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PhD Thesis of

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Social Context and Cost Sharing in Congestion Games

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Στους γονείς μου,
Νίκη και Παναγιώτης.

To my parents,
Niki and Panagiotis.

Contents

Abstract

Congestion games are one of the most prominent classes of games in non-cooperative game theory as they model a large collection of important applications in networks, such as selfish routing in traffic or telecommunications. For this reason, congestion games have been a driving force in recent research and my thesis lies on two major extensions of this class of games.

The first extension considers congestion games embedded in a social network where players are not necessarily selfish and might care about others. We call this class *social context congestion games* and study how the social interactions among players affect it. In particular, we study existence of approximate pure Nash equilibria and our main result is the following. For any given set of cost functions, we provide a threshold value such that: for the class of social context congestion games with cost functions within the given set, sequences of improvement steps of players, are guaranteed to converge to an approximate pure Nash equilibrium if and only if the improvement step factor is larger than this threshold value.

The second topic considers weighted congestion games under a fair cost sharing system which depends on the weight of each player, the (weighted) Shapley values. This class considers weighted congestion games where (weighted) Shapley values are used as an alternative (to proportional shares) for distributing the total cost of each resource among its users. We study the efficiency of this class of games in terms of the price of anarchy and the price of stability. Regarding the price of anarchy, we show general tight bounds, which apply to general equilibrium concepts. For the price of stability, we prove an upper bound for the special case of Shapley values. This bound holds for general sets of cost functions and is tight in special cases of interest, such as bounded degree polynomials. Also for bounded degree polynomials, we show that a slight deviation from the Shapley value has a huge impact on the price of stability. In fact, the price of stability becomes as bad as the price of anarchy. For this model, we also study computation of equilibria. We propose an algorithm to compute approximate pure Nash equilibria which executes a polynomial number of strategy updates. Due to the complex nature of Shapley values, computing a single strategy update is hard, however, applying sampling techniques allow us to achieve polynomial running time.

We generalise the previous model allowing each player to control multiple flows. For this generalised model, we study existence and efficiency of equilibria. We exhibit a separation from the original model (each player controls only one flow) by proving that Shapley values are the only cost-sharing method that guarantees pure Nash equilibria existence in the generalised model. Also, we prove that the price of anarchy and price of stability become no larger than in the original model.

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Chapter 1

Introduction

The general research area of my PhD lies on a relatively new multidisciplinary research field, called *algorithmic game theory*. Before I focus on my PhD topic within this area, I give a brief introduction regarding how algorithmic game theory originated and what the main scope of this emerging area is. To start with, I present an example:

> *Imagine there were 5 intelligent rational pirates, A, B, C, D, E, who found a treasure of 100 gold coins. To share this treasure, the pirates decided to follow an historical hierarchy protocol:*
>
> - *Assume the hierarchy of pirates follows alphabetical order.*
> - *The most senior pirate, A, proposes a distribution plan.*
> - *Then all pirates vote for whether to accept this distribution or not (the proposer has the casting vote).*
> - *If the majority votes YES, the treasure is shared on the proposed plan and the problem is solved.*
> - *If majority votes NO, the proposer is thrown to the sea from the pirate ship and the next most senior pirate, B, proposes a new distribution.*
> - *This process is repeated until a distribution plan is accepted.*
>
> *What is the outcome of this situation?*
>
> THE PIRATE GAME

The above structure is a formal description of a strategic situation where interactive agents (pirates) are called to make decisions. The purpose of this example is to introduce the reader to a strategic state of mind that agents need to follow in order to maximise their share in a strategic environment. Such a situation is called a *game* and each agent involved is called a *player*. The decisions that players make are towards a desirable state. In other words, we can see the series of a player's decisions as the pursuit of a goal. The action of each player has an impact on the interests of others and each player takes this into consideration before his next move. In the Pirate Game, examples of strategic questions include the following. Can a pirate maximise his treasure share by following a specific action? What actions should each pirate avoid? Is the best action to collaborate or get in conflict with others? If to collaborate, then with whom?

Of a similar nature to the Pirate Game, a wide range of situations occur in real life which require solutions. Examples of such situations lie from routing the road traffic to political negotiations and property auctions. To study and solve problems of this nature, mathematical tools are required to model the conflict and cooperation of interactive players. This study topic is called *game theory* [84, 66, 63, 77, 68, 82], and is split in cooperative and noncooperative game theory. Cooperative game thoery studies strategic interactions of coalitions of players where each player wishes to maximise the profit of the coalition she belongs to. This area also studies ways to divide the profit of a coalition among its members [14, 29]. On the other hand, the scope of noncooperative game theory [66] is to formulate and analyse strategic interactions from an individual perspective, where the driving force of each player is its self interest [64]. But what do strategic interactions mean? How can we model such an environment in a mathematical way? And what is the best available tool to do this?

One can think about a strategy as a set of planned actions (decisions) that can probably lead to a desirable state. Applied to real life, this set of actions cannot always follow a predefined structure as external factors usually interfere and cause deviations to paths towards states different than the desirable one. Given such a distraction, a 'plan' (formed from a strategy) adjusts to the new circumstances and re-calculates the next action towards the desirable outcome that was initially set. Here is an interesting description of a strategy,

> *"Strategy is much more than a plan. A plan supposes a sequence of events that allows one to move with confidence from one state of affairs to another. Strategy is required when others might frustrate one's plans because they have different and possibly opposing interests and concerns. The inherent unpredictability of human affairs, due to the chance events as well as the efforts of opponents and the missteps of friends, provides strategy with its challenge and drama. Strategy is often expected to start with a description of a desired end state, but in practice there is rarely an orderly movement to goals set in advance. Instead, the process evolves through a series of states, each one not quite what was anticipated or hoped for, requiring a reappraisal and modification of the original strategy, including ultimate objectives. The picture of strategy... is one that is fluid and flexible, governed by the starting point and not the end point."*
>
> - Sir Lawrence Freedman [80], King's College London.

Since strategies consists of a set of actions/rules that defines a sequence of operations, the ideal tool to express and study strategies, seems to be *algorithms* (for more regarding the framework of algorithms, see [26]). Using algorithms, actions can be performed and applied to a situation. In practice, the more realistic a situation is, the more complex its algorithmic modelling would be. A big part of recent research focuses on using algorithms to make strategic decisions on various aspects of people's lives [55]. This increases the need to combine more and more interdisciplinary areas in modern algorithm design. Real life examples, where algorithms are used for decision making, include route navigation, offering mortgages, hiring job applicants, stock trading, sentencing criminals, preference suggestion and auction resolutions [25]. This trend grows so rapidly and sometimes without debate, which makes it a controversial topic [44, 27, 36, 60].

A catalytical role on viewing game theory as a tool for understanding strategic behavior, was played by the Internet. Why? Since the 1980s, the Internet started creating a new emerging economy by enhancing competition and increasing market contestability. It accelerated the transfer of information over a network of networks consisting of business, academic and government interlinked sectors. As a framework of interactions of many, game theory seemed ideal for studying the strategic behaviour of everyone interacting within it. On the other hand, algorithms seem to be the framework for modelling strategic decision making. The combination and intersection of the above, i.e. *the Internet*, *strategic decision making* and *algorithms*, generated the new research field, *Algorithmic game theory*. Its scope is to design algorithms in strategic environments. Examples of such environments are: *sponsored auctions* in Internet search engines, *load balancing* computing which improves the distribution of workloads across multiple resources, strategic *voting* in elections where a voter might not support his most preferable candidate in order to avoid an unwanted outcome, and *routing* where players select a path among available edges (resources) in a congested network. The goal of each player is to select a strategy that minimises its individual cost.

Our research lies in one of the most important class of games in noncooperative algorithmic game theory, the *congestion games* (see Section 1.3). One of the main reasons of its importance is that they model a large range of applications based on routing. This class has been extensively studied and important results have been presented in the last decades (see Section 1.3 for known results). Our research focuses on extensions of such games, such as *altruistic congestion games* and *cost-sharing in congestion games* (see thesis outline 1.1). The research questions we address regard existence, convergence, efficiency and computation of stable outcomes including *pure Nash equilibria*, where each player in the game is 'happy' with her current strategy, thus no player is incentivised to choose an alternative strategy (see basic game theory definitions in page 12). To summarise the organisation of my thesis, I present a thesis outline in the next section, which also briefly describes the content of each chapter.

1.1 Thesis Outline

Chapter 1

- describes how algorithmic game theory was created,
- introduces the model of congestion games
- introduces *social context congestion games* with examples on the model, related work and a description of the thesis contribution,
- introduces *(weighted) Shapley value congestion games* with examples, related work and a description of the thesis contribution.

CHAPTER 2

· gives in detail the model of social context congestion games

· focuses on existence of approximate pure Nash equilibria and presents the mathematical analysis of our results,

· splits the thesis contribution into *general social context* and *symmetric binary social context*,

· concludes with a discussion on results and open questions,

· can be found in [39].

CHAPTER 3

· gives in detail the model of Shapley value congestion games with polynomial cost functions,

· presents the mathematical analysis of a polynomial running time algorithm that computes approximate equilibria,

· concludes with open questions and a discussion on a comparison of our algorithm to a previously known one which we built on.

· can be found in [33].

CHAPTER 4

· gives in detail a generalisation of the previous model, the (weighted) Shapley value congestion games with multi-commodity players, where each player may control multiple flows instead of one,

· focuses on existence and efficiency of equilbria of this class,

· explains which properties carry over to the generalised model,

· presents the mathematical analysis of our results in these areas,

· concludes with a discussion on results and open questions.

· can be found in [37, 38].

APPENDIX

· presents proofs of technical lemmas.

1.2 Preliminaries

In this section, I present formal definitions of basic terms used in this thesis.

Definition 1. STRATEGY. *For a player i, let $\mathcal{P}_i$ be the finite set of her available strategies and let P_i be her chosen strategy.*

Definition 2. Strategy profile or state or outcome. *Given the strategies of all players in the game, the vector $P = (P_1, P_2, \dots, P_n)$ is called a strategy profile.*

Definition 3. Cost of a player. *For a given strategy profile P, let $X_i : P \to \mathbb{R}$ be a function that assigns a cost to player i.*

Definition 4. Game. *A game is given by a tuple $\Gamma = (N, (P_i)_{i\in N}, (X_i)_{i\in N})$ where N is a finite set of players, $(P_i)_{i\in N}$ the vector with strategies of players and $(X_i)_{i\in N}$ the vector with players' costs.*

Definition 5. Dominant strategy. *For a player i, let P_{-i} be the vector of the strategies pf all players except player's i. Then, a strategy P_i is dominant if $X_i(P) < X_i(P_i', P_{-i})$ for all P_i' different than P_i and all $P_{-i} \in \mathcal{P}_{-i}$.*

Definition 6. Pure Nash equilibrium (PNE). *A strategy profile in which it holds that*

$$X_i(P) \leq X_i(P_i', P_{-i})$$

for each player i and any other strategy P_i'.

Definition 7. Approximate pure Nash equilibrium (ρ-PNE). *A strategy profile in which it holds that, for $\rho \geq 1$,*

$$X_i(P) \leq \rho \cdot X_i(P_i', P_{-i})$$

for each player i and any other strategy P_i'.

Definition 8. Social cost. *The social cost is defined as the sum of the costs of all players in the game, $SC(P) = \sum_{i\in N} X_i(P)$.*

Alternative definitions of social cost exist, defined for example by the minimum or the maximum player cost.

Definition 9. Optimum outcome. *Over all stratgy profiles P, we define as optimum the strategy profile with the minimum social cost, $OPT = \min\limits_{P} SC(P)$.*

Note that it may not be in everyone's self interest to reach this optimum state. For this reason, players are often forced to collaborate (and reach the optimum outcome) by a central authority.

We now define two basic tools which we use to measure the efficiency of equilibria in a game, the price of anarchy and the price of stability.

Definition 10. Price of Anarchy (PoA). *Let $\mathcal{Z}$ be the set of all outcomes and $\mathcal{Z}^{\mathcal{N}}$ the set of pure Nash equilibria of the game. Then the price of anarchy is defined as $PoA = \frac{\max_{P\in\mathcal{Z}^{\mathcal{N}}} SC(P)}{OPT}$.*

Definition 11. ρ-approximate Price of Anarchy (ρ-PoA). *Given a parameter $\rho \geq 1$, let $\mathcal{Z}^{\mathcal{N}}_{\rho}$ the set of approximate pure Nash equilibria of the game. Then the* ρ-approximate price of anarchy *(ρ-PoA) is defined as ρ-$PoA = \frac{\max_{P\in\mathcal{Z}^{\mathcal{N}}_{\rho}} SC(P)}{OPT}$.*

Definition 12. PRICE OF STABILITY (PoS). *The price of stability is defined as* $PoS = \frac{\min_{P \in \mathcal{Z}^{\mathcal{N}}} SC(P)}{OPT}$.

For a *class of games*, the PoA and PoS are defined as the largest such ratios among all games in the class.

The existence of equilibria (PNE) and their quality (PoA, PoS) have been key research questions in various classes of games for decades.

1.3 An Introduction to Congestion Games

The class of congestion games is one of the most prominent classes of games in noncooperative game theory [62, 69] as they model a large collection of important applications in networks, such as selfish routing in traffic or telecommunications [76, 21, 7]. For this reason, they have been a driving force in recent research and their study gave new insights on these practical problems. In a congestion game, there is a set of resources and players compete with each other over the resources. Each player selects a subset of resources as a strategy. Each resource is associated with a cost function which is assumed to be continuous, non-decreasing and nonnegative. A resource cost function expresses the delay on the resource and depends on the number of its users. In general, when the demand (users) on a resource is increased, the quality of the service that users experience on this resource deteriorates. This is a cost minimisation game, in a sense that the goal of each player is to choose the strategy that minimises her cost. Congestion games are split into two categories, the atomic[1] and non-atomic[2] congestion games. In this thesis we focus on the atomic case.

1.3.1 Weighted (atomic) Congestion Games

A game in this class is given by a tuple

$$(N, (w_i)_{i \in N}, E, (\mathcal{P}_i)_{i \in N}, (c_e)_{e \in E}),$$

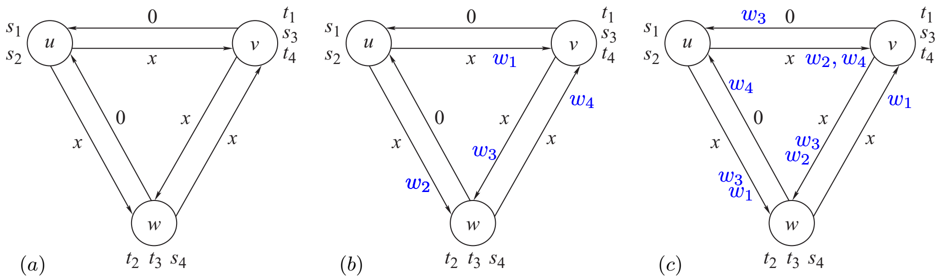
where

- N is the set of players,
- w_i is a positive number that expresses the weight of player $i \in N$,
- E is the set of resources,
- $\mathcal{P}_i \subseteq 2^E$ is the set of strategies of player i,
- c_e is the cost (latency) function of a resource $e \in E$, which is continuous, non-decreasing and non-negative,
- in a profile P, $S_e(P)$ is set of users of a resource e, while the congestion on e is defined as the sum of weights of its users, $f_e(P) = \sum_{i \in S_e(P)} w_i$,
- the cost of a player is given by $X_i(P) = w_i \cdot \sum_{e \in P_i} c_e(f_e(P))$, and each player wishes to minimise her cost (cost minimisation game).

[1] In the atomic case, there are N discrete players in the game.

[2] In the non-atomic case, the players are continuous in a sense of a flow.

Figure 1.1: (a) An atomic (unweighted) network congestion game from [7] with $N = \{1, 2, 3, 4\}$, the cost functions of each resource are mentioned in the middle of each edge, i.e, resource $u \to v$ has a constant function of 0, and resource $v \to u$ has a linear cost function x, (b) its optimum profile and (c) worst equilibrium outcome.



This is the general description of atomic weighted congestion games. However, there are the following important subclasses of these games:

(i) *unweighted* or simply *congestion games* where $w_i = 1$ for all players $i \in N$,

(ii) *symmetric congestion games*, where the sets of available strategies of all players are the same,

(iii) *singleton congestion games*, where each player can use only one resource in each strategy, and

(iv) *network congestion games*, where the congestion occurs on a given network $G = (V, E)$ in a sense that players simultaneously want to find the optimal path for their flows from a source $s_i \in V$ to a destination $t_i \in V$ on that network. The set of edges E equals the set of resources in this subclass.

To clarify the atomic model, I present an unweighted network congestion game[1] (Figure 1.1 (a)) and describe in detail the process of measuring its inefficiency by computing its price of anarchy and price of stability (see p. 12 for definitions).

Example 1. *Consider the unweighted network congestion game illustrated in Figure 1.1(a). What is its PoA and PoS?*

▶ There is a set of players $N = \{1, 2, 3, 4\}$ with identical weights equal to 1 (unweighted case) and each player $i \in N$ wishes to go from s_i to t_i. Observe that each player has two options (available strategies), a single- and a two-edge path. The price of anarchy (see definition 10) is the ratio of the worst equilibrium cost over the optimum. We focus first on finding the optimum, which is the outcome with the minimum social cost. We can find that the social cost is minimised when every player choose her single-edge path from her source to her destination. Thus, player 1 chooses the path $u \to v$, player 2 chooses $u \to w$, player 3 chooses $v \to w$ and player 4 chooses $w \to v$. This strategy

[1]This network congestion game can be found in Figure 1, page 62 of [7].

profile is illustrated in Figure 1.1 (b). Given a strategy profile, the social cost equals to the sum of players' costs (see p. 12 for definition), then

$$OPT = w_1^2 + w_2^2 + w_3^2 + w_4^2 = 4.$$

Second, we compute the cost in a worst pure Nash equilibrium instance. Observe that if each player chooses her two-edge path from her source to her destination, this yields to a pure Nash equilibrium as no player wants to deviate from this instance. The social cost equals to the sum of players costs, thus we get

$$\begin{aligned} SC(P) =& X_1(P) + X_2(P) + X_3(P) + X_4(P) \\ =& [w_1 \cdot (w_1 + w_3) + w_1^2] + [w_2 \cdot (w_2 + w_3) + w_2 \cdot (w_2 + w_4)] + \\ &+ [w_3 \cdot (w_1 + w_3) + w_3^2] + [w_4 \cdot (w_2 + w_4) + w_4^2] = 10. \end{aligned}$$

Therefore the PoA is at least $\frac{5}{2}$. On the other hand, to compute the price of stability, we need to find the equilibrium that minimises the social cost. In this case, observe that the optimum is also an equilibrium, thus $PoS = 1$. ◀

Existence of Equilibria

The class of (unweighted) *congestion games* was first introduced by Rosenthal [69] in 1973, where he shows that such games always possess pure Nash equilibria. In particular, he proves that a series of cost improvement finite steps of players (deviations) always reach a pure Nash equilibrium instead of running into cycles.

Theorem 1.3.1 (Rosenthal 1973)**.** *For every (unweighted) congestion game, every sequence of improvement steps is finite.*

The proof of this result is based on constructing a function, called *exact potential function*, which assigns a value to each outcome of the game and has the following interesting property. If a single player deviates and improves her cost by δ, then also the potential function improves by the same amount δ. More formally, if a player deviates from P_i to another strategy P_i', then

$$\Phi(P) - \Phi(P') = X_i(P) - X_i(P'), \tag{1.1}$$

where $P' = (P_{-i}, P_i')$. The potential function for atomic congestion games that Rosenthal presented is

$$\Phi(P) = \sum_{e \in E} \sum_{k=1}^{|S_e(P)|} c_e(k). \tag{1.2}$$

Example 2. *For a player's deviation in Figure 1.1 (a), show that the potential function property holds for the two different strategy profiles that occur due to this deviation.*

▶ To simpify, set $e_1 = (u, v)$, $e_2 = (v, w)$, $e_3 = (w, u)$, $e_4 = (v, u)$, $e_5 = (u, w)$ and $e_6 = (w, v)$. Focus on player 2 and let P be the outcome where player 1 uses e_1, player 2 uses e_1, e_2, player 3 uses e_2 and player 4 uses e_6. This is an unweighted instance, thus all weights are equal to 1. Observe that the cost of player 2 is equal to $X_2(P) = w_2 \cdot (w_1 + w_2) + w_2 \cdot (w_2 + w_3) = 4$ and that Rosenthal's potential function (1.2) gives

$$\Phi(P) = \sum_{k=1}^{|S_{e_1}(P)|} c_{e_1}(k) + \sum_{k=1}^{|S_{e_2}(P)|} c_{e_2}(k) + \sum_{k=1}^{|S_{e_6}(P)|} c_{e_6}(k).$$

Since $|S_{e_1}(P)| = |S_{e_2}(P)| = 2$, $|S_{e_6}(P)| = 1$ and $c_{e_1}(x) = c_{e_2}(x) = c_{e_6}(x) = x$, we get that $\Phi(P) = 1+2+1+2+1 = 7$. Let $P' = (P'_2, P_{-2})$ be the outcome where players use the same strategies as in outcome P apart from player 2 who deviates to $P'_2 = \{e_5\}$ from the $P_2 = \{e_1, e_2\}$ she was using before. Note that this is the optimum outcome illustrated in Figure 1.1 (b). This move reduces player's 2 cost to $X_2(P') = w_2 = 1$, while the potential function gives a value of $\Phi(P') = 1+1+1+1 = 4$ in this outcome. Observe that the desirable property holds: player's 2 deviation from P_2 to P'_2 reduces the potential function by an amount equal to her cost improvement, $\Phi(P) - \Phi(P') = X_2(P) - X_2(P')$. ◀

Since any improvement step of players would decrease $\Phi(P)$, observe that a series of such improvements will at the end reach a profile where no player can improve anymore. In other words, a minimum of the potential function $\Phi(P)$ is a pure Nash equilibrium. Thus, games admitting an (exact) potential function, always possess a pure Nash equilibrium. Such games are called (exact) *potential games*. Later in 1996, Monderer and Shapley showed that the class of congestion games coincides with the class of finite potential games [62], which implies that every atomic congestion game has a pure Nash equilibrium. However, as soon as we deviate to the weighted congestion games where players can have non-identical weights, then existence of equilibria is not guaranteed anymore. Examples of weighted atomic instances admitting no pure Nash equilibria are described in [59], [35] and [42]. On the positive side, Fotakis et al. [35] prove that by restricting to linear cost functions, weighted (atomic) congestion games are *weighted*[1] *potential games*. To prove this, they use the potential function

$$\Phi(P) = \sum_{i \in [n]} w_i \cdot \sum_{e \in P_i} \left(c_e(f_e(P)) + c_e(w_i) \right).$$

Computation of Equilibria

As mentioned in previous section, pure Nash equilibria in congestion games exist. However, can we reach at them? How difficult is it to compute a pure Nash equilibrium in a congestion game?

To answer this question, Fabrikant et al. [30] relate the complexity of equilibria search in congestion games to the complexity of finding local optima in local search problems. In particular, authors prove that computing a pure Nash equilibrium in atomic congestion games is PLS-complete, even for the symmetric case where all players have the same strategy set. To do this, they show

[1] Similar to the property of exact potential games (1.1), a weighted potential game admits a potential function $\Phi(P)$ with the property: $\Phi(P) - \Phi(P') = w_i \cdot (X_i(P) - X_i(P'))$.

Figure 1.2: Complexity results for PNE computation in atomic congestion games.

	SYMMETRIC GAME	GENERAL GAME
CONGESTION GAME	PLS-complete [30, 1]	PLS-complete [30]
NETWORK CONGESTION GAME	**Polynomial time** [30]	PLS-complete [30]

a reduction from the problem POS-NAE-MAX-3SAT[2]. For the same problem, Ackermann et al.[1] show the same result with a reduction fro a different problem, the Max Cut problem. On the positive side, by restricting to symmetric network congestion games, a pure Nash equilibrium can be computed in polynomial time. This result is shown via a reduction to Min Cost Flow problem [30]. Figure 1.2 gives a table of the main complexity results.

PoA and PoS in Atomic (weighted) Congestion Games

Another important line of research is to measure the quality of an equilibrium. In this section, I present a brief background of how the techniques of measuring equilibria efficiency have been evolved over the recent years.

How bad can the total cost of an equilibrium be? How far is this value from the optimum? The main tools to measure efficiency of equilibria are the *price of anarchy* and *price of stability* (see definitions 10 and 12). The notion of PoA was introduced by Koutsoupias and Papadimitriou [21], and the PoS was firstly proposed by Schulz and Stier Moses [78], and formally defined by Anshelevich et al. [5]. Since then, the study on investigating quality of equilibria has been a driving force in recent research. In the area of congestion games, Christodoulou and Koutsoupias [21] computes the PoA for linear unweighted congestion games and their results are easily extended to the weighted case. Their approach to prove the PoA upper bound follows an interesting technique, which is explained as follows. This technique is based on the next lemma.

Lemma 1. *For every pair of nonnegative integers x, y:* $x \cdot (y+1) \leq \frac{1}{3} \cdot x^2 + \frac{5}{3} \cdot y^2$.

Christodoulou and Koutsoupias [21] use the previous lemma to prove that, for P an equilibrium and P' any other outcome,

$$\sum_i X_i(P_{-i}, P'_i) \leq \frac{1}{3} \cdot SC(P') + \frac{5}{3} \cdot SC(P),$$

which, by rearranging, gives that PoA for linear congestion games is at most $\frac{5}{2}$. Awerbuch et al. [7] independently give the same results by following a similar technique. For polynomial functions, Aland et al. [3] give an exact PoA value. They also use a similar to the above technique, however, they generalise Lemma 1 as follows. The analysis of [3] optimises $\frac{\lambda}{1-\mu}$ such that for all polynomials c with nonnegative coefficients and maximum degree d, and for all nonnegative integers x, y, it must hold that

$$y \cdot c(x+1) \leq \lambda \cdot y \cdot c(y) + \mu \cdot x \cdot c(x). \tag{1.3}$$

[2](Positive - Not all equal - Maximum - 3SAT): Each clause has a positive integer weight and includes three element which can be either positive literals or constants, 0 or 1. A clause is satisfied iff there is at least one element with value 1 and at least one with 0. The goal is to find the assignment that maximises the sum of the weights of the satisfied clauses.

Figure 1.3: (Table 1.1 in [3]) The PoA in atomic congestion games: Comparison of results of [21, 7, 3].

d	Φ_d	Exact value [3]	Upper bound [21]	Lower bound [21]	Upper bound [3]	Lower bound [7]
1	1.618	2.500	2.500	2.500	2.618	2.618
2	2.148	9.583	10.00			9.909
3	2.630	41.54	47.00		47.82	5.000
4	3.080	267.6	269.0	21.33	277.0	15.00
5	3.506	1514	2154	42.67	1858	52.00
6	3.915	12 345	15 187	85.33	14 099	203.0
7	4.309	98 734	169 247	170.7	118 926	877.0
8	4.692	802 603	1 451 906	14 762	1 101 126	4140

To prove their bounds for the unweighted case, [21] sets $\mu = \frac{1}{2}$ in (1.3) and estimates the parameter λ. On the other hand, [3] optimises over both parameters λ, μ and achieves exact PoA values for the weighted and unweighted case closing the bound gaps from [21, 7]. In particular, the PoA for weighted congestion games equals Φ_d^{d+1}, where Φ_d is the unique real solution to $(x+1)^d = x^{d+1}$, and for unweighted congestion games, the PoA equals to

$$\frac{(k+1)^{2\cdot d+1} - k^{d+1}\cdot(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}},$$

where $k = \lfloor \Phi \rfloor_d$. For comparison between [21, 7, 3], see Figure 1.3. Roughgarden generalises this technique for general cost functions and name it as the *smoothness* framework [73]. He also showed that bounds obtained in this way apply to general equilibrium concepts. For the approximate PoA (see definition 11) and linear cost functions, the same author gives tight bounds [71]. For polynomial cost functions, tight bounds were given by Christodoulou et al. [22], however, for the approximate PoS, they give weaker bounds. For weighted atomic congestion games, Bhawalkar et al. [8] characterise the PoA as a function of the allowable resource cost functions.

For PoS in congestion games, results are only known for the unweighted case. Christodoulou and Koutsoupias [21] study linear congestion games, while Christodoulou and Gairing [20] present tight bounds for the more general case of polynomial cost functions. More related work can be found in *Efficiency of equilibria* of Section 1.5.2.

1.4 Social Context in Congestion Games

Game theory deals with the mathematical study of interaction between rational players. A prevalent assumption is that the players are selfish and they act upon their own well-being. But what if they don't? Recently, attention has been given to more general settings where players do not behave entirely selfishly, but they might exhibit altruistic or spiteful behavior. For example, players may partially disregard their own 'happiness' to influence the well-being of others. Studying such alternative behaviours is mainly motivated by the fact that altruism or spite are phenomena that frequently occur in real life. Consequently, it is desirable to incorporate such behaviours in game-theoretical analyses. Despite some recent efforts [54], [6], the impact of social context in fundamental results of noncooperative[1] game theory is not well understood.

One of this thesis goals is to study how social interactions affect the class of congestion games. The next section introduces how social interactions of players (social context) can be embodied in a game. This merge yields to a social context game, which is then adjusted to congestion games (*social context congestion games*). The main research question we address concerns existence of approximate pure Nash equilibria in this class [39].

1.4.1 An Introduction to The Model

Given a game Γ (see definition 4), a *social context game* is generated by considering a neighborhood graph over the players of the game Γ and an aggregation function that determines how the game is affected by that graph. In general, these two terms form a *social context* Ξ and are being described in the next paragraph. Each vertex of this graph represents one player. Every player is connected with other players through the edges of this directed graph. A directed edge between vertices (players) i and j indicates a friendship between players i and j. Otherwise, if there is no edge between two vertices, there is no friendship between the players associated with these vertices. The *perceived cost* $X_i(P, \Xi)$ of a player is defined by the aggregation function applied on the personal costs (original payoffs in the base game) of herself and her friends.

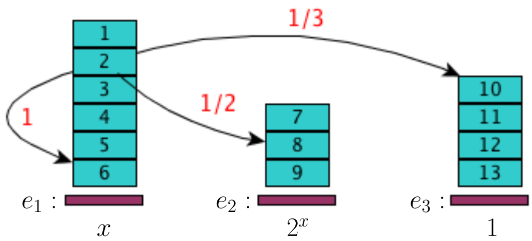
A congestion game is extended by embedding a social context on it and we call such games *social context congestion games.* This extension of standard congestion games (Section 1.3) is also a cost minimisation game in which the difference is that each player wishes to minimise her *perceived cost* instead of her personal cost. In this model, the perceived cost is defined by the following linear combination of all players' personal costs,

$$X_i(P, \Xi) = \sum_{j \in N} \xi_{ij} \cdot X_j(P),$$

where ξ_{ij} monetises how much player i cares about player j. If i cares for j as much as j cares for i, for every pair of players (i, j), then this is a *symmetric social context.* Otherwise, it is called *asymmetric social context.* Throughout

[1]Even though this model allows altruism among players, it differs from cooperative game theory. In the latter [14, 29], a player can join another coalition (group of players collaborating to maximise collective payoff) if this move would increase the payoff share she receives as a member of the current coalition. In this model, social interactions among players are given as input in the game.

Figure 1.4: A social context congestion game with 13 players (each box represent one player), where the interest of player 2 towards player 6, 8 and 10 is expressed by the numbers on the edges of the neighborhood graph.



this model, we assume that for every player i, $\xi_{ii} = 1$. To clarify the model, I present an example on computing the perceived cost of a player.

Example 3. *Computing the perceived cost of player 2 in the social context congestion game of Figure 1.4.*

► There are three resources e_1, e_2, e_3 with $c_{e_1}(x) = x$, $c_{e_2}(x) = 2^x$ and $c_{e_3}(x) = 1$. The users on a resource are represented by the rectangulars assigned to that resource while users' names are given by the numbers inside the boxes. Focus on player 2 who uses resource e_1. Note that she cares for player 6 by a value of 1, for player 8 by a value of $\frac{1}{2}$ and for player 10 by a value of $\frac{1}{3}$. This gives a perceived cost for player 2 of

$$X_2(P, \Xi) = 6 + 1 \cdot 6 + \frac{1}{2} \cdot 2^3 + \frac{1}{3} \cdot 1.$$

◄

1.4.2 Related Work and Contribution

In this section, I present known results in related areas of social context congestion games, and I also present my thesis contribution to this area. For the detailed mathematical analysis of the contribution, see Chapter 2 in p. 35.

What is previously known: The impact of altruism and spite in games has been widely studied [4, 6, 10, 13, 18, 17, 54, 53, 67, 52]. [54] studied the existence of pure Nash equilibria in social context games. They showed that such games admit an exact potential function if and only if they are isomorphic to a social context congestion game with linear cost functions. They also showed that singleton congestion games with binary social context might not admit a pure Nash equilibrium for concave cost functions.[1] For convex cost functions they left the existence of a pure Nash equilibrium as an open problem. [10] studied social context congestion games with different aggregation functions. Altruism in congestion games has also been studied in a different model [52, 53]. The efficiency of Nash equilibria through the price of anarchy in games with social context and in altruistic games has been studied in [4, 13, 18, 17, 67].

Thesis contribution: The existence of pure Nash equilibria is a desirable property of games. Unfortunately, in congestion games this property is very fragile and several generalisations of congestion games do not possess such states

[1] Our analysis contrast this by showing that they admit 4/3-approximate pure Nash equilibria (see Table 2.2.1).

Figure 1.5: Threshold values $\mu(\mathcal{C})$ for some classes of cost functions and comparison to the *anarchy value* $\beta(\mathcal{C})$ by Roughgarden [72]. Polynomials are of maximum degree d and have non-negative coefficients.

Cost functions	$\mathcal{C}$	$\mu(\mathcal{C})$	$\beta(\mathcal{C})$
concave		$4/3 \approx 1.333$	$4/3 \approx 1.333$
polynomials	d		
	0	1	1
	1	$4/3 = 1.333$	$4/3 = 1.333$
	2	$8/5 = 1.6$	$3\sqrt{3}/(3\sqrt{3}-2) \approx 1.626$
	3	$81/43 = 1.884$	$4\sqrt[3]{4}(4\sqrt[3]{4}-3) \approx 1.895$
	4	$243/113 = 2.1504$	$5\sqrt[4]{5}(5\sqrt[4]{5}-4) \approx 2.1505$
	d	$\theta(d/\log d)$	$\theta(d/\log d)$
exponentials	α^x, $\alpha > 1$	α	unbounded

in general, including *social context* congestion games [6]. A natural question to ask is how much we have to relax the equilibrium condition in order to guarantee the existence of pure equilibria. More precisely, we are interested in the existence of a ρ-approximate pure Nash equilibrium, a pure strategy profile in which no player can improve by a factor $\rho > 1$.

For the class of social context congestion games with cost functions from any given set of allowed cost functions, we study the existence of approximate pure Nash equilibria and provide a threshold value such that the following holds. For this subclass of social context congestion games with cost functions from a given set, sequences of players' deviations, where players improve by at least a factor ρ, are guaranteed to converge to a ρ-approximate pure Nash equilibrium if and only if the improvement factor ρ is larger than the threshold value.Last, we make an interesting observation: our threshold value is related and always upper bounded by Roughgarden's anarchy value [72] (see Figure 2.2.1). For more details, see Chapter 2.

1.5 Cost-Sharing in Weighted Congestion Games

The second part of my research lies on weighted congestion games, an extension of congestion games where each player has a weight that expresses how much she 'harms' the network with her presence or the cost she adds to the total cost of the network. Depending on the weight of each player, we investigate distribution methods of the resource cost among her users. The cost sharing methods we study is the (weighted) Shapley values and the research questions we address concern computation, existence and efficiency of equilibria [33], [37], [38].

1.5.1 An Introduction to The Model

A weighted congestion game is described as follows. There is a set N of players and a set E of resources. Each player $i \in N$ has a positive weight w_i and she gets to select the subset of the resources she prefers to use. The resources she can pick are given by her set of possible strategies $\mathcal{P}_i$. When all players decide, each resource $e \in E$ generates a *joint cost* $C_e(f_e)$, where f_e is the total weight of the users of e and C_e is the cost function of e. The joint cost of a resource is covered by its users S_e in a sense that $\sum_{i \in S_e} \chi_{ie} = C_e(f_e)$, where χ_{ie} is the *cost share* of player i on resource e.

Cost-sharing method. Given a class of games, the way the total cost is distributed to players is given by a cost-sharing method. For example in a weighted congestion game, each resource has a total cost *(joint cost)* that depends on the number of players who use this resource. A cost-sharing method distributes the joint cost among the players who use this resource, thus the cost share each player pays is determined by the chosen method. Certain examples of cost-sharing methods include *proportional sharing* (PS) and the *weighted Shapley values* (SV). In PS (see Example 4), the cost share of a player is proportional to her flow weight, i.e., $\chi_{ie} = (w_i/f_e) \cdot C_e(f_e)$.

Example 4. *Compute the proportional shares of the red and blue players of the routing game in Figure 1.6 (a).*

► Focus on the red player. If α is the fraction of her weight over the total weight on the resource, then α is the fraction of the total recourse cost she would pay, her proportional share. Her weight over the total weight is $\frac{1}{4}$, thus her proportional share is

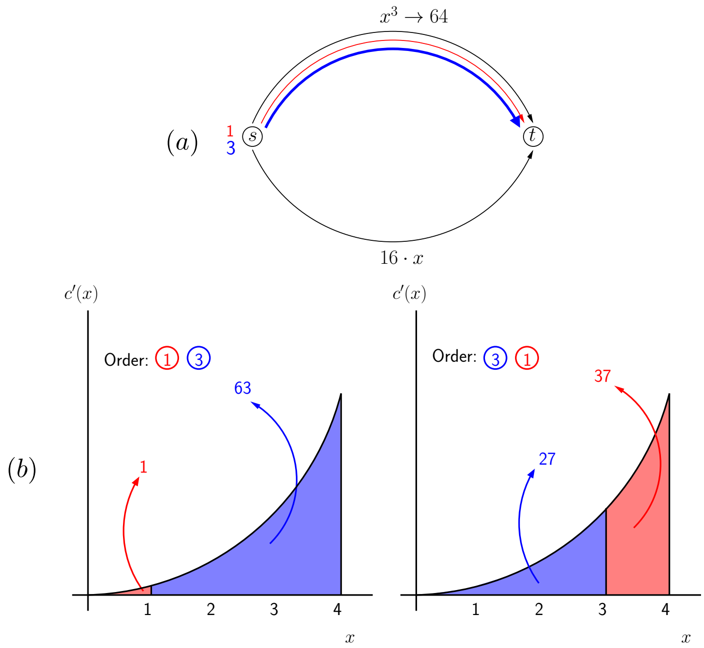
$$\begin{aligned} X_{\text{red}}(P) = \sum_{e \in P_i} \chi_{ie}(P) = \chi_{ie_1}(P) &= \frac{w_{\text{red}}}{w_{\text{red}} + w_{\text{blue}}} \cdot C_{e_1}(w_{\text{red}} + w_{\text{blue}}) \\ &= \frac{1}{4} \cdot 64 = 16. \end{aligned}$$

Similar, we compute the proportional share of the blue player,

$$\begin{aligned} X_{\text{blue}}(P) = \sum_{e \in P_i} \chi_{ie}(P) = \chi_{ie_1}(P) &= \frac{w_{\text{blue}}}{w_{\text{red}} + w_{\text{blue}}} \cdot C_{e_1}(w_{\text{red}} + w_{\text{blue}}) \\ &= \frac{3}{4} \cdot 64 = 48. \end{aligned}$$

◄

Figure 1.6: (a) A routing game with two players, red and blue: the red player controls a flow with weight $w_1 = 1$ and the blue player controls a flow with weight $w_2 = 3$. Let e_1, e_2 be the top and the bottom link, accordingly, with $c_{e_1}(x) = x^3$ and $c_{e_2}(x) = 16 \cdot x$. (b) The marginal cost contributions of both players when they enter the resource.



Another cost-sharing method is the weighted SV (see Example 5), which is the expected marginal cost contribution of a player over all orderings of players. To clarify, I first explain what a marginal cost contribution is and then I focus on the computation of its expectation. Let π be an ordering of the users of resource e. This ordering is the order in which players enter the resource. Then the marginal cost contribution of a player $i \in S_e$ expresses the increase in the resource total cost occured by the entrance of player i. Define as $F_e^{i,\pi}(P) = \sum_{j \in S_e(P)}^{j<i} w_j$ the sum of the weights of the players that enter the resource before player i in ordering π. Then, formally, the marginal cost contribution is given by

$$C_e(F_e^{i,\pi}(P) + w_i) - C_e(F_e^{i,\pi}(P)).$$

For a given distribution Π over orderings, the weighted Shapley value equals to the expectation of the marginal contributions, which is given by

$$E_{\pi \sim \Pi}[C_e(F_e^{i,\pi}(P) + w_i) - C_e(F_e^{i,\pi}(P))].$$

The distribution Π is given by a sampling parameter λ_i for each player i. The last player in the ordering is picked proportional to the sampling parameters λ_i. For example, the first drawn from the distribution player goes last in the ordering, the second drawn player goes second from the last, and this process is repeated iteratively for the remaining players.

The cost-sharing methods used in this thesis is a large subclass of weighted Shapley values where the sampling weight of a player i is given by $(w_i)^\gamma$, where w_i is player i's weight and γ is any number.

Example 5. *Given that the sampling weight of a player equals its weight, what are the weighted Shapley values of the red and blue players of the routing game in Figure 1.6 (a)?*

▶ By assumption, $\lambda_{\text{red}} = w_{\text{red}} = 1$ and $\lambda_{\text{blue}} = w_{\text{blue}} = 3$. Recall that the first player being sampled (drawn by the distribution) is positioned at the end of the ordering, the second sampled player is positioned second from the last, etc. The only difference is that the distribution over the orderings depends on the sampling weights. Thus, the weighted Shapley value for each player is given by the following sum. The first term expresses the probability of the red player being first in the ordering (red player is drawn last) which equals the probability of the blue player being last in the ordering (drawn first), since there are only two players. This is then multiplied by red player's marginal contribution when she enters first the resource. The second term expresses the probability that the red player is being drawn first (thus she is last in the ordering) multiplied with her marginal contribution in this case. For simplicity, I omit from the formula the outcome symbol, P. We have:

$$\begin{aligned} X_{\text{red}} = \chi_{ie_1} &= \frac{\lambda_{\text{blue}}}{\lambda_{\text{blue}} + \lambda_{\text{red}}} \cdot (C_{e_1}(1) - 0) + \frac{\lambda_{\text{red}}}{\lambda_{\text{blue}} + \lambda_{\text{red}}} \cdot (C_{e_1}(4) - C_{e_1}(3)) \\ &= \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 37 = 10. \end{aligned}$$

Similar, we compute the weighted Shapley value of the blue player, and get

$$\begin{aligned} X_{\text{blue}} = \chi_{ie_1} &= \frac{\lambda_{\text{red}}}{\lambda_{\text{blue}} + \lambda_{\text{red}}} \cdot (C_{e_1}(3) - 0) + \frac{\lambda_{\text{blue}}}{\lambda_{\text{blue}} + \lambda_{\text{red}}} \cdot (C_{e_1}(4) - C_{e_1}(1)) \\ &= \frac{1}{4} \cdot 27 + \frac{3}{4} \cdot 63 = 54. \end{aligned}$$

To confirm the computation, note that $X_{\text{blue}} + X_{\text{red}} = C_e(f_e) = 64$ for all three examples in this section. ◀

For $\gamma = 0$, this class reduces to a special case where λ_i are equal for all i and the cost share of a player is her average marginal cost increase over all orderings of the users of e (see Example 8). This special case is called *Shapley values.*

Example 6. *What are the Shapley values of the red and blue players of the routing game in Figure 1.6 (a)?*

▶ Since there are only two players, the total possible players' orderings are two. If red player enters first the resource, her marginal cost contribution in the resource is $C_{e_1}(1) - C_{e_1}(0)$. If the red player enters second, her marginal cost contribution is $C_{e_1}(4) - C_{e_1}(3)$. Then the Shapley value of the red player equals

$$X_{\text{red}} = \sum_{e \in P_i} \chi_{ie} = \chi_{ie_1} = \frac{C_{e_1}(1) - 0}{2} + \frac{C_{e_1}(4) - C_{e_1}(3)}{2} = \frac{1}{2} + \frac{37}{2} = 19.$$

Similar, we compute the Shapley value of the blue player, and get

$$X_{\text{blue}} = \sum_{e \in P_i} \chi_{ie} = \chi_{ie_1} = \frac{C_{e_1}(3) - 0}{2} + \frac{C_{e_1}(4) - C_{e_1}(1)}{2} = \frac{27}{2} + \frac{63}{2} = 45.$$

Figure 1.6 (b) illustrates the marginal cost contributions for both players. ◀

Despite the fact that Shapley and proportional sharing follow different computational procedures, there is a class of cost functions for which they coincide.

Proposition 1.1. *The cost-sharing method of Shapley values is identical to proportional sharing if and only if the resource cost functions are affine.*

I would like to note that the statement of this proposition was mentioned as an observation in [57], however, no proof was given. For that reason, I present a detailed proof.

Proof. Consider a resource with cost function $c(x) = \alpha \cdot x + \beta$. Let $x^{<i}$ be a random variable for the total weight on the resource before player i in the ordering and F^{-i} the total weight of all players excluding player i. We prove that Shapley values with affine functions equals the proportional shares. By definition, the Shapley value of a player i is the expected value over all orderings of her marginal contribution, that is

$$\begin{aligned}
&\mathbf{E}[(x^{<i} + w_i) \cdot c(x^{<i} + w_i) - x^{<i} \cdot c(x^{<i})] \\
&= \mathbf{E}[(x^{<i} + w_i) \cdot \big(\alpha \cdot (x^{<i} + w_i) + \beta\big) - x^{<i} \cdot \big(\alpha \cdot x^{<i} + \beta\big)] \\
&= \mathbf{E}[\alpha(x^{<i} + w_i)^2 + \beta \cdot (x^{<i} + w_i) - \alpha \cdot (x^{<i})^2 - \beta \cdot x^{<i}] \\
&= \mathbf{E}[2 \cdot \alpha \cdot x^{<i} \cdot w_i + \alpha \cdot w_i^2 + \beta \cdot w_i] \\
&= 2 \cdot \alpha \cdot \mathbf{E}[x^{<i}] \cdot w_i + \alpha \cdot w_i^2 + \beta \cdot w_i \\
&= w_i \cdot (2 \cdot \alpha \cdot \mathbf{E}[x^{<i}] + \alpha \cdot w_i + \beta).
\end{aligned} \tag{1.4}$$

Note that variable $x^{<i}$ can take values in the interval $[0, F^{-i}]$ with the maximum value occuring when player i is last in the ordering. Therefore the expected value of $x^{<i}$ equals to $\frac{F^{-i}-0}{2}$. Substituting this to (1.4), we get that

$$\begin{aligned}
\mathbf{E}[(x^{<i} + w_i) \cdot c(x^{<i} + w_i) - x^{<i} \cdot c(x^{<i})] &= w_i \cdot \big(\alpha \cdot \big(F^{-i} + w_i\big) + \beta\big) \\
&= w_i \cdot c(w_i + F^{-i}),
\end{aligned}$$

which is the proportional sharing. In the remaining part of the proof, we show by contradiction that affine functions c are the only functions satisfying equality between Shapley values and proportional shares of players. Consider two arbitrary players, say players 1 and 2 with weights w_1 and w_2. Assume that their Shapley values equal to their proportional shares, then for player 1, we have

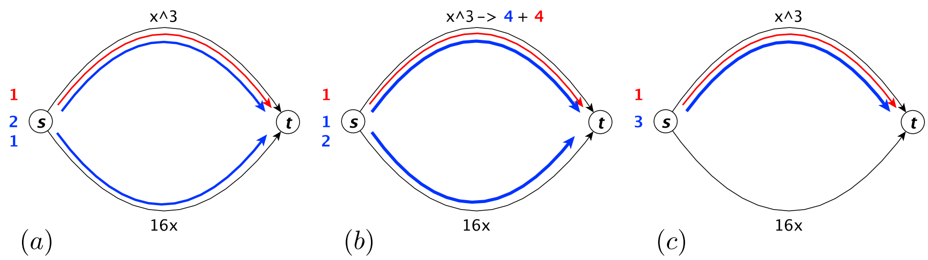
$$\begin{aligned}
&\frac{1}{2} \cdot (w_1 \cdot c(w_1) + (w_1 + w_2) \cdot c(w_1 + w_2) - w_2 \cdot c(w_2)) = w_1 \cdot c(w_1 + w_2) \\
&\Leftrightarrow w_1 \cdot c(w_1) + w_2 \cdot c(w_1 + w_2) - w_2 \cdot c(w_2) = w_1 \cdot c(w_1 + w_2) \\
&\Leftrightarrow \frac{w_1}{w_2} = \frac{c(w_1 + w_2) - c(w_2)}{c(w_1 + w_2) - c(w_1)}.
\end{aligned} \tag{1.5}$$

Since w_1, w_2 are arbitrary numbers, assume w.l.o.g. that $w_1 \leq w_2$ and that c is not an affine function, therefore it has a non-linear progression (the slope of c from w_1 to w_2 is smaller than its slope from w_2 to $w_1 + w_2$). That is,

$$\frac{c(w_2) - c(w_1)}{w_2 - w_1} < \frac{c(w_1 + w_2) - c(w_2)}{w_1} \Rightarrow \frac{w_1}{w_2} < \frac{c(w_1 + w_2) - c(w_2)}{c(w_1 + w_2) - c(w_1)},$$

which contradicts (4.7). $\square$

Figure 1.7: A routing game with two players, r and b (red and blue, accordingly): Player r controls a commodity with weight $w_r = 1$ and player b controls two commodities $b1$, $b2$ with weights $w_{b1} = 1$ and $w_{b2} = 2$. Let e_1, e_2 be the top and the bottom link, accordingly, with $c_{e_1}(x) = x^3$ and $c_{e_2}(x) = 16 \cdot x$. (a) Player r with the large commodity $b2$ of player b use e_1, while the small commodity $b1$ of player b use e_2, (b) player b swaps her commodities compare to a, (c) both players use e_1.



Generalised Model (Multi-Commodity per Player)

My research on cost-sharing is extended to a generalisation of the previous model: Consider a graph where multiple flows want to reach their destination starting from a single source. Each flow has its own source, destination and weight. I refer to such flows as *commodities*. The generalisation in this model is that a player can control either one or multiple commodities in the graph. The players compete between them for resources and each player chooses the resources she will use for each commodity she controls. The total cost on a resource is called *joint cost* and is given by $C_e(f_e) = f_e \cdot c_e(f_e)$ where f_e is the total flow on e and c_e the resource cost function. Compare to the single-commodity model, there is a slight different notation with f_e^i to be the sum of the commodities' flow player i assigns on e. The joint cost is paid by the users of resource e, the players who assign positive flow on e, i.e., $\sum_{i:f_e^i>0} \chi_{ie} = C_e(f_e)$. For details of the multi-commodity model, see Section 4.1. Settings such as routing road traffic which is controlled by competing ride-sharing applications or routing in communication networks where connections are operated by competing service providers lie within this framework. For the rest of the thesis, I refer to the original model and the gerenalised model as *single-* and as *multi-commodity* per player model, respectively.

Example 7. *Focus on proportional sharing method and compute the cost of both players of the routing game instance in Figure 1.7(a). Is this a PNE?*

► Following the approach in the original model (see Example 4), the cost of each player is,

$$X_{\mathrm{r}}(P) = \sum_{e \in P_{\mathrm{r}},} \chi_{\mathrm{r},e}(P) = \chi_{\mathrm{r},e_1}(P) = \frac{w_{\mathrm{r}}}{w_{\mathrm{r}} + w_{\mathrm{b2}}} \cdot C_{e_1}(w_{\mathrm{r}} + w_{\mathrm{b2}}) =$$
$$\frac{1}{1+2} \cdot C_{e_1}(1+2) = \frac{1}{3} \cdot 27 = 9.$$

and for the blue player,

$$\begin{aligned} X_{\mathrm{b}}(P) = \sum_{e \in P_{\mathrm{b}}} \chi_{\mathrm{b},e}(P) &= \chi_{\mathrm{b},e_1}(P) + \chi_{\mathrm{b},e_2}(P) \\ &= \frac{w_{b2}}{w_r + w_{b2}} \cdot C_{e_1}(w_r + w_{b2}) + C_{e_2}(w_{b1}) \\ &= \frac{2}{1+2} \cdot C_{e_1}(1+2) + C_{e_2}(1) \\ &= \frac{2}{3} \cdot 27 + 16 = 34. \end{aligned}$$

Note that if player r deviates to e_2, her cost increases to 16. Similar for player b, any other allocation of her commodities increases her current cost of 34. Thus this outcome is a PNE. ◄

Example 8. *What is the optimal outcome of the instance in Figure 1.7 (a)?. Is this also a PNE?*

► The optimal outcome is illustrated in Figure 1.7 (b), where the social cost is minimised to $4 + 4 + 32 = 40$. Note that this is not an equilibrium as player b would swap her commodities, improving her cost to 34 from 36. ◄

Example 9. *Consider the outcome where both commodities of player b use the top resource. Is this a PNE?*

► This instance is given by Figure 1.7 (c). The difference with Figure 1.6 (a) is that blue player can reassign part of her total flow 3. By Example 4, we know that $X_{\mathrm{r}}(P) = 16$ and $X_{\mathrm{b}}(P) = 48$. If the blue player assigns her commodity $b1$ of weight 1 to e_2, then we get Figure 1.7 (a), where the cost of player b reduces to 34 as opposed to 48. Thus this is not a PNE. ◄

1.5.2 Related Work and Contribution

In this section, I present the contribution of my thesis on cost-sharing in weighted congestion games with single- and multi-commodity players. The results lie on three different aspects (described in the following order):

- Computation of approximate pure Nash equilibria
- Existence of equilibria
- Efficiency of equilibria

In the beginning of each section, it is indicated whether results hold for single- or multi-commodity players.

Computation of Approximate Pure Nash Equilibria

Our results regarding computation of equilibria hold only for the original model where each player controls a single commodity.

What is previously known: Kollias and Roughgarden [57] prove that Shapley value weighted congestion games restore the existence of a potential function and therefore the existence of pure Nash equilibria to such games. Potential functions immediately give rise to a simple and natural search procedure to find an equilibrium by performing iterative improvement steps starting from an arbitrary state. Unfortunately, this process may take exponentially many steps, even in the simple case of unweighted congestion games[1] and linear cost functions [1]. Moreover, computing a pure Nash equilibrium in these games is intractable as the problem is PLS-complete [30], even with only three players [2] or for affine linear cost functions [1]. Note that the latter result directly carries over to our game class with Shapley cost-sharing due to Proposition 1.1, which states that Shapley values and proportional shares coincide for affine cost functions.

Given these intractability results, it is natural to ask for approximation which is formally captured by the concept of an ρ-approximate pure Nash equilibrium. Chien and Sinclair [19] study the convergence towards $(1+\epsilon)$-approximate pure Nash equilibria in symmetric congestion games in polynomial time under a mild assumption on the cost functions. In contrast, Skopalik and Vòcking show that this result cannot be generalised to asymmetric games and that computing a ρ-approximate pure Nash equilibrium is PLS-hard in general [81]. Of special interest to our work is an algorithm proposed by Caragiannis et al. [11], which computes a $(2+\epsilon)$-approximate equilibrium for linear cost functions and a $d^{O(d)}$-approximate equilibrium for polynomial cost functions with degree of d. Authors of [11] extend their algorithm for the weighted case [12] of congestion games and get the following results. For linear cost functions, they compute $\left(\frac{3+\sqrt{5}}{2}+\epsilon\right)$-approximate equilibria while, for polynomial cost functions, their computable approximation factor increases to $d^{2d+o(d)}$.

Thesis contribution: We focus on SV weighted congestion games with polynomial resource cost functions and propose an algorithm with a polynomial

[1]Note that in the unweighted case of congestion games, proportional sharing and Shapley cost sharing coincide.

Figure 1.8: Results for computation of approximate pure Nash equilibria in congestion games with standard, proportional and Shapley value player costs (on a resource), for linear and polynomial rsource cost functions of maximum degree d. The thesis contribution is in blue colour.

	Proportional		Shapley values
	Unweighted	Weighted	
	$C_e(n_e(P))$	$w_i \cdot C_e(f_e(P))$	$E_{\pi\sim\Pi}[C_e(f_e^{i,\pi} + w_i) - C_e(f_e^{i,\pi})]$
linear	$2+\epsilon$, [11]	$\frac{3+\sqrt{5}}{2} + O(\gamma)$, [12]	equal to proportional (Proposition 1.1)
polynomial	$d^{O(d)}$, [11]	$d^{2\cdot d+o(d)}$, [12] $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$, [33]	$\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$, [33]

number of strategy updates for computing ρ-approximate pure Nash equilibria. This is the first algorithmic result regarding computation of equilibria in this class, however, our algorithm builds on the algorithmic ideas of [12] as follows.

The algorithm of [12] computes approximate pure Nash equilibria in polynomial weighted congestion games (proportional sharing). To use this algorithm, it is neccessary to bound the potential function of this class of games. However, it is known that weighted congestion games with polynomial cost functions $d > 1$ may not possess a pure Nash equilibrium [59, 35, 42], thus they do not admit a potential function. To overcome this, [12] constructs a new class of potential games, the Ψ games, which they use to 'approximate' the class of weighted congestion games. This additional approximation is embedded in their final computable approximation factor. More specifically, they approximate a state of Ψ games with its corresponding weighted congestion game by showing that

$$X_i^{\text{Prop}}(P) \leq X_i^{\Psi}(P) \leq d! \cdot X_i^{\text{Prop}}(P),$$

where $X_i^{\text{Prop}}(P)$ is the cost of player i under proportional sharing and $X_i^{\Psi}(P)$ her cost in a Ψ game. At the end, [12] achieves computation of $d^{2d+o(d)}$-approximate pure Nash equilibria with a polynomial time of strategy updates.

The way to adjust this algorithm in our model is the following. Since Shapley value weighted congestion games are potential games [57], the novel idea is to approximate this class of games to weighted congestion games (proportional sharing). With algebraic computations, we exhibit an interesting general relation between the Shapley cost share of a player and her proportional cost share. In particular, we prove that for a player i and state P,

$$\frac{2}{d+1} \cdot X_i^{\text{SV}}(P) \leq X_i^{\text{Prop}}(P) \leq \frac{d+3}{4} \cdot X_i^{\text{SV}}(P). \tag{1.6}$$

where d is the maximum polynomial degree. Using this relation, for SV weighted congestion games with polynomial cost functions of degree at most d, our algorithm achieves computation of approximate pure Nash with an approximation factor asymptotically close to $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$. Similar to [12], our algorithm computes a sequence of improvement steps of polynomial length reaching a ρ-approximate Nash equilibrium. Hence, it performs only a polynomial number of strategy updates. We show that our algorithm can also be used to compute ρ-approximate pure Nash equilibria for weighted congestion games (with proportional sharing) which improves the factor of $d^{2\cdot d+o(d)}$ in [12] to $\left(\frac{d}{\ln(2)}\right)^d \cdot poly(d)$.

Since for each improvement step, we need to compute the Shapley values (which is known to be computationally hard), we do not have a real polynomial time. To resolve this, we show that by applying sampling techniques, it allow us to derive a randomized polynomial running time algorithm which computes an approximate pure Nash equilibrium with high probability.

As a byproduct of this work, we derive bounds on the approximate price of anarchy in Shapley values congestion games which we use to bound the inefficiency of approximate stable outcomes.

Existence of Equilibria

We focus on the genaralised model, where each player may control multiple commodities and study existence of pure Nash equilibria. We even investigate the case where a player can ditribute the flow of a commodity among her available resources (splittable case). Figure 1.9 gives a comparison of our results to the previously known (single-commodity per player).

What is previously known: On weighted congestion games, the most common method to share the resource's total cost among her users is the proportional sharing. However, PS method lacks a desirable property, it does not always guarantee existence of pure Nash equilibria. Assume that $\mathcal{C}$ is the set of resource cost functions. It is known that if $\mathcal{C}$ consists of only affine or only exponential functions, then existence of equilibria is guaranteed [47]. If $\mathcal{C}$ is restricted to only affine functions, then [48] shows that weighted congestion games are exact potential games.

Regarding existence of approximate equilibria under proportional sharing, a recent study is by Hansknecht, Klimm and Skopalik [45], who develop techniques to obtain ρ-approximate potential functions (which prove convergence of ρ-improvement steps). For concave functions, they show upper bounds for the approximation factor of $\frac{3}{2}$ and for polynomials of maximum degree d, an upper bound of $d+1$.

Using Shapley values instead of proportinal sharing, Kollias and Roughgarden [57] show that such games are potential games, therefore they guarantee existence of pure Nash equilibria [57]. A characterisation for a much wider class of cost sharing methods was given by Gopalakrishnan et al. [43]. More specifically, [43] shows that the only cost sharing method that guarantees pure Nash equilibria in games is a generalisation of weighted Shapley values. In our model, this result applies as follows. Weighted SV with sampling weights $\lambda_i = \left(f_e^i\right)^\gamma$ is the only cost sharing method that guarantees pure Nash equilibria in single-commodity weighted congestion games for general convex cost functions. However, it is unknown whether this property carries over to the multi-commodity per player case.

Thesis contribution: As soon as we extend to the multi-commodity per player model [38], we prove that the only cost sharing method that admits

Figure 1.9: Results for existence of pure Nash equilibria in atomic (weighted) SV congestion games with single- and multi-commodity players for continuous and non-decreasing (convex) cost functions. The thesis contribution is in blue colour.

Commodities:	SINGLE Unsplittable	MULTIPLE Unsplittable	MULTIPLE Splittable
SV $\lambda_i = \left(f_e^i\right)^\gamma, \gamma = 0$	Potential Games [57]	Potential Games [38]	Potential Games [38]
WEIGHTED SV $\lambda_i = \left(f_e^i\right)^\gamma, \gamma \in \mathbb{R}$	PNE [43]	No PNE [38]	No PNE [38]

pure Nash equilibria restricts to Shapley values, where all players have identical sampling weights. This is the first result giving this separation from the single-commodity model regarding existence of equilibria. We also focus on a splittable version of the multi-commodity model where each commodity can split and distribute her flow in any way among her strategies. We show that such games under SV remain potential games, while, under weighted SV, we give instances with no pure Nash equilibria. Figure 1.9 exhibits a summary of these results.

Efficiency of Equilibria

We study the efficiency of equilibria in this class of games by computing the *price of anarchy* and the *price of stability*. The PoA is a ratio which compares the worst equilibrium to the optimum. On the other hand, the PoS compares the best pure Nash equilibrium to the optimum. Our results hold for the multi-commodity per player model, thus also for the single-commodity one.

Figure 1.10: Results for efficiency of equilibria in atomic congestion games under various cost-sharing methods with single- and multi-commodity players, and cost functions satisfying natural assumptions (see page 78). The thesis contribution is in blue colour.

Commodities:	SINGLE		MULTIPLE	
	PoA	PoS	PoA	PoS
SV	tight bounds [57]	upper bound (tight for polynomials) [37]	tight bounds [38]	upper bound (tight for polynomials) [38]
GENERAL COST-SHARING METHODS	tight bounds [37]		tight bounds [38]	

WHAT IS PREVIOUSLY KNOWN: Using Shapley values in weighted congestion games (as an alternative to proportional sharing), was introduced in [57] where authors give tight bounds on PoA. Gkatzelis et al. [41] show that, among all cost-sharing methods that guarantee existence of pure Nash equilibria in weighted congestion games, Shapley values minimise the worst PoA and proportional sharing is near optimal in general, for convex cost functions and single-commodity players. Authors in [56], [75] also discuss the optimality of SV for the extended model with non-anonymous costs by using set functions. Network cost sharing games with fixed resource costs was studied in [16] while variants of it were studied in [83] and [23]. Note that constant resource costs allow concave resource cost functions which we disallow in our model. For example, let the cost function of a resource e be $c_e(n_e) = 2$, where $n_e > 0$ the number of users of e. Then $c_e(0) = 0$ and $c_e(n_e) = 2$ for any $n_e > 0$, which gives a concave function. Looking at the same setting as my contribution but for other than Shapley cost sharing methods, [49] gives efficiency bounds that depend on cost functions and cost sharing methods. The methods they study are (a) *average cost sharing* where $X_i(P) = \frac{f_e^i(P) \cdot C_e(f_e^i(P))}{f_e(P)}$, (b) *marginal cost pricing* where $X_i(P) = f_e^i(P) \cdot C_e'(f_e^i(P))$ and (c) *incremental cost sharing* where $X_i(P) = C_e(f_e(P)) - C_e(f_e(P) - f_e^i(P))$.

For network cost-sharing games, the price of Stability was introduced in [5], which was further studied in [15, 16, 57] for weighted players and various cost-sharing methods. For undirected network games with Shapley cost sharing, [5] gives a PoS bound of H_k[1], which was later improved by [28]. In a more general setting with non-anonymous but submodular cost functions, [75] bounds the PoS for Shapley values. Klimm and Schmand [56] focus on non-anonymous costs allowing any cost function and parameterising by the number of players. For arbitrary cost functions, they prove that PoS equals to $\Theta(nlogn)$ and, for supermodular functions, equals to n.

THESIS CONTRIBUTION: We study the efficiency of (weighted) SV weighted congestion games through price of anarchy and price of stability. We give the recipe for computing the price of anarchy (PoA) and the price of stability (PoS)

[1] H_k is the harmonic sum: $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

of cost-sharing methods in weighted congestion games with multi-commodity players. In addition to the multi-commodity player extension, we greatly generalise our PoA results to general cost sharing methods and general (convex) cost functions satisfying natural assumptions. In particular, we present general tight price of anarchy bounds, which are robust, thus, they also apply to general equilibrium concepts. We then turn to the price of stability and prove an upper bound for the Shapley values cost-sharing method, which holds for general sets of (convex) cost functions and which is tight in special cases of interest, such as bounded degree polynomials. To prove this upper bound, we use the potential of such games[2]. Also, for bounded degree polynomials, we have a somehow surprising result, showing that a slight deviation from the Shapley value has a huge impact on the price of stability. In fact, as soon as you deviate to weight dependant sampling weights (weighted Shapley values), the price of stability becomes as bad as the price of anarchy. Figure 1.10 summarises these results.

[2]Shapley values (SV) congestion games with multi-commodity players are exact potential games (see Theorem 4.2.1 in page 79).

Chapter 2

Social Context in Congestion Games

This chapter formally presents an analysis of my thesis contribution in the class of social context congestion games, which was introduced in Section 1.3.1. We study the existence of approximate pure Nash equilibria in social context congestion games. This class of games consists of congestion games embedded in a social context in a sense that players may express altruistic behaviors towards others. For any given set of allowed cost functions $\mathcal{C}$, we provide a threshold value $\mu(\mathcal{C})$, and show that for the class of social context congestion games with cost functions from $\mathcal{C}$, ρ-Nash dynamics are guaranteed to converge to ρ-approximate pure Nash equilibrium if and only if $\rho > \mu(\mathcal{C})$. Interestingly, $\mu(\mathcal{C})$ is related and always upper bounded by Roughgarden's anarchy value [72].

2.1 The Model

Let N be a non-empty finite set of n players and let E be a non-empty finite set of m resources. A congestion game is a tuple $(N, E, (P_i)_{i \in N}, (c_e)_{e \in E})$, where each player chooses her pure strategy $P_i \subseteq E$ from a given set of available strategies $\mathcal{P} \subseteq 2^E$. A state or strategy profile $P = (P_1, \ldots, P_n)$ specifies a strategy for every player. Each resource $e \in E$ is associated with a cost or delay function $c_e : N \rightarrow E^+$. The load $n_e(P)$ of resource e in a state P is the number of players using resource e, i.e., $n_e(P) = |i \in N : e \in P_i|$. The personal cost of a player i is given by $X_i(P) = \sum_{e \in P_i} c_e(n_e(P))$.

We extend the definition of congestion games by embedding in it a social context. A social context is defined by an $n \times n$ matrix $\Xi = (\xi_{ij})_{i,j \in N}$ where ξ_{ij}¿0 expresses player i's interest towards player j. A social context is symmetric, if $\xi_{ij} = \xi_{ji}$ for all $i, j \in N$. Otherwise, the context assymetric. Throughout, we assume that $\xi_{ii} \geq \xi_{ij}$ for all $i, j \in N$ and that the self-interest value for every player i is scaled to $\xi_{ii} = 1$. This implies that $\xi_{ij} \leq 1$.

The perceived cost of a player $i \in N$ is given by a linear combination of his personal cost and a weighted sum of personal costs of the remaining players,

$$X_i(P, \Xi) = X_i(P) + \sum_{j \in N, j \neq i} \xi_{ij} \cdot X_j(P) = \sum_{j \in N} \xi_{ij} \cdot X_j(P).$$

The social context extension of congestion games is classified as a cost minimisation game where every player wishes to minimise his perceived cost.

Improvement move (better response). In a social context congestion game, we call a unilateral deviation of some player $i \in N$ from P to (P'_i, P_{-i}) an improvement move if

$$X_i(P_i, P_{-i}, \Xi) > X_i(P'_i, P_{-i}, \Xi).$$

ρ-improvement move. For a $\rho \geq 1$, such a unilateral deviation is called a ρ-improvement move for player i if

$$X_i(P_i, P_{-i}, \Xi) > \rho \cdot X_i(P'_i, P_{-i}, \Xi).$$

ρ-Nash dynamics. A sequence of ρ-improvement moves is called ρ-Nash dynamics. Thus, for each step in ρ-Nash dynamics, there exists a player i who decreases his perceived cost by more than a factor ρ.

Pure Nash equilibrium. A pure Nash equilibrium in a game with social context is a state $P \in \mathcal{P}$ where no player can improve its perceived costs, i.e.,

$$X_i(P_i, P_{-i}, \Xi) \leq X_i(P'_i, P_{-i}, \Xi)$$

for every player i and alternative strategy P'_i.

ρ-approximate pure Nash equilibrium. For a $\rho \geq 1$, a ρ-approximate pure Nash equilibrium is a state $P \in \mathcal{P}$ where

$$X_i(P_i, P_{-i}, \Xi) \leq \rho \cdot X_i(P'_i, P_{-i}, \Xi)$$

for every player i and alternative strategy P'_i.

ρ-potential function. For $\rho \geq 1$, a function Φ is called a ρ-potential function if $X_i(P_i, P_{-i}, \Xi) > \rho \cdot X_i(P_i, P_{-i}, \Xi)$ implies $\Phi(P) > \Phi(P'_i, P_{-i})$. For $\rho = 1$, this definition coincides with the definition of generalised potential functions in [62]. Such a function is called an exact potential function if $X_i(P) - X_i(P'_i, P_{-i}) = \Phi(P) - \Phi(P'_i, P_{-i})$ for every state P, player $i \in N$ and strategy $P'_i \in \mathcal{P}_i$.

2.2 General Social Context

This section focuses on congestion games with general[1] social context, where we study existence of approximate pure Nash equilibria.

2.2.1 The Threshold Value $\mu(\mathcal{C})$

For any given set of allowed cost functions $\mathcal{C}$, we provide a threshold value $\mu(\mathcal{C})$, and show that for the class of social context congestion games with cost functions from $\mathcal{C}$, ρ-Nash dynamics are guaranteed to converge to ρ-approximate pure Nash equilibrium if and only if $\rho > \mu(\mathcal{C})$. The threshold value $\mu(\mathcal{C})$ is defined as follows.

[1]There are no restrictions on values ξ_{ij} other than $\xi_{ij} \in [0, 1]$.

Figure 2.1: Threshold values $\mu(\mathcal{C})$ for some classes of cost functions and comparison to the *anarchy value* $\beta(\mathcal{C})$ by Roughgarden [72]. Polynomials are of maximum degree d and have non-negative coefficients.

Cost functions	$\mathcal{C}$	$\mu(\mathcal{C})$	$\beta(\mathcal{C})$
concave		$4/3 \approx 1.333$	$4/3 \approx 1.333$
polynomials	d		
	0	1	1
	1	$4/3 = 1.333$	$4/3 = 1.333$
	2	$8/5 = 1.6$	$3\sqrt{3}/(3\sqrt{3}-2) \approx 1.626$
	3	$81/43 = 1.884$	$4\sqrt[3]{4}(4\sqrt[3]{4}-3) \approx 1.895$
	4	$243/113 = 2.1504$	$5\sqrt[4]{5}(5\sqrt[4]{5}-4) \approx 2.1505$
	d	$\theta(d/\log d)$	$\theta(d/\log d)$
exponentials	α^x, $\alpha > 1$	α	unbounded

Definition 13. *For a set of allowed cost functions $\mathcal{C}$, define*

$$\mu(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x\in\mathbb{N}} \left\{ \frac{x \cdot c(x)}{(x-1)\cdot c(x-1) + c(x)} \right\}.$$

An interesting observation is that $\mu(\mathcal{C})$ is related to the *anarchy value* $\beta(\mathcal{C})$ introduced by Roughgarden [72]. He showed that the *price of anarchy* of non-atomic congestion games with cost functions in $\mathcal{C}$ is upper bounded by

$$\beta(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x,y\geq 0} \left\{ \frac{x \cdot c(x)}{y\cdot c(y) + (x-y)\cdot c(x)} \right\}.$$

Observe that $\mu(\mathcal{C}) \leq \beta(\mathcal{C})$ for all sets of cost functions $\mathcal{C}$, since $\mu(\mathcal{C})$ is more restrictive, i.e., it requires $y = x - 1$ and $x \in N$. For some sets of functions $\mathcal{C}$ such as polynomials of maximum degree d with non-negative coefficients, $\mu(\mathcal{C}) \approx \beta(\mathcal{C})$. However, $\mu(\mathcal{C})$ can also be significantly better, i.e., for exponential cost functions. Figure 2.2.1 summarises specific values of $\mu(\mathcal{C})$ and $\beta(\mathcal{C})$ for certain cost functions $\mathcal{C}$.

For general cost functions, $\mu(\mathcal{C})$ can also be unbounded. Consider, for example, a convex function where $c(x-1) = 0$ and $c(x) = 1$. For such functions, one can easily adapt our analysis to show that ρ-improvements cycle for some $\rho = \Theta(n)$. In section 2.3, we show that if we restrict to *symmetric* social context, there can be a cycle of $\Theta(\sqrt{n})$-improvements even for singleton games with binary context on two resources with convex cost functions. We also show that in this case $\Theta(\sqrt{n})$ is the worst possible.

2.2.2 Upper Bound

We start with the upper bound, i.e., by showing convergence for any $\alpha > \mu(\mathcal{C})$.

Theorem 2.2.1. *In social context congestion games with cost functions in $\mathcal{C}$, ρ-Nash dynamics converge to an ρ-approximate Nash equilibrium for any $\rho > \mu(\mathcal{C})$.*

Proof. Denote $\phi_i^e(P) = \sum_{j\in[n]:e\in P_j} \xi_{ij}$. Assume by way of contradiction that there exists a cycle of ρ-improving steps. On this cycle fix a step $P \to P'$, where

$P' = (P_{-i}, P'_i)$ for some player i, such that

$$X_i(P) = \sum_{e \in P_i} c_e(n_e(P)) \leq X_i(P') = \sum_{e \in P'_i} c_e(n_e(P')). \tag{2.1}$$

The step $P \to P'$ with $P' = (P_{-i}, P'_i)$, for player i, must exist, since otherwise each step in the cycle reduces her cost, thus each step improves Rosenthal's potential function [69], which leads in a pure Nash equilibrium. This contradicts the cycle assumption we make in the beginning of the proof. In this step, player i improves by a factor

$$\frac{\sum_{e \in E} \phi_i^e(P) \cdot c_e(n_e(P))}{\sum_{e \in E} \phi_i^e(P') \cdot c_e(n_e(P'))},$$

where the numerator expresses i's perceived cost in outcome P, and the denominator her perceived cost in P'. By splitting the resources in the sum of $e \in P_i \setminus P'_i$ and $e \in P'_i \setminus P_i$, we can upperbound this factor by

$$\leq \frac{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P) \cdot c_e(n_e(P)) + \sum_{e \in P'_i \setminus P_i} \phi_i^e(P) \cdot c_e(n_e(P))}{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P') \cdot c_e(n_e(P')) + \sum_{e \in P'_i \setminus P_i} \phi_i^e(P') \cdot c_e(n_e(P'))}.$$

Since player i makes this step (due to our assumption) which does not reduce her personal cost, we have that for all $e \in P'_i \setminus P_i$, $c_e(n_e(P')) \geq c_e(n_e(P))$ and $\phi_i^e(P') = \phi_i^e(P) + 1$ (resource e in P' includes player i as a user). Similarly, for $e \in P_i \setminus P'_i$, we have $c_e(n_e(P')) = c_e(n_e(P) - 1)$ and $\phi_i^e(P') = \phi_i^e(P) - 1$. Moreover, (2.1) implies $\sum_{e \in P_i \setminus P'_i} c_e(n_e(P)) \leq \sum_{e \in P'_i \setminus P_i} c_e(n_e(P'))$. Using these, we can upper bound our factor by

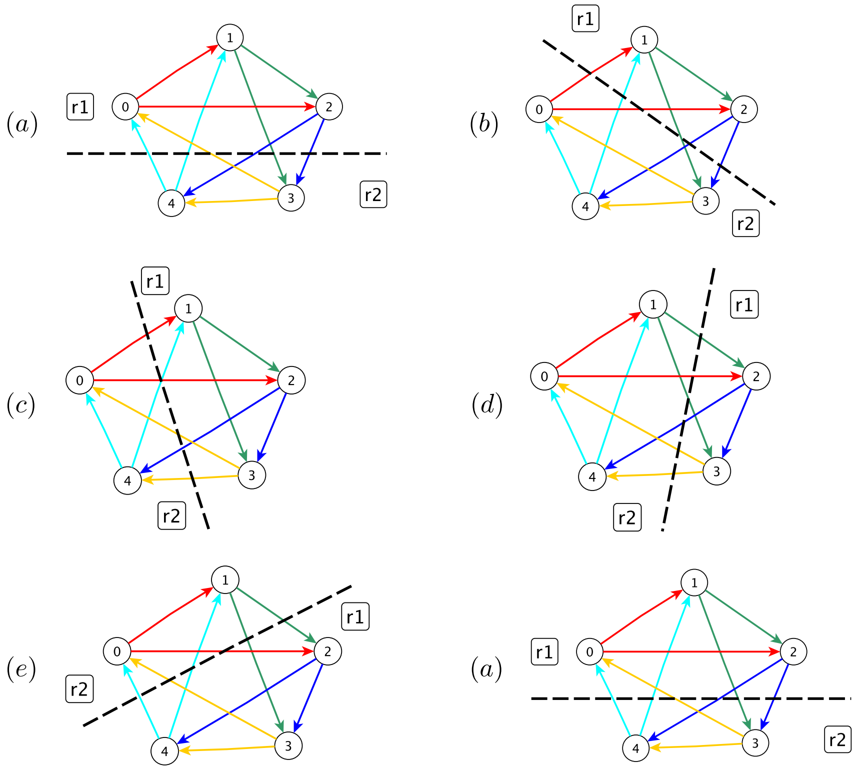
$$\begin{aligned}
&\frac{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P) \cdot c_e(n_e(P)) + \sum_{e \in P'_i \setminus P_i} \phi_i^e(P) \cdot c_e(n_e(P))}{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P') \cdot c_e(n_e(P')) + \sum_{e \in P'_i \setminus P_i} (\phi_i^e(P) + 1) \cdot c_e(n_e(P'))} \\
&\leq \frac{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P) \cdot c_e(n_e(P))}{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P') \cdot c_e(n_e(P')) + \sum_{e \in P'_i \setminus P_i} c_e(n_e(P'))} \\
&\leq \frac{\sum_{e \in P_i \setminus P'_i} \phi_i^e(P) \cdot c_e(n_e(P))}{\sum_{e \in P_i \setminus P'_i} (\phi_i^e(P) - 1) \cdot c_e(n_e(P) - 1) + \sum_{e \in P_i \setminus P'_i} c_e(n_e(P))} \\
&\leq \max_{e \in P_i \setminus P'_i} \frac{\phi_i^e(P) \cdot c_e(n_e(P))}{(\phi_i^e(P) - 1) \cdot c_e(n_e(P) - 1) + c_e(n_e(P))}.
\end{aligned}$$

where we used (2.1) in the second to last step. Observe, that this expression is increasing in $\phi_i^e(P)$ and for each $e \in P_i$, we have $\phi_i^e(P) = \sum_{j \in [n]: e \in P_j} \xi_{ij} \leq \sum_{j \in [n]: e \in P_j} 1 = n_e(P)$. Thus

$$\begin{aligned}
&\max_{e \in P_i \setminus P'_i} \frac{\phi_i^e(P) \cdot c_e(n_e(P))}{(\phi_i^e(P) - 1) \cdot c_e(n_e(P) - 1) + c_e(n_e(P))}. \\
&\leq \max_{e \in P_i \setminus P'_i} \frac{n_e(P) \cdot c_e(n_e(P))}{(n_e(P) - 1) \cdot c_e(n_e(P) - 1) + c_e(n_e(P))} \\
&\leq \mu(\mathcal{C}),
\end{aligned}$$

which is a contradiction to $\rho > \mu(\mathcal{C})$. $\square$

Figure 2.2: For $x = 3$, the lower bound cycling instance: $(a) \rightsquigarrow (b) \rightsquigarrow (c) \rightsquigarrow (d) \rightsquigarrow (e) \rightsquigarrow (a)$. Directed arrows express friendships of the corresponding vertex (player). The dashed line indicates a separation on users depending on which resource they use, r_1 or r_2.



2.2.3 Lower Bound

We proceed by providing a matching lower bound.

Theorem 2.2.2. *Given a set of cost functions $\mathcal{C}$, one can construct a singleton congestion game with (asymmetric) binary social context with cost functions from $\mathcal{C}$, where $\mu(\mathcal{C})$-Nash dynamics cycle, even for $|E| = 2$ resources with identical latency functions.*

Proof. Given $\mathcal{C}$ we construct a congestion game with asymmetric binary social context as follows. Let $c \in \mathcal{C}$ be the cost function and $x \in \mathbb{N}$ the integer that achieves $\mu(\mathcal{C})$ in Definition 13. Construct a game with two resources with identical cost function d and a cyclic ordered set $N = \{0, \ldots, 2x-2\}$ of $n = 2 \cdot x - 1$ players. The asymmetric binary social context Ξ is defined as follows: Each player $i \in N$ considers the next $x - 1$ in N as friend, i.e., $\xi_{ij} = 1$ if $j \in [i, i + x - 1]$ and $\xi_{ij} = 0$, otherwise. Here intervals are considered modulo $2x - 1$.

Define the initial state P where the x players in $\{0, \ldots, x-1\}$ are assigned to one resource and the remaining $x - 1$ players $\{x, \ldots, 2x-2\}$ to the other resource. In this profile, the perceived cost of player 0 is $X_0(P) = x \cdot c(x)$.

By deviating to the other resource, player 0 can achieve a perceived cost of $c(x) + (x-1) \cdot c(x-1)$. Thus, this is a ρ-improving step for

$$\rho = \frac{x \cdot c(x)}{(x-1) \cdot c(x-1) + c(x)} = \mu(\mathcal{C}).$$

By symmetry of social context Ξ, the remaining players can also iteratively improve in the order $\{1, \ldots, 2x-2\}$ by the same factor. We end up in a state similar to P, except that each player is now assigned to the other resource. Then Theorem 2.2.2 follows. Figure 2.2 illustrates this cycle for $x = 3$. $\square$

2.3 Symmetric Binary Social Context

This section restricts to symmetric binary[1] social context. We consider convergence of ρ-Nash dynamics in the class of social context congestion games.

2.3.1 Lower Bound

We start with the following lower bound which is a singleton congestion game with symmetric binary social context.

Theorem 2.3.1. *There exists a singleton congestion game with symmetric binary social context, where ρ-Nash dynamics cycle for $\rho = \frac{1}{3\sqrt{2}} \cdot \sqrt{n}$, even for $|E| = 2$ resources with identical convex latency functions.*

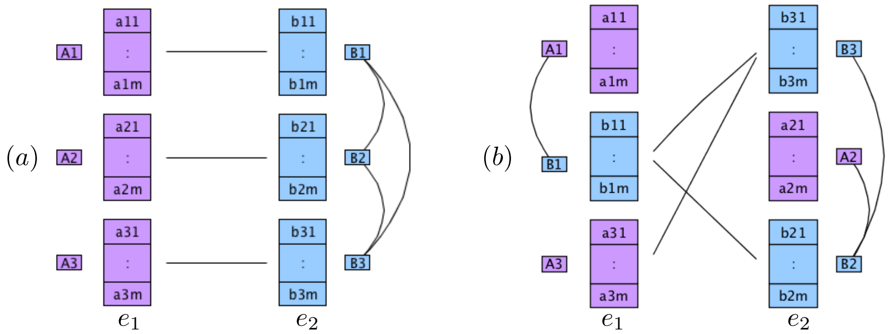
Proof. We prove the theorem by constructing an instance of singleton congestion games with binary social context Ξ and identical cost functions, where ρ-Nash dynamics cycles.

Instance. Let m be any positive integer and consider a congestion game Γ consisting of a set N of $n = 6 \cdot m$ players and a set of two resources $E = \{e_1, e_2\}$. Each player i has two strategies, $P_i = \{\{e_1\}, \{e_2\}\}$. The cost functions of the resources are identical: $c_{e_1}(x) = c_{e_2}(x) = c(x)$ with $c(x) = 0$ for $x \leq 3 \cdot m - 1$, $c(3 \cdot m) = \frac{1}{\sqrt{3 \cdot m + 1}}$ and $c(3 \cdot m + 1) = 1$. The friendships among players are described in Figure 2.3 (a). More specifically, we partition the players in 6 sets of m players each, such that $A_i = \{a_{i1}, \ldots, a_{im}\}$ and $B_i = \{b_{i1}, \ldots, b_{im}\}$, for $i \in \{1, 2, 3\}$. Then, the friendship edges form (i) a complete bipartite graph between sets A_i and B_i for all $i \in \{1, 2, 3\}$, and (ii) a complete bipartite graph between every pair of sets B_1, B_2, B_3.

Cycle. In the initial state, the players from A_1, B_1 and A_3 use resource e_1 and the players from B_3, A_2 and B_2 use resource e_2. This is illustrated in Figure 2.3 (b). We show that we can incrementally swap the players from A_1 and B_3 ending up in a state where B_1, A_3 and B_3 use resource e_1 and A_2, B_2 and A_1 use resource e_2. Since the cost functions are identical this state is symmetric to the initial state and we can proceed in the same way by swapping A_2 with B_1 and afterwards A_3 with B_2.

[1] A social context where $\xi_{ij} = \xi_{ji}$, for every pair of players i, j and ξ_{ij} can be either 0 or 1. Note that in this work, we have not investigated the symmetric non-binary.

Figure 2.3: (a) Illustration of players' friendships (by undirected arcs) in the lower bound instance as described in Theorem's 2.3.1 proof, (b) the initial state of the cycle, as described in the same proof.



Observe that there are no friendship edges between any pair of players in $A_1 \cup B_3$. Thus, all we need to show is that player a_{11} can improve by at least a factor ρ by deviating to resource e_2 and afterwards b_{31} by deviating to e_1. In the initial state a_{11} has m internal and 0 external friends[2]. Thus deviating to e_2 will change its perceived cost by a factor

$$\frac{(m+1)\cdot c(3\cdot m)}{c(3\cdot m+1)+m\cdot c(3\cdot m-1)} = \frac{m+1}{\sqrt{3\cdot m+1}} > \sqrt{\frac{m}{3}}.$$

After this move, b_{31} can deviate to e_1, improving its perceived cost by a factor

$$\frac{(m+1)\cdot c(3\cdot m+1)+2\cdot m\cdot c(3\cdot m-1)}{(3\cdot m+1)\cdot c(3\cdot m)} = \frac{m+1}{\sqrt{3\cdot m+1}} > \sqrt{\frac{m}{3}}.$$

The Theorem 2.3.1 follows, since $m = \frac{n}{6}$. □

Note that although this instance has a cycle of ρ-improvement moves for $\rho = \Omega(\sqrt{n})$, the state where A_1, A_2 and A_3 use resource e_1 and B_1, B_2 and B_3 use resource e_2 is a pure Nash equilibrium. To see this, observe that the perceived cost of some player $a \in A_1 \cup A_2 \cup A_3$ is $\frac{m+1}{\sqrt{3m+1}}$ and deviating would result in a larger perceived cost of $m+1$. Similarly, the perceived cost some player $b \in B_1 \cup B_2 \cup B_3$ is $\sqrt{3m+1}$ and deviating would result in a perceived cost of $m+1 \geq \sqrt{3m+1}$.

The conclusion is that even if there is a pure Nash equilibrium, we didn't achieve to prove convergence to this state. However, we provide the threshold value for which, ρ-Hash Dynamics with ρ larger than the threshold converge to a ρ-approximate pure Nash equilibrium.

2.3.2 Upper Bound

In the following theorem, shows that the existence of a ρ-improvement cycle (starting from a specific configuration) implies that $\rho \leq O(\sqrt{n})$. Our analysis builds on ideas from the lower bound.

[2]As friends of a player i, we consider any other player j for which it holds that $\xi_{ij} > 0$. Internal friends of i are the friends of i who use the same resource as i, while her external friends are her friends who use any other resource excepte the ones that i uses.

Theorem 2.3.2. *In social context congestion games for $|E| = 2$, for every cycle (starting from a specific configuration) of ρ-improvement moves, it holds that $\rho \leq \sqrt{2} \cdot \sqrt{n}$.*

Proof. Assuming that a cycle exists, we prove an upper bound on the improvement factor of the game. In particular, we firstly prove the existence of a special pair of moves, called a *balance-pair* of moves. For every case, we compute and maximise the improvement factors of the moves that form the balance-pair, say ρ_1 and ρ_2. The main improvement factor ρ of the game is equal to the minimum of factors ρ_1, ρ_2 and is computed by setting ρ_1 and ρ_2 equal.

Existence of a *balance-pair* of moves. We define by $int(P)$ a numerical value that expresses how much players using the same resource care for each other, that is, $int(P) = \sum_{i,j:P_i=P_j} \xi_{ij}$, and we call this value *total internal interest* of players in a profile P. Assume now that we have a cycle starting from the profile with the minimum total internal interest, say P_0. Choose the next profile in the cycle, say P_l, such that $n_1(P_l) = n_1(P_0)$ and $n_2(P_l) = n_2(P_0)$. Notice that the number of improving steps between profile P_0 and P_l is even and define that number with l. By definition of P_0, we have

$$int(P_0) \leq int(P_l) \tag{2.2}$$

which implies that these l steps do not decrease the total internal interest.

The number of steps from e_1 to e_2 equals to the number of steps from e_2 to e_1 which equals to $\frac{l}{2}$. For each step from e_1 to e_2, there exists a step from e_2 to e_1 that satisfies the following. The load vector (S_{e_1}, S_{e_2}) before the e_1 to e_2 step is the same as after the e_2 to e_1 step. A pair of steps satisfying this property is called a *balance-pair* of moves.

Claim 1. *In a cycle, there exists a balance-pair of moves that does not decrease the total internal interest.*

Proof. Assume that every balance-pair, between profiles P_0 and P_l, decreases the total internal interest. Then the total internal interest in P_0 must be lower than the total internal interest in P_l, which is contradicted by the definition of P_0, ineq.(2.2). Therefore there should exist at least one balance-pair of moves, between P_0 and P_l, that do not decrease the total internal interest, which proves our claim. □

Computing the improvement factor ρ. Consider a congestion game under social context with two resources $1, 2$. The delay functions are given by c_1, c_2 and the loads by S_1, S_2, accordingly. Assume now that we have a cycle. Then by Claim (1), there is a *balance-pair* of moves that do not decrease the total internal interest. Assume w.l.o.g. that the first deviating move of the *balance-pair* starts from resource 1. Then define $A = c_1(S_1)$, $B = c_1(S_1 + 1)$, $C = c_2(S_2)$ and $D = c_2(S_2 + 1)$. There are two cases: (1) the first step increases and the second decreases the total interest and, (2) the first step decreases and the second step increases the total internal interest. The 'third' case where both steps increase the total internal interest is actually a subcase of the case (1).

Case 1: *Increasing first step / decreasing second step in total internal interest.*

Name as player 1, the player who deviates from resource 1 to resource 2, and as player 2, the one who deviates from resource 2 to resource 1. Assume that player 1 cares about his internal friends (in resource 1) by a value l, and, about his external friends (in resource 2) by a value $l + k'$, where $k' \geq 0$. Notice that, by deviating, he increases the total internal interest by k'. Player 2 cares about his external friends by a value m, and, about his internal friends by a value $m + k$, where $k \leq k'$. Notice that dy deviating, player 2 decreases the total internal interest by k. The improvement factor of a deviating step $P_i \to P_i'$ of a player i, equals to his perceived cost $X(P_i, \mathsf{P}_{-i}, \Xi)$ over his new perceived cost $X(P_i', P_{-i}, \Xi)$, where $X(P_i, P_{-i}, \Xi) > X(P_i', P_{-i}, \Xi)$. The improvement factors of player 1 and player 2 are given by ρ_1 and ρ_2, accordingly,

$$\rho_1 = \frac{l \cdot (B + C) + B + k' \cdot C}{l \cdot (A + D) + (k' + 1) \cdot D}, \qquad \rho_2 = \frac{m \cdot (A + D) + (k + 1) \cdot D}{m \cdot (B + C) + B + k \cdot C}. \tag{2.3}$$

Subcase $(i): B - A \leq D - C \Rightarrow B + C \leq D + A$. Observe that ρ_1 is decreasing in l and since $\rho_1 > 1$, it is maximised for $l = 0$,

$$\begin{aligned} \rho_1 &\leq \frac{B + k' \cdot C}{(k' + 1) \cdot D} \\ &\leq \frac{B + k \cdot C}{(k + 1) \cdot D} \qquad \text{(since } C \leq D \text{ and } k \leq k'). \end{aligned} \tag{2.4}$$

Player 2 cares about his internal and external friends by a value $2 \cdot m + k$. Since $\xi_{ij} \leq 1$, he has at least $2 \cdot m + k$ friends. Therefore the number of players can be lower bounded by $2 \cdot m + k + 1$ and we derive an upper bound for m, $m \leq \frac{1}{2} \cdot (n - k - 1)$. Therefore ρ_2 in (2.3), is maximised as follows,

$$\begin{aligned} \rho_2 &\leq \frac{\frac{1}{2} \cdot (n - k - 1) \cdot (A + D) + (k + 1) \cdot D}{\frac{1}{2} \cdot (n - k - 1) \cdot (B + C) + B + k \cdot C} \\ &\leq \frac{(n - k - 1) \cdot D + 2 \cdot (k + 1) \cdot D}{(n - k - 1) \cdot C + 2 \cdot B + 2 \cdot k \cdot C} \qquad (A \leq B) \\ &= \frac{(n + k + 1) \cdot D}{(n + k - 1) \cdot C + 2 \cdot B}. \end{aligned} \tag{2.5}$$

By setting the maximised factors ρ_1, ρ_2 equal and solving for D, we have

$$(2.4) = (2.5): \qquad D = \sqrt{\frac{(B + k \cdot C) \cdot (C \cdot (n + k - 1) + 2 \cdot B)}{(k + 1) \cdot (n + k + 1)}}.$$

Substituting D to the maximised factor ρ_2 in (2.5) (or to ρ_1 in (2.4)), we have the main factor of the game

$$\begin{aligned} \rho &\leq \sqrt{\frac{(B + k \cdot C) \cdot (n + k + 1)}{(k + 1) \cdot (C \cdot (n + k - 1) + 2 \cdot B)}} \\ &\leq \sqrt{\frac{(B + (n - 1) \cdot C) \cdot ((n + n - 1 + 1))}{2 \cdot B + (n - 1) \cdot C}} \qquad \big(n \geq 2 \cdot m + k + 1 \Rightarrow n \geq k + 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow 0 \leq k \leq n - 1\big) \\ &\leq \sqrt{2 \cdot n}. \end{aligned}$$

Subcase (ii) : $B - A > D - C \Rightarrow B + C > D + A$. Factor ρ_2 is decreasing in m and since $\rho_2 > 1$, it is maximised for $m = 0$,

$$\rho_2 \leq \frac{(k+1) \cdot D}{B + k \cdot C}. \tag{2.6}$$

Player 1 cares about his internal and external friends by a value $2 \cdot l + k$. Since $\xi_{ij} \leq 1$, he has at least $2 \cdot l + k$ friends. Therefore the number of players can be lower bounded by $2 \cdot l + k + 1$ and we derive an upper bound for l, $l \leq \frac{1}{2} \cdot (n - k - 1)$. Therefore ρ_1 in (2.3), is maximised as follows,

$$\begin{aligned}
\rho_1 &\leq \frac{\frac{1}{2} \cdot (n - k' - 1) \cdot (B + C) + B + k' \cdot C}{\frac{1}{2} \cdot (n - k' - 1) \cdot (A + D) + (k' + 1) \cdot D} && (l \leq \frac{1}{2} \cdot (n - k' - 1)) \\
&\leq \frac{B \cdot (n - k' + 1) + C \cdot (n + k' - 1)}{D \cdot (n + k' + 1)} && (A \geq 0) \\
&\leq \frac{B \cdot (n - k + 1) + C \cdot (n + k - 1)}{D \cdot (n + k + 1)}. && (\text{since } C \leq D \text{ and } k \leq k')
\end{aligned} \tag{2.7}$$

By setting the maximised factors ρ_1, ρ_2 equal and solving for D, we have

$$(2.7) = (2.6): \qquad D = \sqrt{\frac{(B \cdot (n - k + 1) + C \cdot (n + k - 1)) \cdot (B + k \cdot C)}{(k + 1) \cdot (n + k + 1)}}.$$

Substituting D to the maximised factor ρ_2 in (2.6) (or to ρ_1 in (2.7)), we have the main factor ρ of the game

$$\begin{aligned}
\rho &= \sqrt{\frac{(B \cdot (n - k + 1) + C \cdot (n + k - 1)) \cdot (k + 1)}{(n + k + 1) \cdot (B + k \cdot C)}} \\
&\leq \sqrt{\frac{(B \cdot (n + k + 1) + C \cdot (n + k + 1)) \cdot n}{(n + k + 1) \cdot (B + k \cdot C)}} && (0 \leq k \leq n - 1) \\
&= \sqrt{\frac{(B + C) \cdot n}{B + k \cdot C}} \\
&\leq \sqrt{n} \qquad \text{for} \quad k \geq 1.
\end{aligned}$$

For $0 \leq k < 1$, we show that $\rho < 2$. By (2.7), we have

$$\begin{aligned}
\rho_1 &\leq \frac{B \cdot (n + 1) + D \cdot (n + 1)}{D \cdot (n + 1)} && (D \geq C) \\
&= 1 + \frac{B}{D},
\end{aligned} \tag{2.8}$$

while for factor ρ_2, by (2.6), we have

$$\rho_2 \leq \frac{2 \cdot D}{B}. \tag{2.9}$$

Then either ρ_1 or ρ_2 is less than 2, which implies $\rho = \min(\rho_1, \rho_2) < 2$ and completes the proof of Case 1. $\square$

The proof for the other two cases is very similar.

Case 2: *Decreasing first step / increasing second step in total internal interest.*

In this case, the deviation of player 1 decreases the total internal interest. Since this is a *balance-pair*, the second step (of player 2) must increase the total internal interest. Player 1 cares about his external friends by a value l, and, about his internal friends by a value $l+k$, where $k \geq 0$. Player 2 cares about his external friends by a value m, and, about his internal friends by a value $m+k'$, where $k' \geq k$. The factors are given by

$$\rho_1 = \frac{l \cdot (B+C) + (k+1) \cdot B}{l \cdot (A+D) + k \cdot A + D} \qquad \rho_2 = \frac{m \cdot (A+D) + k' \cdot A + D}{m \cdot (B+C) + (k'+1) \cdot B}. \tag{2.10}$$

Subcase (i) : $B - A \leq D - C \Rightarrow B + C \leq D + A$, which implies that factor ρ_1 is maximised for $l = 0$,

$$\rho_1 = \frac{(k+1) \cdot B}{k \cdot A + D}. \tag{2.11}$$

Similar to the other cases, by $n \geq 2 \cdot m + k' + 1$, factor ρ_2 is maximised for $m \leq \frac{1}{2} \cdot (n - k' - 1)$,

$$\begin{aligned} \rho_2 &\leq \frac{(n-k'-1) \cdot (A+D) + 2 \cdot k' \cdot A + 2 \cdot D}{(n-k'-1) \cdot (B+C) + (k'+1) \cdot 2 \cdot B} \\ &\leq \frac{(n+k'-1) \cdot A + (n-k'+1) \cdot D}{B \cdot (n+k'+1)} && (C=0) \\ &\leq \frac{(n+k-1) \cdot A + (n-k+1) \cdot D}{B \cdot (n+k+1)} && (\text{since } A \leq B \text{ and } k \leq k'). \end{aligned} \tag{2.12}$$

By setting factors ρ_1, ρ_2 equal and solving for B, we have

$$(2.11) = (2.12) \Rightarrow \qquad B = \sqrt{\frac{((n+k-1) \cdot A + (n-k+1) \cdot D) \cdot (k \cdot A + D)}{(k+1) \cdot (n+k+1)}}.$$

Substituting B to the maximised factor ρ_1 in (2.11) (or to ρ_2 in (2.12)), we have

$$\rho \leq \sqrt{\frac{(k+1) \cdot ((n+k-1) \cdot A + (n-k+1) \cdot D)}{(k \cdot A + D) \cdot (n+k+1)}} \tag{2.13}$$

and we examine the following two cases.

- $k \leq 1$: By computing the derivative, factor ρ is maximised for $A = 0$. Therefore,

$$\begin{aligned} \rho &\leq \sqrt{\frac{(k+1) \cdot (n-k+1) \cdot D}{D \cdot (n+k+1)}} \\ &\leq \sqrt{\frac{(\frac{n}{2}+1) \cdot (\frac{n}{2}+1) \cdot D}{D \cdot (n+1)}} && \left(\frac{\text{numerator: } k = n/2}{\text{denominator: } k = 0}\right) \\ &\leq \frac{(\frac{n}{2}+1)}{\sqrt{n}}, \end{aligned}$$

which is $O(\sqrt{n})$.

- $0 \leq k < 1$:

$$(2.13) \Rightarrow \qquad \rho \leq \frac{2 \cdot (n \cdot A + (n+1) \cdot D)}{D \cdot (n+1)} \leq 2 \cdot (1 + \frac{A}{D}). \tag{2.14}$$

By (2.12), we have

$$\begin{aligned} \rho_2 &\leq \frac{(n+k+1) \cdot A + (n-k+1) \cdot D}{A \cdot (n+k+1)} && \text{(since } A \leq B\text{)} \\ &\leq 1 + \frac{D}{A}. \end{aligned} \tag{2.15}$$

Therefore, if $D \geq A$ then $\frac{A}{D} < 1 \overset{(2.14)}{\Longrightarrow} \rho < 4$. If $D < A$ then $\frac{D}{A} < 1 \overset{(2.15)}{\Longrightarrow}$ $\alpha_2 < 2 \Rightarrow \rho = \min(\rho_1, \rho_2) < 2$.

Subcase (ii) : $B - A > D - C \Rightarrow B + C > D + A$. Smilarly, factor ρ_2 is decreasing in m, therefore is maximised for $m = 0$,

$$\rho_2 = \frac{k' \cdot A + D}{(k'+1) \cdot B} \leq \frac{D}{B}. \tag{2.16}$$

By inequality $n \geq 2 \cdot m + k' + 1$, factor ρ_1 is maximised for $l = \frac{1}{2} \cdot (n - k' - 1)$. Therefore, substituting l in ρ_1 in (2.10), we have

$$\begin{aligned} \rho_1 &\leq \frac{\frac{1}{2} \cdot (n-k'-1) \cdot (B+C) + (k'+1) \cdot B}{\frac{1}{2} \cdot (n-k'-1) \cdot (A+D) + k' \cdot A + D} \\ &= \frac{(n-k'-1) \cdot (B+C) + 2 \cdot (k'+1) \cdot B}{(n-k'-1) \cdot (A+D) + 2 \cdot k' \cdot A + 2 \cdot D} \\ &\leq \frac{(n-k'-1) \cdot B + 2 \cdot (k'+1) \cdot B}{(n-k'-1) \cdot A + 2 \cdot k' \cdot A + 2 \cdot D} && (C \leq D) \\ &= \frac{(n+k'+1) \cdot B}{(n+k'-1) \cdot A + 2 \cdot D} \\ &\leq \frac{(n+(n-1)+1) \cdot B}{(n-1) \cdot A + 2 \cdot D} && (0 \leq k' \leq n-1) \\ &\leq \frac{2 \cdot n \cdot B}{(n-1) \cdot A + D} \\ &\leq \frac{2 \cdot n \cdot B}{D}. \end{aligned} \tag{2.17}$$

By setting factors ρ_1, ρ_2 equal and solving for B, we have

$$(2.17) = (2.16) \Rightarrow \qquad B = \frac{D}{\sqrt{2 \cdot n}}.$$

Substituting B to the maximised factor ρ_1 in (2.17) (or to ρ_2 in (2.16)), we have that the main factor is upperbounded by

$$\rho \leq \frac{2 \cdot n \cdot D}{D \cdot \sqrt{2 \cdot n}} = \sqrt{2 \cdot n}. \tag{2.18}$$

□

Case 3: *Both steps increases the total internal interest.*

In this case, player 1 cares about his internal friends by a value l, and for his external by a value $l + k'_1$, where $k'_1 \geq 0$. Player 2 cares for his internal friends by m, while for his external friends for $m + k'_2$, where $k'_2 \geq 0$. Then the factors are given by

$$\rho_1 = \frac{l \cdot (B+C) + k'_1 \cdot C + B}{l \cdot (A+D) + (k'_1 + 1) \cdot D}, \qquad \rho_2 = \frac{m \cdot (A+D) + k'_2 \cdot A + D}{m \cdot (B+C) + (k'_2 + 1) \cdot B}.$$

Since $C \leq D$ and $A \leq B$, both factors are maximised as follows,

$$\rho_1 = \frac{l \cdot (B+C) + B}{l \cdot (A+D) + D}, \qquad \rho_2 = \frac{m \cdot (A+D) + D}{m \cdot (B+C) + B}. \tag{2.19}$$

But this case is covered by Case (1) for $k' = k = 0$. □

This completes the proof of the three cases, thus the proof of Theorem 2.3.2. □

2.4 Conclusion

The study on existence of approximate pure Nash equilibria focuses on general social context (Section 2.2) and symmetric binary context (Section 2.3). We allow ρ-improvement steps, that is, an improving step for a player would occur only if she had a multiplicative ρ-gain, and prove convergence to ρ-approximate pure Nash equilibria. For general social context, we give a specific value, called the *threshold value* $\mu(\mathcal{C})$, which characterises convergence of ρ-Nash dynamics: the latter converge if and only if $\rho > \mu(\mathcal{C})$. This threshold value depends on the given set of allowable cost functions. Interestingly, $\mu(\mathcal{C})$ is upper bounded but close to the price of anarchy of the Wardrop model, especially for polynomial cost functions.

For symmetric binary social context, the bounds on the factor ρ depend on the number of players: existence of ρ-improvement cycle implies that $\rho \leq \sqrt{2} \cdot \sqrt{n}$, while non convergence of ρ-Nash dynamics is shown for $\rho = \frac{1}{3\sqrt{2}} \cdot \sqrt{n}$. These results are for two resources and identical convex functions. Extending the results for n resources would be an interesting follow up to this work.

Another interesting future direction is to find how much social interactions of players affect the efficiency of such games through PoA and PoS. In particular, it would be interesting to improve already existing PoA/PoS results. For example, in [9], they present a lower bound of $1 + \frac{1}{\sqrt{2}}$ and an upper bound of 2 for the PoS for linear congestion games under altruistic[1] social context. In [4], they prove an upper bound of 7 for the PoA, also for linear congestion games under a restricted altruistic social context. In addition to these thoughts, an interesting idea would be to see how we could connect the players through friendships in order to achieve best or worst Nash equilibrium.

[1] A social context can be altruistic or spiteful, where in the altruistic case a player cares only positively for another player.

Chapter 3

Computation of Approximate Equilibria

This chapter presents a detailed analysis of the contribution for the model introduced in Section 1.5, p. 23. We study the computation of approximate pure Nash equilibria in Shapley value (SV) weighted congestion games, introduced in [57]. This class of games considers weighted congestion games where Shapley values are used as an alternative (to proportional shares) for distributing the total cost of each resource among its users. We focus in the interesting subclass of such games with polynomial resource cost functions and present an algorithm that computes approximate pure Nash equilibria with a polynomial number of strategy updates. Since computing a single strategy update is hard, we apply sampling techniques which allow us to achieve polynomial running time. The algorithm (Section 3.5) builds on the algorithmic ideas of [12], however, to the best of our knowledge, this is the first algorithmic result on computation of approximate equilibria using other than proportional shares as player costs in this setting. We present a novel relation that approximates the Shapley value of a player by her proportional share and vice versa (Section 3.3). As side results, we upper bound the approximate price of anarchy of such games (Section 3.4) and significantly improve the best known factor for computing approximate pure Nash equilibria in weighted congestion games of [12].

3.1 The Model

A weighted congestion game is given by $\mathcal{G} = (N, E, (w_i)_{i\in N}, (\mathcal{P}_i)_{i\in N}, (c_e)_{e\in E})$, where N is the set of players, E the set of resources, w_i is the positive weight of player i, $\mathcal{P}_i \subseteq 2^E$ the strategy set of player i and c_e the cost function of resource e (drawn from a set $\mathcal{C}$ of allowable cost functions). In this work, $\mathcal{C}$ is the set of polynomial functions with maximum degree d and non-negative coefficients.

Strategies. The set of outcomes of this game is given by $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ and, for an outcome, we write $P = (P_1, \ldots, P_n) \in \mathcal{P}$, where $P_i \in \mathcal{P}_i$. Let (P_{-i}, P'_i) be the outcome that results when only player i changes her strategy from P_i to P'_i and let $(P_A, P'_{N\setminus A})$ be the outcome that results when players $i \in A$ play their strategies in P and players $i \in N \setminus A$ the strategies in P'.

Resource Load. For an outcome P, the set of users of resource e is defined by $S_e(P) = \{i : e \in P_i\}$ and the total weight on e by $f_e(P) = \sum_{i \in S_e(P)} w_i$. Furthermore, let $S_e^A(P) = \{i \in A : e \in P_i\}$ and $f_e^A(P) = \sum_{i \in S_e^A(P)} w_i$ be variants of these definitions for a subset of players $A \subseteq N$.

Cost Shares. The Shapley value of a player i on a resource e is given as a function of the player's identity, the resource cost function and her users A, i.e., $\chi_e(i, A, c_e)$. For simplicity, I write $\chi_e(i, A)$, or $\chi_{ie}(P)$ when all users of e are considered in a state P. Let $C_e(x) = x \cdot c_e(x)$. Then, the *joint cost* on a resource e is given by $C_e(f_e(P)) = f_e(P) \cdot c_e(f_e(P))$ and the costs of players are such that $C_e(f_e(P)) = \sum_{i \in S_e(P)} \chi_{ie}(P)$. The *total cost* of a player i equals the sum of her costs in the resources she uses, i.e. $X_i(P) = \sum_{e \in P_i} \chi_{ie}(P)$. The social cost is the total cost of the game which is given by

$$SC(P) = \sum_{e \in E} f_e(P) \cdot c_e(f_e(P)) = \sum_{e \in E} \sum_{i \in S_e(P)} \chi_{ie}(P) = \sum_{i \in N} X_i(P). \tag{3.1}$$

For an $A \subseteq N$, the social cost of the set A equals $SC_A(P) = \sum_{i \in A} X_i(P)$.

The cost-sharing method is important for our analysis, as it defines how the joint cost on a resource e, is distributed among her users. In this paper, the methods we focus on are the Shapley value and the proportional cost-sharing, which we introduce in detail.

Shapley Values. For a set of players A, let $\Pi(A)$ be the set of orderings $\pi : A \to \{1, \ldots, |A|\}$. For a $\pi \in \Pi(A)$, define as $A^{<i,\pi} = \{j \in A : \pi(j) < \pi(i)\}$ the set of players preceding player i in π and as $W_A^{<i,\pi} = \sum_{j \in A : \pi(j) < \pi(i)} w_j$ the sum of their weights. For the uniform distribution over $\Pi(A)$, the Shapley value of a player i on resource e is given by the expectation of her marginal contributions (marginal cost increases caused by i),

$$\chi_e(i, A) = E_{\pi \sim \Pi(A)} \left[C_e\left(W_A^{<i,\pi} + w_i\right) - C_e\left(W_A^{<i,\pi}\right) \right].$$

Proportional Sharing. The cost of a player i on a resource under proportional sharing is given by $\chi_{ie}^{\text{Prop}}(P) = w_i \cdot c_e(f_e(P))$. For the rest of the paper, we write $X_i^{\text{Prop}}(P) = \sum_{e \in E} \chi_{ie}^{\text{Prop}}(P)$ to indicate when switch to proportional sharing.

ρ-Approximate Pure Nash Equilibrium. Given a parameter $\rho \geq 1$ and an outcome P, we call as ρ-*move* a deviation from P_i to P_i' where the player improves her cost by at least a factor ρ, formally $X_i(P) \geq \rho \cdot X_i(P_{-i}, P_i')$. We call the state P an ρ-*approximate pure Nash equilibrium* (ρ-PNE) if and only if no player is able to perform a ρ-move, formally it holds for every player i and any other strategy $P_i' \in \mathcal{P}_i$ that $X_i(P) \leq \rho \cdot X_i(P_{-i}, P_i')$.

ρ-Approximate Price of Anarchy. Given a parameter $\rho \geq 1$, let $\mathcal{Z}$ be the set of all outcomes and $\mathcal{Z}^{\mathcal{N}}$ the set of approximate pure Nash equilibria of the game. Then the ρ-*Approximate price of anarchy* (ρ-PoA) is defined as $\rho\text{-PoA} = \frac{\max_{P \in \mathcal{Z}^{\mathcal{N}}} SC(P)}{\min_{P \in \mathcal{Z}} SC(P)}$.

Kollias and Roughgarden [57] prove that weighted congestion games under Shapley values are potential games. To do this, they use the following potential[1] function:

Potential Function. Given an outcome P and an arbitrary ordering τ of the players in N, the potential is given by

$$\Phi(P) = \sum_{e \in E} \Phi_e(P) = \sum_{e \in E} \sum_{i \in S_e(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}). \tag{3.2}$$

where $\{j : \tau(j) \leq \tau(i), j \in S_e(P)\}$ is the set of players j who are users of resource e and before player i in the ordering τ.

A-Limited Potential. We now restrict this potential function by allowing only a subset of players $A \subseteq N$ to participate and define the *A-limited potential* as

$$\Phi^A(P) = \sum_{e \in E} \Phi_e^A(P) = \sum_{e \in E} \sum_{i \in S_e^A(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^A(P)\}). \tag{3.3}$$

B-Partial Potential. Consider sets A and B such that $B \subseteq A \subseteq N$. Then the *$B$-partial potential* of set A is defined by

$$\Phi_B^A(P) = \Phi^A(P) - \Phi^{A \setminus B}(P) = \sum_{e \in E} \Phi_{e,B}^A(P) = \sum_{e \in E} \Phi_e^A(P) - \Phi_e^{A \setminus B}(P). \tag{3.4}$$

If the set B contains only one player, i.e., $B = \{\{i\}\}$, then we write $\Phi_i^A(P) = \Phi_B^A(P)$. In case of $A = N$, $\Phi_B^N(P) = \Phi_B(P) = \sum_{e \in E} \Phi_{e,B}(P)$. Intuitively, $\Phi_B^A(P)$ is the value that the players in $B \subseteq A$ contribute to the A-limited potential.

ρ-Stretch. Similar to ρ-PoA, we define a ratio with respect to the potential function. Let $\hat{P}$ be the outcome that minimises the potential, i.e., $\hat{P} = \min_{P' \in \mathcal{P}} \Phi(P')$. Then the *$\rho$-stretch* is defined as

$$\rho\text{-}\Omega = \max_{P \in \rho\text{-PNE}} \frac{\Phi(P)}{\Phi(\hat{P})}. \tag{3.5}$$

A-Limited ρ-Stretch. Additionally, we define a ρ-stretch restricted to players in a subset $A \subseteq N$. Let $\rho\text{-PNE}_A \subseteq \mathcal{P}$ be the set of ρ-approximate pure Nash equilibria where only players in A participate. The rest of the players have a fixed strategy $\bar{P}_{N \setminus A}$. Then we define the *A-limited ρ-stretch* as

$$\rho\text{-}\Omega_A = \max_{P \in \rho\text{-PNE}_A} \frac{\Phi(P)}{\Phi(\hat{P})} = \max_{P \in \rho\text{-PNE}_A} \frac{\Phi(P_A, \bar{P}_{N \setminus A})}{\Phi(\hat{P}_A, \bar{P}_{N \setminus A})}. \tag{3.6}$$

3.1.1 Algorithmic Approach and Outline

Our algorithm is based on ideas by Caragiannis et al. [12]. Intuitively, we partition the players' costs into intervals $[b_1, b_2], [b_2, b_3], \ldots, [b_{m-1}, b_m]$ in decreasing order. The cost values in one interval are within a polynomial factor. Note that

[1] The proof of why this is a potential function can be found in [57].

this ensures that every sequence of ρ-moves for $\rho > 1$ of players with costs in one or two intervals converges in polynomial time.

After an initialization, the algorithm proceeds in phases r from 1 to $m-1$. In each phase r, players with costs in the interval $[b_r, +\infty]$ do α-approximate moves where α is close to the desired approximation factor. Players with costs in the interval $[b_{r+1}, b_r]$ make $1+\gamma$-moves for some small $\gamma > 0$. After a polynomial number of steps no such moves are possible and we freeze all players with costs in $[b_r, +\infty]$. These players will never be allowed to move again. We then proceed with the next phase. Note that at the time players are frozen, they are in an α-approximate equilibrium. The purpose of the $1+\gamma$-moves of players of the neighboring interval is to ensure that the costs of frozen players do not change significantly in later phases. To that end we utilize a potential function argument. We argue about the potential of *sub games* among a subset of players. We can bound the potential value of an arbitrary q-approximate equilibrium with the minimal potential value (using the *stretch*). Compared to the approach in [12], we directly work with the exact potential function of the game which significantly improves the results, but also requires a more involved analysis. We show that the potential of the sub game in one phase is significantly smaller than b_r. Therefore, the costs experienced by players moving in phase r are considerably lower than the costs of any player in the interval $[b_1, b_{r-1}]$.

The analysis heavily depends on the stretch of the potential function which we analyze in Section 3.4. The proof there is based on the technique of Section 3.3 in which we approximate the Shapley with proportional cost sharing. For the technical details in both sections we need some structural properties of costs-shares and the restricted potentials which we show in the next section.

3.2 Shapley Value Weighted Congestion Games

The purpose of this section is to exploit properties of Shapley values (Section 3.2.1) and properties of its potential function (Section 3.2.2), which are important for our analysis.

3.2.1 Properties of Shapley Values

The following properties of the Shapley values are extensively used in our proofs.

Proposition 3.1. *Fix a resource e. Then for any set of players S and $i \in S$, we have for $j, j_1, j_2, j', j_1', j_2', i_1, i_2 \notin S$:*

a. $\chi_e(i, S) \leq \chi_e(i, S \cup \{j\})$,

b. $\chi_e(i, S \cup \{j'\}) \geq \chi_e(i, S \cup \{j_1, j_2\})$, *with* $j' \neq i$ *and* $w_{j'} = w_{j_1} + w_{j_2}$,

c. $\chi_e(i, S \cup \{j_1, j_2\}) \geq \chi_e(i, S \cup \{j_1', j_2'\})$, *with* $w_{j_1'} = w_{j_2'} = \frac{w_{j_1} + w_{j_2}}{2}$,

d. $\chi_e(i, S) \geq \chi_e(i_1, S \backslash \{i\} \cup \{i_1\}) + \chi_e(i_2, S \backslash \{i\} \cup \{i_1, i_2\})$,
with $w_{i_1} = w_{i_2} = \frac{w_i}{2}$.

Proof. (a) By the definition of Shapley values, we have

$$\begin{aligned}
\chi_e(i, S \cup \{j\}) &= \frac{1}{(k+1)!} \sum_{\pi \in \Pi(S \cup \{j\})} \left(C_e\left(W_{S \cup \{j\}}^{<i,\pi} + w_i \right) - C_e\left(W_{S \cup \{j\}}^{<i,\pi} \right) \right) \\
&\geq \frac{1}{(k+1)!} \sum_{\pi \in \Pi(S \cup \{j\})} \left(C_e\left(W_S^{<i,\pi} + w_i \right) - C_e\left(W_S^{<i,\pi} \right) \right) \\
&= \frac{1}{k!} \sum_{\pi \in \Pi(S)} \left(C_e\left(W_S^{<i,\pi} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_i \right) \right) \\
&= \chi_e(i, S).
\end{aligned}$$

For (b) and (c), consider $\chi_e\left(i, S \cup \{j_1, j_2\}\right)$. Observe, that only for permutations $\pi \in \Pi(S \cup \{j_1, j_2\})$ where either $j_1 < i < j_2$ or $j_2 < i < j_1$ the corresponding contribution to $\chi_e\left(i, S \cup \{j_1, j_2\}\right)$ changes if we change the weight of j_1, j_2 but keep their sum the same. Fix a permutation $\pi \in \Pi(S \cup \{j_1, j_2\})$ with $j_1 < i < j_2$ and pair it with the corresponding permutation $\hat{\pi}$ where only j_1 and j_2 are swapped. Then the contribution of π and $\hat{\pi}$ to $\chi_e\left(i, S \cup \{j_1, j_2\}\right)$ is

$$\begin{aligned}
&\frac{1}{(k+2)!} \cdot \left(C_e\left(W_S^{<i,\pi} + w_{j_1} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_{j_1} \right) \right) \\
&\qquad + C_e\left(W_S^{<i,\pi} + w_{j_2} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_{j_2} \right) \Big). \qquad (3.7)
\end{aligned}$$

Since $C_e(x + w_i) - C_e(x)$ is convex in x, we get that

$$\begin{aligned}
(3.7) \geq \frac{1}{(k+2)!} \cdot &\left(C_e\left(W_S^{<i,\pi} + w_{j_1'} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_{j_1'} \right) \right. \\
&\left. + C_e\left(W_S^{<i,\pi} + w_{j_2'} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_{j_2'} \right) \right),
\end{aligned}$$

and

$$\begin{aligned}
(3.7) \leq \frac{1}{(k+2)!} \cdot &\left(C_e\left(W_S^{<i,\pi} + w_{j_1} + w_{j_2} + w_i \right) - C_e\left(W_S^{<i,\pi} + w_{j_1} + w_{j_2} \right) \right. \\
&\left. + C_e\left(W_S^{<i,\pi} + 0 + w_i \right) - C_e\left(W_S^{<i,\pi} + 0 \right) \right).
\end{aligned}$$

Part (c) and (b) follow, respectively. Part (d) of the proposition is shown in Lemma 24 of Chapter 4 for a generalisation of this model. □

3.2.2 Properties of Limited and Partial Potentials

We proceed to the properties of the restricted types of the potential function defined before (Proposition 3.2 and 3.3). At the end, we give relation between partial potential and Shapley values (Lemma 2).

Proposition 3.2. *Let A and B be sets of players such that $B \subseteq A \subseteq N$, P and P' outcomes of the game such that the players in $A \subseteq N$ use the same strategies in both P and P', and $z \in N$ an arbitrary player. Then* *(a)* $\Phi_B^A(P) \leq \Phi_B(P)$, *(b)* $\Phi_B^A(P) = \Phi_B^A(P')$, *(c)* $\Phi_z(P) = X_z(P)$.

Proof. **(a)** For each $e \in E$, let $I_e(P) = \Phi_e^A(P) - \Phi_e^{A\setminus B}(P)$. By definition of the B-partial potential (3.4), we have

$$\Phi_B^A(P) = \Phi^A(P) - \Phi^{A\setminus B}(P) = \sum_{e\in E} I_e(P). \tag{3.8}$$

By the definition of limited potential (3.3), for an arbitrary τ, define $I_e(P)$, $\forall e \in E$, as

$$\sum_{i\in S_e^A(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^A(P)\}) - \sum_{i\in S_e^{A\setminus B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{A\setminus B}(P)\}). \tag{3.9}$$

Kollias and Roughgarden [57] proved that the potential is independent of the ordering τ that players are considered. As mentioned before, $\Phi^A(P)$ is a restriction of $\Phi(P)$ where only players in A participate. Thus, independence from τ also applies to the limited potential.

Firstly, we focus on the first term of (3.9) and choose an ordering where the players in set A are first. Then we observe that by substituting $S_e^A(P)$ with $S_e(P)$, the cost share remains the same. This is due to the fact that any player coming after the players in set A in the ordering has no impact in the cost computation. These are the players who belong in set $N \setminus A$ (since we assume players in A are first). Therefore, the first term of (3.9) equals to

$$\sum_{i\in S_e^A(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}).$$

Following the same technique for the second term of (3.9), we choose an ordering in which the players in $A \setminus B$ are first. Then we can substitute $S_e^{A\setminus B}(P)$ with $S_e^{N\setminus B}(P)$ without affecting the term's value. Therefore, (3.9) is equivalent to

$$\sum_{i\in S_e^A(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}) - \sum_{i\in S_e^{A\setminus B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{N\setminus B}(P)\}). \tag{3.10}$$

For each $e \in E$, define $I'_e(P)$ to be equal to

$$\sum_{i\in S_e^{N\setminus A}(P)} \Big(\chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}) - \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{N\setminus B}(P)\})\Big). \tag{3.11}$$

Note that $I'_e(P) \geq 0$, $\forall e \in E$. Intuitively, the first term computes the cost with respect to all players using resource e, $S_e(P)$. Regarding the second term, if we take away some of these players, i.e., players in B, then due to convexity the costs of the remaining players either remain the same or are reduced. This depends on the position players in B had in the ordering. To simplify, for the rest of this proof, let

$$\chi_i^N(P) = \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}), \tag{3.12}$$

$$\chi_i^{N\setminus B}(P) = \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{N\setminus B}(P)\}). \tag{3.13}$$

Since $I'_e(P) \geq 0$, we get that for each $e \in E$,

$$I_e(P) \leq I_e(P) + I'_e(P)$$

which, by (3.10), (3.11), (3.12) and (3.13), is equivalent to

$$\begin{aligned}\sum_{i\in S_e^A(P)} \chi_i^N(P) - \sum_{i\in S_e^{A\setminus B}(P)} \chi_i^{N\setminus B}(P) \leq \\ \leq \sum_{i\in S_e^A(P)} \chi_i^N(P) - \sum_{i\in S_e^{A\setminus B}(P)} \chi_i^{N\setminus B}(P) + \sum_{i\in S_e^{N\setminus A}(P)} \left(\chi_i^N(P) - \chi_i^{N\setminus B}(P)\right).\end{aligned} \tag{3.14}$$

By the assumption $B \subseteq A \subseteq N$, we get that $(N \setminus A) \cup (A \setminus B) = N \setminus B$. Thus, inequality (62) becomes

$$\sum_{i\in S_e^A(P)} \chi_i^N(P) - \sum_{i\in S_e^{A\setminus B}(P)} \chi_i^{N\setminus B}(P) \leq \sum_{i\in S_e(P)} \chi_i^N(P) - \sum_{i\in S_e^{N\setminus B}(P)} \chi_i^{N\setminus B}(P).$$

Substituting $\chi_i^N(P)$, $\chi_i^{N\setminus B}(P)$ from (3.12), (3.13), respectively, and using (3.10) we get that the previous is equivalent to

$$I_e(P) \leq \Phi_e(P) - \Phi_e^{N\setminus B}(P) \quad \Leftrightarrow \quad \sum_{e\in E} I_e(P) \leq \sum_{e\in E} \Phi_e(P) - \Phi_e^{N\setminus B}(P).$$

By (3.8), we conclude to the desirable $\Phi_B^A(P) \leq \Phi_B(P)$. □

(b) By definition (3.4) of partial potential, we have

$$\Phi_B^A(P) = \Phi^A(P) - \Phi^{A\setminus B}(P) = \sum_{e\in E} \left(\Phi_e^A(P) - \Phi_e^{A\setminus B}(P)\right). \tag{3.15}$$

For each $e \in E$ and any $A' \subseteq A$, observe that $S_e^{A'}(P) = S_e^{A'}(P')$, thus

$$\begin{aligned}&\sum_{i\in S_e^A(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^A(P)\}) \\ &= \sum_{i\in S_e^A(P')} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^A(P')\}).\end{aligned}$$

Therefore, $\Phi_e^A(P) = \Phi_e^A(P')$. Similarly, we prove that $\Phi_e^{A\setminus B}(P) = \Phi_e^{A\setminus B}(P')$. Using (3.15), we have

$$\Phi_B^A(P) = \sum_{e\in E} \left(\Phi_e^A(P') - \Phi_e^{A\setminus B}(P')\right) = \Phi_B^A(P'). \quad \square$$

(c) Let P be an outcome of the game. The contribution $\Phi_z(P)$ of player z in the potential value is given by

$$\Phi(P) - \Phi^{N\setminus\{z\}}(P) = \sum_{e\in E}\Big(\Phi_e(P) - \Phi_e^{N\setminus\{z\}}(P)\Big) = \sum_{e\in E} I_e(P), \tag{3.16}$$

where $I_e(P)$ equals

$$\sum_{i\in S_e(P)} \chi_e(i, \{j : \tau(j) \le \tau(i), j \in S_e(P)\}) \\ - \sum_{i\in S_e^{N\setminus\{z\}}(P)} \chi_e(i, \{j : \tau(j) \le \tau(i), j \in S_e^{N\setminus\{z\}}\}).$$

Since the potential is independent of the players' ordering, we choose the τ such that player z is last. Then (3.16) equals to

$$\begin{aligned} &\sum_{e\in E}\chi_e(z, \{j : \tau(j) \le \tau(z), j \in S_e(P)\}) \\ &= \sum_{e\in E}\chi_e(z, j : j \in S_e(P)) \\ &= \sum_{e\in E}\chi_{ze}(P) = X_z(P). \quad \square \end{aligned}$$

This completes the proof of Proposition 3.2. □

Next, we show that the general potential property also holds for the partial potential.

Proposition 3.3. *Consider a subset $B \subseteq N$ and a player $i \in B$. Given two states, P and P', that differ only in the strategy of player i, then*

$$\Phi_B(P) - \Phi_B(P') = X_i(P) - X_i(P'). \tag{3.17}$$

Proof. By definition of the partial potential (3.4),

$$\begin{aligned} \Phi_B(P) - \Phi_B(P') &= \Phi(P) - \Phi^{N\setminus B}(P) - \Big(\Phi(P') - \Phi^{N\setminus B}(P')\Big) \\ &= \Phi(P) - \Phi(P'). \end{aligned}$$

Since the underlying game (considering all players in N) is a potential game [57], we have that $\Phi(P) - \Phi(P') = X_i(P) - X_i(P')$, which completes the proof. □

In the following lemma, we give a relation between partial potential and Shapley values.

Lemma 2. *Given an outcome P of the game, a resource e and a subset $B \subseteq N$, it holds that*

$$\Phi_{e,B}(P) \le \sum_{i\in B}\chi_{ie}(P) \le \Phi_{e,B}(P)\cdot(d+1),$$

where d is the maximum degree of the polynomial cost function.

Proof. By definition (3.4), we have

$$\begin{aligned}\Phi_{e,B}(P) &= \Phi_e(P) - \Phi_e^{N\setminus B}(P)\\ &= \sum_{e\in E}\Big(\Phi_e(P) - \Phi_e^{N\setminus B}(P)\Big) = I_e(P).\end{aligned} \tag{3.18}$$

where $I_e(P)$ equals to

$$\sum_{i\in S_e(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}) \\ - \sum_{i\in S_e^{N\setminus B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{N\setminus B}\}). \tag{3.19}$$

Then we break the first term of (3.19) to the sum of

$$\sum_{i\in S_e^{N\setminus B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}) \\ + \sum_{i\in S_e^{B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}).$$

We choose an ordering τ in which all players in $N \setminus B$ come first. Then the previous sum is equivalent to

$$\sum_{i\in S_e^{N\setminus B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e^{N\setminus B}(P)\}) \\ + \sum_{i\in S_e^{B}(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\}).$$

Substituting the previous sum to (3.19) (first term) gives that

$$\begin{aligned}I_e(P) &= \sum_{i\in S_e^B(P)} \chi_e(i, \{j : \tau(j) \leq \tau(i), j \in S_e(P)\})\\ &\leq \sum_{i\in S_e^B(P)} \chi_e(i, j : j \in S_e(P))\\ &= \sum_{i\in S_e^B(P)} \chi_{ie}(P)\\ &= \sum_{i\in B} \chi_{ie}(P).\end{aligned}$$

Equation (3.18) completes the proof of the lower bound. For the upper bound consider a fixed ordering of the players in D. The partial potential can be written as

$$\begin{aligned}
\Phi_{e,B}(P) &= \left(\Phi_e(P) - \Phi_e^{N\setminus B}(P)\right) \\
&= \sum_{i \in S_e^B(P)} \chi_e\left(i, \{j : \tau(j) \leq \tau(i), j \in S_e^B(P)\} \cup S_e^{N\setminus B}(P)\right) \\
&\geq \int_{f_e^{N\setminus B}(P)}^{f_e^N(P)} c_e(x)dx \\
&\geq \left[\frac{x \cdot c_e(x)}{d+1}\right]_{f_e^{N\setminus B}(P)}^{f_e^N(P)} \\
&= \frac{f_e^N(P) \cdot c_e\left(f_e^N(P)\right) - f_e^{N\setminus B}(P) \cdot c_e\left(f_e^{N\setminus B}(P)\right)}{d+1} \\
&= \frac{f_e(P) \cdot c_e(f_e(P))}{d+1} - \frac{f_e^{N\setminus B}(P) \cdot c_e\left(f_e^{N\setminus B}(P)\right)}{d+1} \\
&= \frac{\sum_{i\in N} \chi_{ie}(P)}{d+1} - \frac{f_e^{N\setminus B}(P) \cdot c_e\left(f_e^{N\setminus B}(P)\right)}{d+1}, \qquad (3.20)
\end{aligned}$$

where the first inequality follows by repeatedly applying Proposition 3.1(c) and 3.1(d) and adding additional players of weight 0 (which do not change the cost shares). The second inequality holds, since c_e is a polynomial of maximum degree d with non-negative coefficients.

Observe, that $f_e^{N\setminus B}(P) \cdot c_e\left(f_e^{N\setminus B}(P)\right)$ is the social cost of P on resource e if only the players in $N \setminus B$ are in the game. By Proposition 3.1(a), the cost shares of those players can only increase if the players in B are joining the game, i.e.:

$$f_e^{N\setminus B}(P) \cdot c_e\left(f_e^{N\setminus B}(P)\right) \leq \sum_{i \in N\setminus B} \chi_{ie}(P).$$

Combining this with (3.20) completes the proof of the claim:

$$\Phi_{e,B}(P) \geq \frac{\sum_{i\in N} \chi_{ie}(P)}{d+1} - \frac{\sum_{i\in N\setminus B} \chi_{ie}(P)}{d+1} = \frac{\sum_{i\in B} \chi_{ie}(P)}{d+1}.$$

□

Summing up over all resources $e \in E$ yields to the next corollary.

Corollary 1. *Given an outcome P of the game and a subset $B \subseteq N$, it holds that*

$$\Phi_B(P) \leq \sum_{i\in B} X_i(P) \leq \Phi_B(P) \cdot (d+1).$$

Proof. By the definition of the partial potential (3.4) and by applying Lemma 2, we directly have

$$\Phi_B(P) = \sum_{e\in E} \Phi_{e,B}(P) \leq \sum_{e\in E} \sum_{i\in B} \chi_{ie}(P) = \sum_{i\in B} X_i(P)$$

and

$$\begin{aligned}\sum_{i\in B} X_i(P) = \sum_{i\in B}\sum_{e\in E}\chi_{ie}(P) &= \sum_{e\in E}\sum_{i\in B}\chi_{ie}(P)\\ &\le \sum_{e\in E}\Phi_{e,B}(P)\cdot(d+1)\\ &= \Phi_B(P)\cdot(d+1).\end{aligned}$$

□

3.3 From Shapley to Proportional Sharing

In this section we approximate a state of Shapley value congestion games with its corresponding weighted congestion game (with proportional sharing). This approximation guarantee plays an important role in our proofs of the stretch and for the computation.

Lemma 3. *For a player i, a resource e and any state P, the following inequality holds between her Shapley and proportional cost,*

$$\frac{2}{d+1}\cdot\chi_{ie}(P) \le \chi_{ie}^{Prop}(P) \le \frac{d+3}{4}\cdot\chi_{ie}(P).$$

For $d=1$, the equality holds (Proposition 1.1).

Proof. Since c_e is a polynomial of maximum degree d with non-negative coefficients, it is equivalent to show the inequalities for all monomial cost functions $c_e(x) = x^r$, with $r = \{0,\ldots,d\}$. Details of this reduction can be found in [31]. Fix some resource e with monomial cost function and a player i assigned to e, i.e., $e\in P_i$. Denote $Y = \{j \ne i : e \in P_j\}$ and $w = w_i$. Define $y = \sum_{j\in Y} w_j$ and $z = \frac{w}{y}$. By Proposition 3.1 (b), we can upper bound $\chi_{ie}(P)$ by replacing Y with a single player of weight y, i.e.,

$$\begin{aligned}\chi_{ie}(P) &\le \frac{1}{2}\cdot\left((y+w)^{r+1} - y^{r+1}\right) + \frac{1}{2}\cdot w^{r+1}\\ &= y^{r+1}\cdot\frac{1}{2}\cdot\left((z+1)^{r+1} - 1 + z^{r+1}\right)\\ &= y^{r+1}\cdot\left(z^{r+1} + \frac{1}{2}\cdot\sum_{j=1}^{r}\binom{r+1}{j}\cdot z^j\right) =: A\end{aligned}$$

Similarly, by repeatedly using Proposition 3.1 (c) and by adding additional players of weight 0, we can lower bound $\chi_{ie}(P)$ by

$$\begin{aligned}&\frac{1}{y}\cdot\int_0^y\left((x+w)^{r+1} - x^{r+1}\right)dx\\ &= \frac{1}{y}\cdot\frac{1}{r+2}\cdot\left((y+w)^{r+2} - y^{r+2} - w^{r+2}\right)\\ &= y^{r+1}\cdot\frac{1}{r+2}\cdot\left((z+1)^{r+2} - 1 - z^{r+2}\right)\\ &= y^{r+1}\cdot\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j}\cdot z^j =: B.\end{aligned}$$

The proportional cost of player i , $\chi_{ie}^{\text{Prop}}(P)$, equals to

$$w \cdot c_e(y+w) = w \cdot (y+w)^r = y^{r+1} \cdot z \cdot (z+1)^r = y^{r+1} \cdot \sum_{j=1}^{r+1} \binom{r}{j-1} \cdot z^j.$$

To complete the proof we give an upper bound on $\frac{A}{\chi_{ie}^{\text{Prop}}(P)}$ and a lower bound on $\frac{B}{\chi_{ie}^{\text{Prop}}(P)}$. We have,

$$\frac{A}{\chi_{ie}^{\text{Prop}}(P)} = \frac{z^{r+1} + \frac{1}{2}\sum_{j=1}^{r}\binom{r+1}{j} \cdot z^j}{\sum_{j=1}^{r+1}\binom{r}{j-1} \cdot z^j} = \frac{z^{r+1} + \frac{1}{2}\sum_{j=1}^{r}\binom{r+1}{j} \cdot z^j}{z^{r+1} + \sum_{j=1}^{r}\binom{r}{j-1} \cdot z^j},$$

which is upper bounded by

$$\frac{A}{\chi_{ie}^{\text{Prop}}(P)} \leq \max\left(1, \max_{1\leq j\leq r} \frac{\binom{r+1}{j}}{2\cdot\binom{r}{j-1}}\right) = \max\left(1, \max_{1\leq j\leq r} \frac{r+1}{2\cdot j}\right) \leq \frac{d+1}{2}. \tag{3.21}$$

This implies the lower bound on $\chi_{ie}^{\text{Prop}}(P)$ in the statement of the lemma. On the other hand, by first order conditions,

$$\frac{B}{\chi_{ie}^{\text{Prop}}(P)} = \frac{\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j} \cdot z^j}{\sum_{j=1}^{r+1}\binom{r}{j-1} \cdot z^j},$$

which achieves its extreme values at the roots of

$$g(z) := \sum_{j=1}^{r+1}\sum_{k=1}^{r+1}(j-k)\binom{r+2}{j}\binom{r}{k-1} \cdot z^{k+j-1}.$$

Lemma 4. [1] *The function $g : z \to \sum_{j=1}^{r+1}\sum_{k=1}^{r+1}(j-k)\binom{r+2}{j}\binom{r}{k-1} \cdot z^{k+j-1}$ has a unique positive real root at $z = 1$.*

By Lemma 4, we conclude that $\frac{B}{\chi_{ie}^{\text{Prop}}(P)}$ is minimized for $z = 1$, i.e.,

$$\frac{B}{\chi_{ie}^{\text{Prop}}(P)} \geq \frac{\frac{1}{r+2}\cdot\sum_{j=1}^{r+1}\binom{r+2}{j}}{\sum_{j=1}^{r+1}\binom{r}{j-1}} = \frac{\frac{1}{r+2}\cdot(2^{r+2}-2)}{2^r} \geq \frac{4}{r+3} \geq \frac{4}{d+3},$$

which completes the proof of the upper bound in the lemma. □

Summing up over all $e \in E$ implies the following corollary for the players' costs.

Corollary 2. *For a player i and any state P, the following inequality holds between her Shapley and proportional cost:*

$$\frac{2}{d+1} \cdot X_i(P) \leq X_i^{Prop}(P) \leq \frac{d+3}{4} \cdot X_i(P).$$

[1]The proof of this lemma can be found in [33].

Lemma 5. *Any ρ-approximate pure Nash equilibrium for a SV weighted congestion game of degree d is a $\frac{(d+3)\cdot(d+1)}{8} \cdot \rho$-approximate pure Nash equilibrium for the weighted congestion game with proportional sharing.*

Proof. Let P be a ρ-approximate pure Nash equilibrium in a SV weighted congestion game. Using the equilibrium condition and Corollary 2, we get

$$\begin{aligned} X_i^{\text{Prop}}(P) \le \frac{d+3}{4} \cdot X_i(P) &\le \frac{d+3}{4} \cdot \rho \cdot X_i(P_{-i}, P_i') \\ &\le \frac{d+3}{4} \cdot \frac{d+1}{2} \cdot \rho \cdot X_i^{\text{Prop}}(P_{-i}, P_i'). \end{aligned}$$

□

Since pure Nash equilibria always exist in Shapley value weighted congestion games [57], for $\rho = 1$ in the last statement implies existence of $\frac{(d+3)\cdot(d+1)}{8}$-approximate equilibria in weighted congestion games.

3.4 Approximate Price of Anarchy and Stretch

The aim of this section is to provide an upper bound to the D-limited ρ-stretch of the potential function of our model (Corollary 3), which is used as parameter in the algorithm. Bounding the stretch, we bound the distance between the potential value in an approximate equilibrium and the minimiser of the potential. Intuitively, the statement in Corollary 3 indicates that the potential values of all the approximate equilibria in a SV congestion game are relatively close.

3.4.1 An Upper Bound on ρ-PoA

First, we upper bound the approximate Price of Anarchy for Shapley value weighted congestion games. This bound is later on used to prove the upper bound of the stretch stated in Corollary 3.

Lemma 6. *Let $\rho \ge 1$ and d the maximum degree of the polynomial cost functions. Then*

$$\rho\text{-}PoA \le \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

Proof. Let P be an ρ-approximate pure Nash equilibrium and P^* the optimal outcome. Then

$$SC(P) = \sum_{i \in N} \sum_{e \in P_i} \chi_e(i, S_e(P)) \overset{\text{Def. } \rho\text{-PNE}}{\le} \rho \cdot \sum_{i \in N} \sum_{e \in P_i^*} \chi_e(i, S_e(P) \cup \{i\}).$$

Due to the convexity of the cost functions, note that the cost share of any player on any resource is always upperbounded by the marginal cost increase she causes to the resource cost when she is last in the ordering,

$$\chi_e(i, S_e(P) \cup \{i\}) \le C_e(f_e(P) + w_i) - C_e(f_e(P)).$$

Thus,

$$\begin{aligned} SC(P) &\leq \rho \cdot \left(\sum_{i \in N} \sum_{e \in P_i^*} C_e(f_e(P) + w_i) - C_e(f_e(P)) \right) \\ &\leq \rho \cdot \left(\sum_{e \in E} \sum_{i : e \in P_i^*} C_e(f_e(P) + w_i) - C_e(f_e(P)) \right) \\ &\leq \rho \cdot \left(\sum_{e \in E} C_e(f_e(P) + f_e(P^*)) - C_e(f_e(P)) \right). \end{aligned} \tag{3.22}$$

The last inequality follows from assumption that C_e is a convex function in players' weights.

Claim 2. *Let* $\lambda = 2^{\frac{d}{d+1}} \cdot \left(2^{\frac{1}{d+1}} - 1\right)^{-d}$ *and* $\mu = 2^{\frac{d}{d+1}} - 1$*, then for* $x, y > 0$ *and* $d \geq 1$*,it holds*

$$(x + y)^{d+1} - x^{d+1} \leq \lambda \cdot y^{d+1} + \mu \cdot x^{d+1}.$$

Using this claim that was proven in [41], inequality (3.22) becomes

$$\begin{aligned} SC(P) &\leq \rho \cdot \left(\sum_{e \in E} \lambda \cdot C_e(f_e(P^*)) + \mu \cdot C_e(f_e(P)) \right) \\ &= \rho \cdot \lambda \cdot SC(P^*) + \rho \cdot \mu \cdot SC(P). \end{aligned}$$

Rearranging and substituting the values for λ and μ we get an upper bound on the ρ-PoA,

$$\begin{aligned} \rho\text{-PoA} &\leq \frac{\rho \cdot \lambda}{1 - \rho \cdot \mu} = \frac{\rho \cdot 2^{\frac{d}{d+1}} \cdot \left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{1 - \rho \cdot \left(2^{\frac{d}{d+1}} - 1\right)} = \rho \cdot \frac{2}{2^{\frac{1}{d+1}}} \cdot \frac{\left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{1 - \rho \cdot \frac{2}{2^{\frac{1}{d+1}}} + \rho} \\ &= \frac{2 \cdot \rho \left(2^{\frac{1}{d+1}} - 1\right)^{-d}}{2^{\frac{1}{d+1}} \cdot (1 + \rho) - 2 \cdot \rho} = \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1 + \rho) - \rho}. \end{aligned}$$

□

3.4.2 An Upper Bound on the Stretch

As we do for the approximate PoA, we now derive an upper bound on the ρ-stretch, which expresses a ratio between a local and the global optimum of the potential function.

Lemma 7. *Let* $\rho \geq 1$ *and* d *the maximum degree of the polynomial cost functions. Then an upper bound for the* ρ*-stretch of polynomial SV weighted congestion games is*

$$\rho\text{-}\Omega \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d} \cdot (d + 1)}{2^{-\frac{d}{d+1}} \cdot (1 + \rho) - \rho}.$$

Proof. Let P be a ρ-approximate equilibrium, P^* the optimal outcome and $\hat{P} = \min_{P' \in \mathcal{P}} \Phi(P')$ the minimizer of the potential which is by definition a pure Nash equilibrium. Then the ρ-approximate price of anarchy equals to

$$\rho\text{-PoA} = \max_{P \in \rho\text{-PNE}} \frac{SC(P)}{SC(P^*)} \geq \max_{P \in \rho\text{-PNE}} \frac{SC(P)}{SC(\hat{P})} \overset{\text{Def. } \Phi}{\geq} \max_{P \in \rho\text{-PNE}} \frac{\Phi(P)}{SC(\hat{P})}.$$

By Lemma 6 and Corollary 1 for $A = N$, the ρ-PoA is bounded as follows

$$\max_{P \in \rho\text{-PNE}} \frac{\Phi(P)}{(d+1) \cdot \Phi(\hat{P})} \leq \rho\text{-PoA} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{\frac{-d}{d+1}} \cdot (1+\rho) - \rho}.$$

Rearranging the terms gives the desired upper bound of the ρ-stretch,

$$\rho\text{-}\Omega = \max_{P \in \rho\text{-PNE}} \frac{\Phi(P)}{\Phi(\hat{P})} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d} \cdot (d+1)}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

□

3.4.3 An Upper Bound on the ρ-Stretch

We now proceed to the upper bound of the D-limited ρ-stretch. To do this, we use the ρ-PoA (Lemma 6) and Lemmas 8, 9, which we prove next.

Lemma 8. *Let $\rho \geq 1$, d the maximum degree of the polynomial cost functions and $\hat{P} = \min_{P' \in \mathcal{P}} \Phi(P')$. Then*

$$\frac{SC(P)}{SC(\hat{P})} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

Proof. Let P be an ρ-approximate equilibrium and P^* the optimal outcome. Let $\hat{P} = \min_{P' \in \mathcal{P}} \Phi(P')$ be the minimizer of the potential and by definition also a pure Nash equilibrium. Then we can lower bound the ρ-approximate Price of Anarchy as follows,

$$\rho\text{-PoA} = \max_{P \in \rho\text{-PNE}} \frac{SC(P)}{SC(P^*)} \geq \max_{P \in \rho\text{-PNE}} \frac{SC(P)}{SC(\hat{P})}. \tag{3.23}$$

Combining (3.23) with Lemma 6, the ρ-PoA is bounded as follows,

$$\max_{P \in \rho\text{-PNE}} \frac{SC(P)}{SC(\hat{P})} \leq \rho\text{-PoA} \leq \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{\frac{-d}{d+1}} \cdot (1+\rho) - \rho},$$

which completes the proof. □

Lemma 9. *Let $\rho \geq 1$, d the maximum degree of the polynomial cost functions and $D \subseteq N$ an arbitrary subset of players. Then*

$$\rho\text{-}\Omega_D \leq \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{SC(P)}{SC(\hat{P})}.$$

Proof. To show the lemma we lower and upper bound the D-partial potential. Let e be an arbitrary resource. By using Lemma 2 and Lemma 3, we get

$$\Phi_{e,D}(P) \leq \sum_{i \in D} \chi_{ie}(P) \leq \frac{d+1}{2} \cdot \sum_{i \in D} \chi_{ie}^{\text{Prop}}(P). \tag{3.24}$$

By definition of the proportional share χ_{ie}^{Prop}, (3.24) becomes

$$\begin{aligned}
\Phi_{e,D}(P) &\leq \frac{d+1}{2} \cdot \sum_{i \in D} w_i \cdot c_e(f_e(P)) = \frac{d+1}{2} \cdot f_e^D(P) \cdot c_e(f_e(P)) \\
&= \frac{d+1}{2} \cdot \frac{f_e^D(P)}{f_e(P)} \cdot f_e(P) \cdot c_e(f_e(P)) \\
&= \frac{d+1}{2} \cdot \frac{f_e^D(P)}{f_e(P)} \cdot \sum_{i \in N} \chi_{ie}(P). \qquad (3.25)
\end{aligned}$$

Rearranging (3.25) gives the following relation of the per unit contribution to Φ_D and Φ,

$$\frac{\Phi_{e,D}(P)}{f_e^D(P)} \leq \frac{d+1}{2} \cdot \frac{\sum_{i \in N} \chi_{ie}(P)}{f_e(P)}$$

and by summing up over all resources e, we get

$$\frac{\Phi_D(P)}{W_D} \leq \frac{d+1}{2} \cdot \frac{SC(P)}{W}, \tag{3.26}$$

where $W = \sum_{i \in N} w_i = \sum_{e \in E} f_e(P)$ and $W_D = \sum_{i \in D} w_i = \sum_{e \in E} f_e^D(P)$.

Similar to (3.25), we lower bound the D-partial potential with

$$\begin{aligned}
\Phi_{e,D}(P) \geq \frac{1}{d+1} \cdot \sum_{i \in D} \chi_{ie}(P) &\geq \frac{4}{(d+1) \cdot (d+3)} \cdot \sum_{i \in D} w_i \cdot c_e(f_e(P)) \\
&= \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{f_e^D(P)}{f_e(P)} \cdot \sum_{i \in N} \chi_{ie}(P).
\end{aligned}$$

The first inequality uses Lemma 2 and the second uses Lemma 3. Again we get a per unit contribution to Φ_D and Φ on one resource and in the whole game,

$$\begin{aligned}
& \frac{\Phi_{e,D}(P)}{f_e^D(P)} \geq \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{\sum_{i \in N} \chi_{ie}(P)}{f_e(P)} \\
\Leftrightarrow \; & \frac{\Phi_D(P)}{W_D} \geq \frac{4}{(d+1) \cdot (d+3)} \cdot \frac{SC(P)}{W}. \qquad (3.27)
\end{aligned}$$

Combining (3.26) with (3.27) and rearranging the terms proves the lemma,

$$\begin{aligned}
\frac{\Phi_D(P)}{\Phi_D(\hat{P})} &\leq \frac{d+1}{2} \cdot \frac{SC(P)}{W} \cdot \frac{W_D}{1} \cdot \frac{(d+1) \cdot (d+3)}{4} \cdot \frac{W}{SC(\hat{P})} \cdot \frac{1}{W_D} \\
&= \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{SC(P)}{SC(\hat{P})}.
\end{aligned}$$

□

By Lemma 8 and Lemma 9, we get the following desirable corollary.

Corollary 3. *For $\rho \geq 1$, d the maximum degree of the polynomial cost functions and $D \subseteq N$ an arbitrary subset of players,*

$$\rho\text{-}\Omega_D \leq \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{\rho \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+\rho) - \rho}.$$

3.5 Computation of Equilibria

To compute ρ-approximate pure Nash equilibria in Shapley cost-sharing games we construct an algorithm based on the algorithmic idea by Caragiannis et al. [12]. The main idea is to separate the players in different blocks depending on their costs. The players who are processed first are the ones with the largest costs followed by the smaller ones. The size of the blocks and the distance between them is polynomially bounded by the number of players n and the maximum degree d of the polynomial cost functions c_e. Formally, we define $X_{\max} = \max_{i \in N} X_i(P)$ as the maximum cost among all players before running the algorithm. Let $\mathcal{BR}_i(0)$ be a state of the game in which only player i participates and plays her best move. Then, define as $X_{\min} = \min_{i \in N} X_i(\mathcal{BR}_i(0))$ the minimum possible cost in the game. Let γ be an arbitrary constant such that $\gamma > 0$, $m = \log\left(\frac{X_{\max}}{X_{\min}}\right)$ the number of different blocks and $b_r = X_{\max} \cdot g^{-r}$ the block size for any $r \in [0, m]$, where $g = 2 \cdot n \cdot (d+1) \cdot \gamma^{-3}$.

The algorithm is now executed in $m-1$ phases. Let P be the current state of the game and, for each phase $r \in [1, m-1]$, let P^r be the state before phase r. All players i with $X_i(P) \in [b_r, +\infty]$ perform an s-move with $s = \left(\frac{1}{t\text{-}\Omega_D} - 2\gamma\right)^{-1}$ (almost t-Ω_D-approximate moves), while all players i with $X_i(P) \in [b_{r+1}, b_r]$ perform a t-move with $t = 1 + \gamma$ (almost pure moves). Let $\mathcal{BR}_i(P)$ be the best response of player i in state P. The phase ends when the first and the second group of players are in an s- and t-approximate equilibrium, respectively. At the end of the phase, players with $X_i(P) > b_r$ have irrevocably decided their strategy and have been added in the list of finished players. In addition, before the described phases are executed, there is an initial phase in which all players with $X_i(P) \geq b_1$ can perform a t-move to prepare the first real phase.

For the analysis, let D_r be the set of deviating players in phase r and $P^{r,i}$ denote the state after player $i \in D_r$ has done her last move within phase r.

Theorem 3.5.1. *An α-approximate pure Nash equilibrium with $\alpha \in \left(\frac{d}{\ln 2}\right)^d \cdot$ poly(d) can be computed with a polynomial number of improvement steps.*

Proof of Theorem 3.5.1. The main argument follows from bounding the D-partial potential of the moving players in each phase (see Lemma 11). To that end, we first prove that the partial potential is bounded by the sum of the costs of players when they did their last move (Lemma 10).

Lemma 10. *For every phase r, it holds that $\Phi_{D_r}(P^r) \leq \sum_{i \in D_r} X_i\left(P^{r,i}\right)$.*

Proof. Let $D_r^i \subseteq D_r$ the set of players who still have to perform their last move after player i in phase r. Then by definition of the partial potential 3.2,

Algorithm 1 Computation of approximate pure Nash equilibria

$X_{\max} = \max_{i \in N} X_i(P)$, $X_{\min} = \min_{i \in N} X_i(\mathcal{BR}_i(0))$, $m = \log\left(\frac{X_{\max}}{X_{\min}}\right)$
$\gamma > 0$, $g = 2 \cdot n \cdot (d+1) \cdot \gamma^{-3}$, $b_r = X_{\max} \cdot g^{-r} \forall \in [0, m]$
$t = 1 + \gamma$, $s = \left(\frac{1}{t\text{-}\Omega_D} - 2\gamma\right)^{-1}$
while there is a player $i \in N$ with $X_i(P) \geq b_1$ and who can perform a t-move **do**
  $P \leftarrow (P_{-i}, \mathcal{BR}_i(P))$
end while
for all phases r from 1 to $m-1$ **do**
  while there is a non-finished player $i \in N$ either with $X_i(P) \in [b_r, +\infty]$ and who can perform a s-move or with $X_i(P) \in [b_{r+1}, b_r]$ and who can perform a t-move **do**
    $P \leftarrow (P_{-i}, \mathcal{BR}_i(P))$
  end while
  Add all players $i \in N$ with $X_i(P) \geq b_r$ to the set of finished players.
end for

$\Phi_{D_r}(P^r)$ equals to

$$\Phi^N(P^r) - \Phi^{N \setminus D_r}(P^r) = \sum_{i=1}^{|D_r|} \left(\Phi^{N \setminus D_r^i}(P^r) - \Phi^{N \setminus D_r^{i-1}}(P^r)\right) = \sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(P^r). \tag{3.28}$$

For each player i, her strategy in state P^r is identical to her strategy in $P^{r,i}$. By Proposition 3.2 ((a)), 3.2 ((b)) and 3.2 ((c)), we upperbound (3.28) by

$$\sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(P^r) = \sum_{i=1}^{|D_r|} \Phi_i^{N \setminus D_r^i}(P^{r,i}) \leq \sum_{i=1}^{|D_r|} \Phi_i(P^{r,i}) = \sum_{i=1}^{|D_r|} X_i\left(P^{r,i}\right).$$

□

Bounding The Potential of Deviating Players

We now proceed to the key property of the algorithm. Using Lemma 10 and the stretch of the previous section, we bound the potential of the moving players by a multiplicative factor of the according block size as stated in Lemma 11.

Lemma 11. *For every phase r, it holds that $\Phi_{D_r}\left(P^{r-1}\right) \leq \frac{n}{\gamma} \cdot b_r$.*

Proof. We show the lemma by contradiction. Thus, assume that $\Phi_{D_r}\left(P^{r-1}\right) > \frac{n}{\gamma} \cdot b_r$. Let $S_r, T_r \subseteq D_r$, be the set of players whose last move is an s-move and a t-move, accordingly, such that $S_r \cup T_r = D_r$. First, we focus on the players in S_r. Let $i \in S_r$ be an arbitrary player. By definition of an s-move, player i decreases her costs in her last move during phase r by at least $(s-1) \cdot X_i\left(P^{r,i}\right)$. By Proposition 3.3, any such improvement step also decreases the i-partial potential by the same amount. Summing up over all players $i \in S_r$, we get a lower bound on the total decrease of the D_r-partial potential between states P^{r-1} and P^r:

$\Phi_{D_r}(P^{r-1}) - \Phi_{D_r}(P^r) \geq (s-1) \cdot \sum_{i \in S_r} X_i(P^{r,i})$. Rearranging, we upper bound the partial potential as follows,

$$\begin{aligned}
\Phi_{D_r}(P^r) &\leq \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \sum_{i \in S_r} X_i(P^{r,i}) \\
&\leq \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \left(\sum_{i \in D_r} X_i(P^{r,i}) - \sum_{i \in T_r} X_i(P^{r,i}) \right) \\
&\leq \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \left(\sum_{i \in D_r} X_i(P^{r,i}) - n \cdot b_r \right) \\
&\leq \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \left(\Phi_{D_r}(P^r) - n \cdot b_r \right) \\
&\leq \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \left(\Phi_{D_r}(P^r) - \gamma \cdot \Phi_{D_r}\left(P^{r-1}\right) \right) \\
&\leq (1 + (s-1) \cdot \gamma) \cdot \Phi_{D_r}(P^{r-1}) - (s-1) \cdot \Phi_{D_r}(P^r),
\end{aligned}$$

where the third inequality follows from the fact that the cost of a player $i \in T_r$ is upper bounded by the block border b_r, the fourth inequality by Lemma 10 and the fifth one by the assumption. Rearranging the terms gives

$$\Phi_{D_r}(P^r) \leq \frac{1 + (s-1) \cdot \gamma}{s} \cdot \Phi_{D_r}(P^{r-1}). \tag{3.29}$$

Let $\bar{P}$ be an intermediate state between P^{r-1} and P^r such that all players in S_r have already finished their s-move and play their strategies in P^r, while the moving players in T_r play their strategies in P^{r-1}. Consider a player $i \in T_r$. The difference in her cost after her t-move is at most b_r. This is due to the fact that her initial cost is at most b_r (by the block construction) and the minimum cost she can improve to is zero. Then, by Proposition 3.3, the difference in the cost of player i equals to the difference in the i-partial potential, that is, $\Phi_i(\bar{P}) - \Phi_i(P^r) = X_i(P) - X_i(P') \leq b_r$. Summing up over all players in T_r, we get that the difference in the D_r-partial potential among states $\bar{P}$ and P^r can be at most $n \cdot b_r$. Then, we get the following upper bound on the partial potential in state $\bar{P}$,

$$\begin{aligned}
\Phi_{D_r}(\bar{P}) \leq \Phi_{D_r}(P^r) + n \cdot b_r &\leq \frac{1 + (s-1) \cdot \gamma}{s} \cdot \Phi_{D_r}(P^{r-1}) + \gamma \cdot \Phi_{D_r}\left(P^{r-1}\right) \\
&= \left(\frac{1-\gamma}{s} + 2 \cdot \gamma \right) \cdot \Phi_{D_r}(P^{r-1}) < \left(\frac{1}{s} + 2 \cdot \gamma \right) \cdot \Phi_{D_r}(P^{r-1}),
\end{aligned}$$

where the second inequality holds by (3.29) and our assumption. Substituting s, we get

$$\Phi_{D_r}(\bar{P}) < \frac{1}{t\text{-}\Omega_D} \cdot \Phi_{D_r}(P^{r-1}) \Rightarrow \frac{\Phi_{D_r}(P^{r-1})}{\Phi_{D_r}(\bar{P})} > t\text{-}\Omega_D \tag{3.30}$$

Let $\hat{P}$ be the D_r-partial potential minimiser. Since state P^{r-1} is an approximate pure Nash equilibrium for the players in D_r, we upper bound (3.30) as follows,

$$\max_{P \in \rho-\text{PNE}} \frac{\Phi_{D_r}(P)}{\Phi_{D_r}(\hat{P})} > \frac{\Phi_{D_r}(P^{r-1})}{\Phi_{D_r}(\bar{P})} > t\text{-}\Omega_D$$

which, for $\rho = t$, contradicts the definition of ρ-Stretch. Thus, our initial assumption does not hold and Lemma 11 is proved. □

Bounding The Running Time

It remains to show that the running time is bounded and that the approximation factor holds. For the first, since the partial potential is bounded and each deviation decreases the potential, we can limit the number of possible improvement steps (Lemma 12).

Lemma 12. *The algorithm uses a polynomial number of improvement steps.*

Proof. At the beginning of the algorithm's execution, the sum of all players' costs is at most $n \cdot X_{\max}$. By Corollary 1, the potential is also upper bounded by the same amount. In the initial phase, each deviating player makes a t-move, therefore her cost improves by at least $(t-1) \cdot b_1$ (since her cost is at most b_1). The potential function also decreases by at least $(t-1) \cdot b_1$ in each step. Using the definition of b_1, we get that $(t-1) \cdot b_1 = \gamma \cdot g^{-1} \cdot X_{\max}$. Using both observations, we can compute the maximum number of improvement steps in the first phase,

$$\begin{aligned}\frac{n \cdot X_{\max}}{\gamma \cdot g^{-1} \cdot X_{\max}} = n \cdot \gamma^{-1} \cdot g &= n \cdot \gamma^{-1} \cdot \frac{2 \cdot n \cdot (d+1)}{\gamma^3} \\ &= 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-4}.\end{aligned}$$

Consider an arbitrary phase $r \geq 1$. By Lemma 11, $\Phi_{D_r}(P^{r-1}) \leq \frac{n}{\gamma} \cdot b_r$. Again, we look at the possible cost improvement in a deviation which equals to the potential decrease in this step. In this case, the cost improvement is at least $(t-1) \cdot b_{r+1}$. By definition of b_{r+1}, we have that $(t-1) \cdot b_{r+1} = b_r \cdot g^{-1} \cdot \gamma$. Similar, the maximum number of improvement moves in this phase is

$$\frac{\frac{n}{\gamma} \cdot b_r}{b_r \cdot g^{-1} \cdot \gamma} = \frac{n \cdot g}{\gamma^2} = \frac{2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-3}}{\gamma^2} = 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-5}.$$

In total, we have at most

$$\begin{aligned}&2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-4} + \log\left(\frac{X_{\max}}{X_{\min}}\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-5} \\ &= \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}\end{aligned}$$

improvement steps. □

Approximation Guarantee

We show next that every player who has already finished her moves will not get much worse costs at the end of the algorithm (see Lemma 13) and that there is no alternative strategy which is more attractive at the end (see Lemma 14).

Lemma 13. *Let i be a player who makes her last move in phase r of the algorithm. Then,*

$$X_i\left(P^{m-1}\right) \leq (1+\gamma^2) \cdot X_i\left(P^r\right).$$

Proof. We first show by contradiction the following. For $j \geq r$, the increase in the cost of player i from an arbitrary state P^j to state P^{j+1} is upper bounded by $\frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. Thus, assume that

$$X_i\left(P^{j+1}\right) - X_i\left(P^j\right) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}.$$

Since player i does not deviate during phase $j+1$, the increase in her cost is caused by other players deviating to the resources she uses. Thus, there exists a set of resources $E' \subseteq E$ such that each resource in E' is used by player i and by at least one player in D_{j+1} at state P^{j+1}. This yields to

$$\begin{aligned}
&\sum_{e\in E'} \chi_{ie}\left(P^{j+1}\right) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1} \\
&\Rightarrow \frac{\sum_{e\in E'} f_e(P^{j+1})\cdot c_e(f_e(P^{j+1}))}{d+1} > \frac{n}{\gamma}\cdot b_{j+1} \\
&\Leftrightarrow \frac{SC_{D_{j+1}}(P^{j+1})}{d+1} > \frac{n}{\gamma}\cdot b_{j+1} \\
&\Rightarrow \Phi_{D_{j+1}}(P^{j+1}) > \frac{n}{\gamma}\cdot b_{j+1}.
\end{aligned}$$

The last step uses Corollary 1. Since the potential decreases during the execution of the algorithm, we get

$$\Phi_{D_{j+1}}(P^j) \geq \Phi_{D_{j+1}}(P^{j+1}) > \frac{n}{\gamma}\cdot b_{j+1},$$

which contradicts Lemma 11. Therefore $X_i\left(P^{j+1}\right) \leq X_i\left(P^j\right) + \frac{n(d+1)}{\gamma}\cdot b_{j+1}$, which we use to show the lemma as follows,

$$\begin{aligned}
X_i\left(P^{m-1}\right) &\leq X_i\left(P^{m-2}\right) + \frac{n\cdot(d+1)}{\gamma}\cdot b_{m-1} \\
&\leq X_i\left(P^r\right) + \frac{n\cdot(d+1)}{\gamma}\sum_{j=r+1}^{m-1} b_j \\
&= X_i\left(P^r\right) + \frac{n\cdot(d+1)}{\gamma}\sum_{j=r+1}^{m-1} X_{\max}\cdot g^{-j} \\
&= X_i\left(P^r\right) + \frac{n\cdot(d+1)}{\gamma}\sum_{j=r+1}^{m-1} b_r\cdot g^{r-j} \\
&\leq X_i\left(P^r\right) + \frac{n\cdot(d+1)}{\gamma}\cdot 2\cdot b_r\cdot g^{-1} \\
&\leq X_i\left(P^r\right) + \frac{2\cdot n\cdot(d+1)}{\gamma\cdot g}\cdot X_i\left(P^r\right) \\
&= \left(1+\frac{2\cdot n\cdot(d+1)}{\gamma\cdot g}\right)\cdot X_i\left(P^r\right) = \left(1+\gamma^2\right)\cdot X_i\left(P^r\right).
\end{aligned}$$

□

Lemma 14. *Let i be a player who makes her last move in phase r and let P'_i be an arbitrary strategy of i. Then,*

$$X_i\left(P^{m-1}_{-i}, P'_i\right) \geq (1-\gamma) \cdot X_i\left(P^r_{-i}, P'_i\right).$$

Proof. Similarly to previous lemma, we first show by contradiction the following. For two arbitrary successive phases j and $j+1$ and an arbitrary alternative strategy P'_i of player i, $X_i\left(P^{j+1}_{-i}, P'_i\right) \geq X_i\left(P^j_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$. To contradict this, assume that

$$X_i\left(P^j_{-i}, P'_i\right) - X_i\left(P^{j+1}_{-i}, P'_i\right) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}.$$

Since player i does not deviate during phase $j+1$, the increase in her costs is caused by other players deviating to the resources she uses. Thus, there exists a set of resources $E' \subseteq E$ such that each resource in E' is used by player i and by at least one player in D_{j+1} at state P^{j+1}. Therefore

$$\begin{aligned}
&\sum_{e\in E'} \chi_{ie}\left(P^j_{-i}, P'_i\right) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1} \\
&\Rightarrow \sum_{e\in E'} \chi_{ie}\left(P^j_{-i}, P_i\right) > \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}.
\end{aligned}$$

Following exactly the same steps as in proof of Lemma 13, the previous yields to a contradiction of Lemma 11. Thus, $X_i\left(P^{j+1}_{-i}, P'_i\right) \geq X_i\left(P^j_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot b_{j+1}$, which we use to show the lemma's statement as follows,

$$\begin{aligned}
X_i\left(P^{m-1}_{-i}, P'_i\right) &\geq X_i\left(P^{m-2}_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot b_{m-1} \\
&\geq X_i\left(P^r_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} b_j \\
&= X_i\left(P^r_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} X_{\max} \cdot g^{-j} \\
&= X_i\left(P^r_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot \sum_{j=r+1}^{m-1} b_r \cdot g^{r-j} \\
&\geq X_i\left(P^r_{-i}, P'_i\right) - \frac{n\cdot(d+1)}{\gamma} \cdot 2 \cdot b_r \cdot g^{-1} \\
&\overset{b_r}{=} X_i\left(P^r_{-i}, P'_i\right) - \frac{2\cdot n\cdot(d+1)}{\gamma\cdot g} \cdot X_i\left(P^r\right) \\
&\overset{g}{=} X_i\left(P^r_{-i}, P'_i\right) - \gamma^2 \cdot X_i\left(P^r\right) \\
&\overset{\gamma\leq\frac{1}{s}}{\geq} X_i\left(P^r_{-i}, P'_i\right) - \frac{\gamma}{s} \cdot X_i\left(P^r\right) \\
&\geq X_i\left(P^r_{-i}, P'_i\right) - \gamma \cdot X_i\left(P^r_{-i}, P'\right) = (1-\gamma)\cdot X_i\left(P^r_{-i}, P'_i\right).
\end{aligned}$$

The second last inequality holds due to the s-approximate equilibrium for player i in P^r. □

Last,we bound the approximation factor of the whole algorithm (next lemma).

Lemma 15. *After the last phase of the algorithm, every player i is in an α-approximate pure Nash equilibrium with $\alpha = (1 + O(\gamma)) \cdot t\text{-}\Omega_D$.*

Proof. Let i be an arbitrary player who took her last move in phase r and let P'_i be an arbitrary other strategy of player i. We use Lemma 13 and Lemma 14 and the fact that player i has no incentive to make a s-move in phase r (by definition of the algorithm):

$$\begin{aligned}\frac{X_i\left(P^{m-1}\right)}{X_i\left(P^{m-1}_{-i}, P'_i\right)} &\leq \frac{(1+\gamma^2)\cdot X_i\left(P^r\right)}{(1-\gamma)\cdot X_i\left(P^r_{-i}, P'_i\right)}\\ &\leq \left(\frac{1+\gamma^2}{1-\gamma}\right)\cdot\left(\frac{1}{t\text{-}\Omega_D} - 2\gamma\right)^{-1}\\ &\leq \left(\frac{1+\gamma^2}{1-\gamma}\right)\cdot\left(\frac{1}{t\text{-}\Omega_D} - 2\gamma\right)^{-1}\end{aligned}$$

By minimizing the first part, we can get arbitrary close to 1. For the second part, we need to fix a γ with $\gamma < \frac{1}{2t\text{-}\Omega_D}$. Therefore, the expression can be simplified to $\alpha = (1 + O(\gamma)) \cdot t\text{-}\Omega_D$. □

The polynomial running time and the approximation factor of $\alpha = (1 + O(\gamma)) \cdot t\text{-}\Omega_D$ follow directly from Lemma 12 and Lemma 15. By Corollary 3,

$$t\text{-}\Omega_D \leq \frac{(d+1)^2\cdot(d+3)}{8}\cdot\frac{t\cdot(2^{\frac{1}{d+1}}-1)^{-d}}{2^{-\frac{d}{d+1}}\cdot(1+t)-t}$$

with $t = 1 + \gamma$, which gives an approximation factor of

$$\alpha = (1+O(\gamma))\cdot\frac{(d+1)^2\cdot(d+3)}{8}\cdot\frac{t\cdot(2^{\frac{1}{d+1}}-1)^{-d}}{2^{-\frac{d}{d+1}}\cdot(1+t)-t}.$$

The order of factor α follows from the next lemma (proof in Appendix) and this completes the proof of the main theorem in this section, Theorem 3.5.1.

Lemma 16. *The approximation factor α is in the order of $\left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$.*

□

Approximation of Weighted Congestion Games

This algorithm can be used to compute also approximate pure Nash equilibria in weighted congestion games (with proportional sharing). Such a game can now be approximated by a Shapley game losing only a factor of $\frac{(d+3)\cdot(d+1)}{8}$ (by Lemma 5), which is included in $poly(d)$.

Corollary 4. *For any weighted congestion game with proportional sharing, an α-approximate pure Nash equilibrium with $\alpha \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$ can be computed within a polynomial number of improvement steps.*

Algorithm 2 Approximation of the Shapley Value by Sampling

(r: phases of the algorithm)
for all r from 1 to $\log\left(2n^{c+3} \cdot \max_{i \in N} |\mathcal{P}_i| \cdot |E| \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot \frac{d+1}{\gamma^9}\right)$ **do**
  for all j from 1 to $k = \frac{4(|S_e(P)|-1)}{\mu^2}$ **do**
    Pick uniformly at random a permutation π of the players $S_e(P)$ using resource e
    Compute the marginal contribution $MC_{ie}^j(P) = C_e\left(W_{S_e(P)}^{<i,\pi} + w_i\right) - C_e\left(W_{S_e(P)}^{<i,\pi}\right)$
  end for
  Let $\overline{MC}_{ie}(P) = \frac{1}{k}\sum_{j=1}^{k} MC_{ie}^j(P)$
end for
Return the median of all $\overline{MC}_{ie}(P)$

3.5.1 Sampling Shapley Values

The previous section gives a polynomial running time algorithm with respect to the number of improvement steps. However, each improvement step requires computation of multiple Shapley values, which can often be a computationally hard problem [32]. For this reason, one can instead compute an approximated Shapley value with sampling methods.

Theorem 3.5.2. *For a constant γ, an α-approximate pure Nash equilibrium with $\alpha \in \left(\frac{d}{\ln 2}\right)^d \cdot poly(d)$ can be computed in polynomial time with high probability.*

Proof. To achieve this, we use sampling techniques from [58, 61], which we adjust to our setting. More specifically, the theorem follows from that combination of the Lemmas 17, 18 and 19. The sampling algorithm is stated in Figure 2. The general idea of this algorithm is the following. For a player i, it runs for a specific number of phases ($\log(\ldots)$) of Algorithm 1 (note that this number depends on the cardinality of the strategy set of i, $|\mathcal{P}_i|$). For each phase r it picks at random k permutations of the players and for each permutation it computes the marginal cost contribution of i. Then it computes the average value of the k marginal contributions and at the end returns the median of all average values (which have been computed for each phase r).

Computing Approximated SV with High Probability

Lemma 17. *For a state P and an arbitrary but fixed constant c, Algorithm 2 computes a μ-approximation of $\chi_{ie}(P)$ for any player i in polynomial running time with probability at least*

$$1 - \left(n^c \cdot n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E| \cdot \left(1 + \log\left(\frac{X_{max}}{X_{min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}\right)^{-1}.$$

Proof. The proof follows the analysis in [58]. Let X be the marginal contribution of player i in a random permutation. Since C_e is a polynomial of degree d and monotone, we have $X \geq 0$. By the definition of the Shapley value, $\chi_{ie}(P) =$

$E[X]$. By the definition of the cost functions, the maximum possible value of X is achieved when i is the last player in the ordering. This happens in $1/|S_e(P)|$ fraction of the permutations. X achieves the maximum value with probability at least $1/|S_e(P)|$ and the maximum value is at most $|S_e(P)| \cdot \chi_{ie}(P)$. This is because of the following. Since the minimum value is 0, if the maximum value was larger (or smaller) than $|S_e(P)| \cdot \chi_{ie}(P)$, then the average value of X would be larger (or smaller) than its expectation $\chi_{ie}(P) = E[X]$.

To upper bound the variance of X, we define a second random variable Y whose value is $|S_e(P)| \cdot \chi_{ie}(P)$ with probability $1/n$ and 0 otherwise. Then,

$$\begin{aligned} Var(X) \leq & Var(Y) \\ & = E[Y^2] - E[Y]^2 \\ & = \frac{1}{2}|S_e(P)|^2 \cdot \chi_{ie}(P)^2 - \left(\frac{1}{2}|S_e(P)| \cdot \chi_{ie}(P)\right)^2 \\ & = (|S_e(P)| - 1) \cdot \chi_{ie}(P)^2. \end{aligned}$$

Since

$$\begin{aligned} \overline{MC}_{ie}(P) &= \frac{1}{k}\sum_{j=1}^{k} MC_{ie}^{j}(P), \\ E\left[\overline{MC}_{ie}(P)\right] &= E[X] = \chi_{ie}(P), \end{aligned}$$

and the single permutations are independent of each other (random), we get

$$\begin{aligned} Var\left(\overline{MC}_{ie}(P)\right) = & \frac{Var(X)}{k} \\ & \leq \frac{1}{k} \cdot (|S_e(P)| - 1) \cdot \chi_{ie}(P)^2. \end{aligned}$$

Using Chebyshev's inequality, we get

$$\begin{aligned} Pr\left[|\overline{MC}_{ie}(P) - \chi_{ie}(P)| \geq \mu \cdot \chi_{ie}(P)\right] &\leq \frac{Var\left(\overline{MC}_{ie}(P)\right)}{\chi_{ie}(P)^2 \cdot \mu^2} \\ &\leq \frac{(|S_e(P)| - 1) \cdot \chi_{ie}(P)^2}{k \cdot \chi_{ie}(P)^2 \cdot \mu^2} \\ &= \frac{|S_e(P)| - 1}{k \cdot \mu^2} \end{aligned}$$

Let $k = \frac{4(|S_e(P)|-1)}{\mu^2}$, then $\overline{MC}_{ie}(P)$ is a μ-approximation for $\chi_{ie}(P)$ with probability at least $3/4$. By repeating this for

$$\log\left(2n^{c+3} \cdot \max_{i \in N} \mathcal{P}_i \cdot |E| \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot (d+1) \cdot \gamma^{-9}\right)$$

times, using the median value of all runs and applying Chernoff bounds, we directly get a result with failure probability at most

$$\frac{1}{n^c \cdot n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E| \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}}.$$

□

Computation of Improvement Steps with Sampling

Using sampling to compute an improvement step, an approximated Shapley value has to be computed for each alternative strategy of a player and for each[1] resource in the strategy. In the worst case, these computations has to be computed for all players (until an improvement step is available).

Lemma 18. *For a state P, running Algorithm 2 at most $n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E|$ times computes an improvement step for an arbitrary player with probability at least*

$$1 - \left(n^c \cdot \left(1 + \log\left(\frac{X_{max}}{X_{min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}\right)^{-1}.$$

Proof. The result follows directly by applying the union bound:

$$\begin{aligned}
&Pr[\exists i \in N : \exists P_i' \in \mathcal{P}_i : \exists e \in P_i' : |\overline{MC}_{ie}(P_{-i}, P_i') - \chi_{ie}(P_{-i}, P_i')| \geq \mu \cdot \chi_{ie}(P_{-i}, P_i')] \\
&\leq \frac{n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E|}{n^c \cdot n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E| \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}} \\
&\leq \frac{1}{n^c \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}}.
\end{aligned}$$

□

Bounding The Sampling Times

By Lemma 12 in page 68, we give a bound on the number of improvement steps. Thus, we can bound the total number of samplings[2] during Algorithm 1.

Lemma 19. *During the whole execution of Algorithm 1, the sampling algorithm for $\mu = 1 + \gamma$ is applied at most*

$$n \cdot \max_{i \in N} \mathcal{P}_i \cdot |E| \cdot \left(1 + \log\left(\frac{X_{max}}{X_{min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}$$

times and the computation of the approximate pure Nash equilibrium is correct with probability at least $1 - n^{-c}$ for an arbitrary constant c.

Proof. The result follows directly by applying the union bound:

$$\begin{aligned}
&Pr[\exists \text{ an improvement step in which the sampling fails}] \\
&\leq \frac{\left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}}{n^c \cdot \left(1 + \log\left(\frac{X_{\max}}{X_{\min}}\right)\right) \cdot 2 \cdot n^2 \cdot (d+1) \cdot \gamma^{-9}} \\
&\leq \frac{1}{n^c}.
\end{aligned}$$

□

[1] The computation of a Shapley value of a player is a local process in a sense that it is first computed separately on each resource that the player uses and then we take the sum of these values.

[2] We can always use the sampling algorithm for $\mu = 1 + \gamma$.

Summing up, we show that a μ-approximation of one Shapley value can be computed in polynomial running time with high probability (Lemma 17) and the sampling algorithm is running at most a polynomial number of times (Lemma 19). Then Theorem 3.5.2 follows. $\square$

3.6 Conclusion

This chapter gives a polynomial running time algorithm with respect to the improvement steps of players for computing approximate equilibria in Shapley value weighted congestion games. Our algorithm builds on the algorithm of [12], however adjusting [12]'s idea on our model requires a very careful and technical adaptation due to the complex nature of Shapley values. This algorithm adjustment helps us to provide new insights on the structure of approximate equilibria of SV weighted congestion games. Another challenge for our model was that each improvement step, during our algorithm's execution, requires the computation of multiple Shapley values, which is a computationally hard problem. We address this issue by computing approximated Shapley values in polynomial running time using sampling techniques. This additional approximation factor in the improvement steps can be embedded in the factor ρ of the ρ-Nash dynamics. Thus, an execution of the algorithm with approximate steps has a negligible impact on the final result.

An extra feature for our algorithm is that it can be also used for computing approximate equilibria in weighted congestion games with a significantly better approximation factor than that of [12]. More specifically, in [12], authors approximate Ψ-games (potential games) to weighted congestion games (non-potential games) and achieve computation of $d^{2d+o(d)}$-approximate pure Nash equilibria. As they also state, their work reveals the following interesting open problem. Is it possible to find better[1] approximation guarantee for approximate equilibria computed in polynomial time? Our contribution closes this open problem as follows. Our approach approximates SV weighted congestion games (potential games) to weighted congestion games (non-potential games), which allow us to compute approximate equilibria with the significantly better approximation factor of $O\left(\left(\frac{d}{\ln 2}\right)^d\right)$. We also note that a significant improvement of the approximation factor below $O\left(\left(\frac{d}{\ln 2}\right)^d\right)$ would require new algorithmic ideas as the lower bound of the PoA in [40] immediately yields a corresponding lower bound on the stretch.

As mentioned, the adaptation of [12] to our model requires a very technical work. For this reason, it would be interesting to create a general framework for this algorithm as a blackbox function. Then, for any class of games that fulfils the neccessary criteria, this blackbox function tool would directly yield a polynomial running time algorithm for computing approximate equilibria. Another interest lies on exploiting more the notion of the stretch. We believe this ratio is a promising future tool for other than its current usage, however no other applications of it have been found yet. Last, as stated in [12], exploring approximations of potential games to non-potential ones keeps the interest high.

[1] A conjecture of the authors of [12] is that their technique is tight only for the linear weighted congestion games.

Chapter 4

Cost Sharing in Generalised Selfish Routing

This chapter formally presents the model and contribution in a generalised version of the model of Chapter 3, which has been briefly introduced in Section 1.5.1, page 27. More specifically, this work studies the design of cost-sharing methods in a generalised selfish routing model where each player may control multiple flows in the graph. We require that our cost-sharing method and set of cost functions satisfy certain natural conditions and we characterize the Shapley value as the unique method that guarantees the existence of a pure equilibrium. Focusing on the inefficiency of equilibria, we present general tight price of anarchy bounds, which are robust and apply to general equilibrium concepts. We then turn to the price of stability and prove an upper bound for the Shapley value cost-sharing method, which holds for general sets of cost functions and which is tight in special cases of interest, such as bounded degree polynomials. Also for bounded degree polynomials, we conclude the paper with a somewhat surprising result, showing that a slight deviation from the Shapley value has a huge impact on the price of stability. In fact, for this case, the price of stability becomes as bad as the price of anarchy. Our tight price of anarchy and price of stability bounds apply both to single and multi-commodity selfish routing.

4.1 The Model

In this section we present the notation and preliminaries for our model in terms of a multi-commodity player congestion game. In such a game, there is a set Q of k commodities which are partitioned into $n \leq k$ non-empty and disjoint subsets $Q_1, Q_2, \ldots Q_n$. Each set of commodities Q_i, for $i = 1, 2, \ldots, n$, is controlled by an independent player. Denote $N = \{1, 2, \ldots, n\}$ the set of players. The players in N share access to a set of resources E. Each resource $e \in E$ has a flow-dependent cost function $C_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Strategies. Each commodity $q \in Q$ has a set of possible strategies $\mathcal{P}^q \subseteq 2^E$. Associated with each commodity q is a weight w_q, which has to be allocated to a strategy in $\mathcal{P}^q$. For a player i, a strategy $P_i = (P_q)_{q \in Q_i}$ defines the strategy for each commodity q player i controls. An outcome $P = (P_1, P_2, \ldots, P_n)$ is a tuple of strategies of the n players.

Resource Load. For an outcome P, the flow $f_e^i(P)$ of a player i on resource e equals the sum of the weights of all her commodities using e, i.e., $f_e^i(P) = \sum_{q \in Q_i, e \in P_q} w_q$. For any $T \subseteq N$ with $|T| \geq 2$, let $f_e^T(P)$ be the vector with the flows that each player in set T assigns to resource e. The total flow on a resource e is given by $f_e(P) = \sum_{i \in N} f_e^i(P)$, while the set of users of e (players who assign positive flow on e) for an outcome P is given by $S_e(P)$.

Cost Shares. The cost sharing method of the game determines how the flow-dependent joint cost of a resource $C_e(f_e(P))$ is divided among its users. The cost of a player on a resource is the share of $C_e(f_e(P))$ assigned to her by the chosen cost-sharing method and is defined as a function of the player's identity, the resource's cost function and the vector of flows assigned to e, i.e., $\chi_e(i, f_e^N(P), C_e)$. For simplicity, let $\chi_e(i, f_e^N(P), C_e) = \chi_{ie}(P)$ when all players are considered in a state P, otherwise, for players in $T \subseteq N$, we restrict to the notation $\chi_e(i, f_e^T(P))$. For a resource e, the cost shares are such that $\sum_{i \in S_e(P)} \chi_{ie}(P) = C_e(f_e(P))$. The total cost of a player is $X_i(P) = \sum_{e \in P_i} \chi_{i.e.}(P)$ and the social cost of the game is given by the sum of the player costs,

$$SC(P) = \sum_{i \in N} \sum_{e \in E} \chi_{ie}(P) = \sum_{i \in N} \sum_{e \in E} \chi_e(i, f_e^N(P), C_e) = \sum_{e \in E} C_e(f_e(P)). \quad (4.1)$$

On the *cost-sharing method* and the set of allowable cost functions, we make the following natural assumptions:

1. The cost functions of the game are drawn from a given set $\mathcal{C}$ of allowable cost functions, such that every $C \in \mathcal{C}$ must be continuous, increasing and convex. We also make the mild technical assumption that $\mathcal{C}$ is closed under dilation, i.e., that if $C(x) \in \mathcal{C}$, then also $C(a \cdot x) \in \mathcal{C}$ for $a > 0$. Without loss of generality, every $\mathcal{C}$ is also closed under scaling, i.e., if $C(x) \in \mathcal{C}$, then also $a \cdot C(x) \in \mathcal{C}$ for $a > 0$ (this is given by simple scaling and replication arguments).

2. The second assumption states that the cost-sharing method only charges players based on how they contribute to the joint cost (there is no other discrimination among the players). To clarify, we give two examples: (i) Consider a resource e with $C_e(x) = x^2$ and two players $1, 2$ with flows $f_e^1 = 1$, $f_e^2 = 2$. We now modify them such that the cost function and their weights become $\bar{C}_e(x) = x^2/4$ and $\bar{f}_e^1 = 2$, $\bar{f}_e^2 = 4$. Our second assumption asks that the new cost shares of the players remain the same as before the modification, i.e., $\chi_{ie} = \bar{\chi}_{ie}$ for $i = 1, 2$. (ii) Assume now we scale the joint cost on a resource e by a positive factor α, i.e., $\bar{C}_e(f_e(P)) = \alpha \cdot C_e(f_e(P))$. Given that the same players assign the same flow on e, the new cost shares of the players would be a scaled by factor α version of their initial cost shares, i.e., $\bar{\chi}_{ie} = \alpha \cdot \chi_{ie}$. In both examples we use simple scaling and replication arguments.

3. Last, we make a fairness-related assumption which states that the cost share of a player on a resource is a continuous, increasing and convex function of her flow. This is something to expect from a reasonable cost-sharing method, given that the joint cost on the resource is a continuous increasing convex function of the resource's total flow $f_e(P)$. For example,

if the cost share of a player i would increase in a slower than convex way in the player's weight, then (given that the joint cost is convex in resource's total flow) the rest of the users of this resource need to be charged additionally to cover the increase in the joint cost from the increase of i's weight.

We now define a class of cost-sharing methods, the weighted Shapley values, for this generalised model.

Weighted Shapley values. The *weighted Shapley value* defines how the cost $C_e(\cdot)$ of resource e is distributed among the players using it. Given an ordering π of N, let $F_e^{<i,\pi}(P)$ be the sum of flows of players preceding i in π. Then the marginal cost increase caused by a player i's flow is

$$C_e(F_e^{<i,\pi}(P) + f_e^i(P)) - C_e(F_e^{<i,\pi}(P)).$$

For a given distribution Π over orderings, the cost share of player i on e is

$$E_{\pi\sim\Pi}[C_e(F_e^{<i,\pi}(P) + f_e^i(P)) - C_e(F_e^{<i,\pi}(P))].$$

For the weighted Shapley value, the distribution over orderings is given by a sampling parameter $\lambda_e^i(P)$ for each player i. The last flow in the ordering is picked proportional to the sampling parameters $\lambda_e^i(P)$. This process is then repeated iteratively for the remaining players.

As in [41], we study a parameterized class of weighted Shapley values defined by a parameter γ. For this class $\lambda_e^i(P) = f_e^i(P)^\gamma$ for all players i and edges e. For $\gamma = 0$, this reduces to the (unweighted) Shapley value, where we have a uniform distribution over orderings.

4.2 Existence of Pure Nash Equilibria

In this section we focus on the existence of pure Nash equilibria. We require that our cost-sharing method and our set of cost functions satisfy certain natural conditions and we characterize the Shapley value as the unique method that guarantees the existence of a pure Nash equilibrium.

In addition, at the end of this section, we present two 'extensions' of this model and our results regarding PNE existence. Section 4.2.3 considers an alternative weighted SV method where the cost shares are computed based on the commodities weights (as opposed to our main model which is based on the total flows of players). In section 4.2.4, we study a splittable variable of our main model where each commodity can split and distribute her flow among her strategies.

4.2.1 Shapley Values

Our first result proves that applying the Shapley value (when every sampling parameter equal to 1), induces a potential game.

Theorem 4.2.1. *Using the Shapley value to share player costs in multi-commodity congestion games yields a potential game.*

Proof. Consider any ordering π of the players in N and let $f_e^{\leq i,\pi}(P)$ denote the vector that we get after truncating $f_e^N(P)$ by removing all entries for players that succeed i in π. We prove that the following is a potential function of the game.

$$\Phi(P) = \sum_{e \in E} \sum_{i \in N} \chi_e(i, f_e^{\leq i,\pi}(P), C_e). \tag{4.2}$$

Note that (4.2) can be also written as

$$\Phi(P) = \sum_{e \in E} \sum_{i \in N} \chi_e(i, \{j : \pi\left(f_e^j(P)\right) \leq \pi\left(f_e^i(P)\right), j \in S_e(P)\}), \tag{4.3}$$

which is a generalisation of the potential function (3.2) of the single-commodity per player model introduced in [57].

Hart and Mas-Colell [50] proved that (4.2) is independent of the ordering π in which players's flows are considered. Let $P' = (P'_i, P_{-i})$. It suffices to show that $\Phi(P) - \Phi(P')$ equals the change in the cost of player i. Focus on a single resource e and let π be one of the orderings that places flow of player i, $f_e^i(P)$, in the last position. Then, the potential on e loses a term equal to

$$\chi_e(i, f_e^{\leq i,\pi}(P), C_e) = \chi_e(i, f_e^N(P), C_e)$$

and gains a term equal to

$$\chi_e(i, f_e^{\leq i,\pi}(P'_i, P_{-i}), C_e) = \chi_e(i, f_e^N(P'_i, P_{-i}), C_e),$$

which is precisely what happens to the cost of player i on e. Summing over all edges completes the proof. $\square$

4.2.2 Weighted Shapley Values

One might expect that, similarly to standard congestion games, the same potential function argument would apply to weighted Shapley values as well. However, as we prove next, this is not the case.

Theorem 4.2.2. *There is a multi-commodity congestion game with no PNE for any weighted Shapley value defined by sampling weights of the form $f_e^i(P)^\gamma$ with $\gamma > 0$ or $\gamma < 0$.*

Proof. We prove this theorem by showing two examples admitting no PNE, for $\gamma > 0$ and $\gamma < 0$. We start with the $\gamma > 0$ case. Consider two players, 1 and 2, who compete for two parallel (meaning each commodity must pick exactly one of them) resources e,e' with identical cost functions $C_e(x) = C_{e'}(x) = x^{1+\delta}$ with $\delta > 0$ and $\frac{\gamma}{\delta}$ a large positive number (note that for $\delta = 0$, we have linear cost functions where in this case we have an equilibrium. As soon as we deviate from linearity, we use convexity to construct an example with no equilibrium). Player 1 controls a unit commodity $p \in Q_1$. Player 2 controls two commodities $q, q' \in Q_2$, with $w_{q'} = 1$ and $w_q = k$, for k a very large number. Recall, that the sampling weight of a player i on a resource e is given by $\lambda_e^i = (f_e^i)^\gamma$. This means that smaller weights are favoured when constructing the weighted Shapley ordering.

We first prove the following lemma which we use in the instance afterwards.

Lemma 20. *For any positive δ and ϵ, there exists some k sufficiently large such that*

$$(1+\delta)\cdot k^{\delta}+k^{1+\delta}\leq (k+1)^{1+\delta}\leq (1+\epsilon)\cdot(1+\delta)\cdot k^{\delta}+k^{1+\delta}.$$

Proof. Both inequalities follow by the fact that

$$\begin{aligned}\lim_{k\to\infty}\frac{(k+1)^{1+\delta}-k^{1+\delta}}{(1+\delta)\cdot k^{\delta}} &= \lim_{k\to\infty}\frac{k^{1+\delta}\left(\left(1+\frac{1}{k}\right)^{1+\delta}-1\right)}{(1+\delta)\cdot k^{\delta}}\\ &= \lim_{k\to\infty}\frac{\left(1+\frac{1}{k}\right)^{1+\delta}-1^{1+\delta}}{(1+\delta)\cdot\frac{1}{k}}\\ &= \frac{(x^{1+\delta})'|_{x=1}}{1+\delta}=1.\end{aligned}$$

□

Suppose, without loss of generality, that player 1 places commodity p on edge e. Then the best response of player 2 is to place the large commodity q alone on e' and the small commodity q' on e. To see this, note that player 2's cost on this outcome will be

$$\begin{aligned}&\frac{1}{2}\cdot(2^{1+\delta}-1)+\frac{1}{2}+k^{1+\delta}\\ &=2^{\delta}+k^{1+\delta}.\end{aligned}\tag{4.4}$$

Then we show that any other outcome results in a larger cost for player 2. First, let her large commodity be assigned with commodity p of player 1 on resource e. This would result in a cost of at least

$$\frac{1}{1+k^{\gamma}}\cdot k^{1+\delta}+\frac{k^{\gamma}}{1+k^{\gamma}}\cdot((k+1)^{1+\delta}-1)+1.\tag{4.5}$$

By Lemma 20, we get that

$$\begin{aligned}(4.5)\geq &\frac{k^{1+\delta}+k^{\gamma}\cdot\left((1+\delta)\cdot k^{\delta}+k^{1+\delta}-1\right)+1+k^{\gamma}}{1+k^{\gamma}}\\ &=k^{1+\delta}+\frac{(1+\delta)\cdot k^{\gamma+\delta}+1}{1+k^{\gamma}}.\end{aligned}\tag{4.6}$$

Note that for our k and ϵ, the second term is larger than 2^{δ}. Therefore (4.6)> (4.4). If now both commodities of player 2 were assigned on e', this would result in a cost of

$$\begin{aligned}(k+1)^{1+\delta}&\overset{Lemma\ 20}{\geq}\left(k^{1+\delta}+(1+\delta)\cdot k^{\delta}\right)\\ &>k^{1+\delta}+2^{\delta}=(4.4).\end{aligned}$$

We focus now on player 1. Her cost on resource e, given player 2's best strategy, is

$$2^{\delta},\tag{4.7}$$

which is a fixed number larger than 1, since $\delta > 0$. We show that player 1 would prefer to assign her commodity p on resource e', together with the large commodity q of player 2. In this outcome, player 1's cost would be

$$\frac{k^\gamma}{1+k^\gamma} \cdot 1 + \frac{1}{1+k^\gamma} \cdot \left((k+1)^{1+\delta} - k^{1+\delta}\right)$$
$$\overset{Lemma\ 20}{\leq} \frac{k^\gamma}{1+k^\gamma} + \frac{(1+\epsilon)\cdot(1+\delta)\cdot k^\delta}{1+k^\gamma}. \tag{4.8}$$

which approaches 1 for large enough k, since $\gamma > \delta$. Thus, (4.8) < (4.7) which proves the desirable deviation. Therefore there is no equilibrium.

We now switch to the case with $\gamma < 0$. Consider players $i = 1, 2, \ldots, k$, who compete for two parallel resources e_1, e_2 with identical cost functions $C_{e_1}(x) = C_{e_2}(x) = x^3$. Player k controls two commodities $p, q \in Q_k$ with weights $w_p = k$ and $w_q = 1$. Each player $i < k$ controls only one commodity $r_i \in Q_i$ with $w_{r_i} = 1$. The sampling weight of a player i on a resource e is given by $\lambda_e^i = (f_e^i)^\gamma$, for $\gamma < 0$. Assume player k assigns commodity p to resource e_1 and commodity q to e_2. In this case, note that it is a dominant strategy for players $1, \ldots, k-1$ to use resource e_2. For player k, this gives a cost share of

$$\frac{1}{k} \cdot C_{e_2}(f_{e_2}) + C_{e_1}(f_{e_1}) = k^2 + k^3. \tag{4.9}$$

On the other hand, if she assigns commodity p to e_2 and q to e_1, by Lemma 24(b) in page 92, her cost can be arbitrarily close to $1 + k^3$, for large k, which is strictly smaller than her previous cost (4.9). Therefore there is no PNE, which completes the proof of Theorem 4.2.2. □

4.2.3 Alternative Weighted Shapley Value Method.

One might consider a different way of generalising weighted Shapley values to multi-commodity congestion games: Apply a weighted Shapley value on the commodity weights by charging a player the sum of the weighted Shapley values of the commodities controlled by her. These cost-sharing methods coincide when all commodities have unit weights, which is equivalent to proportional cost-sharing, i.e., every player pays a cost-share that is proportional to her flow on any given resource. Below we use one such instance with unit commodities to prove that all these methods do not guarantee pure Nash equilibrium existence.

One should note that a similar example has already been given by Rosenthal [70]. However, Rosenthal's example uses concave cost functions, which we disallow in our setting. In contrast, our example only uses convex functions.

Lemma 21. *There is a congestion game with multi-commodity players and cubic cost functions that admits no pure Nash equilibrium under weighted Shapley value applied on commodity weights.*

Proof. We prove the theorem by constructing an instance with unit commodities such that best-response dynamics from any initial configuration cycles. This cycle is based on an example in [34], where Fotakis et al. prove that network unweighted congestion games with linear delays and equal cardinality coalitions do not have the *finite improvement property*, therefore they admit no potential function. Their model translates to a setting of our model where each player

Table 4.1: Players' costs in example of Lemma 21.

	P_1	P_3
P_2	$20+8$ $20+8$	$5+(13-9\cdot\epsilon)$ $(16-9\cdot\epsilon)+(13-9\cdot\epsilon)$
P_4	$20+7$ $20+(2-\epsilon)$	$5+13$ $(17-4\cdot\epsilon)+(5-4\cdot\epsilon)$

controls an equal number of unit commodities. We strengthen their result by proving non-existence of a pure Nash equilibria for (network) congestion games with multi-commodity players and cubic cost functions.

Consider two players, 1 and 2, who control two commodities each, $p, q \in Q_1$ and $p', q' \in Q_2$ with $w_p = w_q = w_{p'} = w_{q'} = 1$. There is a set of resources $E = \{1, 2, \ldots, 6\}$ with cost functions $C_1(x) = C_5(x) = C_6(x) = 5 \cdot x^3$, $C_2(x) = x^3$, $C_4(x) = 2 \cdot x^3$ and $C_3(x) = (1-\epsilon) \cdot x^3$, for small ϵ. The strategy sets for each commodity are: $\mathcal{P}^p = \{P_1\}$, $\mathcal{P}^q = \{P_2, P_4\}$, $\mathcal{P}^{p'} = \{P_1, P_3\}$ and $\mathcal{P}^{q'} = \{P_2\}$, where $P_1 = \{1\}$, $P_2 = \{2, 3\}$, $P_3 = \{3, 4, 5\}$ and $P_4 = \{4, 6\}$. Note that each player has a fixed strategy for the first commodity and two possible strategies for the second. Therefore there are four possible states. We model this instance as a bimatrix game in Table 4.1, where player 1 is the row player and 2 the column player. The costs are described as the sum of commodities' costs for each player. We claim that for any $\epsilon \in \left(\frac{1}{18}, \frac{3}{2}\right)$, there is no pure Nash equilibrium. □

We stregthen this result by showing non-existence of pure Nash equilibria even for a network congestion game under this setting. The proof uses the network in Figure 4.1, page 84.

Corollary 5. *There is even a network congestion game with multi-commodity players and cubic cost functions that admits no pure Nash equilibrium under weighted Shapley value applied on commodity weights.*

Proof. Consider the same set of players, commodities, resources and the associated cost functions, with those described in example of Lemma 21. In addition, for each commodity z there is a source-destination pair (s_z, t_z) with at least one path between them. In total, we have the following seven paths: $P_1 = (s_p, 1, t_p)$, $P_2 = (s_{p'}, 1, t_{p'})$, $P_3 = (s_{p'}, 4, 5, 3, t_{p'})$, $P_4 = (s_{q'}, 2, 3, t_{q'})$, $P_5 = (s_q, 2, 3, t_q)$, $P_6 = (s_q, 4, 5, 3, t_q)$, $P_7 = (s_q, 4, 6, t_q)$ and we see that the strategy sets for each commodity are: $\mathcal{P}^p = \{P_1\}$, $\mathcal{P}^q = \{P_5, P_6, P_7\}$, $\mathcal{P}^{p'} = \{P_2, P_3\}$ and $\mathcal{P}^{q'} = \{P_4\}$. A graphical interpretation of this network is given in Figure 4.1. Note that commodity $q \in Q_1$ has an additional strategy choice compared to the example of Lemma 21, therefore the total states are increased to six. We now model this instance as a bimatrix game, given by Table 4.2, where player 1 is the row player and 2 the column player. Observe, that the additional strategy P_7 for commodity p is strictly dominated by P_2 and P_4, which implies the corollary. □

4.2.4 Splittable Games

We conclude this chapter with results regarding equilibria existence on cost-sharing in the *splittable* version [24, 46, 51, 65, 74] of congestion games with

Figure 4.1: The network congestion game in Corollary 5.

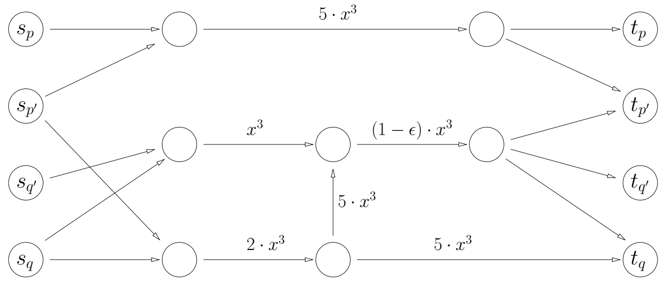


Table 4.2: Players' costs in example of Corollary 5.

	P_1	P_3
P_2	$20+8$ $20+8$	$5+(13-9\epsilon)$ $(16-9\epsilon)+(13-9\epsilon)$
P_4	$20+7$ $20+(2-\epsilon)$	$5+13$ $(17-4\epsilon)+(5-4\epsilon)$
P_3	$20+(11-4\epsilon)$ $20+(5-4\epsilon)$	$5+(37-9\epsilon)$ $(37-9\epsilon)+(10-9\epsilon)$

multi-commodity players. In the splittable version of such games, the weight w_q of a commodity $q \in Q$ can be split among its strategies in $\mathcal{P}^q$; i.e., a fractional strategy of commodity $q \in Q$ is a vector $P_q = (w_{q,P})_{P \in \mathcal{P}^q} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}^q|}$ with $\sum_{P \in \mathcal{P}^q} w_{q,P} = w_q$. For the unsplittable version, vector P_q has only one non-zero and equal to w_q component, which is not necessarily the case for the splittable games. For the single-commodity per player model, it is known that the proportional sharing method, having players paying a cost share proportional to their flows on each resource, guarantees existence of a pure Nash equilibrium [65].

Below we present two results with respect to the pure Nash equilibrium existence property of cost-sharing methods.

Theorem 4.2.3. *Using the Shapley value to share player costs in splittable congestion games with multi-commodity players yields a potential game.*

Proof. We prove this theorem by showing that these games are potential games. We use the potential function that was used for the unsplittable case

$$\Phi(P) = \sum_{e \in E} \sum_{i \in N} \chi(i, f_e^{\leq i, \pi}(P), C_e).$$

The only difference for the splittable case, is that any fraction of the weight of a commodity can deviate to another strategy. In our main model, Shapley value is applied on the total flows players assign on a resource, $f_e^i(P)$, which equals to the sum of the weight-fractions of the commodities of each player on the resource, i.e., $\sum_{q \in Q_i, e \in P_i} w_{q,P}$, as opposed to the $\sum_{q \in Q_i, e \in P_i} w_q$ (for the unsplittable case). But this difference has no affect in the proof steps of Theorem 4.2.1, therefore the same proof steps as in Theorem 4.2.1 apply on splittable games. $\square$

Theorem 4.2.4. *For a weighted Shapley value with parameter γ large enough, there exists a splittable congestion game with single-commodity per player admitting no PNE.*

Proof. We provide a sketch of the proof. For simplicity of the proof, we only proof the case of $\gamma = \infty$. This case captures the main idea of our proof. However, our instance also provides a counter-example if γ is finite, but large enough.

Consider two players 1 and 2 who control one commodity each, $q \in Q_1$ and $q' \in Q_2$ with $w_q = w_{q'} = 1$. There is a set of three parallel resources $E = \{1, 2, 3\}$ with associated cost functions $C_1(x) = C_3(x) = x$ and $C_2(x) = x^3$. The strategy set of commodity q and q' is $\mathcal{P}^q = \{\{1\}, \{2\}\}$ and $\mathcal{P}^{q'} = \{\{2\}, \{3\}\}$, respectively. Each commodity splits her flow of 1 between the available resources. Since each player controls only one commodity, for the rest of the proof we refer to players instead of the commodities.

First, consider the case where players assign different flows on the common resource 2. Let y and z be these flows for player 1 and player 2, respectively. Assume that both players are in a pure Nash equilibrium and, without loss of generality, player 1 assigns more flow than player 2 on resource 2, that is, $y > z$. Since $\gamma = \infty$, player 1 is certain to come last in the Shapley value ordering. This gives the following costs for the players:

$$\begin{aligned}\chi_{1,1} + \chi_{1,2} &= 1 - y + (y + z)^3 - z^3 \\ &= 1 - y + \left(y^3 + 3 \cdot z^2 \cdot y + 3 \cdot z \cdot y^2\right), \end{aligned} \tag{4.10}$$

$$\chi_{2,3} + \chi_{2,2} = 1 - z + z^3. \tag{4.11}$$

Note that the cost share of player 2 depends only on her flow z and observe that (4.11) is minimized for $z = \frac{1}{\sqrt{3}}$. Thus, either $y \leq \frac{1}{\sqrt{3}}$, in which case player 2 can improve by slightly increasing x, or $y > \frac{1}{\sqrt{3}}$ in which case we must have $z = \frac{1}{\sqrt{3}}$ by the Nash equilibrium condition. Substituting $z = \frac{1}{\sqrt{3}}$ in (4.10), we get $y^3 + \sqrt{3} \cdot y^2$, which is increasing in y. Thus, in this case, player 1 can improve by slightly reducing y. Therefore there is no equilibrium when players assign different flows on resource 2, which completes the first part of the proof.

Focus now on the case where players assign the same flow ($y = z$) on resource 2. Then their cost shares of both players are the same and equal to

$$\chi_{1,1} + \chi_{1,2} = 1 - y + \frac{1}{2} \cdot (2 \cdot y)^3 = 1 - y + 4 \cdot y^3. \tag{4.12}$$

We claim that there is at least one player who can improve her cost decreasing her flow on resource 2 by an ϵ small. Let player 1 be the player who deviates. Then she comes first in the ordering and her cost becomes

$$\chi_{1,1} + \chi_{1,2} = 1 - y + \epsilon + (y - \epsilon)^3,$$

which for small enough ϵ is less than her intial cost (4.12). This completes the second part of the proof. $\square$

4.3 Inefficiency of Equilibria

In this section, we focus on the PoA and PoS of multi-commodity per player congestion games and present general tight price of anarchy bounds (Section

4.3.1), which are robust. We then turn to the price of stability and prove an upper bound for the Shapley value cost-sharing method (Section 4.3.2).

4.3.1 Tight PoA Bounds for General Cost-Sharing

We first generalize the (λ, μ)-smoothness framework of [73] to accommodate any cost-sharing method and set of possible cost functions. For any set S, define F^S to be the sum of components of the flow vector f^S, that is $F^S = \sum_{i \in S}(f^S)_i$, $\forall S$. Suppose we identify positive parameters λ and $\mu < 1$ such that for every cost function in our allowable set $C \in \mathcal{C}$, and every pair of sets of players T and T^*, we get

$$\sum_{i \in T^*} \chi(i, f^{T \cup \{i\}}, C) \leq \lambda \cdot C(F^{T^*}) + \mu \cdot C(F^T). \tag{4.13}$$

Then, for P a PNE and P^* the optimal solution, we would get

$$\begin{aligned} SC(P) &\stackrel{(4.1)}{=} \sum_{i \in N} \sum_{e \in E} \chi_e(i, f_e^N(P), C_e) \\ &\stackrel{(??)}{\leq} \sum_{i \in N} \sum_{e \in E} \chi_e(i, f_e^N(P_i^*, P_{-i}), C_e). \end{aligned} \tag{4.14}$$

To simplify, let T_e and T_e^* be the sets of players who assign positive flow on resource e on an equilibrium outcome P and the optimal P^*, accordingly. Thus, $T_e = S_e(P)$ and $T_e^* = S_e(P^*)$. Then

$$\begin{aligned} (4.14) &= \sum_{e \in E} \sum_{i \in N} \chi_e(i, f_e^{T_e \cup \{i\}}(P_i^*, P_{-i}), C_e) \\ &\stackrel{(4.13)}{\leq} \sum_{e \in E} \lambda \cdot C_e(F^{T_e^*}) + \mu \cdot C_e(F^{T_e}) \\ &= \sum_{e \in E} \lambda \cdot C_e \left(\sum_{i \in T_e^*} \left(f^{T_e^*}\right)_i \right) + \mu \cdot C_e \left(\sum_{i \in T_e} \left(f^{T_e}\right)_i \right) \\ &= \sum_{e \in E} \lambda \cdot C_e(f_e(P^*)) + \mu \cdot C_e(f_e(P)) \\ &\stackrel{(4.1)}{=} \lambda \cdot SC(P^*) + \mu \cdot SC(P). \end{aligned} \tag{4.15}$$

Rearranging (4.15) yields a $\lambda/(1-\mu)$ upper bound on the POA. The same bound can be easily shown to apply to MNE and more general concepts (correlated and coarse correlated equilibria), though we omit the details (see, e.g., [73] for more). We then get the following lemma.

Lemma 22. *Consider the following optimization program with variables λ, μ.*

$$\textit{Minimize} \quad \frac{\lambda}{1-\mu} \tag{4.16}$$

$$\textit{Subject To} \quad \mu \leq 1 \tag{4.17}$$

$$\sum_{i \in T^*} \xi(i, f^{T \cup \{i\}}, C) \leq \lambda \cdot C(F^{T^*}) + \mu \cdot C(F^T), \tag{4.18}$$

where constraint (4.18) *needs to hold for any function* $C \in \mathcal{C}$*, pair of sets* $T, T^* \subseteq N$ *and any positive flow vector* f^S*. Every feasible solution yields a* $\lambda/(1-\mu)$ *upper bound on the POA of the cost sharing method given by* $\chi(i, f^S, C)$ *and the set of cost functions* $\mathcal{C}$*.*

The upper bound holds for any cost-sharing method and set of cost functions. We now proceed to show that the optimal solution to Program (4.16)-(4.18) gives a tight upper bound when our assumptions described in Section 3.1 hold.

Theorem 4.3.1. *Let* (λ^*, μ^*) *be the optimal point of Program* (4.16)-(4.18)*. The POA of the cost-sharing method given by* $\chi(i, f^S, C)$ *and the set of cost functions* $\mathcal{C}$ *is precisely* $\lambda^*/(1-\mu^*)$*.*

Proof. Define $\zeta(y, x, C)$ for $y, x > 0$ as

$$\zeta(y, x, C) = \max_{T^*: F^{T^*} = y,\ T: F^T = x} \sum_{i \in T^*} \chi(i, f^{T \cup \{i\}}, C). \tag{4.19}$$

With this definition, we can rewrite Program (4.16)-(4.18) as

$$\begin{aligned} &\textit{Minimize} \quad \frac{\lambda}{1-\mu} && (4.20)\\ &\textit{Subject To} && \\ &\qquad \mu \le 1 && (4.21)\\ &\qquad \zeta(y, x, C) \le \lambda \cdot C(y) + \mu \cdot C(x), \quad \forall C \in \mathcal{C} \text{ and } x, y \in \mathbb{R}_{>0}. && (4.22) \end{aligned}$$

Observe that for every constraint, we can scale the weights of the players by a factor a, dilate the cost function by a factor $1/a$ and scale it by an arbitrary factor, and keep the constraint intact (by Assumption 2). This suggests we can assume that every constraint has $y = 1$ and $C(1) = 1$. Then we rewrite Program (4.20)-(4.22) as

$$\begin{aligned} &\textit{Minimize} \quad \frac{\lambda}{1-\mu} && (4.23)\\ &\textit{Subject To} \quad \mu \le 1 && (4.24)\\ &\qquad \zeta(1, x, C) \le \lambda + \mu \cdot C(x), \quad \forall C \in \mathcal{C} \text{ and } x \in \mathbb{R}_{>0}. && (4.25) \end{aligned}$$

The Lagrangian dual of Program (4.23)-(4.25) is

Maximize

$$\inf_{\lambda,\mu} \left\{ \frac{\lambda}{1-\mu} + \sum_{C \in \mathcal{C}, x > 0} z_{Cx} \cdot (\zeta(1, x, C) - \lambda - \mu \cdot C(x)) + z_\mu \cdot (\mu - 1) \right\} \tag{4.26}$$

$$\textit{Subject To} \qquad z_{Cx}, z_\mu \ge 0. \tag{4.27}$$

Our primal is a semi-infinite program with an objective that is continuous, differentiable, and convex in the feasible region, and with linear constraints. We get that strong duality holds (see also [79, 85] for a detailed treatment of strong duality in this setting). We first treat the case when the optimal value of the primal is finite and is given by point (λ^*, μ^*). Before concluding our proof we will explain how to deal with the case when the primal is infinite or infeasible. The KKT conditions yield for the optimal $\lambda^*, \mu^*, z^*_{Cx}$:

$$\frac{1}{1-\mu^*} = \sum_{c\in\mathcal{C},x>0} z^*_{Cx}, \tag{4.28}$$

$$\frac{\lambda^*}{(1-\mu^*)^2} = \sum_{C\in\mathcal{C},x>0} z^*_{Cx}\cdot C(x). \tag{4.29}$$

Calling $\eta_{Cx} = z^*_{Cx}/\sum_{C\in\mathcal{C},x>0} z^*_{Cx}$ and dividing (4.29) with (4.28) we get

$$\frac{\lambda^*}{1-\mu^*} = \sum_{C\in\mathcal{C},x>0} \eta_{Cx}\cdot C(x). \tag{4.30}$$

By (4.30) and the fact that all constraints for which $z^*_{Cx} > 0$ are tight (by complementary slackness), we get

$$\sum_{C\in\mathcal{C},x>0} \eta_{Cx}\cdot\zeta(1,x,C) = \sum_{C\in\mathcal{C},x>0} \eta_{Cx}\cdot C(x). \tag{4.31}$$

We now proceed to our matching lower bound which holds even for the single-commodity per player model.

Lower bound construction. Let $\mathcal{T} = \{(C,x) : z^*_{Cx} > 0\}$. The construction starts off with a single-commodity player i, who has flow 1 and, in the PNE, uses a single resource e_i by herself. The cost function of resource e_i is an arbitrary function from $\mathcal{C}$ such that $C_{e_i}(1) \neq 0$ (it is easy to see that such a function exists, since $\mathcal{C}$ is closed under dilation, unless all function are 0, which is a trivial case) scaled so that $C_{e_i}(1) = \sum_{(C,x)\in\mathcal{T}} \eta_{Cx}\cdot\zeta(1,x,C)$. The other option of player i is to use a set of resources, one for each $(C,x) \in \mathcal{T}$ with cost functions $\eta_{Cx}\cdot C(\cdot)$. The resource corresponding to each (C,x) is used in the PNE by a player set that is equivalent to the T that maximizes the expression in (4.19) for the corresponding C, x. We now prove that player i does not gain by deviating to her alternative strategy. The key point is that due to convexity of the cost shares (assumption 3, page 78), the worst case T^* in definition (4.19) will always be a single player. Then we can see that the cost share of i on each (C,x) resource in her potential deviation will be $\eta_{Cx}\cdot\zeta(1,x,C)$. It then follows that she is indifferent between her two strategies. Note that the PNE cost of i is $\sum_{(C,x)\in\mathcal{T}} \eta_{Cx}\cdot\zeta(1,x,C)$, which by (4.30) and (4.31) is equal to $\lambda^*/(1-\mu^*)$. Also note that if player i could use her alternative strategy by herself, her cost would be 1.

We now make the following observation which allows us to complete the lower bound construction: Focus on the players and resources of the previous paragraph. Suppose we scale the weight of player i, as well as the weights of the users of the resources in her alternative strategy by the same factor $a > 0$. Then, suppose we dilate the cost functions of all these resources (the one used by i in the PNE and the ones in her alternative strategy) by a factor $1/a$ so that the costs generated by the players go back to the values they had in the previous paragraph. Finally, suppose we scale the cost functions by an arbitrary factor $b > 0$. We observe that the fact that i has no incentive to deviate is preserved (by assumption 2, page 78) and the ratio of PNE cost versus alternative cost for i remains the same, i.e., $\lambda^*/(1-\mu^*)$. This suggests that for every player generated by our construction so far in the PNE, we can repeat these steps by looking at her weight and PNE cost and appropriately constructing her alternative strategy and the users therein. After repeating this construction for a large number of layers $M \to \infty$, we complete the instance by creating a single resource for each of the players in the final layer. The cost functions of these resources are arbitrary nonzero functions from $\mathcal{C}$ scaled and dilated so that each one of these players is indifferent between her PNE strategy and using the newly constructed resource.

Consider the outcome that has all players play their alternative strategies and not the ones they use in the PNE. Every player other than the ones in the final layer would have a cost $\lambda^*/(1-\mu^*)$ smaller, as we argued above. We can now see that, by (4.31), the cost of every player in the PNE is the same as that of the players in the resources of her alternative strategy. This means the cost across levels of our construction is identical and the final layer is negligible, since $M \to \infty$. This proves that the cost of the PNE is $\lambda^*/(1-\mu^*)$ times larger than the outcome that has all players play their alternative strategies, which gives the tight lower bound.

Note on case with primal infeasibility. Recall that during our analysis we assumed that the primal program (4.23)-(4.25) had a finite optimal solution. Now suppose the program is either infeasible or $\mu = 1$, which means the minimizer yields an infinite value. This implies that, if we set μ arbitrarily close to 1, then there exists some $C \in \mathcal{C}$, such that, for any arbitrarily large λ, there exists $x > 0$ such that $\zeta(1, x, C) > \lambda + \mu \cdot C(x)$. We can rewrite this last expression as $\zeta(1, x, C)/C(x) > \mu + \lambda/C(x)$, which shows we have C, x values such that $\zeta(1, x, C)$ is arbitrarily close to $C(x)$ or larger (since μ is arbitrarily close to 1). We can then replace λ with λ' such that the constraint becomes tight. It is not hard to see that these facts give properties parallel to (4.30) and (4.31) by setting $\eta_{Cx} = 1$ for our C, x and every other such variable to 0. Then our lower bound construction goes through for this arbitrarily large $\lambda'/(1-\mu)$, which shows we can construct a lower bound with as high POA as desired. This completes the proof of Theorem 4.3.1. $\square$

4.3.2 PoS for Shapley Values ($\gamma = 0$).

In this section we study the POS for the class of standard Shapley values, where the sampling parameter of each player i is defined by $\lambda_i(P) = (f_e^i(P))^\gamma$ for $\gamma = 0$ and any outcome P.

Upper Bound

We start with an upper bound on the POS for the case that $\gamma = 0$, i.e., for the *standard Shapley value* (SV) cost-sharing method.

Theorem 4.3.2. *The POS of the SV with $\mathcal{C}$ the set of allowable cost functions is at most* $\max_{C \in \mathcal{C}, x > 0} \frac{C(x)}{\int_0^x \frac{C(x')}{x'} dx'}$.

Proof. We begin with the potential function of the game,

$$\Phi(P) = \sum_{e\in E}\sum_{i\in N}\chi(i, f_e^{\leq i,\pi}(P), C_e)$$

and we prove the following lemma which is the main tool for proving our upper bound on the POS. Briefly, the lemma states the following. For any instance with N players and any strategy profile, we can construct a new instance with $N+1$ players by splitting one player in half into two new players. Then this can only reduce the potential value of the game. More precisely, we do this by splitting in half the flow of each commodity controlled by a player i on a resource creating two new commodities, which we assign to the new two players, say i' and i''.

Lemma 23. *Consider an outcome P of the game and assume that on a resource e, we substitute the total flow of a player i with the flows of two other players i',i'' such that $f_e^{i'}(\hat{P}) = f_e^{i''}(\hat{P}) = \frac{f_e^i(P)}{2}$. Then we claim that*

$$\Phi_e(P) \geq \Phi'_e(\hat{P}),$$

where $\Phi'_e(\hat{P})$ is the potential value of resource e after the substitution.

Proof. First, rename the flows such that the substituted one $f_e^i(P)$ to have the highest index. Assign indices i' and i'' to the new ones, with $i < i' < i''$ in ordering π. Then, for any resource e, the new potential value equals to

$$\begin{aligned}\Phi'_e(\hat{P}) = \sum_{j=1}^{i-1}\chi_e(j, f_e^{\leq j,\pi}(P)) \ + \ \chi_e(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}))) \\ + \ \chi_e(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P}))).\end{aligned}$$

Note that the contribution to the potential value of the flows before player's i flow is the same as before the substitution. Therefore it is enough to show that

$$\begin{aligned}\chi_e(i, f_e^N(P)) \ \geq \chi_e(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}))) + \\ \chi_e(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P}))).\end{aligned} \tag{4.32}$$

To simplify, in what follows in this proof, call

$$\begin{aligned}\chi &= \chi_e(i, f_e^N(P)),\\ \chi' &= \chi_e(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}))),\\ \chi'' &= \chi_e(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P}))).\end{aligned}$$

Define as $S_e^i(\pi)$ the set of players preceding player i in π. Then, for every ordering π and permutation τ^i of set $S_e^i(\pi)\cup\{i\}$, define as $F_e^{<i,\pi,\tau^i}(P)$ the sum of players' flows who precede i in both π and τ^i. Let now $|S_e(P)| = r$. By definition of SV, we get

$$\chi = \frac{1}{r!}\sum_{\tau^i}\left(C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^i(P)\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P))\right)\right), \tag{4.33}$$

$$\chi' = \frac{1}{r!}\sum_{\tau^i}\left(C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P})\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P)\right)\right). \tag{4.34}$$

For χ'', since the position of $f_e^{i'}(\hat{P})$ in the ordering is unspecified, we give an upper bound for this value as follows. For any permutation τ, let $A(\tau)$ be the marginal cost increase caused by $f_e^{i''}(\hat{P})$ when she precedes $f_e^{i'}(\hat{P})$ in π, and $B(\tau)$ when she succeeds. That is

$$\begin{aligned} A(\tau) &= C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i''}(\hat{P})\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P)\right), \\ B(\tau) &= C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) + f_e^{i''}(\hat{P})\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P})\right). \end{aligned} \tag{4.35}$$

Let now p equal the probability of $f_e^{i'}(\hat{P})$ preceding $f_e^{i''}(\hat{P})$. Then SV definition gives

$$\chi'' = (1-p)\cdot\frac{1}{r!}\cdot\sum_{\tau^i} A(\tau) + p\cdot\frac{1}{r!}\cdot\sum_{\tau^i} B(\tau). \tag{4.36}$$

Due to convexity, $A(\tau) \leq B(\tau)$. Therefore, by substituting $A(\tau)$ with $B(\tau)$ in definition (4.36), we get the following upper bound for χ'',

$$\chi'' \leq \frac{1}{r!}\sum_{\tau^i} B(\tau). \tag{4.37}$$

Using (4.34) and (4.37), we upperbound $\chi' + \chi''$ by

$$\begin{aligned} &\frac{1}{r!}\sum_{\tau^i} C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i''}(\hat{P})\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P)\right) + \\ &\qquad + C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) + f_e^{i''}(\hat{P})\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P})\right). \end{aligned}$$

Since $f_e^{i'}(\hat{P}) = f_e^{i''}(\hat{P}) = \frac{f_e^i(P)}{2}$, we get

$$\chi' + \chi'' \leq \frac{1}{r!}\sum_{\tau^i}\left(C_e\left(F_e^{<i,\pi,\tau^i}(P) + f_e^i(P)\right) - C_e\left(F_e^{<i,\pi,\tau^i}(P)\right)\right) \overset{(4.33)}{=} \chi,$$

which proves the desirable inequality (4.32) and completes Lemma's 23 proof. □

We now continue with the PoS upper bound. By repeatedly applying Lemma 23, we can break the total flow on each resource in identical flows of infinitesimal size without increasing the value of the potential. This implies that

$$\Phi_e(P) \geq \int_0^{f_e(P)} \frac{C_e(x)}{x} dx. \tag{4.38}$$

Now call P^* the optimal outcome and $P = \arg\min_{P'} \Phi(P')$ the minimizer of the potential function, which is, by definition, also a PNE. Then

$$\begin{aligned} SC(P^*) &\overset{(4.2)}{\geq} \Phi(P^*) \overset{\text{Def.}P}{\geq} \Phi(P) \overset{(4.38)}{\geq} \sum_{e\in E}\int_0^{f_e(P)} \frac{C_e(x)}{x} dxr \\ &= \frac{\sum_{e\in E}\int_0^{f_e(P)} \frac{C_e(x)}{x}dx}{\sum_{e\in E} C_e(f_e(P))}\cdot SC(P) \geq \min_{e\in E}\frac{\int_0^{f_e(P)} \frac{C_e(x)}{x}dx}{C_e(f_e(P))}\cdot SC(P). \end{aligned}$$

Rearranging yields the upper bound $PoS \leq \max_{C\in\mathcal{C},x>0} \frac{C(x)}{\int_0^x \frac{C(x')}{x'}dx'}$, which completes the proof of Theorem 4.3.2. $\square$

Corollary 6. *For polynomials with non-negative coefficients and degree at most d, the POS of the SV is at most d+1, which asymptotically matches the lower bound of [20] for the unweighted single-commodity per player case.*

4.3.3 PoS for Weighted Shapley Values ($\gamma \neq 0$).

In this section we study the PoS for the class of weighted Shapley values, where the sampling parameter of each player i is defined by $\lambda_i(P) = (f_e^i(P))^\gamma$ for any $\gamma \neq 0$ and outcome P.

We show that this linear dependence on the maximum degree d of the polynomial cost functions is very fragile. More precisely, for all values $\gamma \neq 0$, we show an exponential (in d) lower bound which matches the corresponding lower bound on the PoA in [41]. Our bound for $\gamma > 0$ even matches the upper bound on the PoA [41], which holds for the weighted Shapley value in general. Our constructions modify the corresponding instances in [41], making sure that they have a unique Nash equilibrium.

Theorem 4.3.3. *For polynomial cost functions with non-negative coefficients and maximum degree d, the POS for the class of weighted Shapley values with sampling parameters $\lambda_e^i(P) = (f_e^i(P))^\gamma$ is at least*

(a) $(2^{\frac{1}{d}} - 1)^{-d}$*, for all* $\gamma > 0$*, and*

(b) d^d*, for all* $\gamma < 0$*.*

In the following we prove Theorem 4.3.3. To do so, we use Lemma 24[1], which will be crucial for proving our lower bounds on the POS. We then introduce an instance in Example 10 and show in Theorem 4.3.4 that it gives the lower bound for $\gamma > 0$. Afterwards, Example 11 and Theorem 4.3.5, provide the corresponding lower bound for $\gamma < 0$.

Lemma 24. *Consider a resource e, a player i with flow f_e^i and a set T of k players with total weight F_e^T, where $f_e^t = \frac{F_e^T}{k}$ for $t \in T$. Assume that the set of players $\{i\} \cup T'$, for some $T' \subseteq T$, is using a resource where players' cost shares are computed by weighted Shapley values with sampling weights $\lambda_z(f_e^z) = (f_e^z)^\gamma$, for each player z. Let $j \in T'$. Then* (a) *for* $\gamma > 0$

$$\lim_{k\to\infty} \chi_{ie} = C_e\left(|T'| \cdot \frac{F_e^T}{k} + f_e^i\right) - C_e\left(|T'| \cdot f_e^j\right), \tag{4.39}$$

$$\lim_{k\to\infty} \chi_{je} = \frac{1}{|T'|} \cdot C_e\left(|T'| \cdot \frac{F_e^T}{k}\right), \quad \textit{if } \ |T'| \neq \emptyset, \tag{4.40}$$

and, (b) *for* $\gamma < 0$

$$\lim_{k\to\infty} \chi_{ie} = C_e(f_e^i), \tag{4.41}$$

$$\lim_{k\to\infty} \chi_{je} = \frac{1}{|T'|} \cdot \left(C_e\left(|T'| \cdot \frac{F_e^T}{k} + f_e^i\right) - C_e\left(f_e^i\right)\right), \quad \textit{if } \ |T'| \neq \emptyset. \tag{4.42}$$

[1]The proof of Lemma 24 is quite technical and provided in Appendix.

PoS Lower Bound for Positive γ.

In this section, we give the proof for part (a) of Theorem 4.3.3. First, we describe the example that gives our lower bound, for $\gamma > 0$, which is stated in Theorem 4.3.4 afterwards.

Example 10. *Consider a complete k-ary tree $G = (E, N)$ with l levels, where the root is positioned at level 0. Each vertex of the tree corresponds to a resource and each edge to a player. For $0 \le j \le l-1$, let $E_j \subset E$ be the set of resources (nodes) at level j and, for $0 \le j \le l-2$, let $N_j \subset N$ be the set of players (edges) between levels j and $j+1$ of the tree. The strategies of every player are the endpoints of the edge she is associated with, i.e. a player associated with edge $(j, j-1)$ has strategies $\{\{j\}, \{j-1\}\}$. For $0 \le j \le l-2$, the flow f^i of a player $i \in N_j$ is given by ϕ^{l-j-2}, where*

$$\phi = k \cdot \left(2^{\frac{1}{d}} - 1\right). \tag{4.43}$$

Define $\alpha = \frac{k}{\phi^d}$ and let $0 < \epsilon < 1$. The cost function of resources $e \in E_j$, is given by

$$C_e(x) = \left(1 + \epsilon^{l-j}\right) \cdot \alpha^{l-j-2} \cdot x^d, \qquad \textit{for } 0 \le j \le l-2,$$

$$\textit{and } \; C_e(x) = (1 + \epsilon) \cdot k^{d-1} \cdot x^d, \qquad \textit{for } j = l-1.$$

Theorem 4.3.4. *For polynomial cost functions with non-negative coefficients and maximum degree d, the POS for the class of weighted Shapley values with sampling parameters $\lambda_i(f^i) = (f^i)^\gamma$, with $\gamma > 0$, is at least*

$$\left(2^{\frac{1}{d}} - 1\right)^{-d}.$$

Proof. Choose the instance in Example 10. Then, let P be the outcome where all players use the resource closer to the root, and P^* be the outcome where all players use the resource further from the root. We prove by induction that P is a unique Nash equilibrium with a total cost equal to $(2^{\frac{1}{d}} - 1)^{-d}$ times the total cost of outcome P^*.

Consider a player $i \in N_{l-2}$, i.e. a player connected to a leaf resource and assume she uses the leaf resource, $e \in E_{l-1}$. Note that she is the only player who can use this resource. Then player i's cost share equals to

$$\chi_{ie} = (1 + \epsilon) \cdot k^{d-1}. \tag{4.44}$$

For $k' \le k$, consider k' players from set N_{l-2} (including player i) and assume they use the resource e closer to the root, $e \in E_{l-2}$. Choose the (worst) case where a player $b \in N_{l-3}$ also uses resource e. We show that even in this case, player i prefers the resource closer to the root. Using (59) of Lemma 24, player i's cost share is given by

$$\begin{aligned} \lim_{k \to \infty} \chi_{ie} &= \frac{1}{k'} \cdot \left(1 + \epsilon^2\right) \cdot \alpha \cdot (k' \cdot \phi)^d \\ &\overset{k' \le k}{\le} \left(1 + \epsilon^2\right) \cdot k^{d-1}, \end{aligned} \tag{4.45}$$

where (4.45) is strictly smaller than (4.44), since $0 < \epsilon < 1$.

For $l-1 \geq j' \geq j+1$, assume that each player of set $N_{j'}$ uses the strategy closer to the root, $e \in E_{j'}$. Then consider a player $i \in N_j$ who uses the resource e further from the root, $e \in E_{j+1}$. By assumption, this resource is also used by k players from set N_{j+1}. Then, using (58) of Lemma 24, player i's cost share is given by

$$\begin{aligned}
\lim_{k\to\infty} \chi_{ie} &= \left(1+\epsilon^{l-j-1}\right)\cdot \alpha^{l-j-3}\cdot \left(\left(k\cdot\phi^{l-j-3}+\phi^{l-j-2}\right)^d - \left(k\cdot\phi^{l-j-3}\right)^d\right) \\
&= \left(1+\epsilon^{l-j-1}\right)\cdot \alpha^{l-j-3}\cdot k^d\cdot \phi^{d\cdot(l-j-3)}\cdot \left(\left(1+\frac{\phi}{k}\right)^d - 1^d\right). \\
&= \left(1+\epsilon^{l-j-1}\right)\cdot \left(\alpha\cdot\phi^d\right)^{l-j-3}\cdot k^d.
\end{aligned}$$

By substituting α, we have

$$\lim_{k\to\infty} \chi_{ie} = (1+\epsilon^{l-j-1})\cdot (k^{l-j-3+d}). \tag{4.46}$$

For $k' \leq k$, consider k' players from set N_j (including player i) and assume they use the resource e closer to the root, $e \in E_j$. Choose the (worst) case where one player $b \in N_{j-1}$ also uses resource e. Then we show that player i still prefers to use resource $e \in E_j$. By (59) of Lemma 24, i's cost share equals to

$$\begin{aligned}
\lim_{k\to\infty} \chi_{ie} &= \frac{1}{k'}\cdot\left(1+\epsilon^{l-j}\right)\cdot\alpha^{l-j-2}\cdot\left(k'\cdot\phi^{l-j-2}\right)^d \\
&= \left(1+\epsilon^{l-j}\right)\cdot\alpha^{l-j-2}\cdot(k')^{d-1}\cdot\phi^{d\cdot(l-j-2)} \\
&\overset{k'\leq k}{\leq} \left(1+\epsilon^{l-j}\right)\cdot\left(\alpha\cdot\phi^d\right)^{l-j-2}\cdot k^{d-1}.
\end{aligned} \tag{4.47}$$

By substituting α, we have

$$\lim_{k\to\infty} \chi_{ie} \leq \left(1+\epsilon^{l-j}\right)\cdot k^{l-j-3+d},$$

which is strictly smaller than (4.46), since $0 < \epsilon < 1$.

PoS. Let P and P^* be the outcomes where each player chooses the strategy closer and further from the root accordingly. In this section, we compute the total costs of outcomes P and P^*, and present a lower bound to the price of Stability (PoS).

For outcome P, since every player chooses the resource closer to the root, no player uses any resource $e \in E_{l-1}$, therefore the cost at the leaves of the tree is zero. For $0 \leq j \leq l-2$, we proved that each player using a resource $e \in E_j$ incurs a cost of $\left(1+\epsilon^{l-j}\right)\cdot k^{l-j-3+d}$. Since each of the k^j resources in E_j is used by k players, the total cost at level j equals to

$$\begin{aligned}
\sum_{e\in E_j} C_e(f_e(P)) &= k^{j+1}\cdot\left(1+\epsilon^{l-j}\right)\cdot k^{l-j-3+d} \\
&= \left(1+\epsilon^{l-j}\right)\cdot\left(k^{l-2+d}\right).
\end{aligned}$$

Summing up for the l levels, we get that the social cost in outcome P equals to

$$SC(P) = \sum_{j=0}^{l-1}\sum_{e\in E_j} C_e(f_e(P)) = k^{l-2+d}\cdot\left(l-1+\sum_{i=2}^{l}\epsilon^i\right). \tag{4.48}$$

For outcome P^*, we now compute the cost of each player on each level. Since every player chooses to use the resource further from the root, the cost of resource $e \in E_0$ is zero. For $1 \leq j \leq l-2$, each player using a resource $e \in E_j$ incurs a cost of

$$\begin{aligned}\left(1+\epsilon^{l-j}\right) \cdot \alpha^{l-j-2} \cdot \phi^{(l-j-1) \cdot d} &= \left(1+\epsilon^{l-j}\right) \cdot \left(\alpha \cdot \phi^{d}\right)^{l-j-2} \cdot \phi^{d} \\ &= \left(1+\epsilon^{l-j}\right) \cdot k^{l-j-2} \cdot \phi^{d}.\end{aligned}$$

In this case, each of the k^j resources in E_j is used by only one player, therefore the total cost at level j equals to

$$\sum_{e \in E_j} C_e(f_e(P)) = k^j \cdot \left(1+\epsilon^{l-j}\right) \cdot \left(k^{l-j-2}\right) = \left(1+\epsilon^{l-j}\right) \cdot k^{l-2} \cdot \phi^{d}. \tag{4.49}$$

Last, the cost of each player using a leaf resource, $e \in E_{l-1}$, equals to $(1+\epsilon) \cdot k^{d-1}$. Since there are k^{l-1} leaf resources, the total cost at level $l-1$ equals to

$$\sum_{e \in E_{l-1}} C_e(f_e(P)) = k^{l-1} \cdot (1+\epsilon) \cdot k^{d-1} = (1+\epsilon) \cdot \left(k^{l-2+d}\right). \tag{4.50}$$

Summing up for the l levels, we have that the social cost in outcome P^* equals to

$$\begin{aligned}SC(P^*) &= \sum_{j=0}^{l-1} \sum_{e \in E_j} C_e(f_e(P)) \\ &= \sum_{j=1}^{l-2} \sum_{e \in E_j} C_e(f_e(P)) + \sum_{e \in E_{l-1}} C_e(f_e(P)) \\ &\overset{(4.49)}{\underset{(4.50)}{=}} k^{l-2} \cdot \phi^{d} \cdot \left(l-2+\sum_{i=2}^{l} \epsilon^{i}\right) + (1+\epsilon) \cdot k^{l-2+d} \\ &\overset{(4.43)}{=} k^{l-2+d} \cdot \left(\left(2^{\frac{1}{d}}-1\right)^{d} \cdot \left(l-2+\sum_{i=2}^{l} \epsilon^{i}\right) + 1 + \epsilon\right). \end{aligned} \tag{4.51}$$

Using (4.48) and (4.51), the *price of Stability* (PoS) is lower bounded by

$$\frac{SC(P)}{SC(P^*)} = \frac{l-1+\sum_{i=2}^{l} \epsilon^{i}}{\left(2^{\frac{1}{d}}-1\right)^{d} \cdot \left(l-2+\sum_{i=2}^{l-1} \epsilon^{i}\right) + 1 + \epsilon}$$

As $l \to \infty$ and ϵ is arbitrary small, this ratio goes to $\left(2^{\frac{1}{d}}-1\right)^{-d}$, as desired. □

PoS Lower Bound for Negative γ.

In this section, we give the proof for part (b) of Theorem 4.3.3. First, we describe the example that gives our lower bound, for $\gamma < 0$, which is stated in Theorem 4.3.5 afterwards.

Example 11. *Consider a complete k-ary tree $G = (E, N)$ with l levels, where the root is positioned at level 0. Each vertex of the tree corresponds to a resource and each edge to a single-commodity player. For $0 \le j \le l-1$, let $E_j \subset E$ be the set of resources (nodes) at level j, and, for $0 \le j \le l-2$, let $N_j \subset N$ be the set of players (edges) between levels j and $j+1$ of the tree. The strategies of every player are the endpoints of the edge she is associated with, i.e. a player associated with edge $(j, j-1)$ has strategies $\{\{j\}, \{j-1\}\}$. The flow of a player $i \in N_j$ is given by ϕ^j, where*

$$\phi = \frac{1}{k \cdot d}. \tag{4.52}$$

Define $\alpha = k^{d-1} \cdot d^d$. The cost function of resources $e \in E_j$, is given by

$$C_e(x) = (1+\epsilon) \cdot x^d, \qquad \text{for } j = 0,$$

$$\text{and } \; C_e(x) = \left(1+\epsilon^{j+1}\right) \cdot \alpha^{j-1} \cdot x^d, \qquad \text{for } 1 \le j \le l-1.$$

where ϵ is an arbitrarily small parameter.

Theorem 4.3.5. *For polynomial cost functions with non-negative coefficients and maximum degree d, the PoS for the class of weighted Shapley values with sampling parameters $\lambda_i(f^i) = (f^i)^\gamma$, with $\gamma < 0$, is at least d^d.*

Proof. Choose the instance described in Example 11. Then, let P be the outcome where all players use the resource further from the root, and P^* be the outcome where all players use the resource closer to the root. We prove by induction that P is a unique Nash equilibrium with a total cost equal to d^d times the total cost of outcome P^*.

For $k' \le k$, consider k' players from set N_0 icluding a player i and assume they use the resource on the root, $e \in E_0$. Then player i's cost share equals to

$$\lim_{k \to \infty} \chi_{ie} = \frac{1}{k'} \cdot C_e(f_e(P)) = \frac{1}{k} \cdot (1+\epsilon) \cdot (k')^d \overset{k' \ge 1}{\ge} 1+\epsilon. \tag{4.53}$$

Assume now that player i deviates to resource e further from the root, $e \in E_1$, and she uses this resource together with k' players from set N_1, for $k' \le k$. By (60) of Lemma 24, player i's cost share is given by

$$\lim_{k \to \infty} \chi_{ie} = \left(1+\epsilon^2\right) \cdot 1^d = 1 + \epsilon^2$$

for large enough k, which is strictly lower than (4.53).

For $0 \le j' \le j-2$, assume that each player of set $N_{j'}$ uses the strategy further from the root, $e \in E_{j'+1}$. For $k' \le k$, consider k' players from set N_{j-1} including a player i, and assume they use the resource e closer to the root, $e \in E_{j-1}$. By assumption, this resource is also used by one player $b \in N_{j-2}$. We show that player i prefers to deviate. Using (61) of Lemma 24, player i incurs a cost of

$$\begin{aligned}
\lim_{k \to \infty} \chi_{ie} &= \frac{1}{k'} \cdot \left(1+\epsilon^j\right) \cdot \alpha^{j-2} \cdot \left(\left(k' \cdot \phi^{j-1} + \phi^{j-2}\right)^d - \phi^{(j-2)\cdot d}\right) \\
&\overset{k' \ge 1}{\ge} \left(1+\epsilon^j\right) \cdot \alpha^{j-2} \cdot \left(\left(\phi^{j-1} + \phi^{j-2}\right)^d - \phi^{(j-2)\cdot d}\right) \\
&= \left(1+\epsilon^j\right) \cdot \alpha^{j-2} \cdot \phi^{(j-2)\cdot d} \cdot \left((\phi+1)^d - 1\right).
\end{aligned} \tag{4.54}$$

Define function $g(x) = x^d$. Since g is convex and $\phi \to 0$, we have that

$$(\phi + 1)^d - 1 \geq \phi \cdot d.$$

Using the above inequality, we get

$$\begin{aligned}
\lim_{k\to\infty} \chi_{ie} \geq \ (4.54) \ &\geq \left(1+\epsilon^j\right) \cdot (\alpha \cdot \phi^d)^{j-2} \cdot \phi \cdot d \\
&= \left(1+\epsilon^j\right) \cdot \left(\frac{1}{k}\right)^{j-2} \cdot \frac{1}{k \cdot d} \cdot d \\
&= \left(1+\epsilon^j\right) \cdot \frac{1}{k^{j-1}}.
\end{aligned} \tag{4.55}$$

Assume now that player i deviates to resource e further from the root, $e \in E_j$. Choose the case where k' players from set N_j also use this resource e, for a $k' \leq k$. Then by (60) of Lemma, player i's cost share is given by

$$\begin{aligned}
\lim_{k\to\infty} \chi_{ie} &= \left(1+\epsilon^{j+1}\right) \cdot \alpha^{j-1} \cdot \left(\phi^{j-1}\right)^d \\
&= \left(1+\epsilon^{j+1}\right) \cdot \frac{1}{k^{j-1}},
\end{aligned}$$

which is strictly lower than (4.55).

PoS. First, we compute the total cost of the outcome P. Since every player chooses the resource further from the root, the cost of resource $e \in E_0$ (root) is zero. As we proved, each player using a resource $e \in E_j$ incurs a cost of $\left(1+\epsilon^{j+1}\right) \cdot \frac{1}{k^{j-1}}$, for $1 \leq j \leq l-1$. Each of the k^j resources in E_j is used by only one player. Therefore the total cost at level j equals to

$$\sum_{e\in E_j} C_e(f_e(P)) = k^j \cdot \left(1+\epsilon^{j+1}\right) \cdot \frac{1}{k^{j-1}} = \left(1+\epsilon^{j+1}\right) \cdot k.$$

Computing the sum for the l levels, we have that the cost of outcome P equals

$$SC(P) = \sum_{j=0}^{l-1} \sum_{e\in E_j} C_e(f_e(P)) = k \cdot \left(l - 1 + \sum_{i=2}^{l} \epsilon^i\right). \tag{4.56}$$

Now let P^* be the outcome where each player chooses the resource closer to the root. The cost of level $l-1$ (leaves) of the tree is zero. The cost at level 0 is $(1+\epsilon) \cdot k^d$. For $1 \leq j \leq l-2$, each resource $e \in E_j$ incurs a cost of

$$\begin{aligned}
(1+\epsilon^{j+1}) \cdot \alpha^{j-1} \cdot \left(k \cdot \phi^j\right)^d &= (1+\epsilon^{j+1}) \cdot \alpha^{j-1} \cdot k^d \cdot (\phi^{j-1} \cdot \phi)^d \\
&= (1+\epsilon^{j+1}) \cdot (\alpha \cdot w^d)^{j-1} \cdot (k \cdot \phi)^d \\
&= (1+\epsilon^{j+1}) \cdot \frac{1}{k^{j-1} \cdot d^d}.
\end{aligned}$$

Since there are k^j resources in E_j, the total cost of level j of the tree is given by

$$\sum_{e\in E_j} C_e(f_e(P)) = k^j \cdot (1+\epsilon^{j+1}) \cdot \frac{1}{k^{j-1} \cdot d^d} = \left(1+\epsilon^{j+1}\right) \cdot \frac{k}{d^d}.$$

Summing up for all levels, we get that the social cost in outcome P^* equals to

$$SC(P^*) = \sum_{j=0}^{l-1} \sum_{e \in E_j} C_e(f_e(P)) = \sum_{e \in E_0} C_e(f_e(P)) + \sum_{j=1}^{l-1} \sum_{e \in E_j} C_e(f_e(P))$$
$$= (1+\epsilon) \cdot k^d + \left(l - 1 + \sum_{i=2}^{l} \epsilon^i \right) \cdot \frac{k}{d^d}. \quad (4.57)$$

Using (4.56) and (4.57), the *price of Stability* (PoS) is lower bounded by the following ratio

$$\frac{SC(P)}{SC(P^*)} = \frac{k \cdot \left(l - 1 + \sum_{i=2}^{l} \epsilon^i \right)}{(1+\epsilon) \cdot k^d + \left(l - 1 + \sum_{i=2}^{l} \epsilon^i \right) \cdot \frac{k}{d^d}}$$
$$= \frac{l - 1 + \sum_{i=2}^{l} \epsilon^i}{(1+\epsilon) \cdot k^{d-1} + \left(l - 1 + \sum_{i=2}^{l} \epsilon^i \right) \cdot \frac{1}{d^d}}.$$

With $l \to \infty$ and $\epsilon \to 0$, this ratio goes to d^d for any k, which completes Theorem's 4.3.5 proof. $\square$

4.4 Conclusion

The class of generalised weighted Shapley values are the only methods that guarantee existence of a PNE in games with single-commodity players [13]. We prove that the special case of SV still keeps this desirable property even in games with multi-commodity players. However this is not the case, as soon as we introduce weight dependant sampling weights. We then exhibit a separation from the single-commodity case by proving that a large subclass of weighted Shapley values do not guarantee existence of a PNE in multi-commodity routing games. In fact, we show that among all cost-sharing methods satisfying assumptions on p. 78, the SV is the unique method which induces games that guarantee PNE.

The majority of research on price of anarchy in weighted congestion games has been focused in proportional sharing (see Section 1.5.2, page 29). Our POA results greatly generalise the work on cost-sharing methods for weighted congestion games and give a recipe for tight bounds in a large array of applications. We parameterize the POA by (i) the set of allowable cost functions and (ii) the cost-sharing method. Our upper bounds are robust and apply to general equilibrium concepts. Our upper bound applies to games with multi-commodity players, while our lower bound uses an instance with single-commodity players, which proves the POA coincides for the two models.

Results on the price of stability, with respect to congestion games, are only known for polynomial *unweighted* games and single-commodity players, for which [20] provides exact bounds. Work in [56, 75] studies the PoS of the SV in related settings. Our PoS upper bound is the first for weighted congestion games that applies to any class of convex costs. The approach we follow allows an infinite number of players for our bounds to hold and parameterize by the set of possible cost functions. For example, for polynomials of degree at most d, we show that the PoS is at most d, even when $n \to \infty$. Observe that for unweighted games proportional shares and Shapley values are identical (see Proposition 1.1). Thus, the lower bound in [20], which approaches d, also applies to our setting, showing that our bound for polynomials is asymptotically tight. Our lower bounds on the PoS for the parameterized class of weighted Shapley values build on the corresponding lower bounds on the PoA in [41]. Our construction matches these bounds by ensuring that the instance possesses a unique Nash equilibrium. Together with our upper bound this shows an interesting contrast: For the special case of Shapley values, the PoS is exponentially better than the PoA, but as soon as we give some weight dependent priorities to the players, the PoA and the PoS essentially coincide.

For a splittable version of our model (Section 4.2.4), we give results regarding PNE existence. A result from Orda et al. [65] implies the existence of pure Nash equilibria in the multi-commodity splittable model, if the cost share of a player on a resource is a convex function of her flow on the resource. This result [65] is based on the Kakutani Fixed Point theorem. This immediately gives us existence of pure Nash equilibria for the standard Shapley cost sharing. We strengthen this result by showing that such games are exact potential games [62], thus best response dynamics converge to a pure Nash equilibrium. Understanding the POA of other cost-sharing methods both in the single- and multi-commodity models is an interesting open question. Similarly, it is interesting to further explore questions pertaining to the existence of pure Nash equilibria in such games.

Appendix

Proof of Lemma 16, page 71

Lemma. *The approximation factor α is in the order of $\left(\frac{d}{\ln(2)}\right)^d \cdot poly(d)$.*

Proof. We have that this factor α equals to

$$\alpha = (1 + O(\gamma)) \cdot \frac{(d+1)^2 \cdot (d+3)}{8} \cdot \frac{t \cdot (2^{\frac{1}{d+1}} - 1)^{-d}}{2^{-\frac{d}{d+1}} \cdot (1+t) - t},$$

where γ is a small positive constant and $t = 1 + \gamma$. Observe that factor α is essentially in the order of

$$\Theta(d^3) \cdot \left(\frac{1}{2^{\frac{1}{d+1}} - 1}\right)^d.$$

To compute the order of the above, we focus on the second part $\frac{1}{2^{\frac{1}{d+1}}-1}$ and claim that it is assymptotically similar to $\frac{d}{\ln(2)}$. This follows from

$$\lim_{d\to\infty} \frac{\frac{1}{d}}{2^{\frac{1}{d+1}} - 1} = \lim_{d\to\infty} \frac{-\frac{1}{d^2}}{-\frac{2^{\frac{1}{d+1}} \cdot \ln(2)}{(d+1)^2}} = \frac{1}{\ln(2)},$$

which completes the proof. □

Proof of Lemma 24, page 92

Lemma. *Consider a resource e, a player i with flow f_e^i and a set T of k players with total weight F_e^T, where $f_e^t = \frac{F_e^T}{k}$ for $t \in T$. Assume that the set of players $\{i\} \cup T'$, for some $T' \subseteq T$, is using a resource where players' cost shares are computed by weighted Shapley values with sampling weights $\lambda_z(f_e^z) = (f_e^z)^\gamma$, for each player z. Let $j \in T'$. Then (a) for $\gamma > 0$*

$$\lim_{k\to\infty} \chi_{ie} = C_e\left(|T'| \cdot \frac{F_e^T}{k} + f_e^i\right) - C_e\left(|T'| \cdot f_e^j\right), \quad (58)$$

$$\lim_{k\to\infty} \chi_{je} = \frac{1}{|T'|} \cdot C_e\left(|T'| \cdot \frac{F_e^T}{k}\right), \quad \text{if } \; |T'| \neq \emptyset, \quad (59)$$

and, (b) for $\gamma < 0$

$$\lim_{k\to\infty} \chi_{ie} = C_e(f_e^i), \tag{60}$$

$$\lim_{k\to\infty} \chi_{je} = \frac{1}{|T'|} \cdot \left(C_e \left(|T'| \cdot \frac{F_e^T}{k} + f_e^i \right) - C_e\left(f_e^i\right) \right), \quad \textit{if} \ \ |T'| \neq \emptyset. \tag{61}$$

Proof. First, notice that if $|T'| = \emptyset$, equations (58) and (60) of the lemma follow immediately, since only player i uses the resource. We now prove the lemma for the case where $|T'| \neq 0$.

(a) Assume that the sampling weights are given by $\lambda_z(f^z) = (f^z)^\gamma$ for $\gamma > 0$ and let $k \to \infty$. We show that player i is the last player who enters the resource with probability $1 - o(1)$, i.e., she is among the first $\delta \cdot (|T'| + 1)$ players being drawn by the sampling procedure, for any arbitrary small $\delta > 0$. Consider now the probability p that player i is not among the first $\delta \cdot (|T'|+1)$ drawn players. Then p is upper bounded by the probability that player i is not drawn among everyone in set $\{i\} \cup T'$ for $\delta \cdot (|T'| + 1)$ times,

$$p \leq \left(1 - \frac{f_e^i}{f_e^i + |T'| \cdot \left(\frac{F_e^T}{k}\right)^\gamma} \right)^{\delta\cdot(|T'|+1)}. \tag{62}$$

Let $\beta = \frac{|T'|}{|T|} = \frac{|T'|}{k}$ where $\beta \in (0, 1]$. By substituting in (62), we have

$$\left(1 - \frac{f_e^i}{f_e^i + \beta \cdot k \cdot \left(\frac{F_e^T}{k}\right)^\gamma} \right)^{\delta\cdot(\beta\cdot k+1)} = \left(1 - \frac{f_e^i}{f_e^i + \beta \cdot k^{1-\gamma} \cdot (F_e^T)^\gamma} \right)^{\delta\cdot(\beta\cdot k+1)}. \tag{63}$$

For $\gamma \geq 1$, we have

$$(63) \leq \left(1 - \frac{f_e^i}{f_e^i + (F_e^T)^\gamma} \right)^{\delta\cdot(\beta\cdot k+1)}$$

which goes to 0 as $k \to \infty$. Since $(1 - x) \leq e^{-x}$, we have for all $0 < \gamma < 1$,

$$\begin{aligned}(63) &\leq \exp\left(-\delta \cdot \beta \cdot k \cdot \frac{f_e^i}{f_e^i + \beta \cdot k^{1-\gamma} \cdot (F_e^T)^\gamma} \right) \\ &= \exp\left(-\delta \cdot \beta \cdot \frac{f_e^i}{\frac{f_e^i}{k} + \beta \cdot \left(\frac{F_e^T}{k}\right)^\gamma} \right),\end{aligned}$$

which also goes to 0 as $k \to \infty$. Therefore probability p is upper bounded by an arbitrarily small ϵ.

According to the weighted Shapley value method, the cost share of a player equals the expected marginal contribution she causes to the resource cost. Thus her cost share is affected by any player who enters resource before her. For any small $\delta > 0$, player i is with probability $1 - o(1)$ among the last $\delta \cdot (|T'| + 1)$

players who enter the resource. Therefore her cost share is affected by the players who enter the resource earlier. Thus player i incurs a cost of

$$\lim_{k\to\infty} \chi_{ie} = C_e(|T'| \cdot f_e^j + f_e^i) - C_e(|T'| \cdot f_e^j)$$
$$= C_e\left(|T'| \cdot \frac{F_e^T}{k} + f_e^i\right) - C_e\left(|T'| \cdot \frac{F_e^T}{k}\right).$$

In contrast, the cost of each player $j \in T'$ is not affected by the flow of player i. Therefore for each $j \in T'$, we have

$$\lim_{k\to\infty} \chi_{je} = \frac{1}{|T'|} \cdot C_e(|T'| \cdot f_e^j) = \frac{1}{|T'|} \cdot C_e\left(|T'| \cdot \frac{F_e^T}{k}\right)$$

which completes the first part of the proof.

(b) Assume now that the sampling weights are given by $\lambda(f^z) = (f^z)^\gamma$ for $\gamma < 0$ and let $k \to \infty$. We prove that, for any $\delta > 0$, player i is, with probability $1 - o(1)$, among the first $\delta \cdot (|T'| + 1)$ players entering the resource. This probability equals the probability that player i is not drawn for the first $(1 - \delta) \cdot (|T'| + 1)$ sampling rounds. The following formula gives the probability that a player i is not drawn for the first q sampling rounds:

$$\prod_{r=1}^{q} \left(1 - \frac{f_e^i}{f_e^i + (|T'| - (r-1)) \cdot \left(\frac{F_e^T}{k}\right)^\gamma}\right). \tag{64}$$

The probability of a player i being drawn increases with the number of sampling rounds. This implies that the probability of player i not being drawn becomes the smallest in the last sampling round. That is, when $r = q = (1-\delta)\cdot(|T'|+1)$, we get the smallest term of (64). Thus we can lower bound (64) by

$$\left(1 - \frac{f_e^i}{f_e^i + (\delta \cdot |T'| + 1) \cdot \left(\frac{F_e^T}{k}\right)^\gamma}\right)^{(1-\delta)\cdot(|T'|+1)}. \tag{65}$$

Define $\beta = \frac{|T'|}{|T|} = \frac{|T'|}{k}$ where $\beta \in (0, 1]$. By substituting in (65), we have

$$\left(1 - \frac{f_e^i}{f_e^i + (\delta \cdot \beta \cdot k + 1) \cdot \left(\frac{F_e^T}{k}\right)^\gamma}\right)^{(1-\delta)\cdot(\beta\cdot k+1)}$$

which is lower bounded by

$$\left(1 - \frac{f_e^i}{f_e^i + \delta \cdot \beta \cdot k^{1-\gamma} \cdot (F_e^T)^\gamma}\right)^{2\cdot\beta\cdot k} \tag{66}$$

since $\beta \geq \frac{1}{k}$ (due to $|T'| \neq \emptyset$ in this case). By letting $k \to \infty$, we get that

$$\lim_{k\to\infty} (66) \geq \lim_{k\to\infty} \left(1 - \frac{f_e^i}{\delta \cdot \beta \cdot k^{1-\gamma} \cdot (F_e^T)^\gamma}\right)^{2\cdot\beta\cdot k^{1-\gamma}\cdot k^\gamma}. \tag{67}$$

To simplify, let $x = \frac{f_e^i}{\delta \cdot (F_e^T)^\gamma}$. Then (67) equals to

$$\lim_{k\to\infty}\left(\left(1-\frac{x}{\beta\cdot k^{1-\gamma}}\right)^{\beta\cdot k^{1-\gamma}-1}\cdot\left(1-\frac{x}{\beta\cdot k^{1-\gamma}}\right)\right)^{2\cdot k^{\gamma}}. \tag{68}$$

Since $(1-\frac{x}{n})^{n-1} \geq e^{-x}$ for any positive x and n, we lowerbound (68) by

$$\lim_{k\to\infty}\left(e^{-x}\cdot\left(1-\frac{x}{\beta\cdot k^{1-\gamma}}\right)\right)^{2\cdot k^{\gamma}}$$

which equals to 1 for any $\gamma < 0$.

As we mention in part (a) of the proof, the cost share of a player under the weighted Shapley value method equals to the marginal contribution she causes to the resource cost. As we proved, player i is the first player who enters the resource with probability $1-o(1)$. Therefore her cost share is not affected by any player who enters after her. Even if some of the players in T' are introduced before player i in the resource, they will infinitesimally affect player i's cost. This is due to the fact that $f_e^j = \frac{F_e^T}{k} \to 0$ for any player $j \in T'$, since $k \to \infty$. Therefore the cost share of player i is given by

$$\chi_{ie} = C_e(f_e^i)$$

while the cost share of any player $j \in T'$ is given by

$$\begin{aligned}\lim_{k\to\infty}\chi_{je} &= \frac{1}{|T'|}\cdot\left(C_e\left(f_e^i+|T'|\cdot f_e^j\right)-C_e\left(f_e^i\right)\right)\\ &= \frac{1}{|T'|}\cdot\left(C_e\left(f_e^i+|T'|\cdot\frac{F_e^T}{k}\right)-C_e\left(f_e^i\right)\right),\end{aligned}$$

since they enter resource after player i, which completes the proof of the Lemma 24. □

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