2p$, then $[n]^{(p)}$ has the fixed intersection property.*

What if $[n]^{(p)}$ does not have the fixed intersection property? The simplest such case is $n = 4, p = 2$. The main part of this chapter is devoted to showing how the trivial failure of the fixed intersection property of $[4]^{(2)}$ can be 'lifted' to a failure of Question 7.5 for the case $r = 4, p = 2$.

Theorem 7.12. *There exists a* 4*-uniform hypergraph* H *satisfying:*

(a) there exists a 6-colouring of $V(H)^{(2)}$ *such that* H *does not have monochromatic edges*

(b) for every orientation of H *there is a pair of vertices* u, v *such that* $d_I(u, v) > 0$ *for all* $I \in [4]^{(2)}$.

In particular, the answer to Question 7.5 is negative for $r = 4, p = 2$.

This theorem completes our main result. For its proof we obtain the 4-uniform hypergraph H from a graph with certain Ramsey properties, using the observation that the elements of $V(H)^{(2)}$ can be treated as edges of the complete graph on $V(H)$. Our work relies on a new version of the amalgamation technique, which is a well-known tool in sparse (structural) Ramsey theory.

3 Fixed intersection property of $[n]^{(p)}$ for n large

Here we prove Theorem 7.9 which says that for any fixed p if n is sufficiently large then $[n]^{(p)}$ has the fixed intersection property. First, we extend the definition of the fixed intersection property to slightly greater generality.

Definition 7.13. Let $p \geq 1$ be an integer and S, T sets. We say that a function $f\colon S^{(p)} \to T^{(p)}$ *fixes an intersection* if there exist distinct $x, y \in S^{(p)}$ such that $|f(x) \cap f(y)| = |x \cap y|$. Moreover, we say that $S^{(p)}$ has the *fixed intersection property* if every non-constant function $f\colon S^{(p)} \to S^{(p)}$ fixes an intersection.

We can now describe our strategy. First, we use Ramsey's theorem to show that $\mathbb{N}^{(p)}$ has the fixed intersection property. Next, we use compactness to deduce that $[n]^{(p)}$ also has this property for sufficiently large n.

We start with a technical lemma.

Lemma 7.14. *Let* $f : S^{(p)} \to \mathbb{N}^{(p)}$ *be a non-constant function where* S *is a subset of* $\mathbb{N}$ *and* p *is a positive integer. If there exists a set* $M \subset S$ *of size at least* $2p - 1$ *such that* f *is constant on* $M^{(p)}$*, then* f *fixes an intersection.*

Proof. Suppose for contradiction that f does not fix an intersection. Note that $M \neq S$ since f is non-constant. This allows us to define $i_0 = \min(S \setminus M)$. We will show that, in fact, f is constant on $M_1^{(p)}$ where $M_1 = M \cup \{i_0\}$.

Take any $x \in M_1^{(p)}$ of the form $x = x' \cup \{i_0\}$ with $x' \in M^{(p-1)}$. We define $a = f(y)$ for any $y \in M^{(p)}$ and consider two cases: we have either $f(x) = a$ or $|f(x) \cap a| \leq p-1$. In the latter case we can choose $y \in M^{(p)}$ such that $|x' \cap y| = |f(x) \cap a|$ (this is possible because $|M| \geq 2p-1$). However, then $|x \cap y| = |x' \cap y| = |f(x) \cap f(y)|$, so f fixes an intersection, which contradicts our initial assumption. We conclude that $f(x) = a$ and so f is constant on $M_1^{(p)}$.

We repeat this argument to obtain a possibly infinite chain of sets $M_1 \subset M_2 \subset M_3 \subset \cdots$ whose union is S and such that f is constant on $M_i^{(p)}$ for all i. This contradicts the assumption that f is non-constant on $S^{(p)}$ and so we are done. $\square$

We use this lemma to achieve our first goal.

Theorem 7.15. *For any positive integer p, $\mathbb{N}^{(p)}$ has the fixed intersection property.*

Proof. We use induction on p. It is clear that the theorem holds for $p = 1$, so it is enough to consider the case $p \geq 2$. Let $f : \mathbb{N}^{(p)} \to \mathbb{N}^{(p)}$ be a non-constant function. Since $\mathbb{N}$ can be reordered without having any impact on the problem, we are free to choose the value to which f maps $[p]$, say, $f([p]) = [p]$. Now, we define a finite colouring of $\mathbb{N}^{(p)}$ by setting $c(x) = f(x) \cap [p]$, $x \in \mathbb{N}^{(p)}$, where subsets of $[p]$ are the colours. Ramsey's theorem tells us that there exists an infinite set $M \subset \mathbb{N}$ such that c is constant on $M^{(p)}$. Say, $c(x) = a \subset [p]$ for all $x \in M^{(p)}$.

If $a = \emptyset$ then we pick an arbitrary $x \in (M \setminus [p])^{(p)}$. In this case $|x \cap [p]| = 0 = |f(x) \cap f([p])|$, so f fixes an intersection. If $a = [p]$ then $f(x) = [p]$ for all $x \in M^{(p)}$ and we are done by Lemma 7.14. It remains to consider the case where a is a proper subset of $[p]$. We write $s = |a|$ and note that $1 \leq s \leq p-1$. For any fixed $z \in M^{(s)}$ we define a function $f^*\colon (M \setminus z)^{(p-s)} \to (\mathbb{N} \setminus a)^{(p-s)}$ by setting $f^*(x') = f(x' \cup z) \setminus a$ for all $x' \in (M \setminus z)^{(p-s)}$. Now, either f is constant on $M^{(p)}$, in which case we are done by Lemma 7.14, or f is non-constant on $M^{(p)}$ and we can choose z so that f^* is also non-constant. Then, by the induction hypothesis, $|f^*(x') \cap f^*(y')| = |x' \cap y'|$ for some distinct $x', y' \in (M \setminus z)^{(p-s)}$ and so $|f(x' \cup z) \cap f(y' \cup z)| = |f^*(x') \cap f^*(y')| + s = |x' \cap y'| + |z| = |(x' \cup z) \cap (y' \cup z)|$. $\square$

A compactness argument extracts the result for finite domains.

Corollary 7.16. *For any positive integer p there exists an integer $n \geq p+1$ such that every non-constant function $f : [n]^{(p)} \to \mathbb{N}^{(p)}$ fixes an intersection.*

Proof. Suppose that there exists a value for p for which this statement is not true. Then for any integer $n \geq p+1$ there exists a non-constant function $f_n\colon [n]^{(p)} \to \mathbb{N}^{(p)}$ that does not fix an intersection. Trivially, for any fixed n and any $s \geq p$ the sets $f_n(x)$, where x ranges over all p-subsets of $[s]$, cover at most $p\binom{s}{p}$ elements of $\mathbb{N}$. Therefore, after reordering $\mathbb{N}$ if necessary we can achieve that for all n and for all $x \in \mathbb{N}^{(p)}$ the set $f_n(x)$ only contains integers that are less than or equal to $p\binom{\max x}{p}$. The point here is that for any $s \geq p$ there are only finitely many possibilities for f_n on elements of $[s]^{(p)}$.

We define $f\colon \mathbb{N}^{(n)} \to \mathbb{N}^{(n)}$ as follows. By the pigeonhole principle, there exists an infinite set of indices $S_{p+1} \subset \mathbb{N}$ such that the functions f_n, indexed by $n \in S_{p+1}$, are identical on $[p+1]^{(p)}$. We define f on $[p+1]^{(p)}$ to be the same as any f_n with $n \in S_{p+1}$. Now, there is an infinite set $S_{p+2} \subset S_{p+1}$ such that the functions f_n indexed by S_{p+2} agree on $[p+2]^{(p)}$. We extend the definition of f to $[p+2]^{(p)}$ by making it be the same as any f_n with $n \in S_{p+2}$. By repeating this process indefinitely we obtain a function $f : \mathbb{N}^{(n)} \to \mathbb{N}^{(n)}$ with the property that for every $s \geq p$ there exists an index $n(s)$ such that f is the same as $f_{n(s)}$ on $[s]^{(p)}$. In particular, f is not constant since if it were then $f_{n(2p-1)}$ would be constant on $[2p-1]^{(p)}$, which is impossible by Lemma 7.14. By Theorem 7.15, there exist distinct $x, y \in \mathbb{N}^{(p)}$ such that $|f(x) \cap f(y)| = |x \cap y|$. But $x, y \in [s]^{(p)}$ for some s and we have $|f_{n(s)}(x) \cap f_{n(s)}(y)| = |x \cap y|$. This means that $f_{n(s)}$ fixes an intersection, which contradicts our initial assumptions. $\square$

We get Theorem 7.9 as an immediate corollary.

Theorem 7.9. *For every integer $p \geq 1$ there exists a constant $n_0 = n_0(p)$ such that if $n \geq n_0$ then $[n]^{(p)}$ has the fixed intersection property.* $\square$

4 Sparse Ramsey type counterexample

4.1 Overview of the construction

In light of Conjecture 7.11 and Proposition 7.8, we seek $p \geq 2$ such that Question 7.5 has a negative answer when $r = 2p$. It turns out that $p = 2$ works. We recall the exact statement that we prove.

Theorem 7.12. *There exists a 4-uniform hypergraph H satisfying:*

(a) there exists a 6-colouring of $V(H)^{(2)}$ such that H does not have monochromatic edges

(b) for every orientation of H there is a pair of vertices u, v such that $d_I(u, v) > 0$ for all $I \in [4]^{(2)}$.

In particular, the answer to Question 7.5 is negative for $r = 4, p = 2$.

A useful idea here is to consider the elements elements of $V(H)^{(2)}$ as the edges of the complete graph on vertices $V(H)$. It allows us to deduce Theorem 7.12 from a statement about graphs rather than hypergraphs.

Let G be a graph. We can form a 4-uniform hypergraph H by taking $V(H) = V(G)$ and $E(H) = \{A \in V(G)^{(4)} : A^{(2)} \subset E(G)\}$. In other words, the edges of H are the 4-cliques of G. With this setup, property (a) translates to the condition that G admits a 6-edge-colouring without monochromatic 4-cliques. We now seek a condition that would guarantee (b). Suppose (b) is false. Then there exists an edge-colouring $c : E(G) \to [4]^{(2)}$ such that $d_{c(e)}(e) = 0$ for all $e \in E(G)$. Restricting to any 4-clique A of G, this induces a function $c_A \colon V(A)^{(2)} \to [4]^{(2)}$ with the property that there exists a bijection $\sigma : V(A) \to [4]$ such that $c_A(\{u, v\}) \neq \{\sigma(u), \sigma(v)\}$ for all $\{u, v\} \in V(A)^{(2)}$. We know that such σ can exist only if c_A fixes an intersection. Therefore, to ensure (b), it is enough to require that for every edge-colouring of G with colours $[4]^{(2)}$ there is a 4-clique $A \subset G$ such that the induced colouring $V(A)^{(2)} \to [4]^{(2)}$ does not fix an intersection. The following definition describes such a function as a 6-colouring of the edges of a 4-clique.

Definition 7.17. Take six colours and partition them into three pairs. Call two colours *opposing* if they are in the same pair. Let A be a 4-clique in a graph G whose edges are coloured with these six colours. We say that A is *special* if there is a pair of opposing colours c_1, c_2 such that A consists of a 4-cycle of colour c_1 and two independent edges of colour c_2.

We will prove the following result, which immediately implies Theorem 7.12.

Lemma 7.18. *There exists a graph G satisfying:*

(a) it is possible to colour $E(G)$ with six colours without forming a monochromatic 4-clique

(b) it is not possible to colour $E(G)$ with six colours without forming a monochromatic or a special 4-clique.

Proof of Theorem 7.12 (assuming Lemma 7.18). Let G be a graph as in Lemma 7.18 and label the six colours by distinct elements of $[4]^{(2)}$ in such a way that $\{\{1, 2\}, \{3, 4\}\}$,

$\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$ are the pairs of opposite colours. We define a 4-uniform hypergraph H on the same set of vertices as G, where we take a 4-set $A \subset V(G)$ as an edge of H if A spans a clique in G. Condition (a) of Theorem 7.12 follows directly from the correspond condition of Lemma 7.18, and so it remains to establish Condition (b). Assume for contradiction that H admits an orientation such that for every pair of vertices u, v there exists a set $I \in [4]^{(2)}$ such that $d_I(u,v) = 0$. This gives a colouring $c : V(H)^{(2)} \to [4]^{(2)}$ (which maps each pair $\{u,v\}$ to an arbitrary corresponding set I), whose restriction to $E(G)$ is a 6-colouring of $E(G)$. By Condition (b) of Lemma 7.18, there exists a 4-clique in G which is either monochromatic or special. Let $\{v_1, v_2, v_3, v_4\}$ be such a clique. It cannot be monochromatic, since in every ordering of $\{v_1, v_2, v_3, v_4\}$ every pair of positions is occupied by a pair of vertices. Thus, the clique has to be special. Hence, we may assume that in the orientation that $\{v_1, v_2, v_3, v_4\}$ gets from H the pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$ do not take positions $\{1,2\}$, while all other pairs of vertices do not take the positions $\{3,4\}$. However, this is impossible – either by simple case analysis or by our earlier observations regarding functions that fix an intersection. □

4.2 Amalgamation

The proof of Lemma 7.18 is based on a new amalgamation-type method. Amalgamation technique (also known in literature as *partite construction*) was introduced by Nešetřil and Rödl [50–52]. In this subsection we review a basic form of this technique and apply it to prove a few classical results which we will later use as tools.

Suppose that we are interested in a certain Ramsey property for graphs. For example, we may be interested in graphs G that are *c-edge-Ramsey* for H, meaning that every colouring of $E(G)$ with c colours produces a monochromatic copy of H. Or, if we replace edge colourings by vertex colourings, then we have the property of a graph being *c-vertex-Ramsey* for H. Whichever Ramsey property we choose, intuitively we think that if a graph G has that property, then G is 'dense'. A typical problem in sparse Ramsey theory is to construct graphs that are 'dense' in this sense but 'sparse' in some other sense. Amalgamation is useful for constructing such graphs.

As a concrete example, let us construct a graph G with the following two properties, where k, c are fixed positive integers with $k \geq 2$:

- G is c-vertex-Ramsey for K_{k-1}; we call this property `D`

- G does not contain a clique on k vertices; we call this property $\mathtt{S}$.

For this construction we fix a large integer t and consider t-partite graphs (in fact, G itself will be t-partite). Moreover, whenever we mention t-partite graphs, we implicitly assume that the vertex classes are labelled 1 through t.

A key idea is to consider the following weaker versions of the property $\mathtt{D}$ for t-partite graphs. Let F be a t-partite graph. For any $i \in \{0, \ldots, t\}$ we say that a colouring of $V(F)$ is *i-simple* if for every $j \in \{i+1, \ldots, t\}$ all vertices in the jth vertex class have the same colour. We say that F has property $\mathtt{D}_i$ if every i-simple colouring of $V(F)$ with c colours produces a monochromatic clique on k vertices. We note that $\mathtt{D}_t$ is exactly the same property as $\mathtt{D}$.

Now, our strategy is to construct t-partite graphs $G_0, \ldots, G_t$ that all have property $\mathtt{S}$ and such that G_i has property $\mathtt{D}_i$ for every i. If we are successful then we can simply take G to be G_t. Constructing the starting graph G_0 is straightforward. Indeed, we can take G_0 to consist of disjoint copies of K_{k-1} with the property that for every $k-1$ vertex classes there exists a copy of K_{k-1} that intersects them all. Provided that t is greater than $c(k-2)$, which we are free to assume, every 0-simple colouring of $V(G_0)$ with c colours produces at least $k-1$ vertex classes that get the same colour and so there must be a monochromatic copy of K_{k-1}. Therefore, G_0 has property $\mathtt{D}_0$ and it also has property $\mathtt{S}$ by construction.

Suppose that for some $i \in \{1, \ldots, t\}$ we have constructed a t-partite graph G_{i-1} satisfying properties $\mathtt{S}$ and $\mathtt{D}_{i-1}$. We denote the ith vertex class of G_{i-1} by V and take a new set W whose size is much bigger than that of V. We take $\binom{|W|}{|V|}$ disjoint copies of G_{i-1} and label them as G_{i-1}^A where A runs over all $|V|$ element subsets of W. We construct G_i by gluing these copies of G_{i-1} at the ith vertex class: more precisely, for every A we identify the ith vertex class of G_{i-1}^A with the set A. This produces a t-partite graph G_i whose ith vertex class is W and whose every other vertex class is a disjoint union of $\binom{|W|}{|V|}$ copies of the corresponding vertex class of G_{i-1}. To see that G_i has property $\mathtt{S}$, let us assume that $C \subset V(G_i)$ is a clique on k vertices. Since at most one vertex of C belongs to the ith vertex class of G_i, there is a vertex $v \in C$ which is in another vertex class. Such v belongs to a unique copy of G_{i-1}, say G_{i-1}^A. Since C is a clique and all of the neighbours of v are in G_{i-1}^A, the whole clique C belongs to G_{i-1}^A. However, this contradicts the assumption that G_{i-1} satisfies property $\mathtt{S}$. So no such C exists.

It remains to check that G_i has property $\mathtt{D}_i$. Consider any i-simple colouring of $V(G_i)$ with c colours. Provided that the size of W exceeds $(|V|-1)c$, which we are free to assume, there must be a monochromatic $|V|$ element set $B \subset W$. Now, the restriction of the colouring to G_{i-1}^B is an $(i-1)$-simple colouring of a graph which is isomorphic to G_{i-1}. As a result, there must be a monochromatic copy of K_{k-1}. We conclude that G_i has properties $\mathtt{S}$ and $\mathtt{D}_i$ and hence we are done with the construction of G.

We now list a few classical results that can be proved by amalgamation technique. These results will be our main technical tools in the proof of Lemma 7.18.

Lemma 7.19. *Let k, c be positive integers with $k \geq 2$ and H a graph that does not contain a clique on k vertices. Then there exists a graph G satisfying*

- *G is c-vertex-Ramsey for H*
- *G does not contain a clique on k vertices.*

Lemma 7.20. *Let k, b, c be positive integers with $k \geq 2$ and suppose that H is a graph that is not b-edge-Ramsey for K_k. Then there exists a graph G satisfying*

- *G is c-vertex-Ramsey for H*
- *G is not b-edge-Ramsey for K_k.*

Lemma 7.21. *Let k, c be positive integers with $k \geq 2$ and suppose that H is a graph that does not contain a clique on k vertices. Then there exists a graph G satisfying*

- *G is c-edge-Ramsey for H*
- *G does not contain a clique on k vertices.*

We sketch the proofs for completeness. Lemmas 7.19 and 7.21 are completely standard [41]. Lemma 7.20 is less well-known but its proof is no more difficult than that of Lemma 7.19.

Proof of Lemma 7.19. We run the construction described earlier in this subsection, the only difference being that now G_0 consists of disjoint copies of H rather than K_{k-1}. $\square$

Proof of Lemma 7.20. We run the same construction as in Lemma 7.19. We have to check that the final graph G_t is not b-edge-Ramsey for K_k. It is clear that G_0 is not b-edge-Ramsey for K_k, so it suffices to prove that if G_{i-1} is not b-edge-Ramsey for K_k then neither is G_i.

Let ω be a colouring of $E(G_{i-1})$ with b colours that does not produce a monochromatic K_k. By construction, G_i can be partitioned into edge-disjoint copies of G_{i-1} in a natural way, and so ω naturally extends to a colouring of the edges of G_i. Suppose that $C \subset V(G_i)$ is a monochromatic clique on k vertices. Then some vertex $v \in C$ does not belong to the ith vertex class of G_i and so v belongs to a unique copy of G_{i-1}, say G_{i-1}^A. Since C is a clique, it must all belong to G_{i-1}^A but this contradicts the assumption that ω does not produce a monochromatic K_k in G_{i-1}. Therefore, no such C exists. $\square$

The proof of Lemma 7.21 is somewhat more complicated. For this proof we need the following technical proposition.

Proposition 7.22. *Let G be a bipartite graph with vertex classes X, Y and let c be a positive integer. Then there exists a bipartite graph H with vertex classes X', Y' such that every colouring of $E(H)$ with c colours produces a monochromatic induced copy of G. Moreover, in that copy of G, the vertex class corresponding to X is contained in X' and the vertex class corresponding to Y is contained in Y'.*

Proof. Choose a large number n and define H to have vertex classes $X' = X^n, Y' = Y^n$ and edges $E(G)^n$; that is, we join $(x_1, \dots, x_n) \in X', (y_1, \dots, y_n) \in Y'$ by an edge if $x_1y_1, \dots, x_ny_n$ are edges in G. Suppose that the edges of H are coloured with c colours. Provided that n is sufficiently large, it follows from the Hales-Jewett theorem [34] that $E(G)^n$ contains a monochromatic combinatorial line L. We may assume without loss of generality that $L = \{(e, \dots, e, f_{l+1}, \dots, f_n) : e \in E(G)\}$ for some $1 \le l \le n$ and some fixed edges $f_{l+1}, \dots, f_n$. Let us write, for every $l + 1 \le i \le n$, $f_i = a_ib_i$ where $a_i \in X$ and $b_i \in Y$. Moreover, let us define sets

$$A = \{(x, \dots, x, a_{l+1}, \dots, a_n) : x \in X\},$$
$$B = \{(y, \dots, y, b_{l+1}, \dots, b_n) : y \in Y\}.$$

Clearly, H induces a copy of G on $A \cup B$. Moreover, L is precisely the set of edges of H spanned by $A \cup B$. Therefore, that induced copy of G is monochromatic. Finally, we have $A \subset X'$ and $B \subset Y'$. $\square$

Proof of Lemma 7.21. Once again we start by fixing a large integer t. We consider t-partite graphs with the vertex classes labelled 1 through t. In this proof, we will always respect the labels of the vertex classes. In particular, if G', G'' are graphs with labelled vertex classes, then we say that G'' contains G' if it is possible to embed G' into G'' in a way that preserves these labels. We list all $\binom{t}{2}$ pairs (i, j) with $1 \le i < j \le t$ in an arbitrary fixed order $(i_1, j_1), \dots, (i_{\binom{t}{2}}, j_{\binom{t}{2}})$.

Let G be a t-partite graph. We say that a colouring of the edges of G is *n-simple* if, for all m with $n+1 \leq m \leq \binom{t}{2}$, all edges between the i_mth and the j_mth vertex classes have the same colour. We say that G has property $\mathtt{D}_n$ if every n-simple colouring of $E(G)$ with c colours produces a monochromatic copy of H. So, our aim is to find a graph G which does not contain a clique on k vertices but which has property $\mathtt{D}_{\binom{t}{2}}$.

Our strategy is to construct t-partite graphs $G_0, \dots, G_{\binom{t}{2}}$ which do not contain a clique on k vertices and such that, for all n, G_n has property $\mathtt{D}_n$. Having achieved this, we take G to be $G_{\binom{t}{2}}$. We take the starting graph G_0 to be a union of disjoint copies of H such that for every choice of $|H|$ vertex classes of G_0 there exists a copy of H that intersects them all. Provided that the parameter t was chosen to be sufficiently large, namely, at least as large than the c-colour Ramsey number for a clique on $|H|$ vertices, every 0-simple colouring of $E(G)$ with c colours produces a monochromatic copy of H. Moreover, G_0 trivially does not contain a clique on k vertices.

Now, suppose that for some $1 \leq n \leq \binom{t}{2}$ we have a t-partite graph G_{n-1} with the desired properties. Let X and Y be the i_nth and j_nth vertex classes of G_{n-1}, and let $H = G_{n-1}[X \cup Y]$ be the bipartite graph induced by G_{n-1} on $X \cup Y$. By Proposition 7.22 there exists a bipartite graph F, with vertex classes X' and Y', such that every colouring of $E(F)$ with c colours produces a monochromatic induced copy of H such that the vertex classes corresponding to X, Y are contained in X', Y', respectively.

This is how we construct G_n. First, we take F as above and declare that G_n induces F on the union of its i_nth and j_nth vertex classes. In particular, these vertex classes of G_n are exactly the sets X', Y', respectively. Let $\mathcal{H}$ be the family of all induced copies of H in F, with vertex classes corresponding to X, Y contained in X', Y', respectively. Now, we take $|\mathcal{H}|$ disjoint copies of G_{n-1}, labelled G_{n-1}^A where $A \in \mathcal{H}$, and glue them to F by identifying the graph induced by the union of the i_nth and j_nth vertex classes of G_{n-1}^A with A.

It is easy to see that G_n has property $\mathtt{D}_n$. Indeed, it follows from the choice of F that given any n-simple colouring of $E(G_n)$ with c colours there exist a monochromatic $A \in \mathcal{H}$. The restriction of this colouring to G_{n-1}^A is an $(n-1)$-simple colouring, and so there exists a monochromatic induced copy of H.

It remains to check that G_n does not contain a clique on k vertices. If $k = 2$, then G_{n-1} is empty and so G_n is empty. We now suppose that $k \geq 3$ and that $C \subset V(G_n)$ is a clique on k vertices. Then there exists a vertex $v \in C$ that does not belong to the i_nth and j_nth vertex classes of G_n. There exists a unique A such that v belongs to G_{n-1}^A and, since C is a clique, all of C belongs to G_{n-1}^A. However, then G_{n-1} contains a clique on k vertices, which contradicts our assumption. $\square$

4.3 Proof of Lemma 7.18

In this subsection we prove Lemma 7.18. Our proof uses a new amalgamation-type method. The main novelty is that, instead of working with t-partite graphs, we base our construction on graphs whose vertices are partitioned into sets $V_1, \ldots, V_t$ that span sparse subgraphs. In exchange, we ensure that the structure of the cross-edges is simple. More precisely, we work with blowups of a fixed graph.

Definition 7.23. Let G be a graph on n vertices, labelled 1 through n. Given an n-tuple of graphs $\mathcal{F} = (F_1, \ldots, F_n)$, we define the *$\mathcal{F}$-blowup* of G, denoted $G(\mathcal{F})$, to be the graph obtained by the following procedure. First, we take graphs $F_1, \ldots, F_n$ on disjoint vertex sets. Then, we add all possible edges between F_i and F_j for all i, j that are adjacent in G; we do not add any edges between F_i and F_j for i, j that are not adjacent. If i, j are the endpoints of an edge $e \in E(G)$, then the edges added between F_i and F_j are called *e-cross-edges* of $G(\mathcal{F})$.

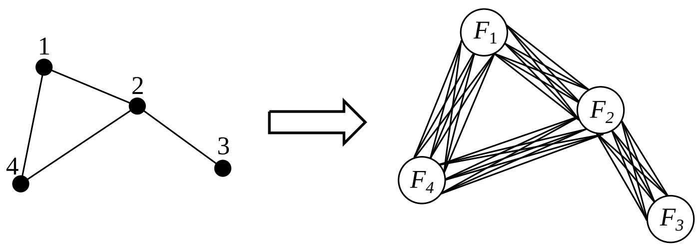

Figure 7.1: Construction of blowup.

The structure of the proof is as follows. We fix a graph $\hat{G}$ on n vertices which has certain properties. Then, we consider the $\mathcal{F}$-blowup of $\hat{G}$, where $\mathcal{F}$ is an n-tuple of fairly simple graphs. We keep replacing graphs in $\mathcal{F}$ by bigger (but still sparse) graphs until eventually the blowup $\hat{G}(\mathcal{F})$ satisfies some properties that we need.

The following technical lemma is the tool that allows us to replace the graphs in $\mathcal{F}$ by bigger graphs, two at a time, in a way that meaningfully affects the blowup $\hat{G}(\mathcal{F})$.

Definition 7.24. Let G, H be graphs. The *join* of G and H, denoted $G + H$, is the graph obtained by taking G and H on disjoint vertex sets and adding all possible edges between G and H. In other words, $G + H = K_2(G, H)$. The edges in $G + H$ that have one endpoint in G and one in H are called *cross-edges*.

Proposition 7.25. *Let k and s be positive integers and suppose G_0 and H_0 are graphs that do not contain K_s. Then there exist graphs G and H satisfying*

(i) G and H do not contain K_s, and

(ii) every colouring of the cross-edges of $G + H$ with k colours produces a copy of $G_0 + H_0$ with $G_0 \subset G$, $H_0 \subset H$ whose cross-edges all have the same colour.

Proof. By Lemma 7.19, there exists a graph H that is k-vertex-Ramsey for H_0 but contains no copies of K_s. There also exists a graph G that is $k^{|H|}$-vertex-Ramsey for G_0 but contains no copies of K_s.

Let c by a colouring of the cross-edges of $G+H$ with colours $1, \dots, k$. It induces a vertex-colouring $c_G : V(G) \to [k]^{V(H)}$ where, for every $x \in V(G)$, the value of $c_G(x)$ is the function which assigns to every $y \in V(H)$ the colour $c(xy)$. By our choice of G, there exists a copy of G_0 in G which is monochromatic with respect to c_G. That is, the colour of any cross-edge of $G_0 + H$ (where G_0 on the left stands for the aforementioned copy of G_0 in G) depends only on its endpoint in H and not on the one in G_0. We define a vertex colouring $c_H : V(H) \to [k]$ by letting each vertex of H have the colour of any edge joining it to G_0. By our choice of H, there exists a copy of H_0 in H that is monochromatic under c_H, which means exactly that all cross-edges of $G_0 + H_0$ have the same colour. □

We now have the technical tools needed to prove Lemma 7.18. We recall the statement of this lemma and then prove it.

Lemma 7.18. *There exists a graph G satisfying:*

(a) it is possible to colour $E(G)$ with six colours without forming a monochromatic 4-clique

(b) it is not possible to colour $E(G)$ with six colours without forming a monochromatic or a special 4-clique.

Proof. Let $\hat{G}$ be a fixed graph. At the moment $\hat{G}$ can be any graph, but as the proof builds up it will become clear what properties $\hat{G}$ needs to satisfy. Let the vertices of $\hat{G}$ be labelled 1 through n, where n is the order of $\hat{G}$. We enumerate the edges of $\hat{G}$ in any fixed order $e_1, \dots, e_m$.

Lemma 7.21 gives us a graph F which is 6-edge-Ramsey for K_3 and which does not contain a K_4. We define $\mathcal{F}_0$ to be the n-tuple $(F, \dots, F)$, consisting of n copies of F. We construct further n-tuples $\mathcal{F}_1, \dots, \mathcal{F}_m$ by considering the edges of $\hat{G}$ in the predefined order and at each step carrying out the following procedure. Suppose that we have constructed the n-tuple $\mathcal{F}_{i-1} = (F_1, \ldots, F_n)$, where $1 \leq i \leq m$. We consider the edge e_i and write $e_i = uv$. By Proposition 7.25, there exist graphs F', F'' satisfying

- F' and F'' contain no copies of K_4, and
- whenever the cross-edges of $F' + F''$ are coloured with six colours, we can find a copy of F_u in F' and a copy of F_v in F'' such that all cross-edges of $F_u + F_v$ have the same colour.

We replace F_u by F' and F_v by F'' to obtain the new n-tuple $\mathcal{F}_i$. We run this procedure for $i = 1, \ldots, m$ until we obtain the n-tuple $\mathcal{F}_m$. We define $G = \hat{G}(\mathcal{F}_m)$. Our aim now is to choose $\hat{G}$ such that G has the desired properties.

Let c be a colouring of $E(G)$ with six colours. By construction of G, there exists a copy of $\hat{G}(\mathcal{F}_{m-1})$ such that only one colour is used for the e_m-cross-edges of $\hat{G}(\mathcal{F}_{m-1})$. Similarly, this copy of $\hat{G}(\mathcal{F}_{m-1})$ contains a copy of $\hat{G}(\mathcal{F}_{m-2})$ in which only one colour is used for the e_{m-1}-cross-edges (and the same still holds for the e_m-cross-edges). We continue in this manner and, eventually, we obtain a copy of $\hat{G}(\mathcal{F}_0)$ such that, for every edge of $\hat{G}$, the corresponding cross-edges of $\hat{G}(\mathcal{F}_0)$ have the same colour. We denote this copy of $\hat{G}(\mathcal{F}_0)$ by A. Now, the restriction of c to A gives rise to a colouring of $E(\hat{G})$, which we denote by $\hat{c}$, where any $e \in E(\hat{G})$ gets the colour of the e-cross-edges in A. To make $\hat{c}$ well defined, if there are multiple ways to obtain A, then we fix one of them.

Recall that $\hat{G}(\mathcal{F}_0)$ was obtained by taking n disjoint copies of F, corresponding to the vertices of $\hat{G}$, and joining some of them by cross-edges. We now consider the colouring of these copies of F in A. Since F is 6-edge-Ramsey for K_3, each copy of F contains a monochromatic triangle. We extend the edge-colouring $\hat{c}$ to the vertices of $\hat{G}$ (making it a total-colouring) by giving each $v \in V(\hat{G})$ the colour of a monochromatic triangle within the corresponding copy of F in A. In every situation where multiple monochromatic triangles could be chosen, we make an arbitrary choice and fix it.

Notice that if $\hat{c}$ produces at least one of the following:

(i) a K_4 with all edges of one colour

(ii) a vertex and an incident edge of one colour

then c produces a monochromatic K_4. Moreover, if $\hat{c}$ produces

(iii) an edge e with endpoints having the colour that opposes the colour of e

then c produces a special K_4.

Conversely, suppose that $\hat{c}$ is a total colouring of $\hat{G}$ with six colours. It induces an edge colouring c of G with six colours where for all $e \in E(G)$ we define

$$c(e) = \begin{cases} \hat{c}(v) & \text{if } e \text{ is an edge in the subgraph that corresponds to a vertex } v \in \hat{G} \\ \hat{c}(e') & \text{if } e \text{ is an } e'\text{-cross-edge.} \end{cases}$$

A monochromatic copy of K_4 appears in c if and only if (i) or (ii) appears in $\hat{c}$.

Putting everything together, it suffices to find a graph $\hat{G}$ that admits a total colouring with six colours with no (i) and (ii) but whose every total colouring with six colours produces at least one of (i), (ii) and (iii). We now construct such $\hat{G}$.

Let H be the smallest complete graph that is 4-edge-Ramsey for K_4. Then H is not 5-edge-Ramsey for K_4. It follows from Lemma 7.20 that there exists a graph $\hat{G}$ that is 6-vertex-Ramsey for H but not 5-edge-Ramsey for K_4. We will show that the graph $\hat{G}$ has the desired properties. First, if we use five colours for the edges of $\hat{G}$ avoiding a monochromatic K_4 and a sixth colour for the vertices, then we create a total colouring of $\hat{G}$ that avoids (i) and (ii). Conversely, given any total colouring of $\hat{G}$ with six colours, there must exist a copy of H whose vertex set is monochromatic. Say, a is the colour of these vertices. If some edge in this copy of H has colour a or the colour opposing a, then we can find (ii) or (iii). Otherwise, the edges in the aforementioned copy of H are coloured with four colours, and so we have (i).

This concludes the proof of Lemma 7.18. □

5 Ordered degrees

In this section we define and briefly examine the notion of 'ordered' degrees for sets of multiple vertices. Let $D(G)$ be an orientation of an r-uniform hypergraph G. Given a pair of vertices u, v, we can define $d^*_{12}(u, v)$ to be the number of edges e such that u is in the first position of $D(e)$ and v is in the second. For example, if $E(G) = \{(4, 5, 1), (4, 1, 3), (1, 4, 2)\}$ then $d^*_{12}(1, 4) = 1$.

More generally, for an ordered p-tuple of distinct vertices $A = (v_1, \ldots, v_p)$ and an ordered p-tuple $I = (i_1, \ldots, i_p) \in [r]^p$ with distinct elements, the *ordered I-degree* of A, denoted by $d^*_I(A)$, is the number of edges whose orientations have vertices $v_1, \ldots, v_p$ in this order occupying positions labelled by I. More formally, $d^*_I(A)$ is the number of edges e such that if we write $D(e) = (x_1, \ldots, x_r)$ then $x_{i_1} = v_1, \ldots, x_{i_p} = v_p$.

In the remainder of this section we reserve the term *p-tuple* to mean an ordered

p-set without repeated elements. For any set S we denote by $S^{\underline{p}}$ the family of all p-tuples consisting of elements of S. For example, $[3]^{\underline{2}} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$.

Following the spirit of Theorems 7.1, 7.2 and Question 7.3, we can ask when an r-uniform hypergraph G can be given an orientation such that $d^*_{(1,2,\ldots,p)}(A) \leq k$ for all p-tuples of vertices A, where $k \geq 0$ is a fixed integer.

It is easy to find a necessary condition: for any collection of p-sets $U \subset V^{(p)}$ there be at most $kp!|U|$ edges e such that $e^{(p)} \subset U$. Indeed, every such edge contributes 1 to the sum

$$\sum_{A \in U} \sum_{\substack{A^* \text{ an} \\ \text{ordering of } A}} d^*_{(1,2,\ldots,p)}(A^*)$$

and this sum does not exceed $kp!|U|$.

The proof of sufficiency is based on Hall's marriage theorem.

Theorem 7.26. *Fix integers $k \geq 0, 1 \leq p \leq r$ and let G be an r-uniform hypergraph. Suppose that for any $U \subset V^{(p)}$ there are at most $kp!|U|$ edges e such that $e^{(p)} \subset U$. Then G admits an orientation such that $d^*_{(1,2,\ldots,p)}(A) \leq k$ for every p-tuple $A \subset V$.*

Proof. Construct a bipartite graph H with vertex classes $X = E(G)$ and $Y = V(G)^{\underline{p}}$. Join $e \in X$ and $A \in Y$ by an edge if e contains all elements of A.

Take any $S \subset X$ and let $\Gamma(S) \subset Y$ be the neighbourhood of S in H. If we treat members of $\Gamma(S)$ as p-sets rather than p-tuples, then we obtain a family of p-sets $U \subset V^{(p)}$ that contains all p-sets of all edges in the family S. Therefore, $|S| \leq kp!|U|$. Moreover, it is clear that $|U| = |\Gamma(S)|/p!$ and so we have $k|\Gamma(S)| \geq |S|$. By Hall's marriage theorem, it is possible to assign an element of Y to every element of X in such a way that each element of Y is used at most k times. Now, we give each edge $e \in E(G) = X$ an orientation such that the initial p positions of that orientation form the p-tuple from Y assigned to e. This produces an orientation of G with the desired property. $\square$

A version of Question 7.5 can be asked for ordered degrees. When does an r-uniform hypergraph G admit an orientation such that for each p-tuple of vertices A there is a p-tuple $I \in [r]^{\underline{p}}$ such that $d^*_I(A) = 0$? If $p = 1$ then this is covered by Theorem 7.4 so let us assume that $p \geq 2$. In contrast to the notion of 'unordered' degrees, it turns out that every G admits such an orientation. In fact, this can be achieved by a simple explicit construction.

Theorem 7.27. *Let $2 \leq p \leq r$ be integers and G an r-uniform hypergraph. Then G admits an orientation such that for each p-tuple of vertices A there is a p-tuple $I \in [r]^{\underline{p}}$ such that $d^*_I(A) = 0$.*

Proof. Without loss of generality we assume that $V(G) = [n]$. For each edge e, we order its vertices in the increasing order. Let $A = (a_1, \ldots, a_p)$ be a p-tuple of vertices. If $a_1 < \cdots < a_p$ then $d^*_{(p,p-1,\ldots,1)}(A) = 0$. Otherwise, $d^*_{(1,2,\ldots,p)}(A) = 0$. □

6 Concluding remarks and open problems

We begin by completing the answer to Question 7.3.

Theorem 7.28. *Let k, p, r be integers with $k \geq 0, r \geq p \geq 1$ and let G be an r-uniform hypergraph. If for every $U \subset V^{(p)}$ the number of edges e satisfying $e^{(p)} \subset U$ does not exceed $k|U|$, then G admits an orientation such that $d_{[p]}(A) \leq k$ for all $A \in V^{(p)}$.*

Proof. We construct a bipartite graph H with vertex classes $X = E(G)$ and $Y = V(G)^{(p)}$ by joining $e \in X$ and $A \in Y$ by an edge if $A \subset e$. In other words, we join each edge of G to its p-subsets.

Given $S \subset E(G)$, we define $\Gamma(S) \subset Y$ to be the neighbourhood of S in H. By the assumed property of G, we have $|S| \leq k|\Gamma(S)|$. By Hall's marriage theorem it is possible to assign, to every $e \in X$, an element $y(e) \in Y$ in such a way that each element of Y is used at most k times. Now, for every edge $e \in E(G) = X$ we assign an orientation $D(e)$ such that the first p positions of $D(e)$ form the set $y(e)$. This gives an orientation of G with the desired property. □

We proved that the answer to Question 7.5 is positive for pairs (r, p) for which $[r]^{(p)}$ has the fixed intersection property. Moreover, the answer is false for $(r, p) = (4, 2)$, which is the smallest pair such that $[r]^{(p)}$ does not have this property. It would be interesting to know if the answer to Question 7.5 is positive precisely for those pairs (r, p) for which $[r]^{(p)}$ has the fixed intersection property.

Conjecture 7.29. *Let p, r be integers such that $r \geq p \geq 2$. If $[r]^{(p)}$ does not have the fixed intersection property, then the answer to Question 7.5 is no.*

There are several interesting questions related to the fixed intersection property. We recall the statement of our main conjecture, which says that if $r \neq 2p$, then $[r]^{(p)}$ has the fixed intersection property.

Conjecture 7.11. *Let n, p be positive integers. If $n > 2p$, then $[n]^{(p)}$ has the fixed intersection property.*

A similar question considers functions $f : [r]^{(p)} \to \mathbb{N}^{(p)}$.

Question 7.30. *For what choices of r and p does every non-constant function $f\colon [r]^{(p)} \to \mathbb{N}^{(p)}$ fix an intersection?*

Obviously, Conjecture 7.11 is true for all pairs (r, p) that satisfy the condition of Question 7.30. Moreover, we have proved that this condition is satisfied for any fixed p as long as r is sufficiently large. However, it is not possible to resolve Conjecture 7.11 solely by answering Question 7.30, because there exists a non-constant function $f : [11]^{(5)} \to \mathbb{N}^{(5)}$ that does not fix an intersection. We found one such function via brute-force computer search.

It is also interesting to consider functions $f : [r]^{(p)} \to [r]^{(p)}$ that have some condition imposed on them. Here is a natural question.

Question 7.31. *For what choices of r and p does every bijection $f\colon [r]^{(p)} \to [r]^{(p)}$ fix an intersection?*

There is a simple counting argument that shows that Question 7.31 is satisfied by pairs (r, p) where $r = \Omega(p^2)$. Suppose that $f\colon [r]^{(p)} \to [r]^{(p)}$ is a bijection and assume for convenience that $f([p]) = [p]$. There are exactly $\binom{r-p}{p}$ elements of $[r]^{(p)}$ that do not intersect $[p]$ and $\binom{r}{p} - \binom{r-p}{p}$ elements that intersect $[p]$. So if there is no $x \in [r]^{(p)}$ such that $|f(x) \cap [p]| = |x \cap [p]| = 0$ then $\binom{r}{p} > 2\binom{r-p}{p}$ which is equivalent to $(1 + p/(r-p))\,(1 + p/(r-p-1)) \cdots (1 + p/(r-1)) > 2$. But then $\exp\left(p/(r-p) + \cdots + p/(r-1)\right) > 2$. If $r \geq cp^2$ then the left hand side is at most $e^{1/(c-1)}$ which is not greater than 2, provided that c is sufficiently large.

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