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Linear Waves in the Interior of Extremal Black Holes II

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Abstract. We consider solutions to the linear wave equation in the interior region of extremal Kerr black holes. We show that axisymmetric solutions can be extended continuously beyond the Cauchy horizon and, moreover, that if we assume suitably fast polynomial decay in time along the event horizon, their local energy is finite. We also extend these results to non-axisymmetric solutions on slowly rotating extremal Kerr–Newman black holes. These results are the analogues of results obtained in Gajic (Commun Math Phys 353(2), 717–770, 2017) for extremal Reissner–Nordström and stand in stark contrast to previously established results for the subextremal case, where the local energy was shown to generically blow up at the Cauchy horizon.

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Birkhäuser

1. Introduction

In the precursor [23] of this paper, we established the following results for the linear wave equation,

$$\Box_g \phi = 0, \tag{1.1}$$

in the black hole interior of extremal Reissner–Nordström spacetimes, which describe maximally charged ($e^2 = M^2$), stationary, spherically symmetric black holes (see [24] for an overview of the geometry of Reissner–Nordström spacetimes):

(A) Uniform boundedness and extendibility of ϕ in C^0 across the Cauchy horizon (Theorem 1 of [23]).

(B) Extendibility of ϕ in H^1_{loc} across the Cauchy horizon (Theorems 2 and 3 of [23]).

(C) Extendibility of ϕ in $C^{0,\alpha}$, with $0 < \alpha < 1$, across the Cauchy horizon (Theorem 5 of [23]).

(D) Extendibility of spherically symmetric ϕ in C^1 across the Cauchy horizon (Theorem 4 of [23]).

(E) Extendibility of spherically symmetric ϕ in C^2 across the Cauchy horizon (Theorem 6 of [23]).

For result (A), we considered Cauchy initial data for ϕ on an asymptotically flat spacelike hypersurface intersecting the event horizon, which decay suitably fast towards spacelike infinity. For results (B), (C) and (D) we imposed stronger decay estimates in affine time on ϕ and its tangential derivatives along the event horizon than those that had previously been established in [6,7] for ϕ arising from Cauchy data. The required decay estimates have been obtained in [3] for suitable Cauchy data. For result (E) we assumed more precise asymptotics of ϕ along the event horizon, which are motivated by the numerical results in [27] and have not yet been shown to hold for ϕ arising from generic, suitably decaying Cauchy data in a mathematically rigorously setting.

In this paper, we shall prove the analogues of (A), (B) and (C) for axisymmetric solutions ϕ to (1.1) in the black hole interior of extremal Kerr–Newman spacetimes; see Theorem 1–4 below. The Kerr–Newman spacetimes are a three-parameter family, characterised by a mass M, a rotation parameter a and a charge e [31]. Extremal Kerr–Newman spacetimes constitute a two-parameter subfamily of spacetimes, satisfying the constraint $M^2 = a^2 + e^2$; they can be viewed as a continuous family that connects the extremal Reissner–Nordström solutions ($a^2 = 0$) to the extremal Kerr solutions ($a^2 = M^2$). For an overview of the geometry of Kerr–Newman spacetimes, see [10].

In [8], polynomial decay in affine time of axisymmetric ϕ and its tangential derivatives was shown to hold along the event horizon of extremal Kerr ($a^2 = M^2$) for suitably decaying Cauchy initial data. To obtain the analogue of (A) for axisymmetric ϕ in the extremal Kerr interior, we will assume the decay rates that follow from [8]. For the analogue of (A) for axisymmetric ϕ in extremal Kerr–Newman spacetimes with $a^2 < M^2$ and, moreover, for the analogues of (B) and (C) for axisymmetric ϕ in *any* extremal Kerr–Newman spacetime,

we assume polynomial decay in time of ϕ along the event horizon that is *conjectured*, but has not yet been *proved*, to hold.

Note that the methods involved in proving results (D) and (E) rely fundamentally on the spherical symmetry of ϕ and the background spacetime. For this reason, they do not carry over to extremal Kerr–Newman.

In addition, we will show that we can drop the axisymmetry assumption on ϕ and prove the analogues of (A), (B) and (C) in extremal Kerr–Newman spacetimes that are sufficiently close to extremal Reissner–Nordström, i.e. with a sufficiently small rotation parameter a; see Theorems 5–7 below. We refer to this subfamily of extremal Kerr–Newman as *slowly rotating* extremal Kerr–Newman. We *assume*, again, the decay for ϕ along the event horizon that is expected to hold for suitably decaying Cauchy initial data in this setting. This assumption is now also necessary for the analogue of (A), as the required polynomial decay has not yet been proved to hold for ϕ (without axisymmetry) along the event horizon of slowly rotating extremal Kerr–Newman.

The analogue of (A) has recently been obtained for the wave equation in *sub*extremal Reissner–Nordström ($e^2 < M^2$) [21] and subextremal Kerr ($a^2 < M^2$) [20] by Franzen (see also the results of Hintz [25] in the very slowly rotating setting, where $a^2 \ll M^2$), whereas the analogue of (B) has been shown to *fail* in subextremal Reissner–Nordström for generic Cauchy data [29] by Luk–Oh. See also related results concerning instabilities in subextremal Kerr [19,30].

The results of this paper are related to Christodoulou's formulation of the strong cosmic censorship conjecture [14]. Indeed, the analogue of the H^1_{loc} extendibility result (B) in extremal Kerr, if also applicable in the context of the vacuum Einstein equations, would provide a construction of dynamical black hole interiors arising from perturbations of extremal Kerr spacetimes which are *extendible* beyond their Cauchy horizons with Christoffel symbols that are locally L^2 with respect to spacetime integration, which is precisely the regularity class considered by Christodoulou. As such, the corresponding initial data would not lie in the class of initial data to which the strong cosmic censorship applies (and would therefore certainly be expected to be *non*-generic). See also related conjectures in the introduction of [23].

1.1. Linear Waves in the Exterior Region of Extremal Kerr

We will review in this section several results for the wave Eq. (1.1) in the *exterior* region of extremal Kerr.

Aretakis considered in [8] axisymmetric solutions ϕ to (1.1) in the exterior region of extremal Kerr, arising from Cauchy data on a spacelike hypersurface Σ intersecting the event horizon $\mathcal{H}^+$; see Fig. 1. He established polynomial decay in time for ϕ everywhere in the exterior, including along $\mathcal{H}^+$.

In [4], he, moreover, proved the existence of conserved quantities, the *Aretakis constants*, along $\mathcal{H}^+$ for solutions ϕ (that need not be axisymmetric). If non-vanishing, these constants constitute an obstruction to the decay of either ϕ itself or its transversal derivative. Since axisymmetric solutions ϕ have been shown to decay along $\mathcal{H}^+$, this means that, generically, their transversal

Figure 1. The Penrose diagram of the maximally analytic extension of extremal Kerr–Newman

derivatives *cannot* decay. Furthermore, higher-order transversal derivatives will generically blow up in infinite time along $\mathcal{H}^+$. These non-decay and blow-up results have been dubbed "the Aretakis instability" in the literature [27].

Lucietti–Reall generalised the Aretakis constants to higher-spin equations in extremal Kerr in [28]. In particular, they showed that conserved quantities also form an obstruction to the decay of solutions to the Teukolsky equation, which governs the evolution of perturbations of certain components of the curvature tensor in the context of the linearised Einstein equations.

In [9] Aretakis extended the results of [8] to show non-decay and blow-up of higher-order derivatives of ϕ even in the case of data with vanishing Aretakis constants. There is still no proof of pointwise and energy boundedness or decay for non-axisymmetric ϕ in the exterior region of extremal Kerr (cf. a complete

picture of the boundedness and decay properties of the linear wave equation in the full *sub*extremal range of Kerr–Newman) spacetimes has recently been obtained in [15,17]).

The main difficulty when studying non-axisymmetric ϕ in the exterior of extremal Kerr is that the geometric phenomena of *superradiance*, the *trapping of null geodesics* and the degeneration of the *local red-shift* effect at $\mathcal{H}^+$ are strongly coupled (in contrast with the subextremal case); see the discussion in the introduction of [8] for more details. Based on numerical studies of quasi-normal modes on extremal Kerr [2,12], the expectation is that ϕ with a *fixed* azimuthal number $m \neq 0$ arising from initial data supported away from $\mathcal{H}^+$ will decay slower than axisymmetric ϕ (for which $m = 0$) both away from $\mathcal{H}^+$ and along $\mathcal{H}^+$.

1.2. Linear Waves in the Interior Region of Extremal Kerr–Newman

In this section, we will give an overview of the main theorems proved in this paper; we will state more detailed versions of the theorems in Sect. 3. In Sect. 2 we will give the precise definitions of the spacetime regions of interest in extremal Kerr–Newman that are mentioned in the paragraphs below and we will present the construction of double-null coordinates that cover these regions.

In this paper, we will restrict to a spacetime rectangle $\mathcal{D}_{u_0,v_0}$, which is a subset of $\mathcal{M} \cup \mathcal{CH}^+$, where $\mathcal{M}$ denotes the extremal Kerr–Newman manifold and $\mathcal{CH}^+$ is the inner horizon of extremal Kerr–Newman. We take $\mathcal{D}_{u_0,v_0}$ to be the intersection of the causal future of the event horizon segment $\mathcal{H}^+ \cap \{v \geq v_0\}$ and the causal past of the inner horizon segment $\mathcal{CH}^+ \cap \{u \leq u_0\}$, with respect to the manifold-with-boundary $\mathcal{M} \cup \mathcal{CH}^+$, where v_0 and u_0 are chosen suitably, such that restriction of $\mathcal{D}_{u_0,v_0}$ to the interior region is entirely contained within the domain of the (u, v) Eddington–Finkelstein-type double-null coordinates; see Fig. 1. Note that we have defined $\mathcal{D}_{u_0,v_0}$ to include a segment of $\mathcal{CH}^+$.

We can employ an ingoing null coordinate $U(u)$ in $\mathcal{M} \cap \mathcal{D}_{u_0,v_0}$, which can be extended across $\mathcal{H}^+$, and an outgoing null coordinate $\widetilde{V}(v)$, which can be extended beyond $\mathcal{CH}^+$, to express $\mathcal{D}_{u_0,v_0}$ as the following set:

$$\mathcal{D}_{u_0,v_0} = \{0 \leq U \leq U(u_0),\ \widetilde{V}(v_0) \leq \widetilde{V} \leq 0,\ (U, V) \neq (0, 0)\},$$

where $U = 0$ at $\mathcal{H}^+$ and $\widetilde{V} = 0$ at $\mathcal{CH}^+$.

We equip $\mathcal{H}^+$ and $\underline{H}_{v_0}$, the ingoing null hypersurface in $\mathcal{D}_{u_0,v_0}$ which is a subset of $\{v = v_0\}$, with characteristic initial data for the wave Eq. (1.1).

We can also consider solutions ϕ arising from Cauchy initial data for (1.1) on an asymptotically flat spacelike hypersurface Σ in extremal Kerr–Newman. We will choose a hypersurface Σ that has a non-trivial intersection with the black hole interior; see the discussion in [5] for why this is a natural choice. As a consequence of the geometry of the interior of extremal Kerr–Newman, Σ must be incomplete; see Fig. 1. We restrict to the future domain of dependence of Σ, which we denote by $D^+(\Sigma)$. The inner horizon $\mathcal{CH}^+$ contains part of the boundary of $D^+(\Sigma)$, so we will sometimes refer to $\mathcal{CH}^+$ as the *Cauchy horizon*.

By choosing u_0 and v_0 appropriately, the rectangle $\mathcal{D}_{u_0,v_0}$ is a subset of $D^+(\Sigma) \cup \mathcal{CH}^+$. The characteristic data on $\mathcal{H}^+ \cup \underline{H}_{v_0}$ can therefore be taken to be *compatible* with the decay of ϕ and its tangential derivatives along $\mathcal{H}^+ \cup \underline{H}_{v_0}$, that is expected to hold generically for ϕ arising from suitable Cauchy initial data on Σ.

In Sect. 1.2.1 below we only consider axisymmetric solutions to (1.1) on extremal Kerr–Newman. In Sect. 1.2.2 we instead consider slowly rotating extremal Kerr–Newman spacetimes, i.e. extremal Kerr–Newman spacetimes with a rotation parameter $0 \leq a^2 < a_c^2$, where $0 < a_c = a_c(M) < M$ can be explicitly obtained by solving a quadratic equation. In this case, we *do not* impose axisymmetry on the solutions to (1.1). See Sect. 1.3 for the precise definition of a_c.

1.2.1. Axisymmetric Solutions. We first formulate an analogue of Theorem 1 of [23] for axisymmetric solutions ϕ to (1.1) in extremal Kerr–Newman, where ϕ arises from characteristic initial data along $\mathcal{H}^+ \cup \underline{H}_{v_0}$.

Theorem 1 (L^∞ boundedness and C^0 extendibility for axisymmetric solutions). *Let ϕ be an axisymmetric solution to* (1.1) *in extremal Kerr–Newman arising from suitably regular characteristic initial data on $\underline{H}_{v_0} \cup \mathcal{H}^+$, such that for some $\epsilon > 0$,*

$$\sup_{v_0 \leq v < \infty} \sum_{|k| \leq 2} \int_{S^2_{-\infty,v}} |\nabla\!\!\!\!/\,^k \phi|^2 < \infty,$$

$$\sum_{0 \leq j_1 + j_2 \leq 4} \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^{1+\epsilon} \left| \nabla\!\!\!\!/\,^{j_1} \partial_v^{j_2+1} \phi \right|^2 + \left| \nabla\!\!\!\!/\,^{j_1+1} \partial_v^{j_2} \phi \right|^2 < \infty.$$

where $\nabla\!\!\!\!/\,$ denotes derivatives tangential to 2-spheres $S^2_{-\infty,v}$ that foliate $\mathcal{H}^+$. Then, there exists a constant $C = C(M, a, \epsilon) > 0$ and a natural norm $D_0 > 0$ on initial data for ϕ, such that

$$|\phi| \leq C D_0,$$

everywhere in $\mathcal{M} \cap \mathcal{D}_{u_0,v_0}$. Moreover, ϕ admits a C^0 extension beyond $\mathcal{CH}^+$.

Theorems 3.5 and 3.6 together form a more precise version of Theorem 1.

In view of the decay results along $\mathcal{H}^+$ in [8] for ϕ in extremal Kerr arising from Cauchy initial data on a spacelike hypersurface Σ, we can reformulate Theorem 1 if we restrict to the subfamily of extremal Kerr spacetimes, where we consider suitably regular Cauchy data along Σ in accordance with the results of [8]:

Theorem 2 (L^∞ boundedness and C^0 extendibility for axisymmetric solutions in extremal Kerr). *Let ϕ be an axisymmetric solution to* (1.1) *in extremal Kerr arising from suitably regular and decaying data on Σ. Then, there exists a constant $C = C(M, \Sigma) > 0$ and a natural norm $D_0 > 0$ on initial data for ϕ, such that*

$$|\phi| \leq C D_0,$$

everywhere in $D^+(\Sigma)$. Moreover, ϕ admits a C^0 extension beyond $\mathcal{CH}^+$.

Furthermore, we obtain the analogue of Theorem 2 of [23]:

Theorem 3 (H^1_{loc} extendibility for axisymmetric solutions). *Let ϕ be an axisymmetric solution to (1.1) in extremal Kerr–Newman arising from suitably regular characteristic initial data on $\underline{H}_{v_0} \cup \mathcal{H}^+$, such that*

$$\int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^2 (\partial_v \phi)^2 + |\slashed{\nabla} \phi|^2 < \infty. \tag{1.2}$$

Then, ϕ admits an extension beyond $\mathcal{CH}^+$ that is H^1_{loc} with respect to spacetime integration.

In [23] we reformulated Theorem 2 of [23] by imposing Cauchy data on a spacelike hypersurface instead of characteristic data on the event horizon to obtain Theorem 3 of [23]. We made use of the improved decay results along the event horizon of extremal Reissner–Nordström that are proved in [3]. However, as the decay estimates for ϕ along $\mathcal{H}^+$ that are necessary for (1.2) to hold have not yet been obtained for suitable data on Σ in any extremal Kerr–Newman spacetime with $a \neq 0$, we cannot yet reformulate Theorem 3 above by imposing Cauchy data on Σ.[1] Theorem 3 follows from Theorem 3.2 after applying the estimates (1.3) and (2.19).

We can further conclude that ϕ can be extended beyond $\mathcal{CH}^+$ in the Hölder space $C^{0,\alpha}$ with $\alpha < 1$. This result is the analogue of Theorem 5 of [23].

Theorem 4 ($C^{0,\alpha}$ extendibility of axisymmetric solutions). *Let $\alpha < 1$. Let ϕ be an axisymmetric solution to (1.1) in extremal Kerr–Newman arising from suitably regular and decaying characteristic initial data on $\underline{H}_{v_0} \cup \mathcal{H}^+$. Then, ϕ admits a $C^{0,\alpha}$ extension beyond $\mathcal{CH}^+$.*

The precise necessary initial decay requirements along $\mathcal{H}^+$ appear in Theorem 3.7.

1.2.2. Slowly Rotating Extremal Kerr–Newman. We now restrict to the slowly rotating subfamily of extremal Kerr–Newman spacetimes, satisfying $0 \leq |a| < a_c$, where a_c is the parameter described above. In particular, this subfamily *excludes* extremal Kerr. We will state analogues of the results from Sect. 1.2.1 in slowly rotating extremal Kerr–Newman <u>without</u> the restriction to axisymmetric solutions of (1.1).

In slowly rotating extremal Kerr–Newman we can obtain L^∞ boundedness and C^0 extendibility without an axisymmetry assumption on ϕ.

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[1] For Kerr–Newman spacetimes with $a^2 < a_c^2$, the Hawking vector field, which is a Killing vector field that is null along $\mathcal{H}^+$ and is precisely defined in Sect. 2.4, will also be timelike in the exterior region in a neighbourhood of $\mathcal{H}^+$. We expect this geometric property would significantly simplify the difficulties in the analysis of non-axisymmetric solutions in the exterior region of slowly rotating Kerr–Newman spacetimes compared to the extremal Kerr case and would lead to better decay estimates for ϕ along the event horizon than expected in extremal Kerr; see also the discussion in Sect. 1.1.

Theorem 5 (L^∞-boundedness and C^0-extendibility in slowly rotating extremal Kerr–Newman). *Let ϕ be a solution to (1.1) in extremal Kerr–Newman with $|a| < a_c$, for $0 < a_c < M$ suitably small, arising from suitably regular and decaying characteristic data on $\underline{H}_{v_0} \cup \mathcal{H}^+$. Then, there exists a constant*

$$C = C(M, a, \Sigma) > 0$$

and a natural norm $D_0 > 0$ on initial data for ϕ, such that

$$|\phi| \leq CD_0,$$

everywhere in $\mathcal{D}_{u_0,v_0}$. Moreover, ϕ admits a C^0 extension beyond $\mathcal{CH}^+$.

See Theorems 3.5 and 3.6 in Sect. 3 for precise requirements for the initial data on $\underline{H}_{v_0} \cup \mathcal{H}^+$. As there are presently no decay results available for non-axisymmetric solutions in the exterior region of slowly rotating extremal Kerr–Newman, we do not reformulate Theorem 5 by imposing Cauchy data on Σ instead of characteristic data on $\underline{H}_{v_0} \cup \mathcal{H}^+$.

We also obtain an analogue of Theorem 3 for ϕ, without the assumption of axisymmetry, in slowly rotating extremal Kerr–Newman.

Theorem 6 (H^1_{loc}-extendibility in slowly rotating Kerr–Newman). *Let ϕ be a solution to (1.1) in extremal Kerr–Newman with $|a| < a_c$, for $0 < a_c < M$ suitably small, arising from suitably regular and decaying characteristic data on $\underline{H}_{v_0} \cup \mathcal{H}^+$. Then, ϕ admits a H^1_{loc} extension beyond $\mathcal{CH}^+$.*

Here, we require decay of more derivatives in the initial data along the event horizon, compared to Theorem 3. See Theorems 3.5 and 3.4 in Sect. 3 for the precise decay rates. Theorem 6 follows from Theorems 3.2, 3.4 and 3.5, together with the estimate (2.19).

Finally, we obtain an analogue of Theorem 4 without the assumption of axisymmetry for ϕ in slowly rotating extremal Kerr–Newman:

Theorem 7 ($C^{0,\alpha}$ extendibility of ϕ in slowly rotating extremal Kerr–Newman). *Let $\alpha < 1$. Let ϕ be a solution to (1.1) in extremal Kerr–Newman with $|a| < a_c$, for $0 < a_c < M$ suitably small, arising from suitably regular and decaying characteristic data on $\underline{H}_{v_0} \cup \mathcal{H}^+$. Then, ϕ admits a $C^{0,\alpha}$ extension beyond $\mathcal{CH}^+$.*

The precise necessary initial decay requirements along $\mathcal{H}^+$ appear in Theorem 3.7.

1.3. Main Ideas in the Proofs of Theorems 1–7

In this section, we will outline the main steps in the proofs of Theorem 1–7. We will restrict to the region $\mathcal{D}_{u_0,v_0}$ in extremal Kerr–Newman with $0 \leq |a| \leq M$ by default, unless specifically mentioned otherwise, and consider appropriate characteristic initial data for ϕ on $\underline{H}_{v_0} \cup \mathcal{H}^+$. We will highlight new difficulties that arise in extremal Kerr–Newman when $a \neq 0$, compared to extremal Reissner–Nordström (where $a = 0$), which was treated in [23].

1.3.1. Part 0: Constructing a Double-Null Foliation. Before carrying out any estimates involving the wave equation, we first construct a suitable double-null foliation of the interior region of extremal Kerr–Newman. As Kerr–Newman spacetimes with $a \neq 0$ are not spherically symmetric, in contrast with Reissner–Nordström spacetimes, the existence of global double-null coordinates in the interior region is not immediate. In [22], a suitable global double-null foliation of extremal Kerr–Newman is constructed, which covers both the exterior and interior regions, following ideas of [32]. We will use the results of [22] here.

1.3.2. Part 1: Vector Field Multipliers and Energy Estimates (Theorem 3). We obtain uniform bounds on weighted L^2 norms of ϕ along null hypersurfaces by means of energy estimates. Energy estimates are derived by using the vector field method; see for example [26] for a general overview and the discussion in Sect. 2.5 for further particulars. Energy estimates for axisymmetric ϕ are obtained very similarly to the energy estimates in extremal Reissner–Nordström in [23]; we use the following vector field multiplier:

$$N_{p,q} = u^p \partial_u + v^q \partial_v,$$

with $p = q = 2$, where u and v are double-null coordinates obtained in Part 0 that are akin to the Eddington–Finkelstein double-null coordinates in extremal Reissner–Nordström. See Sect. 2.2 for an overview of the construction and main properties of the Eddington–Finkelstein-type double-null coordinates u and v in extremal Kerr–Newman, and see Sect. 2.5 for more details regarding $N_{p,q}$.

As in extremal Reissner–Nordström, the energy estimates rely crucially on the following polynomial decay rate of the g_{uv} component of the metric in Eddington–Finkelstein-type double-null coordinates:

$$g_{uv} \sim (v + |u|)^{-2};$$

see Sect. 2.3 for the corresponding estimates. The above bounds play an important role in the proof of Theorem 3; see Sect. 4.

If we drop the axisymmetry assumption on ϕ, we have to take into account additional error terms in the energy estimates; most notably, extra error terms arise that involve the non-vanishing *torsion* of the double-null foliation, denoted by ζ. The torsion can be expressed as a commutator,

$$\zeta = \frac{1}{4}\Omega^{-2}[L, \underline{L}],$$

where L and $\underline{L}$ are vector fields that are tangent to null generators of the outgoing and ingoing null hypersurfaces, respectively. See Sect. 2.3.

In the $a = 0$ case, L and $\underline{L}$ are coordinate vector fields, so they commute, and ζ vanishes everywhere. If $a \neq 0$, ζ does not vanish. It turns out, however, that axisymmetric ϕ still satisfy $\zeta(\phi) = 0$ if $a \neq 0$, so the error terms involving ζ do not form an obstruction for axisymmetric ϕ.

In the case of non-axisymmetric ϕ, we can estimate the error terms involving ζ by invoking the *Hawking vector field*, which we denote by H. In Eddington–Finkelstein-type (u, v) coordinates, we can express

$$H = \frac{1}{2}(\partial_u + \partial_v).$$

The vector field H is Killing and null along $\mathcal{H}^+$ and $\mathcal{CH}^+$. Moreover, it extends as a timelike vector field near the horizons, if $|a| < a_c < M$; in fact, it is precisely the requirement of a timelike H near the horizons that determines a_c. See Sect. 2.4 for more details. The timelike character of H implies that the error terms appearing in energy estimates with respect to the weighted vector field

$$Y_p = |u|^p H,$$

with $p \geq 0$, have a good sign. We can still employ $N_{p,q}$ in suitable neighbourhoods of the $\mathcal{H}^+$ and $\mathcal{CH}^+$ that have a finite spacetime volume, but we use Y_p in their complement in $\mathcal{D}_{u_0,v_0}$. This method gives rise to an ϵ-loss in the exponents of the u and v weights that appear in the energies, compared to the case of axisymmetric ϕ, which prevents us from directly inferring Theorem 6. See Sect. 5.[2]

1.3.3. Part 2: Commutation Vector Fields and Pointwise Estimates (Theorems 1 and 5).
We subsequently use the uniformly bounded weighted L^2 norms from Part 1 to obtain a uniform bound for the L^∞ norm of ϕ everywhere in the interior and to prove continuous extendibility across $\mathcal{CH}^+$. For this purpose, we apply standard Sobolev inequalities on the spheres $S^2_{u,v}$ corresponding to the double-null foliation, i.e. we can estimate

$$||\phi||_{L^\infty\left(S^2_{u,v}\right)} \leq C \sum_{|k| \leq 2} ||\nabla\!\!\!\!/^{\,k}\phi||_{L^2\left(S^2_{u,v}\right)},$$

where $\nabla\!\!\!\!/$ denotes the covariant derivative restricted to $S^2_{u,v}$. Moreover, we apply the fundamental theorem of calculus along the null generators of ingoing null hypersurfaces, together with a (weighted) Cauchy–Schwarz inequality, to arrive at the following estimate:

$$\int_{S^2_{u,v}} \phi^2 \, \mathrm{d}\mu_{\not g} \leq \int_{S^2_{-\infty,v}} \phi^2 \, \mathrm{d}\mu_{\not g} + \int_{-\infty}^u |u'|^{-p} \, \mathrm{d}u' \int_{-\infty}^u \int_{S^2_{u',v}} |u'|^p (\underline{L}\phi)^2 \, \mathrm{d}\mu_{\not g} \, \mathrm{d}u', \tag{1.3}$$

with $p > 1$; see also the proof of Proposition 7.1. The second term on the right-hand side of the inequality can be controlled by a weighted energy along an ingoing null hypersurface.

In order to estimate $||\nabla\!\!\!\!/^{\,k}\phi||_{L^2\left(S^2_{u,v}\right)}$ with $k \geq 1$, we also need to consider appropriately weighted energies for angular derivatives of ϕ. Replacing ϕ by $\nabla\!\!\!\!/^{\,k}\phi$ (or $\partial^k_{\vartheta^A}\phi$, where ϑ^A, with $A = 1, 2$, are coordinates on the spheres $S^2_{u,v}$)

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[2] An interesting question that remains open is whether a loss of derivatives in the initial energies is necessary to prevent the ϵ-loss in the $a \neq 0$ case. That is to say, whether it is possible to construct a sequence of suitably regular and decaying initial data along $\mathcal{H}^+$ and $\underline{H}_{v_0}$ for which the uniform constant appearing in energy estimates without a loss of derivatives blows up as we move along the sequence. Note that this construction is not possible if we restrict to axisymmetric solutions.

in the estimates from Part 1 results in error terms that cannot be controlled using the methods mentioned in Part 1.

Obtaining estimates for angular derivatives of ϕ in $L^2(S^2_{u,v})$ turns out not to be a problem in extremal Reissner–Nordström, as the spacetime is spherically symmetric, which means that the angular momentum operators O_i with $i = 1, 2, 3$, which are Killing vector fields generating the isometries of spherical symmetry, control all derivatives tangential to the round spheres of the double-null foliation; see for example Sect. 2.1 of [23] for explicit expressions of O_i with respect to spherical polar coordinates. Since the vector fields O_i are Killing, they commute with the operator $\Box_g$, so the functions $O_i(\phi)$ are also solutions to (1.1). Any energy estimate for ϕ therefore automatically holds for $O_i(\phi)$.

In extremal Kerr–Newman with $a \neq 0$, however, the only angular momentum operator that remains a Killing vector field is Φ, the generator of rotations about the axis of symmetry. Fortuitously, there exists a *second-order* operator Q, the *Carter operator*, which also commutes with $\Box_g$. This operator is closely related to the conserved *Carter constant*; see [11]. See also Andersson–Blue [1], for example, for more details on the Carter constant and operator, and for applications of the commutation property of Q.

The operator Q, together with the vector fields Φ and T, the Killing vector field corresponding to time-translation symmetry, controls the derivatives of ϕ that are tangent to the spheres of the Boyer–Lindquist foliation of Kerr–Newman. To obtain control over derivatives tangent to the spheres $S^2_{u,v}$, (which do <u>not</u> coincide with the Boyer–Lindquist spheres if $a \neq 0$), we need to additionally commute $\Box_g$ with the vector fields L and $\underline{L}$.

In contrast with the error terms arising from commuting $\Box_g$ with $\slashed{\nabla}$, or ∂_{ϑ^A}, the error terms corresponding to a commutation with L and $\underline{L}$ *can* be controlled via the methods of Part 1 by using profusely the Killing property of Q, Φ and T; see the estimates in Sect. 6.2. As a result, we are able to prove Theorems 1 and 5; see Sects. 7.1 and 7.2.

1.3.4. Part 3: Decay Estimates (Theorems 4, 6 and 7). In the final step, we consider the difference function $\psi = \phi - \phi|_{\mathcal{H}^+}$, such that ψ vanishes along $\mathcal{H}^+$. The function ψ has the advantage that it can be shown to decay uniformly in u. By treating the wave equation as a transport equation for $L\phi$ along ingoing null generators, we can use the u-decay of ψ to obtain v-decay of $||L\phi||_{L^2(S^2_{u,v})}$ with the rate $v^{-2+\epsilon}$, for any $\epsilon > 0$. If ϕ is axisymmetric, we can in fact improve this decay rate to $v^{-2} \log(v)$. By commuting further with L and $\underline{L}$ and applying standard Sobolev inequalities on $S^2_{u,v}$, this allows one to obtain *pointwise* decay for $|L\phi|$ with the rates $v^{-2+\epsilon}$ and $v^{-2} \log(v)$, respectively.

The outgoing derivative corresponding to double-null coordinates that cover the region *beyond* $\mathcal{CH}^+$ in the maximal analytic extension of extremal Kerr–Newman, denoted by $\partial_{\widetilde{V}}$, is related to L as follows:

$$|\partial_{\widetilde{V}} \phi| \sim v^2 |L\phi|.$$

Since we cannot remove the ϵ in the decay rate $v^{-2+\epsilon}$ of $|L\phi|$, we are unable to infer boundedness of $\partial_{\widetilde{V}}\phi$ at $\mathcal{CH}^+$ or C^1 extendibility of ϕ at $\mathcal{CH}^+$. We *can*, nevertheless, infer that ϕ is extendible as a $C^{0,\alpha}$ function beyond $\mathcal{CH}^+$, for any $\alpha < 1$, if the initial data along $\underline{H}_{v_0} \cup \mathcal{H}^+$ are suitably regular and decaying, thereby proving Theorems 4 and 7; see Sect. 7.3.

Moreover, we can integrate the v-decaying $L^2(S^2_{u,v})$ norm of $L\phi$ in the v-direction, in slowly rotating extremal Kerr–Newman, to obtain boundedness of $\int_{H_u} v^2(L\phi)^2$ and also $\int_{\underline{H}_v} v^2\Omega^2|\slashed{\nabla}\phi|^2$. In this way we get rid of some of the ϵ-loss in the weights that was present in the energy estimates of Part 1 and arose from the obstruction of ζ to the energy estimates for ϕ without the axisymmetry assumption. This improvement comes at the expense of requiring decay of higher-order derivatives in the initial data, compared to the estimates in Part 1. See Sect. 7.3 for more details. In particular, we can infer Theorem 6.

1.4. Outline

In Sect. 2 we introduce some notation and state estimates relating to the double-null foliation of the interior of extremal Kerr–Newman (Part 0 of Sect. 1.3) that are relevant in the rest of the paper. We state the theorems that are proved in the paper in Sect. 3. We prove energy estimates for axisymmetric solutions ϕ to (1.1) in extremal Kerr–Newman in Sect. 4. Subsequently, we prove energy estimates in slowly rotating extremal Kerr–Newman in Sect. 5, completing Part 1 of Sect. 1.3. In Sect. 6, we commute with L and $\underline{L}$ to arrive at energy estimates for higher-order derivatives of ϕ. Finally, we use the higher-order energy estimates to prove pointwise estimates of ϕ (Part 2 of Sect. 1.3). Moreover, we obtain pointwise decay in v of $L\phi$ in Sect. 7 by making use of higher-order energy estimates, completing Part 3 of Sect. 1.3.

2. The Geometry of Extremal Kerr–Newman

We will first introduce the extremal Kerr–Newman spacetimes in Boyer–Lindquist and Kerr-star coordinates and subsequently present more convenient double-null coordinates, by foliating the spacetime with suitable ingoing and outgoing null hypersurfaces, covering both the exterior and interior regions of extremal Kerr–Newman. Sections 2.2 and 2.3 are based on a more elaborate discussion on double-null foliations of Kerr–Newman that can be found in [22].

2.1. Boyer–Lindquist and Kerr-star Coordinates

Fix the mass parameter $M > 0$ and the rotation parameter $a \in \mathbb{R}$, such that $|a| \leq M$, and let, moreover, the charge parameter e satisfy $e^2 = M^2 - a^2$.

We define the *exterior region of extremal Kerr–Newman* as a manifold $\mathcal{M}_{\text{ext}}$, together with a metric g, where $\mathcal{M}_{\text{ext}} = \mathbb{R}\times(M,\infty)\times\mathbb{S}^2$ can be equipped with the Boyer–Lindquist coordinate chart (t,r,θ,φ), with $t \in \mathbb{R}$, $r \in (M,\infty)$,

$\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$.[3] In these coordinates, the metric g in $\mathcal{M}_{\text{ext}}$ is given by

$$g = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\varphi^2 - 2\frac{(2Mr - e^2)a\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi. \tag{2.1}$$

Here,

$$\Delta := r^2 - 2Mr + a^2 + e^2 = (M - r)^2,$$
$$\rho^2 := r^2 + a^2\cos^2\theta,$$
$$R^2 := r^2 + a^2 + a^2\frac{(2Mr - e^2)\sin^2\theta}{\rho^2}.$$

We define the *interior region of extremal Kerr–Newman* as the manifold $\mathcal{M}_{\text{int}} = \mathbb{R} \times (0, M) \times \mathbb{S}^2$ equipped with a metric that we also denote by g, which can similarly be covered by *Boyer–Lindquist coordinates* (t, r, θ, φ), where now $t \in \mathbb{R}$, $r \in (0, M)$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$. The components of the metric g in $\mathcal{M}_{\text{int}}$ with respect to Boyer–Lindquist coordinates are also given by (2.1).

We define the ingoing *Kerr-star coordinates* $(t_{\text{KS}})_*$ and $(\varphi_{\text{KS}})_*$ on $\mathcal{M}_{\text{ext}}$ or $\mathcal{M}_{\text{int}}$ in the following way:

$$(t_{\text{KS}})_*(t, r) = t + (r_{\text{KS}})_*(r), \tag{2.2}$$

$$(\varphi_{\text{KS}})_*(r, \varphi) = \varphi + \int_{r_0}^{r} \frac{a}{\Delta(r')}\,\mathrm{d}r', \tag{2.3}$$

where

$$(r_{\text{KS}})_*(r) = \int_{r_0}^{r} \frac{r'^2 + a^2}{\Delta(r')}\,\mathrm{d}r', \tag{2.4}$$

and $r_0 > 0$ is a constant.

We can change from Boyer–Lindquist coordinates to ingoing Kerr-star coordinates

$$((t_{\text{KS}})_*, r, \theta, (\varphi_{\text{KS}})_*)$$

to show that the spacetime $\mathcal{M}_{\text{int}}$ can be smoothly patched to the spacetime $\mathcal{M}_{\text{ext}}$, such that $\mathcal{M}_{\text{int}}$ embeds as the region $\{0 < r < M\}$ of the patched spacetime, and $\mathcal{M}_{\text{ext}}$ embeds as the region $\{r > M\}$. The boundary of $\mathcal{M}_{\text{ext}}$ and $\mathcal{M}_{\text{int}}$ inside the patched spacetime is given by the level set $\{r = M\}$. This boundary is called the *event horizon* and is denoted by $\mathcal{H}^+$. It lies in the causal past of $\mathcal{M}_{\text{int}}$; see also Fig. 2. We denote the patched manifold by $\mathcal{M} := \mathcal{M}_{\text{int}} \cup \mathcal{M}_{\text{ext}} \cup \mathcal{H}^+$. We can write

$$\mathcal{M} = \mathbb{R} \times (0, \infty) \times \mathbb{S}^2,$$

where $(t_{\text{KS}})_* \in \mathbb{R}$, $r \in (0, \infty)$, $\theta \in (0, \pi)$ and $(\varphi_{\text{KS}})_* \in ((\varphi_{\text{KS}})_*(r, 0), (\varphi_{\text{KS}})_*(r, 0) + 2\pi)$.

---

[3] Note that the coordinates (θ, φ) do not cover the full spheres of constant t and constant r and, moreover, the metric degenerates as one approaches the poles. This can be remedied by passing for (θ, φ) to a different chart in a neighbourhood of a great circle segment connecting the poles, as is the case with spherical coordinates on the unit round sphere $\mathbb{S}^2$.
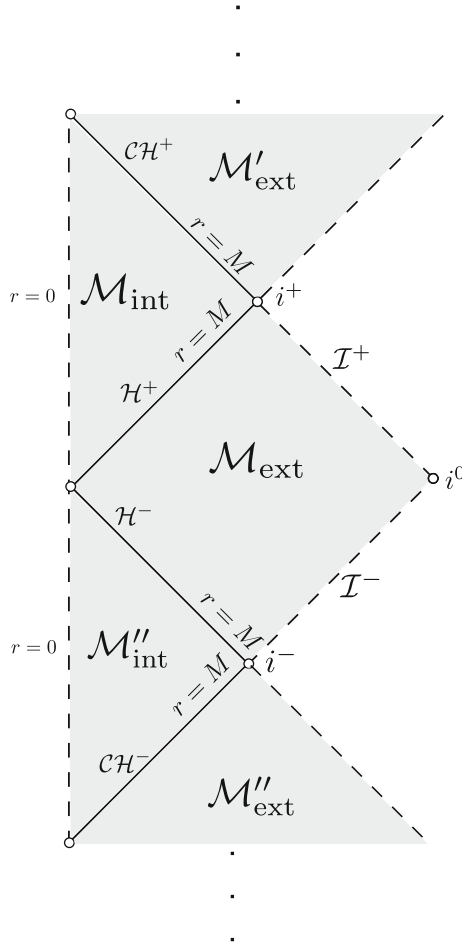
FIGURE 2. The Penrose diagram of $\widetilde{\mathcal{M}}$

We can similarly introduce outgoing Kerr-star coordinates $((\tilde{t}_{\mathrm{KS}})_*, r, \theta, (\varphi_{\mathrm{KS}})_*)$, where

$$(\tilde{t}_{\mathrm{KS}})_* = (t_{\mathrm{KS}})_* - 2(r_{\mathrm{KS}})_*.$$

In these coordinates it is easy to see that $\mathcal{M}_{\mathrm{int}}$ can be smoothly embedded into a bigger spacetime $\mathcal{M}'$, by patching $\mathcal{M}_{\mathrm{int}}$ to a spacetime $\mathcal{M}'_{\mathrm{ext}}$ that is isometric to $\mathcal{M}_{\mathrm{ext}}$. The manifold $\mathcal{M}_{\mathrm{in}}$ is embedded in the patched spacetime as the region $\{0 < r < M\}$ and $\mathcal{M}'_{\mathrm{ext}}$ is embedded as the region $\{r > M\}$. The corresponding boundary $\{r = M\}$ of $\mathcal{M}_{\mathrm{int}}$ and $\mathcal{M}'_{\mathrm{ext}}$ in the patched spacetime lies in the causal future of $\mathcal{M}_{\mathrm{int}}$ and is denoted by $\mathcal{CH}^+$. We refer to this boundary as the *inner horizon*. We can write $\mathcal{M}' = \mathcal{M}_{\mathrm{int}} \cup \mathcal{M}'_{\mathrm{ext}} \cup \mathcal{CH}^+$, or

$$\mathcal{M}' = \mathbb{R} \times (0, \infty) \times \mathbb{S}^2,$$

where $(\tilde{t}_{\mathrm{KS}})_* \in \mathbb{R}$, $r \in (0, \infty)$, $\theta \in (0, \pi)$ and $(\varphi_{\mathrm{KS}})_* \in ((\varphi_{\mathrm{KS}})_*(r, 0), (\varphi_{\mathrm{KS}})_* (r, 0) + 2\pi)$.

As $\mathcal{M}'_{\mathrm{ext}}$ is isometric to $\mathcal{M}_{\mathrm{ext}}$, we can repeat the above procedure ad infinitum to extend the manifold $\mathcal{M} \cup \mathcal{M}'$ further and form an infinite sequence of patched manifolds containing regions isometric to either $\mathcal{M}_{\mathrm{ext}}$ or $\mathcal{M}_{\mathrm{int}}$, glued across horizons. The resulting spacetime $\widetilde{\mathcal{M}}$ is called *maximal analytically extended extremal Kerr–Newman*, and it is depicted in Fig. 2. For the remainder of this paper we will, however, mainly direct our attention to the subset $\mathcal{M} \cup \mathcal{CH}^+$.

2.2. Double-Null Coordinates

In the sections below, we will consider energy fluxes along ingoing and outgoing null hypersurfaces in $\mathcal{M}$. It is therefore more natural to work in double-null coordinates in $\mathcal{M}$ rather than Kerr-star coordinates.

We first consider $\mathcal{M}_{\mathrm{int}}$, covered by Kerr-star coordinates. If we can construct a *tortoise function* r_* to be of the form $r_*(r, \theta)$, such that the functions

$$2v = t + r_*,$$
$$2u = t - r_*,$$

satisfy the eikonal equations

$$g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \quad \text{and} \quad g^{\alpha\beta} \partial_\alpha v \partial_\beta v = 0,$$

then the level sets $\{u = \text{constant}\}$ and $\{v = \text{constant}\}$ are null hypersurfaces. We will follow a construction of r_* that was introduced by Pretorius–Israel in [32] and allows for suitable, double-null coordinates. We will assume that $a \neq 0$. In the $a = 0$ case we consider Eddington–Finkelstein double-null coordinates; see Sect. 2 of [23].

In [22] the construction of r_* from [32] is used to extend the local double-null coordinates in $\mathcal{M}_{\mathrm{int}}$ to obtain a smooth, *global* Eddington–Finkelstein-type double-null foliation of $\mathcal{M}_{\mathrm{int}} \cap \{r > \frac{e^2}{2M}\}$,[4] such that the 2-surfaces

$$S^2_{u', v'} = \{u = u'\} \cap \{v = v'\}$$

are diffeomorphic to 2-spheres and we, moreover, obtain quantitative bounds on the metric components in double-null coordinates (see Sect. 2.3).

The metric g on $\mathcal{M}_{\mathrm{int}} \cap \{r > \frac{e^2}{2M}\}$ can then be written as follows:

$$g = -4\Omega^2 \mathrm{d}u\mathrm{d}v + \not{g}_{AB}(\mathrm{d}\vartheta^A - b^A \mathrm{d}v)(\mathrm{d}\vartheta^B - b^B \mathrm{d}v), \tag{2.5}$$

where $2u = t - r_*$, $2v = t + r_*$, $\vartheta^1 = \theta_*$ and $\vartheta^2 = \varphi_*$, with $u, v \in \mathbb{R}$, $\theta_* \in (0, \pi)$ and $\varphi_* \in (0, 2\pi)$. The metric components in (2.5) are given by

$$\Omega^2 = \Delta R^{-2},$$

$$b^{\theta_*} = 0, \quad b^{\varphi_*} = \frac{4Mar}{\rho^2 R^2} - \frac{4Mar}{\rho^2 R^2}\bigg|_{r=M},$$

---

[4] If $e \neq 0$ (i.e. $0 < |a| < M$), we cannot cover the entire region $\mathcal{M}_{\mathrm{int}}$ by the double-null coordinates introduced in this section, as we cannot exclude the corresponding null generators forming caustics in the region $\{0 < r < \frac{e^2}{2M}\}$.

$$g_{\theta_*\theta_*} = f_1^2 f_2^2 (\partial_{\theta_*} F)^2 R^{-2} + (\partial_{\theta_*} f_4)^2 R^2 \sin^2 \theta,$$
$$g_{\theta_*\varphi_*} = (\partial_{\theta_*} f_4) R^2 \sin^2 \theta,$$
$$g_{\varphi_*\varphi_*} = R^2 \sin^2 \theta;$$

they are of the same form as the metric components with respect to the double-null coordinates considered in [18]. A precise definition of the functions f_i is given in [22], but for the purposes of this paper we only need the estimates on the metric components that are stated in Sect. 2.3 and are derived in [22].

Note, moreover, that we can express:

$$\det g = f_1^2 f_2^2 (\partial_{\theta_*} F)^2 \sin^2 \theta. \tag{2.6}$$

In the formal limit $a \to 0$ (with $e \neq 0$ fixed) the double-null coordinates $(u, v, \theta_*, \varphi_*)$ become simply Eddington–Finkelstein double-null coordinates on (the interior of) extremal Reissner–Nordström.

As we approach $\mathcal{H}^+$ along constant v hypersurface, the coordinate u goes to $-\infty$. We can, however, introduce a rescaled ingoing null coordinate in order to further extend the double-null coordinates and additionally cover the region $\mathcal{M}_{\text{ext}} \cup \mathcal{H}^+$.

Fix $v_0 \in \mathbb{R}$ and define the function $U : \mathbb{R} \to (0, \infty)$ by $U(u) = M - r(u, v_0, \theta_* = \frac{\pi}{2})$. We can interpret U as a smooth, negative function $U : \mathcal{M}_{\text{int}} \cap \{r > \frac{e^2}{2M}\} \to (0, \infty)$.

In [22] it is shown function $U : \mathcal{M}_{\text{int}} \cap \{r > \frac{e^2}{2M}\} \to \mathbb{R}$ extends smoothly with respect to Kerr-star coordinates to the bigger manifold $\mathcal{M} \cap \{r > \frac{e^2}{2M}\}$, such that $U = 0$ along $\mathcal{H}^+$ and $U < 0$ in $\mathcal{M}_{\text{ext}}$ and moreover, the metric is well defined and non-degenerate with respect to the chart $(U, v, \theta_*, \varphi_*)$ on $\mathcal{M} \cap \{r > \frac{e^2}{2M}\}$:

$$g = -4 \frac{\Omega^2(u, v, \theta_*)}{(r^2\Omega^2)\left(u, v = v_0, \theta_* = \frac{\pi}{2}\right)} dUdv + g_{AB}(d\vartheta^A - b^A dv)(d\vartheta^B - b^B dv).$$

Consequently, $(U, v, \theta_*, \varphi_*)$ defines a smooth coordinate chart on $\mathcal{M} \cap \{r > \frac{e^2}{2M}\}$.

By introducing another function f_5 (see [22]) we can shift the angular coordinate φ_* to a new coordinate $\widetilde{\varphi}_* \in (0, 2\pi)$ and the metric can be written in $(\tilde{u}, \tilde{v}, \widetilde{\theta}_*, \widetilde{\varphi}_*)$ coordinates,

$$g = -4\Omega^2(\tilde{u}, \tilde{v}) d\tilde{u} d\tilde{v} + \widetilde{g}_{AB}(d\widetilde{\vartheta}^A - \tilde{b}^A d\tilde{u})(d\widetilde{\vartheta}^B - \tilde{b}^B d\tilde{u}), \tag{2.7}$$

where $2\tilde{u} = 2u = t - r_*$, $2\tilde{v} = 2v = t + r_*$, $\widetilde{\vartheta}^1 = \widetilde{\theta}_*$, $\widetilde{\vartheta}^2 = \widetilde{\varphi}_*$ and moreover,
$$\Omega^2(\tilde{u}, \tilde{v}) = \Omega^2(r_*) = \Delta R^{-2},$$

$$\tilde{b}^{\widetilde{\theta}_*} = 0, \quad \tilde{b}^{\widetilde{\varphi}_*} = \frac{4Mar}{\rho^2 R^2} - \frac{4Mar}{\rho^2 R^2}\bigg|_{r=M},$$
$$\widetilde{g}_{\widetilde{\theta}_*\widetilde{\theta}_*} = f_1^2 f_2^2 (\partial_{\theta_*} F)^2 R^{-2} + (\partial_{\theta_*} f_5)^2 R^2 \sin^2 \theta,$$
$$\widetilde{g}_{\widetilde{\theta}_*\widetilde{\varphi}_*} = (\partial_{\theta_*} f_5) R^2 \sin^2 \theta,$$
$$\widetilde{g}_{\widetilde{\varphi}_*\widetilde{\varphi}_*} = R^2 \sin^2 \theta.$$

To distinguish these coordinates from the previous double-null coordinates, we have denoted them with tildes $(\tilde{u}, \tilde{v}, \widetilde{\vartheta})$, where $\tilde{v} = v$, $\tilde{u} = u$ and $\widetilde{\theta}_* = \theta_*$.

Now, fix $u_0 \in \mathbb{R}$ and define the function $\widetilde{V} : \mathcal{M}_{\text{int}} \cap \{r > \frac{e^2}{2M}\}$ by $\widetilde{V} = (r - M)(u_0, v, \theta_* = \frac{\pi}{2})$. In [22] it is shown that we can extend $\widetilde{V}$ as a smooth function to the bigger manifold $\mathcal{M}' \cap \{r > \frac{e^2}{2M}\}$, such that $\widetilde{V} = 0$ along $\mathcal{CH}^+$ and $\widetilde{V} > 0$ in $\mathcal{M}'_{\text{ext}}$ and moreover, the metric is well defined and non-degenerate with respect to the chart $(\tilde{u}, \widetilde{V}, \widetilde{\theta}_*, \widetilde{\varphi}_*)$ on $\mathcal{M}' \cap \{r > \frac{e^2}{2M}\}$:

$$g = -4 \frac{\Omega^2(\tilde{u}, \tilde{v}, \widetilde{\theta}_*)}{(r^2\Omega^2)\left(\tilde{u} = u_0, \tilde{v}, \widetilde{\theta}_* = \frac{\pi}{2}\right)} \mathrm{d}\tilde{u}\mathrm{d}\widetilde{V} + \not{g}_{AB}(\mathrm{d}\widetilde{\vartheta}^A - \tilde{b}^A \mathrm{d}\tilde{u})(\mathrm{d}\widetilde{\vartheta}^B - \tilde{b}^B \mathrm{d}\tilde{u}).$$

We will use the notation $(u, v, \theta_*, \varphi_*) = (-\infty, v_0, \theta_*, \varphi_*)$ and $(\tilde{u}, \tilde{v}, \theta_*, \varphi_*) = (u_0, \infty, \theta_*, \varphi_*)$, with $u_0, v_0 < \infty$, for points on $\mathcal{H}^+$ and $\mathcal{CH}^+$, respectively, for the sake of convenience. These points lie in the domain of either the (U, v) or $(\tilde{u}, \widetilde{V})$ double-null coordinates.

In $\mathcal{M}_{\text{int}} \cap \left\{r > \frac{e^2}{2M}\right\} \cup \mathcal{H}^+ \cup \mathcal{CH}^+$ we restrict to the region

$$\mathcal{D}_{u_0,v_0} = \left\{x \in \mathcal{M}_{\text{int}} \cap \left\{r > \frac{e^2}{2M}\right\} \cup \mathcal{H}^+ \cup \mathcal{CH}^+ \colon U(x) \in [0, U(u_0)],\right.$$

$$\left. \widetilde{V}(x) \in \left[\widetilde{V}(v_0), 0\right], (U(x), \widetilde{V}(x)) \neq (0,0)\right\}.$$

Let $v' \in [v_0, \infty)$ and $u' = [-\infty, u_0]$. We will consider the following null hypersurfaces:

$$\underline{H}_{v'} := \{x \in \mathcal{M} \; : \; U(x) \in [0, U(u_0)], \; v(x) = v'\},$$
$$H_{u'} := \{x \in \mathcal{M} \; : \; U(x) = U(u'), \; v(x) \in [v_0, \infty)\},$$

and we refer to the hypersurfaces $\underline{H}_{v'}$ and $H_{u'}$ as ingoing and outgoing null hypersurfaces, respectively.

We will fix $|u_0|$ and v_0 to be suitable large such that

$$\underline{H}_{v_0} \cup H_{u_0} \subset \left(\mathcal{M}_{\text{int}} \cap \left\{r > \frac{e^2}{2M}\right\}\right) \cup \mathcal{H}^+.$$

Consider the null vector fields L and $\underline{L}$, which are tangent to the generators of the outgoing and ingoing null hypersurfaces, respectively, and satisfy $Lv = 1$ and $\underline{L}u = 1$.

The vector field $\underline{L}$ can be naturally expressed in the chart $(u, v, \theta_*, \varphi_*)$. Indeed,

$$L = \partial_v + b^A \partial_{\vartheta^A},$$
$$\underline{L} = \partial_u.$$

Note that we can alternatively express $\underline{L}$ in $(U, v, \theta_*, \varphi_*)$ coordinates:

$$\underline{L} = \frac{\mathrm{d}U}{\mathrm{d}u} \partial_U.$$

From the above expression it is clear that $\underline{L}$ can be extended smoothly as a vector field across $\mathcal{H}^+$ (where it vanishes).

The vector field L can similarly be expressed in the chart $(\tilde{u}, \widetilde{v}, \widetilde{\theta}_*, \widetilde{\varphi}_*)$:

$$L = \partial_{\widetilde{v}},$$
$$\underline{L} = \partial_{\tilde{u}} + \tilde{b}^A \partial_{\widetilde{\vartheta}_A}.$$

Note that we can also express L in $(u, \widetilde{V}, \widetilde{\theta}_*, \widetilde{\varphi}_*)$ coordinates:

$$L = \frac{\mathrm{d}\widetilde{V}}{\mathrm{d}\widetilde{v}} \partial_{\widetilde{V}}.$$

From the above expression it is clear that, analogously to $\underline{L}$, L can be extended smoothly as a vector field across $\mathcal{CH}^+$ (where it vanishes).

2.3. Estimates for Metric Components and Connection Coefficients in Double-Null Coordinates

In this section, we will present an overview of relevant estimates for the metric components $g_{\alpha\beta}$ in Eddington–Finkelstein-type double-null coordinates, their derivatives and components (and derivatives) of the Jacobian matrix relating Eddington–Finkelstein-type double-null coordinates to Boyer–Lindquist coordinates. All these estimates are obtained in [22].

We first define the following notation to separate out leading-order terms in $v + |u|$.

Definition 2.1. Let $f : \mathcal{M} \cap \mathcal{D}_{u_0, v_0} \to \mathbb{R}$ be a C^0 function. We say that $f \in \mathcal{O}((v + |u|)^{-l})$, where u and v are Eddington–Finkelstein-type double-null coordinates in $\mathcal{M}_{\mathrm{int}} \cap \{r > r_0 > \frac{e^2}{2M}\}$, if there exists a constant $C = C(M, a, r_0, \Sigma) > 0$, such that

$$|f|(u, v, \theta_*, \varphi_*) \le C(v + |u|)^{-l}.$$

We obtain in [22] the following estimates for the metric components $g_{\alpha\beta}$ in $\mathcal{M}_{\mathrm{int}} \cap \{r > r_0 > \frac{e^2}{2M}\}$ in the Eddington–Finkelstein-type coordinates $(u, v, \theta_*, \varphi_*)$ introduced above:

Theorem 2.1 (Estimates for metric components in double-null coordinates, [22]). *Let $r_0 > \frac{e^2}{2M}$ and consider $\mathcal{M}_{\mathrm{int}} \cap \{r > r_0\}$ covered by the double-null coordinates $(u, v, \theta_*, \varphi_*)$ introduced in Sect. 2.2.*

(i) *There exist constants $c = c(r_0, a, M) > 0$ and $C = C(N, r_0, a, M) > 0$, such that for all $n \in \mathbb{Z}$, with $n \le N$, where $N \in \mathbb{N}_0$,*

$$|\partial_{r_*}^n \slashed{g}_{\theta_* \theta_*}| \le C(v + |u|)^{-2n},$$
$$|\partial_{r_*}^n \slashed{g}_{\theta_* \varphi_*}| \le C(v + |u|)^{-2n},$$
$$|\partial_{r_*}^n \slashed{g}_{\varphi_* \varphi_*}| \le C \sin^2\theta (v + |u|)^{-2n},$$
$$|\partial_{\theta_*} \slashed{g}_{\theta_* \theta_*}| \le C,$$
$$|\partial_{\theta_*} \slashed{g}_{\theta_* \varphi_*}| \le C \sin^2\theta,$$

$$|\partial_{\theta_*} \not{g}_{\varphi_* \varphi_*}| \le C \sin \theta,$$
$$c \sin^2 \theta \le \det \not{g} \le C \sin^2 \theta.$$

(ii) *We can expand*

$$v + |u| = r_*(r, \theta) = \frac{a^2 + M^2}{M - r} + 2M \log(M - r) + \mathcal{O}(1),$$

$$\Omega^{-2} = \frac{1}{M^2 + a^2 \cos^2 \theta} \left[(v + |u|)^2 + 4M(v + |u|) \log(M - r) \right] + \mathcal{O}(v + |u|),$$

$$b^{\varphi_*} = \frac{4Ma}{(M^2 + a^2)^2} (3M^2 - a^2)(v + |u|)^{-1} + \log(v + |u|) \mathcal{O}((v + |u|)^{-2})$$

and estimate for $n \le N$, *with* $N \in \mathbb{N}$,

$$|\partial_{r_*}^{n+1} b^{\varphi_*}| \le C(v + |u|)^{-2(n+1)},$$
$$|\partial_{r_*}^n \partial_{\theta_*} b^{\varphi_*}| \le C(v + |u|)^{-2-2n} \log(v + |u|),$$
$$|\partial_{r*} \Omega^2| \le C(v + |u|)^{-2},$$
$$|\partial_{r*} \partial_{\theta_*} \Omega^2| \le C(v + |u|)^{-2},$$

where $C = C(N, r_0, a, M) > 0$.

(iii) *There exist* $c = c(r_0, a, M) > 0$ *and* $C = C(r_0, a, M) > 0$ *such that for all* $n \in \mathbb{Z}$, *with* $n \le N$, *where* $N \in \mathbb{N}_0$:

$$|\partial_{r_*} \theta| \le C(v + |u|)^{-2} \sin \theta, \tag{2.8}$$
$$|\partial_{r_*} r| \le C(v + |u|)^{-2}, \tag{2.9}$$
$$c(v - u)^{-2n} \le |\partial_{r_*}^n \partial_\theta \theta_*| \le C(v + |u|)^{-2n}, \tag{2.10}$$
$$|\partial_{r_*}^n \partial_\theta^2 \theta_*| \le C(v + |u|)^{-2n}, \tag{2.11}$$
$$|\partial_{r_*}^n \partial_\theta r_*| \le C \sin \theta (v + |u|)^{-2n} \le C \sin \theta_* (v + |u|)^{-2n}, \tag{2.12}$$
$$|\partial_\theta^2 r_*| \le C. \tag{2.13}$$

We define the connection coefficients

$$\chi_{AB} := g(\nabla_{\partial_{\vartheta_A}} e_4, \partial_{\vartheta_B}),$$
$$\underline{\chi}_{AB} := g(\nabla_{\partial_{\vartheta_A}} e_3, \partial_{\vartheta_B}),$$
$$\omega := -\frac{1}{4} g(\nabla_{e_4} e_3, e_4),$$
$$\underline{\omega} := -\frac{1}{4} g(\nabla_{e_3} e_4, e_3),$$
$$\zeta_A := \frac{1}{2} g(\nabla_{\partial_{\vartheta_A}} e_4, e_3),$$

where $A = 1, 2$ and $e_3 = \Omega^{-1} \underline{L}$ and $e_4 = \Omega^{-1} L$ are renormalised null vector fields, such that $g(e_3, e_4) = -2$. We have the following relations between connection coefficients and metric derivatives:

$$2\Omega \chi_{AB} = L(\not{g}_{AB}) + \partial_A b^C \not{g}_{CB} + \partial_B b^C \not{g}_{CA},$$
$$2\Omega \underline{\chi}_{AB} = \underline{L}(\not{g}_{AB}),$$

$$\Omega \operatorname{tr} \chi = \frac{L(\sqrt{\det \not g})}{\sqrt{\det \not g}} + \partial_C b^C,$$

$$\Omega \operatorname{tr} \underline{\chi} = \frac{\underline{L}(\sqrt{\det \not g})}{\sqrt{\det \not g}},$$

$$4\Omega\omega = \Omega^{-2} L(\Omega^2),$$

$$4\Omega\underline{\omega} = \Omega^{-2}\underline{L}(\Omega^2),$$

$$\zeta^A = \frac{1}{4}\Omega^{-2}[L,\underline{L}]^A = \Omega^{-2}\partial_v b^A,$$

where we also have that $\partial_C b^C = 0$ in (sub)extremal Kerr–Newman.

See "Appendix A" for the derivations of the above identities and for further properties the connection coefficients and their expressions in terms of derivatives of $g_{\alpha\beta}$.

Theorem 2.2 (Estimates for connection coefficients in double-null coordinates, [22]). *Let $r_0 > \frac{e^2}{2M}$ and consider $\mathcal{M}_{\mathrm{int}} \cap \{r > r_0\}$.*

(i) *Let $A, B = 1, 2$ and $n \in \mathbb{Z}$, with $n \le N$, where $N \in \mathbb{N}_0$. There exists a constant $C = C(N, r_0, a, M) > 0$, such that*

$$|\Omega\hat{\chi}_{AB}| \le C(v + |u|)^{-2} \log(v + |u|)|\not g_{AB}|,$$

$$|\Omega\hat{\underline{\chi}}_{AB}| \le C(v + |u|)^{-2}|\not g_{AB}|,$$

$$0 < \Omega \operatorname{tr} \chi \le C(v + |u|)^{-2},$$

$$0 < -\Omega \operatorname{tr} \underline{\chi} \le C(v + |u|)^{-2},$$

$$|\partial_{r_*}^n (\Omega \operatorname{tr} \chi)| \le C(v + |u|)^{-2-n},$$

where we made use of the following notation

$$\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}\not g_{AB} \operatorname{tr} \chi,$$

$$\hat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2}\not g_{AB} \operatorname{tr} \underline{\chi}.$$

(ii) *Moreover, we can expand*

$$4\Omega\omega = -\frac{2}{v + |u|} + \log(v + |u|)\mathcal{O}((v + |u|)^{-2}),$$

$$4\Omega\underline{\omega} = \frac{2}{v + |u|} + \log(v + |u|)\mathcal{O}((v + |u|)^{-2}),$$

$$\Omega^2\zeta^{\varphi_*} = \frac{Ma}{(M^2 + a^2)^2}(3M^2 - a^2)(v + |u|)^{-2} + \log(v + |u|)\mathcal{O}((v + |u|)^{-3}),$$

$$\Omega^2\zeta^{\theta_*} = 0.$$

2.4. Killing Vector Fields

The vector field $T = \frac{\partial}{\partial(t_{\mathrm{KS}})_*}$ in $\mathcal{M}$, as expressed in Kerr-star coordinates ($T = \partial_t$ in Boyer–Lindquist coordinates on $\mathcal{M}_{\mathrm{int}}$), is a Killing vector field;

it corresponds to time-translation symmetry in extremal Kerr–Newman. Note that T is not causal everywhere in $\mathcal{M}$. The subset of $\mathcal{M}_{\text{ext}}$ in which T is not causal is called the *ergoregion*. Similarly, there is a subset of $\mathcal{M}_{\text{int}}$ in which T fails to be causal everywhere (cf. T is timelike *everywhere* away from the horizons in extremal Reissner–Nordström, where $a = 0$).

We denote the Killing vector field corresponding to axial symmetry in extremal Kerr–Newman by Φ. In Kerr-star coordinates, we can write $\Phi = \frac{\partial}{\partial(\varphi_{\text{KS}})_*}$. However, we can also write $\Phi = \partial_{\varphi_*}$ in Eddington–Finkelstein-type double-null coordinates, or $\Phi = \partial_\varphi$ in Boyer–Lindquist coordinates.

The Carter operator is a second-order differential operator that can be expressed as follows:

$$Q = \Delta_{\mathbb{S}^2} + (a^2 \sin \theta)T^2 - \Phi^2,$$

where $\Delta_{\mathbb{S}^2}$ is the Laplacian with respect to the metric on the round sphere (of area radius 1). Since T and Φ are Killing vector fields, we have that

$$[\Box_g, T] = [\Box_g, \Phi] = 0.$$

It turns out that the Carter operator also commutes with the wave operator:

$$[\Box_g, Q] = 0.$$

See [1] for a derivation of the above commutator identity.

We can define the *Hawking vector field H* in $\mathcal{D}_{u_0, v_0}$ by

$$H = \frac{1}{2}(\partial_u + \partial_v) = \frac{1}{2}(\underline{L} + L - b^{\varphi_*}\Phi).$$

We can also express H by as a linear combination of the Killing vector fields T and Φ,

$$H = T + \omega_{\mathcal{H}^+}\Phi,$$

where

$$\omega_{\mathcal{H}^+} := \left.\frac{2Mar}{\rho^2 R^2}\right|_{r=r_+} = \frac{2aM^2}{(M^2 + a^2)^2}.$$

Indeed, in $\mathcal{M}_{\text{int}} \cap \{r > \frac{e^2}{2M}\}$ we can write

$$T = \partial_t = \partial_t u \partial_u + \partial_t v \partial_v + \partial_t \varphi_* \partial_{\varphi_*}$$

$$= \frac{1}{2}\left[\partial_u + \partial_v - 2\omega_{\mathcal{H}^+}\Phi\right].$$

In the literature, the constant $\omega_{\mathcal{H}^+}$ is commonly referred to as the *angular velocity of the Kerr–Newman black hole*.

In order for the energy fluxes with respect to H along null hypersurfaces to be non-negative definite, we need H to be causal. We have that

$$g(H, H) = g(L + \underline{L} - b^{\varphi_*}\partial_{\varphi_*}, L + \underline{L} - b^{\varphi_*}\partial_{\varphi_*}) = -4\Omega^{-2} + (b^{\varphi_*})^2 R^2 \sin^2 \theta.$$

The maximum value of $R^2 \sin^2 \theta$ is obtained at $\theta = \frac{\pi}{2}$,

$$R^2 \sin^2 \theta|_{\theta=\frac{\pi}{2}} = \frac{(M^2 + a^2)^2}{M^2} + \mathcal{O}((v + |u|)^{-1}).$$

Consequently, by applying the estimates in Theorem 2.1, we obtain

$$
\begin{aligned}
g(H,H)|_{\theta=\frac{\pi}{2}} &= \left[-8M^2 + R^2 \left(\frac{4Ma}{(M^2+a^2)^2}(3M^2-a^2) \right)^2 \right] (v+|u|)^{-2} \\
&\quad + \log(v+|u|)\mathcal{O}((v+|u|)^{-3}) \\
&= \left[-8M^2 + \left(\frac{4a}{M^2+a^2}(3M^2-a^2) \right)^2 \right] (v+|u|)^{-2} \\
&\quad + \log(v+|u|)\mathcal{O}((v+|u|)^{-3}).
\end{aligned}
$$

Therefore, $g(H,H) \leq 0$ everywhere for $v+|u|$ suitably large, or equivalently, $M-r$ suitably small, if

$$
2a^2(3M^2-a^2)^2 - M^2(M^2+a^2)^2 < 0.
$$

We rescale $x = \left(\frac{a}{M}\right)^2$, with $x \in [0,1]$, to obtain an equivalent inequality:

$$
2x(3-x)^2 - (1+x)^2 < 0.
$$

One can solve the above cubic equation to obtain $0 < a_c(M) < M$, such that $g(H,H) < 0$ for all $0 \leq |a| < a_c$ and $v+|u|$ suitably large.

We define *slowly rotating extremal Kerr–Newman spacetimes* to be the subfamily of extremal Kerr–Newman spacetimes satisfying $0 \leq |a| < a_c$. Note that extremal Kerr ($|a| = M$) is *not* a slowly rotating extremal Kerr–Newman spacetime.

2.5. The Divergence Theorem and Integration Norms

In this section, we will introduce some basic notation regarding integration in $\mathcal{M} \cap \mathcal{D}_{u_0,v_0}$. We will state the divergence theorem, which is the main ingredient of the vector field method; see also the discussion in Sect. 1.3 of [23].

Let V be a vector field in a Lorentzian manifold $(\mathcal{N}, g)$. We consider the stress–energy tensor $\mathbf{T}[\phi]$ corresponding to (1.1), with components

$$
\mathbf{T}_{\alpha\beta}[\phi] = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial^\gamma \phi \partial_\gamma \phi.
$$

Let $J^V[\phi]$ denote the energy current corresponding to V, which is obtained by applying V as a *vector field multiplier*, i.e. in components

$$
J^V_\alpha[\phi] = \mathbf{T}_{\alpha\beta}[\phi]V^\beta.
$$

An energy flux is an integral of $J^V[\phi]$ contracted with the normal to a hypersurface with the natural volume form corresponding to the metric induced on the hypersurface. We apply the divergence theorem to relate the energy flux along the boundary of a spacetime region to the spacetime integral of the divergence of the energy current J^V. If the boundary has a null segment, there is no natural volume form or normal; these are assumed compatible with the divergence theorem.

That is to say, if we take $-\infty \le u_1 < u_2 \le u_0$ and $v_0 \le v_1 < v_2 \le \infty$, the divergence theorem in the open rectangle $\{u_1 < u < u_2, \ v_1 < v < v_2\}$ in $\mathcal{M}_{\mathrm{int}} \cap \{r > \frac{e^2}{2M}\}$ gives the following identity:

$$\int_{\{u_1<u<u_2,\, v_1<v<v_2\}} \mathrm{div} J^V[\phi]$$

$$= -\int_{H_{u_2}\cap\{v_1\le v\le v_2\}} J^V[\phi]\cdot L + \int_{H_{u_1}\cap\{v_1\le v\le v_2\}} J^V[\phi]\cdot L$$

$$-\int_{\underline{H}_{v_2}\cap\{u_1\le u\le u_2\}} J^V[\phi]\cdot \underline{L} + \int_{\underline{H}_{v_1}\cap\{u_1\le u\le u_2\}} J^V[\phi]\cdot \underline{L}. \quad (2.14)$$

Here, we introduced the following notation:

$$J^V[\phi]\cdot W = \mathbf{T}(V,W),$$

for vector fields V and W. Moreover, in the notation on the left-hand side of (2.14), we integrate over spacetime with respect to the standard volume form, i.e. let $f : \mathcal{M} \cap \mathcal{D}_{u_0,v_0} \to \mathbb{R}$ be a suitably regular function and U an open subset of $\mathcal{M}$, then

$$\int_U f := \int_U f(u,v,\theta_*,\varphi_*)\sqrt{-\det g}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}u\mathrm{d}v$$

$$= \int_U f(u,v,\theta_*,\varphi_*)\,2\Omega^2\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}u\mathrm{d}v,$$

where $\det \slashed{g}$ is expressed in (2.6).

When integrating over H_u and $\underline{H}_v$ we used the following convention in the notation on the right-hand side of (2.14):

$$\int_{\underline{H}_v} f := \int_{-\infty}^{u_0}\int_{S^2_{u,v}} f\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}u,$$

$$\int_{H_u} f := \int_{v_0}^{\infty}\int_{S^2_{u,v}} f\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}v.$$

Note that by a change of variables we can alternatively express the above integrals in terms of (U,θ_*,φ_*) or $(\widetilde{V},\widetilde{\theta}_*,\widetilde{\varphi}_*)$ coordinates, respectively:

$$\int_{-\infty}^{u_0}\int_{S^2_{u,v}} f\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}u = \int_0^{U(u_0)}\int_{S^2_{U,v}} f\frac{\mathrm{d}u}{\mathrm{d}U}\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}U,$$

$$\int_{v_0}^{\infty}\int_{S^2_{u,v}} f\sqrt{\det \slashed{g}}\,\mathrm{d}\theta_*\mathrm{d}\varphi_*\mathrm{d}v = \int_{\widetilde{V}(v_0)}^{0}\int_{S^2_{u,\widetilde{V}}} f\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}\widetilde{V}}\sqrt{\det \slashed{g}}\,\mathrm{d}\widetilde{\theta}_*\mathrm{d}\widetilde{\varphi}_*\mathrm{d}\widetilde{V}.$$

In the notation of [13] we decompose the divergence term appearing in (2.14) in the following way:

$$\mathrm{div} J^V[\phi] = K^V[\phi] + \mathcal{E}^V[\phi],$$

where

$$K^V[\phi] := \mathbf{T}^{\alpha\beta}[\phi]\nabla_\alpha V_\beta,$$

$$\mathcal{E}^V[\phi] := V(\phi)\Box_g\phi.$$

In particular, $\mathcal{E}^V[\phi] = 0$ if ϕ is a solution to (1.1). We can also replace ϕ by $W\phi$, where W is a vector field that is referred to as a *commutation vector field*. The expression $\mathcal{E}^V[W\phi]$ now does not need to vanish.

Furthermore, we can write

$$\int_0^{U(u_0)} \int_{S^2_{U,v}} (\partial_U f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}U$$

$$= \int_{-\infty}^{u_0} \int_{S^2_{u,v}} \frac{\mathrm{d}u}{\mathrm{d}U} (\partial_u f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}u = \int_{\underline{H}_v} \frac{\mathrm{d}u}{\mathrm{d}U} (\partial_u f)^2, \qquad (2.15)$$

$$\int_{\widetilde{V}(v_0)}^0 \int_{S^2_{u,v}} (\partial_{\widetilde{V}} f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}\widetilde{V}$$

$$= \int_{v_0}^{\infty} \int_{S^2_{u,v}} \frac{\mathrm{d}\tilde{v}}{\mathrm{d}\widetilde{V}} (\partial_v f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}v = \int_{H_u} \frac{\mathrm{d}v}{\mathrm{d}\widetilde{V}} (\partial_v f)^2. \qquad (2.16)$$

We can estimate in $\mathcal{M} \cap \mathcal{D}_{u_0,v_0}$, with $|u_0|, v_0 \geq 1$ without loss of generality,

$$C_1 u^2 \leq \frac{\mathrm{d}u}{\mathrm{d}U} \leq C_2 u^2,$$

$$C_1 v^2 \leq \frac{\mathrm{d}\tilde{v}}{\mathrm{d}\widetilde{V}} \leq C_2 v^2,$$

for $C_1 = C_1(a, M, u_0, v_0) > 0$ and $C_2 = C(a, M, u_0, v_0) > 0$ uniform constants.

We rewrite the estimates above by using the following notation:

$$\frac{\mathrm{d}u}{\mathrm{d}U} \sim u^2, \qquad (2.17)$$

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\widetilde{V}} \sim v^2, \qquad (2.18)$$

so that

$$\int_{\underline{H}_v} u^2 (\partial_u f)^2 \sim \int_0^{U(u_0)} \int_{S^2_{U,v}} (\partial_U f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}U,$$

$$\int_{H_u} v^2 (\partial_v f)^2 \sim \int_{\widetilde{V}(v_0)}^0 \int_{S^2_{u,v}} (\partial_{\widetilde{V}} f)^2 \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_* \mathrm{d}\widetilde{V}.$$

Let use introduce the following natural L^2 norms:

$$||f||^2_{L^2(S^2_{u,v})} := \int_{S^2_{u,v}} f^2 \mathrm{d}\mu_{\mathring{g}}, \quad \text{where } \mathrm{d}\mu_{\mathring{g}} := \sqrt{\det \mathring{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_*,$$

$$||f||^2_{L^2(H_u)} := \int_{H_u} \frac{\mathrm{d}u}{\mathrm{d}U} f^2,$$

$$||f||^2_{L^2(\underline{H}_v)} := \int_{\underline{H}_v} \frac{\mathrm{d}\tilde{v}}{\mathrm{d}\widetilde{V}} f^2.$$

Now consider a compact subset $\mathcal{K} \subset \mathcal{M}' \cap \{r > \frac{e^2}{2M}\}$, such that, moreover, $\mathcal{K} \subset \mathcal{D}_{u_0,v_0}$. Then, we define the following spacetime L^2 norms:

$$||f||_{L^2(\mathcal{K})}^2 := \int_{\mathcal{K} \cap \mathcal{M}_{\text{int}}} f^2,$$

$$||\partial f||_{L^2(\mathcal{K})}^2 := \int_{\mathcal{K} \cap \mathcal{M}_{\text{int}}} (\partial_{\widetilde{V}} f)^2 + (\partial_U f)^2 + |\nabla\!\!\!/\, f|^2,$$

where $\nabla\!\!\!/$ denotes the induced covariant derivative on $S^2_{u,v}$.

We can, in particular, estimate

$$||\partial f||_{L^2(\mathcal{K})}^2 = \int_{\mathcal{K} \cap \mathcal{M}_{\text{int}}} (\partial_{\widetilde{V}} f)^2 + (\partial_U f)^2 + |\nabla\!\!\!/\, f|^2$$

$$\leq \int_{u_\mathcal{K}}^{u_0} \int_{\widetilde{V}_\mathcal{K}}^0 \int_{\mathbb{S}^2} \left[(\partial_{\widetilde{V}} f)^2 + (\partial_U f)^2 + |\nabla\!\!\!/\, f|^2 \right] 2\Omega^2 \partial_{\widetilde{V}} \tilde{v} d\mu_{g\!\!\!/} d\widetilde{V} du$$

$$\leq C|\widetilde{V}_\mathcal{K}||u_\mathcal{K}|^2 \sup_{\tilde{v}(\widetilde{V}_\mathcal{K}) \leq v < \infty} \int_{\underline{H}_v \cap \{|u| \leq |u_\mathcal{K}|\}} (\partial_u f)^2 + \Omega^2 v^2 |\nabla\!\!\!/\, f|^2$$

$$+ C|u_0 - u_\mathcal{K}| \sup_{u_\mathcal{K} \leq u < u_0} \int_{H_u} v^2 (\partial_v f)^2, \qquad (2.19)$$

where $\mathcal{K} \subset [u_\mathcal{K}, u_0] \times [\widetilde{V}_\mathcal{K}, 0] \times \mathbb{S}^2$, with $-\infty < u_\mathcal{K} < u_0$, $\widetilde{V}(v_0) < \widetilde{V}_\mathcal{K} < 0$ and $C = C(u_0, v_0) > 0$.

We define the weighted null-directed vector field $N_{p,q}$ in $\mathcal{M}_{\text{int}} \cap \mathcal{D}_{u_0,v_0}$ as follows:

$$N_{p,q} = |u|^p \underline{L} + v^q L = |u|^p \partial_u + v^q (\partial_v + b^A \partial_{\vartheta_A}) = |\tilde{u}|^p (\partial_{\tilde{u}} + b^A \partial_{\widetilde{\vartheta}_A}) + \tilde{v}^p \partial_{\tilde{v}},$$

with $0 \leq p, q \leq 2$. In particular, in (U, v, ϑ) coordinates, we can express

$$N_{p,q} = |u(U)|^p (r^2 \Omega^2)\big|_{v=v_0, \, \theta_* = \frac{\pi}{2}} \partial_U + v^q (\partial_v + b^A \partial_{\vartheta_A}).$$

If $p \leq 2$, $N_{p,q}$ can be extended as a smooth vector field across $\mathcal{H}^+$ into $\mathcal{M}_{\text{ext}}$. In $(\tilde{u}, \widetilde{V}, \vartheta)$ coordinates, we have that

$$N_{p,q} = |\tilde{u}|^p (\partial_{\tilde{u}} + b^A \partial_{\widetilde{\vartheta}_A}) + \tilde{v}^q(\widetilde{V}) (r^2 \Omega^2)\big|_{u=u_0, \, \theta_* = \frac{\pi}{2}} \partial_{\widetilde{V}}.$$

If $q \leq 2$, $N_{p,q}$ can be extended as a smooth vector field beyond $\mathcal{CH}^+$ in $\mathcal{M}'_{\text{ext}}$. The energy currents with respect to the constant u and constant v null hypersurfaces are given by

$$J^{N_{p,q}}[\phi] \cdot L = v^q \mathbf{T}(L, L) + |u|^p \mathbf{T}(L, \underline{L}) = v^q (L\phi)^2 + |u|^p \Omega^2 |\nabla\!\!\!/\, \phi|^2,$$

$$J^{N_{p,q}}[\phi] \cdot \underline{L} = v^q \mathbf{T}(L, \underline{L}) + |u|^p \mathbf{T}(\underline{L}, \underline{L}) = |u|^p (\underline{L}\phi)^2 + v^q \Omega^2 |\nabla\!\!\!/\, \phi|^2,$$

where we inserted the expressions for $\mathbf{T}_{\alpha\beta}$ from "Appendix A".

In "Appendix A" we show that the current $K^{N_{p,q}}$, compatible to $J^{N_{p,q}}$, is given by

$$K^{N_{p,q}}[\phi] = K^{N_{p,q}}_{\text{null}}[\phi] + K^{N_{p,q}}_{\text{angular}}[\phi] + K^{N_{p,q}}_{\text{mixed}}[\phi],$$

with

$$K_{\text{null}}^{N_{p,q}}[\phi] = \frac{1}{2}\Omega^{-2}(v^q\Omega\,\text{tr}\,\chi + |u|^p\Omega\,\text{tr}\,\underline{\chi})L\phi\underline{L}\phi, \tag{2.20}$$

$$K_{\text{angular}}^{N_{p,q}}[\phi] = -\frac{1}{2}\left[-p|u|^{p-1} + qv^{q-1} + 4\Omega(v^q\omega + |u|^p\underline{\omega})\right]|\slashed{\nabla}\phi|^2 \tag{2.21}$$

$$+ \left[v^q\Omega\hat{\chi}^{AB} + |u|^p\Omega\underline{\hat{\chi}}^{AB}\right](\partial_A\phi)(\partial_B\phi),$$

$$K_{\text{mixed}}^{N_{p,q}}[\phi] = 2[v^q(L\phi) - |u|^p(\underline{L}\phi)]\zeta^{\varphi_*}\partial_{\varphi_*}\phi. \tag{2.22}$$

In extremal Kerr–Newman spacetimes with $|a| < a_c$, we consider, more-over, the vector field Y_p in the region $\mathcal{M}_{\text{int}} \cap (\{v \geq v_1\} \cup \{u \leq u_1\})$, which is defined by

$$Y_p = |u|^p H.$$

From Sect. 2.4 it follows that H is timelike in $\mathcal{M}_{\text{int}} \cap (\{v \geq v_1\} \cup \{u \leq u_1\})$, if $|u_1|$ and v_1 are chosen suitably large. We use, moreover, that H is a Killing vector field to easily obtain an expression for K^{Y_p},

$$K^{Y_p}[\phi] = g^{\alpha\beta}\nabla_\beta(J_\alpha^{Y_p}) = g^{\alpha\beta}\nabla_\beta(|u|^p)J_\alpha^H[\phi] + |u|^p K^H[\phi]$$

$$= \frac{p}{2}\Omega^{-2}|u|^{p-1}J^H[\phi] \cdot L \geq 0,$$

where non-negativity, in the case that ϕ is not axisymmetric, follows from the timelike character of H.

3. Precise Statements of the Main Theorems

In this section we present more precise versions of the main results proved in this paper, which are stated in Sect. 1.2. Let $\mathcal{M}$ denote extremal Kerr–Newman with $0 \leq |a| \leq M$, unless otherwise stated. We first give a formulation of the standard global existence and uniqueness for the characteristic initial value problem for (1.1) in $\mathcal{M} \cap \mathcal{D}_{u_0,v_0}$.

Proposition 3.1. *Let ϕ be a continuous function on the union of null hypersurfaces*

$$\left(\mathcal{H}^+ \cap \{v \geq v_0\}\right) \cup \underline{H}_{v_0},$$

such that the restriction to $\mathcal{H}^+$ and the restriction to $\underline{H}_{v_0}$ are smooth functions. Then, there exists a unique, smooth extension of ϕ to $\mathcal{M}_{\text{int}} \cup \mathcal{H}^+ \cap \mathcal{D}_{u_0,v_0}$ that satisfies (1.1) in extremal Kerr–Newman. We also denote this extension by ϕ. We refer to the restriction $\phi|_{(\mathcal{H}^+\cap\{v\geq v_0\})\cup\underline{H}_{v_0}}$ as characteristic initial data. If $\phi|_{(\mathcal{H}^+\cap\{v\geq v_0\})\cup\underline{H}_{v_0}}$ is axisymmetric, the extension ϕ to $\mathcal{M}_{\text{int}} \cup \mathcal{H}^+ \cap \mathcal{D}_{u_0,v_0}$ must also be axisymmetric.

The above proposition can be proved by reducing the characteristic initial value problem to a Cauchy problem with initial data on a spacelike hypersurface, as done in [33], and then appealing to a global existence and uniqueness result for the standard Cauchy problem for linear wave equations; see for example [16].

Observe that Proposition 3.1 does not provide any information about the asymptotic behaviour of ϕ towards $\mathcal{CH}^+$. We will state in the subsections below further quantitative and qualitative properties of ϕ, relating to boundedness and extendibility of ϕ and its derivatives beyond $\mathcal{CH}^+$, under the assumption of suitable additional decay requirements along $\mathcal{H}^+$.

3.1. Energy Estimates Along Null Hypersurfaces

Consider solutions ϕ to (1.1) that arise from the characteristic initial data in Proposition 3.1. We will first show that we can prove boundedness of weighted L^2 norms for ϕ along null hypersurfaces, under additional assumptions on suitable initial L^2 norms along $\mathcal{H}^+$. We will treat separately the case of axisymmetric solutions ϕ on extremal Kerr–Newman with $0 \leq |a| \leq M$, and the case of general solutions ϕ on slowly rotating extremal Kerr–Newman, with $0 \leq |a| < a_c$. In the next section, we will give an overview of the theorems regarding L^∞ estimates for ϕ. Unless specified differently, we consider (1.1) on an extremal Kerr–Newman background with $0 \leq |a| \leq M$.

Theorem 3.2. *Take $0 < q \leq 2$. Let ϕ be a solution to (1.1) corresponding to axisymmetric initial data from Proposition 3.1 satisfying*

$$E_q[\phi] := \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^q (L\phi)^2 + |\not\nabla\phi|^2 + \int_{\underline{H}_{v_0}} |u|^2 (\underline{L}\phi)^2 + \Omega^2 |\not\nabla\phi|^2 < \infty.$$

Then, there exists a constant $C = C(a, M, u_0, v_0, q) > 0$ such that for all H_u and $\underline{H}_v$

$$\int_{H_u} v^q (L\phi)^2 + |u|^2 |\not\nabla\phi|^2 + \int_{\underline{H}_v} |u|^2 (\underline{L}\phi)^2 + \Omega^2 v^q |\not\nabla\phi|^2 \leq C E_q[\phi].$$

Theorem 3.2 is proved in Proposition 4.2. Theorem 3 follows immediately by using, moreover, Theorem 3.5 below and the estimate (2.19). Note, moreover, that by using (2.15) and (2.17) one can easily see that the assumption of ϕ along $\underline{H}_{v_0}$ is certainly satisfied if ϕ is smooth along $\mathcal{H}^+$ with respect to (U, θ_*, φ_*), which, in particular, is the case if one considers ϕ arising from smooth initial data along a hypersurface Σ intersecting $\mathcal{H}^+$.

Theorem 3.3. *Let ϕ be a solution to (1.1), with $|a| < a_c$, corresponding to initial data from Proposition 3.1 satisfying*

$$E_q[\phi] := \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^q (L\phi)^2 + |\not\nabla\phi|^2 + \int_{\underline{H}_{v_0}} |u|^2 (\underline{L}\phi)^2 + \Omega^2 |\not\nabla\phi|^2 < \infty,$$

for some $0 < q \leq 2$.

Let $0 \leq p < 2$ and let $\epsilon > 0$ be arbitrarily small. Then, there exists a constant $C = C(a, M, u_0, v_0, p, q, \epsilon) > 0$, such that for all H_u and $\underline{H}_v$,

$$\int_{H_u} v^{q-\epsilon} (L\phi)^2 + |u|^p |\not\nabla\phi|^2 + \int_{\underline{H}_v} |u|^p (\underline{L}\phi)^2 + \Omega^2 v^{q-\epsilon} |\not\nabla\phi|^2 \leq C E_q[\phi].$$

Theorem 3.3 is proved in Proposition 5.3.

We can remove the ϵ in Theorem 3.3 at the cost of losing derivatives on the right-hand side of the estimate.

Theorem 3.4. *Let ϕ be a solution to* (1.1), *with* $|a| < a_c$, *corresponding to initial data from Proposition* 3.1 *and denote*

$$D = ||\partial_U \phi||^2_{L^\infty\left(\underline{H}_{v_0}\right)} + ||\slashed{\nabla}\phi||^2_{L^\infty\left(\underline{H}_{v_0}\right)}.$$

Assume further that

$$E_{\text{extra};\eta}[\phi] := \int_{\mathcal{H}^+\cap\{v\geq v_0\}} v^2(L\phi)^2 + \sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}}$$

$$\times v^\eta\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2 + |\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right) < \infty,$$

for $\eta > 0$ arbitrarily small. Then, there exists a constant $C = C(a, M, v_0, u_0, \eta)$ > 0 such that,

$$\int_{H_u} v^2(L\phi)^2 + u^2\Omega^2|\slashed{\nabla}\phi|^2 + \int_{\underline{H}_v} v^2\Omega^2|\slashed{\nabla}\phi|^2 \leq C(D + E_{\text{extra};\eta}[\phi]).$$

Theorem 3.4 is proved in Corollary 7.7. Theorem 6 now follows from Theorem 3.4, combined with Theorem 3.5 below and the estimate (2.19).

3.2. Pointwise Estimates and Continuous Extendibility Beyond $\mathcal{CH}^+$

We can use the energy estimates in the subsection above to obtain L^∞ estimates in $\mathcal{M}\cap\mathcal{D}_{u_0,v_0}$, and we can, moreover, show that ϕ is continuously extendible beyond $\mathcal{CH}^+$. Here, we treat the restriction to axisymmetric ϕ and the restriction to slowly rotating extremal Kerr–Newman simultaneously.

Theorem 3.5. *Either take $0 < p < 2$ and let ϕ be a solution to* (1.1), *with* $|a| < a_c$, *corresponding to initial data from Proposition* 3.1 *without any symmetry assumptions, or take $0 \leq p \leq 2$ and let ϕ be a solution to* (1.1), *with $0 \leq |a| \leq M$, corresponding to axisymmetric initial data from Proposition* 3.1.

Assume that, for $\epsilon > 0$ arbitrarily small,

$$\sup_{v_0\leq v<\infty}\sum_{|k|\leq 2}\int_{S^2_{-\infty,v}}|\slashed{\nabla}^k\phi|^2 < \infty,$$

$$\sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}} v^\epsilon|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2 < \infty.$$

Then, there exists a constant $C = C(a, M, v_0, u_0, \epsilon) > 0$ such that

$$\phi^2(u, v, \theta_*, \varphi_*) \leq \sum_{|k|\leq 2}\int_{S^2_{-\infty,v}}|\slashed{\nabla}^k\phi|^2 + C|u|^{1-p}$$

$$\times \sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}} v^\epsilon|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2.$$

Theorem 3.5 follows from Proposition 7.1.

Theorem 3.6. *Let ϕ be a solution to* (1.1), *with* $|a| < a_c$, *corresponding to initial data from Proposition* 3.1 *without any symmetry assumptions, or let ϕ be a solution to* (1.1), *with $0 \leq |a| \leq M$, corresponding to axisymmetric initial data from Proposition* 3.1.

Assume furthermore that

$$\sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}} v^q|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2 < \infty, \qquad (3.1)$$

for some $q > 1$.

Then, ϕ can be extended as a C^0 function beyond $\mathcal{CH}^+$.

Theorem 3.6 is proved in Proposition 7.2. We can infer Theorems 1 and 5 from Theorems 3.5 and 3.6.

As Theorem 2 is formulated in terms of Cauchy initial data for ϕ on an asymptotically flat hypersurface Σ in extremal Kerr, we also need to appeal to the decay estimates in the exterior of extremal Kerr. In particular, boundedness of a non-degenerate energy and $\tau^{-1-\epsilon}$-decay of the (degenerate) T-energy for axisymmetric solutions, with respect to a suitable spacelike foliation Σ_τ of the extremal Kerr exterior, which are proved in Theorems 2 and 3 of [8], are sufficient to show that (3.1) holds for suitable Cauchy data for ϕ, so that Theorem 2 can be viewed as a corollary of Theorem 1.

Finally, we obtain v-decay estimates for $\int_{S^2_{u,v}}(L\phi)^2\,\mathrm{d}\mu_{\slashed{g}}$, which are needed to show that ϕ can be extended as a $C^{0,\alpha}$ (with $\alpha < 1$) across $\mathcal{CH}^+$:

Theorem 3.7. *Let ϕ be a solution to* (1.1) *corresponding to initial data from Proposition* 3.1 *without any symmetry assumptions. Let $k \in \mathbb{N}_0$ and denote*

$$D_{2k} := \sum_{j_1+j_2+2j_3+j_4\leq 2k}||\partial_U L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})}$$

$$+ \sum_{j_1+2j_2\leq n}||\partial_U\Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}\Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})}.$$

Assume that

$$\int_{S^2_{-\infty,v}} v^4(L\phi)^2 + v^2|\slashed{\nabla}\phi|^2 + v^2|\slashed{\nabla}^2\phi|^2\,\mathrm{d}\mu_{\slashed{g}} < \infty.$$

(i) *Let $|a| < a_c$ and assume also that*

$$\sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}} v\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2 + |\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right) < \infty.$$

Then, we can estimate

$$\int_{S^2_{u,v}} v^4(L\phi)^2(u,v,\theta_*,\varphi_*)\,\mathrm{d}\mu_{\slashed{g}}$$

$$\leq \int_{S^2_{-\infty,v}} v^4(L\phi)^2\,\mathrm{d}\mu_{\slashed{g}} + C\int_{S^2_{-\infty,v}} v^2|\slashed{\nabla}\phi|^2 + v^2|\slashed{\nabla}^2\phi|^2\,\mathrm{d}\mu_{\slashed{g}}$$

$$+ Cv^\epsilon\left[D_2 + \sum_{0\leq j_1+j_2\leq 4}\int_{\mathcal{H}^+\cap\{v\geq v_0\}}\right.$$

$$\left.\times v\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2 + |\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2 + |\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right)\right].$$

(ii) *Restrict to axisymmetric data from Proposition 3.1 and assume that*

$$\sum_{0\le j_1+j_2\le 4}\int_{\mathcal{H}^+\cap\{v\ge v_0\}} v^{1+\epsilon}\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2+|\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2+|\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right)<\infty,$$

for $\epsilon>0$ arbitrarily small. Then, we can estimate

$$\int_{S^2_{u,v}} v^4(L\phi)^2(u,v,\theta_*,\varphi_*)\,\mathrm{d}\mu_{\slashed{g}}\le\int_{S^2_{-\infty,v}} v^4(L\phi)^2\,\mathrm{d}\mu_{\slashed{g}}$$

$$+C\int_{S^2_{-\infty,v}} v^2|\slashed{\nabla}\phi|^2+v^2|\slashed{\nabla}^2\phi|^2\mathrm{d}\mu_{\slashed{g}}+C\log\left(\frac{v+|u|}{|u|}\right)$$

$$\cdot\left[D_2+\sum_{0\le j_1+j_2\le 4}\int_{\mathcal{H}^+\cap\{v\ge v_0\}} v^{1+\epsilon}\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2+|\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2+|\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right)\right].$$

(iii) *Either restrict to axisymmetric data in from Proposition 3.1, or let $|a|<a_c$. Assume that*

$$\int_{S^2_{-\infty,v}} v^4\sum_{j_1+j_2\le 2}(LL^{j_1}\underline{L}^{j_2}\phi)^2+v^4\sum_{j_1+j_2\le 1}\sum_{\Gamma\in\{\Phi,\Phi^2,T^2,Q\}}(LL^{j_1}\underline{L}^{j_2}\Gamma\phi)^2\,\mathrm{d}\mu_{\slashed{g}}<\infty,$$

$$\sum_{j_1+j_2\le 2}\int_{S^2_{-\infty,v}} v^2\left(|\slashed{\nabla}L^{j_1}\underline{L}^{j_2}\phi|^2+|\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|^2\right)\,\mathrm{d}\mu_{\slashed{g}}<\infty,$$

$$\sum_{0\le j_1+j_2\le 8}\int_{\mathcal{H}^+\cap\{v\ge v_0\}} v\left(|\slashed{\nabla}^{j_1}L^{j_2+1}\phi|^2+|\slashed{\nabla}^{j_1+1}L^{j_2}\phi|^2+|\slashed{\nabla}^{j_1+2}L^{j_2}\phi|^2\right)<\infty.$$

Then, ϕ can be extended in $C^{0,\alpha}$, for all $\alpha<1$.

Theorem 3.7 follows from Propositions 7.6 and 7.8 and implies Theorems 4 and 7.

4. Energy Estimates for Axisymmetric Solutions

We will first restrict to axisymmetric solutions to (1.1) on extremal Kerr–Newman spacetimes with $0\le|a|\le M$. In this section we will always use ϕ to denote a solution to (1.1), with $0\le|a|\le M$, corresponding to <u>axisymmetric</u> initial data from Proposition 3.1.

We will frequently make use of a Grönwall-type lemma.

Lemma 4.1. *Let $-\infty\le u_1<u_2\le\infty$ and $-\infty\le v_1<v_2\le\infty$. Consider continuous, non-negative functions $f,g:[u_1,u_2]\times[v_1,v_2]\to\mathbb{R}$ and continuous, non-negative functions $h:[u_1,u_2]\to\mathbb{R}$ and $k:[v_1,v_2]\to\mathbb{R}$. Suppose*

$$f(u,v)+g(u,v)\le A+B\left[\int_{u_1}^u h(u')f(u',v)\,\mathrm{d}u'+\int_{v_1}^v k(v')g(u,v')\,\mathrm{d}v'\right],\quad(4.1)$$

for all $u\in[u_1,u_2]$ and $v\in[v_1,v_2]$, where $A,B>0$ are constants. Then,

$$f(u,v)+g(u,v)\le(1+\eta)Ae^{\beta B\left[\int_{u_1}^u h(u')\,\mathrm{d}u'+\int_{v_1}^v k(v')\,\mathrm{d}v'\right]},\quad(4.2)$$

for all $u\in[u_1,u_2]$ and $v\in[v_1,v_2]$, where $\eta>0$ can be taken arbitrarily small and $\beta\ge\frac{2(1+\eta)}{\eta}$.

Proof. See Sect. 4 of [23]. □

We can use the vector field $N_{p,q}$ with $p = 2$, defined in Sect. 2.5, as a vector field multiplier to obtain weighted energy estimates.

Proposition 4.2. *Fix $p = 2$ and let $0 < q \leq 2$. There exists a constant $C = C(a, M, u_0, v_0, q) > 0$ such that for all H_u and $\underline{H}_v$ in $\mathcal{D}_{u_0,v_0}$*

$$\int_{H_u} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_v} J^{N_{2,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_{2,q}}[\phi] \cdot \underline{L} \right] =: C E_q[\phi]. \quad (4.3)$$

Proof. By applying the divergence theorem from Sect. 2.5 in $\mathcal{D}_{u_0,v_0}$, we can estimate

$$\int_{H_u} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_v} J^{N_{2,q}}[\phi] \cdot \underline{L}$$

$$= \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_{2,q}}[\phi] \cdot \underline{L}$$

$$- \int_{\mathcal{D}_{u_0,v_0}} K^{N_{p,q}}_{\text{null}}[\phi] + K^{N_{p,q}}_{\text{angular}}[\phi] + K^{N_{p,q}}_{\text{mixed}}. \quad (4.4)$$

By the assumption that ϕ is axisymmetric, we have that $K^{N_{p,q}}_{\text{mixed}} = 0$.

We first consider $K^{N_{p,q}}_{\text{null}}[\phi]$ and apply the estimates of Sects. 2.3–(2.20) to find that

$$\Omega^2 |K^{N_{p,q}}_{\text{null}}[\phi]| \leq C(v + |u|)^{-2}(v^q - |u|^p)|L\phi||\underline{L}\phi|.$$

By applying a (weighted) Cauchy–Schwarz inequality, we can further estimate for $\eta > 0$,

$$v^q(v + |u|)^{-2}|L\phi||\underline{L}\phi| \leq C v^q (v + |u|)^{-1-q-\eta}|u|^p(\underline{L}\phi)^2$$

$$+ C|u|^{-p}(v + |u|)^{q+\eta-3}v^q(L\phi)^2$$

$$\leq C v^q \sup_{u \leq u' \leq u_0} \left[(v + |u'|)^{-1-q-\eta} \right] |u|^p(\underline{L}\phi)^2$$

$$+ C|u|^{-p} \sup_{v_0 \leq v' \leq v} \left[(v' + |u|)^{q+\eta-3} \right] v^q(L\phi)^2$$

$$\leq C v^{-1-\eta}|u|^p(\underline{L}\phi)^2 + C|u|^{q+\eta-p-3}v^q(L\phi)^2,$$

for $\eta < 3 - q$.

Similarly, by reversing the roles of u and v, we obtain

$$|u|^p(v + |u|)^{-2}|L\phi||\underline{L}\phi| \leq C|u|^{-1-\eta}v^q(L\phi)^2 + C v^{p+\eta-q-3}|u|^p(\underline{L}\phi)^2,$$

for $\eta < 3 - p$.

We will now estimate $K_{\mathrm{angular}}^{N_{p,q}}$ by applying the estimates of Sect. 2.3 to (2.21). We obtain

$$
\begin{aligned}
K_{\mathrm{angular}}^{N_{p,q}} &= -\frac{1}{2}\left[qv^{q-1} - p|u|^{p-1} + 4\Omega\omega(v^q - |u|^p)\right]|\not\nabla\phi|^2 \\
&\quad + (v^q + |u|^p)\log(v + |u|)\mathcal{O}((v + |u|)^{-2})|\not\nabla\phi|^2.
\end{aligned}
\tag{4.5}
$$

Recall from (ii) of Theorem 2.2 that we can expand

$$
4\Omega\omega = -\frac{2}{v + |u|} + \log(v + |u|)\mathcal{O}((v + |u|)^{-2}).
$$

Consequently, we can rewrite (4.5) to obtain

$$
\begin{aligned}
K_{\mathrm{angular}}^{N_{p,q}} &= -\frac{1}{2}\left[\left(q - 2\frac{v^{\frac{q}{2}} + |u|^{\frac{p}{2}}}{v + |u|}v^{1-\frac{q}{2}}\right)v^{q-1} + \left(2\frac{v^{\frac{q}{2}} + |u|^{\frac{p}{2}}}{v + |u|}|u|^{1-\frac{p}{2}} - p\right)|u|^{p-1}\right]|\not\nabla\phi|^2 \\
&\quad + (v^q + |u|^p)\log(v + |u|)\mathcal{O}((v + |u|)^{-2})|\not\nabla\phi|^2.
\end{aligned}
\tag{4.6}
$$

First, let $0 \le p < 2$. Then, the term between square brackets in front of $|\not\nabla\phi|^2$ will become positive in the region $|u| > v$, as we approach $\mathcal{H}^+$, which means that $K_{\mathrm{angular}}^{N_{p,q}}$ will be negative, and we are not able to control it. We therefore restrict to $p = 2$.

If $p = 2$ and $q < 2$, the term inside the square brackets is negative for suitably large v, so we can estimate

$$
K_{\mathrm{angular}}^{N_{p,q}} \ge Cv^{q-1}|\not\nabla\phi|^2 + \mathcal{O}((v^q + |u|^2)(v + |u|)^{-2}\log(v + |u|))|\not\nabla\phi|^2.
$$

If $p = 2$ and $q = 2$, a cancellation occurs in the leading-order terms between square brackets, so we can estimate

$$
K_{\mathrm{angular}}^{N_{p,q}} = (v^2 + |u|^2)\log(v + |u|)\mathcal{O}((v + |u|)^{-2})|\not\nabla\phi|^2.
$$

If we fix $p = 2$, we can therefore estimate for all $0 \le q \le 2$,

$$
\begin{aligned}
\Omega^2 K_{\mathrm{angular}}^{N_{p,q}} &\ge (v^q + |u|^2)\log(v + |u|)\mathcal{O}((v + |u|)^{-2})\Omega^2|\not\nabla\phi|^2 \\
&\ge -C_\epsilon|u|^{-2+\epsilon}v^q\Omega^2|\not\nabla\phi|^2 - C_\epsilon v^{-2+\epsilon}|u|^2\Omega^2|\not\nabla\phi|^2,
\end{aligned}
$$

with $\epsilon > 0$ arbitrarily small and $C_\epsilon = C_\epsilon(M, u_0, v_0, \epsilon) > 0$. We will fix $0 < \epsilon < 1$.

We combine the estimates above for $K_{\mathrm{null}}^{N_{2,q}}$ and $K_{\mathrm{angular}}^{N_{2,q}}$ to obtain, for $0 \le q \le 2$,

$$
\begin{aligned}
-\Omega^2\left(K_{\mathrm{angular}}^{N_{2,q}} + K_{\mathrm{null}}^{N_{2,q}}\right) \\
\le C\Big[(v^{-1-\eta} + v^{\eta-q-1})|u|^2(\underline{L}\phi)^2 + (|u|^{-1-\eta} + |u|^{q+\eta-5})v^q(L\phi)^2 \\
+ |u|^{-2+\epsilon}v^q\Omega^2|\not\nabla\phi|^2 + v^{-2+\epsilon}|u|^2\Omega^2|\not\nabla\phi|^2\Big].
\end{aligned}
$$

Finally, we can apply Lemma 4.1 with the choices

$$
A = \int_{\mathcal{H}^+ \cap \{v \ge v_0\}} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_{2,q}}[\phi] \cdot \underline{L},
$$

$$
f(u,v) = \int_{H_u} v^q(L\phi)^2 + |u|^2\Omega^2|\not\nabla\phi|^2,
$$

$$g(u,v) = \int_{\underline{H}_v} |u|^2 (\underline{L}\phi)^2 + v^q \Omega^2 |\slashed{\nabla}\phi|^2,$$

$$h(u) = |u|^{-1-\eta} + |u|^{q+\eta-5} + |u|^{-2+\epsilon},$$

$$k(v) = v^{-1-\eta} + v^{\eta-q-1} + v^{-2+\epsilon},$$

where we use that h and k are integrable for $0 < \eta < \min\{q, 1\}$ and $0 < \epsilon < 1$, to arrive at the estimate in the proposition. We therefore need the restriction $q > 0$ if $p = 2$. □

We have now proved Theorem 3.2.

5. Energy Estimates in Slowly Rotating Extremal Kerr–Newman

We now drop the axisymmetry assumptions on solutions to (1.1) on extremal Kerr–Newman. We do, however, restrict to the subfamily of slowly rotating extremal Kerr–Newman spacetimes, with $0 \le |a| < a_c$; see Sect. 2.5.

In this section we will always use ϕ to denote a solution to (1.1), with $0 \le |a| < a_c$, corresponding to initial data from Proposition 3.1 without symmetry assumptions.

Even without an axisymmetry assumption on ϕ, we can still obtain energy estimates with respect to vector fields $N_{p,q}$ if we restrict to subsets of $\mathcal{D}_{u_0,v_0}$ with a finite spacetime volume. We introduce the hypersurfaces $\underline{\gamma}_\alpha$ and γ_β, with $\alpha \ge 1$ and $\beta \ge 1$, such that

$$\underline{\gamma}_\alpha := \{(u,v,\theta_*,\varphi_*) \in \mathcal{D}_{u_0,v_0} : f_\alpha(u,v) = 0\},$$

$$\gamma_\beta := \{(u,v,\theta_*,\varphi_*) \in \mathcal{D}_{u_0,v_0} : \underline{f}_\beta(u,v) = 0\}.$$

We define $f_\alpha(u,v)$ as follows:

$$f_\alpha(u,v) = |u| - v^\alpha, \quad |u| > |u_1|,$$

$$= h_\alpha(u,v), \quad |u| \le |u_1|,$$

where $|u_1|$ is taken suitably large, such that $-g(df_\alpha, df_\alpha) \ge C$, for $|u| > |u_1|$, with $C > 0$ a constant. Moreover, we can choose h_α such that f_α is a smooth function on $(-\infty, u_0] \times [v_0, \infty)$ and for all (u,v) such that $h_\alpha(u,v) = 0$, we can uniformly bound $-g(dh_\alpha, dh_\alpha)(u,v) \ge C$.

We define $\underline{f}_\beta(u,v)$ as follows:

$$\underline{f}_\beta(u,v) = v - |u|^\beta, \quad v > v_1,$$

$$= \underline{h}_\beta(u,v), \quad v \le v_1,$$

where v_1 is taken suitably large, such that $-g(\mathrm{d}\underline{f}_\beta, \mathrm{d}\underline{f}_\beta) \ge C$, for $v > v_1$, with $C > 0$ a constant. Moreover, we can choose $\underline{h}_\beta$ such that $\underline{f}_\beta$ is a smooth function on $(-\infty, u_0] \times [v_0, \infty)$ and for all (u,v) such that $h_\beta(u,v) = 0$, we can uniformly bound $-g(dh_\beta, dh_\beta)(u,v) \ge C$.
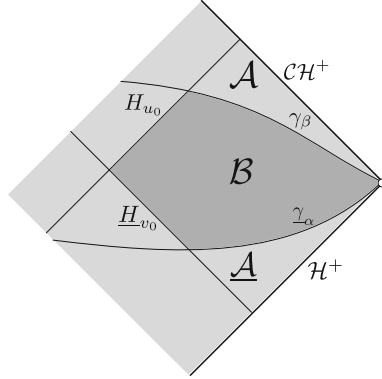
FIGURE 3. The subsets $\mathcal{A}$, $\underline{\mathcal{A}}$ and $\mathcal{B}$ of $\mathcal{D}_{u_0,v_0}$

Consequently, $\underline{\gamma}_\alpha$ and γ_β are spacelike hypersurfaces. Denote

$$\underline{\mathcal{A}} := J^-(\underline{\gamma}_\alpha) \cap \mathcal{D}_{u_0,v_0},$$
$$\mathcal{A} := J^+(\gamma_\beta) \cap \mathcal{D}_{u_0,v_0},$$
$$\mathcal{B} := J^+(\underline{\gamma}_\alpha) \cap J^-(\gamma_\beta) \cap \mathcal{D}_{u_0,v_0}.$$

See Fig. 3. It is easy to verify that the spacetime volumes of $\mathcal{A}$ and $\underline{\mathcal{A}}$ are finite, if we take $\alpha > 1$ and $\beta > 1$.

5.1. Energy Estimates in $\underline{\mathcal{A}}\mathcal{A}$

We first consider energy estimates with respect to $N_{p,q}$. Since ϕ is no longer assumed to be axisymmetric, $K_{\mathrm{mixed}}[\phi]$ does not necessarily vanish. To deal with a non-vanishing $K_{\mathrm{mixed}}[\phi]$, we first consider the regions $\mathcal{A}$ and $\underline{\mathcal{A}}$ of finite spacetime volume.

Proposition 5.1. (i) *Let $p = 2$ and $0 < q < 2$. Fix $\alpha > 1$. Then, there exists a constant $C = C(a, M, u_0, v_0, q, \alpha) > 0$ such that*

$$\int_{H_u \cap \underline{\mathcal{A}}} J^{N_{2,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \underline{\mathcal{A}}} J^{N_{2,q}}[\phi] \cdot \underline{L} + \int_{\underline{\gamma}_\alpha} J^{N_{p,q}}[\phi] \cdot n_{\underline{\gamma}_\alpha}$$

$$\leq C \left[\int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q}}[\phi] \cdot n_{\mathcal{H}^+} + \int_{\underline{H}_{v_0} \cap \{|u| \geq |u_{\underline{\gamma}_\alpha}|(v_0)\}} J^{N_{2,q}}[\phi] \cdot \underline{L} \right].$$
$$(5.1)$$

(ii) *Let $p = 2$ and $0 < q \leq 2$, or let $0 \leq p < 2$ and $0 \leq q < 2$. Fix $\beta > 1$. Then, there exists a constant $C = C(a, M, u_0, v_0, p, q, \beta) > 0$ such that*

$$\int_{H_u \cap \mathcal{A}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{A}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{\gamma_\beta} J^{N_{p,q}}[\phi] \cdot n_{\gamma_\beta} + \int_{\underline{H}_{v_0} \cap \{|u| \leq |u_{\gamma_\beta}|(v_0)\}} J^{N_{p,q}}[\phi] \cdot \underline{L} \right]. \quad (5.2)$$

Proof. We first apply the divergence theorem in region $\underline{\mathcal{A}}$, to obtain

$$\int_{H_u \cap \underline{\mathcal{A}}} J^{N_2,q}[\phi] \cdot L + \int_{\underline{H}_v \cap \underline{\mathcal{A}}} J^{N_2,q}[\phi] \cdot \underline{L}$$

$$= \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_2,q}[\phi] \cdot L + \int_{\underline{H}_{v_0} \cap \{|u| \geq |u_{\underline{\gamma}_\alpha}|(v_0)\}} J^{N_2,q}[\phi] \cdot \underline{L}$$

$$- \int_{\underline{\mathcal{A}}} K_{\text{null}}^{N_{p,q}}[\phi] + K_{\text{angular}}^{N_{p,q}}[\phi] + K_{\text{mixed}}^{N_{p,q}}[\phi]. \tag{5.3}$$

We can estimate $K_{\text{null}}^{N_{p,q}}[\phi]$ and $K_{\text{angular}}^{N_{p,q}}[\phi]$ in exactly the same way as in Proposition 4.2. We are left with estimating $K_{\text{mixed}}^{N_{p,q}}[\phi]$. If we apply the estimates in Sect. 2.3 to (2.22), we can estimate,

$$\Omega^2 |K_{\text{mixed}}^{N_{p,q}}[\phi]| \leq C(v + |u|)^{-2} (v^q |L\phi||\partial_{\varphi_*}\phi| + |u|^p |\underline{L}\phi||\partial_{\varphi_*}\phi|).$$

By applying Cauchy–Schwarz, we can further estimate, for $\eta > 0$,

$$(v + |u|)^{-2} v^q |L\phi||\partial_{\varphi_*}\phi|$$
$$\leq C|u|^{-1-\eta} v^q (L\phi)^2 + C(v + |u|)^{-2} v^q |u|^{1-p+\eta} |u|^p \Omega^2 |\not\nabla\phi|^2$$
$$\leq C|u|^{-1-\eta} v^q (L\phi)^2 + C|u|^{\frac{q}{\alpha}+\eta-p-1} |u|^p \Omega^2 |\not\nabla\phi|^2,$$

where we used that $v \leq C|u|^{\frac{1}{\alpha}}$ in $J^-(\underline{\gamma}_\alpha)$, for some constant $C > 0$.

Similarly, we apply Cauchy–Schwarz to estimate

$$(v + |u|)^{-2} |u|^p |\underline{L}\phi||\partial_{\varphi_*}\phi|$$
$$\leq Cv^{-1-\eta} |u|^p (\underline{L}\phi)^2 + C(v + |u|)^{-2} v^{1+\eta} |u|^p \Omega^2 |\not\nabla\phi|^2$$
$$\leq Cv^{-1-\eta} |u|^p (\underline{L}\phi)^2 + C|u|^{\frac{1+\eta}{\alpha}-2} |u|^p \Omega^2 |\not\nabla\phi|^2.$$

Combined with the estimates from Proposition 4.2 for $K_{\text{angular}}^{N_{p,q}}$ and $K_{\text{null}}^{N_{p,q}}$, we can apply Lemma 4.1 with

$$A = \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_2,q}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_2,q}[\phi] \cdot \underline{L},$$

$$f(u,v) = \int_{H_u \cap \underline{\mathcal{A}}} v^q (L\phi)^2 + |u|^2 \Omega^2 |\not\nabla\phi|^2,$$

$$g(u,v) = \int_{\underline{H}_v \cap \underline{\mathcal{A}}} |u|^2 (\underline{L}\phi)^2 + v^q \Omega^2 |\not\nabla\phi|^2,$$

$$h(u) = |u|^{-1-\eta} + |u|^{q+\eta-5} + |u|^{-2+\epsilon} + |u|^{\frac{q}{\alpha}+\eta-3} + |u|^{\frac{1+\eta}{\alpha}-2},$$

$$k(v) = v^{-1-\eta} + v^{\eta-q-1} + v^{-2+\epsilon},$$

where we use that h and k are integrable for $0 < \eta < \min\{q, 1, 2 - \frac{q}{\alpha}, \alpha - 1\}$ and $0 < \epsilon < 1$. Since we assumed that $\alpha > 1$, we obtain the estimate in (i).

Consider now the region $\mathcal{A}$. The estimates above can be repeated here, where the roles of $|u|$ and v are replaced when estimating $K_{\text{mixed}}^{N_{p,q}}$:

$$\Omega^2 |K^{N_{p,q}}_{\text{mixed}}[\phi]| \le C v^{-1-\eta} |u|^p (\underline{L}\phi)^2$$
$$+ C|u|^{-1-\eta} v^q (L\phi)^2 + C(v^{\frac{p}{\beta}+\eta-q-1} + v^{\frac{1+\eta}{\beta}-2}) v^q \Omega^2 |\slashed{\nabla}\phi|^2.$$

Furthermore, we can actually improve the estimate for $K^{N_{p,q}}_{\text{angular}}$ from Proposition 4.2 when restricted to $\mathcal{A}$, by including the cases $0 < p \le 2$, with $0 \le q < 2$. This improvement will in fact be necessary to prove Proposition 5.3.

Indeed, we can estimate

$$-K^{N_{p,q}}_{\text{angular}} = \frac{1}{2(v+|u|)} \left[((q-2)v + q|u|) v^{q-1} + ((2-p)|u| - pv)|u|^{p-1} \right] |\slashed{\nabla}\phi|^2$$
$$+ \mathcal{O}((v^q + |u|^p)(v+|u|)^{-2} \log(v+|u|)) |\slashed{\nabla}\phi|^2$$
$$\le C(v+|u|)^{-1} |u|^p |\slashed{\nabla}\phi|^2 + \mathcal{O}((v^q + |u|^p)(v+|u|)^{-2} \log(v+|u|))|\slashed{\nabla}\phi|^2,$$

where in the second inequality we used that $(q-2)v + q|u| < 0$ in $\mathcal{A}$ if $v+|u|$ is suitably large and $q < 2$, which follows from the inequality $|u| < \frac{2-q}{q} v$, which holds in $\mathcal{A}$ if $v+|u|$ is suitably large and $\beta > 1$.

We can now apply Lemma 4.1 with

$$A = \int_{\mathcal{H}^+ \cap \{v \ge v_0\}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_{p,q}}[\phi] \cdot \underline{L},$$

$$f(u,v) = \int_{H_u \cap \mathcal{A}} v^q (L\phi)^2 + |u|^p \Omega^2 |\slashed{\nabla}\phi|^2,$$

$$g(u,v) = \int_{\underline{H}_v \cap \mathcal{A}} |u|^p (\underline{L}\phi)^2 + v^q \Omega^2 |\slashed{\nabla}\phi|^2,$$

$$h(u) = |u|^{-1-\eta} + |u|^{q+\eta-3-p} + |u|^{-2+\epsilon},$$

$$k(v) = v^{-1-\eta} + v^{p+\eta-q-3} + v^{-2+\epsilon} + v^{\frac{p}{\beta}+\eta-1-q} + v^{\frac{1+\eta}{\beta}-2},$$

where h and k are integrable for $0 < \eta < \min\{q+2-p, p+2-q, 2-\frac{p}{\beta}, \beta-1\}$, $0 < \epsilon < 1$. For consistency, we therefore require $\beta > 1$, $p < q+2$ and $q < p+2$. In particular, if $p = 2$, we need $q > 0$ and if $q = 2$, we need $p > 0$. $\qquad\square$

5.2. Energy Estimates in $\mathcal{B}$

We are left with proving a suitable energy estimate in the region $\mathcal{B}$. In Kerr–Newman spacetimes with $0 \le |a| < a_c$, we can obtain an energy estimate away from $\mathcal{H}^+$ with respect to the vector field Y_p, defined by

$$Y_p = |u|^p H,$$

if we restrict to a region $\{v \ge v_1\}$, where v_1 is taken suitably large, such that $M - r$ is sufficiently small, so as to ensure that Y_p is a causal vector field everywhere in $\mathcal{B} \cap \{v \ge v_1\}$; see the discussion in Sect. 2.4.

Proposition 5.2. *Let $0 \le p \le 2$ and let $v_1 > v_0$ be suitably large. Then, there exists a constant $C = C(a, M, u_0, v_0, v_1, p) > 0$ such that*

$$\int_{\underline{H}_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y_p}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y_p}[\phi] \cdot \underline{L} + \int_{\gamma_\beta \cap \{v \geq v_1\}} J^{Y_p}[\phi] \cdot n_{\gamma_\beta}$$

$$+ \int_{\mathcal{B} \cap \{v \geq v_1\}} K^{Y_p}[\phi] \leq C \left[\int_{\underline{\gamma}_\alpha \cap \{v \geq v_1\}} J^{Y_p}[\phi] \cdot n_{\mathcal{H}^+} \right.$$

$$\left. + \int_{\underline{H}_{v_1} \cap \{|u| \leq u_{\underline{\gamma}_\alpha}(v_1)\}} J^{Y_p}[\phi] \cdot \underline{L} \right]. \tag{5.4}$$

Proof. We consider the region $\mathcal{B} \cap \{v \geq v_1\}$, where v_1 can be chosen suitably large, such that H is causal for $|a| < a_c$ everywhere in $\mathcal{B} \cup \{v \geq v_1\}$. See Sect. 2.4.

We use that H is a Killing vector field to easily obtain an expression for K^{Y_p},

$$K^{Y_p}[\phi] = g^{\alpha\beta} \nabla_\beta (J^{Y_p}_\alpha) = g^{\alpha\beta} \nabla_\beta (|u|^p) J^H_\alpha[\phi] + |u|^p K^H[\phi]$$

$$= \frac{p}{2} \Omega^{-2} |u|^{p-1} J^H[\phi] \cdot L \geq 0,$$

where non-negativity in the case that ϕ is not axisymmetric requires that H is causal.

Moreover, there exists a constant $C > 0$ such that we can estimate

$$J^H[\phi] \cdot L = (L\phi)^2 + \Omega^2 |\slashed{\nabla}\phi|^2 - b^{\varphi_*} L\phi \partial_{\varphi_*}\phi$$

$$\geq C \left[(L\phi)^2 + \Omega^2 |\slashed{\nabla}\phi|^2 \right].$$

If we apply the divergence theorem in the region $\mathcal{B} \cap \{v \geq v_1\}$, the bulk term is therefore of a good sign. $\qquad\square$

We can now obtain energy estimates in the entire region $\mathcal{D}_{u_0,v_0}$ by combining the results from Propositions 5.1 and 5.2.

Proposition 5.3. *Let* $0 \leq p < 2$ *and* $0 \leq q < 2$*. Then, there exist* $\alpha = \alpha(p,q) > 1$ *and* $\beta = \beta(p,q) > 1$*, such that for all* H_u *and* $\underline{H}_v$ *in* $\mathcal{D}_{u_0,v_0}$*,*

$$\int_{H_u} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q\beta\alpha}}[\phi] \cdot L + \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[\phi] \cdot \underline{L} \right] =: CE_{q\alpha\beta}[\phi],$$

with $0 < q\beta\alpha \leq 2$ *and* $C = C(a, M, u_0, v_0, p, q, \alpha, \beta) > 0$*.*

Proof. We first restrict to the region $\{v \geq v_1\}$. Note that

$$\int_{H_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{H_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot \underline{L} \right],$$

for $p' \geq p$ and $p' \geq q\beta$.

Furthermore,

$$\int_{H_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y'_p}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{Y'_p}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{H_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{2,q''}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{2,q''}}[\phi] \cdot \underline{L} \right],$$

for $q'' \geq p'\alpha$.

Combining Propositions 5.1 and 5.2 we can therefore estimate

$$\int_{H_u \cap \mathcal{A} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{A} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \int_{\gamma_\beta \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot n_{\gamma_\beta} + \int_{\underline{H}_{v_1} \cap \{|u| \leq |u_{\gamma_\beta}(v_1)|\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \int_{\gamma_\beta \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot n_{\gamma_\beta} + C \int_{\underline{H}_{v_1} \cap \{|u| \leq |u_{\gamma_\beta}(v_1)|\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{\underline{\gamma}_\alpha \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot n_{\underline{\gamma}_\alpha} + \int_{\underline{H}_{v_1} \cap \{|u_{\gamma_\beta}(v_1)| \leq |u| \leq |u_{\underline{\gamma}_\alpha}(v_1)|\}} J^{Y_{p'}}[\phi] \cdot \underline{L} \right.$$

$$\left. + \int_{\underline{H}_{v_1} \cap \{|u| \leq |u_{\gamma_\beta}(v_1)|\}} J^{N_{p,q}}[\phi] \cdot \underline{L} \right],$$

where we need $\beta > 1$, $0 \leq p \leq 2$ and $0 < q \leq 2$, or $0 < p \leq 2$ and $0 \leq q \leq 2$. Moreover, we need

$$p' \geq p,$$
$$p' \geq q\beta.$$

Similarly, we can estimate

$$\int_{H_u \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \mathcal{B} \cap \{v \geq v_1\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \left[\int_{\underline{\gamma}_\alpha \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot n_{\underline{\gamma}_\alpha} + \int_{\underline{H}_{v_1} \cap \{|u_{\gamma_\beta}(v_1)| \leq |u| \leq |u_{\underline{\gamma}_\alpha}(v_1)|\}} J^{Y_{p'}}[\phi] \cdot \underline{L} \right],$$

where $p' \geq p$ and $p' \geq q\beta$.

Now we apply Proposition 5.1 in the region $\underline{\mathcal{A}}$ to estimate

$$\int_{\underline{\gamma}_\alpha \cap \{v \geq v_1\}} J^{Y_{p'}}[\phi] \cdot n_{\underline{\gamma}_\alpha}$$

$$\leq C \left[\int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,q''}}[\phi] \cdot L + \int_{\underline{H}_{v_1} \cap \{|u| \geq |u_{\underline{\gamma}_\alpha}(v_1)|\}} J^{N_{2,q''}}[\phi] \cdot \underline{L} \right],$$

where we need $\alpha > 1$, and we require

$$p' \leq 2,$$
$$p'\alpha \leq q'' \leq 2.$$

If we combine the restrictions on p, q, p', q' and q'', we obtain

$$p \leq p' \leq \frac{2}{\alpha},$$

$$q \leq \frac{p'}{\beta} \leq \frac{2}{\alpha\beta}.$$

We now consider the region $\{v_0 \leq v \leq v_1\}$. Since the region $\mathcal{B} \cap \{v \leq v_1\}$ is compact, we do not need to appeal to the estimates with respect to the vector fields Y_p from Proposition 5.2. Instead, we use the vector fields $N_{p,q}$, as in Proposition 5.1, making use of the compactness of $\mathcal{B} \cap \{v \leq v_1\}$ to, in particular, estimate the previously problematic $K_{\mathrm{mixed}}^{N_{p,q}}[\phi]$ error term.

We arrive at the estimate:

$$\int_{H_u \cap \{v \leq v_1\}} J^{N_{p,q}}[\phi] \cdot L + \int_{\underline{H}_v \cap \{v \leq v_1\}} J^{N_{p,q}}[\phi] \cdot \underline{L}$$

$$\leq C \int_{\mathcal{H}^+ \cap \{v \leq v_1\}} J^{N_{p,q}}[\phi] \cdot L + C \int_{\underline{H}_{v_0}} J^{N_{p,q}}[\phi] \cdot \underline{L},$$

for any $0 \leq p, q \leq 2$.

The estimate in the proposition now follows by adding the estimates in $\{v \geq v_1\}$ and $\{v \leq v_1\}$ together. $\qquad\square$

Remark 5.1. For $\epsilon > 0$ arbitrarily small we can always choose α and β in Proposition 5.3 suitably close to 1, so that we can take $p = 2 - \epsilon$ and $q = 2 - \epsilon$.

We have now proved Theorem 3.3.

6. Higher-Order Energy Estimates

In order to obtain L^∞ bounds from the L^2 bounds derived in Sects. 4 and 5, we need to derive similar L^2 bounds for higher-order derivatives of ϕ. In this section we will use ϕ to denote a solution to (1.1) corresponding to initial data from Proposition 3.1. We will always specify whether we are assuming ϕ arises from axisymmetric data in Proposition 3.1, or the rotation parameter a is restricted to the range $0 \leq |a| < a_c$.

6.1. Elliptic Estimates on $S_{u,v}^2$

In this section we will show that the angular derivatives on the Eddington–Finkelstein-type spheres $S_{u,v}^2$ can be controlled by derivatives with respect to the Killing vector field Φ, the null-directed vector fields L and $\underline{L}$ and the Carter operator Q; see Sect. 2.4.

Note that norms of the angular derivatives of functions on Boyer–Lindquist spheres of constant t and r with respect to the corresponding induced spherical metric can easily been seen to be comparable to analogous norms with the induced metric on the Boyer–Lindquist spheres replaced by the metric on the unit round sphere. This follows from the fact that r is bounded

away from zero and infinity in the region of interest in the black hole interior. Similarly, the induced volume form on the Boyer–Lindquist spheres is comparable to the natural volume form on the unit round sphere.

As a preliminary step to considering norms on Eddington–Finkelstein-type spheres $S_{u,v}^2$, we will need that L^2 norms of angular derivatives of any function f restricted to the Boyer–Lindquist spheres with respect to the unit round sphere can be controlled *solely* by L^2 norms of $T(f)$, $\Phi(f)$ and $Q(f)$. Therefore, the L^2 norms with respect to the *actual* induced metric on the Boyer–Lindquist spheres can also be controlled similarly. The lemma below can be found in [8].

Lemma 6.1. *Given a function $f : \mathcal{M} \cap \mathcal{D}_{u_0,v_0} \to \mathbb{R}$, there exists a $C = C(a) > 0$ such that*

$$\int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} f|^2(t, r, \theta, \varphi) + |\nabla_{\mathbb{S}^2}^2 f|^2(t, r, \theta, \varphi)\, \mathrm{d}\mu_{\mathbb{S}^2}$$
$$\leq C \int_{\mathbb{S}^2} (Qf)^2 + (\Phi^2 f)^2 + (T^2 f)^2\, \mathrm{d}\mu_{\mathbb{S}^2}, \tag{6.1}$$

where $\nabla_{\mathbb{S}^2}$ denotes the covariant derivative on $\mathbb{S}^2$ and $\mathrm{d}\mu_{\mathbb{S}^2} = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$.

Proof. By decomposing f into spherical harmonics f_ℓ on $\mathbb{S}^2$, one can show that

$$\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} f)^2\, \mathrm{d}\mu_{\mathbb{S}^2} = \sum_{\ell=0}^{\infty} \int_{\mathbb{S}^2} (\ell(\ell+1))^2 f_\ell^2\, \mathrm{d}\mu_{\mathbb{S}^2} = \sum_{\ell=0}^{\infty} \int_{\mathbb{S}^2} \sum_{1 \leq |k| \leq 2} (O^k f_l)^2\, \mathrm{d}\mu_{\mathbb{S}^2}$$
$$= \sum_{1 \leq |k| \leq 2} \int_{\mathbb{S}^2} (O^k f)^2\, \mathrm{d}\mu_{\mathbb{S}^2} \geq \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} f|^2 + |\nabla_{\mathbb{S}^2}^2 f|^2\, \mathrm{d}\mu_{\mathbb{S}^2},$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplacian on $\mathbb{S}^2$ and O^k denotes the operators of the form $O_1^{j_1} O_2^{j_2} O_3^{j_3}$ with $j_1 + j_2 + j_3 = k$, with O_i angular momentum operators; see for example Sect. 2.1 of [23] for explicit expressions of O_i.

The estimate (6.1) follows by using the definition of Q to rewrite the left-hand side above and applying Cauchy–Schwarz. $\qquad\square$

We would similarly like to control the angular derivatives in the coordinates (θ_*, φ_*) by using the operators Q, T and Φ that commute with $\Box_g$. However, since the tangent spaces to the Boyer–Lindquist spheres and the spheres $S_{u,v}^2$ are not spanned by the same tangent vectors, we need to include L and $\underline{L}$ derivatives in our estimate.

For the sake of convenience, we change from the chart (θ_*, φ_*) on the 2-spheres $S_{u,v}^2$ to the chart (θ_*, φ), because the induced metric on $S_{u,v}^2$ then becomes diagonal:

$$\slashed{g} = f_1^2 f_2^2 (\partial_{\theta_*} F)^2 R^{-2} \mathrm{d}\theta_*^2 + R^2 \sin^2\theta \mathrm{d}\varphi^2.$$

Proposition 6.2. *Given a suitably regular function $f : \mathcal{M} \cap \mathcal{D}_{u_0,v_0} \to \mathbb{R}$, there exists a $C = C(M, a, u_0, v_0) > 0$ such that*

$$\int_{S_{u,v}^2} |\slashed{\nabla} f|^2 + |\slashed{\nabla}^2 f|^2 \sqrt{\det \slashed{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_*$$

$$\leq C \sum_{k=0}^{1} \sum_{\Gamma \in \{id, \Phi, \Phi^2, T^2, Q\}} \int_{S_{u,v}^2} (\Gamma \underline{L}^k f)^2 + (\Gamma L^k f)^2 + (\underline{L}^2 f)^2$$
$$+ (L^2 f)^2 + (L\underline{L}f)^2 + (b^{\varphi_*})^2 [(Q\Phi f)^2 + (T^2 \Phi f)^2$$
$$+ (\Phi^3 f)^2] \sqrt{\det \mathring{g}} d\theta_* d\varphi_*, \tag{6.2}$$

for all $|u| \geq |u_0|$ and $v \geq v_0$.

Let $v_1 > v_0$ and $|u_1| > u_0$ be suitably large. Then, we can estimate

$$\int_{S_{u,v}^2} |\mathring{\nabla}^2 f|^2 \sqrt{\det \mathring{g}} d\theta_* d\varphi_*$$

$$\leq C \int_{S_{u,v}^2} (Qf)^2 + (\Phi^2 f)^2 + (T^2 f)^2 + (L^2 f)^2 + (\underline{L}^2 f)^2$$

$$+ (L\underline{L}f)^2 + (Lf)^2 + (\underline{L}f)^2 + |\mathring{\nabla} Lf|^2 + |\mathring{\nabla} \underline{L}f|^2 + |\mathring{\nabla} f|^2 \sqrt{\det \mathring{g}} d\theta_* d\varphi_*, \tag{6.3}$$

if either $|u| \geq |u_1|$ or $v \geq v_1$, where $C = C(M, a, u_0, v_0, u_1, v_1) > 0$.

Proof. By Theorem 2.1 there exist uniform constants $C, c > 0$ such that

$$c \sin \theta \leq \det \mathring{g} \leq C \sin \theta,$$

where $\det \mathring{g}$ is the determinant with respect to the coordinate basis corresponding to the chart (θ_*, φ_*), which is equal to the determinant of the matrix of $\mathring{g}$ with respect to the coordinate basis corresponding to the chart (θ_*, φ).

Consider the first-order angular derivatives. We can write

$$|\mathring{\nabla} f|^2 = \mathring{g}^{\theta_* \theta_*} (\partial_{\theta_*} f)^2 + \mathring{g}^{\varphi \varphi} (\partial_\varphi f)^2 \leq C \left[(\partial_{\theta_*} f)^2 + \sin^{-2} \theta (\partial_\varphi f)^2 \right].$$

By the chain rule, we have that

$$\partial_\theta f = (\partial_\theta \theta_*) \partial_{\theta_*} f + \frac{1}{2} (\partial_\theta r_*)(\partial_v f - \partial_u f)$$

$$= (\partial_\theta \theta_*) \partial_{\theta_*} f + \frac{1}{2} (\partial_\theta r_*)(Lf - b^{\varphi_*} \partial_{\varphi_*} f - \underline{L}f).$$

By applying (2.10) and (2.12), we find that

$$(\partial_{\theta_*} f)^2 \leq C \left[(\partial_\theta f)^2 + \sin^2 \theta (Lf)^2 + \sin^2 \theta (\underline{L}f)^2 + \sin^2 \theta (b^{\varphi_*})^2 (\Phi f)^2 \right]. \tag{6.4}$$

We can now conclude the following:

$$|\mathring{\nabla} f|^2 \leq C \left[|\nabla_{\mathbb{S}^2} f|^2 + \sin^2 \theta (Lf)^2 + \sin^2 \theta (\underline{L}f)^2 \right]. \tag{6.5}$$

Now consider the second-order angular derivatives. We can estimate

$$|\mathring{\nabla}^2 f|^2 = \mathring{g}^{AB} \mathring{g}^{CD} (\nabla_A \partial_C f)(\nabla_B \partial_D f)$$

$$= (\mathring{g}^{\varphi \varphi})^2 (\partial_\varphi^2 f)^2 + (\mathring{g}^{\theta_* \theta_*})^2 (\partial_{\theta_*}^2 f)^2 + \mathring{g}^{\theta_* \theta_*} \mathring{g}^{\varphi \varphi} (\partial_{\theta_*} \partial_\varphi f)^2$$

$$+ \mathring{g}^{AB} \mathring{g}^{CD} (\mathring{\nabla}_A \partial_C)^E (\mathring{\nabla}_B \partial_D)^F \mathring{\nabla}_E f \mathring{\nabla}_F f,$$

$$\leq C \left[\sin^{-4} \theta (\partial_\varphi^2 f)^2 + (\partial_{\theta_*}^2 f)^2 + \sin^{-2} \theta (\partial_{\theta_*} \partial_\varphi f)^2 \right]$$

$$+ \mathring{g}^{AB} \mathring{g}^{CD} (\mathring{\nabla}_A \partial_C)^E (\mathring{\nabla}_B \partial_D)^F \mathring{\nabla}_E f \mathring{\nabla}_F f.$$

By applying the chain rule we find that

$$\partial_\theta^2 f = \partial_\theta^2 \theta_* \partial_{\theta_*} f + (\partial_\theta \theta_*) \left[\partial_\theta \theta_* \partial_{\theta_*}^2 f + \partial_\theta r_* (\partial_{\theta_*} Lf - \partial_{\theta_*}(b^{\varphi_*}\Phi f) - \partial_{\theta_*}\underline{L}f) \right]$$
$$+ \frac{1}{4}(\partial_\theta r_*)^2((L - b^{\varphi_*}\Phi)^2 f - 2(L - b^{\varphi_*}\Phi)\underline{L}f + \underline{L}^2 f)$$
$$+ \frac{1}{2}\partial_\theta^2 r_*(Lf - b^{\varphi_*}\partial_{\varphi_*}f - \underline{L}f). \tag{6.6}$$

Consequently, by applying the estimates from Theorem 2.1, we obtain

$$(\partial_{\theta_*}^2 f)^2 \le (\partial_{\theta_*} f)^2 + (\partial_\theta^2 f)^2 + \sin^2\theta(\partial_{\theta_*}(\underline{L}f))^2 + \sin^2\theta(\partial_{\theta_*}(b^{\varphi_*}\Phi f))^2 + \sin^2\theta(\partial_{\theta_*}(Lf))^2$$
$$+ (\underline{L}f)^2 + (Lf)^2 + \sin^4\theta(L^2 f)^2 + \sin^4\theta(L\underline{L}f)^2 + \sin^4\theta(\underline{L}^2 f)^2 + \sin^4\theta(b^{\varphi_*})^2(L\Phi f)^2$$
$$+ \sin^4\theta(b^{\varphi_*})^2(\underline{L}\Phi f)^2 + \sin^4\theta(b^{\varphi_*})^2(\underline{L}\Phi f)^2 + (b^{\varphi_*})^4(\Phi^2 f)^2 + (\Phi f)^2, \tag{6.7}$$

where we used, moreover, that the vector field Φ commutes with all the vector fields $L, \underline{L}, \partial_{\theta_*}, \partial_\theta$ and with b^{φ_*}.

We can further estimate

$$(\partial_{\theta_*}(b^{\varphi_*}\Phi f))^2 \le (b^{\varphi_*})^2(\Phi\partial_{\theta_*}f)^2 + (\partial_{\theta_*}b^{\varphi_*})^2(\Phi f)^2$$
$$\le C\left[(\partial_\theta\Phi f)^2 + (L\Phi f)^2 + (\underline{L}\Phi f)^2 + (\Phi^2 f)^2\right]$$
$$\le C\left[(Q\Phi f)^2 + (T^2\Phi f)^2 + (L\Phi f)^2 + (\underline{L}\Phi f)^2 + (\Phi f)^2 + (\Phi^2 f)^2 + (\Phi^3 f)^2\right]. \tag{6.8}$$

We now turn to $(\slashed{\nabla}_A \partial_C)^E$. The only non-vanishing components are given by:

$$\left|(\slashed{\nabla}_{\theta_*}\partial_{\theta_*})^{\theta_*}\right| = \left|\frac{1}{2}g^{\theta_*\theta_*}\partial_{\theta_*}g_{\theta_*\theta_*}\right| \le C,$$
$$\left|(\slashed{\nabla}_\varphi\partial_\varphi)^{\theta_*}\right| = \left|\frac{1}{2}g^{\theta_*\theta_*}\partial_{\theta_*}g_{\varphi\varphi}\right| \le C\sin\theta,$$
$$\left|(\slashed{\nabla}_{\theta_*}\partial_\varphi)^\varphi\right| = \left|\frac{1}{2}g^{\varphi\varphi}\partial_{\theta_*}g_{\varphi\varphi}\right| \le C\sin\theta,$$

where we used the estimates from Theorem 2.1 to arrive at the inequalities on the right-hand sides.

We can now estimate, by applying (6.4),

$$\slashed{g}^{AB}\slashed{g}^{CD}(\slashed{\nabla}_A\partial_C)^E(\slashed{\nabla}_B\partial_D)^F\slashed{\nabla}_E f\slashed{\nabla}_F f$$
$$= \left(\slashed{g}^{\theta_*\theta_*}\right)^2((\slashed{\nabla}_{\theta_*}\partial_{\theta_*})^{\theta_*})^2(\partial_{\theta_*}f)^2 + 2\slashed{g}^{\theta_*\theta_*}\slashed{g}^{\varphi\varphi}((\slashed{\nabla}_{\theta_*}\partial_\varphi)^\varphi)^2(\partial_\varphi f)^2$$
$$+ (\slashed{g}^{\varphi\varphi})^2((\slashed{\nabla}_\varphi\partial_\varphi)^{\theta_*})^2(\partial_{\theta_*}f)^2$$
$$\le C\sin^{-2}\theta(\partial_{\theta_*}f)^2 + C\sin^{-4}\theta(\partial_\varphi f)^2$$
$$\le C\slashed{g}_{\mathbb{S}^2}^{AB}\slashed{g}_{\mathbb{S}^2}^{CD}((\nabla_{\mathbb{S}^2})_A\partial_C)^E((\nabla_{\mathbb{S}^2})_B\partial_D)^F\partial_E f\partial_F f$$
$$+ C(Lf)^2 + C(\underline{L}f)^2 + C(b^{\varphi_*})^2(\Phi f)^2. \tag{6.9}$$

By combining (6.7), the first inequality in (6.8), and (6.9) and making use of the smallness of $|b^{\varphi_*}|$ for suitably large v_1 and $|u_1|$, we obtain (6.3). We

can also estimate

$$\int_{S_{u,v}^2} |\overset{\scriptscriptstyle\nabla}{\nabla} f|^2 + |\overset{\scriptscriptstyle\nabla}{\nabla}^2 f|^2 \sqrt{\det \overset{\scriptscriptstyle\centerdot}{g}} \mathrm{d}\theta_* \mathrm{d}\varphi$$

$$\begin{aligned}
\leq C \sum_{k=0}^{1} \sum_{\Gamma \in \{\mathrm{id}, \Phi, \Phi^2, T^2\}} \int_{S_{u,v}^2} & (\underline{L}^k \Gamma f)^2 + (L^k \Gamma f)^2 + (\underline{L}^2 f)^2 \\
& + (L^2 f)^2 + (L\underline{L} f)^2 + (\Phi^2 f)^2 + (b^{\varphi_*})^2 [(Q\Phi f)^2 + (T^2 \Phi f)^2 \\
& + (\Phi^3 f)^2] + (Qf)^2 + \sin^2 \theta (QLf)^2 \\
& + \sin^2 \theta (QLf)^2 \sqrt{\det \overset{\scriptscriptstyle\centerdot}{g}} \mathrm{d}\theta_* \mathrm{d}\varphi_*.
\end{aligned}$$

We have now obtained the estimate (6.2).

We can easily commute L and $\underline{L}$ with Γ above. $\qquad\square$

Remark 6.1. Observe that there is a loss of derivatives on the right-hand side of (6.2), but no loss of derivatives on the right-hand side of (6.3). Since (6.2) will be used to obtain L^∞ estimates from energy estimates, this (additional) loss of derivatives will be present in the pointwise estimates of Sect. 7. See also Remark 6.2 about the loss of derivatives in the energy estimates themselves.

6.2. Commutator Estimates

We can use the elliptic estimates in Proposition 6.2 to control angular error terms that arise from commuting $\Box_g$ with L and $\underline{L}$. We first derive a general expression for the commutator $[\Box_g, W^m V^n]$, where V and W are vector fields.

Lemma 6.3. *Let V and W be vector fields and $n \geq 1$, then*

$$2\Omega^2 \Box_g (W^m V^n \phi)$$

$$\begin{aligned}
= {} & [W^m V^n, \underline{L}] L\phi + \underline{L}([W^m V^n, L]\phi) + [W^m V^n, L]\underline{L}\phi + L([W^m V^n, \underline{L}]\phi) \\
& \sum_{l=1}^{m} \sum_{k=1}^{n} \binom{m}{l}\binom{n}{k} W^l V^k (\Omega \operatorname{tr} \underline{\chi}) W^{m-l} V^{n-k} L\phi + \Omega \operatorname{tr} \underline{\chi}[W^m V^n, L]\phi \\
& + \sum_{l=1}^{m} \sum_{k=1}^{n} \binom{m}{l}\binom{n}{k} W^l V^k (\Omega \operatorname{tr} \chi) W^{m-l} V^{n-k} \underline{L}\phi + \Omega \operatorname{tr} \chi[W^m V^n, \underline{L}]\phi \\
& - 2\sum_{l=1}^{m} \sum_{k=1}^{n} \binom{m}{l}\binom{n}{k} W^l V^k (\overset{\scriptscriptstyle\centerdot}{g}{}^{AB} \partial_A \Omega^2) W^{m-l} V^{n-k} \partial_B \phi \\
& - 2\overset{\scriptscriptstyle\centerdot}{g}{}^{AB} \partial_A \Omega^2 [W^m V^n, \partial_B]\phi \\
& - 2\sum_{l=1}^{m} \sum_{k=1}^{n} \binom{m}{l}\binom{n}{k} W^l V^k (\Omega^2) W^{m-l} V^{n-k} \overset{\scriptscriptstyle\centerdot}{\Delta}\phi - 2\Omega^2 [W^m V^n, \overset{\scriptscriptstyle\centerdot}{\Delta}]\phi.
\end{aligned}$$

Proof. Use the expression for the wave Eq. (B.2) in "Appendix B", together with

$$W^m V^n (2\Omega^2 \Box_g \phi) = 0. \qquad\square$$

Proposition 6.4. *Either restrict to $|a| < a_c$ and let $0 \leq p < 2$ and $0 \leq q < 2$, or restrict to axisymmetric ϕ, with $p = 2$ and $0 < q \leq 2$. Then, there exist $\alpha = \alpha(p,q) > 1$ and $\beta = \beta(p,q) > 1$, such that for u_1 suitably large and $|u| \geq |u_1|$,*

$$\int_{\underline{H}_v \cap \{|u| \geq |u_1|\}} J^{N_{p,q}}[\underline{L}\phi] \cdot L + J^{N_{p,q}}[L\phi] \cdot \underline{L} + \int_{H_u} J^{N_{p,q}}[\underline{L}\phi] \cdot L + J^{N_{p,q}}[L\phi] \cdot L$$

$$\leq C \Bigg[\int_{\underline{H}_{v_0} \cap \{|u| \geq |u_1|\}} J^{N_{2,q\beta\alpha}}[\underline{L}\phi] \cdot \underline{L} + J^{N_{2,q\beta\alpha}}[L\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q\beta\alpha}}[\underline{L}\phi] \cdot L + J^{N_{2,q\beta\alpha}}[L\phi] \cdot L \Bigg]$$

$$+ C \sum_{\Gamma \in \{\mathrm{id}, \Phi^2, Q\}} \Bigg[\int_{\underline{H}_{v_0} \cap \{|u| \geq |u_1|\}} J^{N_{2,q\beta\alpha}}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,q\beta\alpha}}[\Gamma\phi] \cdot L \Bigg]$$

$$+ C \sum_{\Gamma \in \{\Phi^2, Q\}} \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^{-2+\epsilon} (\Gamma\phi)^2,$$

where $0 < q\beta\alpha \leq 2$, $\epsilon > 0$ can be taken arbitrarily small and

$$C = C(a, M, u_0, v_0, u_1, v_1, p, q, \alpha, \beta, \epsilon) > 0.$$

For axisymmetric ϕ, we can replace $N_{2,q\beta\alpha}$ on the right-hand side by $N_{2,q}$.

Moreover, for $v_1 > v_0$ suitably large and $v \geq v_1$,

$$\int_{\underline{H}_v} J^{N_{p,q}}[\underline{L}\phi] \cdot L + J^{N_{p,q}}[L\phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}}[\underline{L}\phi] \cdot L + J^{N_{p,q}}[L\phi] \cdot L$$

$$\leq C \Bigg[\int_{\underline{H}_{v_1}} J^{N_{2,q\beta\alpha}}[\underline{L}\phi] \cdot \underline{L} + J^{N_{2,q\beta\alpha}}[L\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,q\beta\alpha}}[\underline{L}\phi] \cdot L + J^{N_{2,q\beta\alpha}}[L\phi] \cdot L \Bigg]$$

$$+ C \sum_{\Gamma \in \{\mathrm{id}, \Phi^2, Q\}} \Bigg[\int_{\underline{H}_{v_1}} J^{N_{2,q\beta\alpha}}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,q\beta\alpha}}[\Gamma\phi] \cdot L \Bigg]$$

$$+ C \sum_{\Gamma \in \{\Phi^2, Q\}} \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-2+\epsilon} (\Gamma\phi)^2,$$

where $0 < q\beta\alpha \leq 2$, $\epsilon > 0$ can be taken arbitrarily small and

$$C = C(a, M, u_0, v_0, u_1, v_1, p, q, \alpha, \beta, \epsilon) > 0.$$

For axisymmetric ϕ, we can replace $N_{2,q\beta\alpha}$ on the right-hand side by $N_{2,q}$.

Proof. For any vector field V, we have that

$$\mathcal{E}^{N_{p,q}}[V\phi] = N_{p,q}(\phi)\Box_g(V\phi) = (|u|^p \underline{L} V\phi + v^q L V\phi)\Box_g(V\phi).$$

We can commute $\Box_g$ with L and $\underline{L}$ and apply Lemma 6.3 to obtain

$$2\Omega^2\Box_g(\underline{L}\phi) = \underline{L}([\underline{L}, L]\phi) + [\underline{L}, L]\underline{L}\phi + \underline{L}(\Omega\operatorname{tr}\underline{\chi})L\phi + \Omega\operatorname{tr}\underline{\chi}[\underline{L}, L]\phi$$
$$+ \underline{L}(\Omega\operatorname{tr}\chi)\underline{L}\phi - 2\underline{L}(\cancel{g}^{AB}\partial_A\Omega^2)\partial_B\phi - 2\underline{L}(\Omega^2)\cancel{\Delta}\phi$$
$$- 2\Omega^2[\underline{L}, \cancel{\Delta}]\phi$$

and

$$2\Omega^2\Box_g(L\phi) = L([L, \underline{L}]\phi) + [L, \underline{L}]L\phi + L(\Omega\operatorname{tr}\chi)\underline{L}\phi + \Omega\operatorname{tr}\chi[\underline{L}, L]\phi$$
$$+ L(\Omega\operatorname{tr}\underline{\chi})L\phi - 2L(\cancel{g}^{AB}\partial_A\Omega^2)\partial_B\phi - 2L(\Omega^2)\cancel{\Delta}\phi$$
$$- 2\Omega^2[L, \cancel{\Delta}]\phi - 2\cancel{g}^{AB}\partial_A\Omega^2[L, \partial_B]\phi.$$

Moreover, we have that

$$[\cancel{\Delta}, \underline{L}]\phi = -\underline{L}\left(\frac{1}{\sqrt{\det\cancel{g}}}\right)\partial_A\left(\cancel{g}^{AB}\sqrt{\det\cancel{g}}\partial_B\phi\right) - \frac{1}{\sqrt{\det\cancel{g}}}\partial_A\left(\underline{L}\left(\cancel{g}^{AB}\sqrt{\det\cancel{g}}\right)\partial_B\phi\right),$$

$$[\cancel{\Delta}, L]\phi = -L\left(\frac{1}{\sqrt{\det\cancel{g}}}\right)\partial_A\left(\cancel{g}^{AB}\sqrt{\det\cancel{g}}\partial_B\phi\right) - \frac{1}{\sqrt{\det\cancel{g}}}\partial_A\left(L\left(\cancel{g}^{AB}\sqrt{\det\cancel{g}}\right)\partial_B\phi\right)$$
$$- \frac{1}{\sqrt{\det\cancel{g}}}[L, \partial_A]\left(\cancel{g}^{AB}\sqrt{\det\cancel{g}}\partial_B\phi\right) - \cancel{g}^{AB}\partial_A([L, \partial_B]\phi).$$

By applying the estimates from Sect. 2.3, we obtain

$$|2\Omega^2\Box_g(\underline{L}\phi)| \lesssim (v+|u|)^{-2}|\underline{L}(\partial_{\varphi_*}\phi)| + (v+|u|)^{-3}(|L\phi| + |\underline{L}\phi|)$$
$$+ (v+|u|)^{-3}|\cancel{\nabla}\phi| + (v+|u|)^{-3}|\cancel{\nabla}^2\phi|.$$

and

$$|2\Omega^2\Box_g(L\phi)| \lesssim (v+|u|)^{-2}|L(\partial_{\varphi_*}\phi)| + (v+|u|)^{-3}(|L\phi| + |\underline{L}\phi|)$$
$$+ (v+|u|)^{-3}|\cancel{\nabla}\phi| + (v+|u|)^{-3}|\cancel{\nabla}^2\phi|.$$

Consequently, we can apply Cauchy–Schwarz to obtain

$$2\Omega^2|\mathcal{E}^{N_{p,q}}[\underline{L}\phi]| \lesssim v^{-1-\eta}|u|^p(\underline{L}\underline{L}\phi)^2 + |u|^{-1-\eta}v^q(L\underline{L}\phi)^2$$
$$+ (v^{1+\eta}|u|^p + |u|^{1+\eta}v^q)\Big[(v+|u|)^{-4}|\underline{L}(\partial_{\varphi_*}\phi)|^2$$
$$+ (v+|u|)^{-6}(|L\phi|^2 + |\underline{L}\phi|^2)$$
$$+ (v+|u|)^{-6}|\cancel{\nabla}\phi|^2 + (v+|u|)^{-6}|\cancel{\nabla}^2\phi|^2\Big]$$

and

$$2\Omega^2|\mathcal{E}^{N_{p,q}}[L\phi]| \lesssim v^{-1-\eta}|u|^p(\underline{L}L\phi)^2 + |u|^{-1-\eta}v^q(LL\phi)^2$$
$$+ (v^{1+\eta}|u|^p + |u|^{1+\eta}v^q)\Big[(v+|u|)^{-4}|L(\partial_{\varphi_*}\phi)|^2$$
$$+ (v+|u|)^{-6}(|L\phi|^2 + |\underline{L}\phi|^2)$$
$$+ (v+|u|)^{-6}|\cancel{\nabla}\phi|^2 + (v+|u|)^{-6}|\cancel{\nabla}^2\phi|^2\Big].$$

We obtain similar estimates for $2\Omega^2|\mathcal{E}^{Y_p}[\underline{L}\phi]|$ and $2\Omega^2|\mathcal{E}^{Y_p}[L\phi]|$, where we replace the weight v^q by $|u|^p$.

Using (6.3) of Proposition 6.2, we can further estimate

$$(v+|u|)^{-6} \int_{S^2_{u,v}} |\overset{\scriptscriptstyle\diagup}{\nabla}^2 \phi|^2 \, \mathrm{d}\mu_{\not{g}} \leq C(v+|u|)^{-6}$$

$$\int_{S^2_{u,v}} (Q\phi)^2 + (\Phi^2\phi)^2 + (T^2\phi)^2 + (L^2\phi)^2 + (\underline{L}^2\phi)^2$$

$$+ (L\underline{L}\phi)^2 + (L\phi)^2 + (\underline{L}\phi)^2 + |\overset{\scriptscriptstyle\diagup}{\nabla} L\phi|^2 + |\overset{\scriptscriptstyle\diagup}{\nabla}\underline{L}\phi|^2 + |\overset{\scriptscriptstyle\diagup}{\nabla}\phi|^2 \, \mathrm{d}\mu_{\not{g}}. \qquad (6.10)$$

The $T^2\phi$ term can be absorbed into the remaining terms on the right-hand side of the above inequality. The energy estimates of Propositions 4.2 and 5.3 apply also to $Q\phi$ and $\Phi^2\phi$, so we can estimate, by applying the fundamental theorem of calculus together with Cauchy–Schwarz in the region $\{|u| \geq |u_1|\}$:

$$\int_{S^2_{u,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} \leq \int_{S^2_{-\infty,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} + C|u|^{1-p'} \sup_{v_0 \leq v' \leq v} \int_{\underline{H}'_v \cap \{|u| \geq |u_1|\}} |u|^{p'} (\underline{L}\Gamma\phi)^2$$

$$\leq C|u|^{1-p'} \left[\int_{\underline{H}_{v_0} \cap \{|u| \geq |u_1|\}} J^{N_{2,\eta}}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,\eta}}[\Gamma\phi] \cdot L \right]$$

$$+ \int_{S^2_{-\infty,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}},$$

where $\Gamma \in \{\Phi^2, Q\}$ and we can take $p' = 2$ if ϕ is axisymmetric and $p' = 2 - \eta$, with $\eta > 0$ arbitrarily small, if ϕ is not axisymmetric.

We similarly apply the fundamental theorem of calculus together with Cauchy–Schwarz to obtain:

$$\int_{S^2_{u,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} \leq \int_{S^2_{-\infty,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} + C|u|^{1-p'} \sup_{v_1 \leq v' \leq v} \int_{\underline{H}'_v} |u|^{p'} (\underline{L}\Gamma\phi)^2$$

$$\leq \int_{S^2_{-\infty,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}}$$

$$+ C|u|^{1-p'} \left[\int_{\underline{H}_{v_1}} J^{N_{2,\eta}}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,\eta}}[\Gamma\phi] \cdot L \right],$$

Therefore, we can estimate in $\{|u| \geq |u_1|\}$,

$$\int_{v_0}^{\infty} \int_{-\infty}^{u_1} (v^q |u|^{1+\eta} + |u|^p v^{1+\eta})(v+|u|)^{-6} \int_{S^2_{u,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} \mathrm{d}u \mathrm{d}v$$

$$\leq C \int_{v_0}^{\infty} v^{-4+\eta+\max\{p,q\}} \int_{S^2_{-\infty,v}} (\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}} \mathrm{d}v$$

$$+ C \int_{v_0}^{\infty} \int_{-\infty}^{u_1} (v^q |u|^{\eta} + |u|^{p-1+\eta} v^{1+\eta})(v+|u|)^{-6} \, \mathrm{d}u \mathrm{d}v$$

$$\times \left[\int_{\underline{H}_{v_0} \cap \{|u| \geq |u_1|\}} J^{N_{2,\eta}}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_{2,\eta}}[\Gamma\phi] \cdot L \right]$$

$$\leq C \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} v^{-4+\eta+\max\{p,q\}} (\Gamma\phi)^2$$

$$+ C \left[\int_{\underline{H}_{v_0} \cap \{|u| \geq |u_1|\}} J^{N_2,\eta}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_0\}} J^{N_2,\eta}[\Gamma\phi] \cdot L \right].$$

Similarly, we can estimate in $\{v \geq v_1\}$

$$\int_{v_1}^\infty \int_{-\infty}^{u_0} (v^q|u|^{1+\eta} + |u|^p v^{1+\eta})(v+|u|)^{-6} \int_{S_{u,v}^2} (\Gamma\phi)^2 \, d\mu_{\not g} du dv$$

$$\leq C \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-4+\eta+\max\{p,q\}} (\Gamma\phi)^2$$

$$+ C \left[\int_{\underline{H}_{v_1}} J^{N_2,\eta}[\Gamma\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,\eta}[\Gamma\phi] \cdot L \right].$$

The remaining terms on the right-hand side of (6.10) can be estimated by energy fluxes through H_u and $\underline{H}_v$, multiplied by integrable functions $h(u)$ or $k(v)$. We can apply Lemma 4.1 in the regions $\underline{\mathcal{A}}$ and $\mathcal{B}$, with $|u| \geq |u_1|$ or $v \geq v_1$. Furthermore, we can estimate in a similar way the terms in $\mathcal{E}^{Y_p}[L\phi]$ and $\mathcal{E}^{Y_p}[\underline{L}\phi]$ in the region $\mathcal{B}$, with $|u| \geq |u_1|$ or $v \geq v_1$.

We combine in the $|a| < a_c$ case, as in Proposition 5.3, the estimates with respect to the multipliers $N_{p,q}$ and Y_p. $\qquad\square$

We can easily commute $\Box_g$ with L, $\underline{L}$ and ∂_{θ_*} in the region $\{|u| \leq |u_1|, v \leq v_1\}$. As $|u|$ and v are both finite in this region, we do not need to keep track of the behaviour in $v + |u|$ of the error terms.

Proposition 6.5. *Let* $0 \leq p, q \leq 2$ *and* $v_1 > v_0$, $u_1 < u_0$. *Then, there exists a* $C = C(a, M, p, q, u_0, v_0, u_1, v_1) > 0$ *such that for all* $k \in \mathbb{N}$,

$$\sum_{0 \leq j_1+j_2+j_3+j_4 \leq k} \int_{\underline{H}_v \cap \{|u| \leq |u_1|\}} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot \underline{L}$$

$$+ \int_{H_u \cap \{v \leq v_1\}} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot L$$

$$\leq C \sum_{0 \leq j_1+j_2+j_3+j_4 \leq k} \int_{\underline{H}_{v_1} \cap \{|u| \leq |u_1|\}} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot \underline{L}$$

$$+ \int_{H_{u_1} \cap \{v \leq v_1\}} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot L.$$

Proof. It easily follows that

$$\sum_{0 \leq j_1+j_2+j_3+j_4 \leq k} |\mathcal{E}^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi]| + |K^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi]|$$

$$\lesssim \sum_{0 \leq j_1+j_2+j_3+j_4 \leq k} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot \underline{L}$$

$$+ J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \partial_{\theta_*}^{j_3} \Phi^{j_4} \phi] \cdot L,$$

as the energy fluxes together control all derivatives, since $|u|$ and v are bounded. We can therefore directly apply Lemma 4.1 in $\{|u| \leq |u_1|, \, v \leq v_1\}$ to obtain the estimate in the proposition. $\qquad\square$

We also commute with higher-order derivatives along null vector fields in the region $\{v \geq v_1\} \cup \{|u| \geq |u_1|\}$. In this case, we do need to keep track of the behaviour in $v + |u|$ of the error terms arising from commuting with L and $\underline{L}$.

Lemma 6.6. *Let $n \in \mathbb{N}_0$. Then, there exists a constant $C = C(a, M, v_0, u_0, n) > 0$, such that*

$$
\left| \sum_{j_1 + j_2 = n} 2\Omega^2 \Box_g (L^{j_1} \underline{L}^{j_2} \phi) \right| \leq C(v + |u|)^{-2} \sum_{j_1 + j_2 \leq n} |L^{j_1} \underline{L}^{j_2} \Phi \phi|
$$

$$
+ C(v + |u|)^{-3} \sum_{j_1 + j_2 \leq n-1} |\slashed{\nabla} L^{j_1} \underline{L}^{j_2} \phi| + |\slashed{\nabla}^2 L^{j_1} \underline{L}^{j_2} \phi|
$$

$$
+ C(v + |u|)^{-3} \sum_{j_1 + j_2 + j_3 \leq n-1} |L^{j_1} \underline{L}^{j_2} \Phi^{j_3+1} \phi|. \tag{6.11}
$$

Proof. By Lemma 6.3, we have that

$$
2\Omega^2 \Box_g (L^{j_1} \underline{L}^{j_2} \phi)
$$

$$
= [L^{j_1} \underline{L}^{j_2}, \underline{L}] L\phi + \underline{L}([L^{j_1} \underline{L}^{j_2}, L]\phi) + [L^{j_1} \underline{L}^{j_2}, L]\underline{L}\phi + L([L^{j_1} \underline{L}^{j_2}, \underline{L}]\phi)
$$

$$
\sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l}\binom{j_2}{k} L^l \underline{L}^k (\Omega \operatorname{tr} \underline{\chi}) L^{j_1-l} \underline{L}^{j_2-k} L\phi + \Omega \operatorname{tr} \underline{\chi} [L^{j_1} \underline{L}^{j_2}, L]\phi
$$

$$
+ \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l}\binom{j_2}{k} L^l \underline{L}^k (\Omega \operatorname{tr} \chi) L^{j_1-l} \underline{L}^{n-k} \underline{L}\phi + \Omega \operatorname{tr} \chi [L^{j_1} \underline{L}^{j_2}, \underline{L}]\phi
$$

$$
- 2 \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l}\binom{j_2}{k} L^l \underline{L}^k (\slashed{g}^{AB} \partial_A \Omega^2) L^{j_1-l} \underline{L}^{j_2-k} \partial_B \phi - 2\slashed{g}^{AB} \partial_A \Omega^2 [L^{j_1} \underline{L}^{j_2}, \partial_B]\phi
$$

$$
- 2 \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l}\binom{j_2}{k} L^l \underline{L}^k (\Omega^2) L^{j_1-l} \underline{L}^{j_2-k} \slashed{\Delta}\phi - 2\Omega^2 [L^{j_1} \underline{L}^{j_2}, \slashed{\Delta}]\phi.
$$

We have that

$$
[L, \underline{L}] = L(b^{\varphi_*})\Phi.
$$

It follows that

$$
[L^{j_1} \underline{L}^{j_2}, L](f) = \sum_{k=1}^{j_2} L^{j_1} \underline{L}^{j_2-k} [\underline{L}, L](\underline{L}^{k-1} f)
$$

$$
= - \sum_{k=1}^{j_2} L^{j_1} \underline{L}^{j_2-k} (L(b^{\varphi_*}) \underline{L}^{k-1} \Phi f),
$$

$$[L^{j_1}\underline{L}^{j_2}, \underline{L}](f) = \sum_{k=1}^{j_1} L^{j_1-k}[L, \underline{L}](L^{k-1}\underline{L}^{j_2}f)$$

$$= \sum_{k=1}^{j_1} L^{j_1-k}(L(b^{\varphi_*})L^{k-1}\underline{L}^{j_2}\Phi f).$$

By making use of the estimates for $\partial_{r_*}^n b^{\varphi_*}$ from Theorem 2.1, it follows that

$$\sum_{j_1+j_2=n} |[L^{j_1}\underline{L}^{j_2}, \underline{L}]L\phi + \underline{L}([L^{j_1}\underline{L}^{j_2}, L]\phi) + [L^{j_1}\underline{L}^{j_2}, L]\underline{L}\phi + L([L^{j_1}\underline{L}^{j_2}, \underline{L}]\phi)|$$

$$\leq C(v+|u|)^{-2} \sum_{j_1+j_2\leq n} |L^{j_1}\underline{L}^{j_2}\Phi\phi| + C(v+|u|)^{-3} \sum_{j_1+j_2+j_3\leq n-1} |L^{j_1}\underline{L}^{j_2}\Phi^{j_3+1}\phi|.$$

Furthermore, we have that

$$[L, \partial_{\theta_*}] = -\partial_{\theta_*}b^{\varphi_*}\partial_{\varphi_*},$$

so we obtain:

$$[L^{j_1}\underline{L}^{j_2}, \partial_{\theta_*}](f) = \sum_{k=1}^{j_1} L^{j_1-k}\left(\Phi b^{\varphi_*}\partial_{\theta_*} L^{k-1}\underline{L}^{j_2}f\right).$$

We once again make use of the estimates for $\partial_{r_*}^n\partial_{\theta_*}b^{\varphi_*}$ from Theorem 2.1 to estimate

$$\sum_{j_1+j_2=n} |[L^{j_1}\underline{L}^{j_2}, \partial_{\theta_*}]\phi| \leq C(v+|u|)^{-1} \sum_{j_1+j_2\leq n-1} |L^{j_1}\underline{L}^{j_2}\Phi\phi|^2.$$

Recall from Proposition 6.4 that

$$[\slashed{\Delta}, \underline{L}]\phi = -\underline{L}\left(\frac{1}{\sqrt{\det\slashed{g}}}\right)\partial_A\left(\slashed{g}^{AB}\sqrt{\det\slashed{g}}\partial_B\phi\right) - \frac{1}{\sqrt{\det\slashed{g}}}\partial_A\left(\underline{L}\left(\slashed{g}^{AB}\sqrt{\det\slashed{g}}\right)\partial_B\phi\right),$$

$$[\slashed{\Delta}, L]\phi = -L\left(\frac{1}{\sqrt{\det\slashed{g}}}\right)\partial_A\left(\slashed{g}^{AB}\sqrt{\det\slashed{g}}\partial_B\phi\right) - \frac{1}{\sqrt{\det\slashed{g}}}\partial_A\left(L\left(\slashed{g}^{AB}\sqrt{\det\slashed{g}}\right)\partial_B\phi\right)$$

$$- \frac{1}{\sqrt{\det\slashed{g}}}[L, \partial_A]\left(\slashed{g}^{AB}\sqrt{\det\slashed{g}}\partial_B\phi\right) - \slashed{g}^{AB}\partial_A([L, \partial_B]\phi).$$

We can therefore estimate

$$[L^{j_1}\underline{L}^{j_2}, \slashed{\Delta}](f) = \sum_{k=1}^{j_2} L^{j_1}\underline{L}^{j_2-k}[\underline{L}, \slashed{\Delta}](\underline{L}^{k-1}f) + \sum_{k=1}^{j_1} L^{j_1-k}[L, \slashed{\Delta}](L^{k-1}\underline{L}^{j_1}f).$$

Hence, we obtain by using the estimates for $\partial_{r_*}^k\slashed{g}_{AB}$ from Theorem 2.1,

$$\sum_{j_1+j_2=n} |[L^{j_1}\underline{L}^{j_2}, \slashed{\Delta}]\phi| \leq (v+|u|)^{-1} \sum_{j_1+j_2\leq n-1} |\slashed{\nabla}L^{j_1}\underline{L}^{j_2}\phi| + |\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|.$$

We can easily estimate the remaining terms in $2\Omega^2\Box_g(L^{j_1}\underline{L}^{j_2}\phi)$, applying the estimates from Sect. 2.3, to conclude that (6.11) must hold. □

Proposition 6.7. *Let $k \in \mathbb{N}_0$. Either restrict to $|a| < a_c$ and let $0 \leq p < 2$ and $0 \leq q < 2$, or restrict to axisymmetric ϕ, with $p = 2$ and $0 < q \leq 2$. Then, there exist $\alpha = \alpha(p,q) > 1$ and $\beta = \beta(p,q) > 1$, such that for u_1 suitably large, $|u| \geq |u_1|$:*

$$\sum_{j_1+j_2=2k+1} \int_{\underline{H}_v \cap \{|u| \geq |u_1|\}} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$\leq C \sum_{j_1+j_2+2j_3+j_4 \leq 2k+1} \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k+1} \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L} + \int_{\mathcal{H}^+} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[Q^{j_1+1}\Phi^{j_2}\phi] \cdot \underline{L} + \int_{\mathcal{H}^+} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2}\phi] \cdot L$$

$$+ C \sum_{j_1+2j_2 \leq 2k+2} \int_{\mathcal{H}^+} v^{-2+\epsilon}(\Phi^{j_1}Q^{j_2}\phi)^2,$$

and

$$\sum_{j_1+j_2=2k} \int_{\underline{H}_v \cap \{|u| \geq |u_1|\}} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$\leq C \sum_{j_1+j_2+2j_3+j_4 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L} + \int_{\mathcal{H}^+} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ C \sum_{j_1+2j_2 \leq 2k+1} \int_{\mathcal{H}^+} v^{-2+\epsilon}(\Phi^{j_1}Q^{j_2}\phi)^2,$$

where $0 < q\beta\alpha \leq 2$, $\epsilon > 0$ can be taken arbitrarily small and

$$C = C(k, a, M, u_0, v_0, u_1, v_1, p, q, \alpha, \beta, \epsilon) > 0.$$

For axisymmetric ϕ, we can replace $N_{2,q\beta\alpha}$ on the right-hand side by $N_{2,q}$.
Moreover, for $v_1 > v_0$ suitably large and $v \geq v_1$,

$$\sum_{j_1+j_2=2k+1} \int_{\underline{H}_v} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$\leq C \sum_{j_1+j_2+2j_3+j_4 \leq 2k+1} \int_{\underline{H}_{v_1}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k+1} \int_{\underline{H}_{v_1}} J^{N_{2,q\beta\alpha}}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_1}} J^{N_2,q\beta\alpha}[Q^{j_1+1}\Phi^{j_2}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2}\phi] \cdot L$$

$$+ C \sum_{j_1+2j_2 \leq 2k+2} \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-2+\epsilon}(\Phi^{j_1}Q^{j_2}\phi)^2,$$

and

$$\sum_{j_1+j_2=2k} \int_{\underline{H}_v} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$\leq C \sum_{j_1+j_2+2j_3+j_4 \leq 2k} \int_{\underline{H}_{v_1}} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ C \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_1}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ C \sum_{j_1+2j_2 \leq 2k+1} \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-2+\epsilon}(\Phi^{j_1}Q^{j_2}\phi)^2,$$

where $0 < q\beta\alpha \leq 2$, $\epsilon > 0$ *can be taken arbitrarily small and*

$$C = C(k, a, M, u_0, v_0, u_1, v_1, p, q, \alpha, \beta, \epsilon) > 0.$$

For axisymmetric ϕ, *we can replace* $N_{2,q\beta\alpha}$ *on the right-hand side by* $N_{2,q}$.

Proof. We have that

$$\sum_{j_1+j_2=n} 2\Omega^2 \mathcal{E}^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] = \sum_{j_1+j_2=n} N_{p,q}(L^{j_1}\underline{L}^{j_2}\phi)2\Omega^2\Box_g(L^{j_1}\underline{L}^{j_2}\phi).$$

From (6.11) it follows that

$$\sum_{j_1+j_2=n} 2\Omega^2|\mathcal{E}^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi]| \leq C \left(|u|^p|\underline{L}L^{j_1}\underline{L}^{j_2}\phi| + v^q|LL^{j_1}\underline{L}^{j_2}\phi|\right)$$

$$\cdot \left[(v+|u|)^{-2} \sum_{j_1+j_2 \leq n} |L^{j_1}\underline{L}^{j_2}\Phi\phi| \right.$$

$$+ (v+|u|)^{-3} \sum_{j_1+j_2 \leq n-1} |\nabla\!\!\!/ L^{j_1}\underline{L}^{j_2}\phi| + |\nabla\!\!\!/^2 L^{j_1}\underline{L}^{j_2}\phi|$$

$$\left. + (v+|u|)^{-3} \sum_{j_1+j_2+j_3 \leq n-1} |L^{j_1}\underline{L}^{j_2}\Phi^{j_3+1}\phi| \right].$$

We apply Cauchy–Schwarz to further estimate

$$\sum_{j_1+j_2=n} 2\Omega^2 |\mathcal{E}^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi]|$$

$$\leq C \sum_{j_1+j_2\leq n} v^{-1-\eta}|u|^p(\underline{L}L^{j_1}\underline{L}^{j_2}\phi)^2 + |u|^{-1-\eta}v^q(LL^{j_1}\underline{L}^{j_2}\phi)^2$$

$$+ C\left(v^{1+\eta}|u|^p + |u|^{1+\eta}v^q\right)\left[(v+|u|)^{-4}\sum_{j_1+j_2\leq n}|L^{j_1}\underline{L}^{j_2}\Phi\phi|^2\right.$$

$$+ (v+|u|)^{-6}\sum_{j_1+j_2\leq n-1}|\slashed{\nabla}L^{j_1}\underline{L}^{j_2}\phi|^2 + |\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|^2$$

$$\left.+ (v+|u|)^{-6}\sum_{j_1+j_2+j_3\leq n-1}|L^{j_1}\underline{L}^{j_2}\Phi^{j_3+1}\phi|^2\right].$$

We can apply (6.3) to obtain

$$\sum_{j_1+j_2\leq n-1}\int_{S^2_{u,v}}|\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|^2\,\mathrm{d}\mu_{\slashed{g}}$$

$$\leq C\int_{S^2_{u,v}}\sum_{j_1+j_2\leq n-1}(QL^{j_1}\underline{L}^{j_2}\phi)^2 + (T^2 L^{j_1}\underline{L}^{j_2}\phi)^2 + (L^{j_1}\underline{L}^{j_2}\Phi^2\phi)^2$$

$$+ \sum_{j_1+j_2\leq n+1}(L^{j_1}\underline{L}^{j_2}\phi)^2 + \sum_{j_1+j_2\leq n}|\slashed{\nabla}L^{j_1}\underline{L}^{j_2}\phi|^2\,\mathrm{d}\mu_{\slashed{g}}.$$

We can further estimate

$$\sum_{j_1+j_2\leq n-1}(QL^{j_1}\underline{L}^{j_2}\phi)^2 = \sum_{j_1+j_2\leq n-1}(L^{j_1}\underline{L}^{j_2}Q\phi)^2 + ([Q,L^{j_1}\underline{L}^{j_2}]\phi)^2.$$

We have that

$$[Q,L^{j_1}\underline{L}^{j_2}](f) = \sum_{k=1}^{j_2}L^{j_1}\underline{L}^{j_2-k}[Q,\underline{L}](\underline{L}^{k-1}f) + \sum_{k=1}^{j_1}L^{j_1-k}[Q,L](L^{k-1}\underline{L}^{j_2}f).$$

Furthermore,

$$[Q,\underline{L}](f) = [\Delta_{\mathbb{S}^2},\underline{L}](f) + a^2\underline{L}(\sin^2\theta)T^2 f$$
$$= [\sin\theta^{-1}\partial_\theta(\sin\theta\partial_\theta),\underline{L}]f - \underline{L}(\sin^{-2}\theta)\Phi^2 f + a^2\underline{L}(\sin^2\theta)T^2 f,$$
$$[Q,L](f) = [\Delta_{\mathbb{S}^2},L](f) + a^2 L(\sin^2\theta)T^2 f$$
$$= [\sin\theta^{-1}\partial_\theta(\sin\theta\partial_\theta),L]f - L(\sin^{-2}\theta)\Phi^2 f + a^2 L(\sin^2\theta)T^2 f,$$

with

$$[\underline{L},\partial_\theta]f = \underline{L}(\partial_\theta\theta_*)\partial_{\theta_*}f + \frac{1}{2}\underline{L}(\partial_\theta r_*)(Lf - b^{\varphi_*}\Phi f - \underline{L}f),$$
$$[L,\partial_\theta]f = L(\partial_\theta\theta_*)\partial_{\theta_*}f + \frac{1}{2}L(\partial_\theta r_*)(Lf - b^{\varphi_*}\Phi f - \underline{L}f).$$

By using the estimates from Sect. 2.3, we can estimate

$$|[Q, L^{j_1}\underline{L}^{j_2}](f)| \leq C(v + |u|)^{-2}$$

$$\times \left(\sum_{k+l \leq j_1+j_2} |\slashed{\nabla}^2 L^l \underline{L}^k \phi| + \sum_{k+l \leq j_1+j_2+1} |\slashed{\nabla} L^l \underline{L}^k \phi| \right.$$

$$\left. + \sum_{k+l \leq j_1+j_2+2} |L^l \underline{L}^k \phi| \right).$$

We conclude that, for either $v \geq v_1$ or $|u| \geq |u_1|$, with v_1 and $|u_1|$ suitably large

$$\sum_{j_1+j_2 \leq n-1} \int_{S^2_{u,v}} |\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|^2 \, \mathrm{d}\mu_{\slashed{g}}$$

$$\leq C \int_{S^2_{u,v}} \sum_{j_1+j_2 \leq n-1} (L^{j_1}\underline{L}^{j_2}Q\phi)^2 + (L^{j_1}\underline{L}^{j_2}\Phi^2\phi)^2$$

$$+ \sum_{j_1+j_2 \leq n+1} (L^{j_1}\underline{L}^{j_2}\phi)^2 + \sum_{j_1+j_2 \leq n} |\slashed{\nabla} L^{j_1}\underline{L}^{j_2}\phi|^2 \, \mathrm{d}\mu_{\slashed{g}}.$$

We can therefore use the above estimate to obtain

$$\sum_{j_1+j_2=n} \int_{S^2_{u,v}} 2\Omega^2 |\mathcal{E}^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi]| \, \mathrm{d}\mu_{\slashed{g}}$$

$$\leq C \sum_{j_1+j_2 \leq n} \int_{S^2_{u,v}} 2v^{-1-\eta}|u|^p (LL^{j_1}\underline{L}^{j_2}\phi)^2 + |u|^{-1-\eta}v^q (LL^{j_1}\underline{L}^{j_2}\phi)^2 \, \mathrm{d}\mu_{\slashed{g}}$$

$$+ C \int_{S^2_{u,v}} 2 \left(v^{1+\eta}|u|^p + |u|^{1+\eta}v^q \right) \left[(v+|u|)^{-4} \sum_{j_1+j_2 \leq n} |L^{j_1}\underline{L}^{j_2}\Phi\phi|^2 \right.$$

$$+ (v+|u|)^{-6} \sum_{j_1+j_2 \leq n} |\slashed{\nabla} L^{j_1}\underline{L}^{j_2}\phi|^2$$

$$+ (v+|u|)^{-6} \sum_{j_1+j_2 \leq n-1} (L^{j_1}\underline{L}^{j_2}Q\phi)^2 + (L^{j_1}\underline{L}^{j_2}\Phi^2\phi)^2$$

$$+ (v+|u|)^{-6} \sum_{j_1+j_2+j_3 \leq n-1} \left. |L^{j_1}\underline{L}^{j_2}\Phi^{j_3+1}\phi|^2 \right] \mathrm{d}\mu_{\slashed{g}},$$

for $0 \leq p, q \leq 2$. We obtain similar estimates for $\sum_{j_1+j_2=n} 2\Omega^2 |\mathcal{E}^{Y_p}[L^{j_1}\underline{L}^{j_2}\phi]|$, by replacing the weight v^q above by $|u|^p$.

We apply the divergence theorem, as in the previous propositions, to obtain the following energy estimate:

$$\sum_{j_1+j_2=n} \int_{H_v} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$\leq C \sum_{j_1+j_2=n} \int_{\underline{H}_{v_1}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,q\beta\alpha}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L$$

$$+ C \sup_{v_1 \leq v < \infty} \int_{\underline{H}_v} \sum_{j_1 + j_2 \leq n-2} \sum_{\Gamma \in \{Q, \Phi^2\}} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Gamma \phi] \cdot \underline{L}$$

$$+ \sum_{j_1 + j_2 \leq n-1} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Phi \phi] \cdot \underline{L}$$

$$+ \sum_{j_1 + j_2 + j_3 \leq n-2} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Phi^{j_3+1} \phi] \cdot \underline{L}$$

$$+ C \sup_{-\infty \leq u < u_0} \int_{H_u \cap \{v \geq v_1\}} \sum_{j_1 + j_2 \leq n-2} \sum_{\Gamma \in \{Q, \Phi^2\}} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Gamma \phi] \cdot L$$

$$+ \sum_{j_1 + j_2 \leq n-1} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Phi \phi] \cdot L$$

$$+ \sum_{j_1 + j_2 + j_3 \leq n-2} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \Phi^{j_3+1} \phi] \cdot L.$$

By induction, using the estimates Proposition 6.4, it follows that for odd n:

$$\sum_{j_1 + j_2 = n} \int_{\underline{H}_v} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \phi] \cdot L$$

$$\leq C \sum_{j_1 + j_2 + 2j_3 + j_4 \leq n} \int_{\underline{H}_{v_1}} J^{N_{2,\beta\alpha}} [L^{j_1} \underline{L}^{j_2} Q^{j_3} \Phi^{j_4} \phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,\beta\alpha}} [L^{j_1} \underline{L}^{j_2} Q^{j_3} \Phi^{j_4} \phi] \cdot L$$

$$+ C \sum_{2j_1 + j_2 + j_3 \leq n} \int_{\underline{H}_{v_1}} J^{N_{2,\beta\alpha}} [Q^{j_1} \Phi^{j_2+1} T^{j_3} \phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,\beta\alpha}} [Q^{j_1} \Phi^{j_2+1} T^{j_3} \phi] \cdot L$$

$$+ C \sum_{2j_1 + j_2 + j_3 \leq n-1} \int_{\underline{H}_{v_1}} J^{N_{2,\beta\alpha}} [Q^{j_1+1} \Phi^{j_2} T^{j_3} \phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,\beta\alpha}} [Q^{j_1} \Phi^{j_2} T^{j_3} \phi] \cdot L$$

$$+ C \sum_{j_1 + 2j_2 + j_3 \leq n+1} \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-2+\epsilon} (\Phi^{j_1} Q^{j_2} T^{j_3} \phi)^2,$$

whereas for even n we can estimate

$$\sum_{j_1 + j_2 = n} \int_{\underline{H}_v} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \phi] \cdot \underline{L} + \int_{H_u \cap \{v \geq v_1\}} J^{N_{p,q}} [L^{j_1} \underline{L}^{j_2} \phi] \cdot L$$

$$\leq C \sum_{j_1 + j_2 + 2j_3 + j_4 \leq n} \int_{\underline{H}_{v_1}} J^{N_{2,\beta\alpha}} [L^{j_1} \underline{L}^{j_2} Q^{j_3} \Phi^{j_4} \phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_{2,\beta\alpha}} [L^{j_1} \underline{L}^{j_2} Q^{j_3} \Phi^{j_4} \phi] \cdot L$$

$$+ C \sum_{2j_1+j_2+j_3 \leq n} \int_{\underline{H}_{v_1}} J^{N_2,\beta\alpha}[Q^{j_1}\Phi^{j_2+1}T^{j_3}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} J^{N_2,\beta\alpha}[Q^{j_1}\Phi^{j_2+1}T^{j_3}\phi] \cdot L$$

$$+ C \sum_{j_1+2j_2+j_3 \leq n+1} \int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{-2+\epsilon}(\Phi^{j_1}Q^{j_2}T^{j_3}\phi)^2,$$

The estimates in the region $\{|u| \geq |u_1|\}$ proceed similarly. $\qquad\square$

Let us define the following higher-order energy norms for $k \in \mathbb{N}_0$:

$$E_{q\beta\alpha;2k+1,\epsilon} := \sum_{j_1+j_2+2j_3+j_4 \leq 2k+1} \int_{\underline{H}_{v_0}} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ \sum_{2j_1+j_2 \leq 2k+1} \int_{\underline{H}_{v_0}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_2,q\beta\alpha}[Q^{j_1+1}\Phi^{j_2}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2}\phi] \cdot L$$

$$+ \sum_{2j_1+j_2 \leq 2k+2} \int_{\mathcal{H}^+} v^{-2+\epsilon}(Q^{j_1}\Phi^{j_2}\phi)^2,$$

$$E_{q\beta\alpha;2k,\epsilon} := \sum_{j_1+j_2+2j_3+j_4 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_2,q\beta\alpha}[L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi] \cdot L$$

$$+ \sum_{2j_1+j_2 \leq 2k} \int_{\underline{H}_{v_0}} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot \underline{L}$$

$$+ \int_{\mathcal{H}^+} J^{N_2,q\beta\alpha}[Q^{j_1}\Phi^{j_2+1}\phi] \cdot L$$

$$+ \sum_{2j_1+j_2 \leq 2k+1} \int_{\mathcal{H}^+} v^{-2+\epsilon}(Q^{j_1}\Phi^{j_2}\phi)^2.$$

We combine the results of Propositions 6.7 and 6.5 to obtain an energy estimate in the entire region $\mathcal{D}_{u_0,v_0}$.

Corollary 6.8. *Let $k \in \mathbb{N}_0$. Restrict to $|a| < a_c$ and let $0 \leq p < 2$ and $0 \leq q < 2$, or restrict to axisymmetric ϕ, with $p = 2$ and $0 < q \leq 2$.*

Then, there exist $\alpha = \alpha(p,q) > 1$ *and* $\beta = \beta(p,q) > 1$ *and a constant* $C = C(n, a, M, u_0, v_0, p, q, \alpha, \beta) > 0$, *such that*

$$\sum_{j_1+j_2=2k+1} \int_{\underline{H}_v} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L \leq CE_{q\beta\alpha;2k+1,\epsilon},$$

and

$$\sum_{j_1+j_2=2k} \int_{\underline{H}_v} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot \underline{L} + \int_{H_u} J^{N_{p,q}}[L^{j_1}\underline{L}^{j_2}\phi] \cdot L \leq CE_{q\beta\alpha;2k,\epsilon},$$

Remark 6.2. In the inequalities of Corollary 6.8, integrals of n derivatives of ϕ are estimated by initial integrals of $n+1$ derivatives. The loss of derivatives in the *even* n case arises only because of Φ. Therefore, if ϕ is axisymmetric and we can neglect the Φ derivatives, we <u>do not lose any derivatives</u> in Proposition 6.7 for even n. This fact is important when proving energy estimates for nonlinear wave equations.

7. Pointwise Estimates

7.1. Uniform Boundedness of ϕ

We can use the higher-order energy estimates in the previous section to obtain a uniform pointwise bound on ϕ. As in the previous section, ϕ always denotes a solution to (1.1) arising from initial data prescribed in Proposition 3.1. We will always indicate whether we are assuming axisymmetry of ϕ or the restriction $0 \leq |a| < a_c$ for the rotation parameter a.

Proposition 7.1. *Let* $n \in \mathbb{N}_0$. *Restrict to* $0 \leq |a| < a_c$ *and take* $0 \leq p < 2$, *or restrict to axisymmetric* ϕ *and take* $0 \leq p \leq 2$. *Let* $\epsilon > 0$ *arbitrarily small and take* $0 < q < 2$. *There exists a constant* $C = C(a, M, v_0, u_0, q, \epsilon) > 0$ *such that,*

$$\sum_{j_1+j_2\leq n}(L^{j_1}\underline{L}^{j_2}\phi)^2(u, v, \theta_*, \varphi_*) \leq \sum_{|k|\leq 2}\sum_{j_1+j_2\leq n}\int_{S^2_{-\infty,v}}|\overset{k}{\nabla}L^{j_1}\underline{L}^{j_2}\phi|^2\,\mathrm{d}\mu_{\not g}$$

$$+ C|u|^{1-p}\Bigg[E_{q;n+2,\epsilon}[\phi]$$

$$+ \sum_{\Gamma\in\{\Phi^2,T^2,Q\}}E_{q;n+1,\epsilon}[\Gamma\phi]$$

$$+ \sum_{\Gamma\in\{\Phi^3,T^2\Phi,Q\Phi\}}E_{q;n,\epsilon}[\Gamma\phi]\Bigg].$$

Proof. By the fundamental theorem of calculus applied to integrating along ingoing null geodesics, together with Cauchy–Schwarz, we can estimate

$$\phi^2(u, v, \theta_*, \varphi_*) \leq \phi^2(-\infty, v, \theta_*, \varphi_*) + \left(\int_{-\infty}^{u}|\underline{L}\phi|(u', v, \theta_*, \varphi_*)\,\mathrm{d}u'\right)^2,$$

$$\leq \phi^2(-\infty, v, \theta_*, \varphi_*) + \frac{1}{p+1}|u|^{-1+p}\int_{-\infty}^{u}|u'|^p(\underline{L}\phi)^2(u', v, \theta_*, \varphi_*)\,\mathrm{d}u',$$

for $p > 1$.

We can integrate over the spheres and apply Proposition 5.3 to arrive at

$$\int_{S^2_{u,v}} \phi^2 \, \mathrm{d}\mu_{\not g} \leq \int_{S^2_{-\infty,v}} \phi^2 \, \mathrm{d}\mu_{\not g} + C|u|^{-1+p} \int_{H_v \cap \{|u'| \geq |u|\}} |u|^p (\underline{L}\phi)^2$$

$$\leq \int_{S^2_{-\infty,v}} \phi^2 \, \mathrm{d}\mu_{\not g} + C|u|^{1-p} E_q[\phi],$$

for $q > 0$.

To arrive at a pointwise estimate, we apply the standard Sobolev inequality on the spheres $S^2_{u,v}$, together with Proposition 6.2:

$$||L^{j_1} \underline{L}^{j_2} \phi||^2_{L^\infty(S^2_{u,v})}$$

$$\leq C \int_{S^2_{u,v}} (L^{j_1} \underline{L}^{j_2} \phi)^2 + |\not\nabla L^{j_1} \underline{L}^{j_2} \phi|^2 + |\not\nabla^2 L^{j_1} \underline{L}^{j_2} \phi|^2 \, \mathrm{d}\mu_{\not g}$$

$$\leq C \sum_{k=0}^1 \sum_{\Gamma \in \{\mathrm{id}, \Phi, \Phi^2, T^2, Q\}} \int_{S^2_{u,v}} (\Gamma \underline{L}^k L^{j_1} \underline{L}^{j_2} \phi)^2 + (\Gamma L^k L^{j_1} \underline{L}^{j_2} \phi)^2 + (\underline{L}^2 L^{j_1} \underline{L}^{j_2} \phi)^2$$

$$+ (L^2 L^{j_1} \underline{L}^{j_2} \phi)^2 + (L \underline{L} L^{j_1} \underline{L}^{j_2} \phi)^2 + (b^{\varphi_*})^2 [(Q \Phi L^{j_1} \underline{L}^{j_2} \phi)^2 + (T^2 \Phi L^{j_1} \underline{L}^{j_2} \phi)^2$$

$$+ (\Phi^3 L^{j_1} \underline{L}^{j_2} \phi)^2] \, \mathrm{d}\mu_{\not g}.$$

We now combine the results of Propositions 6.2, 6.4, 6.5 and Corollary 6.8; in particular, we commute Γ in the terms above to act directly on ϕ, in order to arrive at the estimate in the proposition. $\qquad\square$

We have now proved Theorem 3.5.

7.2. Extendibility of ϕ in C^0

We can use Proposition 7.1 to show that ϕ can be extended as a continuous function beyond the Cauchy horizon $\mathcal{CH}^+$. As this extension is independent of the characteristic data, it is non-unique.

Proposition 7.2. *Let the initial data for ϕ satisfy*

$$E_{q;2,\epsilon}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} E_{q;1,\epsilon}[\Gamma \phi] + \sum_{\Gamma \in \{\Phi^3, T^2 \Phi, Q\Phi\}} E_q[\Gamma \phi] < \infty,$$

for some $q > 1$ and $\epsilon > 0$.

Let $x_{\mathcal{CH}^+}$ be a point on $\mathcal{CH}^+$. Then, for any $x \in \mathcal{D}_{u_0,v_0}$,

$$\lim_{x \to x_{\mathcal{CH}^+}} \phi(x)$$

is well defined, so ϕ can be extended as a C^0 function to the region beyond $\mathcal{CH}^+$.

Proof. Consider a sequence of points x_k in $\mathcal{D}_{u_0,v_0} \backslash \mathcal{H}^+$, such that $\lim_{k \to \infty} x_k = x_{\mathcal{CH}^+}$. The sequence $\{x_k\}$ is, in particular, a Cauchy sequence. We will show that the sequence of points $(\phi)(x_k)$ must also be a Cauchy sequence, from which it follows immediately that the sequence converges to a finite number as $k \to \infty$.

Denote $x_k = (\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)$. Let $l > k$, then

$$|\phi(x_l) - \phi(x_k)|^2 \leq |\phi(\tilde{u}_l, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|$$
$$+ |\phi(\tilde{u}_k, \widetilde{V}_l, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2$$
$$+ |\phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_l, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2$$
$$+ |\phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_l) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2.$$

By the fundamental theorem of calculus, a Sobolev inequality on $\mathbb{S}^2$ and Cauchy–Schwarz, we can estimate for $q > 0$

$$\left|\phi(\tilde{u}_l, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)\right|^2$$
$$\leq C \sum_{|s| \leq 2} \left|\int_{u_k}^{u_l} \int_{\mathbb{S}^2} u^{1+\epsilon} |\partial_u \slashed{\nabla}^s \phi|^2(u, v(\widetilde{V}_k), (\theta_*)_k, (\varphi_*)_k) \, d\mu_{\mathbb{S}^2} du\right|$$
$$\leq C \sum_{0 \leq s_1 + s_2 \leq 2} \int_{\underline{H}_{v(\widetilde{V}_k)}} J^{N_{2,q}}[\partial_{\theta_*}^{s_1} \Phi^{s_2} \phi] \cdot \underline{L}.$$

Similarly, we find that for $q > 1$:

$$|\phi(\tilde{u}_k, \widetilde{V}_l, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2$$
$$\leq C \left|(-\widetilde{V}_l)^{q-1} - (-\widetilde{V}_k)^{q-1}\right|$$
$$\sum_{|s| \leq 2} \left|\int_{\widetilde{V}_k}^{\widetilde{V}_l} \int_{\mathbb{S}^2} (-\widetilde{V})^{2-q} |\partial_{\widetilde{V}} \slashed{\nabla}^s \phi|^2(\tilde{u}, \widetilde{V}, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) \, d\mu_{\mathbb{S}^2} d\widetilde{V}\right|$$
$$\leq C \left|(-\widetilde{V}_l)^{q-1} - (-\widetilde{V}_k)^{q-1}\right|$$
$$\sum_{|s| \leq 2} \left|\int_{\widetilde{V}(\widetilde{V}_k)}^{\widetilde{V}(\widetilde{V}_l)} \int_{\mathbb{S}^2} \widetilde{V}^q |\partial_{\widetilde{V}} \slashed{\nabla}^s \phi|^2(\tilde{u}, \widetilde{V}, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k) \, d\mu_{\mathbb{S}^2} d\widetilde{V}\right|$$
$$\leq C \left|(-\widetilde{V}_l)^{q-1} - (-\widetilde{V}_k)^{q-1}\right| \sum_{0 \leq s_1 + s_2 \leq 2} \int_{H_{u_k}} J^{N_{2,q}}[\partial_{\theta_*}^{s_1} \Phi^{s_2} \phi] \cdot L,$$

where we used that $(-\widetilde{V})^{2-q} \sim \widetilde{V}^{q-2}$ and $|\partial_{\widetilde{V}} \slashed{\nabla}^s \phi|^2 d\widetilde{V} \sim \widetilde{V}^2 |\partial_{\widetilde{V}} \slashed{\nabla}^s \phi|^2 d\widetilde{V}$.

Finally, we can estimate by Cauchy–Schwarz on $S^2_{\tilde{u}_k, \widetilde{V}_k}$,

$$|\phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_l, (\widetilde{\varphi}_*)_k) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2$$
$$+ |\phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_l) - \phi(\tilde{u}_k, \widetilde{V}_k, (\widetilde{\theta}_*)_k, (\widetilde{\varphi}_*)_k)|^2$$
$$\leq C \int_{S^2_{\tilde{u}_k, \widetilde{V}_k}} |\slashed{\nabla}\phi|^2(\tilde{u}_k, \widetilde{V}_k, \widetilde{\theta}_*, \widetilde{\varphi}_*) \, d\mu_{\slashed{g}} \leq C \sum_{s_1 + s_2 = 1} \int_{\underline{H}_{v(\widetilde{V}_k)}} J^{N_{2,q}}[\partial_{\theta_*}^{s_1} \Phi^{s_2} \phi] \cdot \underline{L},$$

where we need $q > 0$.

By the above estimates it follows that $\phi(x_k)$ must also be a Cauchy sequence if the energies on the right-hand sides are finite.

Finally, as in Proposition 7.1, we can estimate the energies on the right-hand sides of the above estimates by the initial energy $E_{\Gamma;q}[\phi]$. $\qquad \square$

Remark 7.1. By employing the higher-order energy estimates from Sect. 6.2 the above proposition can in fact easily be repeated to obtain also C^0 extendibility of $|\slashed{\nabla}^{j_1}\underline{L}^{j_2}\phi|^2$ across $\mathcal{CH}^+$ for any $j_1, j_2 \in \mathbb{N}_0$, provided the initial data are taken to be suitably regular (and decaying).

We have now proved Theorem 3.6.

7.3. Decay of $L\phi$

Consider the function $\phi_{\mathcal{H}^+} : \mathcal{M} \cap \mathcal{D}_{u_0,v_0} \to \mathbb{R}$ defined by

$$\phi_{\mathcal{H}^+}(u, v, \theta_*, \varphi_*) := \phi(-\infty, \theta_*, \varphi_*).$$

In particular, $\underline{L}\phi_{\mathcal{H}^+} = 0$.

We consider $\psi := \phi - \phi_{\mathcal{H}^+}$. We can improve the pointwise decay in ψ with respect to $|u|$ and use the wave Eq. (1.1) to obtain decay of $|L\phi|$ in v.

Moreover, we will show in this section that we can obtain boundedness of

$$\int_{H_u} v^2(L\phi)^2 + \int_{\underline{H}_v} v^2\Omega^2|\slashed{\nabla}\phi|^2,$$

which is an improvement over Corollary 5.3. Note that the analogous statement for axisymmetric solutions already follows from Proposition 4.2.

Proposition 7.3. *Denote*

$$D = ||\partial_U\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}\phi||^2_{L^\infty(\underline{H}_{v_0})}.$$

Let $0 < p < 2$, $0 < q < 2$ *and* $0 \le s \le 1$ *if* $|a| < a_c$. *Let* $p = 2$, $0 < q \le 2$ *and* $0 \le s \le 1$ *if* ϕ *is axisymmetric. Then, for every* $\epsilon > 0$, *there exists a constant* $C = C(M, u_0, v_0, p, q, s, \epsilon) > 0$, *such that for all* H_u *and* $\underline{H}_v$ *in* $\mathcal{D}_{u_0,v_0}$,

$$\int_{H_u} J^{N_{p,q}}[\psi] \cdot L + \int_{\underline{H}_v \cap \{u' \le u\}} J^{N_{p,q}}[\psi] \cdot \underline{L}$$

$$\le C|u|^{-s} \left[\int_{\mathcal{H}^+ \cap \{v \ge v_1\}} v^{s+\epsilon} \left[(L\phi)^2 + |\slashed{\nabla}\phi|^2 + |\slashed{\nabla}^2\phi|^2 \right] + D \right]$$

$$:= C|u|^{-s}\widetilde{E}_{\epsilon,s}[\phi].$$

Proof. We have that

$$2\Omega^2\Box_g\psi = -2\Omega^2\Box_g\phi|_{\mathcal{H}^+} = -\Omega\,\mathrm{tr}\,\underline{\chi}L\phi_{\mathcal{H}^+} + 2\slashed{\nabla}\Omega^2 \cdot \slashed{\nabla}\phi_{\mathcal{H}^+} + 2\Omega^2\slashed{\Delta}\phi_{\mathcal{H}^+}.$$

Consequently, we can estimate,

$$2\Omega^2|\Box_g\psi| \le C(v + |u|)^{-2}\left(|L\phi_{\mathcal{H}^+}| + |\slashed{\nabla}\phi_{\mathcal{H}^+}| + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|\right).$$

By applying Stokes' theorem in $\mathcal{D}_{u_0,v_0}$ we obtain the following error term:

$$\left| \int_{\mathcal{D}_{u_0,v_0}} \mathcal{E}^{N_{p,q}}[\psi] \right| \le \int\int\int_{S^2_{u,v}} (v + |u|)^{-2}(|u|^p|\underline{L}\psi| + v^q|L\psi|)$$

$$\left(|L\phi_{\mathcal{H}^+}| + |\slashed{\nabla}\phi_{\mathcal{H}^+}| + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|\right).$$

By Cauchy–Schwarz, we can estimate for $\eta > 0$

$$(v + |u|)^{-2}(|u|^p|\underline{L}\psi| + v^q|L\psi|)\left(|L\phi_{\mathcal{H}^+}| + |\slashed{\nabla}\phi_{\mathcal{H}^+}| + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|\right)$$
$$\lesssim v^{-1-\eta}|u|^p(\underline{L}\psi)^2$$
$$+ |u|^{-1-\eta'}v^q(L\psi)^2 + (v + |u|)^{-4}(|u|^p v^{1+\eta} + v^q|u|^{1+\eta})$$
$$\left[(L\phi_{\mathcal{H}^+})^2 + |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2\right].$$

We further estimate for $0 \le s \le 1$,

$$(v + |u|)^{-4}(|u|^p v^{1+\eta} + v^q|u|^{1+\eta})\left[(L\phi_{\mathcal{H}^+})^2 + |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2\right]$$
$$\lesssim |u|^{-1-s}(v^{p-2+s+\eta} + v^{q-2+s+\eta})\left[(L\phi_{\mathcal{H}^+})^2 + |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2\right].$$

Hence,

$$\int_{\mathcal{D}_{u_0,v_0}} |u|^{-1-s}(v^{p-2+s+\eta} + v^{q-2+s+\eta})\left[(L\phi_{\mathcal{H}^+})^2 + |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2\right]$$
$$\le |u|^{-s}\int_{\mathcal{H}^+ \cap \{v' \ge v_0\}} (v^{p-2+s+\eta} + v^{q-2+s+\eta})\left((L\phi)^2 + |\slashed{\nabla}\phi|^2 + |\slashed{\nabla}^2\phi|^2\right),$$

where we used that

$$(L\phi_{\mathcal{H}^+})^2 + |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2 \sim (L\phi)|^2_{\mathcal{H}^+} + |\slashed{\nabla}\phi|^2|_{\mathcal{H}^+} + |\slashed{\nabla}^2\phi|^2|_{\mathcal{H}^+}.$$

The remaining terms in $\mathcal{E}^{N_{p,q}}[\psi]$ and the terms in $K^{N_{p,q}}$ can be estimated as in Proposition 4.2 and the propositions in Sect. 5. $\qquad\square$

Proposition 7.4. *Let $k \in \mathbb{N}_0$. Denote*

$$D_{2k} := \sum_{j_1+j_2+2j_3+j_4 \le 2k} ||\partial_U L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})}$$
$$+ \sum_{j_1+2j_2 \le 2k} ||\partial_U \Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}\Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})},$$

$$D_{2k+1} := \sum_{j_1+j_2+2j_3+j_4 \le 2k+1} ||\partial_U L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}L^{j_1}\underline{L}^{j_2}Q^{j_3}\Phi^{j_4}\phi||^2_{L^\infty(\underline{H}_{v_0})}$$
$$+ \sum_{j_1+2j_2 \le 2k+1} ||\partial_U \Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}\Phi^{j_1+1}Q^{j_2}\phi||^2_{L^\infty(\underline{H}_{v_0})}$$
$$+ \sum_{j_1+2j_2 \le 2k} ||\partial_U \Phi^{j_1}Q^{j_2+1}\phi||^2_{L^\infty(\underline{H}_{v_0})} + ||\slashed{\nabla}\Phi^{j_1}Q^{j_2+1}\phi||^2_{L^\infty(\underline{H}_{v_0})}.$$

Let $0 < p < 2$, $0 < q \le 2$ and $0 \le s \le 1$ if $|a| < a_c$. Let $p = 2$, $0 < q \le 2$ and $0 \le s \le 1$ if ϕ is axisymmetric.

Let $n \in \mathbb{N}_0$. Then, for every $\epsilon > 0$, there exists a constant $C = C(M, a, n, u_0, v_0, p, q, s, \epsilon) > 0$, such that

$$\sum_{j_1 + j_2 = n} \int_{H_u} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \psi] \cdot L + \int_{\underline{H}_v \cap \{u' \leq u\}} J^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \psi] \cdot \underline{L}$$

$$\leq C|u|^{-s} \sum_{j_1 + 2j_2 + j_3 \leq n} \left[\int_{\mathcal{H}^+ \cap \{v \geq v_1\}} v^{s+\epsilon} \left[(L^{j_1+1} Q^{j_2} \Phi^{j_3} \phi)^2 + |\slashed{\nabla} L^{j_1} Q^{j_2} \Phi^{j_3} \phi|^2 \right. \right.$$

$$\left. \left. + |\slashed{\nabla}^2 L^{j_1} Q^{j_2} \Phi^{j_3} \phi|^2 \right] + D_n \right]$$

$$=: C|u|^{-s} \widetilde{E}_{s,\epsilon;n}[\phi].$$

Proof. We have that

$$2\Omega^2 \Box_g (L^{j_1} \underline{L}^{j_2} \psi)$$

$$= [L^{j_1} \underline{L}^{j_2}, L] \underline{L} \psi + \underline{L}([L^{j_1} \underline{L}^{j_2}, L] \psi) + [L^{j_1} \underline{L}^{j_2}, L] \underline{L} \psi + L([L^{j_1} \underline{L}^{j_2}, \underline{L}] \psi)$$

$$\sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l} \binom{j_2}{k} L^l \underline{L}^k (\Omega \operatorname{tr} \underline{\chi}) L^{j_1-l} \underline{L}^{j_2-k} L \psi + \Omega \operatorname{tr} \underline{\chi} [L^{j_1} \underline{L}^{j_2}, L] \psi$$

$$+ \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l} \binom{j_2}{k} L^l \underline{L}^k (\Omega \operatorname{tr} \chi) L^{j_1-l} \underline{L}^{n-k} \underline{L} \psi + \Omega \operatorname{tr} \chi [L^{j_1} \underline{L}^{j_2}, \underline{L}] \psi$$

$$- 2 \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l} \binom{j_2}{k} L^l \underline{L}^k (\slashed{g}^{AB} \partial_A \Omega^2) L^{j_1-l} \underline{L}^{j_2-k} \partial_B \psi - 2\slashed{g}^{AB} \partial_A \Omega^2 [L^{j_1} \underline{L}^{j_2}, \partial_B] \psi$$

$$- 2 \sum_{l=1}^{j_1} \sum_{k=1}^{j_2} \binom{j_1}{l} \binom{j_2}{k} L^l \underline{L}^k (\Omega^2) L^{j_1-l} \underline{L}^{j_2-k} \slashed{\Delta} \psi - 2\Omega^2 [L^{j_1} \underline{L}^{j_2}, \slashed{\Delta}] \psi$$

$$- L^{j_1} \underline{L}^{j_2} (2\Omega^2 \Box_g \psi).$$

We can repeat the proof of Proposition 6.7, but we have to additionally estimate the contribution of the final term in the above expression for $2\Omega^2 \Box_g (L^{j_1} \underline{L}^{j_2} \psi)$. We can estimate

$$|L^{j_1} \underline{L}^{j_2} (2\Omega^2 \Box_g \psi)| \leq \sum_{j_1 \leq n} |L^{j_1+1} \phi_{\mathcal{H}^+}| + |\slashed{\nabla} L^{j_1} \phi_{\mathcal{H}^+}| + |\slashed{\nabla}^2 L^{j_1} \phi_{\mathcal{H}^+}|.$$

We can therefore deal with the corresponding term in $\mathcal{E}^{N_{p,q}}[L^{j_1} \underline{L}^{j_2} \psi]$ in the same way as in the proof of Proposition 7.3. $\qquad\square$

Proposition 7.5. *Let $s \leq 1$ and $0 \leq p < 2$ for $0 \leq |a| < a_c$. For axisymmetric ϕ we let $0 \leq |a| \leq M$ and we can also take $p = 2$. Then, there exists a constant $C = C(M, v_0, u_0, p, s) > 0$ such that,*

$$\sum_{j_1 + j_2 = n} \int_{S^2_{u,v}} (L^{j_1} \underline{L}^{j_2} \psi)^2 + |\slashed{\nabla} L^{j_1} \underline{L}^{j_2} \psi|^2 + |\slashed{\nabla}^2 L^{j_1} \underline{L}^{j_2} \psi|^2 \, \mathrm{d}\mu_{\slashed{g}}$$

$$\leq C|u|^{1-s-p} \left[\widetilde{E}_{s,\epsilon;n+2}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;n+1}[\Gamma\phi] + \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{s,\epsilon;n}[\Gamma\phi] \right]$$

for $s \leq 1$ and $\epsilon > 0$ arbitrarily small.

Proof. We obtain estimates for angular derivatives along $S_{u,v}^2$ from the higher-order energy estimates in Proposition 7.4 in the same way as in Proposition 7.1.
$\square$

We can now obtain decay for $L\phi$.

Proposition 7.6. *Let $0 \leq |a| < a_c$, $1 < p < 2$ and $0 \leq s \leq 1$. For $\delta, \epsilon, q > 0$ arbitrarily small, there exists a constant $C = C(a, M, v_0, u_0, p, q, \epsilon, \delta, s) > 0$ such that,*

$$
\int_{S_{u,v}^2} (L\phi)^2(u, v, \theta_*, \varphi_*) \, \mathrm{d}\mu_{\not{g}} \leq \int_{S_{-\infty,v}^2} (L\phi)^2 \, \mathrm{d}\mu_{\not{g}}
$$
$$
+ C(v + |u|)^{-4} |u|^{1-p} E_q[\phi]
$$
$$
+ C v^{-4+(1-s)+(2-p)+\delta} \left[\widetilde{E}_{s,\epsilon;2}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;1}[\Gamma\phi] \right.
$$
$$
\left. + \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{s,\epsilon}[\Gamma\phi] \right]
$$
$$
+ C(v + |u|)^{-2} \int_{S_{-\infty,v}^2} |\not{\nabla}\phi_{\mathcal{H}^+}|^2 + |\not{\nabla}^2\phi_{\mathcal{H}^+}|^2 \mathrm{d}\mu_{\not{g}}. \tag{7.1}
$$

Moreover, for axisymmetric ϕ, we have a stronger estimate for all $0 \leq |a| \leq M$,

$$
\int_{S_{u,v}^2} (L\phi)^2(u, v, \theta_*, \varphi_*) \, \mathrm{d}\mu_{\not{g}} \leq \sum_{|k| \leq 2} \int_{S_{-\infty,v}^2} (L\phi)^2 \, \mathrm{d}\mu_{\not{g}} + C(v + |u|)^{-4} |u|^{1-p} E_q[\phi]
$$
$$
+ C v^{-4} \log\left(\frac{v + |u|}{|u|} \right) \left[\widetilde{E}_{1,\epsilon;2}[\phi] \right.
$$
$$
\left. + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{1,\epsilon;1}[\Gamma\phi] + \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{1,\epsilon}[\Gamma\phi] \right]
$$
$$
+ C(v + |u|)^{-2} \int_{S_{-\infty,v}^2} |\not{\nabla}\phi_{\mathcal{H}^+}|^2 + |\not{\nabla}^2\phi_{\mathcal{H}^+}|^2 \mathrm{d}\mu_{\not{g}}.
$$

Proof. We can write the wave equation as a transport equation for $(\det \not{g})^{\frac{1}{4}} L\phi$,

$$
\underline{L}((\det \not{g})^{\frac{1}{4}} L\phi) = (\det \not{g})^{\frac{1}{4}} \left[-\Omega \operatorname{tr}\chi \underline{L}\phi + 2\Omega^2 \zeta^{\varphi_*} \partial_{\varphi_*}\phi + \not{\nabla}\Omega^2 \cdot \not{\nabla}\phi + \Omega^2 \not{\Delta}\phi \right],
$$

see "Appendix B". In particular, we can estimate

$$
|\underline{L}((\det \not{g})^{\frac{1}{4}} L\phi)|(v, u, \theta_*, \varphi_*) \leq C(v + |u|)^{-2} \left(|\underline{L}\phi| + |\not{\nabla}\phi| + |\not{\nabla}^2\phi| \right).
$$

We can split

$$
\int_{S_{u,v}^2} (L\phi)^2 \, \mathrm{d}\mu_{\not{g}} \leq \int_{S_{-\infty,v}^2} (L\phi)^2 \, \mathrm{d}\mu_{\not{g}} + \int_{S_{u,v}^2} (L\psi)^2 \, \mathrm{d}\mu_{\not{g}}.
$$

We now integrate along ingoing null geodesics for fixed θ_* and φ_*, and subsequently integrate in θ_* and φ_* to obtain

$$
\int_{S^2_{u,v}} (L\phi)^2 \, \mathrm{d}\mu_{\slashed{g}}
$$

$$
\leq \int_{S^2_{u,v}} \left(\int_{-\infty}^{u} |\underline{L}((\det \slashed{g})^{\frac{1}{4}} L\phi)| \, \mathrm{d}u' \right)^2 \mathrm{d}\mu_{\slashed{g}}
$$

$$
\leq C \int_{S^2_{u,v}} \left(\int_{-\infty}^{u} (v + |u'|)^{-2} \left(|\underline{L}\phi| + |\slashed{\nabla}\phi| + |\slashed{\nabla}^2\phi| \right) \mathrm{d}u' \right)^2 \mathrm{d}\mu_{\slashed{g}}
$$

$$
\leq C(v + |u|)^{-4}|u|^{-\eta} \int_{H_v \cap \{|u'| \leq |u|\}} |u'|^{1+\eta} (\underline{L}\phi)^2
$$

$$
+ C \int_{-\infty}^{u} |u|^{-s}(v + |u'|)^{-2} \, \mathrm{d}u'
$$

$$
\cdot \int_{-\infty}^{u} |u'|^{s}(v + |u'|)^{-2} \left(\int_{S^2_{u',v}} |\slashed{\nabla}\psi|^2 + |\slashed{\nabla}^2\psi|^2 \mathrm{d}\mu_{\slashed{g}} \right) \mathrm{d}u'
$$

$$
+ C(v + |u|)^{-2} \int_{S^2_{-\infty,v}} |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2 \mathrm{d}\mu_{\slashed{g}}.
$$

We now apply the results of Propositions 7.3 and 7.5 to arrive at

$$
\int_{S^2_{u,v}} (L\psi)^2 \, \mathrm{d}\mu_{\slashed{g}} \leq C(v + |u|)^{-4}|u|^{-\eta} E_q[\phi]
$$

$$
+ C \left(\widetilde{E}_{s,\epsilon;2}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;1}[\Gamma\phi] + \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{s,\epsilon}[\Gamma\phi] \right)
$$

$$
\cdot \int_{-\infty}^{u} |u|^{-s}(v + |u'|)^{-2} \, \mathrm{d}u' \int_{-\infty}^{u} |u|^{1-p}(v + |u'|)^{-2} \, \mathrm{d}u'
$$

$$
+ C(v + |u|)^{-2} \int_{S^2_{-\infty,v}} |\slashed{\nabla}\phi_{\mathcal{H}^+}|^2 + |\slashed{\nabla}^2\phi_{\mathcal{H}^+}|^2 \mathrm{d}\mu_{\slashed{g}}.
$$

where $0 < s < 1$ and $\epsilon' > 0$ suitably small. Moreover, if ψ is axisymmetric we can take $s = 1$.

Now we use that

$$
\int_{-\infty}^{u} |u|^{-1}(v + |u'|)^{-2} \, \mathrm{d}u' \leq C(v + |u|)^{-2} \log \left(\frac{v + |u|}{|u|} \right),
$$

$$
\int_{-\infty}^{u} |u'|^{-s}(v + |u'|)^{-2} \, \mathrm{d}u' \leq C(v + |u|)^{-2+(1-s+\eta)}|u|^{-\eta},
$$

for $0 < s < 1$ and $\eta > 0$ arbitrarily small, to arrive at the statement in the proposition. $\qquad\square$

Corollary 7.7. *Let $|a| < a_c$ and $0 < s \le 1$. Then, there exist $1 < p < 2$, $\epsilon > 0$ and a constant $C = C(a, M, v_0, u_0, p, \epsilon, \delta, s) > 0$ such that,*

$$\int_{H_u} v^2 (L\phi)^2 + u^2 \Omega^2 |\nabla\!\!\!/\, \phi|^2 + \int_{\underline{H}_v} v^2 \Omega^2 |\nabla\!\!\!/\, \phi|^2$$

$$\le C|u|^{-p} E_q[\phi] + C\left[\widetilde{E}_{s,\epsilon;2}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;1}[\Gamma\phi] + \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{s,\epsilon}[\Gamma\phi]\right]$$

$$+ C\int_{\mathcal{H}^+ \cap \{v \ge v_0\}} v^2 (L\phi)^2 + |\nabla\!\!\!/\, \phi|^2 + |\nabla\!\!\!/\,^2 \phi|^2,$$

with $q > 0$ arbitrarily small.

Proof. To estimate the terms involving $\nabla\!\!\!/\, \phi$, we use that

$$\int_{S^2_{u,v}} |\nabla\!\!\!/\, \phi|^2 \, \mathrm{d}\mu_{g\!\!\!/} \le C\int_{S^2_{-\infty,v}} |\nabla\!\!\!/\, \phi|^2 \, \mathrm{d}\mu_{g\!\!\!/} + C\int_{S^2_{u,v}} |\nabla\!\!\!/\, \psi|^2 \, \mathrm{d}\mu_{g\!\!\!/}$$

and apply Proposition 7.5.

To estimate the term involving $L\phi$, we multiply (7.1) by v^2 and integrate from $v = v_0$ to $v = \infty$. $\qquad\square$

Remark 7.2. Recall that Corollary 5.3 gives a bound on

$$\int_{H_u} v^q (L\phi)^2,$$

with the restriction $q < 2$. Corollary 7.7 provides, moreover, an estimate for $q = 2$, at the expense of losing derivatives on the right-hand side.

We have now proved Theorem 3.4.

Proposition 7.8. *Let $0 \le |a| < a_c$ or assume ϕ is axisymmetric with $0 \le |a| \le M$. For $\delta, \epsilon, q > 0$ arbitrarily small and $0 \le s \le 1$, there exists a constant $C = C(a, M, v_0, u_0, \epsilon, \delta, s) > 0$ such that,*

$$v^{4+(s-1)-\delta} ||L\phi||^2_{L^\infty(S^2_{u,v})}$$

$$\le C\int_{S^2_{-\infty,v}} \sum_{j_1+j_2 \le 2} (LL^{j_1} \underline{L}^{j_2} \phi)^2 + \sum_{j_1+j_2 \le 1} \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} (LL^{j_1} \underline{L}^{j_2} \Gamma\phi)^2 \, \mathrm{d}\mu_{g\!\!\!/}$$

$$+ Cv^{2+(s-1)-\delta} \int_{S^2_{-\infty,v}} \sum_{j_1+j_2 \le 2} \left(|\nabla\!\!\!/\, L^{j_1} \underline{L}^{j_2} \phi|^2 + |\nabla\!\!\!/\,^2 L^{j_1} \underline{L}^{j_2} \phi|^2\right) \, \mathrm{d}\mu_{g\!\!\!/}$$

$$+ E_{q;6,\epsilon}[\phi] + E_{q;6,\epsilon}[\Phi\phi] + \widetilde{E}_{s,\epsilon;6}[\phi] + \widetilde{E}_{s,\epsilon;6}[\Phi\phi].$$

Proof. From "Appendix A" it follows that

$$\underline{L}((\det g\!\!\!/)^{\frac{1}{4}} Lf) = (\det g\!\!\!/)^{\frac{1}{4}}$$
$$\left[-\Omega \operatorname{tr} \chi \underline{L}\phi + 2\Omega^2 \zeta^{\varphi_*} \partial_{\varphi_*} \phi + \nabla\!\!\!/\, \Omega^2 \cdot \nabla\!\!\!/\, \phi + \Omega^2 \Delta\!\!\!/\, \phi - \Omega^2 \Box_g(f)\right],$$

for any suitably regular function $f : \mathcal{M} \cap \mathcal{D}_{u_0,v_0} \to \mathbb{R}$. In particular, we can estimate

$$\sum_{j_1+j_2=n} |\underline{L}((\det \not{g})^{\frac{1}{4}} LL^{j_1} L^{j_2} \phi)|(v,u,\theta_*,\varphi_*)$$

$$\leq C \sum_{j_1+j_2=n} (v+|u|)^{-2} \left(|\underline{LL}^{j_1} L^{j_2} \phi| + |\not{\nabla} L^{j_1} L^{j_2} \phi| + |\not{\nabla}^2 L^{j_1} L^{j_2} \phi| \right)$$

$$+ C|\Omega^2 \Box_g (L^{j_1} L^{j_2} \phi)|.$$

Using Lemma 6.6, we therefore obtain

$$\sum_{j_1+j_2=n} |\underline{L}((\det \not{g})^{\frac{1}{4}} LL^{j_1} \underline{L}^{j_2} \phi)|$$

$$\leq C \sum_{j_1+j_2 \leq n} (v+|u|)^{-2} \left(|\underline{LL}^{j_1} \underline{L}^{j_2} \phi| + |\not{\nabla} L^{j_1} \underline{L}^{j_2} \phi| + |\not{\nabla}^2 L^{j_1} \underline{L}^{j_2} \phi| \right)$$

$$+ C \sum_{j_1+j_2+j_3 \leq n-2} |L^{j_1} \underline{L}^{j_2} \Phi^{j_3+1} \phi|.$$

We can therefore repeat the proof of Proposition 7.6, using appropriate higher-order energy estimates, to obtain

$$\sum_{j_1+j_2=n} \int_{S_{u,v}^2} (LL^{j_1} \underline{L}^{j_2} \phi)^2 (u,v,\theta_*,\varphi_*) \, d\mu_{\not{g}}$$

$$\leq \sum_{j_1+j_2=n} \int_{S_{-\infty,v}^2} (LL^{j_1} \underline{L}^{j_2} \phi)^2 \, d\mu_{\not{g}} + C(v+|u|)^{-4} |u|^{1-p} E_{q;n,\epsilon}[\phi]$$

$$+ Cv^{-4+(1-s)+(2-p)+\delta} \Bigg[\widetilde{E}_{s,\epsilon;n+2}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;n+1}[\Gamma \phi]$$

$$+ \sum_{\Gamma \in \{\Phi^3, T^2\Phi, Q\Phi\}} \widetilde{E}_{s,\epsilon;n}[\Gamma \phi] \Bigg] + \sum_{j_1+j_2+j_3 \leq n-2} \widetilde{E}_{s,\epsilon;j_1+j_2+2}[\Phi^{j_3+1} \phi]$$

$$+ C(v+|u|)^{-2} \sum_{j_1+j_2=n} \int_{S_{-\infty,v}^2} |\not{\nabla} L^{j_1} \underline{L}^{j_2} \phi_{\mathcal{H}^+}|^2 + |\not{\nabla}^2 L^{j_1} \underline{L}^{j_2} \phi_{\mathcal{H}^+}|^2 d\mu_{\not{g}}$$

with $q > 0$ arbitrarily small.

We now apply (6.2) together with a standard Sobolev inequality on $S_{u,v}^2$ to obtain the following L^∞ estimate:

$$\|L\phi\|_{L^\infty(S_{u,v}^2)}^2 \leq C \sum_{j_1+j_2 \leq 2} \int_{S_{u,v}^2} (L^{j_1} \underline{L}^{j_2} L\phi)^2 \, d\mu_{\not{g}}$$

$$+ C \sum_{\Gamma \in \{id, \Phi, \Phi^2, T^2, Q\}} \int_{S_{u,v}^2} (\Gamma L\phi)^2 + (\Gamma \underline{LL}\phi)^2 + (\Gamma L^2 \phi)^2 \, d\mu_{\not{g}}$$

$$+ C(v+|u|)^{-2} \sum_{\Gamma \in \{Q, T^2, \Phi^2\}} \int_{S_{u,v}^2} (\Gamma \Phi L\phi)^2 \, d\mu_{\not{g}}.$$

We can bring the operator L in front of ϕ in the above inequality to the front at the expense of including commutators with L:

$$||L\phi||^2_{L^\infty(S^2_{u,v})} \leq C \sum_{j_1+j_2\leq 2} \int_{S^2_{u,v}} (LL^{j_1}\underline{L}^{j_2}\phi)^2 \, d\mu_{\not g}$$

$$+ C \sum_{\Gamma\in\{id,\Phi,\Phi^2,T^2,Q\}} \int_{S^2_{u,v}} (L\Gamma\phi)^2 + (L\Gamma\underline{L}\phi)^2 + (LL\Gamma\phi)^2 \, d\mu_{\not g}$$

$$+ C(v+|u|)^{-2} \sum_{\Gamma\in\{Q,T^2,\Phi^2\}} \int_{S^2_{u,v}} (L\Gamma\Phi\phi)^2 \, d\mu_{\not g} + J,$$

where

$$J := \int_{S^2_{u,v}} J_1 + J_2 + J_3 \, d\mu_{\not g},$$

with

$$J_1 := \sum_{\Gamma\in\{T^2,Q\}} ([\Gamma,L]\phi)^2 + ([\Gamma,L]\underline{L}\phi)^2 + ([\Gamma,L]L\phi)^2$$
$$+ (L[\Gamma,\underline{L}]\phi)^2 + (\underline{L}[\Gamma,L]\phi)^2 + \Gamma([L,\underline{L}]\phi)^2,$$

$$J_2 := \sum_{j_1+j_2\leq 2} ([L^{j_1}\underline{L}^{j_2},L]\phi)^2,$$

$$J_3 := (v+|u|)^{-2} \sum_{\Gamma\in\{T^2,Q\}} ([\Gamma,L]\Phi\phi)^2.$$

We apply the estimates for the above commutators that are derived in the proofs of Lemma 6.6 and Proposition 6.7 to estimate

$$J_1 \leq C(v+|u|)^{-4} \sum_{j_1+j_2\leq 1} |L^{j_1}\underline{L}^{j_2}\Phi\phi|^2,$$

$$J_2 \leq C(v+|u|)^{-4} \left[\sum_{j_1+j_2\leq 2} |\not\nabla^2 L^{j_1}\underline{L}^{j_2}\phi|^2 + \sum_{j_1+j_2\leq 3} |\not\nabla L^{j_1}L^{j_2}\phi|^2 \right.$$

$$\left. + \sum_{j_1+j_2\leq 4} |L^{j_1}\underline{L}^{j_2}\phi|^2 \right],$$

$$J_3 \leq C(v+|u|)^{-6} \left[\sum_{j_1+j_2\leq 1} |\not\nabla^2 L^{j_1}\underline{L}^{j_2}\Phi\phi|^2 \right.$$

$$\left. + \sum_{j_1+j_2\leq 2} |\not\nabla L^{j_1}L^{j_2}\Phi\phi|^2 + \sum_{j_1+j_2\leq 3} |L^{j_1}\underline{L}^{j_2}\Phi\phi|^2 \right].$$

We use the estimates above to obtain

$$||L\phi||^2_{L^\infty(S^2_{u,v})} \leq C(I_1+I_2),$$

where

$$I_1 := C \sum_{j_1+j_2\leq 2} \int_{S^2_{u,v}} (LL^{j_1} \underline{L}^{j_2} \phi)^2 \, \mathrm{d}\mu_{\not{g}}$$

$$+ C \sum_{\Gamma \in \{\mathrm{id}, \Phi, \Phi^2, T^2, Q\}} \int_{S^2_{u,v}} (L\Gamma\phi)^2 + (L\underline{L}\Gamma\phi)^2 + (LL\Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}}$$

$$+ C(v+|u|)^{-2} \sum_{\Gamma \in \{Q, T^2, \Phi^2\}} \int_{S^2_{u,v}} (L\Gamma\Phi\phi)^2 \, \mathrm{d}\mu_{\not{g}},$$

$$I_2 := C(v+|u|)^{-4} \int_{S^2_{u,v}} \sum_{j_1+j_2\leq 1} |\not\nabla^2 L^{j_1} \underline{L}^{j_2} \Phi\phi|^2 + \sum_{j_1+j_2\leq 2} |\not\nabla^2 L^{j_1} \underline{L}^{j_2} \phi|^2$$

$$+ \sum_{j_1+j_2\leq 3} |\not\nabla L^{j_1} \underline{L}^{j_2} \phi|^2 + \sum_{j_1+j_2\leq 4} (L^{j_1} \underline{L}^{j_2} \phi)^2 \, \mathrm{d}\mu_{\not{g}}.$$

By applying Corollary 6.8, as in Proposition 7.1, and, moreover, the estimates for the $L^2(S^2_{u,v})$ norms of the angular derivatives from Proposition 7.1, we can further estimate

$$I_2 \leq C(v+|u|)^{-4} \int_{S^2_{u,v}} E_{q;5,\epsilon}[\phi] + \sum_{\Gamma \in \{\Phi^2, T^2, Q\}} (E_{q;4,\epsilon}[\Gamma\phi] + E_{q;4,\epsilon}[\Gamma\Phi\phi]).$$

Furthermore,

$$I_1 \leq C \int_{S^2_{-\infty,v}} \sum_{j_1+j_2\leq 2} (LL^{j_1} \underline{L}^{j_2} \phi)^2 + \sum_{j_1+j_2\leq 1} \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} (LL^{j_1} \underline{L}^{j_2} \Gamma\phi)^2 \, \mathrm{d}\mu_{\not{g}}$$

$$+ C(v+|u|)^{-2} \int_{S^2_{-\infty,v}} \sum_{j_1+j_2\leq 2} \left(|\not\nabla L^{j_1} \underline{L}^{j_2} \phi|^2 + |\not\nabla^2 L^{j_1} \underline{L}^{j_2} \phi|^2 \right)$$

$$+ \sum_{j_1+j_2\leq 1} \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} \left(|\not\nabla L^{j_1} \underline{L}^{j_2} \Gamma\phi|^2 + |\not\nabla^2 L^{j_1} \underline{L}^{j_2} \Gamma\phi|^2 \right) \, \mathrm{d}\mu_{\not{g}}$$

$$+ C(v+|u|)^{-4} \left(E_{q;2,\epsilon}[\phi] + \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} E_{q;1,\epsilon}[\phi] \right)$$

$$+ Cv^{-4+(1-s)+\delta} \left(\widetilde{E}_{s,\epsilon;4}[\phi] + \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;3}[\Gamma\phi] + \widetilde{E}_{s,\epsilon;3}[\Gamma\Phi\phi] \right.$$

$$\left. + C \sum_{\Gamma' \in \{\Phi, \Phi^2, T^2, Q\}} \sum_{\Gamma \in \{\Phi, \Phi^2, T^2, Q\}} \widetilde{E}_{s,\epsilon;2}[\Gamma\Gamma'\phi] \right).$$

We conclude that
$$v^{4+(s-1)-\delta}||L\phi||^2_{L^\infty(S^2_{u,v})}$$
$$\leq C \int_{S^2_{-\infty,v}} \sum_{j_1+j_2\leq 2} (LL^{j_1}\underline{L}^{j_2}\phi)^2 + \sum_{j_1+j_2\leq 1}\sum_{\Gamma\in\{\Phi,\Phi^2,T^2,Q\}} (LL^{j_1}\underline{L}^{j_2}\Gamma\phi)^2 \, \mathrm{d}\mu_{\slashed{g}}$$
$$+ Cv^{2+(s-1)-\delta} \int_{S^2_{-\infty,v}} \sum_{j_1+j_2\leq 2} \left(|\slashed{\nabla}L^{j_1}\underline{L}^{j_2}\phi|^2 + |\slashed{\nabla}^2 L^{j_1}\underline{L}^{j_2}\phi|^2 \right)$$
$$+ E_{q;6,\epsilon}[\phi] + E_{q;6,\epsilon}[\Phi\phi] + \widetilde{E}_{s,\epsilon;6}[\phi] + \widetilde{E}_{s,\epsilon;6}[\Phi\phi].$$
$\square$

We have now proved Theorem 3.7.

Acknowledgements

Appendix A. Energy Currents in Kerr–Newman

We consider a spacetimes $(\mathcal{N},g)$ equipped with a double-null foliation $(u,v,\vartheta^1,\vartheta^2)$, such that the metric is given by
$$g = -2\Omega^2(u,v)(\mathrm{d}u\otimes\mathrm{d}v + \mathrm{d}v\otimes\mathrm{d}u) + \slashed{g}_{AB}(\mathrm{d}\vartheta^A - b^A\mathrm{d}v)\otimes(\mathrm{d}\vartheta^B - b^B\mathrm{d}v). \quad (A.1)$$
Here, u,v solve the Eikonal equation and the (topological) spheres $(S^2_{u,v},\slashed{g})$ are covered by coordinates ϑ^A, with $A=1,2$, foliate the null hypersurfaces $\{u=\text{const.}\}$ and $\{v=\text{const.}\}$.

Let
$$L = \partial_v + b^A\partial_A,$$
$$\underline{L} = \partial_u.$$
In the $(L,\underline{L},\partial_{\vartheta_A})$ basis, the metric components are given by
$$g(L,L) = 0,$$
$$g(\underline{L},\underline{L}) = 0,$$
$$g(L,\underline{L}) = -2\Omega^2.$$
$$g(L,\partial_{\vartheta_A}) = 0,$$

$$g(\underline{L}, \partial_{\vartheta_A}) = 0.$$

Lemma A.1.

$$\nabla_L L = \Omega^{-2} L(\Omega^2) L, \tag{A.2}$$

$$\nabla_{\underline{L}} \underline{L} = \Omega^{-2} \underline{L}(\Omega^2) \underline{L}, \tag{A.3}$$

$$g(\nabla_L \underline{L}, L) = 0, \tag{A.4}$$

$$g(\nabla_{\underline{L}} L, \underline{L}) = 0, \tag{A.5}$$

$$g(\nabla_{\underline{L}} L, \partial_A) = -g(L, \nabla_A \underline{L}), \tag{A.6}$$

$$g(\nabla_L \underline{L}, \partial_A) = -g(\underline{L}, \nabla_A L), \tag{A.7}$$

$$[L, \underline{L}]_A = 2g(\underline{L}, \nabla_A L) + 2\partial_A \Omega^2, \tag{A.8}$$

$$= -2g(L, \nabla_A \underline{L}) - 2\partial_A \Omega^2. \tag{A.9}$$

Proof. We have that

$$[L, \underline{L}] = -\partial_u b^C \partial_C,$$
$$[L, \partial_A] = -\partial_A b^C \partial_C,$$
$$[\underline{L}, \partial_A] = 0.$$

and

$$[L, \underline{L}] = \nabla_L \underline{L} - \nabla_{\underline{L}} L,$$
$$[L, \partial_A] = \nabla_L \partial_A - \nabla_A L,$$

so

$$g(\nabla_L L, L) = g(\nabla_{\underline{L}} \underline{L}, \underline{L}) = 0,$$
$$g(\nabla_L L, \partial_A) = -g(L, \nabla_L \partial_A) = -g(L, \nabla_A L) = 0,$$
$$g(\nabla_{\underline{L}} \underline{L}, \partial_A) = -g(\underline{L}, \nabla_{\underline{L}} \partial_A) = 0.$$

We obtain

$$g(\nabla_L L, \underline{L}) = L(g(L, \underline{L}) - g(L, \nabla_L \underline{L}) = -2L(\Omega^2),$$
$$g(\nabla_{\underline{L}} \underline{L}, L) = -2\underline{L}(\Omega^2).$$

Equations (A.2) and (A.3) now immediately follow.

Furthermore, by the above identities we can rewrite

$$[L, \underline{L}]_A = g(\nabla_L \underline{L} - \nabla_{\underline{L}} L, \partial_A) = g(\underline{L}, \nabla_L \partial_A) - g(L, \nabla_{\underline{L}} \partial_A)$$
$$= g(\underline{L}, \nabla_A L) - g(L, \nabla_A \underline{L}) = \partial_A(g(L, \underline{L})) - 2g(L, \nabla_A \underline{L})$$
$$= -2g(L, \nabla_A \underline{L}) - 2\partial_A \Omega^2.$$

From (A.2) and (A.3) it follows, moreover, that

$$g(\nabla_L \underline{L}, L) = \nabla_L(g(L, \underline{L}) - g(\underline{L}, \nabla_L L) = -2L(\Omega^2) + 2L(\Omega^2) = 0,$$
$$g(\nabla_{\underline{L}} L, \underline{L}) = 0.$$

Finally, we use that $g(\underline{L}, [L, \partial_A]) = g(L, [\underline{L}, \partial_A]) = 0$ to obtain,

$$g(\nabla_{\underline{L}} L, \partial_A) = -g(L, \nabla_{\underline{L}} \partial_A) = -g(L, \nabla_A \underline{L}),$$

$$g(\nabla_L \underline{L}, \partial_A) = -g(\underline{L}, \nabla_L \partial_A) = -g(\underline{L}, \nabla_A L).$$

□

We can write $L = \Omega^2 L'$, $\underline{L} = \Omega^2 \underline{L}'$, where L' and $\underline{L}'$ are geodesic vector fields, i.e.

$$\nabla_{L'} L' = 0,$$
$$\nabla_{\underline{L}'} \underline{L}' = 0,$$

which follows from (A.2) and (A.3).

We can define a renormalised ingoing null vector e_3 and outgoing null vector e_4, satisfying $g(e_3, e_4) = -2$ by

$$e_3 = \Omega^{-1} \partial_u,$$
$$e_4 = \Omega^{-1}(\partial_v + b^A \partial_A).$$

The inverse metric in the basis (e_3, e_4, ∂_A)

$$g^{-1} = -\frac{1}{2}(e_3 \otimes e_4 + e_4 \otimes e_3) + (\not{g}^{-1})^{AB} \partial_A \otimes \partial_B$$

can therefore be expressed in the double-null coordinate basis as

$$g^{-1} = -\frac{1}{2}\Omega^{-2}(u,v)(\partial_u \otimes \partial_v + \partial_v \otimes \partial_u)$$
$$- \frac{1}{2}\Omega^{-2} b^A(\partial_u \otimes \partial_A + \partial_A \otimes \partial_u) + (\not{g}^{-1})^{AB} \partial_A \otimes \partial_B.$$

With respect to the basis $(L, \underline{L}, \partial_{\vartheta_A})$ the inverse metric is given by

$$g^{-1} = -\frac{1}{2}\Omega^{-2}(u,v)(L \otimes \underline{L} + \underline{L} \otimes L) + (\not{g}^{-1})^{AB} \partial_A \otimes \partial_B.$$

Define the *second fundamental forms* χ_{AB} and $\underline{\chi}$ by

$$\chi_{AB} := g(\nabla_{\partial_A} e_4, \partial_B) = \Omega^{-1} g(\nabla_{\partial_A} L, \partial_B),$$
$$\underline{\chi}_{AB} := g(\nabla_{\partial_A} e_3, \partial_B) = \Omega^{-1} g(\nabla_{\partial_A} \underline{L}, \partial_B).$$

Lemma A.2. *We can express,*

$$L(\not{g}_{AB}) = 2\Omega \chi_{AB} - \partial_A b^C \not{g}_{CB} - \partial_B b^C \not{g}_{CA}, \qquad (A.10)$$
$$\underline{L}(\not{g}_{AB}) = 2\Omega \underline{\chi}_{AB}. \qquad (A.11)$$

Moreover,

$$\frac{L(\sqrt{\det \not{g}})}{\sqrt{\det \not{g}}} = \Omega \operatorname{tr}\chi - \partial_C b^C, \qquad (A.12)$$

$$\frac{\underline{L}(\sqrt{\det \not{g}})}{\sqrt{\det \not{g}}} = \Omega \operatorname{tr}\underline{\chi}. \qquad (A.13)$$

Proof. We use the expression for $[L, \partial_A]$ in Lemma A.1 to obtain

$$
\begin{aligned}
L(\not g_{AB}) &= g(\nabla_L \partial_A, \partial_B) + g(\partial_A, \nabla_L \partial_B) \\
&= g(\nabla_A L, \partial_B) + g(\partial_A, \nabla_B L) - -\partial_A b^C \not g_{CB} - \partial_B b^C \not g_{CA} \\
&= 2\Omega \chi_{AB} - \partial_A b^C \not g_{CB} - \partial_B b^C \not g_{CA},
\end{aligned}
$$

where we used that $\chi_{AB} = \chi_{BA}$, which can easily be shown. Equation (A.11) can be proved similarly.

We can apply the chain rule to obtain

$$
L(\det \not g) = \frac{\partial \det \not g}{\partial \not g_{AB}} L(\not g_{AB}).
$$

By Laplace's formula for the determinant of a matrix, we can express

$$
(\det \not g)\delta_C^B = \not g_{AB}\mathrm{Adj}(\not g)^{BC},
$$

where $\mathrm{Adj}(\not g)^{BC}$ are the components of the adjugate matrix of $\not g_{AB}$. Consequently,

$$
\frac{\partial \det \not g}{\partial \not g_{AB}} = \mathrm{Adj}(\not g)^{AB} = \det \not g \not g^{AB},
$$

so

$$
L(\det \not g) = (\det \not g)\not g^{AB} L(\not g_{AB}).
$$

We can therefore conclude that

$$
\frac{L(\sqrt{\det \not g})}{\sqrt{\det \not g}} = \frac{1}{2}\frac{L(\det \not g)}{\det \not g} = \Omega \operatorname{tr}\chi - \partial_C b^C.
$$

Equation (A.13) can be proved similarly. $\qquad\square$

We introduce additional metric derivatives,

$$
\omega := -\frac{1}{4}g(\nabla_{e_4} e_3, e_4),
$$

$$
\underline{\omega} := -\frac{1}{4}g(\nabla_{e_3} e_4, e_3),
$$

$$
\zeta_A := \frac{1}{2}g(\nabla_A e_4, e_3).
$$

ζ_A are the components of the *torsion tensor*.

Lemma A.3. *We can express ω, $\underline{\omega}$ and ζ_A as follows:*

$$
4\Omega\omega = \Omega^{-2}L(\Omega^2),
$$

$$
4\Omega\underline{\omega} = \Omega^{-2}\underline{L}(\Omega^2),
$$

$$
\zeta_A = \frac{1}{4}\Omega^{-2}[L, \underline{L}]_A = -\frac{1}{4}\Omega^{-2}\not g_{AB}\partial_u b^B.
$$

Proof. We have that

$$4\omega = -\Omega^{-2}g(\nabla_L(\Omega^{-1}\underline{L}), L) = -\Omega^{-2}L(\Omega^{-1})g(L, \underline{L}) - \Omega^{-3}g(\nabla_L\underline{L}, L)$$

$$= \frac{1}{2}\Omega^{-5}L(\Omega^2)g(L, \underline{L}) - \Omega^{-3}g(\underline{L}, \nabla_L L)$$

$$= \Omega^{-3}L(\Omega^2),$$

where we used (A.2) in the last equality. The expression for $\underline{\omega}$ follows similarly.

Moreover, by (A.8),

$$\zeta_A = \frac{1}{2}\Omega^{-1}g(\nabla_A(\Omega^{-1}L), \underline{L})$$

$$= \frac{1}{2}\Omega^{-2}g(\nabla_A L, \underline{L}) - \frac{1}{4}\Omega^{-4}\partial_A\Omega^2 g(L, \underline{L})$$

$$= \frac{1}{4}\Omega^{-2}[L, \underline{L}]_A$$

$$= -\frac{1}{4}\Omega^{-2}\slashed{g}_{AB}\partial_u b^B. \qquad \square$$

Consider the weighted vector field

$$N = N^L L + N^{\underline{L}}\underline{L} = N^u\partial_u + N^v\partial_v + N^A\partial_A,$$

where $N^L = N^L(u, v) = N^v(u, v)$, $N^{\underline{L}} = N^{\underline{L}}(u, v) = N^u(u, v)$ and $N^A = b^A N^L = b^A N^v$. The corresponding compatible current K^N is given by

$$K^N[\phi] := T_{\alpha\beta}[\phi]\,^N\pi^{\alpha\beta},$$

with the components of the deformation tensor $^N\pi_{\alpha\beta} = \frac{1}{2}[\nabla_\alpha N_\beta + \nabla_\beta N_\alpha]$ given by

$$^N\pi^{LL} = g^{L\underline{L}}g^{L\underline{L}}\,^N\pi_{\underline{L}\underline{L}} = \frac{1}{4}\Omega^{-4}g(\nabla_{\underline{L}}N, \underline{L}),$$

$$^N\pi^{\underline{L}\underline{L}} = g^{L\underline{L}}g^{L\underline{L}}\,^N\pi_{LL} = \frac{1}{4}\Omega^{-4}g(\nabla_L N, L),$$

$$^N\pi^{LA} = g^{L\underline{L}}\slashed{g}^{AB}\,^N\pi_{\underline{L}B},$$

$$^N\pi^{\underline{L}A} = g^{L\underline{L}}\slashed{g}^{AB}\,^N\pi_{LB},$$

$$^N\pi^{AB} = \frac{1}{2}(\slashed{g}^{AC}\partial_C N^B + \slashed{g}^{BC}\partial_C N^A$$

$$+ \slashed{g}^{AC}\slashed{g}^{BD}(N^u\partial_u + N^v\partial_v + N^E\partial_E)\slashed{g}_{CD}).$$

We use Eqs. (A.2)–(A.9) to obtain

$$^N\pi^{LL} = \frac{1}{4}\Omega^{-4}g(\nabla_{\underline{L}}N, \underline{L}) = -\frac{1}{2}\Omega^{-2}\underline{L}(N^L),$$

$$^N\pi^{\underline{L}\underline{L}} = \frac{1}{4}\Omega^{-4}g(\nabla_L N, L) = -\frac{1}{2}\Omega^{-2}L(N^{\underline{L}}),$$

$$^N\pi^{L\underline{L}} = \frac{1}{8}\Omega^{-4}\left[g(\nabla_L N, \underline{L}) + g(\nabla_{\underline{L}}N, L)\right]$$

$$= -\frac{1}{4}\Omega^{-2}\left[L(N^L) + \underline{L}(N^{\underline{L}}) + \Omega^{-2}N^L L(\Omega^2) + \Omega^{-2}N^{\underline{L}}\underline{L}(\Omega^2)\right],$$

$$^N\pi^{LA} = -\frac{1}{4}\Omega^{-2}\slashed{g}^{AB}\left[g(\nabla_B N, \underline{L}) + g(\nabla_{\underline{L}} N, \partial_B)\right],$$

$$= -\frac{1}{4}N^L\Omega^{-2}\slashed{g}^{AB}[g(\nabla_B L, \underline{L}) - g(L, \nabla_B \underline{L})] = -\frac{1}{4}N^L\Omega^{-2}[L, \underline{L}]^A$$

$$= -\frac{1}{4}N^L\Omega^{-2}\partial_u b^A,$$

$$^N\pi^{\underline{L}A} = -\frac{1}{4}N^{\underline{L}}\Omega^{-2}[\underline{L}, L]^A,$$

$$= \frac{1}{4}N^{\underline{L}}\Omega^{-2}\partial_u b^A,$$

$$^N\pi^{AB} = \frac{1}{2}\slashed{g}^{AC}\slashed{g}^{BD}(N^u\partial_u + N^v\partial_v)\slashed{g}_{CD} + \frac{1}{2}\slashed{g}^{AC}\slashed{g}^{BD}N^L b^E \partial_E \slashed{g}_{CD}$$

$$+ \frac{1}{2}\left(\slashed{g}^{AC}\partial_C N^B + \slashed{g}^{BC}\partial_C N^A\right)$$

$$= \frac{1}{2}\slashed{g}^{AC}\slashed{g}^{BD}(N^{\underline{L}}\underline{L} + N^L L)\slashed{g}_{CD} + \frac{1}{2}\left(\slashed{g}^{AC}\partial_C N^B + \slashed{g}^{BC}\partial_C N^A\right).$$

Now consider the wave Eq. (1.1) on an extremal Kerr–Newman background in a double-null foliation, with a corresponding energy momentum tensor

$$\mathbf{T}_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(g^{\lambda\kappa}\partial_\lambda\phi\partial_\kappa\phi).$$

We have that

$$g^{\lambda\kappa}\partial_\lambda\phi\partial_\kappa\phi = g^{L\underline{L}}L\phi\underline{L}\phi + \slashed{g}^{AB}\partial_A\phi\partial_B\phi$$

$$= -\frac{1}{2}\Omega^{-2}L\phi\underline{L}\phi + |\slashed{\nabla}\phi|^2.$$

Now we obtain the components

$$\mathbf{T}(L, L) = (L\phi)^2,$$

$$\mathbf{T}(\underline{L}, \underline{L}) = (\underline{L}\phi)^2$$

$$\mathbf{T}(L, \underline{L}) = \Omega^2|\slashed{\nabla}\phi|^2,$$

$$\mathbf{T}(L, \partial_A) = L\phi\partial_A\phi,$$

$$\mathbf{T}(\underline{L}, \partial_A) = \underline{L}\phi\partial_A\phi,$$

$$\mathbf{T}(\partial_A, \partial_B) = (\partial_A\phi)(\partial_B\phi) + \frac{1}{2}\slashed{g}_{AB}(\Omega^{-2}L\phi\underline{L}\phi - |\slashed{\nabla}\phi|^2).$$

Therefore, by using the expressions in Lemmas A.2 and A.3, we obtain

$$K^N = {}^N\pi^{LL}\mathbf{T}(L, L) + {}^N\pi^{\underline{L}\underline{L}}\mathbf{T}(\underline{L}, \underline{L}) + 2\,{}^N\pi^{L\underline{L}}\mathbf{T}(L, \underline{L}) + {}^N\pi^{AB}\mathbf{T}(\partial_A, \partial_B)$$

$$+ 2\,{}^N\pi^{LA}\mathbf{T}(L, \partial_A) + 2\,{}^N\pi^{\underline{L}A}\mathbf{T}(\underline{L}, \partial_A)$$

$$= -\frac{1}{2}\Omega^{-2}\underline{L}(N^L)(L\phi)^2 - \frac{1}{2}\Omega^{-2}L(N^{\underline{L}})(\underline{L}\phi)^2$$

$$+ \frac{1}{4}\Omega^{-2}[\slashed{g}^{AB}N^L L(\slashed{g}_{AB}) + N^{\underline{L}}\underline{L}(\slashed{g}_{AB}) + 2N^L\partial_E b^E]L\phi\underline{L}\phi$$

$$- \frac{1}{2}\left[\underline{L}(N^{\underline{L}}) + L(N^L) + \Omega^{-2}(N^{\underline{L}}\underline{L}(\Omega^2) + N^L L(\Omega^2))\right]|\slashed{\nabla}\phi|^2$$

$$+ \left[\frac{1}{2} \rlap{/}{g}^{AC} \rlap{/}{g}^{BD} (N^L L - N^{\underline{L}} \underline{L}) \rlap{/}{g}_{CD} + N^L \rlap{/}{g}^{AC} \partial_C b^B \right] (\partial_A \phi)(\partial_B \phi)$$

$$- \frac{1}{4} \left[\rlap{/}{g}^{BD} (N^L L - N^{\underline{L}} \underline{L}) \rlap{/}{g}_{BD} + 2N^L \partial_E b^E \right] |\rlap{/}{\nabla}\phi|^2$$

$$- \frac{1}{2} N^L \Omega^{-2} \partial_u b^A L\phi \partial_A \phi$$

$$+ \frac{1}{2} N^{\underline{L}} \Omega^{-2} \partial_u b^A \underline{L}\phi \partial_A \phi$$

$$= -\frac{1}{2} \Omega^{-2} \underline{L}(N^L)(L\phi)^2 - \frac{1}{2} \Omega^{-2} L(N^{\underline{L}})(\underline{L}\phi)^2$$

$$+ \frac{1}{2} \Omega^{-2} (N^L \Omega \operatorname{tr} \chi + N^{\underline{L}} \Omega \operatorname{tr} \underline{\chi}) L\phi \underline{L}\phi$$

$$- \frac{1}{2} \left[\underline{L}(N^{\underline{L}}) + L(N^L) + 4\Omega(N^L \omega + N^{\underline{L}} \underline{\omega}) \right] |\rlap{/}{\nabla}\phi|^2$$

$$+ \left[N^L \Omega \hat{\chi}^{AB} + N^{\underline{L}} \Omega \hat{\underline{\chi}}^{AB} \right] (\partial_A \phi)(\partial_B \phi)$$

$$+ 2[N^L(L\phi) - N^{\underline{L}}(\underline{L}\phi)]\zeta^A \partial_A \phi,$$

where we used the notation

$$\hat{\chi}_{AB} := \chi_{AB} - \frac{1}{2} \rlap{/}{g}_{AB} \operatorname{tr} \chi,$$

$$\hat{\underline{\chi}}_{AB} := \underline{\chi}_{AB} - \frac{1}{2} \rlap{/}{g}_{AB} \operatorname{tr} \underline{\chi}.$$

Appendix B. The Wave Equation in Double-Null Coordinates

Consider the extremal Kerr–Newman metric in Eddington–Finkelstein-type double-null coordinates $(u, v, \theta_*, \varphi_*)$. Then, the wave operator becomes

$$\Box_g \phi = \frac{1}{2\sqrt{\det \rlap{/}{g}}} \Omega^{-2} \partial_\alpha \left(g^{\alpha\beta} 2\Omega^2 \sqrt{\det \rlap{/}{g}} \partial_\beta \phi \right)$$

$$= \frac{1}{2\sqrt{\det \rlap{/}{g}}} \Omega^{-2} \left[-\partial_v \left(\sqrt{\det \rlap{/}{g}} \partial_u \phi \right) - \partial_u \left(\sqrt{\det \rlap{/}{g}} (\partial_v \phi + b^A \partial_A \phi) \right) \right.$$

$$\left. - \partial_A \left(\sqrt{\det \rlap{/}{g}} b^A \partial_u \phi \right) + \partial_A \left(\sqrt{\det \rlap{/}{g}} 2\Omega^2 \rlap{/}{g}^{AB} \partial_B \phi \right) \right]$$

$$= \frac{1}{2\sqrt{\det \rlap{/}{g}}} \Omega^{-2} \left[-L \left(\sqrt{\det \rlap{/}{g}} \underline{L}\phi \right) - \sqrt{\det \rlap{/}{g}} \partial_A b^A \underline{L}\phi - \underline{L} \left(\sqrt{\det \rlap{/}{g}} L\phi \right) \right]$$

$$+ \Omega^{-2} \rlap{/}{g}^{AB} \partial_A \Omega^2 \partial_B \phi + \rlap{/}{\Delta}\phi$$

$$= -\frac{1}{2} \Omega^{-2} \left[L\underline{L}\phi + \underline{L}L\phi + \Omega \operatorname{tr} \chi \underline{L}\phi + \Omega \operatorname{tr} \underline{\chi} L\phi \right] + \Omega^{-2} \rlap{/}{\nabla}\Omega^2 \cdot \rlap{/}{\nabla}\phi + \rlap{/}{\Delta}\phi.$$

$$\tag{B.1}$$

We can write

$$2\Omega^2 \Box_g \phi = -\underline{L}L\phi - L\underline{L}\phi - \Omega\operatorname{tr}\underline{\chi}L\phi - \Omega\operatorname{tr}\chi\underline{L}\phi + 2\slashed{\nabla}\Omega^2 \cdot \slashed{\nabla}\phi + 2\Omega^2\slashed{\Delta}\phi. \tag{B.2}$$

Moreover, we use that $[L, \underline{L}] = L(b^A)\partial_A = 4\Omega^2\zeta^A$ and $\Omega\operatorname{tr}\underline{\chi} = (\det\slashed{g})^{-\frac{1}{2}}\underline{L}((\det\slashed{g})^{\frac{1}{2}})$, to write

$$2\Omega^2\Box_g\phi = -2\underline{L}L\phi - 4\Omega^2\zeta(\phi) - \frac{\underline{L}(\sqrt{\det\slashed{g}})}{\sqrt{\det\slashed{g}}}L\phi - \Omega\operatorname{tr}\chi\underline{L}\phi + 2\slashed{\nabla}\Omega^2 \cdot \slashed{\nabla}\phi + 2\Omega^2\slashed{\Delta}\phi.$$

The wave equation in the above form can easily be rewritten as a transport equation for $(\det\slashed{g})^{\frac{1}{4}}L\phi$:

$$2\underline{L}((\det\slashed{g})^{\frac{1}{4}}L\phi)) = (\det\slashed{g})^{\frac{1}{4}}\Omega\operatorname{tr}\chi\underline{L}\phi + 2(\det\slashed{g})^{\frac{1}{4}}\slashed{\nabla}\Omega^2 \cdot \slashed{\nabla}\phi$$
$$-4(\det\slashed{g})^{\frac{1}{4}}\Omega^2\zeta(\phi) + 2(\det\slashed{g})^{\frac{1}{4}}\Omega^2\slashed{\Delta}\phi. \tag{B.3}$$

Remark B.1. Note that expressions (B.2) and (B.3) are not restricted to extremal Kerr–Newman spacetimes; they in fact hold for any background metric in double-null coordinates of the form (A.1).

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