Mesoscale asymptotic approximations to solutions of mixed boundary value problems in perforated domains

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Abstract

We describe a method of asymptotic approximations to solutions of mixed boundary value problems for the Laplacian in a three-dimensional domain with many perforations of arbitrary shape, with the Neumann boundary conditions being prescribed on the surfaces of small voids. The only assumption made on the geometry is that the diameter of a void is assumed to be smaller compared to the distance to the nearest neighbour. The asymptotic approximation, obtained here, involves a linear combination of dipole fields constructed for individual voids, with the coefficients, which are determined by solving a linear algebraic system. We prove the solvability of this system and derive an estimate for its solution. The energy estimate is obtained for the remainder term of the asymptotic approximation.

1 Introduction

In the present paper we discuss a method for asymptotic approximations to solutions of the mixed problems for the Poisson equation for domains containing a number, possibly large, of small perforations of arbitrary shape. The Dirichlet condition is set on the exterior boundary of the perforated body, and the Neumann conditions are specified on the boundaries of small holes. Neither periodicity nor even local "almost" periodicity constraints are imposed on the position of holes, which makes the homogenization methodologies not applicable (cf. Chapter 4 in [5], and Chapter 5 in [9]). Two geometrical parameters, ε and d, are introduced to characterize the maximum diameter of perforations within the array and the minimum distance between the voids, respectively. Subject to the *mesoscale* constraint, $\varepsilon <$ const d, the asymptotic approximation to the solution of the mixed boundary value problem is constructed. The approximate solution involves a linear combination of dipole fields constructed for individual voids, with the coefficients determined from a linear algebraic system. The formal asymptotic representation is accompanied by the energy estimate of the remainder term. The general idea of mesoscale approximations originated a couple of years ago in [6], where the Dirichlet problem was considered for a domain with multiple inclusions.

The asymptotic methods, presented here and in [6], can be applied to modelling of dilute composites in problems of mechanics, electromagnetism, heat conduction and phase transition. In such models, the boundary conditions have to be satisfied across a large array of small voids, which is the situation fully served by our approach. Being used in the case of a dilute array of small spherical particles, the method also includes the physical models of many point interactions treated previously in [1, 2, 4] and elsewhere. Asymptotic approximations applied to solutions of boundary value problems of mixed type in domains containing many small spherical inclusions were considered in [1]. The point interaction approximations to solutions of diffusion problems in domains with many small spherical holes were analysed in [2]. Modelling of multi-particle interaction in problems of phase transition was considered in [4] where the evolution of a large number of small spherical particles embedded into an ambient medium takes place during the last stage of phase transformation; such a phenomenon where particles in a melt are subjected to growth is referred to as Ostwald ripening. For the numerical treatment of models involving large number N of spherical particles, the fast multipole method, of order $O(N)$, was proposed in [3], and it appears to be efficient for the rapid evaluation of potential and force fields for systems of a large number of particles interacting with each other via the Coulomb law.

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We give an outline of the paper. The notation Ω_N will be used for a domain containing small voids $F^{(j)}$, $j = 1, \ldots, N$, while the unperturbed domain, without any holes, is denoted by Ω. The number N is assumed to be large.

If Ω is a bounded domain in $\mathbb{R}^3$, we introduce $L^{1,2}(\Omega)$ as the space of functions on Ω with distributional first derivatives in $L^2(\Omega)$ provided with the norm

$$\|u\|_{L^{1,2}(\Omega)} = \Big(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big)^{1/2}. \tag{1.1}$$

Here B is a ball at a positive distance from $\partial\Omega$. If Ω is unbounded, by $L^{1,2}(\Omega)$ we mean the completion of the space of functions with $\|\nabla u\|_{L^2(\Omega)} < \infty$, which have bounded supports, in the norm (1.1). The space of traces of functions in $L^{1,2}(\Omega)$ on $\partial\Omega$ will be denoted by $L^{1/2,2}(\partial\Omega)$.

The maximum of diameters of $F^{(j)},\ j = 1, \ldots, N$ is denoted by ε. An array of points $\mathbf{O}^{(j)},\ j = 1, \ldots, N$, is chosen in such a way that $\mathbf{O}^{(j)}$ is an interior point of $F^{(j)}$ for every $j = 1, \ldots, N$. By $2d$ we denote the smallest distance between the points within the array $\{\mathbf{O}^{(j)}\}_{j=1}^N$. It is assumed that there exists an open set $\omega \subset \Omega$ situated at a positive distance from $\partial\Omega$ and such that

$$\cup_{j=1}^N F^{(j)} \subset \omega,\ \operatorname{dist}\big(\cup_{j=1}^N F^{(j)}, \partial\omega\big) \geq 2d,\ \text{and diam } \omega = 1, \tag{1.2}$$

With the last normalization of the size of ω, the parameters ε and d can be considered as non-dimensional. The scaled open sets $\varepsilon^{-1}F^{(j)}$ are assumed to have Lipschitz boundaries, with Lipschitz characters independent of N.

Our goal is to obtain an asymptotic approximation to a unique solution $u_N \in L^{1,2}(\Omega_N)$ of the problem

$$-\Delta u_N(\mathbf{x}) = f(\mathbf{x})\ , \quad \mathbf{x} \in \Omega_N\ , \tag{1.3}$$

$$u_N(\mathbf{x}) = \phi(\mathbf{x})\ , \quad \mathbf{x} \in \partial\Omega\ , \tag{1.4}$$

$$\frac{\partial u_N}{\partial n}(\mathbf{x}) = 0\ , \quad \mathbf{x} \in \partial F^{(j)}\ , j = 1, \ldots, N\ , \tag{1.5}$$

where $\phi \in L^{1/2,2}(\partial\Omega)$ and $f(\mathbf{x})$ is a function in $L^\infty(\Omega)$ with compact support at a positive distance from the cloud ω of small perforations.

We need solutions to certain model problems in order to construct the approximation to u_N; these include

1. v as the solution of the unperturbed problem in Ω (without voids),
2. $\boldsymbol{\mathcal{D}}^{(k)}$ as the vector function whose components are the dipole fields for the void $F^{(k)}$,
3. H as the regular part of Green's function G in Ω.

The approximation relies upon a certain algebraic system, incorporating the field v_f and integral characteristics associated with the small voids. We define

$$\boldsymbol{\Theta} = \left(\frac{\partial v}{\partial x_1}(\mathbf{O}^{(1)}), \frac{\partial v}{\partial x_2}(\mathbf{O}^{(1)}), \frac{\partial v}{\partial x_3}(\mathbf{O}^{(1)}), \ldots, \frac{\partial v}{\partial x_1}(\mathbf{O}^{(N)}), \frac{\partial v}{\partial x_2}(\mathbf{O}^{(N)}), \frac{\partial v}{\partial x_3}(\mathbf{O}^{(N)})\right)^T,$$

and $\mathfrak{S} = [\mathfrak{S}_{ij}]_{i,j=1}^N$ which is a $3N \times 3N$ matrix with 3×3 block entries

$$\mathfrak{S}_{ij} = \begin{cases} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}})\,(G(\mathbf{z}, \mathbf{w}))\Big|_{\substack{\mathbf{z}=\mathbf{O}^{(i)} \\ \mathbf{w}=\mathbf{O}^{(j)}}} & \text{if } i \neq j \\ 0 I_3 & \text{otherwise} \end{cases},$$

where G is Green's function in Ω, and I_3 is the 3×3 identity matrix. We also use the block-diagonal matrix

$$\mathbf{Q} = \operatorname{diag}\{\boldsymbol{\mathcal{Q}}^{(1)}, \ldots, \boldsymbol{\mathcal{Q}}^{(N)}\}, \tag{1.6}$$

where $\boldsymbol{\mathcal{Q}}^{(k)}$ is the so-called 3×3 polarization matrix for the small void $F^{(k)}$ (see [7] and Appendix G of [8]). The shapes of the voids $F^{(j)}, j = 1, \ldots, N$, are constrained in such a way that the maximal and minimal eigenvalues $\lambda^{(j)}_{max},\ \lambda^{(j)}_{min}$ of the matrices $-\boldsymbol{\mathcal{Q}}^{(j)}$ satisfy the inequalities

$$A_1\varepsilon^3 > \max_{1 \leq j \leq N} \lambda^{(j)}_{max}, \quad \min_{1 \leq j \leq N} \lambda^{(j)}_{min} > A_2\varepsilon^3, \tag{1.7}$$

where A_1 and A_2 are positive and independent of ε.

One of the results, for the case when $\Omega = \mathbb{R}^3$, $H \equiv 0$, and when (1.4) is replaced by the condition of decay of u_N at infinity, can be formulated as follows

Theorem 1 *Let*

$$\varepsilon < c\, d\ ,$$

where c is a sufficiently small absolute constant. Then the solution $u_N(\mathbf{x})$ admits the asymptotic representation

$$u_N(\mathbf{x}) = v(\mathbf{x}) + \sum_{k=1}^{N} \boldsymbol{C}^{(k)} \cdot \boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) + \mathcal{R}_N(\mathbf{x})\ , \tag{1.8}$$

where $\boldsymbol{C}^{(k)} = (C_1^{(k)}, C_2^{(k)}, C_3^{(k)})^T$ and the column vector $\mathbf{C} = (C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, \ldots, C_1^{(N)}, C_2^{(N)}, C_3^{(N)})^T$ satisfies the invertible linear algebraic system

$$(\mathbf{I} + \mathfrak{S}\mathbf{Q})\mathbf{C} = -\boldsymbol{\Theta}\ . \tag{1.9}$$

The remainder $\mathcal{R}_N$ satisfies the energy estimate

$$\|\nabla \mathcal{R}_N\|^2_{L_2(\Omega_N)} \leq \text{const}\ \Big\{\varepsilon^{11} d^{-11} + \varepsilon^5 d^{-3}\Big\} \|\nabla v\|^2_{L^2(\Omega)}. \tag{1.10}$$

We remark that since ε and d are non-dimensional parameters, there is no dimensional mismatch in the right-hand side of (1.10).

We now describe the plan of the article. In Section 2, we introduce the multiply-perforated geometry and consider the above model problems. The formal asymptotic algorithm for a cloud of small perforations in the infinite space and the analysis of the algebraic system (1.9) are given in Sections 3 and 4. Section 5 presents the proof of Theorem 1. The problem for a cloud of small perforations in a general domain is considered in Section 6. Finally, in Section 7 we give an illustrative example accompanied by the numerical simulation.

2 Main notations and model boundary value problems

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We shall also consider the case when $\Omega = \mathbb{R}^3$.

The perforated domain Ω_N, is given by

$$\Omega_N = \Omega \backslash \overline{\cup_{j=1}^N F^{(j)}}\ ,$$

where $F^{(j)}$ are small voids introduced in the previous section. Also in the previous section we introduced the notations ε and d for two small parameters, characterizing the maximum of the diameters of $F^{(j)}, j = 1, \ldots, N$, and the minimal distance between the small voids, respectively.

In sections where we are concerned with the energy estimates of the remainders produced by asymptotic approximations we frequently use the obvious estimate

$$N \leq \text{const}\ d^{-3}\ . \tag{2.1}$$

We consider the approximation of the function u_N which is a variational solution of the mixed problem (1.3)-(1.5).

Before constructing the approximation to u_N, we introduce model auxiliary functions which the asymptotic scheme relies upon.

1. *Solution v in the unperturbed domain Ω.* Let $v \in L^{1,2}(\Omega)$ denote a unique variational solution of the problem

$$-\Delta v(\mathbf{x}) = f(\mathbf{x})\ , \quad \mathbf{x} \in \Omega\ , \tag{2.2}$$

$$v(\mathbf{x}) = \phi(\mathbf{x})\ , \quad \mathbf{x} \in \partial\Omega\ . \tag{2.3}$$

2. *Regular part of Green's function in* Ω. By H we mean the regular part of Green's function G in Ω defined by the formula

$$H(\mathbf{x},\mathbf{y}) = (4\pi|\mathbf{x}-\mathbf{y}|)^{-1} - G(\mathbf{x},\mathbf{y}) \ . \tag{2.4}$$

Then H is a variational solution of

$$\Delta_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) = 0 \ , \quad \mathbf{x},\mathbf{y} \in \Omega \ ,$$

$$H(\mathbf{x},\mathbf{y}) = (4\pi|\mathbf{x}-\mathbf{y}|)^{-1} \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega \ .$$

3. *The dipole fields* $\mathcal{D}_i^{(j)}$, $i = 1,2,3$, *associated with the void* $F^{(j)}$. The vector functions $\boldsymbol{\mathcal{D}}^{(j)} = \{\mathcal{D}_i^{(j)}\}_{i=1}^3$, which are called the dipole fields, are variational solutions of the exterior Neumann problems

$$\left.\begin{array}{c} \Delta\boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x}) = \mathbf{O} \ , \quad \mathbf{x} \in \mathbb{R}^3 \setminus \bar{F}^{(j)} \ , \\ \dfrac{\partial \boldsymbol{\mathcal{D}}^{(j)}}{\partial n}(\mathbf{x}) = \boldsymbol{n}^{(j)} \ , \quad \mathbf{x} \in \partial F^{(j)} \ , \\ \boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x}) = O(\varepsilon^3|\mathbf{x}-\mathbf{O}^{(j)}|^{-2}) \quad \text{as} \quad |\mathbf{x}| \to \infty \ , \end{array}\right\} \tag{2.5}$$

where $\boldsymbol{n}^{(j)}$ is the unit outward normal with respect to $F^{(j)}$. In the text below we also use the negative definite polarization matrix $\boldsymbol{\mathcal{Q}}^{(j)} = \{\mathcal{Q}_{ik}^{(j)}\}_{i,k=1}^3$, as well as the following asymptotic result (see [7] and Appendix G in [8]), for every void $F^{(j)}$:

Lemma 1 *For* $|\mathbf{x}-\mathbf{O}^{(j)}| > 2\varepsilon$, *the dipole fields admit the asymptotic representation*

$$\mathcal{D}_i^{(j)}(\mathbf{x}) = \frac{1}{4\pi}\sum_{m=1}^{3} \mathcal{Q}_{im}^{(j)} \frac{x_m - O_m^{(j)}}{|\mathbf{x}-\mathbf{O}^{(j)}|^3} + O\left(\varepsilon^4|\mathbf{x}-\mathbf{O}^{(j)}|^{-3}\right) \ , \quad i = 1,2,3 \ . \tag{2.6}$$

The shapes of the voids $F^{(j)}, j = 1,\ldots,N$, are constrained in such a way that the maximal and minimal eigenvalues $\lambda_{max}^{(j)}$, $\lambda_{min}^{(j)}$ of the matrices $-\boldsymbol{\mathcal{Q}}^{(j)}$ satisfy the inequalities (1.7).

3 The formal approximation of u_N for the infinite space containing many voids

In this section we deduce formally the uniform asymptotic approximation of u_N:

$$u_N(\mathbf{x}) \sim v(\mathbf{x}) + \sum_{k=1}^{N} \boldsymbol{C}^{(k)} \cdot \boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) \ ,$$

for the case $\Omega = \mathbb{R}^3$ and derive an algebraic system for the coefficients $\boldsymbol{C}^{(k)} = \{C_i^{(k)}\}_{i=1}^3$, $k = 1,\ldots,N$.

The function u_N satisfies

$$-\Delta u_N(\mathbf{x}) = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Omega_N \ , \tag{3.1}$$

$$\frac{\partial u_N}{\partial n}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in \partial F^{(j)}, j = 1,\ldots,N \ , \tag{3.2}$$

$$u_N(\mathbf{x}) \to 0 \ , \quad \text{as } |\mathbf{x}| \to \infty \ . \tag{3.3}$$

We begin by constructing the asymptotic representation for u_N in this way

$$u_N(\mathbf{x}) = v(\mathbf{x}) + \sum_{k=1}^{N} \boldsymbol{C}^{(k)} \cdot \mathcal{D}^{(k)}(\mathbf{x}) + \mathcal{R}_N(\mathbf{x}) \tag{3.4}$$

where $\mathcal{R}_N$ is the remainder, and $v(\mathbf{x})$ satisfies

$$-\Delta v(\mathbf{x}) = f(\mathbf{x}) \ , \quad \mathbf{x} \in \mathbb{R}^3 \ ,$$

$$v(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty \ ,$$

and $\mathcal{D}^{(k)}$ are the dipole fields defined as solutions of problems (2.5). The function $\mathcal{R}_N$ is harmonic in Ω_N and

$$\mathcal{R}_N(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty \ . \tag{3.5}$$

Placement of (3.4) into (3.2) together with (2.5) gives the boundary condition on $\partial F^{(j)}$:

$$\frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x}) = -\boldsymbol{n}^{(j)} \cdot \Big\{ \nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + O(\varepsilon) + \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} \nabla(\boldsymbol{C}^{(k)} \cdot \mathcal{D}^{(k)}(\mathbf{x})) \Big\}.$$

Now we use (2.6), for $\mathcal{D}^{(k)}$, $k \neq j$, so that this boundary condition becomes

$$\frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x}) \sim -\boldsymbol{n}^{(j)} \cdot \Big\{ \nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} T(\mathbf{x}, \mathbf{O}^{(k)}) \boldsymbol{\mathcal{Q}}^{(k)} \boldsymbol{C}^{(k)} \Big\} \ , \quad \mathbf{x} \in \partial F^{(j)}, j = 1, \ldots, N \ ,$$

where

$$T(\mathbf{x}, \mathbf{y}) = (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(\frac{1}{4\pi |\mathbf{z} - \mathbf{w}|} \right) \Big|_{\substack{\mathbf{z}=\mathbf{x} \\ \mathbf{w}=\mathbf{y}}} \ . \tag{3.6}$$

Finally, Taylor's expansion of $T(\mathbf{x}, \mathbf{O}^{(k)})$ about $\mathbf{x} = \mathbf{O}^{(j)}$, $j \neq k$, leads to

$$\frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x}) \sim -\boldsymbol{n}^{(j)} \cdot \Big\{ \nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} T(\mathbf{O}^{(j)}, \mathbf{O}^{(k)}) \boldsymbol{\mathcal{Q}}^{(k)} \boldsymbol{C}^{(k)} \Big\} \ , \quad \mathbf{x} \in \partial F^{(j)}, j = 1, \ldots, N \ .$$

To remove the leading order discrepancy in the above boundary condition, we require that the vector coefficients $\mathbf{C}^{(j)}$ satisfy the algebraic system

$$\nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} T(\mathbf{O}^{(j)}, \mathbf{O}^{(k)}) \boldsymbol{\mathcal{Q}}^{(k)} \boldsymbol{C}^{(k)} = \mathbf{O} \ , \quad \text{for } j = 1, \ldots, N \ , \tag{3.7}$$

where the polarization matrices $\boldsymbol{\mathcal{Q}}^{(j)}$ characterize the geometry of $F^{(j)}$, $j = 1, \ldots, N$. Upon solving the above algebraic system, the formal asymptotic approximation of u_N is complete. The next section addresses the solvability of the system (3.7), together with estimates for the vector coefficients $\mathbf{C}^{(j)}$.

4 Algebraic system in the case $\Omega = \mathbb{R}^3$

The algebraic system for the coefficients $\mathbf{C}^{(j)}$ can be written in the form

$$\mathbf{C} + \boldsymbol{\mathcal{S}} \mathbf{Q} \mathbf{C} = -\boldsymbol{\Theta}, \tag{4.1}$$

where

$$\mathbf{C} = ((\mathbf{C}^{(1)})^T, \ldots, (\mathbf{C}^{(N)})^T)^T, \quad \boldsymbol{\Theta} = ((\nabla v(\mathbf{O}^{(1)}))^T, \ldots, (\nabla v(\mathbf{O}^{(N)}))^T)^T,$$

are vectors of the dimension $3N$, and

$$\boldsymbol{\mathcal{S}} = [\mathcal{S}_{ij}]_{i,j=1}^N, \ \mathcal{S}_{ij} = \begin{cases} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(\dfrac{1}{4\pi |\mathbf{z} - \mathbf{w}|} \right) \Big|_{\substack{\mathbf{z}=\mathbf{O}^{(i)} \\ \mathbf{w}=\mathbf{O}^{(j)}}} & \text{if } i \neq j \\ 0 I_3 & \text{otherwise,} \end{cases} \tag{4.2}$$

$$\mathbf{Q} = \operatorname{diag}\{\boldsymbol{\mathcal{Q}}^{(1)}, \ldots, \boldsymbol{\mathcal{Q}}^{(N)}\} \quad \text{is negative definite.} \tag{4.3}$$

These are $3N \times 3N$ matrices whose entries are 3×3 blocks. The notation in (4.2) is interpreted as

$$\mathcal{S}_{ij} = \left\{ \frac{1}{4\pi} \frac{\partial}{\partial z_q} \Big(\frac{z_r - O_r^{(j)}}{|\mathbf{z} - \mathbf{O}^{(j)}|^3} \Big) \Big|_{\mathbf{z}=\mathbf{O}^{(i)}} \right\}_{q,r=1}^3 \quad \text{when } i \neq j.$$

We use the piecewise constant vector function

$$\boldsymbol{\Xi}(\mathbf{x}) = \begin{cases} \boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}, & \text{when } \mathbf{x} \in \overline{B}^{(j)}_{d/4},\ j = 1, \dots, N, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4}$$

where $B^{(j)}_r = \{\mathbf{x} : |\mathbf{x} - \mathbf{O}^{(j)}| < r\}$.

Theorem 2 *Assume that $\lambda_{max} <$ const d^3, where λ_{max} is the largest eigenvalue of the positive definite matrix $-\mathbf{Q}$ and the constant is independent of d. Then the algebraic system* (4.1) *is solvable and the vector coefficients $\mathbf{C}^{(j)}$ satisfy the estimate*

$$\sum_{j=1}^{N} |(\mathbf{C}^{(j)})^T \boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}| \le (1 - \text{const } \frac{\lambda_{max}}{d^3})^{-2} \sum_{j=1}^{N} |(\nabla v(\mathbf{O}^{(j)}))^T \boldsymbol{\mathcal{Q}}^{(j)} \nabla v(\mathbf{O}^{(j)})|. \tag{4.5}$$

We consider the scalar product of (4.1) and the vector $\mathbf{QC}$:

$$\langle \mathbf{C}, \mathbf{QC}\rangle + \langle \boldsymbol{\mathcal{S}}\mathbf{QC}, \mathbf{QC}\rangle = -\langle \boldsymbol{\Theta}, \mathbf{QC}\rangle. \tag{4.6}$$

Prior to the proof of Theorem we formulate and prove the following identity.

Lemma 2 *a) The scalar product $\langle \boldsymbol{\mathcal{S}}\mathbf{QC}, \mathbf{QC}\rangle$ admits the representation*

$$\begin{aligned} \langle \boldsymbol{\mathcal{S}}\mathbf{QC}, \mathbf{QC}\rangle &= \frac{576}{\pi^3 d^6} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{X} - \mathbf{Y}|} (\nabla \cdot \boldsymbol{\Xi}(\mathbf{X}))(\nabla \cdot \boldsymbol{\Xi}(\mathbf{Y})) d\mathbf{Y} d\mathbf{X} \\ &\quad - \frac{16}{\pi d^3} \sum_{j=1}^{N} |\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}|^2. \end{aligned} \tag{4.7}$$

b) The following estimate holds

$$|\langle \boldsymbol{\mathcal{S}}\mathbf{QC}, \mathbf{QC}\rangle| \le \text{const } d^{-3} \sum_{1 \le j \le N} |\mathbf{Q}^{(j)}\mathbf{C}^{(j)}|^2 ,$$

where the constant in the right-hand side does not depend on d.

Remark. Using the notation $\boldsymbol{\mathcal{N}}(\nabla \cdot \boldsymbol{\Xi})$ for the Newton's potential acting on $\nabla \cdot \boldsymbol{\Xi}$ we can interpret the integral in (4.7) as

$$\Big(\boldsymbol{\mathcal{N}}(\nabla \cdot \boldsymbol{\Xi}), \nabla \cdot \boldsymbol{\Xi}\Big)_{L_2(\mathbb{R}^3)},$$

since obviously $\nabla \cdot \boldsymbol{\Xi} \in W^{-1,2}(\mathbb{R}^3)$ and $\boldsymbol{\mathcal{N}}(\nabla \cdot \boldsymbol{\Xi}) \in W^{1,2}(\mathbb{R}^3)$. Here and in the sequel we use the notation (φ, ψ) for the extension of the integral $\int_{\mathbb{R}^3} \varphi(\mathbf{X})\psi(\mathbf{X})d\mathbf{X}$ onto the Cartesian product $W^{1,2}(\mathbb{R}^3) \times W^{-1,2}(\mathbb{R}^3)$.

Proof of Lemma 2. a) By (4.2), (4.3), the following representation holds

$$\langle \boldsymbol{\mathcal{S}}\mathbf{QC}, \mathbf{QC}\rangle = \frac{1}{4\pi} \sum_{j=1}^{N} \Big(\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}\Big)^T \sum_{1 \le k \le N, k \ne j} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(\frac{1}{|\mathbf{z} - \mathbf{w}|}\right)\Big|_{\substack{\mathbf{z}=\mathbf{O}^{(j)} \\ \mathbf{w}=\mathbf{O}^{(k)}}} \Big(\boldsymbol{\mathcal{Q}}^{(k)}\mathbf{C}^{(k)}\Big). \tag{4.8}$$

Using the mean value theorem for harmonic functions we note that when $j \ne k$

$$(\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(\frac{1}{|\mathbf{z} - \mathbf{w}|}\right)\Big|_{\substack{\mathbf{z}=\mathbf{O}^{(j)} \\ \mathbf{w}=\mathbf{O}^{(k)}}} = \frac{3}{4\pi(d/4)^3} \int_{B^{(k)}_{d/4}} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(\frac{1}{|\mathbf{z} - \mathbf{w}|}\right)\Big|_{\mathbf{z}=\mathbf{O}^{(j)}} d\mathbf{w}.$$

Substituting this identity into (4.8) and using definition (4.4) we see that the inner sum on the right-hand side of (4.8) can be presented in the form

$$\frac{48}{\pi d^3} \lim_{\tau \to 0+} \int_{\mathbb{R}^3 \setminus B^{(j)}_{(d/4)-\tau}} \Big\{\frac{\partial}{\partial Y_q}\Big(\frac{Y_r - O^{(j)}_r}{|\mathbf{Y} - \mathbf{O}^{(j)}|^3}\Big)\Big\}^3_{q,r=1} \boldsymbol{\Xi}(\mathbf{Y}) d\mathbf{Y},$$

and further integration by parts gives

$$\begin{aligned}\langle \boldsymbol{\mathcal{S}}\mathbf{QC},\mathbf{QC}\rangle &= -\frac{12}{\pi^2 d^3}\sum_{j=1}^{N}\Big(\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}\Big)^T \\ &\cdot \lim_{\tau\to 0+}\Bigg\{\int_{\mathbb{R}^3\setminus B^{(j)}_{(d/4)-\tau}}\Big\{\frac{Y_r-O_r^{(j)}}{|\mathbf{Y}-\mathbf{O}^{(j)}|^3}\nabla\cdot\boldsymbol{\Xi}(\mathbf{Y})\Big\}_{r=1}^{3}d\mathbf{Y} \\ &+\int_{|\mathbf{Y}-\mathbf{O}^{(j)}|=(d/4)-\tau}\Bigg\{\frac{(Y_r-O_r^{(j)})(Y_q-O_q^{(j)})}{|\mathbf{Y}-\mathbf{O}^{(j)}|^4}\Bigg\}_{r,q=1}^{3}dS_{\mathbf{Y}}\ \boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}\Bigg\},\end{aligned} \tag{4.9}$$

where the integral over $\mathbb{R}^3\setminus B_{(d/4)-\tau}$ in (4.9) is understood in the sense of distributions. The surface integral in (4.9) can be evaluated explicitly, i.e.

$$\int_{|\mathbf{Y}-\mathbf{O}^{(j)}|=(d/4)-\tau}\Bigg\{\frac{(Y_r-O_r^{(j)})(Y_q-O_q^{(j)})}{|\mathbf{Y}-\mathbf{O}^{(j)}|^4}\Bigg\}_{r,q=1}^{3}dS_{\mathbf{Y}}\ \boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}=\frac{4\pi}{3}\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}. \tag{4.10}$$

Once again, applying the mean value theorem for harmonic functions in the outer sum of (4.9) and using (4.10) together with the definition (4.4) we arrive at

$$\begin{aligned}\langle \boldsymbol{\mathcal{S}}\mathbf{QC},\mathbf{QC}\rangle &= -\frac{16}{\pi d^3}\sum_{j=1}^{N}|\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}|^2 \\ -\frac{576}{\pi^3 d^6}&\lim_{\tau\to 0+}\sum_{j=1}^{N}\int_{B^{(j)}_{(d/4)+\tau}}\int_{\mathbb{R}^3\setminus B^{(j)}_{(d/4)-\tau}}\sum_{r=1}^{3}\Xi_r(\mathbf{X})\frac{\partial}{\partial X_r}\Big(\frac{1}{|\mathbf{Y}-\mathbf{X}|}\Big)\nabla\cdot\boldsymbol{\Xi}(\mathbf{Y})d\mathbf{Y}d\mathbf{X},\end{aligned} \tag{4.11}$$

where Ξ_r are the components of the vector function $\boldsymbol{\Xi}$ defined in (4.4).

The last integral is understood in the sense of distributions. Referring to the definition (4.4), integrating by parts, and taking the limit as $\tau\to 0+$ we deduce that the integral term in (4.11) can be written as

$$\frac{576}{\pi^3 d^6}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{Y}-\mathbf{X}|}\Big(\nabla\cdot\boldsymbol{\Xi}(\mathbf{X})\Big)\Big(\nabla\cdot\boldsymbol{\Xi}(\mathbf{Y})\Big)d\mathbf{Y}d\mathbf{X} \tag{4.12}$$

Using (4.11) and (4.12) we arrive at (4.7).

b) Let us introduce a piece-wise constant function

$$\boldsymbol{\mathcal{C}}(\mathbf{x})=\begin{cases}\mathbf{C}^{(j)}\ , & \text{when } \mathbf{x}\in\overline{B^{(j)}_{d/4}}\ ,\quad j=1,\dots,N\ ,\\ 0\ , & \text{otherwise}\ .\end{cases}$$

According to the system (3.7), $\nabla\times\boldsymbol{\mathcal{C}}(\mathbf{x})=\mathbf{O}$, and one can use the representation

$$\boldsymbol{\mathcal{C}}(\mathbf{x})=\nabla W(\mathbf{x}) \tag{4.13}$$

where W is a scalar function with compact support, and (4.13) is understood in the sense of distributions. We give a proof for the case when all voids are spherical, of diameter ε, and hence $\boldsymbol{\mathcal{Q}}^{(j)}=-\frac{\pi}{4}\varepsilon^3 I_3$, where I_3 is the identity matrix. Then according to (4.11) we have

$$\begin{aligned}|\langle \boldsymbol{\mathcal{S}}\mathbf{QC},\mathbf{QC}\rangle| &\le \frac{16}{\pi d^3}\sum_{1\le j\le N}|\mathbf{Q}^{(j)}\mathbf{C}^{(j)}|^2+\frac{36\varepsilon^6}{\pi d^3}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\nabla_{\mathbf{X}}W(\mathbf{X})\cdot\nabla_{\mathbf{X}}\Big(\frac{1}{|\mathbf{Y}-\mathbf{X}|}\Big)\Big)\Delta_{\mathbf{Y}}W(\mathbf{Y})\,d\mathbf{Y}d\mathbf{X}\Big| \\ &\le \frac{16}{\pi d^3}\sum_{1\le j\le N}|\mathbf{Q}^{(j)}\mathbf{C}^{(j)}|^2+\frac{144\varepsilon^6}{d^3}\sum_{1\le j\le N}\int_{B^{(j)}_{d/4}}|\nabla W(\mathbf{Y})|^2\,d\mathbf{Y} \\ &\le \frac{\text{const}}{d^3}\sum_{1\le j\le N}|\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}|^2\ .\end{aligned}$$

□

Proof of Theorem 2. Consider the equation (4.6). The absolute value of its right-hand side does not exceed

$$\langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle^{1/2}\langle \mathbf{\Theta}, -\mathbf{Q}\mathbf{\Theta}\rangle^{1/2}.$$

Using Lemma 1 and part b) of Lemma 2 we derive

$$\langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle - \mathrm{const} d^{-3}\langle -\mathbf{Q}\mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle \le \langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle^{1/2}\langle \mathbf{\Theta}, -\mathbf{Q}\mathbf{\Theta}\rangle^{1/2},$$

leading to

$$\Big(1 - \frac{\mathrm{const}}{d^3}\frac{\langle -\mathbf{Q}\mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle}{\langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle}\Big)\langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle^{1/2} \le \langle \mathbf{\Theta}, -\mathbf{Q}\mathbf{\Theta}\rangle^{1/2},$$

which implies

$$\Big(1 - \mathrm{const}\frac{\lambda_{max}}{d^3}\Big)^2\langle \mathbf{C}, -\mathbf{Q}\mathbf{C}\rangle \le \langle \mathbf{\Theta}, -\mathbf{Q}\mathbf{\Theta}\rangle. \tag{4.14}$$

The proof is complete. □

Assuming that the eigenvalues of the matrices $-\boldsymbol{\mathcal{Q}}^{(j)}$ are strictly positive and satisfy the inequality (1.7), we also find that Theorem 2 yields

Corollary 1 *Assume that the inequalities* (1.7) *hold for* λ_{max} *and* λ_{min}*. Then the vector coefficients* $\mathbf{C}^{(j)}$ *in the system* (4.1) *satisfy the estimate*

$$\sum_{1\le j\le N}|\mathbf{C}^{(j)}|^2 \le \mathrm{const}\ d^{-3}\|\nabla v\|^2_{L^2(\omega)}, \tag{4.15}$$

where the constant depends only on the coefficients A_1 *and* A_2 *in* (1.7)*.*

Proof. According to the inequality (4.5) of Theorem 2 we deduce

$$\lambda_{min}\sum_{1\le j\le N}|\mathbf{C}^{(j)}|^2 \le (1 - \frac{\mathrm{const}}{d^3}\lambda_{max})^{-2}\lambda_{max}\sum_{1\le j\le N}|\nabla v(\mathbf{O}^{(j)})|^2. \tag{4.16}$$

We note that v is harmonic in a neighbourhood of $\overline{\omega}$. Applying the mean value theorem for harmonic functions together with the Cauchy inequality we write

$$|\nabla v(\mathbf{O}^{(j)})|^2 \le \frac{48}{\pi d^3}\|\nabla v\|^2_{L_2(B^{(j)}_{d/4})}.$$

Hence, it follows from (4.16) that

$$\begin{aligned}\sum_{1\le j\le N}|\mathbf{C}^{(j)}|^2 &\le d^{-3}(1 - \frac{\mathrm{const}}{d^3}\lambda_{max})^{-2}\frac{48}{\pi}\frac{\lambda_{max}}{\lambda_{min}}\sum_{1\le j\le N}\|\nabla v\|^2_{L_2(B^{(j)}_{d/4})}\\ &\le d^{-3}\left((1 - \frac{\mathrm{const}}{d^3}\lambda_{max})^{-2}\frac{48}{\pi}\frac{\lambda_{max}}{\lambda_{min}}\right)\|\nabla v\|^2_{L_2(\omega)},\end{aligned} \tag{4.17}$$

which is the required estimate (4.15). □

5 Energy error estimate in the case $\Omega = \mathbb{R}^3$

In this section we prove the result concerning the asymptotic approximation of u_N for the perforated domain $\Omega_N = \mathbb{R}^3\backslash\overline{\cup_{j=1}^N F^{(j)}}$. The changes in the argument, necessary for the treatment of a general domain, will be described in Section 6.

Proof of Theorem 1. *a) Neumann problem for the remainder.* The remainder term $\mathcal{R}_N$ in (1.8) is a harmonic function in Ω_N, which vanishes at infinity and satisfies the boundary conditions

$$\frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x}) = -\Big(\nabla v(\mathbf{x}) + \mathbf{C}^{(j)}\Big)\cdot \boldsymbol{n}^{(j)}(\mathbf{x}) - \sum_{\substack{k\neq j\\ 1\le k\le N}} \mathbf{C}^{(k)}\cdot \frac{\partial}{\partial n}\boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}), \quad \text{when} \quad \mathbf{x}\in \partial F^{(j)}, j=1,\dots,N. \tag{5.1}$$

Since supp f is separated from $F^{(j)}, j=1,\dots,N$, and since $\boldsymbol{\mathcal{D}}^{(j)},\ j=1,\dots,N$, satisfy (2.5) we have

$$\int_{\partial F^{(j)}} \frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x})dS_{\mathbf{x}} = 0,\ j=1,\dots,N. \tag{5.2}$$

b) Auxiliary functions. Throughout the proof we use the notation $B_\rho^{(k)} = \{\mathbf{x} : |\mathbf{x}-\mathbf{O}^{(k)}| < \rho\}$. We introduce auxiliary functions which will help us to obtain (1.10). Let

$$\begin{aligned}\Psi_k(\mathbf{x}) \,=\, & v(\mathbf{x}) - v(\mathbf{O}^{(k)}) - (\mathbf{x}-\mathbf{O}^{(k)})\cdot\nabla v(\mathbf{O}^{(k)}) + \sum_{\substack{1\le j\le N\\ j\neq k}} \boldsymbol{C}^{(j)}\cdot \boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x})\\ & - \sum_{\substack{1\le j\le N\\ j\neq k}} (\mathbf{x}-\mathbf{O}^{(j)})\cdot T(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\boldsymbol{\mathcal{Q}}^{(j)}\boldsymbol{C}^{(j)}\ ,\end{aligned} \tag{5.3}$$

for all $\mathbf{x}\in\Omega_N$ and $k=1,\dots,N$. Every function Ψ_k satisfies

$$-\Delta\Psi_k(\mathbf{x}) = f(\mathbf{x})\ , \quad \mathbf{x}\in\Omega_N\ , \tag{5.4}$$

and since $\omega\cap \text{supp } f = \varnothing$, we see that $\Psi_k,\ k=1,\dots,N$, are harmonic in ω. Since the coefficients $\mathbf{C}^{(j)}$ satisfy system (4.1), we obtain

$$\frac{\partial \Psi_k}{\partial n}(\mathbf{x}) + \frac{\partial \mathcal{R}_N}{\partial n}(\mathbf{x}) = 0\ , \quad \mathbf{x}\in\partial F^{(k)}\ . \tag{5.5}$$

and according to (5.2) the functions Ψ_k have zero flux through the boundaries of small voids $F^{(k)}$, i.e.

$$\int_{\partial F^{(k)}} \frac{\partial \Psi_k}{\partial n}(\mathbf{x})\, d\mathbf{x} = 0\ , k=1,\dots,N\ . \tag{5.6}$$

Next, we introduce smooth cutoff functions

$$\chi_\varepsilon^{(k)} : \mathbf{x}\to \chi((\mathbf{x}-\mathbf{O}^{(k)})/\varepsilon),\ k=1,\dots,N,$$

equal to 1 on $B_{2\varepsilon}^{(k)}$ and vanishing outside $B_{3\varepsilon}^{(k)}$. Then by (5.5) we have

$$\frac{\partial}{\partial n}\Big(\mathcal{R}_N(\mathbf{x}) + \sum_{1\le k\le N}\chi_\varepsilon^{(k)}(\mathbf{x})\Psi_k(\mathbf{x})\Big) = 0 \ \text{ on } \ \partial F^{(j)},\ j=1,\dots,N. \tag{5.7}$$

c) Estimate of the energy integral of $\mathcal{R}_N$ in terms of Ψ_k. Integrating by parts in Ω_N and using the definition of $\chi_\varepsilon^{(k)}$, we write the identity

$$\int_{\Omega_N} \nabla\mathcal{R}_N\cdot\nabla\Big(\mathcal{R}_N + \sum_{1\le k\le N}\chi_\varepsilon^{(k)}\Psi_k\Big)\,d\mathbf{x} = -\int_{\Omega_N}\mathcal{R}_N\Delta\Big(\mathcal{R}_N + \sum_{1\le k\le N}\chi_\varepsilon^{(k)}\Psi_k\Big)\,d\mathbf{x}, \tag{5.8}$$

which is equivalent to

$$\int_{\Omega_N}\big|\nabla\mathcal{R}_N\big|^2 d\mathbf{x} + \sum_{1\le k\le N}\int_{B_{3\varepsilon}^{(k)}\setminus\overline{F}^{(k)}}\nabla\mathcal{R}_N\cdot\nabla\big(\chi_\varepsilon^{(k)}\Psi_k\big)\,d\mathbf{x} = -\sum_{1\le k\le N}\int_{B_{3\varepsilon}^{(k)}\setminus\overline{F}^{(k)}}\mathcal{R}_N\Delta\big(\chi_\varepsilon^{(k)}\Psi_k\big)\,d\mathbf{x}, \tag{5.9}$$

since $\mathcal{R}_N$ is harmonic in Ω_N.

We preserve the notation $\mathcal{R}_N$ for an extension of $\mathcal{R}_N$ onto the union of voids $F^{(k)}$ with preservation of the class $W^{1,2}$. Such an extension can be constructed by using only values of $\mathcal{R}_N$ on the sets $B_{2\varepsilon}^{(k)}\setminus\overline{F}^{(k)}$ in such a way that

$$\|\nabla\mathcal{R}_N\|_{L^2(B_{2\varepsilon}^{(k)})} \le \text{const}\|\nabla\mathcal{R}_N\|_{L^2(B_{2\varepsilon}^{(k)}\setminus\overline{F}^{(k)})}. \tag{5.10}$$

The above fact follows by dilation $\mathbf{x} \to \mathbf{x}/\varepsilon$ from the well-known extension theorem for domains with Lipschitz boundaries (see Section 3 of Chapter 6 in [10]). We shall use the notation $\overline{\mathcal{R}}^{(k)}$ for the mean value of $\mathcal{R}_N$ on $B^{(k)}_{3\varepsilon}$.

The integral on the right-hand side of (5.9) can be written as

$$-\sum_{1\le k\le N}\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}\mathcal{R}_N\Delta\big(\chi^{(k)}_\varepsilon\Psi_k\big)\,d\mathbf{x}=-\sum_{1\le k\le N}\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}(\mathcal{R}_N-\overline{\mathcal{R}}^{(k)})\Delta\big(\chi^{(k)}_\varepsilon\Psi_k\big)\,d\mathbf{x}, \tag{5.11}$$

In the derivation of (5.11) we have used that

$$\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}\Delta\Big(\chi^{(k)}_\varepsilon\Psi_k\Big)\,d\mathbf{x}=\int_{\partial F^{(k)}}\frac{\partial\Psi_k}{\partial n}dS_{\mathbf{x}}=0 \tag{5.12}$$

according to (5.6) and the definition of $\chi^{(k)}_\varepsilon$.

Owing to (5.8) and (5.11), we can write

$$\|\nabla\mathcal{R}_N\|^2_{L^2(\Omega_N)}\ \le\Sigma_1+\Sigma_2, \tag{5.13}$$

where

$$\Sigma_1=\sum_{1\le k\le N}\Big|\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}\nabla\mathcal{R}_N\cdot\nabla\big(\chi^{(k)}_\varepsilon\Psi_k\big)d\mathbf{x}\Big|, \tag{5.14}$$

and

$$\Sigma_2=\sum_{1\le k\le N}\Big|\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}(\mathcal{R}_N-\overline{\mathcal{R}}^{(k)})\Delta\big(\chi^{(k)}_\varepsilon(\Psi_k-\overline{\Psi}_k)\big)\,d\mathbf{x}\Big|, \tag{5.15}$$

where $\overline{\Psi}_k$ is the mean value of Ψ_k over the ball $B^{(k)}_{3\varepsilon}$. Here, we have taken into account that by harmonicity of $\mathcal{R}_N$, (5.2) and definition of $\chi^{(k)}_\varepsilon$

$$\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}\Delta\Big(\mathcal{R}_N-\overline{\mathcal{R}}^{(k)}\Big)\chi^{(k)}_\varepsilon d\mathbf{x}=\int_{B^{(k)}_{3\varepsilon}}\Delta\Big(\mathcal{R}_N-\overline{\mathcal{R}}^{(k)}\Big)\chi^{(k)}_\varepsilon d\mathbf{x}=0.$$

By the Cauchy inequality, the first sum in (5.13) allows for the estimate

$$\Sigma_1\le\Big(\sum_{1\le k\le N}\|\nabla\mathcal{R}_N\|^2_{L^2(B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)})}\Big)^{1/2}\Big(\sum_{1\le k\le N}\Big\|\nabla\big(\chi^{(k)}_\varepsilon\Psi_k\big)\Big\|^2_{L^2(B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)})}\Big)^{1/2}. \tag{5.16}$$

Furthermore, using the inequality

$$\sum_{1\le k\le N}\|\nabla\mathcal{R}_N\|^2_{L^2(B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)})}\le\|\nabla\mathcal{R}_N\|^2_{L^2(\Omega_N)}, \tag{5.17}$$

together with (5.16), we deduce

$$\Sigma_1\le\|\nabla\mathcal{R}_N\|_{L^2(\Omega_N)}\Big(\sum_{1\le k\le N}\big\|\nabla\big(\chi^{(k)}_\varepsilon\Psi_k\big)\big\|^2_{L^2(B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)})}\Big)^{1/2}. \tag{5.18}$$

Similarly to (5.16), the second sum in (5.13) can be estimated as

$$\Sigma_2\le\sum_{1\le k\le N}\Big(\int_{B^{(k)}_{3\varepsilon}}(\mathcal{R}_N-\overline{\mathcal{R}}^{(k)})^2d\mathbf{x}\Big)^{1/2}\Big(\int_{B^{(k)}_{3\varepsilon}\setminus\overline{F}^{(k)}}\big(\Delta(\chi^{(k)}_\varepsilon(\Psi_k-\overline{\Psi}_k))\big)^2\,d\mathbf{x}\Big)^{1/2}. \tag{5.19}$$

By the Poincaré inequality for the ball $B^{(k)}_{3\varepsilon}$

$$\|\mathcal{R}_N-\overline{\mathcal{R}}^{(k)}\|^2_{L^2(B^{(k)}_{3\varepsilon})}\le\text{const }\varepsilon^2\|\nabla\mathcal{R}_N\|^2_{L^2(B^{(k)}_{3\varepsilon})} \tag{5.20}$$

we obtain

$$\Sigma_2 \le \text{const } \varepsilon \Big(\sum_{1\le k\le N} \|\nabla \mathcal{R}_N\|^2_{L^2(B^{(k)}_{3\varepsilon})}\Big)^{1/2} \Big(\sum_{1\le k\le N} \int_{B^{(k)}_{3\varepsilon}\setminus \overline{F}^{(k)}} \big(\Delta(\chi^{(k)}_\varepsilon(\Psi_k - \overline{\Psi}_k))\big)^2\, d\mathbf{x}\Big)^{1/2},$$

which does not exceed

$$\text{const } \varepsilon \, \|\nabla \mathcal{R}_N\|_{L^2(\Omega_N)} \Big(\sum_{1\le k\le N} \int_{B^{(k)}_{3\varepsilon}\setminus \overline{F}^{(k)}} \big(\Delta\big(\chi^{(k)}_\varepsilon(\Psi_k - \overline{\Psi}_k)\big)\big)^2\, d\mathbf{x}\Big)^{1/2}, \tag{5.21}$$

because of (5.10). Combining (5.13)–(5.21) and dividing both sides of (5.13) by $\|\nabla \mathcal{R}_N\|_{L^2(\Omega_N)}$ we arrive at

$$\begin{aligned} \|\nabla \mathcal{R}_N\|_{L^2(\Omega_N)} \ &\le \Big(\sum_{1\le k\le N} \Big\|\nabla\big(\chi^{(k)}_\varepsilon(\Psi_k - \overline{\Psi}_k)\big)\Big\|^2_{L^2(B^{(k)}_{3\varepsilon})}\Big)^{1/2} \\ &\quad + \text{const } \varepsilon \Big(\sum_{1\le k\le N} \int_{B^{(k)}_{3\varepsilon}} \big\{(\Psi_k - \overline{\Psi}_k)\Delta\chi^{(k)}_\varepsilon + 2\nabla\chi^{(k)}_\varepsilon \cdot \nabla\Psi_k\big\}^2\, d\mathbf{x}\Big)^{1/2}, \end{aligned} \tag{5.22}$$

which leads to

$$\|\nabla \mathcal{R}_N\|^2_{L^2(\Omega_N)} \le \text{const} \sum_{1\le k\le N} \Big(\|\nabla\Psi_k\|^2_{L^2(B^{(k)}_{3\varepsilon})} + \varepsilon^{-2}\|\Psi_k - \overline{\Psi}_k\|^2_{L^2(B^{(k)}_{3\varepsilon})}\Big). \tag{5.23}$$

Applying the Poincaré inequality (see (5.20)) for Ψ_k in the ball $B^{(k)}_{3\varepsilon}$ and using (5.23), we deduce

$$\|\nabla \mathcal{R}_N\|^2_{L^2(\Omega_N)} \le \text{const} \sum_{1\le k\le N} \|\nabla\Psi_k\|^2_{L^2(B^{(k)}_{3\varepsilon})}. \tag{5.24}$$

d) Final energy estimate. Here we prove the inequality (1.10). Using definition (5.3) of Ψ_k, $k = 1, \dots, N$, we can replace the preceding inequality by

$$\|\nabla \mathcal{R}_N\|^2_{L^2(\Omega_N)} \le \text{const } \big\{\mathcal{K} + \mathcal{L}\big\}\,, \tag{5.25}$$

where

$$\begin{aligned} \mathcal{K} &= \sum_{1\le k\le N} \|\nabla v(\cdot) - \nabla v(\mathbf{O}^{(k)})\|^2_{L^2(B^{(k)}_{3\varepsilon})}\,, \\ \mathcal{L} &= \sum_{1\le k\le N} \Big\| \sum_{\substack{j\ne k\\ 1\le j\le N}} \Big[\nabla\Big(\boldsymbol{C}^{(j)} \cdot \boldsymbol{\mathcal{D}}^{(j)}(\cdot)\Big) - T(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\boldsymbol{\mathcal{Q}}^{(j)}\boldsymbol{C}^{(j)}\Big]\Big\|^2_{L^2(B^{(k)}_{3\varepsilon})}. \end{aligned} \tag{5.26}$$

The estimate for $\mathcal{K}$ is straightforward and it follows by Taylor's expansions of v in the vicinity of $\mathbf{O}^{(k)}$,

$$\mathcal{K} \le \text{const } \varepsilon^5 d^{-3} \max_{\mathbf{x}\in\overline{\omega}, 1\le i,j\le 3} \Big|\frac{\partial^2 v}{\partial x_i \partial x_j}\Big|^2. \tag{5.27}$$

Since v is harmonic in a neighbourhood of $\overline{\omega}$, we obtain by the local regularity property of harmonic functions that

$$\mathcal{K} \le \text{const } \varepsilon^5 d^{-3} \big\|\nabla v\big\|^2_{L^2(\mathbb{R}^3)}. \tag{5.28}$$

To estimate $\mathcal{L}$, we use Lemma 1 on the asymptotics of the dipole fields together with the definition (3.6) of the matrix function T, which lead to

$$|\nabla(\boldsymbol{C}^{(j)} \cdot \boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x})) - T(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\boldsymbol{\mathcal{Q}}^{(j)}\boldsymbol{C}^{(j)}| \le \text{const } \varepsilon^4 |\boldsymbol{C}^{(j)}||\mathbf{x} - \mathbf{O}^{(j)}|^{-4}\,, \tag{5.29}$$

for $\mathbf{x} \in B^{(k)}_{3\varepsilon}$. Now, it follows from (5.26) and (5.29) that

$$\mathcal{L} \ \le \ \text{const } \varepsilon^8 \sum_{k=1}^{N} \int_{B^{(k)}_{3\varepsilon}} \Big(\sum_{1\le j\le N, j\ne k} \frac{|\mathbf{C}^{(j)}|}{|\mathbf{x} - \mathbf{O}^{(j)}|^4}\Big)^2 d\mathbf{x}, \tag{5.30}$$

and by the Cauchy inequality the right-hand side does not exceed

$$\text{const}\ \ \varepsilon^8 \sum_{p=1}^{N} |\mathbf{C}^{(p)}|^2 \sum_{k=1}^{N} \sum_{1\le j\le N, j\ne k} \int_{B_{3\varepsilon}^{(k)}} \frac{d\mathbf{x}}{|\mathbf{x}-\mathbf{O}^{(j)}|^8} \le \text{const}\ \varepsilon^{11} \sum_{p=1}^{N} |\mathbf{C}^{(p)}|^2 \sum_{k=1}^{N} \sum_{1\le j\le N, j\ne k} \frac{1}{|\mathbf{O}^{(k)}-\mathbf{O}^{(j)}|^8}$$

$$\le \text{const}\ \frac{\varepsilon^{11}}{d^6} \sum_{p=1}^{N} |\mathbf{C}^{(p)}|^2 \int\int_{\{\omega\times\omega:|\mathbf{X}-\mathbf{Y}|>d\}} \frac{d\mathbf{X}d\mathbf{Y}}{|\mathbf{X}-\mathbf{Y}|^8} \le \text{const}\ \frac{\varepsilon^{11}}{d^8} \sum_{p=1}^{N} |\mathbf{C}^{(p)}|^2. \tag{5.31}$$

Since the eigenvalues of the matrix $-\mathbf{Q}$ satisfy the constraint (1.7), we can apply Corollary 1 and use the estimate (4.15) for the right-hand side of (5.31) to obtain

$$\mathcal{L} \le \text{const}\ \varepsilon^{11} d^{-11} \|\nabla v\|^2_{L^2(\omega)}. \tag{5.32}$$

Combining (5.25), (5.28) and (5.32), we arrive at (1.10) and complete the proof. □

6 Approximation of u_N for a perforated domain

Now we seek an approximation of the solution u_N to the problem (1.3)–(1.5) assuming that Ω is an arbitrary domain in $\mathbb{R}^3$. We first describe the formal asymptotic algorithm and derive a system of algebraic equations, similar to (4.1), which is used for evaluation of the coefficients in the asymptotic representation of u_N.

6.1 Formal asymptotic algorithm for the perforated domain Ω_N

The solution $u_N \in L^{1,2}(\Omega_N)$ of (1.3)–(1.5) is sought in the form

$$u_N(\mathbf{x}) = v(\mathbf{x}) + \sum_{k=1}^{N} \boldsymbol{C}^{(k)} \cdot \Big\{ \boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) - \boldsymbol{\mathcal{Q}}^{(k)} \nabla_{\mathbf{y}} H(\mathbf{x},\mathbf{y})\big|_{\mathbf{y}=\mathbf{O}^{(k)}} \Big\} + R_N(\mathbf{x}) \ , \tag{6.1}$$

where in this instance v solves problem (2.2), (2.3) in Section 2, and R_N is a harmonic function in Ω_N. Here $\boldsymbol{C}^{(k)}$, $k = 1,\dots,N$ are the vector coefficients to be determined.

Owing to the definitions of $\boldsymbol{\mathcal{D}}^{(k)}$, $k = 1,\dots,N$, and H as solutions of Problems 2 and 3 in Section 2, and taking into account Lemma 1 on the asymptotics of $\boldsymbol{\mathcal{D}}^{(k)}$ we deduce that $|R_N(\mathbf{x})|$ is small for $\mathbf{x} \in \partial\Omega$.

On the boundaries $\partial F^{(j)}$, the substitution of (6.1) into (1.5) yields

$$\begin{aligned} \frac{\partial R_N}{\partial n}(\mathbf{x}) \ = \ & -\boldsymbol{n}^{(j)} \cdot \Big\{ \nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + O(\varepsilon) + O(\varepsilon^3 |\boldsymbol{C}^{(j)}|) \\ & + \sum_{\substack{k\ne j \\ 1\le k\le N}} \nabla \Big\{ \boldsymbol{C}^{(k)} \cdot \Big(\boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) - \boldsymbol{\mathcal{Q}}^{(k)} \nabla_{\mathbf{y}} H(\mathbf{x},\mathbf{y})\big|_{\mathbf{y}=\mathbf{O}^{(k)}} \Big) \Big\} \Big\} \ , \quad \mathbf{x} \in \partial F^{(j)}, j = 1,\dots,N \ . \end{aligned}$$

Then, using the asymptotic representation (2.6) in Lemma 1 we deduce

$$\frac{\partial R_N}{\partial n}(\mathbf{x}) \sim -\boldsymbol{n}^{(j)} \cdot \Big\{ \nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + \sum_{\substack{k\ne j \\ 1\le k\le N}} \mathfrak{T}(\mathbf{x},\mathbf{O}^{(k)}) \boldsymbol{\mathcal{Q}}^{(k)} \boldsymbol{C}^{(k)} \Big\} \ , \quad \mathbf{x} \in \partial F^{(j)}, j = 1,\dots,N \ , \tag{6.2}$$

where $\mathfrak{T}(\mathbf{x},\mathbf{y})$ is defined by

$$\mathfrak{T}(\mathbf{x},\mathbf{y}) = (\nabla_{\mathbf{x}} \otimes \nabla_{\mathbf{y}}) G(\mathbf{x},\mathbf{y}) \ , \tag{6.3}$$

with $G(\mathbf{x},\mathbf{y})$ being Green's function for the domain Ω, as defined in Section 2. To compensate for the leading discrepancy in the boundary conditions (6.2), we choose the coefficients $\boldsymbol{C}^{(m)}$, $m = 1,\dots,N$, subject to the algebraic system

$$\nabla v(\mathbf{O}^{(j)}) + \boldsymbol{C}^{(j)} + \sum_{\substack{k\ne j \\ 1\le k\le N}} \mathfrak{T}(\mathbf{O}^{(j)},\mathbf{O}^{(k)}) \boldsymbol{\mathcal{Q}}^{(k)} \boldsymbol{C}^{(k)} = 0, \quad j = 1,\dots,N, \tag{6.4}$$

where $\boldsymbol{\mathcal{Q}}^{(k)}, k = 1,\dots,N$, are polarization matrices of small voids $F^{(k)}$, as in Lemma 1.

Provided system (6.4) has been solved for the vector coefficients $\mathbf{C}^{(k)}$, formula (6.1) leads to the formal asymptotic approximation of u_N:

$$u_N(\mathbf{x}) \sim v(\mathbf{x}) + \sum_{k=1}^{N} \boldsymbol{C}^{(k)} \cdot \Big\{ \boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) - \boldsymbol{\mathcal{Q}}^{(k)} \nabla_{\mathbf{y}} H(\mathbf{x},\mathbf{y})\big|_{\mathbf{y}=\mathbf{O}^{(k)}} \Big\}. \tag{6.5}$$

6.2 Algebraic system

The system (6.4) can be written in the matrix form

$$\mathbf{C} + \mathfrak{S}\mathbf{Q}\mathbf{C} = -\boldsymbol{\Theta}, \tag{6.6}$$

where

$$\mathfrak{S} = [\mathfrak{S}_{ij}]_{i,j=1}^{N}, \ \mathfrak{S}_{ij} = \begin{cases} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) G(\mathbf{z},\mathbf{w})\Big|_{\substack{\mathbf{z}=\mathbf{O}^{(i)} \\ \mathbf{w}=\mathbf{O}^{(j)}}} & \text{if } i \neq j \\ 0 I_3 & \text{otherwise} \end{cases} \tag{6.7}$$

with $G(\mathbf{z},\mathbf{w})$ standing for Green's function in the limit domain Ω, and the block-diagonal matrix $\mathbf{Q}$ being the same as in (1.6). The system (6.6) is similar to that in Section 4, with the only change of the matrix $\boldsymbol{\mathcal{S}}$ for $\mathfrak{S}$. The elements of $\mathfrak{S}$ are given via the second-order derivatives of Green's function in Ω, as defined in (6.3). The next assertion is similar to Corollary 1.

Lemma 3 *Assume that inequalities* (1.7) *hold for* λ_{max} *and* λ_{min}. *Also let* v *be a unique solution of problem* (2.2), (2.3) *in the domain* Ω. *Then the vector coefficients* $\mathbf{C}^{(j)}$ *in the system* (6.4) *satisfy the estimate*

$$\sum_{1\le j\le N} |\mathbf{C}^{(j)}|^2 \le \text{const } d^{-3} \|\nabla v\|^2_{L^2(\Omega)}, \tag{6.8}$$

where the constant depends on the shape of the voids $F^{(j)}, j = 1,\dots,N$.

Proof. The proof of the theorem is very similar to the one given in Section 4. We consider the scalar product of (6.6) and the vector $\mathbf{QC}$:

$$\langle \mathbf{C}, \mathbf{QC}\rangle + \langle \mathfrak{S}\mathbf{QC}, \mathbf{QC}\rangle = -\langle \boldsymbol{\Theta}, \mathbf{QC}\rangle, \tag{6.9}$$

and similarly to (4.7) derive

$$\begin{aligned} \langle \mathfrak{S}\mathbf{QC}, \mathbf{QC}\rangle &= 48^2\, \pi^{-2}\, d^{-6} \int_\Omega \int_\Omega G(\mathbf{X},\mathbf{Y}) (\nabla\cdot \boldsymbol{\Xi}(\mathbf{X}))(\nabla\cdot\boldsymbol{\Xi}(\mathbf{Y})) d\mathbf{Y} d\mathbf{X} \\ &\quad -16\pi^{-1} d^{-3} \sum_{1\le j\le N} |\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}|^2 \\ &\quad + \sum_{1\le j\le N} \Big(\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}\Big)^T (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \left(H(\mathbf{z},\mathbf{w})\right)\Big|_{\substack{\mathbf{z}=\mathbf{O}^{(j)} \\ \mathbf{w}=\mathbf{O}^{(j)}}} \Big(\boldsymbol{\mathcal{Q}}^{(j)}\mathbf{C}^{(j)}\Big), \end{aligned} \tag{6.10}$$

where the integral in the right-hand side is positive, and it is understood in the sense of distributions, in the same way as in the proof of Lemma 2, while the magnitude of the last sum in (6.10) is small compared to the magnitude of the second sum.

Now, the right-hand side in (6.9) does not exceed

$$\langle \mathbf{C}, -\mathbf{QC}\rangle^{1/2} \langle \boldsymbol{\Theta}, -\mathbf{Q}\boldsymbol{\Theta}\rangle^{1/2}.$$

Following the same pattern as in the proof of Theorem 2, we deduce

$$\langle \mathbf{C}, -\mathbf{QC}\rangle - \text{const } d^{-3} \langle -\mathbf{QC}, -\mathbf{QC}\rangle \le \langle \mathbf{C}, -\mathbf{QC}\rangle^{1/2} \langle \boldsymbol{\Theta}, -\mathbf{Q}\boldsymbol{\Theta}\rangle^{1/2},$$

where the constant is independent of d. Furthermore, this leads to

$$\Big(1 - \text{const } d^{-3} \frac{\langle -\mathbf{QC}, -\mathbf{QC}\rangle}{\langle \mathbf{C}, -\mathbf{QC}\rangle}\Big) \langle \mathbf{C}, -\mathbf{QC}\rangle^{1/2} \le \langle \boldsymbol{\Theta}, -\mathbf{Q}\boldsymbol{\Theta}\rangle^{1/2},$$

which implies

$$\Big(1-\text{const}\ d^{-3}\lambda_{max}\Big)^2\langle\mathbf{C},-\mathbf{Q}\mathbf{C}\rangle\le\langle\mathbf{\Theta},-\mathbf{Q}\mathbf{\Theta}\rangle, \tag{6.11}$$

where λ_{max} is the largest eigenvalue of the positive definite matrix $-\mathbf{Q}$. Then using the same estimates (4.16) and (4.17) as in the proof of Corollary 1 we arrive at (6.8). □

6.3 Energy estimate for the remainder

Theorem 3 *Let the parameters ε and d satisfy the inequality*

$$\varepsilon < c\,d\ ,$$

where c is a sufficiently small absolute constant. Then the solution $u_N(\mathbf{x})$ of (1.3)–(1.5) is represented by the asymptotic formula

$$u_N(\mathbf{x})=v(\mathbf{x})+\sum_{k=1}^{N}\boldsymbol{C}^{(k)}\cdot\{\boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x})-\boldsymbol{\mathcal{Q}}^{(k)}\nabla_{\mathbf{y}}H(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=\mathbf{O}^{(k)}}\}+R_N(\mathbf{x})\ , \tag{6.12}$$

where $\boldsymbol{C}^{(k)}=(C_1^{(k)},C_2^{(k)},C_3^{(k)})^T$ solve the linear algebraic system (6.4). The remainder R_N in (6.12) satisfies the energy estimate

$$\|\nabla R_N\|^2_{L_2(\Omega_N)}\le\text{const}\ \Big\{\varepsilon^{11}d^{-11}+\varepsilon^5d^{-3}\Big\}\|\nabla v\|^2_{L_2(\Omega)}\ . \tag{6.13}$$

Proof. Essentially, the proof follows the same steps as in Theorem 1. Thus, we give an outline indicating the obvious modifications, which are brought by the boundary $\partial\Omega$.

a) Auxiliary functions. Let us preserve the notations $\chi_\varepsilon^{(k)}$ for cutoff functions used in the proof of Theorem 1. We also need a new cutoff function χ_0 to isolate $\partial\Omega$ from the cloud of holes. Namely, let $(1-\chi_0)\in C_0^\infty(\Omega)$ and $\chi_0=0$ on a neighbourhood of $\overline{\omega}$. A neighbourhood of $\partial\Omega$ containing supp χ_0 will be denoted by $\mathcal{V}$. Instead of the functions Ψ_k defined in (5.3), we introduce

$$\begin{aligned}\Psi_k^{(\Omega)}(\mathbf{x})\ =\ &v(\mathbf{x})-v(\mathbf{O}^{(k)})-(\mathbf{x}-\mathbf{O}^{(k)})\cdot\nabla v(\mathbf{O}^{(k)})+\sum_{\substack{j\ne k\\1\le j\le N}}\boldsymbol{C}^{(j)}\cdot\boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x})\\&-\sum_{\substack{j\ne k\\1\le j\le N}}(\mathbf{x}-\mathbf{O}^{(j)})\cdot\mathfrak{T}(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\boldsymbol{\mathcal{Q}}^{(j)}\boldsymbol{C}^{(j)}-\sum_{j=1}^{N}\boldsymbol{C}^{(j)}\cdot\boldsymbol{\mathcal{Q}}^{(j)}\nabla_{\mathbf{y}}H(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=\mathbf{O}^{(j)}}\ ,\end{aligned} \tag{6.14}$$

where the matrix $\mathfrak{T}$ is defined in (6.3) via second-order derivatives of Green's function in Ω. Owing to (6.12) and the algebraic system (6.4) we have

$$\frac{\partial}{\partial n}\Big(\Psi_k^{(\Omega)}(\mathbf{x})+R_N(\mathbf{x})\Big)=0,\quad \mathbf{x}\in\partial F^{(k)}. \tag{6.15}$$

We also use the function

$$\Psi_0(\mathbf{x})=\sum_{j=1}^{N}\boldsymbol{C}^{(j)}\cdot\big\{\mathcal{D}^{(j)}(\mathbf{x})-\boldsymbol{\mathcal{Q}}^{(j)}\frac{(\mathbf{x}-\mathbf{O}^{(j)})}{4\pi|\mathbf{x}-\mathbf{O}^{(j)}|^3}\big\}\ , \tag{6.16}$$

which is harmonic in Ω_N. It follows from (6.12) that

$$R_N(\mathbf{x})+\Psi_0(\mathbf{x})=-\sum_{1\le j\le N}\mathbf{C}^{(j)}\cdot\boldsymbol{\mathcal{Q}}^{(j)}\Big\{\frac{(\mathbf{x}-\mathbf{O}^{(j)})}{4\pi|\mathbf{x}-\mathbf{O}^{(j)}|^3}-\nabla_{\mathbf{y}}H(\mathbf{x},\mathbf{Y})\Big|_{\mathbf{Y}=\mathbf{O}^{(j)}}\Big\}=0\ ,\quad \mathbf{x}\in\partial\Omega\ . \tag{6.17}$$

b) The energy estimate for R_N. We start with the identity

$$\int_{\Omega_N} \nabla\Big(R_N + \chi_0\Psi_0\Big)\cdot\nabla\Big(R_N + \sum_{1\le k\le N}\chi_\varepsilon^{(k)}\Psi_k^{(\Omega)}\Big)\,d\mathbf{x}$$
$$= -\int_{\Omega_N}\Big(R_N + \chi_0\Psi_0\Big)\Delta\Big(R_N + \sum_{1\le k\le N}\chi_\varepsilon^{(k)}\Psi_k^{(\Omega)}\Big)\,d\mathbf{x}, \tag{6.18}$$

which follows from (6.15), (6.17) by Green's formula. According to the definitions of χ_0 and $\chi_\varepsilon^{(k)}$, we have $\operatorname{supp}\chi_0\cap\operatorname{supp}\chi_\varepsilon^{(k)}=\emptyset$ for all $k=1,\dots,N$. Hence the integrals in (6.18) involving the products of χ_0 and $\chi_\varepsilon^{(k)}$ or their derivatives are equal to zero. Thus, using that $\Delta R_N=0$ on Ω_N, we reduce (6.18) to the equality

$$\int_{\Omega_N}|\nabla R_N|^2 d\mathbf{x} + \sum_{1\le k\le N}\int_{B_{3\varepsilon}\setminus\overline{F}^{(k)}}\nabla R_N\cdot\nabla\Big(\chi_\varepsilon^{(k)}\Psi_k^{(\Omega)}\Big)d\mathbf{x} + \int_{\Omega_N\cap\mathcal{V}}\nabla R_N\cdot\nabla\Big(\chi_0\Psi_0\Big)d\mathbf{x} \tag{6.19}$$
$$= -\sum_{1\le k\le N}\int_{\Omega_N} R_N\Delta\left(\chi_\varepsilon^{(k)}\Psi_k^{(\Omega)}\right)d\mathbf{x},$$

which differs in the left-hand side from (5.9) only by the integral over $\Omega_N\cap\mathcal{V}$.

Similarly to the part (b) of the proof of Theorem 1 we deduce

$$\|\nabla\mathcal{R}_N\|^2_{L^2(\Omega_N)} \le \text{const}\Big\{\|\nabla\Psi_0\|^2_{L^2(\Omega\cap\mathcal{V})} + \|\Psi_0\|^2_{L^2(\Omega\cap\mathcal{V})} + \sum_{1\le k\le N}\|\nabla\Psi_k\|^2_{L^2(B^{(k)}_{3\varepsilon})}\Big\}. \tag{6.20}$$

Similar to the steps of part (d) of the proof in Theorem 1, the last sum is majorized by

$$\text{const }(\varepsilon^{11}d^{-11} + \varepsilon^5 d^{-3})\|\nabla v\|^2_{L^2(\Omega)}. \tag{6.21}$$

It remains to estimate two terms in (6.20) containing Ψ_0. Using (5.29), together with (6.8) we deduce

$$\|\Psi_0\|^2_{L^2(\Omega\cap\mathcal{V})} \le \text{const }\varepsilon^8\sum_{1\le j\le N}\int_{\Omega\cap\mathcal{V}}\frac{|C^{(j)}|^2 d\mathbf{x}}{|\mathbf{x}-\mathbf{O}^{(j)}|^6}$$
$$\le \text{const }\varepsilon^8\sum_{1\le j\le N}|C^{(j)}|^2 \le \text{const}\frac{\varepsilon^8}{d^3}\|\nabla v\|^2_{L^2(\Omega)}, \tag{6.22}$$

and

$$\|\nabla\Psi_0\|^2_{L^2(\Omega\cap\mathcal{V})} \le \text{const }\varepsilon^8\sum_{1\le j\le N}\int_{\Omega\cap\mathcal{V}}\frac{|C^{(j)}|^2 d\mathbf{x}}{|\mathbf{x}-\mathbf{O}^{(j)}|^8}$$
$$\le \text{const }\varepsilon^8\sum_{1\le j\le N}|C^{(j)}|^2 \le \text{const}\frac{\varepsilon^8}{d^3}\|\nabla v\|^2_{L^2(\Omega)}. \tag{6.23}$$

Combining (6.20)–(6.23) we complete the proof. □

7 Illustrative example

Now, the asymptotic approximation derived in the previous section is applied to the case of a relatively simple geometry, where all the terms in the formula (6.12) can be written explicitly.

7.1 The case of a domain with a cloud of spherical voids

Let Ω_N be a ball of a finite radius R, with the centre at the origin, containing N spherical voids $F^{(j)}$ of radii ρ_j with the centres at $\mathbf{O}^{(j)}, j=1,\dots,N$, as shown in Fig. 1. The radii of the voids are assumed to be smaller than the distance between nearest neighbours. We put $\phi\equiv 0$ and

$$f(\mathbf{x}) = \begin{cases} 6 & \text{when } |\mathbf{x}|<\rho,\\ 0 & \text{when } \rho<|\mathbf{x}|<R.\end{cases} \tag{7.1}$$

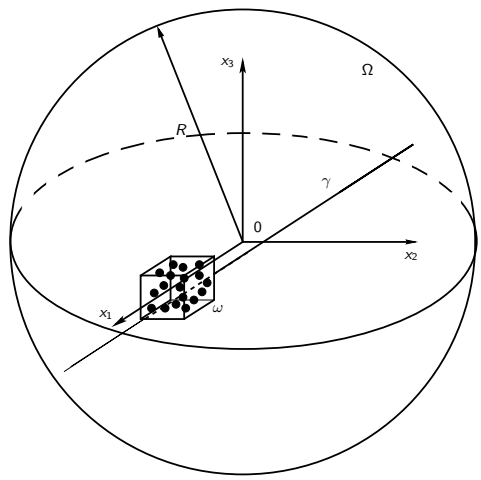


Figure 1: Example configuration of a sphere containing a cloud of spherical voids in a the cube ω.

Here, it is assumed that $\rho + b < |\mathbf{O}^{(j)}| < R - b,\ 1 \le j \le N$, where ρ and b are positive constants independent of ε and d.

The function u_N is the solution of the mixed boundary value problem for the Poisson equation:

$$\Delta u_N(\mathbf{x}) + f(\mathbf{x}) = 0, \quad \text{when } \mathbf{x} \in \Omega_N, \tag{7.2}$$

$$u_N(\mathbf{x}) = 0, \quad \text{when } |\mathbf{x}| = R, \tag{7.3}$$

$$\frac{\partial u_N}{\partial n}(\mathbf{x}) = 0, \quad \text{when } |\mathbf{x} - \mathbf{O}^{(j)}| = \rho_j,\ j = 1, \dots, N. \tag{7.4}$$

In this case, u_N is approximated by (6.12), where the solution of the Dirichlet problem in Ω is given by

$$v(\mathbf{x}) = \begin{cases} \rho^2(3 - 2\rho R^{-1}) - |\mathbf{x}|^2 & \text{when } |\mathbf{x}| < \rho, \\ 2\rho^3(|\mathbf{x}|^{-1} - R^{-1}) & \text{when } \rho < |\mathbf{x}| < R. \end{cases} \tag{7.5}$$

In turn, the dipole fields $\boldsymbol{\mathcal{D}}^{(j)}$ and the dipole matrices $\boldsymbol{\mathcal{Q}}^{(j)}$ have the form

$$\boldsymbol{\mathcal{D}}^{(j)}(\mathbf{x}) = -\rho_j^3 \frac{\mathbf{x} - \mathbf{O}^{(j)}}{|\mathbf{x} - \mathbf{O}^{(j)}|^3}, \quad \boldsymbol{\mathcal{Q}}^{(j)} = -2\pi\rho_j^3 I_3, \tag{7.6}$$

where I_3 is the 3×3 identity matrix.

The regular part $H(\mathbf{x}, \mathbf{y})$ of Green's function in the domain Ω (see (2.4)) is

$$H(\mathbf{x}, \mathbf{y}) = \frac{R}{4\pi|\mathbf{y}||\mathbf{x} - \hat{\mathbf{y}}|}, \quad \hat{\mathbf{y}} = \frac{R^2}{|\mathbf{y}|^2}\mathbf{y}. \tag{7.7}$$

The coefficients $\mathbf{C}^{(j)},\ j = 1, \dots, N$, in (6.12) are defined from the algebraic system (6.4), where Green's function $G(\mathbf{x}, \mathbf{y})$ is given by

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} - \frac{R}{4\pi|\mathbf{y}||\mathbf{x} - \hat{\mathbf{y}}|}. \tag{7.8}$$

7.2 Finite elements simulation versus the asymptotic approximation

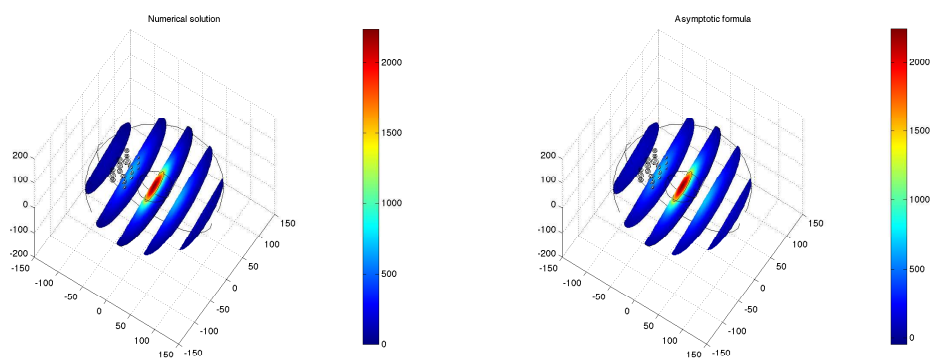
The explicit representations of the fields $v, \boldsymbol{\mathcal{D}}^{(j)}, H, G$, given above, are used in the asymptotic formula (6.12). Here, we present a comparison between the results of an independent Finite Element computation, produced in COMSOL, and the mesoscale asymptotic approximation (6.12).

For the computational example, we set $R = 120$, and consider a cloud of $N = 18$ spherical voids arranged into a cloud of a parallelipiped shape. The position of the centre and radius of each void is included in Table 1. The support of the function f (see (7.1)), is chosen to be inside the sphere with radius $\rho = 30$ and centre at the origin, as stated in (7.1).

Void	Centre	ρ_j/R	Void	Centre	ρ_j/R
$F^{(1)}$	(-50, 0, 0)	0.0417	$F^{(10)}$	(-72, 0, 0)	0.0417
$F^{(2)}$	(-50, 0, 22)	0.0333	$F^{(11)}$	(-72, 0, 22)	0.0458
$F^{(3)}$	(-50, 22, 0)	0.0292	$F^{(12)}$	(-72, 22, 0)	0.0292
$F^{(4)}$	(-50, 0, -22)	0.0375	$F^{(13)}$	(-72, 0, -22)	0.0375
$F^{(5)}$	(-50, -22, 0)	0.0458	$F^{(14)}$	(-72, -22, 0)	0.0417
$F^{(6)}$	(-50, 22, 22)	0.0292	$F^{(15)}$	(-72, 22, 22)	0.0333
$F^{(7)}$	(-50, 22, -22)	0.025	$F^{(16)}$	(-72, 22, -22)	0.05
$F^{(8)}$	(-50, -22, 22)	0.0375	$F^{(17)}$	(-72, -22, 22)	0.0333
$F^{(9)}$	(-50, -22, -22)	0.0375	$F^{(18)}$	(-72, -22, -22)	0.0375

Table 1: Data for the voids $F^{(j)}$, $j = 1, \ldots, 18$.

Figure 2 shows the asymptotic solution u_N of the mixed boundary value problem (part (b) of the figure) and its numerical counterpart obtained in COMSOL 3.5 (part (a) of the figure). This computation has been produced for a spherical body containing 18 small voids defined in Table 1. The relative error for the chosen configuration does not exceed 2%, which confirms a very good agreement between the asymptotic and numerical results, which are visually indistinguishable in Fig. 2a and Fig. 2b.



(a) (b)

Figure 2: Perforated domain containing 18 holes: (a) Numerical solutions produced in COMSOL; (b) Asymptotic approximation.

The computation was performed on Apple Mac, with 4Gb of RAM, and the number $N = 18$ was chosen because any further increase in the number of voids resulted in a large three-dimensional computation, which exceeded the amount of available memory. Although, increase in RAM can allow for a larger computation, it is evident that three-dimensional finite element computations for a mesoscale geometry have serious limitations. On the other hand, the analytical asymptotic formula can still be used on the same computer for sigtnificantly larger number of voids.

In the next subsection, we show such an example where the number of voids within the mesoscale cloud runs upto $N = 1000$, which would simply be unachievable in a finite element computation in COMSOL 3.5 with the same amount of RAM available.

7.3 Non-uniform cloud containing a large number of spherical voids

Here we consider the same mixed boundary value problem as in Section 7.1, but the cloud of voids is chosen in such a way that the number N may be large and voids of different radii are distributed in a non-uniform arrangement. For different values of N, the overall volume of voids is preserved - examples of the clouds used here are shown in Fig. 1.

The results are based on the numerical implementation of formula (6.12) in MATLAB.

The cloud ω is assumed to be the cube with side length $\frac{1}{\sqrt{3}}$ and the centre at $(3, 0, 0)$. Positioning of voids

is described as follows. Assume we have $N = m^3$ voids, where $m = 2, 3, \ldots$. Then ω is divided into N smaller cubes of side length $h = \frac{1}{\sqrt{3}m}$, and the centres of voids are placed at

$$\mathbf{O}^{(p,q,r)} = \Big(3 - \frac{1}{2\sqrt{3}} + \frac{2p-1}{2}h, -\frac{1}{2\sqrt{3}} + \frac{2q-1}{2}h, -\frac{1}{2\sqrt{3}} + \frac{2r-1}{2}h\Big)$$

for $p, q, r = 1, \ldots, m$, and we assign their radii $\rho_{p,q,r}$ by

$$\rho_{p,q,r} = \begin{cases} \dfrac{h}{5} & \text{if } p > q\ , \\ \dfrac{\alpha h}{2} & \text{if } p < q\ , \\ \dfrac{h}{4} & \text{if } p = q\ , \end{cases}$$

where $\alpha < 1$, and it is chosen in such a way that the overall volume of all voids within the cloud remains constant for different N. An elementary calculation suggests that there will be m^2 voids with radius $\frac{h}{4}$ and equal number $\frac{m^2(m-1)}{2}$ of voids with radius $\frac{h}{5}$ or $\frac{\alpha h}{2}$.

Assuming that the volume fraction of all voids within the cube is equal to β, we have

$$\frac{4\pi h^3}{3}\Big(\frac{m^2(m-1)(8+125\alpha^3)}{2000} + \frac{m^2}{64}\Big) = \beta\frac{1}{3\sqrt{3}}\ ,$$

and hence

$$\alpha^3 = \frac{16m}{m-1}\Big\{\frac{3}{4\pi}\beta - \frac{125+32(m-1)}{8000m}\Big\}\ . \tag{7.9}$$

In particular, if $N \to \infty$, the limit value α_∞ becomes

$$\alpha_\infty = \Big\{\frac{12}{\pi}\beta - \frac{8}{125}\Big\}^{1/3}\ . \tag{7.10}$$

In the numerical computation of this section, $\beta = \pi/25$.

Taking $R = 7$ and $\rho = 2$, we compute the leading order approximation of $u_N - v$, as defined in the asymptotic formula (6.12), along the line γ at the intersection of the planes $x_2 = -1/(2\sqrt{3})$ and $x_3 = -1/(2\sqrt{3})$, for $N = 8, 125, 1000$. Fig. 3 below shows the configuration of the cloud of voids for a) $N = 8$ and b) $N = 125$. For a large number of voids ($N = 1000$), Fig. 4a) shows the cloud and Fig 4b) includes the graph of α versus N. The plot of $u_N - v$ given by (6.12) for $2 \le x_1 \le 4$ is shown in Fig. 5. The asymptotic correction has been computed along the straight line $\gamma = \{x_1 \in \mathbb{R}, x_2 = -1/(2\sqrt{3}), x_3 = -1/(2\sqrt{3})\}$. Dipole type fluctuations are clearly visible on the diagram. Beyond $N = 1000$ the graphs are visually indistinguishable and hence the values $N = 8, 125, 1000$, as in Figures 3 and 4 have been chosen in the computations. The algorithm is fast and does not impose periodicity constraints on the array of small voids.

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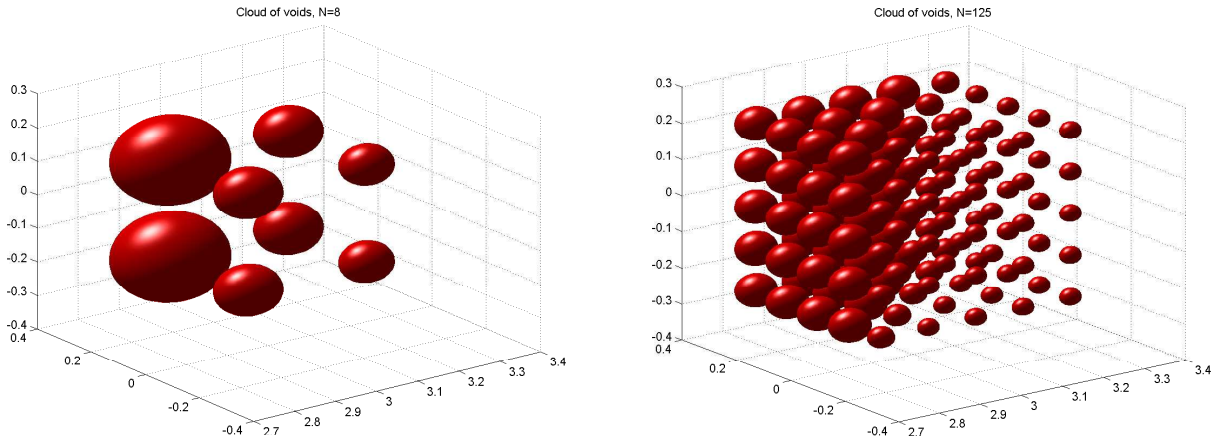
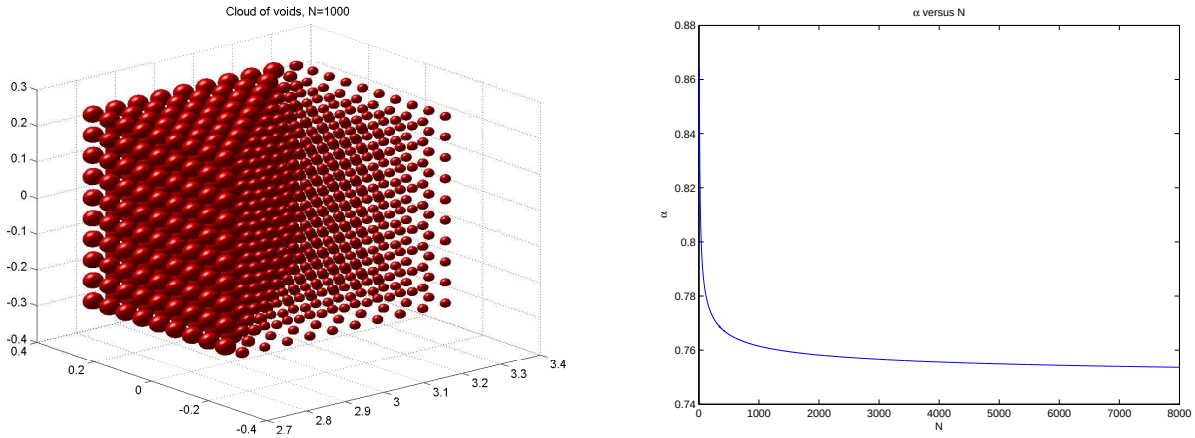


a) b)

Figure 3: The cloud of voids for the cases when a) $N = 8$ and b) $N = 125$.



a) b)

Figure 4: a) The cloud of voids for the cases when $N = 1000$, b) The graph of α versus N given by formula (7.9) when $\beta = \pi/25$, for large N we see that α tends to 0.7465 which is predicted value present in (7.10).

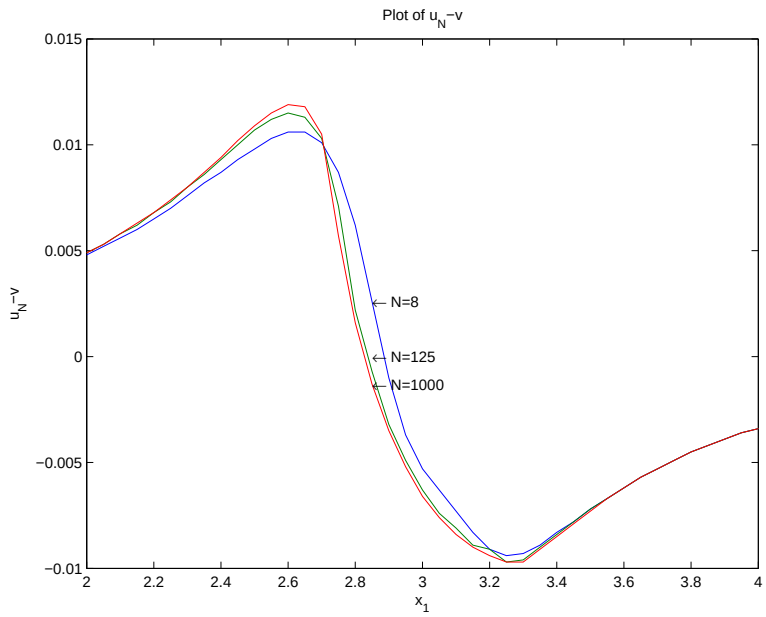


Figure 5: The graph of $u_N - v$ given by (6.12), for $2 \leq x_1 \leq 4$ plotted along the straight line γ adjacent to the cloud of small voids.

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