Birational geometry of algebraic varieties, fibred into Fano double spaces

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We develop the quadratic technique of proving birational rigidity of Fano-Mori fibre spaces over a higher-dimensional base. As an application, we prove birational rigidity of generic fibrations into Fano double spaces of dimension $M \geqslant 4$ and index one over a rationally connected base of dimension at most $\frac{1}{2}(M-2)(M-1)$. An estimate for the codimension of the subset of hypersurfaces of a given degree in the projective space with a positive-dimensional singular set is obtained, which is close to the optimal one.

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Introduction

0.1. Statement of the main result. In [9] birational rigidity was shown for two large classes of higher-dimensional Fano-Mori fibre spaces: generic fibrations into double spaces of index one and dimension $M \geqslant 5$ when the dimension of the base does not exceed $\frac{1}{2}(M-4)(M-1)-1$ and generic fibrations into hypersurfaces of index one and dimension $M-1 \geqslant 9$ when the dimension of the base does not exceed $\frac{1}{2}(M-7)(M-6)-6$ (in both cases under the assumption of sufficient twistedness over the base). For Fano-Mori fibre spaces over the projective line the question of birational rigidity is studied well enough, see [8, Chapter 5]. However, one should note that almost all results on birational rigidity of Fano-Mori fibre spaces over the line were obtained by means of the *quadratic* technique (that is, via analysis of the singularities of the self-intersection of a mobile linear system, defining the birational map), whereas the main result of [9] was obtained by means of the *linear* technique (that is, via direct analysis of the singularities of the linear system itself, without using the quadratic operation of taking the self-intersection). The quadratic technique requires less restrictions on the variety underconsideration and for that reason makes it possible to embrace a considerably large class of rationally connected varieties. In many respects it is more efficient (at least, at the present stage of of the theory of birational rigidity).

The aim of the present paper is to develop the quadratic technique of studying birational geometry of Fano-Mori fibre spaces over a higher-dimensional base and apply it to fibrations into Fano double spaces of index one, considerably improving the result of [9] for that class of varieties: we show birational rigidity of generic fibrations into Fano double spaces of dimension $M \geqslant 4$ and index one over a rationally connected base of dimension at most $\frac{1}{2}(M-2)(M-1)$. This result, considerably increasing the admissible dimension of the base of the fibre space, is obtained by means of the quadratic technique of counting multiplicities (see [8, Chapter 5]), which was not used in [9].

Let us make the precise statements.

We consider a Fano-Mori fibre space $\pi\colon V \to S$, where the base S is non-singular, the variety V is factorial and has at most terminal singularities, the antocanonical class $(-K_V)$ is relatively ample and

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S.$$

We say that a fibre $F = F_s = \pi^{-1}(s)$, $s \in S$, satisfies the *condition* (h), if for any irreducible subvariety $Y \subset F$ of codimension 2 and any point $o \in Y$ the inequality

$$\frac{\operatorname{mult}_o Y}{\deg Y} \leqslant \frac{4}{\deg F}$$

holds, where the degrees are understood in the sense of the anticanonical class, that is,

$$\deg Y = \left(Y \cdot (-K_V)^{\dim Y}\right)$$

and

$$\deg F = \left(F \cdot (-K_V)^{\dim F}\right),$$

and the *condition* (hd), if for any mobile linear system $\Delta \subset |-n(K_V|_F)|$ and any irreducible subvariety $Y \subset F$ of codimension 2 the inequality

$$\operatorname{mult}_Y \Delta \leqslant n$$

holds. Further, we say that a fibre F satisfies the *condition* (v), if for any prime divisor $Y \subset F$ and any point $o \in F$ of this fibre the inequality

$$\frac{\operatorname{mult}_o Y}{\deg Y} \leqslant \frac{2}{\deg F}$$

holds. Finally, we say that the fibre space V/S satisfies the K-*condition*, if for any mobile family $\overline{\mathcal{C}}$ of curves on the base S, sweeping out S, and a general curve $\overline{C} \in \overline{\mathcal{C}}$ the class of algebraic cycle

$$-N(K_V \cdot \pi^{-1}(\overline{C})) - F$$

of dimension $\dim F$ for any $N \geqslant 1$ is not effective, that is, it is not rationally equivalent to an effective cycle of dimension $\dim F$, and the K^2-*condition*, if for any mobile family $\overline{\mathcal{C}}$ of curves on the base S, sweeping out S, and a general curve $\overline{C} \in \overline{\mathcal{C}}$ the class of algebraic cycle

$$N(K_V^2 \cdot \pi^{-1}(\overline{C})) - H_F$$

of dimension $\dim F - 1$ is not effective for any $N \geqslant 1$, where $H_F = (-K_V \cdot F)$ is the class of the anticanonical section of the fibre.

The following claim is the main result of the present paper.

Theorem 0.1. *Assume that* $\dim F \geqslant 4$ *and every fibre* F *of the projection* π *is a variety with at most quadratic singularities of rank at least 4, and moreover* $\mathrm{codim}(\mathrm{Sing}\, F \subset F) \geqslant 4$. *Assume further that every fibre* F *satisfies the conditions (h), (hd) and (v), whereas the fibre space* V/S *satisfies the* K*-condition* K^2*-condition.*

Then the fibre space V/S *is birationally rigid: every birational map* $\chi\colon V \dashrightarrow V'$ *onto the total space of rationally connected fibre space* V'/S' *is fibre-wise, that is, there is a rational dominant map* $\beta\colon S \dashrightarrow S'$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{\beta}{\dashrightarrow} & S'. & \end{array}$$

(Recall that a morphism of projective algebraic varieties $\pi'\colon V' \to S'$ is a rationally connected fibre space if the base S' and the general fibre $\pi'^{-1}(s')$, $s' \in S'$, are rationally connected.)

Theorem 0.1 implies immediately the following claim.

Corollary 0.1. *In the assumptions of Theorem 0.1 on the variety* V *there are no structures of a rationally connected fibre space over a base of dimension higher than* $\dim S$. *In particular, the variety* V *is non-rational. Any birational self-map of the variety* V *is fibre-wise and induces a birational self-map of the base* S, *so that there is a natural homomorphism of groups* $\rho\colon \mathrm{Bir}\, V \to \mathrm{Bir}\, S$, *the kernel of which* $\mathrm{Ker}\, \rho$ *is the group* $\mathrm{Bir}\, F_\eta = \mathrm{Bir}(V/S)$ *of birational self-maps of the generic fibre* F_η *(over the non-closed generic point* η *of the base* S*), whereas the group* $\mathrm{Bir}\, V$ *is an extension of the normal subgroup* $\mathrm{Bir}\, F_\eta$ *by the group* $\Gamma = \rho(\mathrm{Bir}\, V) \subset \mathrm{Bir}\, S$*:*

$$1 \to \mathrm{Bir}\, F_\eta \to \mathrm{Bir}\, V \to \Gamma \to 1.$$

Recall that in [9] the following fact was shown.

Theorem 0.2. *Assume that a Fano-Mori fibre space* $\pi\colon V \to S$ *satisfies the following conditions:*

(i) *every fibre* $F_s = \pi^{-1}(s)$, $s \in S$, *is a factorial Fano variety with at most terminal singularities and the Picard group* $\mathrm{Pic}\, F_s = \mathbb{Z}K_{F_s}$, *where* F_s *has complete intersection singularities and* $\mathrm{codim}(\mathrm{Sing}\, F \subset F) \geqslant 4$,

(ii) *for every effective divisor* $D \in |-nK_{F_s}|$ *on an arbitrary fibre* F_s *the pair* $(F_s, \frac{1}{n}D)$ *is log canonical, and for any mobile linear system* $\Sigma_s \subset |-nK_{F_s}|$ *the pair*

$(F_s,\frac{1}{n}\Sigma_s)$ *is canonical (that is, the pair* $(F_s,\frac{1}{n}D)$ *is canonical for a general divisor* $D\in\Sigma_s$*),*

(iii) *for any mobile family* $\overline{\mathcal{C}}$ *of curves on the base* S*, sweeping out* S*, and a general curve* $\overline{C}\in\overline{\mathcal{C}}$ *the class of algebraic cycle of dimension* $\dim F$ *for any positive* $N\geqslant 1$

$$-N(K_V\cdot\pi^{-1}(\overline{C}))-F$$

(where F *is the fibre of the projection* π*) is not effective, that is, it is not rationally equivalent to an effective cycle of dimension* $\dim F$*.*

Then any birational map $\chi\colon V\dashrightarrow V'$ *onto the total space of a rationally connected fibre space* V'/S' *is fibre-wise, that is, there is a rational dominant map* $\beta\colon S\dashrightarrow S'$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{\beta}{\dashrightarrow} & S'. & \end{array}$$

Let us compare the assumptions of Theorems 0.1 and 0.2. The canonicity of the pair $(F_s,\frac{1}{n}D)$ in the condition (ii) of Theorem 0.2 (that is, essentially the birational superrigidity of the fibre F_s) follows from the conditions (h) and (hd) in Theorem 0.1 (and is actually equivalent to them as the main method of proving birational superrigidity of a primitive Fano variety is the application of the $4n^2$-inequality combined with the exclusion of maximal subvarieties of codimension two, see [8, Chapter 2]). The log canonicity of the pair $(F_s,\frac{1}{n}D)$ in the condition (ii) of Theorem 0.2 is replaced in Theorem 0.1 by the condition (v), which for certain classes of Fano varieties is much easier to check. Finally, in Theorem 0.1 a new global condition for the Fano-Mori fibre space V/S is added, the K^2-condition, which is easy to check.

Theorem 0.1 will be applied to fibrations into double spaces of index one, when the conditions (h) and (v) hold automatically by the equality $\deg F=2$.

0.2. Fibrations into double spaces of index one. We use the notations of subsection 0.2 of [9]: the symbol $\mathbb{P}$ stands for the projective space $\mathbb{P}^M$, $M\geqslant 4$, and $\mathcal{W}=\mathbb{P}(H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(2M)))$ is the space of hypersurfaces of degree $2M$ in $\mathbb{P}$. The following general fact is true.

Theorem 0.3. *The closed algebraic subset of homogeneous polynomials* f *of degree* d *in* $(N+1)$ *variables, such that the hypersurface* $\{f=0\}\subset\mathbb{P}^N$ *has a singular set of positive dimension, is of codimension at least* $(d-2)N$ *in the space* $H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(d))$*.*

Proof is given in §3.

The following theorem is immediately implied by Theorem 0.3.

Theorem 0.4. *There is a Zariski open subset* $\mathcal{W}_{\rm reg}\subset\mathcal{W}$*, such that any hypersurface* $W\in\mathcal{W}_{\rm reg}$ *has finitely many singular points, each of which is a quadratic singularity of rank at least 3, and, moreover, the following estimate holds:*

$$\mathrm{codim}((\mathcal{W}\setminus\mathcal{W}_{\rm reg})\subset\mathcal{W})\geqslant\frac{(M-2)(M-1)}{2}+1.$$

Proof. Setting in Theorem 0.3 $d = 2M$ and $N = M$, we obtain, that in the complement to a closed subset of codimension $2M(M-1)$ in $\mathcal{W}$ any hypersurface W has finitely many singular points. It is easy to check that the closed set of hypersurfaces W with a quadratic singular point of rank at most 2 or with a singularity $o \in W$ of multiplicity $\mathrm{mult}_o W \geqslant 3$, is of codimension $\frac{1}{2}(M-2)(M-1)+1$ in the space $\mathcal{W}$. This proves the theorem.

If $F \to \mathbb{P}$ is a double cover, branched over a hypersurface $W \in \mathcal{W}_{\mathrm{reg}}$, then F is a factorial Fano variety with terminal singularities (see [1], Subsection 2.1 in [9] and Proposition 1.4 below), satisfying the conditions (h) and (v) by the equality $\deg F = 2$. The condition (hd) is easy to show by the standard methods (see [8, Chapter 2]; for $M \geqslant 5$ it holds in a trivial way, because for any irreducible subvariety $Y \subset F$ of codimension 2 the inequality $\deg Y \geqslant 2$ holds). Thus in order to apply Theorem 0.1, it is sufficient to require every fibre F_s, $s \in S$, to be branched over a regular hypersurface $W_s \in \mathcal{W}_{\mathrm{reg}}$, and the fibre space V/S to satisfy the K-condition and the K^2-condition.

In the notations of Subsection 0.2 of [9] let S be a non-singular rationally connected variety of dimension $\dim S \leqslant \frac{1}{2}(M-2)(M-1)$. Let $\mathcal{L}$ be a locally free sheaf of rank $M+1$ on S and $X = \mathbb{P}(\mathcal{L}) = \mathbf{Proj} \mathop{\oplus}\limits_{i=0}^{\infty} \mathcal{L}^{\otimes i}$ the corresponding $\mathbb{P}^M$-bundle. We may assume that $\mathcal{L}$ is generated by its global sections, so that the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(1)$ is also generated by the global sections. Let $L \in \mathrm{Pic}\, X$ be the class of that sheaf, so that

$$\mathrm{Pic}\, X = \mathbb{Z}L \oplus \pi_X^* \mathrm{Pic}\, S,$$

where $\pi_X \colon X \to S$ is the natural projection. Take a general divisor $U \in |2(ML + \pi_X^* R)|$, where $R \in \mathrm{Pic}\, S$ is some class. If that system is sufficiently mobile, then by the assumption about the dimension of the base S and by Theorem 0.4 we may assume that for any point $s \in S$ the hypersurface $U_s = U \cap \pi_X^{-1}(s) \in \mathcal{W}_{\mathrm{reg}}$, and for that reason the double space, branched over U_s, satisfies the conditions of Theorem 0.1. Let $\sigma \colon V \to X$ be the double cover branched over U. Set $\pi = \pi_X \circ \sigma \colon V \to S$, so that V is a fibration into Fano double spaces of index one over S. Recall that the divisor $U \in |2(ML + \pi_X^* R)|$ is assumed to be sufficiently general.

Theorem 0.5. *Assume that the variety V is general in the sense of the construction described above and the divisorial class $(K_S + R)$ is pseudo-effective. Then for the fibre space $\pi \colon V \to S$ the claims of Theorem 0.1 and Corollary 0.1 are true. In particular,*

$$\mathrm{Bir}\, V = \mathrm{Aut}\, V = \mathbb{Z}/2\mathbb{Z}$$

is the cyclic group of order 2.

Proof. Since the class L is numerically effective and

$$\left((\sigma^* L)^M \cdot F\right) = \left((\sigma^* L)^{M-1} \cdot H_F\right) = 2,$$

it is sufficient to check the inequalities

$$\left((\sigma^* L)^M \cdot K_V \cdot \pi^{-1}(\overline{C})\right) \geqslant 0 \quad \text{and} \quad \left((\sigma^* L)^{M-1} \cdot K_V^2 \cdot \pi^{-1}(\overline{C})\right) \leqslant 0.$$

As it was noted in Subsection 0.2 in [9], the first of these inequalities up to a positive factor is the inequality $((K_S+R)\cdot\overline{C})\geqslant 0$, which holds because the class (K_S+R) is pseudo-effective and the family of curves $\overline{\mathcal{C}}$ is mobile and sweeps out the base S. As for the second inequality, then elementary computations show that up to a positive factor it can be written as the inequality

$$2((K_S+R)\cdot\overline{C})+((\det\mathcal{L})\cdot\overline{C})\geqslant 0,$$

which is the more so true because the locally free sheaf $\mathcal{L}$ is generated by global sections. Q.E.D. for the theorem.

Remark 0.1. For fibrations into double spaces of index one the K^2-condition follows from the K-condition. Theorem 0.5 makes Theorem 0.3 of [9] stronger in respect of the genericity conditions which should be satisfied for every fibre of the fibre space V/S: in Theorem 0.5 these conditions are weaker, and for that reason the set $\mathcal{W}_{\rm reg}$ is larger. This makes it possible to prove birational rigidity for fibre spaces over a base of higher dimension, and in particular, for fibrations onto four-dimensional double spaces.

0.3. The structure of the paper. The present paper is organized in the following way. §1 contains mostly the first part of the proof of Theorem 0.1: we construct a modification of the base $S^+\to S$ such that on the pull back $\pi_+\colon V^+\to S^+$ of the original fibre space onto S^+, the centre of each maximal singularity covers a divisor on S^+ (this procedure is often referred to as *flattening* the maximal singularities). These arguments are similar to the arguments of §1 in [9], however, in contrast to [9], here they give no proof of the main theorem, but only show the existence of a *supermaximal singularity* (under the assumption that the claim of Theorem 0.1 does not hold). The latter concept plays an important role in the proof of birational superrigidity of fibre spaces over $\mathbb{P}^1$, see [8, Chapter 5]; here we extend it to the case of fibrations over a base of arbitrary dimension. We complete §1, studying quadratic singularities, the rank of which is bounded from below (we need this to claim factoriality and terminality of the modified fibre space $\pi_+\colon V^+\to S^+$).

In §2 we complete the proof of Theorem 0.1: we exclude the supermaximal singularity, the existence of which has been shown in §1, whence the claim of Theorem 0.1 follows immediately. The excluding is achieved by means of the standard technique of counting multiplicities (see [8, Chapter 5]), adjusted to the situation under consideration.

In §3 we obtain an estimate for the codimension of the closed set of hypersurfaces of degree d in $\mathbb{P}^N$ with a singular set of positive dimension, in the space of all hypersurfaces of degree d in $\mathbb{P}^N$. The estimate is close to the optimal one. This is a general and quite useful result, proved by elementary (but non-trivial) methods of algebraic geometry; as far as the author knows, this estimate was not known earlier.

0.4. Historical remarks and acknowledgements. The history of the problems connected with birational rigidity of Fano-Mori fibre spaces over a base of positive dimension, has been reviewed in the introduction to [9] in a detailed enough way, and we will not consider it here. We note, however, that the Sarkisov theorem on conic bundles [10, 11] was proved by the quadratic method (the self-intersection of the mobile linear system, defining the birational map, was considered), although for that class of varieties the quadratic technique of counting multiplicities is not needed.

The problem of birational rigidity for del Pezzo fibrations over a base of dimension higher than one is entirely open. However, it is clear that for that class of varieties it is the quadratic techniques that is needed, although it is possible that a combination of the linear and the quadratic method will be successful. In the direction of computing the possible values of log canonical thresholds on del Pezzo surfaces a lot of work has been recently done, see [12, 13, 14, 15].

Finally, let us point out the recent work [4], where by means of the results of [6, 7] (see also [8, Chapter 7]) the problem of existence of rationally connected varieties that are non-Fano type varieties, stated in [3], was solved.

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1 Maximal and supermaximal singularities

The contents of this section is the first part of the proof of Theorem 0.1. In Subsection 1.1 we modify the fibre space V/S: this procedure is similar to §1 in [9]. As a result, we obtain a new Fano-Mori fibre space V^+/S^+, satisfying all assumptions of Theorem 0.1 and an additional condition: the centre on V^+ of any maximal singularity covers a divisor on S^+. In Subsection 1.2 we consider the self-intersection of the mobile linear system Σ, related to the birational map χ, and show the existence of a supermaximal singularity. In Subsection 1.3 we make the information about quadratic singularities of a bounded rank more precise.

1.1. Modification of the fibre space V/S. In the notations of Theorem 0.1 fix a birational map $\chi\colon V \dashrightarrow V'$. Repeating the arguments of Subsection 1.1 in [9], consider an arbitrary very ample linear system $\overline{\Sigma'}$ on S'. Let $\Sigma' = (\pi')^*\overline{\Sigma'}$ be its pull back onto V', so that the divisors $D' \in \Sigma'$ are composed from the fibres of the projection π', and for that reason for any curve $C \subset V'$ that is contracted by the projection π', we have $(D' \cdot C) = 0$; the linear system Σ' is obviously mobile. Set

$$\Sigma = (\chi^{-1})_*\Sigma' \subset |-nK_V + \pi^*Y|$$

to be its strict transform on V, where $n \in \mathbb{Z}_+$. Obviously, the map χ is fibre-wise if and only if $n = 0$. Therefore, if $n = 0$, then the claim of Theorem 0.1 holds. So let us assume that $n \geq 1$ and show that this assumption leads to a contradiction.

As was shown in [9, Lemma 1.1], for any mobile family of curves $\overline{C} \in \overline{\mathcal{C}}$ on S, sweeping out S, the inequality $(\overline{C} \cdot Y) \geqslant 0$ holds.

Following [9], we call a prime divisor E over V a *maximal singularity* of the birational map χ, if its image on V' is a prime divisor, covering the base S', and the Noether-Fano inequality holds:

$$\varepsilon(E) = \mathrm{ord}_E \Sigma - na(E) > 0,$$

where $a(E)$ is the discrepancy of E with respect to V. In [9, Proposition 1.1] it was shown that maximal singularities exist. Let $\mathcal{M}$ be the (finite) set of all maximal singularities.

In the proof of the existence of maximal singularities an important role is played by a *very mobile* family $\mathcal{C}'$ of rational curves on the variety V'. Recall [9, Subsection 1.1], that a family of rational curves $\mathcal{C}'$ on V' is very mobile if the curves $C' \in \mathcal{C}'$ are contracted by the projection π', sweep out a dense open subset in V', do not intersect the set of indeterminancy of the map $\chi^{-1}\colon V' \dashrightarrow V$, and a general curve $C' \in \mathcal{C}'$ intersects the image of each maximal singularity $E \in \mathcal{M}$ transversally at points of general position. Let us fix a very mobile family of curves on V'. Its strict transform on V we denote by the symbol $\mathcal{C}$, and its projection $\pi(\mathcal{C})$ on S by the symbol $\overline{\mathcal{C}}$. Further, the following fact is true.

Proposition 1.1. *For every maximal singularity $E \subset \mathcal{M}$ its centre*

$$\mathrm{centre}(E, V) = \varphi(E)$$

on V does not cover the base: $\pi(\mathrm{centre}(E, V)) \subset S$ is a proper closed subset of the variety S.

Proof. Although the statement of this proposition repeats the statement of Proposition 1.2 in [9] word for word, a new proof is needed, since the assumptions are different. Again it is sufficient to show that the restriction $\Sigma|_F$ of the linear system Σ onto a fibre $F = \pi^{-1}(s)$ of general position has no maximal singularities (in the standard, weaker sense, see [8, Chapter 2]). This follows immediately from the conditions (h) and (hd), which are satisfied for the variety V. Q.E.D. for the proposition.

Now let us construct, following [9, Subsection 1.2], a modification of the base $\sigma_S\colon S^+ \to S$ and the corresponding modification of the total space

$$\sigma_S\colon V^+ = V \times_S S^+ \to V$$

of the fibre space V/S, such that the new fibre space $\pi_+\colon V^+ \to S$ satisfies the following conditions:

- the base S^+ is non-singular,

- for every singularity E of the birational map $\chi\circ\sigma\colon V^+ \dashrightarrow V'$, which is realized on V' by a divisor, covering the base S', its centre on V^+ covers a divisor on S^+, that is,
$$\operatorname{codim}(\pi_+(\operatorname{centre}(E,V^+))\subset S^+)=1.$$

The modification σ_S is constructed as a sequence of blow ups with non-singular centres. Since the fibre of the fibre space V^+/S^+ over a point $p\in S^+$ is naturally isomorphic to the fibre of the original fibre space V/S over the point $\sigma_S(p)\in S$, and the base S^+ is non-singular, by the assumption about the singularities of the fibres of the original fibre space V/S, the variety V^+ has at most quadratic (in particular, hypersurface) singularities of rank at least 4, and moreover, $\operatorname{codim}(\operatorname{Sing} V^+\subset V^+)\geqslant 4$, so that the variety V^+ is factorial and terminal. Obviously,

$$\operatorname{Pic} V^+ = \mathbb{Z}K_+\oplus \pi_+^*\operatorname{Pic} S^+,$$

so that V^+/S^+ is again a Fano-Mori fibre space. Let $\overline{\mathcal{T}}$ be the set of all σ_S-exceptional prime divisors on S^+ and $\mathcal{T}$ the set of all σ-exceptional prime divisors on V^+. The map
$$\mathcal{T}\ni T\mapsto \pi_+(T)=\overline{T}\ni\overline{\mathcal{T}}$$
is a bijection between $\mathcal{T}$ and $\overline{\mathcal{T}}$, the inverse map is

$$\overline{\mathcal{T}}\ni\overline{T}\mapsto\pi_+^{-1}(\overline{T})=T\in\mathcal{T}.$$

Obviously, $\operatorname{Pic} S^+=\sigma_S^*\operatorname{Pic} S\bigoplus\bigoplus\limits_{\overline{T}\in\overline{\mathcal{T}}}\mathbb{Z}\overline{T}$ and a similar equality is true for $\operatorname{Pic} V^+$.

Proposition 1.2. *For the Fano-Mori fibre space V^+/S^+ the K-condition and the K^2-condition hold.*

Proof. Let $\overline{\mathcal{R}}$ be a mobile family of curves on S^+, sweeping out S^+, and $\overline{R}\in\overline{\mathcal{R}}$ a general curve. Then, obviously, $\sigma_S(\overline{\mathcal{R}})$ is a mobile family of curves on S, sweeping out S, and $\sigma_S(\overline{R})$ is a general curve in that family. We have

$$K_{S^+}=\sigma_S^*K_S+\sum_{\overline{T}\in\overline{\mathcal{T}}}a_T\overline{T}$$

and, respectively,
$$K_+=\sigma^*K_V+\sum_{T\in\mathcal{T}}a_TT$$

(the discrepancies of the prime divisors $\overline{T}$ and $T=\pi_+^{-1}(\overline{T})$ with respect to S and V, are obviously equal), and moreover, $a_T>0$ for all $T\in\mathcal{T}$. Let us consider the class of an algebraic cycle

$$\sigma_*[-N(K_+\cdot\pi_+^{-1}(\overline{R}))-F]=-N(K_V\cdot\pi^{-1}(\sigma_S(\overline{R})))-\alpha F,$$

where $\alpha=N\sum\limits_{\overline{T}\in\overline{\mathcal{T}}}a_T(\overline{T}\cdot\overline{R})+1\geqslant 1$. Since for the fibre space V/S the K-condition is satisfied, we can see from here that it is satisfied for V^+/S^+, too. Let us consider the K^2-condition. Writing out explicitly K_+^2, we get: $(K_+^2 \cdot \pi_+^{-1}(\overline{R})) =$

$$= (\sigma^* K_V^2 \cdot \pi_+^{-1}(\overline{R})) + 2(\sigma^* K_V \cdot \left(\sum_{T \in \mathcal{T}} a_T T\right) \cdot \pi_+^{-1}(\overline{R})) =$$

$$= (\sigma^* K_V^2 \cdot \pi_+^{-1}(\overline{R})) - 2 \sum_{\overline{T} \in \overline{\mathcal{T}}} a_T (\overline{T} \cdot \overline{R}) H_F$$

(since $((\sum_{T \in \mathcal{T}} a_T T)^2 \cdot \pi_+^{-1}(\overline{R})) = 0$). Therefore,

$$\sigma_*[N(K_+^2 \cdot \pi^{-1}(\overline{R})) - H_F] = N(K_V^2 \cdot \pi^{-1}(\sigma_S(\overline{R}))) - \beta F,$$

where $\beta = 2N \sum_{\overline{T} \in \overline{\mathcal{T}}} a_T (\overline{T} \cdot \overline{R}) + 1 \geqslant 1$. Since the fibre space V/S satisfies the K^2-condition, this implies that V^+/S^+ satisfies it as well. Q.E.D. for the proposition.

Since obviously the map $\chi \circ \sigma \colon V^+ \dashrightarrow V$ is fibre-wise with respect to the projections π_+, π' if and only if the map χ is fibre-wise, we will prove Theorem 0.1 for the Fano-Mori fibre space V^+/S^+. The fibres of that fibre space by construction are the fibres of the original fibre space V/S, so that for V^+/S all assumptions of Theorem 0.1 are satisfied.

From now on, in order to simplify the notations, we assume that V^+/S^+ is the original Fano-Mori fibre space V/S, which now has a new property: every singularity E of the map χ (which is still not fibre-wise), the centre of which on V' is divisorial and covers the base S', has on the variety V the centre centre(E, V), covering a prime divisor on S. In particular, this is true for every maximal singularity $E \in \mathcal{M}$. In order not to make the text more difficult to read using by new symbols, we will use the symbols $\mathcal{T}$, $\overline{\mathcal{T}}$ in the new sense: $\overline{\mathcal{T}}$ is the set of such prime divisors $\overline{T}$ on the base S, that for some maximal singularity $E \in \mathcal{M}$ we have $\pi(\text{centre}(E, V)) = \overline{T}$, and $\mathcal{T}$ is the set of preimages $T = \pi^{-1}(\overline{T})$ of those divisors on V. The projection π gives a one-to-one correspondence between the sets $\mathcal{T}$ and $\overline{\mathcal{T}}$. Let

$$\tau \colon \mathcal{M} \to \mathcal{T}$$

be the map, relating to a maximal singularity $E \in \mathcal{M}$ the divisor $T \in \mathcal{T}$, containing its centre centre(E, V), and $\overline{\tau} = \pi \circ \tau \colon \mathcal{M} \to \overline{\mathcal{T}}$, that is to say, $\overline{\tau}(E) = \pi(\text{centre}(E, V)) \subset S$. For $T \in \mathcal{T}$ set $\mathcal{M}_T = \tau^{-1}(T)$, so that

$$\mathcal{M} = \bigsqcup_{T \in \mathcal{T}} \mathcal{M}_T.$$

Remark 1.1. In the situation considered in [9], the modification of the base completes the proof of birational rigidity of the fibre space, since by the assumption about the global log canonical threshold of every fibre, no maximal singularity, the centre of which covers a divisor on the base, can exist. In this paper the assumption about the global log canonical threshold is missing, and for that reason the main part of the proof of Theorem 0.1 starts when the base is modified and the centre of every maximal singularity covers a divisor on the base. In the next subsection we carry out some preparatory work for the subsequent exclusion of maximal singularities.

1.2. Supermaximal singularities. For any maximal singularity $E \in \mathcal{M}$ set

$$t_E = \mathrm{ord}_E\, T,$$

where $T = \tau(E)$. By construction, $T \supset \mathrm{centre}(E, V)$, so that $t_E \geqslant 1$. Let $\varphi\colon \widetilde{V} \to V$ be a birational morphism, resolving the singularities of the map χ. Every maximal singularity $E \in \mathcal{M}$ is realized on the variety $\widetilde{V}$ by a prime divisor, which we will denote by the same symbol E. By the definition of the numbers t_E we get: the divisor

$$\varphi^* T - \sum_{E \in \mathcal{M}_T} t_E E$$

is effective and contains none of the maximal singularities $E \in \mathcal{M}$ as a component. Now let us consider the strict transform $\widetilde{\mathcal{C}}$ on $\widetilde{V}$ of the mobile family of curves $\mathcal{C}$, which was fixed in Subsection 1.1. For $\widetilde{C} \in \widetilde{\mathcal{C}}$ we have:

$$\left(\left(\varphi^* T - \sum_{E \in \mathcal{M}_T} t_E E\right) \cdot \widetilde{C}\right) \geqslant 0.$$

Set $\nu_E = \mathrm{ord}_E\, \Sigma$ and let $a_E \geqslant 1$ be the discrepancy of E with respect to V. By the symbol $\widetilde{K}$ we denote the canonical class $K_{\widetilde{V}}$, so that for the strict transform $\widetilde{\Sigma}$ of the linear system Σ on $\widetilde{V}$ we have

$$\widetilde{\Sigma} \subset |-n\widetilde{K} + \widetilde{Y} + \Xi|,$$

where $\widetilde{Y} = \varphi^*\pi^* Y - \sum\limits_{E \in \mathcal{M}} \varepsilon(E) E$ (recall that $\varepsilon(E) = \nu_E - n a_E$) and Ξ is a linear combination of φ-exceptional divisors E', such that either the centre of E' on V' is a subvariety of codimension at least 2, or $\varepsilon(E') \leqslant 0$, and for that reason $(\widetilde{C} \cdot \Xi) \geqslant 0$. Therefore, the following inequality holds:

$$\sum_{E \in \mathcal{M}} \varepsilon(E)(E \cdot \widetilde{C}) > (\overline{Y} \cdot \overline{C}). \tag{1}$$

Recall that by the K-condition $(Y \cdot \overline{C}) \geqslant 0$. On the other hand, as we could see a bit earlier, the estimate

$$(\overline{T} \cdot \overline{C}) = (\varphi^* T \cdot \widetilde{C}) \geqslant \sum_{E \in \mathcal{M}_T} t_E (E \cdot \widetilde{C}) \tag{2}$$

holds.

Now let us consider the self-intersection $Z = (D_1 \circ D_2)$ of the mobile linear system Σ (where $D_1, D_2 \in \Sigma$ are general divisor which do not have common components due to the mobility). Let us write this effective algebraic cycle of codimension 2 in the following way:

$$Z = Z^h + Z^v + Z^\emptyset,$$

where in the sub-cycle Z^h are collected all components Z, covering the base (*the horizontal part* of Z), in the sub-cycle Z^v are collected all components of the cycle Z that are contained in the divisors $T \in \mathcal{T}$ and cover $\overline{T}$ (*the vertical part* of Z), and in the sub-cycle $Z^\emptyset$ are collected all the other components of the cycle Z (and that part of the cycle Z is inessential for us). Obviously, we have the presentation

$$Z^v = \sum_{T \in \mathcal{T}} Z^v_T,$$

where Z^v_T consists of those components of the vertical part, which are contained in the divisor T and cover $\overline{T}$.

Let $F = F_s = \pi^{-1}(s)$ be the fibre over a point of general position $s \in \overline{T}$. Since $\operatorname{Pic} F = \mathbb{Z} K_F$, we have

$$Z^v_T|_F \sim -\lambda_T K_F$$

for some $\lambda_T \in \mathbb{Z}_+$. Therefore,

$$(Z^v_T \cdot \pi^{-1}(\overline{C})) = \lambda_T (\overline{T} \cdot \overline{C}) H_F.$$

Definition 1.1. A maximal singularity $E \in \mathcal{M}_T$ is said to be *supermaximal*, if the inequality

$$2n\varepsilon(E) > \lambda_T \operatorname{ord}_E T \tag{3}$$

holds.

This definition is modelled on the definition of a supermaximal singularity for Fano fibre spaces over $\mathbb{P}^1$, see [8, Chapter 5], and plays the same role.

Proposition 1.3. *A supermaximal singularity exists.*

Proof. Since

$$Z \sim n^2 K_V^2 + 2n((-K_V) \cdot \pi^* Y) + \pi^*(Y^2),$$

we have

$$(Z \cdot \pi^{-1}(\overline{C})) = n^2 (K_V^2 \cdot \pi^{-1}(\overline{C})) + 2n(Y \cdot \overline{C}) H_F,$$

as obviously $(\pi^*(Y^2) \cdot \pi^{-1}(\overline{C})) = 0$. On the other hand,

$$(Z \cdot \pi^{-1}(\overline{C})) = (Z^h \cdot \pi^{-1}(\overline{C})) + \left(\sum_{T \in \mathcal{T}} \lambda_T (\overline{T} \cdot \overline{C}) \right) H_F + \lambda_\emptyset H_F$$

for some $\lambda_\emptyset \in \mathbb{Z}_+$. By the K^2-condition we get the inequality

$$2n(Y \cdot \overline{C}) \geqslant \sum_{T \in \mathcal{T}} \lambda_T (\overline{T} \cdot \overline{C}). \tag{4}$$

Combining the inequalities (1), (2) and (4), we get

$$2n \sum_{E \in \mathcal{M}} \varepsilon(E)(E \cdot \widetilde{C}) > \sum_{T \in \mathcal{T}} \lambda_T \left(\sum_{E \in \mathcal{M}_T} t_E (E \cdot \widetilde{C}) \right).$$

Taking into account that the set of maximal singularities $\mathcal{M}$ is a disjoint union of the subsets M_T, $T \in \mathcal{T}$, we see that in the last inequality every maximal singularity appears only once. Therefore, for some singularity $E \in \mathcal{M}_T$ the inequality

$$2n\varepsilon(E)(E \cdot \widetilde{C}) > \lambda_T t_E (E \cdot \widetilde{C})$$

holds. Since $(E \cdot \widetilde{C}) > 0$ for all $E \in \mathcal{M}$, this implies the inequality (3). Q.E.D. forthe proposition.

1.3. A remark on quadratic singularities. In [2, Theorem 4] and [9, . 2.1] it was shown that the quadratic singularities of rank at least $r \geqslant 1$ are stable with respect to blow ups. This fact can be made more precise in the following way.

Proposition 1.4. *Assume that an algebraic variety X has at most quadratic singularities of rank at least r, and moreover, the inequality*

$$\mathrm{codim}(\mathrm{Sing}\, X \subset X) \geqslant r$$

holds. Then for any irreducible subvariety $B \subset X$ there is a Zariski open subset $U \subset X$, such that $U \cap B \neq \emptyset$ and the blow up $\widetilde{U} \to U$ along the subvariety $B_U = B \cap U$ has at most quadratic singularities of rank at least r, and the following inequality holds:

$$\mathrm{codim}(\mathrm{Sing}\, \widetilde{U} \subset \widetilde{U}) \geqslant r. \qquad (5)$$

Remark 1.2. In [2, 9] the following obvious fact was used:if a variety X has at most quadratic singularities of rank at least r, then the inequality $\mathrm{codim}(\mathrm{Sing}\, X \subset X) \geqslant r - 1$ holds. Therefore, the codimension of the singular set $\mathrm{Sing}\, \widetilde{U}$ is at least $r - 1$. The proposition stated above makes the results of [2, 9] more precise: the property of the singular set of the variety X to have codimension at least r is also stable with respect to blow ups.

Proof of Proposition 1.4. By [2, Theorem 4] and [9, Subsection 2.1] we only need to show the inequality (5). Obviously, we may assume that $B \subset \mathrm{Sing}\, X$. Arguing as in Subsection 2.1 of [9], consider a Zariski open subset $U \subset X$, such that B_U is a non-singular subvariety, and moreover the rank of quadratic points $b \in B_U$ is constant and equal to $r_1 \geqslant r$. Let $E_U \subset \widetilde{U}$ be the exceptional divisor of the blow up $\varphi_B \colon \widetilde{U} \to U$ of the subvariety B_U. Obviously, $\varphi_B|_{E_U} \colon E_U \to B_U$ is a fibration into quadrics of rank r_1. It is clear that the set of singular points $\mathrm{Sing}(\widetilde{U} \backslash E_U)$ is of codimension at least r. However, a quadric of rank r_1 has a singular set of codimension $r_1 - 1$. Therefore,

$$\mathrm{codim}(\mathrm{Sing}\, E_U \subset E_U) = r_1 - 1$$

so that the more so

$$\operatorname{codim}((\operatorname{Sing}\widetilde{U}\cap E_U)\subset E_U)\geqslant r_1-1$$

and for that reason

$$\operatorname{codim}((\operatorname{Sing}\widetilde{U}\cap E_U)\subset U)\geqslant r_1\geqslant r.$$

Q.E.D. for the proposition.

2 Exclusion of supermaximal singularities

In this section we complete the proof of Thorem 0.1: we show that a maximal singularity can not exist. For that purpose, we use the technique of counting multiplicities (Subsection 2.1) in a modified form, adjusted to varieties with quadratic singularities. We prove that the multiplicities of the self-intersection of the mobile linear system Σ along the centres of the supermaximal singularity satisfy a certain quadratic inequality, which is impossible, as our computations in Subsection 2.2 show. This contradiction completes the proof of Theorem 0.1. In Subsection 2.3 we correct a small issue in [2].

2.1. The technique of counting multiplicities. Let us fix a supermaximal singularity E and the corresponding divisor $T=\pi^{-1}(\overline{T})$. To simplify the notations, we write Z^v instead of Z^v_T and λ instead of λ_T: the other singularities and divisors $T'\in\mathcal{T}$ take no part in the subsequent arguments. Let

$$V_K\to\ldots\to V_i\to V_{i-1}\to\ldots\to V_0=V$$

be the resolution of the singularity E, that is, the sequence of blow ups $\varphi_{i,i-1}\colon V_i\to V_{i-1}$ of irreducible subvarieties $B_{i-1}=\operatorname{centre}(E,V_{i-1})$ with exceptional divisors $E_i=\varphi^{-1}_{i,i-1}(B_{i-1})$, where the last exceptional divisor E_K is the supermaximal singularity E. The set of indices $I=\{1,\ldots,K\}$, parameterizing the blow ups, is the disjoint union

$$I=I_0\sqcup I_1\sqcup\ldots\sqcup I_{M-1},$$

where $M=\dim F$ is the dimension of the fibre and $i\in I_k$ if and only if $\dim B_{i-1}=\dim S-1+k$ (obviously, $\pi\circ\varphi_{i-1,0}(B_{i-1})=T$, so that $\dim B_{i-1}\geqslant\dim T$; here

$$\varphi_{i,j}=\varphi_{j+1,j}\circ\ldots\circ\varphi_{i,i-1}\colon V_i\to V_j$$

is a composition of elementary blow ups). Certain sets I_k can be empty. By Proposition 1.4 for $j\in I_{M-2}\cup I_{M-1}$ we have $B_{j-1}\not\subset\operatorname{Sing}V_{j-1}$. For $j\in I$ set

$$\mu_j=\operatorname{mult}_{B_{j-1}}V_{j-1}\in\{1,2\},$$

so that for $j\in I_{M-2}\cup I_{M-1}$ we have $\mu_j=1$. The strict transform of a subvariety, an effective divisor or a linear system on V_j we denote by adding the upper index j. For a general divisor $D\in\Sigma$ write

$$D^j=\varphi^*_{j,j-1}(D^{j-1})-\nu_jE_j.$$

Let $Z=(D_1\circ D_2)$ be the self-intersection of the mobile system Σ. Writing in the usual way (see [8, Chapter 2])

$$(D_1^i\circ D_2^i)=(D_1^{i-1}\circ D_2^{i-1})^i+Z_i,$$

where Z_i is an effective cycle of codimension 2 with the support inside the exceptional divisor E_i, we define the *degree* d_i of the cycle Z_i in the following way. If $B_{i-1}\not\subset \operatorname{Sing} V_{i-1}$, then for a point $p\in B_{i-1}$ of general position $\varphi_{i,i-1}^{-1}(p)$ is the projective space $\mathbb{P}^{\delta_i}$ and

$$d_i=\deg(Z_i|_{\varphi_{i,i-1}^{-1}(p)})$$

is the degree of an effective divisor in that projective space. If $B_{i-1}\subset \operatorname{Sing} V_{i-1}$, then for a general point $p\in B_{i-1}$ the fibre $\varphi_{i,i-1}^{-1}(p)$ is an irreducible quadric in the projective space $\mathbb{P}^{\delta_i+2}$ and d_i is the degree of the effective cycle $Z_i|_{\varphi_{i,i-1}^{-1}(p)}$ in that projective space. In both cases δ_i means the elementary discrepancy $\operatorname{codim}(B_{i-1}\subset V_{i-1})-\mu_i$. As usual, we break the set I into the *lower part*

$$I_l=I_0\sqcup\ldots\sqcup I_{M-2}$$

and the *upper part* $I_u=I_{M-1}$, and set

$$L=\max\{i\in I_l\}.$$

Finally for $0\leqslant i<j\leqslant L$ set

$$m_{i,j}=\operatorname{mult}_{B_{j-1}}(Z_i^{j-1}),$$

for $i=0$ we write simply m_j. The technique of counting multiplicities ([8, Chapter 2]) gives the system of equalities

$$\begin{array}{rcl}
\mu_1\nu_1^2+d_1 & = & m_1,\\
\mu_2\nu_2^2+d_2 & = & m_2+m_{1,2},\\
 & \ldots & \\
\mu_i\nu_i^2+d_i & = & m_i+m_{1,i}+\ldots+m_{i-1,i},\\
 & \ldots & \\
\mu_L\nu_L^2+d_L & = & m_L+m_{1,L}+\ldots+m_{L-1,L}.\\
\cdot & &
\end{array}$$

Besides, we have the estimate

$$d_L\geqslant\sum_{i=L+1}^{K}\nu_i^2.$$

Let Γ be the oriented graph of the resolution of the singularity E, that is, the graph with the set of vertices I and an oriented edge (arrow) joins the vertices i and j (notation: $i\to j$) if and only if $i>j$ and $B_{i-1}\subset E_j^{i-1}$. Recall [8, Chapter 2, Definition 2.1], that a function $a\colon I_l \to \mathbb{R}_+$ is *compatible with the structure of the graph* Γ, if

$$a(i) \geqslant \sum_{I_l \ni j \to i} a(j)$$

for every $i \in I_l$.

Proposition 2.1. *The function*

$$r_i = r(i) = \mathrm{ord}_E\, \varphi^*_{K,i} E_i$$

is compatible with the structure of the graph Γ.

Proof. The cartier divisor

$$\varphi^*_{K,i} E_i - \sum_{I_l \ni j \to i} \varphi^*_{K,j} E_j$$

is effective, which immediately implies the claim of the proposition. Q.E.D.

Now [8, Chapter 2, Proposition 2.4] gives the inequality

$$\sum_{i=1}^{L} r_i m_i \geqslant \sum_{i=1}^{L} r_i \mu_i \nu_i^2 + r_L \sum_{i=L+1}^{K} \nu_i^2.$$

Extending the definition of the numbers r_i to $i \in I_{M-2}$ and using the obvious fact that r_i is non-increasing as a function of i, we get finally:

$$\sum_{i=1}^{L} r_i m_i \geqslant \sum_{i=1}^{K} r_i \mu_i \nu_i^2. \tag{6}$$

Remark 2.1. Let p_{ai} be the number of paths in the oriented graph Γ from the vertex a to the vertex i for $a \neq i$ (so that $p_{ai} = 0$ for $a < i$); set $p_{ii} = 1$ for all $i \in I$. Usually (see [8, Chapter 2]) the technique of counting multiplicities makes use of the numbers p_{Ki} istead of r_i in the inequalities of the type (6), and it is easy to see that for $\mu_1 = 1$ the equality $r_i = p_{Ki}$ holds. If $\mu_1 = 2$, then $r_1 \geqslant p_{K1}$ (see below). The inequality (6) remains true, if we replace r_i by p_{Ki}, however such a modification is hard to use, since it is the coefficients r_i that appear both in the explicit form of the Noether-Fano inequality, and in the explicit expression for $\mathrm{ord}_E\, \varphi^*_{K,0} T$.

Set $L_{\mathrm{sing}} = \max\{1 \leqslant i \leqslant L \mid \mu_i = 2\}$.

Proposition 2.2. (i) *For* $i \geqslant 1 + L_{\mathrm{sing}}$ *the equality* $r_i = p_{Ki}$ *holds.* (ii) *For* $1 \leqslant i \leqslant L_{\mathrm{sing}}$ *the inequality*

$$p_{Ki} \leqslant r_i \leqslant 2 p_{Ki}$$

holds.

Proof. The claim (i) is obvious, since for $i \geqslant 1 + L_{\mathrm{sing}}$ the exceptional divisor E_i is non-singular over a general point of the subvariety B_{i-1}, so that

$$\mathrm{ord}_E\, \varphi^*_{K,i} E_i = \sum_{j \to i} \mathrm{ord}_E\, \varphi^*_{K,j} E_j$$

and the decreasing induction gives the equality $r_i = p_{Ki}$. For $i \leqslant L_{\text{sing}}$ the fibre of the exceptional divisor E_i over a point of general position on B_{i-1} is a quadric of rank at least 4. If for $j \leqslant L_{\text{sing}}$, $j > i$, we have $j \to i$, then, obviously,

$$\varphi^*_{j,j-1}(E_i^{j-1}) = E_i^j + E_j, \tag{7}$$

as in the non-singular case. If $j \to i$ for some $j \geqslant 1 + L_{\text{sing}}$, then two cases are possible:

1) $B_{j-1} \not\subset \operatorname{Sing} E_i^{j-1}$, and then again the equality (7) holds,

2) $B_{j-1} \subset \operatorname{Sing} E_i^{j-1}$, and then the equality

$$\varphi^*_{j,j-1}(E_i^{j-1}) = E_i^j + 2E_j \tag{8}$$

holds.

We emphasize that if the equality (8) holds, then $j > L_{\text{sing}}$, so that

$$\operatorname{ord}_E \varphi^*_{K,j} E_j = p_{Kj}.$$

For that reason, every path in the graph Γ from the top vertex K to the vertex i gives an input into the number r_i, which is equal to 1 or 2, and the latter takes place if and only if the path is of the form

$$i = j_0 \leftarrow j_1 \leftarrow \ldots \leftarrow j_k \leftarrow j_{k+1} \leftarrow \ldots \leftarrow j_m = K,$$

where $j_k \leqslant L_{\text{sing}}$, $j_{k+1} > L_{\text{sing}}$ and for the arrow $j_{k+1} \to j_k$ the case 2), described above, is realized. Q.E.D. for the proposition.

2.2. End of the proof of Theorem 0.1. Recall that above we defined the elementary discrepancies $\delta_i = \operatorname{codim}(B_{i-1} \subset V_{i-1}) - \mu_i$ for $i = 1, \ldots, K$. Set

$$L_{\text{fibre}} = \max\{1 \leqslant i \leqslant K \mid B_{i-1} \subset T^{i-1}\}.$$

For $1 \leqslant i \leqslant L_{\text{fibre}}$ we define the numbers $\gamma_i \in \mathbb{Z}$ by the equalities

$$\varphi^*_{i,i-1}(T^{i-1}) = T^i + \gamma_i E_i,$$

so that $\gamma_i \in \{1, 2\}$.

Proposition 2.3. *The following equalities hold:*

(i) *the multiplicity of the linear system Σ with respect to E satisfies the relation*

$$\operatorname{ord}_E \Sigma = \sum_{i=1}^{K} r_i \nu_i, \tag{9}$$

(ii) *the multiplicity of the divisor T with respect to E satisfies the relation*

$$\operatorname{ord}_E T = \sum_{i=1}^{K} r_i \gamma_i, \tag{10}$$

(iii) *the discrepancy of E satisfies the relation*

$$a(E)=\sum_{i=1}^{K}r_i\delta_i. \tag{11}$$

Proof repeats the arguments in the non-singular case (see [8, Chapter 2]) word for word, just the number of paths p_{Ki} should be replaced by the new coefficients r_i. We will show the equality (9); in the other cases the arguments are similar. We use the induction on $K\geqslant 1$. If $K=1$, then the equality (9) is obvious. Let $K\geqslant 2$. For a general divisor $D\in\Sigma$ write:

$$\varphi_{1,0}^*D=D^1+\nu_1E_1,$$

so that $\varphi_{K,0}^*D=\varphi_{K,1}^*D^1+\nu_1\varphi_{K,1}^*E_1$ and for that reason

$$\mathrm{ord}_E\,\Sigma=\mathrm{ord}_E\,D=\mathrm{ord}_E\,D^1+r_1\nu_1.$$

For D^1 the claim of the proposition holds by the induction hypothesis. The proof is complete.

Set $L^*=\min(L,L_{\mathrm{fibre}})$ and

$$m_i^h=\mathrm{mult}_{B_{i-1}}(Z^h)^{i-1}$$

for $i=1,\dots,L$, and

$$m_i^v=\mathrm{mult}_{B_{i-1}}(Z^v)^{i-1}$$

for $i=1,\dots,L^*$. Now the left-hand side of the inequality (6) rewrites in the form

$$\sum_{i=1}^{L}r_im_i^h+\sum_{i=1}^{L^*}r_im_i^v. \tag{12}$$

The first component in this sum does not exceed

$$4n^2\sum_{i=1}^{L}r_i,$$

since the sequence of multiplicities m_i^h is not increasing, and

$$m_i^h=\mathrm{mult}_{B_0}\,Z^h\leqslant\mathrm{mult}_{B_0}(Z^h\circ T)\leqslant 4n^2$$

by the condition (h). The "vertical" component in the sum (12) by the condition (v) does not exceed the number

$$2\lambda\sum_{i=1}^{L^*}r_i\leqslant 2\lambda\,\mathrm{ord}_E\,T$$

(see the equality (10)), and the right hand side of the last inequality is strictly smaller than $4ne$, where $e = \varepsilon(E)$, by the definition of a supermaximal singularity (the inequality (3)). Combining these estimates, we get that the left hand side of the inequality (6) is strictly smaller than the expression

$$4n^2 \sum_{i=1}^{L} r_i + 4ne.$$

Let us consider now the right hand side of the inequality (6). By the definition of the number $\varepsilon(E)$ we have:

$$\sum_{i=1}^{K} r_i \nu_i = n \sum_{i=1}^{K} r_i \delta_i + e \tag{13}$$

(so that in these notations the Noether-Fano inequality takes the form of the estimate $e > 0$). Using the standard methods, it is easy to check that the minimum of the right hand side of the inequality (6) on the hyperplane in the space $\mathbb{R}^K_{(\nu_1,\dots,\nu_K)}$, given by the equation (13), is attained for $\nu_i = \theta/\mu_i$, where θ can be found from the equation (13). We introduce the following notations:

$$\Sigma_l = \sum_{i=1}^{L} r_i, \quad \Sigma_u = \sum_{i=L+1}^{K} r_i, \quad \Sigma_{\text{sing}} = \sum_{i=1}^{L_{\text{sing}}} r_i, \quad \Sigma_{\text{non-sing}} = \sum_{i=L_{\text{sing}}+1}^{K} r_i.$$

In these notations the inequality (6) implies the estimate

$$4n^2\Sigma_l + 4ne > 2\,\frac{(2n\Sigma_l + n\Sigma_u + e)^2}{\Sigma_{\text{sing}} + 2\Sigma_{\text{non-sing}}}.$$

Taking into account that $\Sigma_{\text{sing}} + \Sigma_{\text{non-sing}} = \Sigma_l + \Sigma_u$, after easy computations we get:

$$2n^2\Sigma_l\Sigma_{\text{non-sing}} + 2ne\Sigma_{\text{non-sing}} > 2n^2\Sigma_l^2 + 2n^2\Sigma_l\Sigma_u + n^2\Sigma_u^2 + 2ne\Sigma_l + e^2.$$

However, $\Sigma_{\text{non-sing}} \leqslant \Sigma_l + \Sigma_u$, so that the previous inequality implies the estimate

$$2ne\Sigma_u > n^2\Sigma_u^2 + e^2,$$

which can not be true. This contradiction excludes the supermaximal singularity and completes the proof of Theorem 0.1.

2.3. Birationally rigid Fano hypersurfaces. In the context of the proceedings performed in this subsection, let us consider the problem of estimating the codimension of the set of non-rigid hypersurfaces of degree M in $\mathbb{P}^M$, which was set and solved in [2]. Working on the present paper, the author detected an incorrectness in that paper in the proof of the $4n^2$-inequality for Fano hypersurfaces with quadratic singularities of rank at least 5 for $M \geqslant 5$ ([2, Section 3]). In this subsection we explain what was incorrect and how it should be corrected. Note that the main claim of ([2, Proposition 1]), and the method of its proof are valid.

Recall that in [2, Section 3] the following local fact was shown. Let X be an algebraic variety with quadratic (in particular, hypersurface) singularities of rank at least 5 (so that the set of singular points $\mathrm{Sing}\, X$ is of codimension at least 4 and for that reason the variety X is factorial), $B \subset \mathrm{Sing}\, X$ an irreducible subvariety, Σ a mobile linear system on X, and moreover, for some $n \geqslant 1$ the pair $(X, \frac{1}{n}\Sigma)$ is not canonical; more precisely, it has a non canonical singularity E with the centre at B. Then the self-intersection $Z = (D_1 \circ D_2)$, $D_i \in \Sigma$ are general divisors, satisfies the inequality

$$\mathrm{mult}_B\, Z > 4n^2.$$

(The multiplicity is understood in the usual sense, see [8, Chapter 2].) In fact, the assumptions can be somewhat relaxed. The following claim is true.

Proposition 2.4. *Let X be a variety with quadratic singularities of rank at least 4, and assume that* $\mathrm{codim}(\mathrm{Sing}\, X \subset X) \geqslant 4$. *Assume further that a certain divisor E over X is a non canonical singularity of the pair $(X, \frac{1}{n}\Sigma)$ with the centre $B \subset \mathrm{Sing}\, X$, where Σ is a mobile linear system. Then the self-intersection Z of the system Σ satisfies the inequality*

$$\mathrm{mult}_B\, Z > 4n^2.$$

Proof. We only point out what should be modified in the arguments of [2, Section 3]. It follows from Proposition 1.4 that the technique of counting multiplicities works without changes under the relaxed assumptions about the rank of quadratic singularities. Furthermore, in [2, Section 3] it is claimed erroneously that the Noether-Fano inequality has the form

$$\sum_{i=1}^{K} p_i \nu_i > n \left(\sum_{i=1}^{K} p_i \delta_i \right), \tag{14}$$

where p_i is the number of paths in the oriented graph of the resolution of the singularity E from the top vertex to the vertex i (the meaning of all notations is exactly the same as in Subsection 2.1 of the present section). In fact, in the inequality (14) instead of p_1 the coefficients r_i, introduced in Subsection 2.1, must be used. After the replacement of the coefficients p_i by the coefficients r_i all the arguments in [2, Section 3] work as they are and prove Proposition 2.4.

3 Hypersurfaces with non-isolated singularities

In this section we prove Theorem 0.3. The procedure of estimating the codimension of the set of hypersurfaces in the projective space with a singular set of a positive dimension, depends on the type of that singular set. In Subsection 3.1 we consider some simple cases (for instance, when the singular set is a line), where the codimension of the set of hypersurfaces with a singular set of the given type can be directly estimated or explicitly computed. In Subsection 3.2 we develop a technique that makes it possible to estimate the codimension of the set of hypersurfaces with at least finite, but sufficiently large set of singular points. In Subsection 3.3 we apply this technique and complete the proof of Theorem 0.3.

3.1. The sets of singular hypersurfaces. Let $\mathbb{P}^N$ be the projective space with homogeneous coordinates $(x_0 : x_1 : \ldots : x_N), N \geqslant 3$, and $\mathcal{P}_{N,d} = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ the linear space of homogeneous polynomials of degree d. For $f \in \mathcal{P}_{N,d}$ the set of singular points of the hypersurface $\{f = 0\}$ we denote by the symbol $\mathrm{Sing}(f)$. Set

$$\mathcal{P}^{(i)}_{N,d} = \{f \in \mathcal{P}_{N,d} \,|\, \dim \mathrm{Sing}(f) \geqslant i\}.$$

These are closed subsets of the space $\mathcal{P}_{N,d}$, and for $i \geqslant j$ we have $\mathcal{P}^{(j)}_{N,d} \subset \mathcal{P}^{(i)}_{N,d}$.

Example 3.1. Let $\mathcal{P}^{\mathrm{line}}_{N,d}$ be the closed subset of the space $\mathcal{P}_{N,d}$, consisting of polynomials f such that the set $\mathrm{Sing}(f)$ contains a line in $\mathbb{P}^N$. Fixing a line $L \subset \mathbb{P}^N$, we may assume that $L = \{x_2 = \ldots = x_N = 0\}$, so that the condition $L \subset \mathrm{Sing}(f)$ is equivalent to the set of equalities

$$\left.\frac{\partial f}{\partial x_0}\right|_L \equiv \left.\frac{\partial f}{\partial x_1}\right|_L \equiv \ldots \equiv \left.\frac{\partial f}{\partial x_N}\right|_L \equiv 0,$$

whence, taking into account the dimension of the Grassmanian of lines in $\mathbb{P}^N$, we obtain the equality

$$\mathrm{codim}(\mathcal{P}^{\mathrm{line}}_{N,d} \subset \mathcal{P}_{N,d}) = (d-2)N + 3.$$

The following claim is true, which immediately implies Theorem 0.3.

Theorem 3.1. *The following inequality holds:*

$$\mathrm{codim}(\mathcal{P}^{(1)}_{N,d} \subset \mathcal{P}_{N,d}) \geqslant (d-2)N.$$

Remark 3.1. It seems that the inequality of Theorem 3.1 can be improved, replacing its right hand side by $(d-2)N + 3$, after which it would become precise. However, the proof below is insufficient for that purpose. In any case the claim of Theorem 3.1 is much stronger than what we need in this paper.

Proof of Theorem 3.1. Let $\mathcal{P}^{(i,k)}_{N,d} \subset \mathcal{P}_{N,d}$ be the closure of the set, consisting of polynomials f such that $\mathrm{Sing}(f)$ contains an irreducible component C of dimension $i \geqslant 1$, the linear span of which $\langle C \rangle$ is a k-plane in $\mathbb{P}^N$, $k \geqslant i$. For instance, $\mathcal{P}^{(1,1)}_{N,d} = \mathcal{P}^{\mathrm{line}}_{N,d}$. Obviously,

$$\mathcal{P}^{(i)}_{N,d} = \bigcup_{k=i}^{N} \mathcal{P}^{(i,k)}_{N,d},$$

so that in order to estimate the codimension of the set $\mathcal{P}^{(i)}_{N,d}$, it is sufficient to estimate the codimension of each set $\mathcal{P}^{(i,k)}_{N,d}$, $k = i, \ldots, N$. Furthermore, for a k-plane $P \subset \mathbb{P}^N$

consider the set $\mathcal{P}^{(i,k)}_{N,d}(P)$, which is the closure of the subset, consisting of polynomials f such that Sing f contains an irreducible component C of dimension i and $\langle C\rangle = P$. The following fact is obvious.

Proposition 3.1. *The following inequality holds:*

$$\operatorname{codim}\left(\mathcal{P}^{(i,k)}_{N,d} \subset \mathcal{P}_{N,d}\right) \geqslant \operatorname{codim}(\mathcal{P}^{(i,k)}_{N,d}(P) \subset \mathcal{P}_{N,d}) - (k+1)(N-k).$$

Finally, let $\mathcal{P}^{(i,k;l)}_{N,d}(P) \subset \mathcal{P}^{(i,k)}_{N,d}(P)$ be the closure of the subset, consisting of such f that (in terms of the definition of the set $\mathcal{P}^{(i,k)}_{N,d}(P)$) the set of singular points $\operatorname{Sing}(f|_P)$ contains an irreducible component B of dimension l such that $C \subset B \subset P$. In particular, $l \geqslant i$ and

$$\mathcal{P}^{(i,k)}_{N,d}(P) = \bigcup_{l=i}^{k} \mathcal{P}^{(i,k;l)}_{N,d}(P).$$

Now by Proposition 3.1 the claim of Theorem 3.1 follows from the system of inequalities

$$\operatorname{codim}(\mathcal{P}^{(1,k;l)}_{N,d}(P) \subset \mathcal{P}_{N,d}) \geqslant (d-2)N + (k+1)(N-k), \tag{15}$$

which we will prove for all $1 \leqslant l \leqslant k \leqslant N$ and a fixed k-plane $P \subset \mathbb{P}^N$, given by the equations $\{x_{k+1} = \ldots = x_N = 0\}$.

Example 3.2. Consider the case $l = k = 2$. In that case P is a plane, $P \subset \{f = 0\}$ and the closed set $\operatorname{Sing}(f)$ contains an irreducible plane curve $C \subset P$ of degree $q \geqslant 2$. This gives $(d+1)(d+2)/2$ independent conditions on the coefficients of the polynomial $f|_P$ (they all vanish) and $(N-2)$ polynomials

$$\left.\frac{\partial f}{\partial x_3}\right|_P, \ldots, \left.\frac{\partial f}{\partial x_N}\right|_P$$

vanish on the curve C. Note that the coefficients of the polynomials $f|_P$, $\partial f/\partial x_i|_P$, $i = 3, \ldots, N$ up to a non-zero integral factor are distinct coefficients of the polynomial f. We may assume that at least one of the polynomials $\partial f/\partial x_i|_P$ is not identical zero, say, $\partial f/\partial x_3|_P \not\equiv 0$. Then the curve C is an irreducible component of the plane curve $\{\partial f/\partial x_3|_P = 0\}$. Fixing the polynomial $\partial f/\partial x_3|_P$, we finally obtain

$$\frac{(d+1)(d+2)}{2} + (N-3)\left(qd - \frac{q(q-1)}{2}\right)$$

independent conditions on the coefficients of the polynomial f, where $2 \leqslant q \leqslant d-1$. It ie easy to see that this number satisfies the inequality (15).

Example 3.3. Consider the case $l = 1, k = 2$. In that case $\operatorname{Sing}(f)$ contains an irreducible plane curve $C \subset P$ of degree $q \geqslant 2$, but $f|_P \not\equiv 0$, so that $\{f|_P = 0\}$ is a reducible plane curve of degree d, containing C as a double component, so that $2q \leqslant d$. An easy dimension count gives

$$\frac{1}{2}(5q^2 - (4d+3)q + d^2 + 3d + 4)$$

independent conditions on the coefficients of the polynomial $f|_P$. The minimum of the last expression is attained for $q=2$. Now the fact that the polynomials $\partial f/\partial x_i|_P$, $i=3,\dots,N$, vanish on the curve C, gives in addition at least $(N-2)(2d+1)$ independent conditions on the coefficients of f. As a result, we get

$$\operatorname{codim}\left(\mathcal{P}^{(1,2;1)}_{d,N}(P)\right) \geqslant \frac{d(d-5)}{2} + (N-2)(2d+1)+9,$$

and it is easy to check that the inequality (15) is satisfied.

Starting from this moment, we assume that $k=3$. Recall the following

Definition 3.1. (See [5, Section 3] or [8, Chapter 3]). A sequence of homogeneous polynomials $g_1,\dots,g_m$ of arbitrary degrees on the projective space $\mathbb{P}^e$, $e \geqslant m+1$, is called a *good sequence*, and an irreducible subvariety $W \subset \mathbb{P}^e$ of codimension m is its *associated subvariety*, if there exists a sequence of irreducible subvarieties $W_j \subset \mathbb{P}^e$, $\operatorname{codim} W_j = j$ (in particular, $W_0 = \mathbb{P}^e$), such that:

- $g_{j+1}|_{W_j} \not\equiv 0$ for $j=0,\dots,m+1$,
- W_{j+1} is an irreducible component of the closed algebraic set $g_{j+1}|_{W_j}=0$,
- $W_m = W$.

A good sequence can have more than one associated subvarieties, but their number is bounded from above by a constant depending on the degrees of the polynomials q_j only (see [5, Section 3]).

Let us consider two more examples, similar to Examples 3.2 and 3.3.

Example 3.4. Let us consider the case $l=k$. This case generalizes Example 3.2. We have $f|_P \equiv 0$, which gives $\binom{k+d}{d}$ independent conditions on the coefficients of f. Since the polynomials

$$\left.\frac{\partial f}{\partial x_{k+1}}\right|_P, \dots, \left.\frac{\partial f}{\partial x_N}\right|_P$$

vanish identically on C and the curve C is an irreducible component of the set $\operatorname{Sing}(f)$, from those polynomials we can choose $(k-1)$ ones that form a good sequence with the curve C as an associated subvariety (in particular, $N-k \geqslant k-1$). Fixing these polynomials, for each of the remaining $(N+1-2k)$ polynomials we get the condition

$$\left.\left(\left.\frac{\partial f}{\partial x_i}\right|_P\right)\right|_C \equiv 0, \tag{16}$$

where the curve C, as one of the associted subvarieties of the fixed good sequence, can be assumed to be fixed. In [5, Section 3] it was shown that the condition (16) defines a closed subset of codimension at least $(d-1)k+1$. Therefore,

$$\operatorname{codim}(\mathcal{P}^{(i,k;k)}_{N,d} \subset \mathcal{P}_{N,d}) \geqslant \binom{k+d}{d} + (N+1-2k)((d-1)k+1),$$

and elementary computations show that the inequality (15) holds.

Example 3.5. Let us consider the case $l = k-1$. This case generalizes Example 3.3. Here the hypersurface $\{f|_P = 0\}$ has a multiple irreducible non-degenerate component of degree q, where $2q \leqslant d$, so that the coefficients of the polynomial $f|_P$ belong to a closed subset of codimension

$$\binom{k+d}{k} - \binom{k+d-2q}{k} - \binom{k+q}{k}$$

in the space $\mathcal{P}_{d,k}$. Furthermore, since the curve C is an irreducible component of the set $\mathrm{Sing}(f)$, from the set of polynomials

$$f|_P, \left.\frac{\partial f}{\partial x_{k+1}}\right|_P, \dots, \left.\frac{\partial f}{\partial x_N}\right|_P$$

we can choose a good sequence, starting with $f|_P$, for which the curve C will be an associated subvariety. In particular, the estimate $N+2 \geqslant 2k$ holds. Fixing the polynomials of that sequence, we may assume the curve C to be fixed. Now we argue as in Example 3.4 and obtain, in addition to the conditions on the coefficients of the polynomial $f|_P$, also $(N+2-2k)((d-1)k+1)$ more independent conditions on the coefficients of the polynomial f. An elementary, although tedious, check shows that the inequality (15) is satisfied.

In order to prove the inequality (15) in the case $l \leqslant k-2$, we need a new technique, which is developed below.

3.2. Linearly independent points. The following claim is true.

Lemma 3.1. *Assume that $d \geqslant 3$. For any set of m linearly independent points $p_1, \dots, p_m \in \mathbb{P}^N$, $m \leqslant N+1$, the condition*

$$\{p_1, \dots, p_m\} \subset \mathrm{Sing}(g),$$

$g \in \mathcal{P}_{N,d}$, defines a linear subspace of codimension $m(N+1)$ in $\mathcal{P}_{N,d}$.

Proof. We may assume that

$$p_1 = (1:0:0\dots:0), \quad p_2 = (0:1:0:\dots:0)$$

etc. correspond to the first m vectors of the standard basis of the linear space $\mathbb{C}^{N+1}$. The condition $p_i \in \mathrm{Sing}(g)$ means vanishing the coefficients at the monomials $x^d_{i-1}, x^{d-1}_{i-1}x_j$, for all $j \neq i-1$. For $d \geqslant 3$ all those $m(N+1)$ monomials are distinct. Q.E.D. for the lemma.

Now let us consider an arbitrary linear subspace $\Pi \subset \mathbb{P}^N$ of codimension $r+1$, where $r \geqslant 1$, given by a system of $r+1$ equaltions

$$l_0(x) = 0,\ l_1(x) = 0, \dots, l_r(x) = 0,$$

where $l_0, \ldots, l_r$ are linearly independent forms. For every $i = 1, \ldots, r$ let us fix an arbitrary tuple of distinct constants $\lambda_{i0}, \ldots, \lambda_{i,d-1} \in \mathbb{C}$; we assume that $\lambda_{i0} = 0$ for all $i = 1, \ldots, r$. Now for any integer point

$$\underline{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r_+, \quad e_i \leqslant d - 1,$$

by the symbol $\Theta(\underline{e})$ we denote the linear subspace

$$\{l_i(x) - \lambda_{i,e_i} l_0(x) = 0 \,|\, i = 1, \ldots, r\} \subset \mathbb{P}^N$$

of codimension r. Obviously, $\Theta(\underline{e}) \supset \Pi$. Set

$$|\underline{e}| = e_1 + \ldots + e_r \in \mathbb{Z}_+.$$

For each tuple $\underline{e} \in \mathbb{Z}^r_+$ with $|\underline{e}| \leqslant d - 3$ consider an arbitrary set

$$S(\underline{e}) = \{p_1(\underline{e}), \ldots, p_m(\underline{e})\} \subset \Theta(\underline{e}) \backslash \Pi$$

of m linearly independent points (so that $m \leqslant N - r + 1$).

Proposition 3.2. *The set of conditions*

$$S(\underline{e}) \subset \mathrm{Sing}(\mathrm{g}|_{\Theta(\underline{\mathrm{e}})}),$$

$\underline{e} \in \mathbb{Z}^r_+$, $|\underline{e}| \leqslant d - 3$, *defines a linear subspace of codimension*

$$m(N - r + 1)|\Delta|$$

in $\mathcal{P}_{N,d}$, *where*

$$\Delta = \{e_1 \geqslant 0, \ldots, e_r \geqslant 0, e_1 + \ldots + e_r \leqslant d - 3\} \subset \mathbb{R}^r$$

is an integral simplex and $|\Delta|$ *means the number of integral points in that simplex,* $|\Delta| = \sharp(\Delta \cap \mathbb{Z}^r)$.

Proof. We may assume that $l_0 = x_0$, $l_1 = x_1, \ldots, l_r = x_r$. In order to simplify the formulas, we will prove the affine version of the proposition: set $v_1 = x_1/x_0, \ldots, v_r = x_r/x_0$ and $u_i = x_{r+i}/x_0$, $i = 1, \ldots, N - r$. In the affine space $\mathbb{A}^N \subset \mathbb{P}^N$, $\mathbb{A}^N = \mathbb{P}^N \backslash \{x_0 = 0\}$ with coordinates $(u, v) = (u_1, \ldots, u_{N-r}, v_1, \ldots, v_r)$ the affine spaces $A(\underline{e}) = \Theta(\underline{e}) \backslash \Pi$ are contained entirely:

$$A(\underline{e}) = \Theta(\underline{e}) \cap \mathbb{A}^N,$$

so that $S(\underline{e}) \subset A(\underline{e})$ for all $\underline{e}$. Obviously,

$$A(\underline{e}) = \{v_1 = \lambda_{1,e_1}, \ldots, v_r = \lambda_{r,e_r}\} \subset \mathbb{A}^N$$

is a $(N-r)$-plane, which is parallel to the coordinate $(N-r)$-plane $(u_1, \ldots, u_{N-r}, 0, \ldots, 0)$. Let us write the polynomial g in terms of the affine coordinates (u, v) in the following way:

$$g(u, v) = \sum_{\underline{e} \in \mathbb{Z}^r_+, |\underline{e}| \leqslant d} g_{e_1,\ldots,e_r}(u) \prod_{i=1}^{r} \prod_{j=0}^{e_i - 1} (v_i - \lambda_{ij})$$

(if $e_i = 0$, then the corresponding product is assumed to be equal to 1). Here $g_{\underline{e}}(u) = g_{e_1,\dots,e_r}(u)$ is an affine polynomial in $u_1, \dots, u_{N-r}$ of degree $\deg g_{\underline{e}} \leqslant d - |e|$. For the fixed λ_{ij} this presentation is unique. By Lemma 3.1, the condition

$$S(\underline{0}) = S(0, \dots, 0) \subset \mathrm{Sing}(g|_{A(\underline{0})})$$

defines a linear subspace of codimension $m(N - r + 1)$ in the space of polynomials $\mathcal{P}_{N-r,d}$. However, it is easy to see that

$$g|_{A(\underline{0})} = g_{0,\dots,0}(u),$$

since for $\underline{e} \neq \underline{0}$ in the product

$$\prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (v_i - \lambda_{ij})$$

there is at least one factor $(v_i - \lambda_{i0}) = v_i$, which vanishes when restricted onto the $(N - r)$-plane $A(\underline{0})$. Therefore, the condition $S(\underline{0})\,\mathrm{Sing}(g|_{A(\underline{0})})$ imposes on the coefficients of the polynomial $g_{0,\dots,0}(u)$ precisely $m(N - r + 1)$ independent linear conditions, whereas the polynomials $g_{\underline{e}}(u)$ for $\underline{e} \neq \underline{0}$ can be arbitrary.

Now let us complete the proof of Proposition 3.2 by induction on $|\underline{e}|$. More precisely, for any $a \in \mathbb{Z}_+$ set

$$\Delta_a = \{e_1 \geqslant 0, \dots, e_r \geqslant 0,\ e_1 + \dots + e_r \leqslant a\} \subset \mathbb{R}^r,$$

so that $\Delta = \Delta_{d-3}$, and let us prove the claim of Proposition 3.2 in the following form: for every $a = 0, \dots, d - 3$

$(*)_a$ the set of conditions

$$S(\underline{e}) \subset \mathrm{Sing}(g|_{\Theta(\underline{e})}),$$

$\underline{e} \in \mathbb{Z}^r_+$, $|\underline{e}| \leqslant a$, defines a linear subspace of codimension $m(N-r+1)|\Delta_a|$ in $\mathcal{P}_{N,d}$, where the restrictions are imposed on the coefficients of the polynomials $g_{\underline{e}}(u)$ for $\underline{e} \in \Delta_a$, whereas for $\underline{e} \notin \Delta_a$ the polynomials $g_{\underline{e}}(u)$ can be arbitrary.

The case $a = 0$ has already been considered, so we assume that $a \leqslant d - 4$ and the claims $(*)_j$ have been shown for $j = 0, \dots, a$. Let us show the claim $(*)_{a+1}$. Let $\underline{e} \in \mathbb{Z}^r_+$ be an arbitrary multi-index, $|\underline{e}| = a + 1$. The restriction onto the affine subspace $A(\underline{e})$ means the substitution $v_1 = \lambda_{1,e_1}, \dots, v_r = \lambda_{r,e_r}$. For that reason the polynomial $g_{\underline{e}(u)}$ comes into the restriction $g|_{A(\underline{e})}$ with a non-zero coefficient

$$\alpha_e = \prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (\lambda_{i,e_i} - \lambda_{ij}).$$

On the other hand, for $\underline{e}' \neq \underline{e}$, $|\underline{e}'| \geqslant a+1$ the product

$$\prod_{i=1}^{r} \prod_{j=0}^{e'_i - 1} (\lambda_{i,e_i} - \lambda_{ij}).$$

is equal to zero, as for at least one index $i \in \{1, \dots, r\}$ we have $e'_i > e_i$ and therefore that product contains a zero factor. So $g|_{A(\underline{e})}$ is the sum of the polynomial $\alpha_e g_{\underline{e}}$ and a linear combination of the polynomials $g_{\underline{e}'}$ with $|\underline{e}'| \leqslant a$ with constant coefficients. Now, fixing the polynomials $g_{\underline{e}'}$ $|\underline{e}'| \leqslant a$, we see that the condition

$$S(\underline{e}) \subset \mathrm{Sing}(g|_{A(\underline{e})})$$

defines an *affine* (generally speaking, not a linear) subspace of codimension $m(N-r+1)$ of the space of polynomials $g_{\underline{e}}(u_1, \dots, u_{N-r})$ of degree at most $d - |e|$, the corresponding linear space of which is given by the condition

$$S(\underline{e}) \subset \mathrm{Sing}\, g_{\underline{e}}(u).$$

Note that on the coefficients of other polynomials $g_{\underline{e}'}$ with $|\underline{e}'| = a+1$ no restrictions are imposed.

This completes the proof of the claim $(*)_a$ for all $a = 0, \dots, d-3$. Q.E.D. for Proposition 3.2.

3.3. End of the proof of Theorem 3.1. Let

$$\Theta = \Theta[l_0, \dots, l_r; \lambda_{i,j}, i = 1, \dots, r, j = 0, \dots, d-1] = \{\Theta(\underline{e}) \,|\, \underline{e} \in \Delta\}$$

be a set of linear subspaces of codimension r in $\mathbb{P}^N$, considered in Proposition 3.2. We define the subset

$$\mathcal{P}_{N,d}(\Theta) \subset \mathcal{P}_{N,d}$$

by the following condition: for every subspace $\Theta(\underline{e})$ with $|\underline{e}| \leqslant d-3$ there is a set $S(\underline{e}) \subset \Theta(\underline{e}) \backslash \Pi$, consisting of m linearly independent points, such that $S(\underline{e}) \subset \mathrm{Sing}(g|_{\Theta(\underline{e})})$.

Proposition 3.3. *The following inequality is true:*

$$\mathrm{codim}(\mathcal{P}_{N,d}(\Theta) \subset \mathcal{P}_{N,d}) \geqslant m|\Delta|.$$

Proof is obtained by the obvious dimension count: the subspaces $\Theta(\underline{e})$ are fixed, so that every point $p_i(\underline{e})$ varies in a $(N-r)$-dimensional family. Q.E.D. for the proposition.

Let us complete, finally, the proof of Theorem 3.1. Consider the set $\mathcal{P}^{(1,k;l)}_{N,d}(P)$, where $P \subset \mathbb{P}^N$ is a fixed k-plane $\{x_{k+1} = \ldots = x_N = 0\}$, and $l \leqslant k-2$. We apply proposition 3.3 to the space P instead of $\mathbb{P}^N$ and to the space of polynomials $\mathcal{P}_{k,d}$ instead of $\mathcal{P}_{N,d}$. For an arbitrary set $\Theta = \{\Theta(\underline{e}) \,|\, \underline{e} \in \Delta\}$ of linear subspaces of codimension l in $P = \mathbb{P}^k$ let

$$\mathcal{P}^{(1,k;l)}_{N,d}(P, \Theta) \subset \mathcal{P}^{(1,k;l)}_{N,d}$$

be the set of polynomials $f \in \mathcal{P}^{(1,k;l)}_{N,d}$ such that the set $\mathrm{Sing}(f|_P)$ has an irreducible component Q of dimension l, containing a curve $C \subset \mathrm{Sing}(f)$, and such that it is in general position with the subspaces from the set Θ: for all $\underline{e} \in \Delta$ the set $\Theta(\underline{e}) \cap Q$ contains $(k-l+1)$ linearly independent points. Since $\langle Q\rangle = \langle C\rangle = P$, the subset $\mathcal{P}^{(1,k;l)}_{N,d}(P,\Theta)$ is a Zariski open subset of the set $\mathcal{P}^{(1,k;l)}_{N,d}$, so that the inequality (15) will be shown is we show it for $\mathcal{P}^{(1,k;l)}_{N,d}(P,\Theta)$ instead of $\mathcal{P}^{(1,k;l)}_{N,d}$. By Proposition 3.3, applied to the space P, the condition $f \in \mathcal{P}^{(1,k;l)}_{N,d}(P,\Theta)$ imposes on the coefficients of the polynomial $f|_P$ at least $(k-l+1)|\Delta|$ independent conditions. Furthermore, from the set of $(N+1)$ polynomials

$$\left.\frac{\partial f}{\partial x_0}\right|_P, \dots, \left.\frac{\partial f}{\partial x_N}\right|_P$$

we may select a good sequence of $(k-1)$ polynomials, with a certain curve C, $\langle C\rangle = P$, as an associated subvariety, and moreover, this can be done in such a way that the first $(k-l)$ polynomials in that sequence are chosen among the polynomials

$$\left.\frac{\partial f}{\partial x_0}\right|_P, \dots, \left.\frac{\partial f}{\partial x_k}\right|_P$$

(and some subvariety $Q \supset C$, $Q \subset P$ of dimension l is an associated subvariety of that subsequence), whereas the following $(l-1)$ polynomials are chosen among the polynomials

$$\left.\frac{\partial f}{\partial x_{k+1}}\right|_P, \dots, \left.\frac{\partial f}{\partial x_N}\right|_P.$$

Fixing the polynomial $f|_P$ and the other polynomials of the good sequence, we may assume the curve $C \subset \mathrm{Sing}(f)$ of singular points to be fixed. Now the condition $\partial f/\partial x_i|_C \equiv 0$ for every $i \in \{k+1, \dots, N\}$, which did not get into the good sequence, give in addition $(N+1-k-l)((d-1)k+1)$ independent conditions on the coefficients of the polynomial f. An elementary, although tedious, check shows that the inequality

$$(k-l+1)|\Delta| + (N+1-k-l)((d-1)k+1) \geqslant (d-2)N + (k+1)(N-k)$$

holds for all the values k, l under consideration, which completes the proof of the inequality (15) and of Theorem 3.1, and therefore, of Theorem 0.3.

Remark 3.2. It is easy to see that the worst estimate for the codimension of $\mathcal{P}^{(1,k;l)}_{N,d}(P)$ corresponds to the case $k = N$ and $l = 1$, that is, the hypersurface $\{f = 0\}$ has a non-degenerate curve of singular points. In that case Proposition 3.3 yields the inequality

$$\mathrm{codim}(\mathcal{P}^{(1,k;1)}_{N,d} \subset \mathcal{P}_{N,d}) \geqslant (d-2)N.$$

It seems hardly probable that the presence of a non-degenerate curve of singular points imposes on the coefficients of the polynomial f less (although slightly less) independent conditions than the presence of a line consisting of singular points (when the estimate for the codimension is precise). And indeed, when we apply Proposition 3.3, we essentially replace a curve, consisting of singular points, by a finite set of singular points (although it is quite a large set). Probably, the technique used in the proof of Theorem 3.1 can be improved and for the case of a non-degenerate curve of singular points a more precise estimate could be obtained. This is what was meant in Remark 3.1.

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