Article:

Shi, H., Worden, K. orcid.org/0000-0002-1035-238X and Cross, E.J. (2018) A regime-switching cointegration approach for removing environmental and operational variations in structural health monitoring. Mechanical Systems and Signal Processing, 103. pp. 381-397. ISSN 0888-3270

https://doi.org/10.1016/j.ymssp.2017.10.013

A Regime-Switching Cointegration Approach for Removing Environmental and Operational Variations in Structural Health Monitoring

Haichen Shi, Keith Worden, Elizabeth J. Cross

Dynamics Research Group, Department of Mechanical Engineering, University of Sheffield[a]

[a]*Mappin Street, Sheffield, S1 3JD, United Kingdom*

Abstract

Cointegration is now extensively used to model the long term common trends among economic variables in the field of econometrics. Recently, cointegration has been successfully implemented in the context of structural health monitoring (SHM), where it has been used to remove the confounding influences of environmental and operational variations (EOVs) that can often mask the signature of structural damage. However, restrained by its linear nature, the conventional cointegration approach has limited power in modelling systems where measurands are nonlinearly related; this occurs, for example, in the benchmark study of the Z24 Bridge, where nonlinear relationships between natural frequencies were induced during a period of very cold temperatures. To allow the removal of EOVs from SHM data with nonlinear relationships like this, this paper extends the well-established cointegration method to a nonlinear context, which is to allow a breakpoint in the cointegrating vector. In a novel approach, the augmented Dickey-Fuller (ADF) statistic is used to find which position is most appropriate for inserting a breakpoint, the Johansen procedure is then utilised for the estimation of cointegrating vectors. The proposed approach is examined with a simulated case and real SHM data from the Z24 Bridge, demonstrating that the EOVs can be neatly eliminated.

*Corresponding email: haichen.shi@sheffield.ac.uk

Preprint submitted to Mechanical System and Signal Processing *August 31, 2017*

Keywords: structural health monitoring, environmental and operational variation, cointegration

1. Introduction

Due to the fact that many structural and mechanical systems are approaching or exceeding their original design life, structural health monitoring (SHM) has been continually developing over the past decades. However, one of the main obstacles that has kept SHM from practical implementation in industry is the effect of environmental and operational variations (EOVs). EOVs are variations induced by temperature, wind, humidity, traffic, etc. Because EOVs are influencing the systems constantly, any measurements of the system responses can be contaminated by EOVs, thus any potential damage information may be masked falsely [1]. For instance, when Farrar *et al.* considered different levels of damage on the I-40 Bridge in New Mexico, USA, it turned out that changes of the first modal frequency caused by damage were from 2% to 6% depending on different damage levels [2]; however, numerous investigations have suggested that variations induced by ambient temperature can range from 5% to 10% [3]. As such, it is crucial for *in situ* SHM to discriminate the changes in the features extracted from sensor readings which are caused by structural damage from those changes caused by benign EOVs, such a process is termed as *data normalisation.* Many approaches have been developed to address data normalisation issues, including regression modelling, machine learning approaches, projection methods and so on [1]. When the measurements of the EOVs are available, regression models are common methods employed to explicitly model the dependency between the EOVs and system response or damage-sensitive features [4]. Alternatively, principal component analysis (PCA) is a projection method to find EOV-insensitive features which are damage-sensitive at the same time. The fact that EOVs are accountable for the most of the variance in the sensor readings, meaning that, by projecting the features onto the space spanned by the eigenvectors associated with the smallest eigenvalues, one can obtain features that are not influenced by

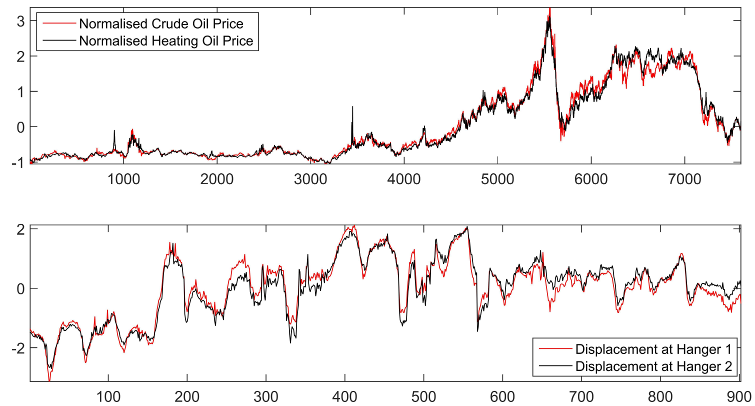

Figure 1: Upper panel: normalised price index of crude oil price and household heating oil price of US; Lower panel: normalised series of displacement measurements from the Tamar Bridge.

EOVs but are potentially still sensitive to damage [5]. For further information on these methods, readers are suggested to refer to [6] as a comparatively recent review.

Recently, *cointegration* has been adopted successfully to address the challenge of EOVs in structural health monitoring [7]. As a routine method for dealing with nonstationary time series in econometric studies, cointegration is now widely used in statistical arbitrage, macroeconomic analysis, and fiscal policy research. However, what is the link between an econometric method and EOVs in SHM? The answer is the existence of stochastic common trends. Consider Figure 1 for example; the upper panel shows two normalised price indices (heating oil and crude oil in the US) during a certain time period; the lower panel exhibits two time series of two hanger displacements of the Tamar Bridge measured during a certain time history [8]. By visual inspection of these two images, two common characteristics can be observed immediately: each pair of time series is nonstationary; each pair shares some long-term common trend. These characteristics are not hard to understand, that economic time series are simultaneously affected by markets, monetary policies etc., while displacement

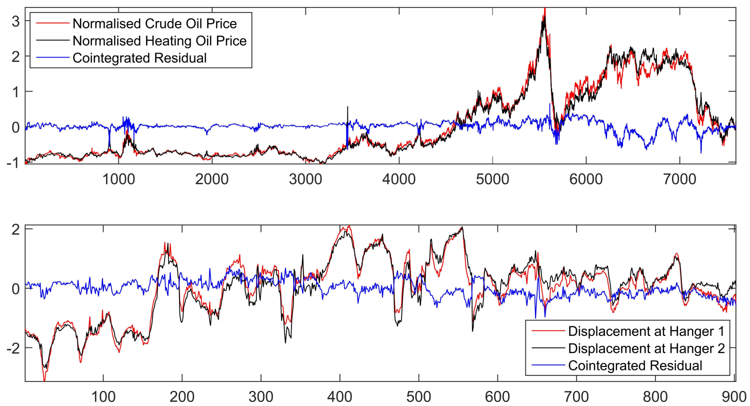

Figure 2: Cointegrated residual series plotted with normalised price index of crude oil price and household heating oil price of US.

of each bridge hanger is significantly influenced by temperature, traffic etc., or in the terminology of this paper - EOVs. Nonstationary series are said to be *cointegrated* if there exists a linear combination of them that is stationary. Denote the two time series in the upper panel of Figure 1 by x_t and y_t ; they can be found to be cointegrated if some linear combination of them:

$$\varepsilon_t = x_t + \alpha y_t \tag{1}$$

is stationary (confirmed by performing a stationarity hypothesis test). The residual series for the oil price series and the displacement series are plotted in Figure 2, which shows that the residual series are purged of common trends and become largely stationary. Once the underlying equilibrium between the displacement series is built, the stationary residual series can serve as a damage indicator that is immune to EOVs. It is worth noting at this point that the cointegrated residual series of the oil series seems to behave differently before and after approximately point 5000. This interesting phenomenon can be seen as a regime change in the market, which will be elaborated more in the latter part of this paper.

Based on the concept of cointegration, several applications on the issue of

EOVs in SHM have been attempted. Cross *et al.* [7] proposed the whole framework of the cointegration approach for SHM; the method was validated with a time-varying multi-degree-freedom lumped mass system, and also a real SHM problem, which was to perform damage detection of a composite plate subjected to cyclic temperature variations. An efficient maximum likelihood estimation method - the Johansen procedure (technical details of which will be covered shortly in Section 2) - was used to estimate the most stationary cointegrating relationship; environmental variability could be significantly suppressed in the stationary residual series and damage could be successfully detected. One major benefit of employing the cointegration method was that by building the inner relationship of the monitored variables, direct measurements of EOVs were not necessary. Furthermore, Cross *et al.*[5] compared the cointegration method with another two conventional methods: outlier analysis and principal component analysis (PCA), utilising the same experimental data from [7]. Although outlier analysis may produce an acceptably low number false-positive damage indications, some points that are furthest away from the spectrum peak were observed to be less sensitive to damage; PCA produced very similar results to cointegration analysis, but PCA may be restricted by the fact that principal components are decomposed by the rule of orthogonality. After the first principal component is determined, the minor component used for damage detection is, therefore, restricted by this orthogonality condition. The cointegrating relationship, however, determined by the maximum likelihood method - the Johansen procedure - is not restrained by other conditions, therefore the authors of [7] concluded that cointegration may outperform PCA. A similar study can be found in [9], where Dao and Staszewski applied the Johansen procedure and a stationarity test also in the context of Lamb-wave-based SHM. As Lamb waves can be easily corrupted by undesired temperature effects, instead of using the Lamb wave response directly, they used the cointegrated residual series and the stationarity test statistic as an indicator of damage presence and severity respectively. More recently, in an effort to enhance the damage sensitivity of the cointegration method, Worden *et al.* [10] explored the potential connection between multiresolution (discrete wavelet) analysis and cointegration. By using a discrete wavelet transform (DWT), the original nonstationary signal could be decomposed into several components at different levels. One interesting finding was that the "degree of nonstationarity" decreased as the level number increased. This provided the possibility of extracting the most nonstationary components from variables, and then constructing an enhanced cointegrating relationship. Despite the result that this method improved the damage sensitivity significantly, the decomposed components were mostly denoised. This may contradict a prerequisite of the Johansen procedure, therefore regularisation to the decomposed component by adding noise was still necessary. A more recent review paper presented by Worden *et al.* [11] reviewed some of the latest developments in algorithms based on nonstationary time series analysis. Statistical control chart methods, cointegration and Bayesian mixture of experts models were reviewed with examples; they were proved to be efficient in removing benign environmental changes and detecting anomalies. The authors also brought forward two open issues in cointegration analysis, which are heteroskedasticity and nonlinearity, and this paper will attempt to address the latter one.

Although cointegration has started to play an important role in modelling nonstationary SHM data, its linear nature may still obstruct its implementation in a more general sense. Many real world engineering systems have nonlinear responses to environmental and operational variability, which may further cause nonlinear relationships among monitored variables; here, the conventional cointegration method becomes much less effective, or inappropriate [12]. The current authors have carried out exploratory studies on nonlinear methods of cointegration [13]; however, the methods produced a heteroskedastic residual. Zolna et al., [14] attempted to remedy this issue via a scaling transformation which produced a residual stationary in the variance; however, this approach appeared to move the nonstationarity into the tails of the residual distribution, thus invalidating any thresholds set using standard statistical process control. The current authors also investigated a nonlinear cointegration approach in the

Engle-Granger framework, with Gaussian process (GP) regression performing as a cointegrating regression function [15]. The GP is a powerful nonparametric regression model that could provide robust estimation of the distributions of cointegrated residuals, rather than the point estimates from Ordinary Least Squares. Real engineering data from the Z24 Bridge was used to validate the model; it was shown that EOVs could be robustly eliminated with the GP, which still maintained sensitivity to damage. More recently, the authors of the current paper have presented an exploratory work to further extend the aforementioned work. A novel nonlinear cointegration method named *regime-switching cointegration* has been presented, which will allow cointegrating relationships to switch according to certain criteria [16]. This paper will be a much extended treatment of [16], considerably more technical detail will be presented and a real-world example will be used to validate the proposed approach and the effects of allowing multiple switching points will be considered.

This paper is organised as follows. First of all, the background theory of unit roots and cointegration is briefly reviewed, and then a synthetic example is presented; the *regime-switching cointegration* method is illustrated with this example. The proposed method is briefly summarised in Section 3. A real example from the benchmark study of the Z24 Bridge is used to examine the proposed method. Finally, discussions and conclusions are presented.

2. Background theory

2.1. Unit roots and unit root tests

In terms of nonstationarity, econometricians have developed various tools for testing for it. The unit root process is one of the most popular and well established modelling methods for nonstationary time series. Consider the first order autoregressive model of a time series x_t:

$$x_t = \alpha x_{t-1} + \xi_t, (t = 1, 2, ...N) \tag{2}$$

where ξ_t is a stationary process with zero mean and variance σ^2, α is a real number that determines the stationarity of x_t: if $|\alpha| < 1$, x_t is stationary; if

$|\alpha| > 1$, then x_t is nonstationary, and its variance grows explosively with time; if $|\alpha| = 1$, then the variance of x_t will be $t\sigma^2$, which will grow with time, thus the process is nonstationary. Such a data generating process is termed a *unit root process*, or integrated of order 1, $I(1)$. Generally, a process is integrated of order d, $I(d)$, if it becomes stationary after differencing d times. In the context of SHM, it is not common to observe time series integrated of 2 or more, a detailed discussion on this can be found in [17]. Therefore, nonstationary series are treated as $I(1)$ processes in this paper, if not explicitly stated otherwise.

Unit root tests are still an ongoing popular research topic in econometrics, good surveys can be found in [18] and [19]. The unit root test adopted in this paper is perhaps the most commonly used, the Dickey-Fuller(DF) test. To illustrate how the DF test works, the argument will start from the simplest form of unit root as expresssed in equation (2), this model can be reformulated as:

$$\Delta x_t = (\alpha - 1)x_{t-1} + \xi_t = \pi x_{t-1} + \xi_t, (t = 1, 2, ...N) \tag{3}$$

where Δ is a differencing operator such that $\Delta x_t = x_t - x_{t-1}$, and $\pi = \alpha - 1$. Based on this form, testing the null hypothesis $H_0 : \pi = 0$ is equivalent to testing the hypothesis $\alpha = 1$; the alternative hypothesis H_1 is $\pi < 0$. The test for the null is simply a t test:

$$\hat{\tau} = \frac{\hat{\pi}}{se(\hat{\pi})} \tag{4}$$

where $\hat{\pi}$ is the least-squares estimate of π, and $se(\hat{\pi})$ is the standard error of $\hat{\pi}$. However, Dickey and Fuller investigated that under the null, the least-squares estimation $\hat{\pi}$ is not consistent with the true value, thus the usual t test would be inappropriate for testing the null. They further investigated the asymptotic distribution of the $t-$statistic, and gave corrected tables based on Monte Carlo simulations [20][21].

One may notice that in (2) and (3), the disturbance term ξ_t is a zero-mean stationary series, which is still a strong assumption for many cases. To allow potential serial correlation in the disturbance term, the augmented Dickey-Fuller

(ADF) test was developed, which is based on the following form:

$$\Delta x_t = \pi x_{t-1} + \sum_{j=1}^{m} \gamma_j \Delta x_{t-j} + \varepsilon_t \tag{5}$$

where the γ_j are the coefficients of the autoregressive terms, m is the lag number. Detailed deviations from equation (3) to (5) can be found in Chapter 6 of [12]. In this regression, a sufficient number of lags should be included to achieve a white noise residual term ε_t; an information criteria is a common choice for determining the lag number. Clearly, the ADF test has greater robustness and flexibility than the DF test, thus it is most widely used in unit root testing [18].

Similarly to the previous, the null of the ADF test is $H_0 : \pi = 0$ with alternative: $H_1 : \pi < 0$. The $t-$statistic is the same form as in (4), critical values are given in [21]. The null hypothesis is rejected if $\hat{\tau}$ is smaller than the corresponding critical value, and accepted otherwise. The ADF test can also be easily extended by adding shift terms and/or trend terms:

$$\Delta x_t = \mu + \nu t + \pi x_{t-1} + \sum_{j=1}^{m} \gamma_j \Delta x_{t-j} + \varepsilon_t \tag{6}$$

where μ and νt are the shift and trend terms respectively. Details of the model (6) and further extensions can be found in [18].

Having reviewed the fundamentals of unit root processes and their statistical tests, one can now ascertain the nonstationarity of a series through these procedures. It is not difficult to find that the test statistic is the key ingredient in the unit root test, therefore in this paper, the power of the test statistic will be explored, and attempts to measure the degree of stationarity with it, and determine the best possible model form, will be carried out.

2.2. Cointegration and Johansen procedure

As reviewed in Section 1, cointegration is a powerful tool to understand nonstationary data. As previously stated, two or more nonstationary series are cointegrated if a linear combination of them can be found to be stationary. The example in (1) is the simplest case, bivariate cointegration: x_t and y_t are cointegrated, and $(1, \alpha)'$ is called a cointegrating vector. However, cointegration can be extended to the multivariate context by using vector notation. Let $\mathbf{x_t} = (x_{1t}, x_{2t}, ...x_{mt})$ denote an $m-$variate time series and suppose there exists a vector $\boldsymbol{\beta}$ that makes

$$u_t = \boldsymbol{\beta} \cdot \mathbf{x}_t \tag{7}$$

a univariate stationary time series. Here, the vector $\boldsymbol{\beta} = (\beta_1, \beta_2, ...\beta_m)'$ is referred to as a *cointegrating vector*. Usually there are more than one possible cointegrating relationships for a multivariate series $\mathbf{x}_t$, and many methods to estimate the cointegrating vectors are available in the literature. The Engle-Granger two-stage method for example [20], is a simple regression method, which is to check if single-equation estimates of equilibrium error achieve stationarity. The main drawback of the Engle-Granger two-stage method, however, is that it only allows estimation of one cointegrating relationship every time, and the specification of the regression form is somewhat arbitrary. Hence alternatively, the Johansen procedure, an efficient maximum likelihood (ML) estimator, is adopted in dealing with engineering data; successful applications can be found in [7], [9] and [10].

The Johansen procedure offers an efficient framework that not only estimates multiple cointegrating vectors at the same time, but also produces a test statistic for determining the number of cointegrating vectors. In the SHM context, it is more of interest to estimate the cointegrating vectors than to perform tests on the number of cointegrating vectors, because it is the most stationary combination that one looks for to eliminate the EOV-induced nonstationary components in the data. For the sake of simplicity, this paper will only give details of the estimation part of the procedure, readers who are interested in the cointegration statistical test can find reviews and details from [22].

To perform the Johansen procedure, one starts from a vector autoregressive (VAR) model, which has the form:

$$\mathbf{X}_t = \Phi_1 \mathbf{X}_{t-1} + \Phi_2 \mathbf{X}_{t-2} + ... + \Phi_p \mathbf{X}_{t-p} + \mathbf{u}_t = \sum_{j=1}^{p} \Phi_j \mathbf{X}_{t-j} + \mathbf{u}_t \tag{8}$$

where $\mathbf{X}_t$ is a $m-$dimensional vector time series, Φ_j is a $m \times m$ coefficient matrix, $\mathbf{u_t}$ is a $m-$dimensional vector Gaussian noise series, and the autoregressive order p can be determined via information criteria. From a VAR(p), there will always exist a corresponding vector error correction (VEC) model (by substituting $\mathbf{X}_t = \mathbf{X}_{t-1} + \Delta\mathbf{X}_t, \mathbf{X}_{t-1} = \mathbf{X}_{t-2} + \Delta\mathbf{X}_{t-1}, ..., \mathbf{X}_{t-p} = \mathbf{X}_{t-p-1} + \Delta\mathbf{X}_{t-p}$ into (8) and a few rearrangements), which has the following expression:

$$\Delta\mathbf{X}_t = \Pi\mathbf{X}_{t-1} + \sum_{j=1}^{p-1} \Psi_j \Delta\mathbf{X}_{t-j} + \mathbf{u}_t = AB^T\mathbf{X}_{t-1} + \sum_{j=1}^{p-1} \Psi_j \Delta\mathbf{X}_{t-j} + \mathbf{u}_t \quad (9)$$

where $\Pi = -(I - \Phi_1 - ... - \Phi_p)$, $\Psi_j = -(\Phi_{j+1} + ... \Phi_p)$, and $\mathbf{u_t}$ is a $m-$dimensional vector Gaussian noise series, $\mathbf{u}_t \sim N(\mathbf{0}, \Omega)$. A and B are two $m \times r$ matrices, where r is the rank of the matrix Π. Matrix B is the cointegration vector matrix to be found, consisting of r cointegrating vectors. Matrix A is the adjustment matrix. Expression (9) is also referred to as the *Granger representation theorem*, which explicitly depicts the dynamics between the long run equilibrium (cointegration) and short term adjustments. Because $\Delta\mathbf{X}_t$, $\Delta\mathbf{X}_{t-j}$ and $\mathbf{u}_t$ are stationary, in order to make both sides of (9) equivalent, $\Pi\mathbf{X}_{t-1}$ has to be stationary as well. When the matrix Π is full rank, $r = m$, $\mathbf{X}_t$ will be a stationary vector series, which violates the preassumption of nonstationarity; When Π is zero rank, then $\Pi = 0$, which means $\mathbf{X}_t$ has no cointegration. Consequently, matrix Π has to be rank deficient such that $0 < r < m$ [18].

To find B, Johansen proposed a maximum likelihood method. One can first break the VEC regression in (9) into the following three smaller regressions:

$$\Delta\mathbf{X}_t = \sum_{j=1}^{p-1} C_j \Delta\mathbf{X}_{t-j} + \mathbf{r}_{0t} \quad (10)$$

$$\mathbf{X}_{t-1} = \sum_{j=1}^{p-1} D_j \Delta\mathbf{X}_{t-j} + \mathbf{r}_{1t} \quad (11)$$

$$\mathbf{r}_{0t} = \Pi\mathbf{r}_{1t} + \mathbf{u}_t = AB^T\mathbf{r}_{1t} + \mathbf{u}_t \quad (12)$$

where A and B are equivalent to those in (9). Based on the regression in (12) and the assumption that $\mathbf{u}_t$ is *iid* Gaussian noise $\mathbf{u}_t \sim N(\mathbf{0}, \Omega)$, one can have the logarithm likelihood function:

$$lnL(A,B,\Omega \mid \mathbf{X}_t) = -\frac{mN}{2}ln(2\pi) - \frac{N}{2}ln\left|\Omega\right| - \frac{1}{2}\sum_{t=1}^{N}\mathbf{u}_t^T\Omega^{-1}\mathbf{u}_t \tag{13}$$

$$= -\frac{mN}{2}ln(2\pi) - \frac{N}{2}ln\left|\Omega\right| - \frac{1}{2}\sum_{t=1}^{N}(\mathbf{r}_{0t} - AB^T\mathbf{r}_{1t})^T\Omega^{-1}(\mathbf{r}_{0t} - AB^T\mathbf{r}_{1t}) \tag{14}$$

where N is the sample size.

The next step is to find the parameters that maximise the log likelihood function (14) and to estimate the residuals $\mathbf{r}_{0t}$ and $\mathbf{r}_{1t}$. However, the details of derivation are omitted here, one can find the theory and proofs behind it in [22].

Finally, the optimisation problem turns into solving the following characteristic equation:

$$\left|\lambda_i S_{11} - S_{11}S_{10}^{-1}S_{01}\right| = 0 \tag{15}$$

where $S_{hk} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{r}_{ht}\cdot\mathbf{r}_{kt}, (h,k = 0,1)$. Assuming that $(\lambda_1, \lambda_2, ...\lambda_r)$ are the r eigenvalues of equation (15), and they are arranged in the order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$, then the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ can form the estimate of the cointegrating vector matrix as follows:

$$\hat{B} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r) \tag{16}$$

As the first cointegrating vector $\mathbf{v}_1$ corresponds to the largest eigenvalue λ_1, so it is natural to select $\mathbf{v}_1$ as the "most stationary" cointegrating vector, so as to make the stationary residual series.

So far, the background theory of cointegration has been reviewed. Although it is far from thorough, it is sufficient for the method that will be proposed shortly, which will largely be built around the theories above. As they are already relatively mature methods in the field of econometrics, the implementation of the unit root test and the Johansen procedure is fully integrated in various software platforms, such as Matlab (Econometric Toolbox), R and Eviews.

3. Illustration of the regime-switching cointegration method with a simulated example

As discussed in the introduction, the conventional cointegration method may sometimes fall short because of the involvement of nonlinearity. To illustrate the issue more specifically, a simple system is simulated, and the proposed new method will be illustrated with it.

Consider a four degree of freedom (DOF) spring-mass system, where four lumped masses ($2kg$ each) are in a chain with both ends connected to ground, as shown in Figure 3. To mimic the effect of EOVs, temperature particularly in this case, a changing thermal field is applied to the system. 10000 real temperature measurement data from the SHM campaign of the Tamar Bridge are used as the thermal field. The temperature data ranges approximately from -10°C to 20°C, which is fully displayed in the lower panel of Figure 4, representing readings from about one year [8]. To introduce artificial nonlinearity to the system, the springs in the system are all set to have nonlinear influences from temperature, the third spring is set to have a slightly different effect from temperature. The explicit expressions of their stiffness versus temperature T are given as follows:

$$k_1 = k_2 = k_4 = k_5 = \begin{cases} -0.15 \times T + 4, & \text{if } T < 0 \\ -0.05 \times T + 4, & \text{if } T \geq 0 \end{cases} \tag{17}$$

$$k_3 = \begin{cases} -0.15 \times T + 5, & \text{if } T < 0 \\ -0.25 \times T + 5, & \text{if } T \geq 0 \end{cases} \tag{18}$$

Because of the different behaviour of k_3, the nonlinear effect is introduced into the vibration modes in which the third spring is participating - the second and the fourth mode to be specific. Damage is simulated by letting the stiffness of the second spring k_2 decrease by 20%, at datapoint 5000 in the simulation. The four natural frequencies of the system are obtained at each time instant by solving the equations of motion. Additionally, a small amount of Gaussian white noise $N(0, 0.02)$ is added, to simulate measurement errors.

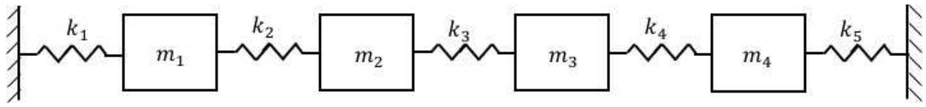

Figure 3: A four-DOF spring mass system.

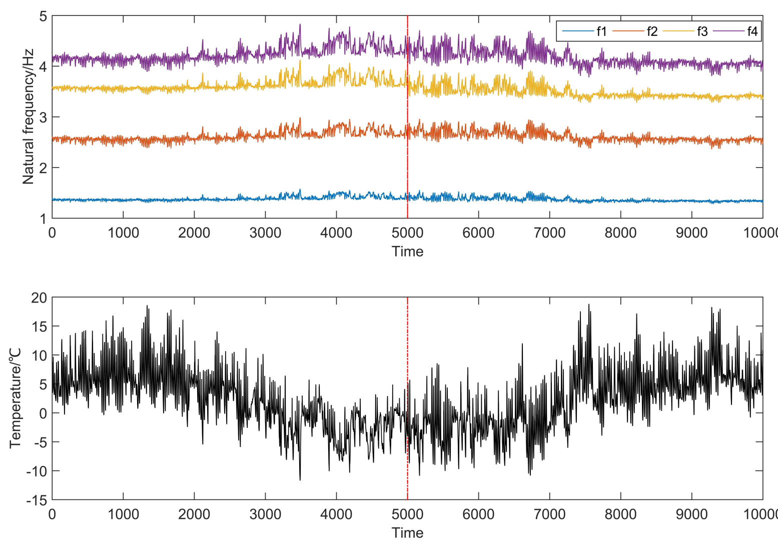

Figure 4: Upper panel: the four natural frequency series of the system in Figure 3 plotted as a function of time; Lower panel: temperature series plotted against time. Red dashed line indicates damage introduction.

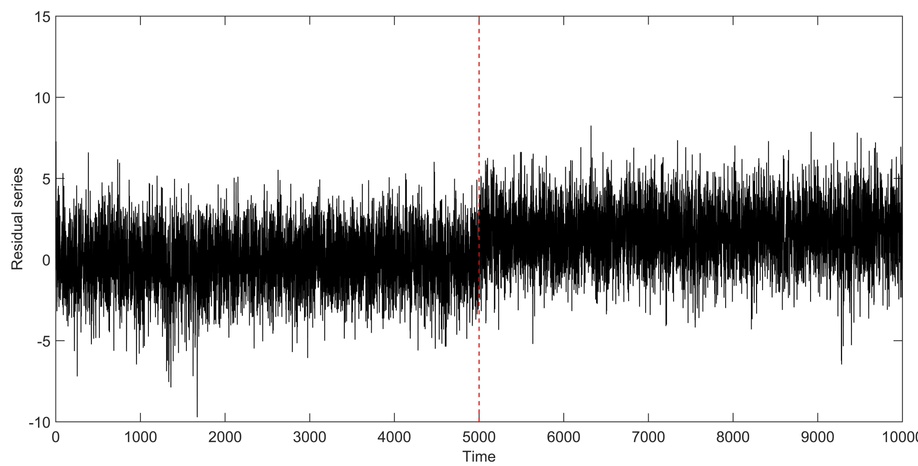

Figure 5: Residual series obtained using the conventional cointegration method, the red dashed line indicates where damage occurs.

The upper panel of Figure 4 shows the identified four natural frequency series, plotted as a function of time. The dashed vertical line indicates where damage is introduced. It is clear from Figure 4 that the effect of temperature is significant. When the damage level is not high enough, damage information may be overwhelmingly masked by the changes caused by temperature. Following the conventional cointegration procedures proposed in [7], the data points ranging from point number 2000 to 4000 are used for establishing the cointegrating vector. From this data one can obtain the residual series as shown in Figure 5; here one can clearly see that the residual series is not very sensitive to the damage occurrence - the underlying cointegration relationship has not been accurately modelled and any conclusion drawn from this may therefore be misleading. Because of the nonlinear effect of the third spring, the mutual correlation between the four natural frequencies will shift from one regime to another, as soon as the temperature crosses the zero degree point. This regime switching may be evidently observed from Figure 6, which shows that the mutual correlations among the four natural frequencies have a distinct bilinear relationship, the knee points in the images correspond to the zero temperature points.

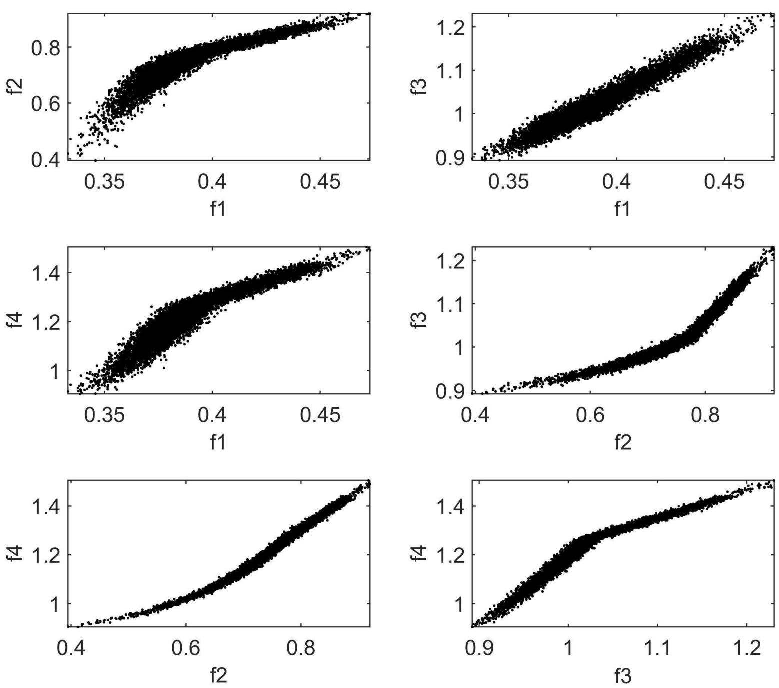

Figure 6: Mutual relationship between natural frequency series.

Regime switching is not an uncommon issue in the economic world, for instance, cointegrated stock indexes might change their inner dynamics from a "bull market" to a "bear market" because of external influences, for example monetary policy intervention, financial crisis or the latest unexpected event e.g. "Brexit". A large body of the econometric literature concerning this falls in the extension to threshold cointegration, first proposed by Balke and Fomby in 1997 [23]. In their framework, the adjustment term in the cointegrating regression is allowed to shift once some indication variable exceeds a threshold. Furthermore, there are several other variants built on the vector error correction (VEC) model, as expressed above in (9); in [24] and [25] for example, they allow a threshold effect on the lag terms and the intercept term respectively. Gregory and Hansen [26], however, take the opposite direction for allowing a cointegrating relationship change, or in their terms, shift regime. More specifically, the cointegrating vector can change its value after a certain breakpoint, after which the system will stabilise itself at another long term equilibrium. The position of the breakpoint is unlikely to be determined in advance, thus they calculate the unit root statistic for each possible regime shift, and evaluate the smallest values across all possible breakpoints. The situation in the four-DOF system above is very similar to their case, as it enters another regime as soon as the temperature drops below zero degrees. Inspired by Gregory and Hansen's work, a regime-switching cointegration method will be adopted to address the issue above. Different from their method however, instead of using the Engle-Granger framework, the more efficient Johansen procedure is implemented to estimate cointegrating vectors.

Coming back to the problem of the four-DOF system, firstly only a small amount of the data are needed for estimating the model, data points from point 2000 to 4000 are extracted for establishing the cointegrating vectors and breakpoints. The training series are rearranged according to the order of temperature, the shuffled series $\mathbf{f}_t = (f_{1t}, f_{2t}, f_{3t}, f_{4t}), t = 1, 2, ...N$, where N is the sample size, is shown in the upper panel of Figure 7 , indexed by the temperature in the lower panel. Even though a breakpoint was simulated to occur at zero degrees

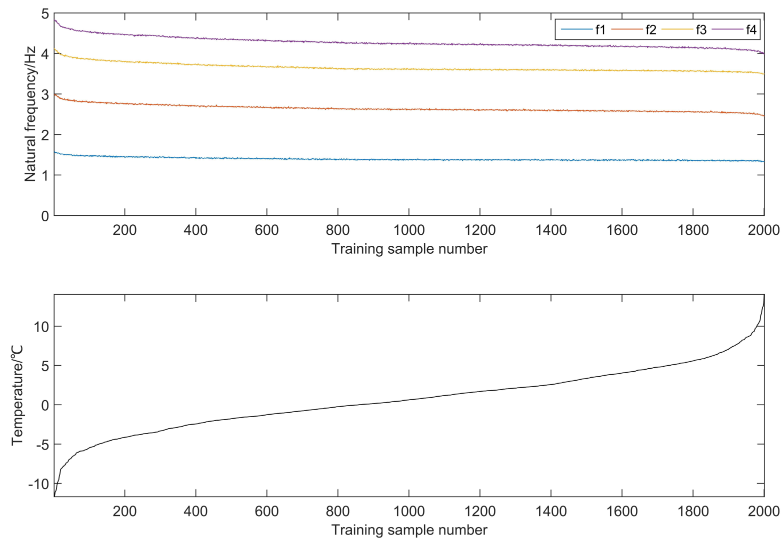

Figure 7: Upper panel: natural frequency series rearranged in the order of temperature; Lower panel: temperature series rearranged in the order of magnitude.

(around point number 900 in Figure 7), there is no clear sign of a shifting regime in the figure. The next step is to ascertain the position of the break point from the training data with the help of a unit root statistic, the ADF $t-$statistic. Assume the current breakpoint is at position τ, then $\mathbf{f}_t(1:N)$ is split into two sets: $\mathbf{f}_{1\tau}(1:\tau)$, $\mathbf{f}_{2\tau}(\tau+1:N)$. One then uses the Johansen procedure presented earlier to estimate the cointegrating vector of each set, say $\boldsymbol{\beta}_{1\tau}$ and $\boldsymbol{\beta}_{2\tau}$, and to construct the residual series at this breakpoint, $e_\tau = (\boldsymbol{\beta}_{1\tau}\mathbf{f}_{1\tau}; \boldsymbol{\beta}_{2\tau}\mathbf{f}_{2\tau})$, where ";" is used to concatenate these two vector series; the subscript τ denotes the fact the residual series depends on the position of the breakpoint. From the residual series e_τ, the ADF statistic can be calculated using (4). Note that not all positions are valid for τ because calculating the ADF statistic demands a small number of samples, therefore in practice, the data sets in the interval $([0.15N], [0.85N])$ are used to evaluate the possible breakpoint. Following the procedures above, the ADF statistic of the four-DOF system is plotted in Figure 8, as a function of the training sample points. The blank space in the

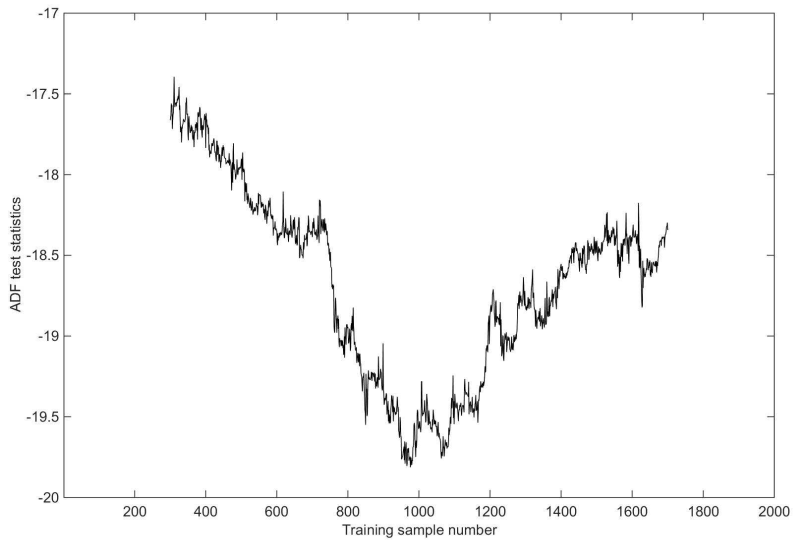

Figure 8: ADF statistics plot of the training sample points, the lowest point position determines the breakpoint position for the regime switch.

beginning and the end of the figure represents the fact that ADF statistics are only evaluated in the interval $([0.15N],[0.85N])$. The smallest value of the curve is at data point 976, corresponding to the temperature 0.4767°C, which is quite close to the simulation assumption. Furthermore, with the estimated best breakpoint and cointegrating vectors correspondingly, one can have the following regime-switching cointegration relationship which is indexed by the value of temperature:

$$\varepsilon_t = \begin{cases} 147.90 \times y_{1t} - 107.29 \times y_{2t} - 122.96 \times y_{3t} + 10.69 \times y_{4t} - 3.54, & \text{if } T \leq 0.4767 \\ -4.51 \times y_{1t} - 84.87 \times y_{2t} - 127.87 \times y_{3t} - 165.07 \times y_{4t} - 24.19, & \text{if } T > 0.4767 \end{cases} \tag{19}$$

Plotting the residual series from (19), as shown in Figure 9, it is clear that the series is stationary before damage introduction, any effect from temperature is effectively eliminated, and the nonlinear behaviour of the frequency response is precisely captured. After 5000 data points, the magnitude of the residual sees a sudden jump, which indicates strongly the occurrence of damage; the overlaid

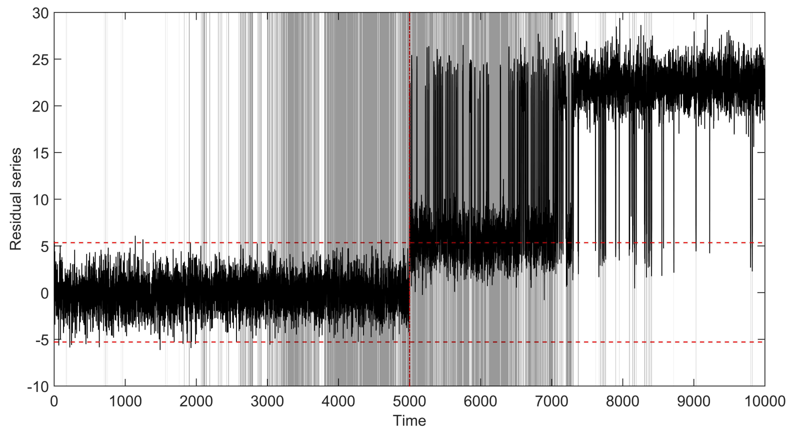

Figure 9: Residual series of the cointegration model, the vertical red dashed line indicates damage introduction, the two horizontal red lines represent the three standard error bars; the grey shaded areas show where cointegration switches regimes.

grey areas show where cointegration switches from one regime to the other. The result can be interpreted by the fact that the regime-switching cointegration is estimated with training data under normal condition, the healthy state of the system has been accurately modelled. Whenever damage occurs, the long term relationship of the variables no longer holds, thus the residual series turns nonstationary immediately.

Despite the fact that the method suggests very good results, one may still argue that shuffling the original series may break the underlying cointegrating relationship, therefore the estimation procedure might be ill-conditioned. This argument is partly true, that rearranging the order of series will surely break the underlying error correction mechanism (as expressed in (9)), but the long term relationship stays the same, or in other words, the rearranged series have the same cointegrating vectors as the original series, because the cointegrating relationships are stacking pointwise in time. One should bear in mind that the final goal here is fundamentally different from the aim of the econometricians, the concern is more about the long term relationship between variables, the short term adjustments are less of interest for the moment. Therefore, it is legitimate to use temperature as a reference series to rearrange the original series, and estimate the cointegrating vectors of the yielded series.

Next, the proposed method will be briefly summarised and then a real engineering example will be used to examine the effectiveness of this method.

4. A brief summary of the regime-switching cointegration method

The procedure of the method is summarised as follows:

1. Rearrange the monitored series in the order of environmental or operational variable.
2. Insert a breakpoint at a position ranging from $([0.15N], [0.85N])$, where N is the sample size.
3. At each possible breakpoint, split the series into two halves, use the Johansen procedure to estimate the cointegrating vectors for each half.
4. With the estimated cointegrating vectors, calculate the residual series of both halves and then merge them into one series, and determine the ADF $t-$statistic of the merged residual series.
5. Repeat procedures from step 2 to 4 at each point from $[0.15N]$ to $[0.85N]$, and construct a plot of all ADF statistics with respect to the breakpoint positions. Pick the minimum value of the curve; the corresponding position represents the optimal breakpoint.
6. With the optimal results from 5, using the environmental or operational variable as an index variable, construct a switching cointegration relationship and a stationary residual series, which should be purged of EOVs and still have the power to detect damage.

This regime switching cointegration method is suitable for dealing with nonstationary SHM data corrupted by EOVs, where system response may have two distinct behaviours with respect to EOVs. For example, bridges may have very different dynamic responses in hot and cold weather because of change of stiffness or boundary conditions. The current approach however, assumes

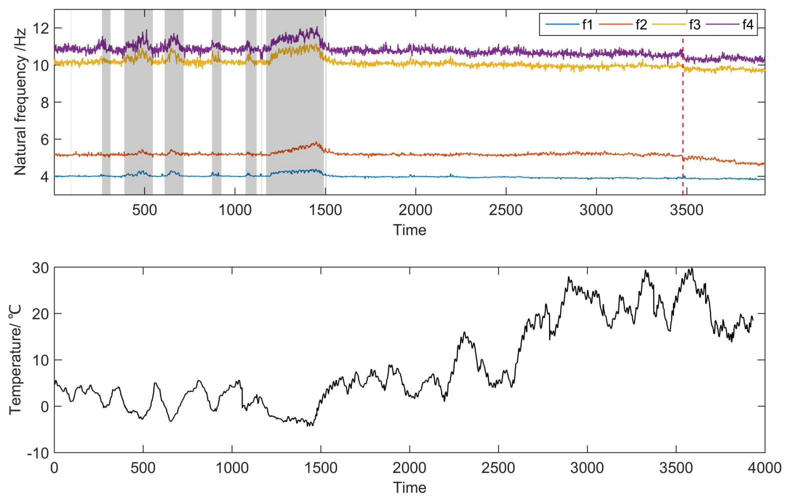

Figure 10: Upper panel: the first four natural frequency series of the Z24 Bridge, the red dashed line indicates damage introduction; Lower panel: the air temperature measurement during this time period.

that the measurements of the EOVs are accessible and only one kind of EOV is driving the nonlinear behaviour of the structure. Likewise, any engineering system with similar behaviour may be suitable for the proposed method, systems that accommodate more regimes can be possibly addressed by inserting more breakpoints in the proposed model.

5. An application to the SHM of the Z24 Bridge

The Z24 Bridge is now a benchmark study in the SHM community. The monitoring campaign spanned one year before the bridge was dismantled, before dismantling, several damage scenarios were implemented [4]. The monitoring campaign also recorded various environmental parameters including temperature, wind speed and humidity. In order to obtain the dynamic properties of the bridge, the natural frequencies were identified from acceleration measurements. The upper panel of Figure 10 illustrates the first four natural frequency series, f_1 to f_4, plotted with respect to time history; the vertical dashed line indicates

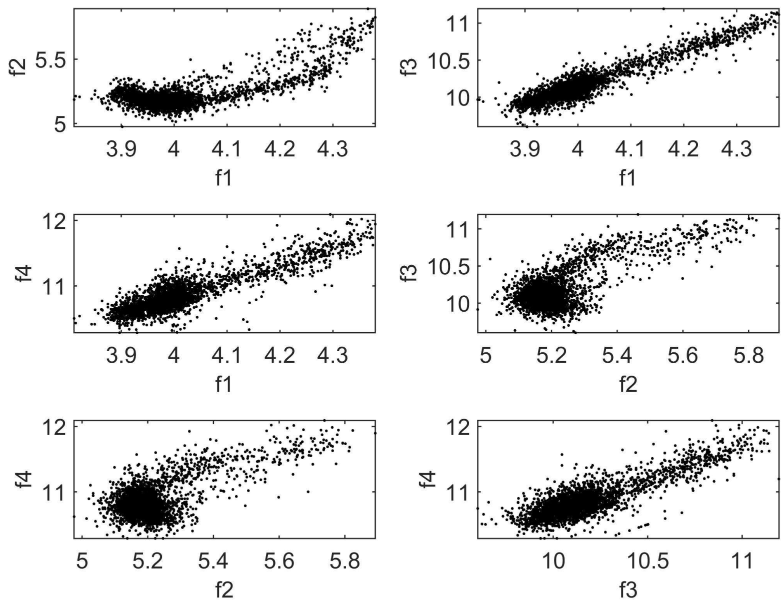

Figure 11: Mutual relationships of the first four natural frequencies.

the position where the first damage scenario was implemented. The temperature readings for this time period are plotted in the lower panel. Note that there are some missing data in the original dataset, thus the points corresponding to time instants when data missing occurs are all removed as a data pre-processing procedure.

On further examining the mutual relationship between the four natural frequencies, as shown in Figure 11, the second natural frequency f_2 has a clear bilinear relationship with the other three. As discussed above, it is a quite similar situation to the four-DOF system, the conventional cointegration method may therefore fail to model this phenomena. Following the cointegration approach proposed in [7], one out of every two from the first 3000 data points are used to estimate the cointegration model, and a residual series is obtained, as demonstrated in Figure 12. Even though the residual becomes largely stationary, the underlying cointegrating relationships are not accurately modelled,

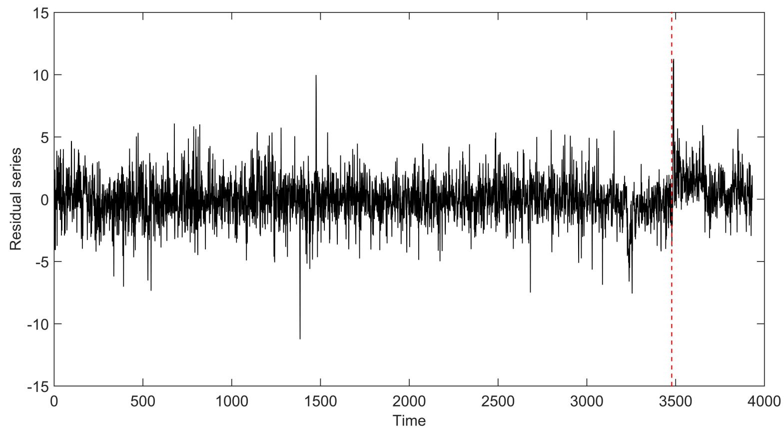

Figure 12: Residual series obtained using the conventional cointegration method, the red dashed line indicates where damage occurs.

therefore the damage information has been smoothed out as well.

The aim is to build a damage indicator based on the healthy state of the bridge, so only the data before the dashed line are used for estimation; the same training data set from above (one out of every two from the first 3000 data points) are used for training purpose. Following the procedures in Section 4, firstly the training series are rearranged in the order of the corresponding temperature series, as exhibited in Figure 13. Then the ADF statistics of all possible breakpoints are plotted in Figure 14, the lowest point of the curve is selected as the best breakpoint, the estimate is 0.98°C, and the estimated switching cointegration has the following form:

$$\varepsilon_t = \begin{cases} 28.54 \times f1 + 6.53 \times f2 - 5.56 \times f3 - 9.62 \times f4 + 13.07, & \text{if } T \leq 0.98 \\ 23.02 \times f1 - 21.86 \times f2 - 1.00 \times f3 - 12.01 \times f4 + 161.15, & \text{if } T > 0.98 \end{cases} \tag{20}$$

Substituting the original series into (20), creates the residual series, which is plotted in Figure 15. The effect of EOVs have been mostly eliminated, the residual series before the dashed line is stationary. Three-sigma error bars are overlaid in the figure; one can see that the undamaged residual series lies predominantly within the confidence intervals; immediately after the damage intro-

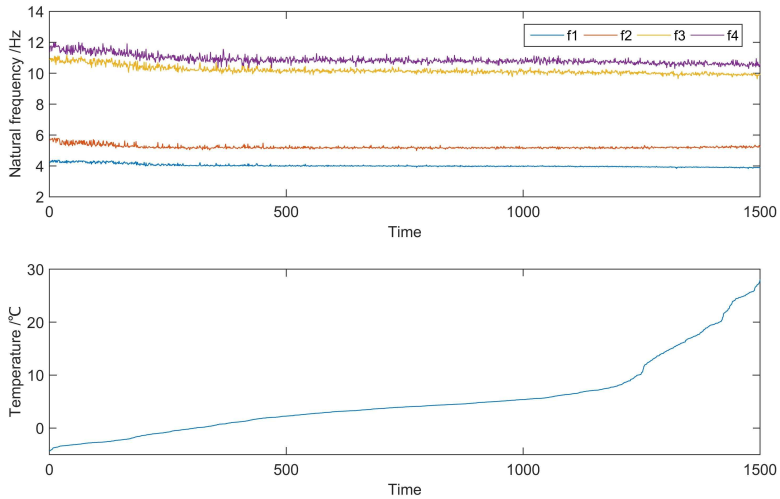

Figure 13: Upper panel: natural frequency series rearranged in the order of temperature; Lower panel: temperature series rearranged in the order of magnitude.

duction, the level of the residual has shifted drastically. To illustrate when the system enters another regime, the cold regime (when temperature drops below 0.98°C) is overlaid with shaded areas, the same shaded areas are duplicated on Figure 10. One can see during winter time that the bridge may switch frequently between two regimes; this may help to explain why conventional linear cointegration fails to model the relationships between the natural frequencies.

However, note that in Figure 15, there are several blips before the dashed line, nonetheless they will not affect the global stationarity. Several reasons may account for these blips. Firstly, as the Johansen procedure is a maximum likelihood method, the cointegrating vectors are all point estimates, thus it is naturally prone to outliers. This may be further improved by putting the cointegration approach in a Bayesian framework, so as to give the posterior distribution of all the parameters. Another possible reason is the effect of the missing values referred to earlier; it may be that this has biased the estimation. Because of sensor faults, a small proportion of the original data are invalid, all the time instants when missing data occurs have been removed, which might

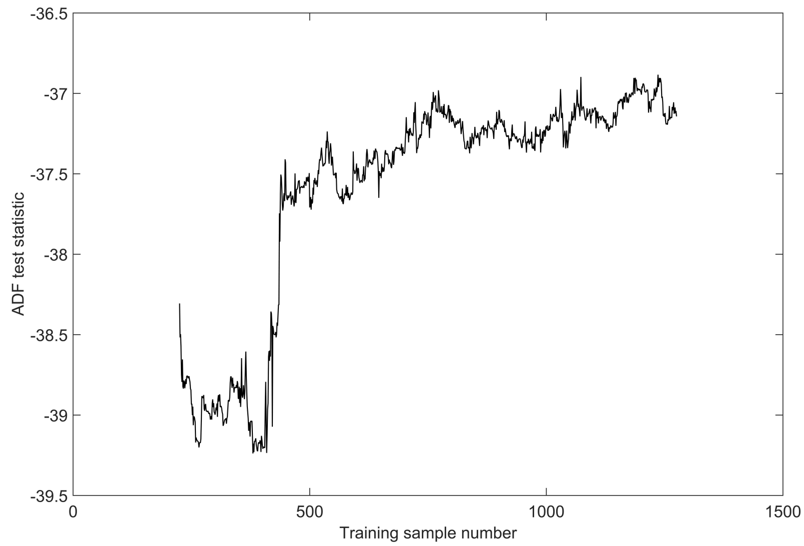

Figure 14: ADF statistics plot of the training sample points, the lowest point position determines the breakpoint position.

bias the estimation of the breakpoint position, causing a small number of data points to enter the wrong regime.

Moreover, despite the fact that most parts of the residual series manifest safely within the error bars, there is a potentially upward trend between data point 3000 to 3500, before damage happens. There may be two main reasons to explain this trend: firstly, this trend might be a local behaviour of the stationary residual series, the local mean value may deviate from global mean sometimes, but it will eventually revert back to the global mean. One may observe that near data point 3500, the residual series has already started to drop back. Another possible explanation for this trend is that the training data used above are from cold seasons, while data point 3000 to 3500 correspond to hot seasons, thus there might be another regime in the hot season. Unfortunately, due to the limited length of data (10 months), the behaviour of the regime-switching cointegration method cannot be evaluated in hotter months.

It is straightforward to apply the proposed method to a three-regime case, it

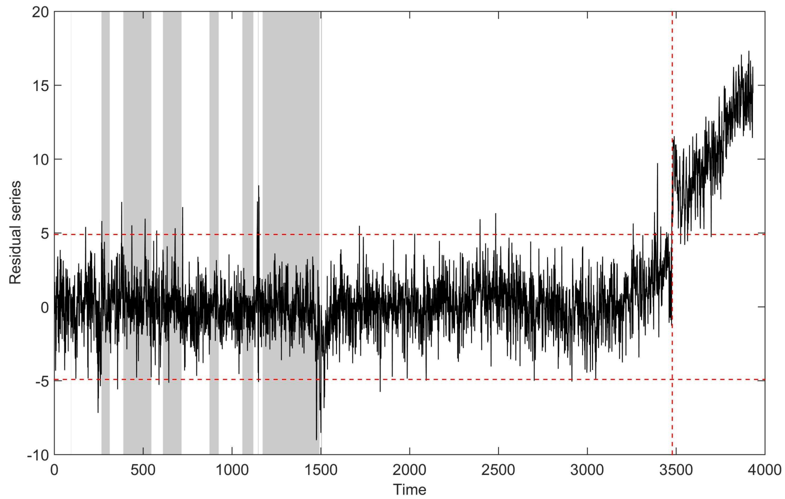

Figure 15: Residual series of the cointegration model, the red vertical dashed line indicates damage introduction, the two horizontal red lines represent the three standard error bars; the grey shaded areas show where cointegration switches regimes.

is attempted here as at least one of the relationships shown in Figure 11 could be described as being more complex than bilinear (particularly that between $f1$ and $f2$). In the procedures of Section 4, one should insert two breakpoints instead of one breakpoint in step 2; and then make the first breakpoint fixed and evaluate the second breakpoint at every possible position; subsequently, move the first breakpoint to the next position, and evaluate the second breakpoint again at every possible position; repeat the previous steps until every possible breakpoint position is evaluated. This is essentially an exhaustive search; it is feasible in the situation considered here as the size of the training set is not too large. In general problems, it would be necessary to use a more sophisticated optimisation/search routine.

Data points 1 to 3000 shown in Figure 10 are used as training samples to estimate the two breakpoints, results are presented in Figure 16, where the vertical and horizontal axes represent the positions of the first and the second breakpoints, the colour indicates the magnitude of the ADF statistic evaluated at the corresponding breakpoints. The darkest point is selected as the optimal breakpoint position. According to Figure 16, two breakpoints are selected at 2.36°C and 3.95°C, and the estimated regime-switching cointegration has the following form:

$$\varepsilon_t = \begin{cases} -4.80 \times f1 + 1.14 \times f2 + 11.31 \times f3 - 9.27 \times f4 - 1.12, & \text{if } T \leq 2.36 \\ 27.76 \times f1 + 13.35 \times f2 + 8.26 \times f3 + 7.82 \times f4 - 348.94, & \text{if } 2.36 < T < 3.95 \\ -20.41 \times f1 + 14.40 \times f2 + 19.28 \times f3 - 5.64 \times f4 - 127.03, & \text{if } T \geq 3.95 \end{cases} \tag{21}$$

Substituting the original series into (21), one can obtain a residual series, as shown in Figure 17. The blue and grey areas shown the first and second regimes respectively, and the left areas are the third regime. As expected, the three-regime-switching cointegration produces a stationary residual series which is still sensitive to damage. Interestingly, the residual shown in Figure 17 appears more stationary during the undamaged period than the results from one switching point. The two breakpoints estimated in this model coincide well with the switching response surface model estimated in [27], where a Bayesian treed linear model is fitted. Comparing to the previous work where Gaussian process regression is used to build the nonlinear cointegration relationship [15] using an Engle-Granger approach, this paper implements a cointegration method using the more powerful framework of the Johansen procedure. A more stationary residual is obtained in this paper; a more interpretable model is presented; and the model itself is even easier to implement in practice.

6. Discussions and conclusions

The contents of this paper are mainly about an exploratory approach aiming to enhance the conventional cointegration method in the context of structural health monitoring. Conventional cointegration methods can be used to remove the common trends in SHM data induced by environmental and operational effects, however, in some circumstances, nonlinearity in the system may undermine the cointegrating relationship; as such, a regime-switching cointegration

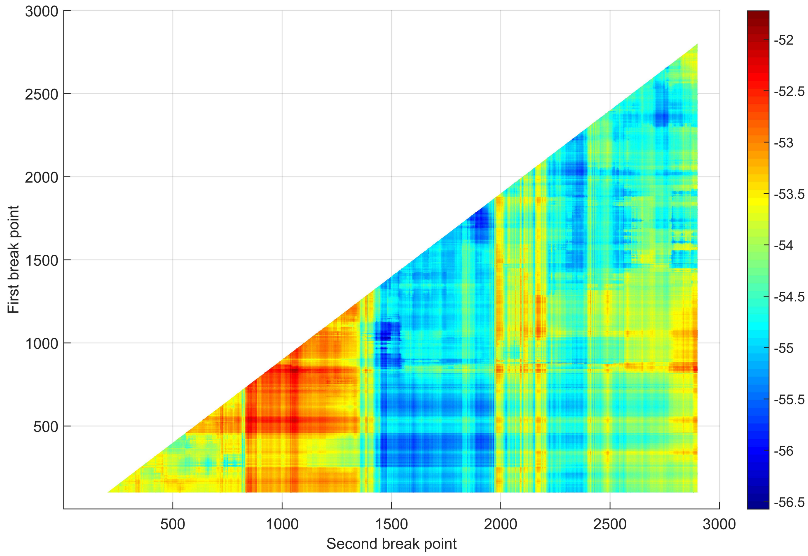

Figure 16: ADF statistic plot of the training sample points, the vertical axis represents the positions of the first breakpoint, the horizontal axis represents the positions of the second breakpoints; the colours in the plot indicate the value of the ADF statistic evaluated at the corresponding breakpoints.

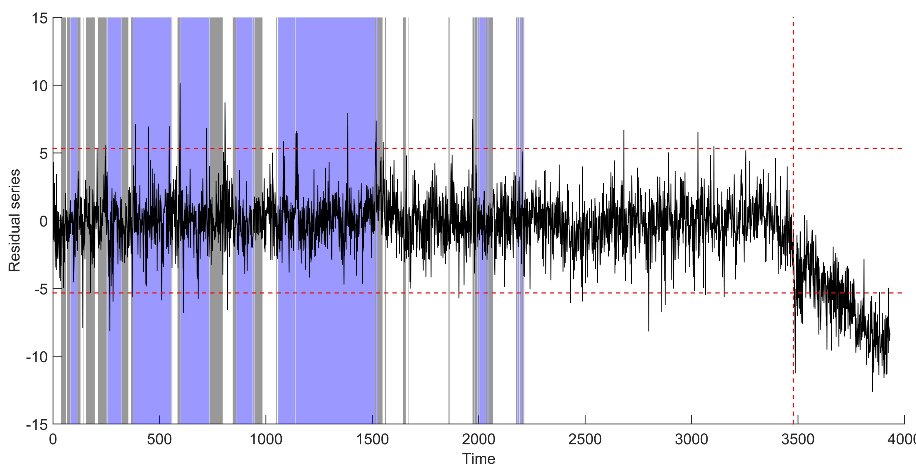

Figure 17: Residual series of the regime-switching cointegration model, the red vertical dashed line indicates damage introduction, the two horizontal red lines indicate the three standard error bars; the grey and blue shaded areas show where cointegration switches regimes.

method has been introduced in this paper to address both nonlinearity and nonstationarity in SHM data. System responses may become nonstationary because of the effect of environmental variation, while sometimes the effect can simultaneously induce a nonlinear relationship between features. The proposed method allows the cointegrating relationship to switch according to the variation of environmental variables, the switching point is called a breakpoint. The position of the breakpoint is not likely to be known beforehand, thus all possible positions are evaluated by inserting a breakpoint at a time and assessing the global nonstationarity property of the residual series, the procedure is repeated throughout all possible breakpoint positions, and the test statistics are compared to find the most probable breakpoint position. The proposed method is employed here in two case studies, a simulated four-DOF system and the benchmark study of the Z24 Bridge; they both give very promising results, showing that all the benign environmental effects have been successfully removed. Once damage occurs, the underlying cointegration relationship no longer holds, therefore the residual series shows a very significant indication of damage, as the residual series become nonstationary again. However, it is important to note that there are still some restrictions of the current approach, which will be future directions for the authors:

- The Johansen procedure implemented is a maximum likelihood method, which gives a crisp estimate of the cointegrating vector. It is known that the maximum likelihood approach can be greatly affected by outliers and dependent on the selection of training data. A possible solution to this issue might be using cross validation to ascertain the model, but the variation of noise level and data missing can be difficult to deal with. Another possible direction would be to put the Johansen procedure in a Bayesian framework, instead of giving point estimates of cointegrating vectors, one could ideally have the whole posterior distribution of them; however the complexity of Johansen procedure will present a big challenge.

- In this paper, environmental measurements of temperature have been used to direct the regime shift of the system. One of the main benefits from the previous cointegration framework is that the measurements of EOVs are unnecessary. However, conventional cointegration is linear in nature, and may not suffice to account for the nonlinear behaviour observed in this paper. To address this, the measurement of temperature is taken into account to build a nonlinear model which still maintains a simple form. In this situation, temperature is the main driving variable of the nonlinear relationship between the natural frequencies, and other EOVs including wind speed, humidity are unnecessary in the analysis.

Strictly speaking, the method presented here should be considered a hybrid regression/cointegration approach. In using measurements of the temperature in order to construct the cointegration regimes, the approach represents a step forward, followed by a small step back; however, there are overall advantages. It is important to note that, if a linear switching behaviour is present, any global model polynomial or otherwise is likely to be *input-dependent*, and may not generalise well away from the training data. On a related issue, global models may need more parameters to explain piecewise-linear behaviour and will be less parsimonious.

An ideal enhancement here would be to allow the choice of a switching point without measurement of the environmental variable. One possible direction would fall in the domain of change point detection, a good reference can be found in [28], where a Bayesian online change point detection algorithm is developed, which might be helpful to identify a switching point purely based on data. Another interesting possibility is provided by the idea of inferential parametrisation [29]. As in [29] one could in principle, infer a proxy for the temperature measurement directly from the natural frequencies themselves. In fact, a preliminary study has shown that this can be accomplished for the Z24 data; however, there is an important issue to overcome before those results can be shown with confidence. The point is this; if the switching parameter (temperature) in this case is learned from data, it has to be learned from training data unaffected by damage. This means that there is no guarantee that, when damage occurs, the previously learned inferential parameter is still accurate. The Z24 data itself is not sufficient to validate the approach because the full range of environmental conditions were not available for any of the damage states of the bridge. The inferential parameterisation approach is a work in progress.

- Another possible enhancement of this method is to incorporate more regimes in the cointegration. In the current method, only two and three regimes are used, which captures well the nonlinear property of icing of the bridge. However in many other cases, there are possibly more than three regimes. The approach itself can be easily extended to the multiple regime context, however the difficulty is how to determine the number of regimes. More specific hypothesis test methods can be developed accordingly.

- Away from the example presented here, it is possible, or indeed likely, that a structure may be influenced by multiple EOVs at the same time. However, not every EOV may induce nonlinear (regime-switching) behaviour in the features of interest. Observing the phenomena of stiffening of the asphalt in the Z24 case, this paper assumes that temperature is the main EOV driving the regime-switching behaviour of that particular structure. This assumption, however, may be violated in more of an operational environment, if, for example, the bridge had been opened to traffic. A challenging scenario in this context would be if multiple EOVs with multiple regimes induced a nonlinear relationship between features of interest. In such a case, an entire embedded submanifold of switching points might be present within the space of EOVs. A possible solution to this issue may be to put the model in the framework of decision tree learning [30], where a high-dimensional input space can be partitioned into finite discrete domains, each domain representing a class of features determining the regime in which the structure is behaving. Again, research is in progress on this matter.

7. Acknowledgements

The work is partially funded by the China Scholarship Council (Grant No.201406830013).

References

[1] C. R. Farrar, K. Worden, Structural Health Monitoring: A Machine Learning Perspective, John Wiley & Sons, 2012.

[2] C. R. Farrar, W. Baker, T. Bell, K. Cone, T. Darling, T. Duffey, A. Eklund, A. Migliori, Dynamic characterization and damage detection in the I-40 bridge over the Rio Grande, Tech. rep., Los Alamos National Lab., NM (United States) (1994).

[3] E. Figueiredo, G. Park, C. R. Farrar, K. Worden, J. Figueiras, Machine learning algorithms for damage detection under operational and environmental variability, Structural Health Monitoring 10 (6) (2011) 559–572.

[4] B. Peeters, G. De Roeck, One-year monitoring of the Z24-Bridge: environmental effects versus damage events, Earthquake Engineering & Structural Dynamics 30 (2) (2001) 149–171.

[5] E. Cross, G. Manson, K. Worden, S. Pierce, Features for damage detection with insensitivity to environmental and operational variations, in: Proc. R. Soc. A, The Royal Society, 2012, p. rspa20120031.

[6] H. Sohn, Effects of environmental and operational variability on structural health monitoring, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 365 (1851) (2007) 539–560.

[7] E. J. Cross, K. Worden, Q. Chen, Cointegration: a novel approach for the removal of environmental trends in structural health monitoring data, in:

Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, The Royal Society, 2011, p. rspa20110023.

[8] E. Cross, K. Koo, J. Brownjohn, K. Worden, Long-term monitoring and data analysis of the Tamar Bridge, Mechanical Systems and Signal Processing 35 (1) (2013) 16–34.

[9] P. B. Dao, W. J. Staszewski, Cointegration approach for temperature effect compensation in lamb-wave-based damage detection, Smart Materials and Structures 22 (9) (2013) 095002.

[10] K. Worden, E. Cross, I. Antoniadou, A. Kyprianou, A multiresolution approach to cointegration for enhanced shm of structures under varying conditions–an exploratory study, Mechanical Systems and Signal Processing 47 (1) (2014) 243–262.

[11] K. Worden, T. Baldacchino, J. Rowson, E. J. Cross, Some recent developments in SHM based on nonstationary time series analysis, Proceedings of the IEEE 104 (8) (2016) 1589–1603.

[12] E. J. Cross, On structural health monitoring in changing environmental and operational conditions, Ph.D. thesis, University of Sheffield (2012).

[13] E. Cross, K. Worden, Approaches to nonlinear cointegration with a view towards applications in SHM, in: Journal of Physics: Conference Series, Vol. 305, IOP Publishing, 2011, p. 012069.

[14] K. Zolna, P. B. Dao, W. J. Staszewski, T. Barszcz, Towards homoscedastic nonlinear cointegration for structural health monitoring, Mechanical Systems and Signal Processing 75 (2016) 94–108.

[15] H. Shi, K. Worden, E. J. Cross, A nonlinear cointegration approach with applications to structural health monitoring, in: Journal of Physics: Conference Series, Vol. 744, IOP Publishing, 2016, p. 012025.

[16] H. Shi, K. Worden, E. J. Cross, An exploratory study on removing environmental and operational effects using a regime-switching cointegration method, in: Dynamics of Civil Structures, Volume 2, Springer, 2017, pp. 329–337.

[17] E. J. Cross, K. Worden, Cointegration and why it works for shm, in: Journal of Physics: Conference Series, Vol. 382, IOP Publishing, 2012, p. 012046.

[18] J. D. Hamilton, Time Series Analysis, Vol. 2, Princeton University Press, 1994.

[19] W. Enders, Applied Econometric Time Series, John Wiley & Sons, 2008.

[20] R. Perman, Cointegration: an introduction to the literature, Journal of Economic Studies 18 (3).

[21] D. A. Dickey, W. A. Fuller, Distribution of the estimators for autoregressive time series with a unit root, Journal of the American Statistical Association 74 (366a) (1979) 427–431.

[22] S. Johansen, Likelihood-Based Inference in Cointegrated Vector Autoregressive Models, Oxford University Press, 1995.

[23] N. S. Balke, T. B. Fomby, Threshold cointegration, International Economic Review (1997) 627–645.

[24] B. E. Hansen, B. Seo, Testing for two-regime threshold cointegration in vector error-correction models, Journal of Econometrics 110 (2) (2002) 293–318.

[25] M. C. Lo, E. Zivot, Threshold cointegration and nonlinear adjustment to the law of one price, Macroeconomic Dynamics 5 (04) (2001) 533–576.

[26] A. W. Gregory, B. E. Hansen, Residual-based tests for cointegration in models with regime shifts, Journal of Econometrics 70 (1) (1996) 99–126.

[27] K. Worden, E. Cross, On switching response surface models, with applications to the structural health monitoring of bridges, Mechanical Systems and Signal Processing 98 (2018) 139–156.

[28] R. P. Adams, D. J. MacKay, Bayesian online changepoint detection, arXiv preprint arXiv:0710.3742.

[29] K. Worden, Inferential parameterisation using principal curves, in: 3rd International Conference on Identification in Engineering Systems, University of Wales Swansea, UK, 2002.

[30] W. Loh, Classification and regression trees, Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery 1 (1) (2011) 14–23.