Heterogeneous stochastic scalar conservation laws with non-homogeneous Dirichlet boundary conditions

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Abstract. We introduce a notion of stochastic entropy solutions for heterogeneous scalar conservation laws with multiplicative noise on a bounded domain with non-homogeneous boundary condition. Using the concept of measure-valued solutions and Kruzhkov's semi-entropy formulations, we show the existence and uniqueness of stochastic entropy solutions. Moreover, we establish an explicit estimate for the continuous dependence of stochastic entropy solutions on the flux function and the random source function.

Keywords: Scalar conservation law; entropy solutions; Itô's formula.

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1 Introduction

Fix $N \in \mathbb{N}$, we let D be a bounded open set in $\mathbb{R}^N$ with boundary ∂D in which we assume the boundary ∂D is Lipschitz in case the space dimension $N > 1$. Let $T > 0$ be arbitrarily fixed. Set $Q = (0,T) \times D$ and $\Sigma = (0,T) \times \partial D$. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ be a given probability set-up. In this paper, we are interested in the first order stochastic conservation laws driven by a multiplicative noise of the following type

$$du - [div(f(t,x,u)) - g(t,x,u)]dt = h(u)dw(t), \quad \text{in } \Omega \times Q, \tag{1.1}$$

with initial condition

$$u(0,\cdot) = u_0(\cdot), \quad \text{in } D, \tag{1.2}$$

and boundary condition

$$u = a, \quad \text{on } \Sigma, \tag{1.3}$$

for a random scalar-valued function $u : (\omega, t, x) \in \Omega \times [0,T] \times D \mapsto u(\omega, t, x) =: u(t,x) \in \mathbb{R}$, where $f = (f_1, ..., f_N) : [0,T] \times D \times \mathbb{R} \to \mathbb{R}^N$ is a differentiable vector field standing for the flux, $g : [0,T] \times D \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are measurable, and $w = \{w(t)\}_{0 \le t \le T}$ is a standard one-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \in [0,T]})$. The initial data $u_0 : D \subset \mathbb{R}^N \to \mathbb{R}$ will be specified later and the boundary data $a : \Sigma \to \mathbb{R}$ is supposed to be measurable.

When $f(t,x,u) = f(u)$, the problem (1.1)-(1.3) is studied by Kobayasi-Noboriguchi [21]. By introducing a notion of kinetic formulations in which the kinetic defect measures on the boundary of domain are turncated, they obtained the well-posedness of (1.1)-(1.3). Lv-Wu [29] revisited the problem (1.1)-(1.3) and obtained the existence and uniqueness of stochastic entropy solution by using the concept of measure-valued solutions and Kruzhkov's semi-entropy formulations.

When $h = 0$ and $f(t,x,u) = f(u)$, the problem (1.1)-(1.3) is well studied by many authors, see [1, 32] for example. In paper [32], the authors studied the problem (1.1)-(1.3) in L^1-setting. In order to deal with unbounded solutions, they defined a notion of renormalized entropy solution which generalizes the definition of entropy solutions introduced by Otto in [31] in the L^∞ frame work. They have proved existence and uniqueness of such generalized solution in the case when f is locally Lipschitz and the boundary data a verifies the following condition: $f_{max}(a) \in L^1(\Sigma)$, where f_{max} is the "maximal effective flux" defined by

$$f_{max}(s) = \{\sup |f(t)|, \quad t \in [-s^-, s^+]\}.$$

They gave an example to illustrate that the assumption $a \in L^1(\Sigma)$ is not enough in order to prove a priori estimates in $L^1(Q)$, and that the assumption should be $f_{max}(a) \in L^1(\Sigma)$. Ammar et al. [1] revisited the problem (1.1)-(1.3) with $h = 0$ and introduced a notion of entropy solution of (1.1)-(1.3). Following [1], an entropy solution of (1.1)-(1.3) is a function $u \in L^\infty(Q)$ satisfying

$$\begin{aligned} -\int_\Sigma \xi \omega^+(x,k,a(t,x)) &\le \int_Q \big[(u-k)^+ \xi_t - \chi_{u>k}(f(u)-f(k)) \cdot \nabla \xi\big] \\ &\quad + \int_D (u_0 - k)^+ \xi(0,\cdot) \quad \text{and} \end{aligned} \tag{1.4}$$

$$\begin{aligned} -\int_\Sigma \xi \omega^-(x,k,a(t,x)) &\le \int_Q \big[(k-u)^+ \xi_t - \chi_{k>u}(f(k)-f(u)) \cdot \nabla \xi\big] \\ &\quad + \int_D (k - u_0)^+ \xi(0,\cdot) \end{aligned} \tag{1.5}$$

for any $\xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$, $\xi \ge 0$ and for all $k \in \mathbb{R}$, where

$$\omega^+(x,k,a) := \max_{k \le r, s \le a \vee k} |(f(r) - f(s)) \cdot \vec{n}(x)|$$
$$\omega^-(x,k,a) := \max_{a \wedge k \le r, s \le k} |(f(r) - f(s)) \cdot \vec{n}(x)|$$

for any $k \in \mathbb{R}$, a.e. $x \in \partial D$, and $\vec{n}$ denoting the unit outer normal to ∂D. Here and in what follows, $a \wedge k := \min\{a,k\}$ and $a \vee k := \max\{a,k\}$. It is remarked that the above definition of entropy solution is a natural extension of the definition of that given by Otto [31].

When $h = 0$, the problem (1.1)-(1.3) is considered by Martin [30]. Using Kruzkov's semi-entropy formulations, they defined a weak entropy solution and obtained an existence and uniqueness result in L^∞ setting, where the solution u satisfies the following entropy inequality

$$\begin{aligned} 0 \ \leq\ & \int_D (u_0-k)^{\pm}\xi(0,x)dx - \int_Q sgn_{\pm}(u-k)[\nabla\cdot f(t,x,k)+g(t,x,u)]\xi dxdt \\ & +\int_Q (u-k)^{\pm}\partial_t\xi + sgn_{\pm}(u-k)(f(t,x,u)-f(t,x,k))\cdot\nabla\xi dxdt \\ & +L_f\int_\Sigma (a-k)^{\pm}\xi dSdt, \quad \forall\xi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N), \forall k\in\mathbb{R}, \end{aligned} \tag{1.6}$$

and L_f is the Lipschitz constant of flux function f. The Cauchy problem of (1.1) is well studied by many authors [11, 22, 23].

To add a stochastic forcing $h(u)dw(t)$ is natural for applications, which appears in wide variety of field as physics, engineering, biology and so on. The Cauchy problem of equation (1.1) with additive noise has been studied in [20]. J. U. Kim [20] proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution. A Kruzhkov-typy method was used to prove the uniqueness. Vallet-Wittbold [33] extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. By using vanishing viscosity method, Young measure techniques and Kruzhkov doubling variables technique, they proved the existence and uniqueness of the stochastic entropy solution.

Concerning multiplicative noise, for Cauchy problem, Feng-Nualart [14] introduced a notion of strong entropy solution in order to prove the uniqueness for the entropy solution. Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solution only in $1D$ case. Chen et al. [9] proved that the multi-dimensional stochastic problem is well-posedness by using a uniform spatial BV-bound. Following the idea of [14, 9], Lv et al. [27] considered the Cauchy problem of stochastic nonlocal conservation law. Bauzet et al.[2] proved a result of existence and uniqueness of the weak measure-valued entropy solution to the multi-dimensional Cauchy problem. Recently, Friz and Gess [15] considered the stochastic scalar conservation laws driven by rough paths.

Using a kinetic formulation, Debussche-Vovelle [12] obtained a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus, (also see [21, 18]).

Just recently, Bauzet et al. [3] studied the problem (1.1)-(1.3) with $f(t,x,u)=f(u)$, $g=0$ and $a=0$. Under the assumptions that the flux function f and h satisfy the global Lipschitz condition, they obtained the existence and uniqueness of measure-valued solution to problem (1.1)-(1.3) with $f(t,x,u)=f(u)$, $g=0$ and $a=0$. Lv et al. [28] extended the result of [3] to the stochastic nonlocal conservation law.

Cautious remarks: we give the following reasons to interpret why we write this paper.

1. The model (1.1)-(1.3) is a general model, which has not been studied so far. Due to the nonlinear terms f, g depending on the time t and the space x, we will define a new stochastic entropy solution, which coincides with the earlier entropy solution (included the deterministic case), see section 2 for more details. Moreover, we obtain the existence of stochastic entropy solution in $L^p\cap BV$, $p\geq 2$.

2. The proof of the uniqueness of stochastic entropy solution in this paper is different from the earlier results [2, 3, 9, 14, 29, 33] because the flux function depends on the space x, see section 4 for more details. The trick used here is from the fact that $|u-v|=|v-u|$. We can see some difference between the deterministic case and the stochastic case, see Remark 4.2 for details.

3. We remove the assumption "flux function f satisfies Lipschitz condition" and only assume that the flux function and its derivative with respect to x have at most polynomial growth. It is worthing noting that the Lipschitz condition is corresponding to L^2-solution, and the polynomial

growth is corresponding to L^p-solution. Thus the definition of stochastic entropy solution is different from that in [2, 3]. Furthermore we want to study the continuous dependence on flux function, we need additional assumptions, see section 2.

4. The earlier results concerning with stochastic law on bounded domain are only well-posedness. In this paper, we are also interested in the continuous dependence on flux function, nonlinear terms and noise term. When $f(t,x,u) = f(u)$ and $g = 0$, the continuous dependence estimate of Cauchy problem (1.1) was obtained by Chen et al. [9]. Relevant continuous dependence results for deterministic conservation laws have been solved in [6, 25] and in [10] for strong degenerate parabolic equations, see also [8, 19]. Just recently, Biswas et al. [5] considered the continuous dependence estimate for conservation laws with Lévy noise.

As an extension, we propose in this paper to prove a result of existence, uniqueness and continuous dependence estimate of stochastic entropy solution to the initial boundary value problem (1.1)-(1.3). A method of artificial viscosity is proposed to prove the existence of a solution. The compactness properties used are based on the theory of Young measures and on measure-valued solutions [7, 34]. An approximation adaptation of the Kruzhkov's doubling variables is proposed to prove the uniqueness of the measure-valued entropy solution. Using bounded variation (BV) estimates for vanishing viscosity approximations, we derive an explicit continuous dependence estimate on the flux function, nonlinear term and noise term.

The paper is organized as follows. In section 2, we introduce the notion of stochastic entropy solution for (1.1)-(1.3) and state out the main results. In section 3, a priori estimate and the existence of a measure-valued entropy solution for (1.1)-(1.3) is proved via a vanishing viscosity approximation. Section 4 is devoted to the proof of uniqueness. In section 5, continuous dependence estimates are obtained.

Before ending up this section, we introduce some notations.

Notations. In general, if $G \subset \mathbb{R}^N$, $\mathcal{D}(G)$ denotes the restriction to G of $\mathcal{D}(\mathbb{R}^N)$ functions u such that $support(u) \cap G$ is compact. Then $\mathcal{D}^+(G)$ will denote the subset of non-negative elements of $\mathcal{D}(G)$. $\|\cdot\|_{BV(D)}$ denotes the bounded variation on domain D. L_f denotes the Lipschitz constant of the function f.

For a given separable Banach space X, we denote by $N_w^2(0,T,X)$ the space of the predictable X-valued processes. This space is the space $L^2((0,T)\times\Omega, X)$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (i.e. the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $(s,t)\times A$ for any $A \in \mathcal{F}_s$).

Denote $\mathcal{E}^+$ as the set of non-negative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \to x^+$ such that $\eta(x) = 0$ if $x \leq 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then η'' has a compact support and η and η' are Lipschitz-continuous functions. $\mathcal{E}^-$ denotes the set $\{\breve{\eta} := \eta(-\cdot), \eta \in \mathcal{E}^+\}$; and for the definition of the entropy inequality. Then, for convenience, denote

$$sgn_0^+(x) = 1 \text{ if } x > 0 \text{ and } 0 \text{ else}; \quad sgn_0^-(x) = -sgn_0^+(-x); \quad sgn_0 = sgn_0^+ + sgn_0^-,$$
$$F(a,b) = sgn_0(a-b)[f(t,x,a) - f(t,x,b)]; F^{+(-)}(a,b) = sgn_0^{+(-)}(a-b)[f(t,x,a) - f(t,x,b)],$$
$$\text{and for any } \eta \in \mathcal{E}^+ \cup \mathcal{E}^-, \quad F^\eta(a,b) = \int_b^a \eta'(\sigma - b)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma.$$

2 Entropy solution

The aim of this section is to give a definition of stochastic entropy solutions. We study certain properties implicitly satisfied by such a solution, and then we present our main result of the paper.

Assume that for any positive ε, u_ε is the solution of the following stochastic nonlinear parabolic problem

$$\begin{cases} du_\varepsilon - [\varepsilon\Delta u_\varepsilon + div(f(t,x,u_\varepsilon)) - g(t,x,u_\varepsilon)]dt = h(u_\varepsilon)dw(t) & \text{in } Q, \\ u_\varepsilon(0,x) = u_{0\varepsilon}(x) & \text{in } D, \\ u_\varepsilon = a_\varepsilon & \text{on } \Sigma, \end{cases} \tag{2.1}$$

where $u_{0\varepsilon}$ and a_ε satisfy the compatibility condition on $\bar{\Sigma} \cap \bar{Q}$. In particular, $u_{0\varepsilon}$ and a_ε should be a restriction on the sets $\{0\} \times D$ and Σ, respectively. It follows from [13, Theorem 2.7] that the solution u_ε of (2.1) with $a_\varepsilon = 0$ belongs to $L^m(\Omega, C^{2+\iota}\bar{Q})$, where $m \geq 2$ and $0 < \iota < 1$. Then by using the technique of [24], we deduce that the solution u_ε of (2.1) also belongs to $L^m(\Omega, C^{2+\iota}\bar{Q})$, see [24, Remark 5.1.14].

In order to propose an entropy formula, let us analyze the viscous parabolic case. For this, we consider $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$, k a real number, and $\eta \in \mathcal{E}$.

Since $\eta(u_\varepsilon - k)\varphi \in L^2(0,T; H^1(D))$ a.s., it is possible to apply Itô's formula to the operator $\Psi(t, u_\varepsilon) := \int_D \eta(u_\varepsilon - k)\varphi dx$ and thus we get

$$\begin{aligned} 0 \leq\ & \int_D \eta(u_\varepsilon(T) - k)\varphi(T)dx \\ =\ & \int_D \eta(u_{0\varepsilon} - k)\varphi(0)dx + \int_Q \eta(u_\varepsilon - k)\partial_t\varphi dxdt \\ & + \int_Q \eta'(u_\varepsilon - k)[\varepsilon\Delta u_\varepsilon + div(f(t,x,u_\varepsilon)) - g(t,x,u_\varepsilon)]\varphi dxdt \\ & + \int_Q \eta'(u_\varepsilon - k)h(u_\varepsilon)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u_\varepsilon - k)h^2(u_\varepsilon)\varphi dxdt \\ =\ & \int_D \eta(u_{0\varepsilon} - k)\varphi(0)dx + \int_Q \eta(u_\varepsilon - k)\partial_t\varphi dxdt - \int_Q \eta'(u_\varepsilon - k)g(t,x,u_\varepsilon)\varphi dxdt \\ & + \int_Q \eta'(u_\varepsilon - k)h(u_\varepsilon)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u_\varepsilon - k)h^2(u_\varepsilon)\varphi dxdt \\ & -\varepsilon\int_Q \eta'(u_\varepsilon - k)\nabla u_\varepsilon \cdot \nabla\varphi dxdt - \int_Q \eta'(u_\varepsilon - k)f(t,x,u_\varepsilon) \cdot \nabla\varphi dxdt \\ & -\varepsilon\int_Q \eta''(u_\varepsilon - k)\varphi|\nabla u_\varepsilon|^2 dxdt - \int_Q \eta''(u_\varepsilon - k)\varphi f(t,x,u_\varepsilon) \cdot \nabla u_\varepsilon dxdt \\ & +\varepsilon\int_\Sigma \eta'(u_\varepsilon - k)\varphi\nabla u_\varepsilon \cdot \vec{n}(x)dxdt + \int_\Sigma \eta'(a_\varepsilon - k)\varphi f(t,x,a_\varepsilon) \cdot \vec{n}(x)dxdt. \end{aligned} \tag{2.2}$$

Since the support of η'' is compact, for any $i = 1, \cdots, N$, $\mathbb{R} \ni r \mapsto \eta''(r-k)f_i(t,x,r)$ is a bounded continuous function uniformly in (t,x) (Here we assume that f_i is a continuous function and $f_i(t,x,0) = 0$). Then, by using the chain-rule Sobolev functions and integrating by part, we have

$$
\begin{aligned}
& -\int_Q \eta'(u_\varepsilon-k)f(t,x,u_\varepsilon)\cdot\nabla\varphi dxdt-\int_Q \eta''(u_\varepsilon-k)\varphi f(t,x,u_\varepsilon)\cdot\nabla u_\varepsilon dxdt\\
= & -\int_Q \eta'(u_\varepsilon-k)f(t,x,u_\varepsilon)\cdot\nabla\varphi dxdt-\int_Q \varphi div\left(\int_k^{u_\varepsilon}\eta''(\sigma-k)f(t,x,\sigma)d\sigma\right)dxdt+\mathcal{A}\\
= & \int_Q \nabla\varphi\left(\int_k^{u_\varepsilon}\eta''(\sigma-k)f(t,x,\sigma)d\sigma-\eta'(u_\varepsilon-k)f(t,x,u_\varepsilon)\right)dxdt\\
& -\int_\Sigma\varphi\int_k^{u_\varepsilon}\eta''(\sigma-k)f(t,x,\sigma)d\sigma\cdot\vec{n}(x)dSdt+\mathcal{A}\\
= & -\int_Q \nabla\varphi\left(\int_k^{u_\varepsilon}\eta'(\sigma-k)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma\right)dxdt-\int_\Sigma\varphi\eta'(u_\varepsilon-k)f(t,x,u_\varepsilon)\cdot\vec{n}(x)dSdt\\
& +\int_\Sigma\varphi\int_k^{u_\varepsilon}\eta'(\sigma-k)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma\cdot\vec{n}(x)dSdt+\mathcal{A}\\
= & -\int_Q F^\eta(u_\varepsilon,k)\nabla\varphi dxdt-\int_\Sigma\varphi\eta'(u_\varepsilon-k)f(t,x,u_\varepsilon)\cdot\vec{n}(x)dSdt\\
& +\int_\Sigma\varphi\int_k^{u_\varepsilon}\eta'(\sigma-k)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma\cdot\vec{n}(x)dSdt+\mathcal{A},
\end{aligned}
\tag{2.3}
$$

where we have used $\eta'(0)=0$, and

$$
\mathcal{A}=\int_Q\varphi\left(\int_k^{u_\varepsilon}\eta''(\sigma-k)\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}(t,x,\sigma)d\sigma\right)dxdt.
$$

Thus we get

$$
\begin{aligned}
0\ \leq\ & \int_D\eta(u_{0\varepsilon}-k)\varphi(0)dx+\int_Q\eta(u_\varepsilon-k)\partial_t\varphi dxdt-\int_Q\eta'(u_\varepsilon-k)g(t,x,u_\varepsilon)\varphi dxdt\\
& +\int_Q\eta'(u_\varepsilon-k)h(u_\varepsilon)\varphi dxdw(t)+\frac{1}{2}\int_Q\eta''(u_\varepsilon-k)h^2(u_\varepsilon)\varphi dxdt\\
& -\varepsilon\int_Q\eta'(u_\varepsilon-k)\nabla u_\varepsilon\cdot\nabla\varphi dxdt-\int_Q F^\eta(u_\varepsilon,k)\nabla\varphi dxdt+\mathcal{A}\\
& +\varepsilon\int_\Sigma\eta'(u_\varepsilon-k)\varphi\nabla u_\varepsilon\cdot\vec{n}(x)dxdt+\int_\Sigma\varphi\int_k^{u_\varepsilon}\eta'(\sigma-k)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma\cdot\vec{n}(x)dSdt.
\end{aligned}
\tag{2.4}
$$

Now, let us assume that as ε tends to 0, the approximation solution u_ε converges in an appropriate sense to a function $u\in N_w^2(0,T;L^2(D))$ such that for any dP-measurable set A

$$
\begin{aligned}
& \varepsilon\mathbb{E}\int_Q 1_A\eta'(u_\varepsilon-k)\nabla u_\varepsilon\cdot\nabla\varphi dxdt\to 0,\qquad\text{as }\varepsilon\to 0,\\
& \varepsilon\mathbb{E}\int_\Sigma\eta'(u_\varepsilon-k)\varphi\nabla u_\varepsilon\cdot\vec{n}(x)dxdt\to 0,\qquad\text{as }\varepsilon\to 0,
\end{aligned}
$$

where we have used $\varepsilon\mathbb{E}\|\nabla u_\varepsilon\|^p_{L^2(\bar{D})}\leq C$, $p\geq 2$ and C does not depend on ε. Since $\eta'(u)=1$ if $u>\delta$ and $\eta'(u)=0$ if $u\leq 0$, and $f\in C^2$, we can assume that f' (here for simplicity, we assume that the function f is a scalar function, and if f is a vector function, we can deal with the component similaryly) keeps sign in $(k,k+\delta)$ for any $k\in\mathbb{R}$. Note that $\eta''\geq 0$. If $\frac{\partial f}{\partial\sigma}\geq 0$ in $(k,k+\delta)$ uniformly with respect to t and x, we have

$$\begin{aligned}\int_k^{u_\varepsilon} \eta'(\sigma-k)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma &= \int_{k+\delta}^{u_\varepsilon\vee(k+\delta)} \frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma + \int_k^{k+\delta} \eta'(\sigma-k)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma \\ &\le \int_{k+\delta}^{u_\varepsilon\vee(k+\delta)} \frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma + \eta'(\delta)\int_k^{k+\delta} \frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma \\ &= (u_\varepsilon-(k+\delta))^+[f(t,x,u_\varepsilon)-f(t,x,k+\delta)] \\ &\quad +\eta'(\delta)[f(t,x,k+\delta)-f(t,x,k)] \\ &\le \eta'(u_\varepsilon-k)[f(t,x,u_\varepsilon)-f(t,x,k)].\end{aligned}$$

If $\frac{\partial f}{\partial \sigma} \le 0$ in $(k, k+\delta)$ uniformly with respect to t and x, we have

$$\begin{aligned}\int_k^{u_\varepsilon} \eta'(\sigma-k)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma &= \int_{k+\delta}^{u_\varepsilon\vee(k+\delta)} \frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma + \int_k^{k+\delta} \eta'(\sigma-k)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma \\ &\le \int_{k+\delta}^{u_\varepsilon\vee(k+\delta)} \frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma \\ &\le (u_\varepsilon-(k+\delta))^+[f(t,x,u_\varepsilon)-f(t,x,k+\delta)],\end{aligned}$$

In order to have the same estimate for the above two inequality, we will take maximum. Combining the above discussion, we get

$$\begin{aligned}&\Big|\int_\Sigma \varphi \int_k^{u_\varepsilon} \eta'(\sigma-k)\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma\cdot\vec{n}(x)dSdt\Big| \\ \le\ & \int_\Sigma \varphi \int_k^{u_\varepsilon} \eta'(\sigma-k)|\frac{\partial f}{\partial \sigma}(t,x,\sigma)d\sigma\cdot\vec{n}(x)|dSdt \\ \le\ & \int_\Sigma \eta'(u_\varepsilon-k)\varphi\hat{\omega}^+(x,k,u_\varepsilon)dxdt,\end{aligned}$$

where

$$\hat{\omega}^+(x,k,a) := \max_{k\le r,s\le a\vee k} |[f(t,x,r)-f(t,x,s)]\cdot\vec{n}(x)|.$$

Here we can see how we can define the boundary effect. We also remark that it coincides with those in [1, 30, 32]. Lastly, we consider $\mathcal{A}$. Integrating by part, we have

$$\begin{aligned}\mathcal{A} &= \int_Q \varphi\left(\int_k^{u_\varepsilon} \eta''(\sigma-k)\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t,x,\sigma)d\sigma\right)dxdt \\ &= -\int_Q \varphi\eta'(u_\varepsilon-k)\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t,x,u_\varepsilon)dxdt \\ &\quad +\int_Q \varphi\left(\int_k^{u_\varepsilon} \eta'(\sigma-k)\sum_{i=1}^N \frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right)dxdt.\end{aligned}$$

Then we may pass to the limit in (2.4) and obtain a family of entropy inequalities satisfied by the limit of u. This observation motivates the definition of entropy solution for the stochastic conservation law (1.1)-(1.3).

Define

$$\hat{\omega}^-(x,k,a) := \max_{a\wedge k\le r,s\le k} |[f(t,x,r)-f(t,x,s)]\cdot\vec{n}(x)|.$$

For convenience, for any function u of $N_w^2(0,T;L^2(D))$, any real number k and any regular function $\eta \in \mathcal{E}^+$, denote dP-a.s. in Ω by $\mu_{\eta,k}$, the distribution in D defined by

$$\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) \;=\;& \int_D \eta(u_0-k)\varphi(0)dx + \int_Q \eta(u-k)\partial_t\varphi - F^{\eta}(u,k)\nabla\varphi dxdt \\
& - \int_Q \varphi\eta'(u-k)\left[\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t,x,u) + g(t,x,u)\right] dxdt \\
& + \int_Q \varphi\left(\int_k^u \eta'(\sigma-k)\sum_{i=1}^N \frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right) dxdt \\
& + \int_Q \eta'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\varphi dxdt \\
& + \int_\Sigma \eta'(a-k)\varphi\hat{\omega}^+(x,k,a(t,x))dxdt; \\
\varphi \mapsto \mu_{\breve{\eta},k}(\varphi) \;=\;& \int_D \breve{\eta}(u_0-k)\varphi(0)dx + \int_Q \breve{\eta}(u-k)\partial_t\varphi - F^{\breve{\eta}}(u,k)\nabla\varphi dxdt \\
& - \int_Q \varphi\breve{\eta}'(u-k)\left[\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t,x,u) + g(t,x,u)\right] dxdt \\
& + \int_Q \varphi\left(\int_k^u \breve{\eta}'(\sigma-k)\sum_{i=1}^N \frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right) dxdt \\
& + \int_Q \breve{\eta}'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \breve{\eta}''(u-k)h^2(u)\varphi dxdt \\
& + \int_\Sigma \breve{\eta}'(a-k)\varphi\hat{\omega}^-(x,k,a(t,x))dxdt.
\end{aligned}$$

Now we propose the following definition of entropy solution of (1.1)-(1.3).

Definition 2.1 *A function $u \in N_w^2(0,T;L^2(D))$ is an entropy solution of stochastic conservation law (1.1) with the initial condition $u_0 \in L^p(D)$ and boundary condition $a \in C(\Sigma)$, if $u \in L^2(0,T;L^2(\Omega;L^p(D)))$, $p = 2,3,\cdots$, and*

$$\mu_{\eta,k}(\varphi) \geq 0, \qquad \mu_{\breve{\eta},k}(\varphi) \geq 0 \quad dP-a.s.,$$

where $\varphi \in \mathcal{D}^+((0,T\times\mathbb{R}^N))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}^+$ and $\breve{\eta} \in \mathcal{E}^-$.

For technical reasons, we need to consider a generalized notion of entropy solution. In fact, in the first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we will be able to deduce the existence of an entropy solution in the sense of Definition 2.1.

Definition 2.2 *A function u of $N_w^2(0,T;L^2(D\times(0,1))) \cap L^\infty(0,T;L^p(\Omega\times D\times(0,1)))$ is a Young measure-valued solution of stochastic conservation law (1.1) with the initial condition $u_0 \in L^p(D)$, $p = 2,3,\cdots$, and boundary condition $a \in C(\Sigma)$, if*

$$\int_0^1 \mu_{\eta,k}(\varphi)d\alpha \geq 0, \qquad \int_0^1 \mu_{\breve{\eta},k}(\varphi)d\alpha \geq 0 \quad dP-a.s.,$$

where $\varphi \in \mathcal{D}^+((0,T\times\mathbb{R}^N))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}^+$ and $\breve{\eta} \in \mathcal{E}^-$.

Remark 2.1 *1. Note that an entropy solution of (1.1)-(1.3) is a.s. a weak solution, see [29] for more details.*

2. Let $a = 0 = g$ *and* $f(t,x,u) = f(u)$*, then we find* $\mu_{\eta,k}(\varphi)$ *will become the "*$\mu_{\eta,k}(\varphi)$*" in Definition 1 of [3].*

3. Let $h = 0 = g$ *and* $f(t,x,u) = f(u)$*, then* $\mu_{\eta,k}(\varphi) \geq 0$ *and* $\mu_{\breve{\eta},k}(\varphi) \geq 0$ *will coincide with (1.4) and (1.5), respectively. That is to say, letting* $\delta \to 0$*, then* $\mu_{\eta,k}(\varphi) \geq 0$ *will converge to (1.4).*

4.Let $h = 0$*, noting that* $|[f(t,x,r) - f(t,x,s)] \cdot \vec{n}(x)| \leq L_f|r-s| \leq L_f(a-k)$*, then* $\mu_{\eta,k}(\varphi) \geq 0$ *and* $\mu_{\breve{\eta},k}(\varphi) \geq 0$ *will coincide with (1.6).*

Therefore, Definition 2.1 is a natural extension of the definition of entropy solution given by [1, 3, 30].

Throughout this paper, we assume that $f = (f_1, \cdots, f_N)$, $p = 2, 3, \cdots$, and

(H_1): The flux functions f and $\frac{\partial f_k}{\partial x_i}$ $(k, i = 1, \cdots, N)$ have at most polynomial growth w.r.t. u, $g : \mathbb{R}_+ \times \bar{D} \times \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous w.r.t. u uniformly in (t,x), $f \in [C^2([0,T] \times \bar{D} \times \mathbb{R})]^N$ with $f(\cdot,\cdot,0) = 0$ and $g \in C^2([0,T] \times \bar{D} \times \mathbb{R})$ with $g(\cdot,\cdot,0) = 0$;

(H_2): $h : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$;

(H_3) $u_0 \in L^p(D)$ and $a \in C(\Sigma)$ for some $p \geq 2$;

(H_3'): $u_0 \in L^p(D) \cap BV(D)$ and $a \in L^\infty(0,T; C^1(\partial D))$;

(H_4): $\frac{\partial^2 f_i}{\partial x_i \partial x_j}$ and $\frac{\partial g}{\partial x_i}$ have at most polynomial growth w.r.t. u, both f and $\frac{\partial f_i}{\partial x_j}$ satisfy the Lipschitz condition, $i, j = 1, 2, \cdots, N$.

The main result of this paper is as follows

Theorem 2.1 *Under assumptions* $(H_1) - (H_3)$*, there exists a unique measure-valued entropy solution* u *in the sense of Definition 2.2, which is obtained by viscous approximation.*

It is unique entropy solution in the sense of Definition 2.1.

If u_1, u_2 *are entropy solutions of (1.1) corresponding to initial data* u_{01}, $u_{02} \in L^p(D)$ *and the boundary data* a_1, $a_2 \in C(\Sigma)$*, respectively, then for any* $t \in (0,T)$

$$\mathbb{E}\int_D |u_1 - u_2| \leq \int_D |u_{01} - u_{02}| dx + \int_\Sigma \max_{min(a_1,a_2) \leq r,s \leq max(a_1,a_2)} |(f(t,x,r) - f(t,x,s)) \cdot \vec{n}(x)|.$$

The proof of Theorem 2.1 is exactly similar to that in [29] except the uniqueness. In section 4, we will prove the uniqueness. Now we focus on another case, that is, the condition (H_3) is replaced by (H_3'), (H_4).

Theorem 2.2 *(Continuous dependence estimates) Under assumptions* $(H_1), (H_2), (H_3'), (H_4)$*, there exists a unique measure-valued entropy solution* u *as stated in Theorem 2.1. Moreover, the solution satisfies*

$$\mathbb{E}\int_0^1 \left[|u(t,\cdot,\alpha)|_{BV(D)} \right] d\alpha \leq C \left(\|u_0\|^p_{L^p(D)} + |u_0|_{BV(D)} + \|a\|_{L^\infty(0,T;C^1(\partial D))} \right).$$

In addition, suppose $(H_1), (H_2), (H_3'), (H_4)$ *hold for the two given data sets* (u_0, a, f, g, h) *and* $(v_0, \hat{a}, \hat{f}, \hat{g}, \hat{h})$*. Let* v *be the solution to the stochastic parabolic problem (2.1). In addition, we assume that either*

$$u, v \in L^\infty(\Omega \times Q) \quad \text{for any } T > 0,$$

or

$$\frac{\partial^2 f}{\partial u^2},\ \frac{\partial f}{\partial u}-\frac{\partial \hat{f}}{\partial u},\ \frac{\partial f_i}{\partial x_i}-\frac{\partial \hat{f}_i}{\partial x_i},\ g-\hat{g},\ h-\hat{h}\in L^\infty.$$

Then, there is a constant $C_T>0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,x,\beta)|\psi(x)d\alpha d\beta dx\right]\\
\leq\ & C_T\left(\int_D |u_0(x)-v_0(x)|\psi(x)dx+\sqrt{t}\|h-\hat{h}\|_{L^\infty}+\|a-\hat{a}\|_{L^\infty}\right.\\
&\left.+\|g-\hat{g}\|_{L^\infty}+|v_0|_{BV(D)}\|\frac{\partial f}{\partial u}-\frac{\partial \hat{f}}{\partial u}\|_{L^\infty}+\|h-\hat{h}\|_{L^\infty}\right),
\end{aligned}$$

where the constant $C_T>0$ is independent of $|v_0|_{BV(D)}$ and $\psi(x)\in\mathcal{D}^+(\mathbb{R}^N)$ is any function satisfying $|\psi|\leq C_0$ and $|\nabla\psi|\leq C_0\psi$ (about the existence of this ψ, see [9]).

Remark 2.2 *1. In order to consider the continuous dependence on the flux function f, we must prove the bounded variation of u can be controlled by the bounded variation of u_0, see section 5 for details.*

2. Thanks to the uniqueness result, we are able to prove that the measure-valued solution is an entropy solution in the sense of Definition 2.1.

3. We remark that the nonlinear term g satisfying the Lipschitz condition is natural. It follows from [26] that if g satisfies the local Lipschitz condition, then the solution of problem (2.1) maybe blow up in finite time.

3 Existence

In this section, we mainly prove the existence of stochastic entropy solution in $L^p\cap BV$. The aim of this section is to prove the following

Theorem 3.1 *Under assumptions $(H_1),(H_2),(H_3'),(H_4)$, there exists a measure-valued entropy solution in the sense of Definition 2.2 satisfying*

$$\mathbb{E}\int_0^1\left[|u(t,\cdot,\alpha)|_{BV(D)}\right]d\alpha\leq C\left(\|u_0\|_{L^p(D)}^p+|u_0|_{BV(D)}+\|a\|_{L^\infty(0,T;C^1(\partial D))}\right).$$

The technique used here is based on uniform spatial BV and the notion of narrow convergence of Young measure. We first consider the spatial BV-estimate.

Lemma 3.1 *Suppose $(H_1),(H_2),(H_3'),(H_4)$ hold. Let u_ε be the solution of (2.1). Then, for any $t\in(0,T)$, there exists a constant $p\geq 2$ such that*

$$\begin{aligned}
\mathbb{E}\int_D|\nabla u_\varepsilon(t,x)|dx\ &\leq\ C\left(\|u_{0\varepsilon}\|_{L^p(D)}^p+\int_D|\nabla u_{0\varepsilon}|dx+\|a_\varepsilon\|_{L^\infty(0,T;C^1(\partial D))}\right)\\
&\leq\ C\left(\|u_0\|_{L^p(D)}^p+|u_0|_{BV(D)}+\|a\|_{L^\infty(0,T;C^1(\partial D))}\right).
\end{aligned}$$

Proof. We assume that $u_{0\varepsilon}\in C^\infty$, $a_\varepsilon\in C^\infty$ such that

$$\|u_{0\varepsilon}\|_{C^1}\leq|u_0|_{BV(D)},\ \mathbb{E}\|a_\varepsilon\|_{L^\infty(0,T;C^1(\partial D))}\leq\|a\|_{L^\infty(0,T;C^1(\partial D))}.$$

Following [21], we have for every $p \geq 2$

$$\mathbb{E} \sup_{0\leq t\leq T} \|u_\varepsilon(t)\|_{L^p(D)}^p + \varepsilon \int_0^T \int_D |\nabla u_\varepsilon|^2 dxds \leq C \left(\|u_0\|_{L^p(D)}^p + \|a\|_{L^\infty(0,T;C^1(\partial D))} \right). \tag{3.1}$$

Taking the derivative of the first equation to (2.1) with respect to x_i, $1 \leq i \leq N$, we obtain

$$\partial_t u_{i\varepsilon} - \varepsilon \Delta u_{i\varepsilon} - div(\vec{f_i} + \partial_u f(t,x,u_\varepsilon)u_{i\varepsilon}) + g_i + \partial_u g(t,x,u_\varepsilon)u_{i\varepsilon} = h'(u_\varepsilon)u_{i\varepsilon}\partial_t w(t),$$

where $v_i = \frac{\partial v}{\partial x_i}$ for $v = u_\varepsilon, g$. Here $\vec{f_i} = (\frac{\partial f_1}{\partial x_i}, \cdots, \frac{\partial f_N}{\partial x_i})$. Applying Itô formula to $\eta_\delta(u_{i\varepsilon})$ yields

$$\begin{aligned} \partial_t \eta_\delta(u_{i\varepsilon}) &= \eta_\delta'(u_{i\varepsilon})[\varepsilon\Delta u_{i\varepsilon} + div(\vec{f_i} + \partial_u f(t,x,u_\varepsilon)u_{i\varepsilon}) - g_i - \partial_u g(t,x,u_\varepsilon)u_{i\varepsilon}] \\ &\quad + \eta_\delta'(u_{i\varepsilon})h'(u_\varepsilon)u_{i\varepsilon}\partial_t w(t) + \frac{1}{2}\eta_\delta''(u_{i\varepsilon})(h'(u_\varepsilon)u_{i\varepsilon})^2. \end{aligned} \tag{3.2}$$

Due to $\eta_\delta \in \mathcal{E}$, we have

$$\varepsilon \eta_\delta'(u_{i\varepsilon})\Delta u_{i\varepsilon} \leq \varepsilon \Delta \eta_\delta(u_{i\varepsilon}).$$

Integrating (3.2) with respect to x and t, and noting that

$$\int_D \int_0^t \eta_\delta'(u_{i\varepsilon})h'(u_\varepsilon)u_{i\varepsilon}dw(s)dx$$

is a martingale, we get

$$\begin{aligned} &\mathbb{E}\left[\int_D \eta_\delta(u_{i\varepsilon}(t,x))dx\right] - \mathbb{E}\left[\int_D \eta_\delta(u_{i\varepsilon}(0,x))dx\right] \\ \leq\ & \mathbb{E}\int_0^t \int_D \eta_\delta'(u_{i\varepsilon})[div(\vec{f_i} + \partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}) - g_i - \partial_u g(s,x,u_\varepsilon)u_{i\varepsilon}]dxds \\ & + \frac{1}{2}\mathbb{E}\int_0^t \int_D \eta_\delta''(u_{i\varepsilon})(h'(u_\varepsilon)u_{i\varepsilon})^2 + \varepsilon \int_0^t \int_{\partial D} \nabla \eta_\delta(u_{i\varepsilon}) \cdot \vec{n}(x) dSds. \end{aligned}$$

Letting $\delta \to 0$ in above inequality and adding the resulting two inequalities for $\eta_\delta \in \mathcal{E}^+$ and $\eta_\delta \in \mathcal{E}^-$, we have

$$\begin{aligned} &\mathbb{E}\left[\int_D |u_{i\varepsilon}(t,x)|dx\right] \\ \leq\ & \mathbb{E}\int_D |u_{i\varepsilon}(0,x)|dx + \lim_{\delta\to 0}\mathbb{E}\int_0^t \int_D [\eta_\delta'(u_{i\varepsilon}) - \breve{\eta}_\delta'(u_{i\varepsilon})][div(\vec{f_i} + \partial_u f(s,x,u_\varepsilon)u_{i\varepsilon})]dxds \\ & - \lim_{\delta\to 0}\mathbb{E}\int_0^t \int_D [\eta_\delta'(u_{i\varepsilon}) - \breve{\eta}_\delta'(u_{i\varepsilon})][g_i + \partial_u g(s,x,u_\varepsilon)u_{i\varepsilon}]dxds \\ & + \lim_{\delta\to 0}\frac{1}{2}\mathbb{E}\int_0^t \int_D [\eta_\delta''(u_{i\varepsilon}) + \breve{\eta}_\delta''(u_{i\varepsilon})](h'(u_\varepsilon)u_{i\varepsilon})^2 dxds \\ & + \varepsilon \lim_{\delta\to 0}\mathbb{E}\int_0^t \int_{\partial D} \nabla[\eta_\delta(u_{i\varepsilon}) + \breve{\eta}_\delta(u_{i\varepsilon})] \cdot \vec{n}(x) dSds \\ =:\ & \int_D |u_{i\varepsilon}(0,x)|dx + I_1 + \cdots + I_4. \end{aligned}$$

For the term I_1, by using the assumption (H_1), we have

$$\begin{aligned} |I_1| &\leq \lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D div\Big([\eta'_\delta(u_{i\varepsilon})-\breve{\eta}'_\delta(u_{i\varepsilon})][\vec{f_i}+\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}]\Big)dxds\Big| \\ &\quad +\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D [\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot[\vec{f_i}+\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}]dxds\Big| \\ &= \lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_{\partial D}[\eta'_\delta(u_{i\varepsilon})-\breve{\eta}'_\delta(u_{i\varepsilon})][\vec{f_i}+\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon})]\cdot\vec{n}(x)dSds\Big| \\ &\quad +\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D [\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot[\vec{f_i}+\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}]dxds\Big| \\ &\leq C\Big(\|u_0\|^p_{L^p(D)}+\|a\|_{L^\infty(0,T;C^1(\partial D))}\Big) \\ &\quad +\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D [\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot[\vec{f_i}+\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}]dxds\Big|, \end{aligned}$$

where constant C depends on L_f and Σ. Notice that

$$[\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]u_{i\varepsilon}\to 0 \quad \text{as } \delta\to 0$$

for almost everywhere (t,x) almost surely and there exists constant $p\geq 2$ such that

$$\Big|[\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot\partial_u f(t,x,u_\varepsilon)u_{i\varepsilon}\Big|\leq C(|\nabla u_{i\varepsilon}|^2+|u_\varepsilon|^p),$$

where the right-side term of the above inequality is integrable and independent of δ. Thus the dominated convergence theorem implies that

$$\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D [\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot\partial_u f(s,x,u_\varepsilon)u_{i\varepsilon}dxds\Big|=0.$$

By the assumption H_4 and utilising (3.1), there exists a constant p such that

$$\begin{aligned} &\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D [\eta''_\delta(u_{i\varepsilon})+\breve{\eta}''_\delta(u_{i\varepsilon})]\nabla u_{i\varepsilon}\cdot\vec{f_i}dxdt\Big| \\ &= \lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_{\partial D}[\eta'_\delta(u_{i\varepsilon})-\breve{\eta}'_\delta(u_{i\varepsilon})]\vec{f_i}\cdot\vec{n}(x)dSdt\Big| \\ &\quad +\lim_{\delta\to 0}\Big|\mathbb{E}\int_0^t\int_D[\eta'_\delta(u_{i\varepsilon})-\breve{\eta}'_\delta(u_{i\varepsilon})]\left(\sum_{k=1}^N\frac{\partial^2 f_k}{\partial x_i\partial x_k}+\frac{\partial^2 f_k}{\partial x_i\partial u}u_{x_k}\right)dxdt\Big| \\ &\leq C\mathbb{E}\|a\|_{L^\infty(0,T;C^1(\partial D))}+C\mathbb{E}\int_0^t\int_D|u_\varepsilon|^p dxds \\ &\quad +CL_{\partial_{x_i}f}\mathbb{E}\int_0^t\int_D|\nabla u|dxds \\ &\leq C\Big(\|u_0\|^p_{L^p(D)}+\|a\|_{L^\infty(0,T;C^1(\partial D))}\Big)+C\mathbb{E}\int_0^t\int_D|\nabla u(s,x)|dxds, \end{aligned}$$

where the constant C does not depend on ε.

For the term I_2, using the assumption (H_4) and (3.1), there exists a constant $p\geq 2$ such that

$$\begin{aligned} |I_2| &= -\lim_{\delta\to 0}\mathbb{E}\int_0^t\int_D[\eta'_\delta(u_{i\varepsilon})-\breve{\eta}'_\delta(u_{i\varepsilon})][g_i+\partial_u g(s,x,u_\varepsilon)u_{i\varepsilon}]dxds \\ &\leq C\int_0^t\mathbb{E}\int_D|u_\varepsilon|^p dxds+L_g\mathbb{E}\int_0^t\int_D|\nabla u(s,x)|dxds \\ &\leq C\Big(\|u_0\|^p_{L^p(D)}+\|a\|_{L^\infty(0,T;C^1(\partial D))}\Big)+C\mathbb{E}\int_0^t\int_D|\nabla u(s,x)|dxds. \end{aligned}$$

Next we consider the term I_3. By the condition (H_2) and the properties of η_δ, we have

$$\left|[\eta_\delta''(u_{i\varepsilon}) + \breve{\eta}_\delta''(u_{i\varepsilon})](h'(u_\varepsilon)u_{i\varepsilon})^2\right| \le C|u_{i\varepsilon}|1_{|u_{i\varepsilon}|\le\delta} \le C|u_{i\varepsilon}| \in L^1((0,T)\times D).$$

We remark that $|u_{i\varepsilon}|$ is integrable and independent of δ, and $|u_{i\varepsilon}|1_{|u_{i\varepsilon}|\le\delta} \to 0$ as $\delta \to 0$ for almost everywhere (t,x) almost surely. Then the dominated convergence theorem implies $|I_3| = 0$.

For the last term I_4, by the condition (H_3') and the properties of η_δ, we have

$$\varepsilon\mathbb{E}\int_0^t\int_{\partial D}\nabla[\eta_\delta(u_{i\varepsilon}) + \breve{\eta}_\delta(u_{i\varepsilon})]\cdot\vec{n}(x)dSds \le C\|a\|_{L^\infty(0,T;C^1(\partial D))}$$

uniformly $\varepsilon \in (0,1]$.

Combining the above estimates, we have

$$\begin{aligned}&\mathbb{E}\left[\int_D |u_{i\varepsilon}(t,x)|dx\right]\\ \le\ & \int_D |u_{i\varepsilon}(0,x)|dx + C\left(\|u_0\|^p_{L^p(D)} + \|a\|_{L^\infty(0,T;C^1(\partial D))}\right) + C\mathbb{E}\int_0^t\int_D |\nabla u(s,x)|dxds.\end{aligned}$$

Summing up the above inequality w.r.t. i from 1 to N, and using the Gronwall inequality, one can obtain the desired result. This completes the proof. □

Proof of Theorem 3.1 Following [2], there exists a unique solution

$$u(t,x,\alpha) \in N_w^2(0,T;L^2(D\times(0,1))) \cap L^\infty(0,T;L^p(\Omega\times D\times(0,1)))$$

$\forall p \ge 2$. Note that the constant in Lemma 3.1 is independent of ε. Letting $\varepsilon \to 0$, we obtain the inequality in Theorem 3.1 by utilising Young measure convergence theorem. This completes the proof. □

4 Uniqueness

The aim of this section is to show the following

Theorem 4.1 *The solution given by Theorem 2.1 is the unique measure-valued entropy solution in the sense of Definition 2.2.*

The following comparison result plays a crucial role in the proof of Theorem 4.1 and of the continuous dependence estimate.

Lemma 4.1 *Suppose $(H_1)-(H_3)$ hold for the two data sets (u_0,a,f,g,h) and $(v_0,\hat{a},\hat{f},\hat{g},\hat{h})$. Let u be a solution of (1.1)-(1.3) in the sense of Definition 2.2. Let v be the solution to the following stochastic parabolic problem*

$$\begin{cases} dv - [\varepsilon\Delta v + div(\hat{f}(t,y,v)) - \hat{g}(t,y,v)]dt = \hat{h}(v)dw(t) & \text{in } Q,\\ v(0,y) = v_0(y) & \text{in } D,\\ v = \hat{a} & \text{on } \Sigma.\end{cases} \tag{4.1}$$

For $\eta_\delta \in \mathcal{E}$, we introduce the associated entropy fluxes for $u,v\in\mathbb{R}$, respectively, as

$$F^{\eta_\delta}(u,v) = \int_v^u \eta_\delta'(\sigma - v)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma, \quad \hat{F}^{\eta_\delta}(u,v) = \int_v^u \eta_\delta'(\sigma - v)\frac{\partial \hat{f}}{\partial\sigma}(t,x,\sigma)d\sigma.$$

Then, for any $\varphi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$*, the following holds*

$$
\begin{aligned}
0 \ \le\ & \mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\partial_t\varphi d\alpha dydxdt\\
&-\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dydxdt\\
&+\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta''(u(t,x,\alpha)-v(t,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\varphi d\alpha dydxdt\\
&+I^{f,\hat{f}}(\varphi)-I^{\hat{f}}(\varphi)+J^{f,\hat{f}}(\varphi)-J^{g,\hat{g}}(\varphi)+\int_\Sigma\int_D\eta_\delta'(a-\hat{a})\varphi\rho_n(y-x)\omega^+(x,\hat{a},a)\\
&+\int_D\int_D\eta_\delta(u_0-v_0)\varphi(0)\rho_n(y-x)dxdy, \qquad (4.2)
\end{aligned}
$$

where $\rho_n(y-x)$ *will be determined later, and*

$$
\begin{aligned}
I^{f,\hat{f}}(\varphi) &= \mathbb{E}\int_Q\int_D\int_0^1\left(F^{\eta_\delta}(u,v)-\hat{F}^{\breve{\eta}_\delta}(v,u)\right)\varphi\nabla_y\rho_n(y-x)d\alpha dydxdt;\\
I^{\hat{f}}(\varphi) &= \mathbb{E}\int_Q\int_D\int_0^1 \hat{F}^{\breve{\eta}_\delta}(v,u)\rho_n(y-x)\nabla_y\varphi d\alpha dydxdt;\\
J^{f,\hat{f}}(\varphi) &= \mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\\
&\quad\times\left(\int_v^u\eta_\delta'(\sigma-v)\sum_{i=1}^N\frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma-\eta_\delta'(u-v)\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}(t,x,u)\right)d\alpha dydxdt\\
&\quad+\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\\
&\quad\times\left(\int_u^v\eta_\delta'(u-\sigma)\sum_{i=1}^N\frac{\partial^2 \hat{f}_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma-\eta_\delta'(u-v)\sum_{i=1}^N\frac{\partial \hat{f}_i}{\partial x_i}(t,x,v)\right)d\alpha dydxdt;\\
J^{g,\hat{g}}(\varphi) &= \mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi[g(t,x,u)-\hat{g}(t,y,v)]d\alpha dydxdt.
\end{aligned}
$$

Remark 4.1 *Similar to Lemma 4.1, one can prove that for any* $\varphi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$*, the following holds*

$$
\begin{aligned}
0 \ \le\ & \mathbb{E}\int_Q\int_D\int_0^1 \breve{\eta}_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\partial_t\varphi d\alpha dydxdt\\
&-\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \breve{\eta}_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dydxdt\\
&+\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \breve{\eta}_\delta''(u(t,x,\alpha)-v(t,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\varphi d\alpha dydxdt\\
&+I^{f,\hat{f}}(\varphi)-I^{\hat{f}}(\varphi)+J^{f,\hat{f}}(\varphi)-J^{g,\hat{g}}(\varphi)+\int_\Sigma\int_D\breve{\eta}_\delta'(a-\hat{a})\varphi\rho_n(y-x)\omega^-(x,\hat{a},a)\\
&+\int_D\int_D\breve{\eta}_\delta(u_0-v_0)\varphi(0)\rho_n(y-x)dxdy,
\end{aligned}
$$

where η *and* $\breve{\eta}$ *will be replaced by* $\breve{\eta}$ *and* η *in* $I^{f,\hat{f}}(\varphi)$*,* $I^{\hat{f}}(\varphi)$*,* $J^{f,\hat{f}}(\varphi)$ *and* $J^{g,\hat{g}}(\varphi)$*, respectively.*

Proof of Lemma 4.1. As usual, we shall use Kruzhkov's technique of doubling variables [22, 23] to show the comparison result. We choose two pairs of variables (t,x) and (s,y) and then we consider u as a function of $(t,x)\in Q$ and v as a function of $(s,y)\in Q$. For any $r>0$, let $\{B_i^r\}_{i=0,\cdots,m_r}$ be a covering of $\bar{D}$ satisfying $B_0^r\cap\partial D=\emptyset$, and such that, for each $i\geq 1$, B_i^r is a ball of diameter $\leq r$, contained in some larger ball $\tilde{B}_i^r$ with $\tilde{B}_i^r\cap\partial D$ is part of the graph of a Lipschitz function. Let $\{\phi_i^r\|_{i=0,\cdots,m_r}$ denote a partition of unity subordinate to the covering $\{B_i^r\}_i$. Let $\varphi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$.

The proof of the following stochastic local inequality is similar to the general case, that is, $\xi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$. And we only prove the general case. For any $\xi\in\mathcal{D}^+(Q)$, one can prove

$$\begin{aligned}0 \leq\ & \mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\partial_t\xi d\alpha dydxdt\\ &-\varepsilon\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\xi\Delta_y v d\alpha dydxdt\\ &+\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta''(u(t,x,\alpha)-v(t,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\xi d\alpha dydxdt\\ &+I^{f,\hat{f}}(\xi)-I^{\hat{f}}(\xi)+J^{f,\hat{f}}(\xi)-J^{g,\hat{g}}(\xi)+\int_\Sigma\int_D\eta_\delta(a-\hat{a})\xi\rho_n(y-x)\omega^+(x,\hat{a},a)\\ &+\int_D\int_D\eta_\delta(u_0-v_0)\xi(0)\rho_n(y-x)dxdy, \end{aligned}\tag{4.3}$$

In particular, (4.3) holds with $\xi=\varphi\phi_0^r$. Now, let $i\in\{1,\cdots,m_r\}$ be fixed in the following. For simplicity, we omit the dependence on r and i and simply set $\phi=\phi_i^r$ and $B=B_i^r$. We choose a sequence of mollifiers $(\rho_n)_n$ in $\mathbb{R}^N$ such that $x\mapsto\rho_n(x-y)\in\mathcal{D}$ for all $y\in B$. $\sigma_n(x)=\int_D\rho_n(x-y)dy$ is an increasing sequence for all $x\in B$ and $\sigma_n(x)=1$ for all $x\in B$ with $dist(x,\mathbb{R}^N\setminus D)>\frac{c}{n}$ for some $c=c(i,r)$ depending on $B=B_i^r$. Let $(\varrho_m)_m$ denote a sequence of mollifiers in $\mathbb{R}$ with $supp\varrho_m\subset(-\frac{2}{m},0)$.

Define the test function

$$\zeta_{m,n}(t,x,s,y)=\varphi(s,y)\phi(y)\rho_n(y-x)\varrho_m(t-s)$$

Note that, for m,n sufficiently large

$$\begin{aligned}&(t,x)\mapsto\zeta_{m,n}(t,x,s,y)\in\mathcal{D}((0,T)\times\mathbb{R}^N), && \text{for any } (s,y)\in Q,\\ &(s,y)\mapsto\zeta_{m,n}(t,x,s,y)\in\mathcal{D}(Q), && \text{for any } (t,x)\in Q.\end{aligned}$$

Let $v(s,y)$ be the solution of (4.1) with initial data v_0 and boundary data $\hat{a}$, and $\eta_\delta\in\mathcal{E}^+$ satisfying $\eta_\delta(\cdot)\mapsto(\cdot)^+$ and $\eta_\delta'(\cdot)\mapsto sgn_0^+(\cdot)$ as $\delta\to 0$. Then taking $\varphi=\zeta_{m,n}(t,x,s,y)$ in Definition 2.2, for a. e. $(t,x)\in Q$, we have

$$\begin{aligned}-\int_\Sigma\eta_\delta'(a-k)\zeta_{m,n}\omega^+(x,k,a)\ \leq\ & \int_0^1\int_Q[\eta_\delta(u-k)(\zeta_{m,n})_t-F^{\eta_\delta}(u,k)\cdot\nabla_x\zeta_{m,n}]\,dxdtd\alpha\\ &+\int_0^1\int_Q\eta_\delta'(u-k)h(u_1)\zeta_{m,n}dxdw(t)d\alpha\\ &-\int_Q\varphi\eta_\delta'(u-k)\left[\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}(t,x,u)+g(t,x,u)\right]dxdt\\ &+\int_Q\varphi\left(\int_k^u\eta_\delta'(\sigma-k)\sum_{i=1}^N\frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right)dxdt\\ &+\frac{1}{2}\int_0^1\int_Q h^2(u)\eta_\delta''(u-k)\zeta_{m,n}\\ &+\int_D\eta_\delta(u_{01}-k)\zeta_{m,n}(0,x,s,y)dx.\end{aligned}$$

Multiplying the above inequality by $\varrho_l(k-v)$ and integrating in k and (t,x) over $\mathbb{R}$ and Q, respectively, and taking expectation, we have

$$\begin{aligned}
0 \leq\ & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D \eta_\delta(u_0-k)\zeta_{m,n}(0,x,s,y)dx\varrho_l(k-v)dkdyds \\
&+\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u-k)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(k-v)dkdxdtdyds \\
&-\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 F^{\eta_\delta}(u,k)\varphi\phi\varrho_m\cdot\nabla_x\rho_n(y-x)d\alpha\varrho_l(k-v)dkdxdtdyds \\
&-\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta'(u-k)\left[\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}(t,x,u)+g(t,x,u)\right]\zeta_{m,n}d\alpha\varrho_l(k-v)dkdxdtdyds \\
&+\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\zeta_{m,n}\int_0^1\left(\int_k^u\eta_\delta'(\sigma-k)\sum_{i=1}^N\frac{\partial^2 f_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right)d\alpha\varrho_l(k-v)dkdxdtdyds \\
&+\frac{1}{2}\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 h^2(u)\eta_\delta''(u-k)\zeta_{m,n}d\alpha\varrho_l(k-v)dkdxdtdyds \\
&+\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta'(u-k)h(u)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k-v)dkdyds \\
&+\mathbb{E}\int_Q\int_{\mathbb{R}}\int_\Sigma \eta_\delta'(a-k)\zeta_{m,n}\omega^+(x,k,a)dSdt\varrho_l(k-v)dkdyds \\
=:\ & I_1+I_2+\cdots+I_8.
\end{aligned}$$

As v is a viscous solution, the Itô formula applied to $\int_D\eta_\delta(k-v)$ yields that for a.e. $(t,x)\in Q$

$$\begin{aligned}
0 \leq\ & \int_D\eta_\delta(k-v)\zeta_{m,n}(t,x,0,y)dy+\int_Q\eta_\delta(k-v)(\zeta_{m,n})_s dyds \\
&-\varepsilon\int_Q\eta_\delta(k-v)\Delta_y v\zeta_{m,n}dyds-\int_Q\hat{F}^{\check{\eta}_\delta}(v,k)\cdot\nabla_y\zeta_{m,n}dyds \\
&-\int_Q\varphi\eta_\delta'(k-v)\left[\sum_{i=1}^N\frac{\partial\hat{f}_i}{\partial x_i}(t,x,u)+\hat{g}(t,x,u)\right]dxdt \\
&+\int_Q\varphi\left(\int_k^v\eta_\delta'(k-\sigma)\sum_{i=1}^N\frac{\partial^2\hat{f}_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right)dxdt \\
&+\frac{1}{2}\int_Q\eta_\delta''(k-v)\hat{h}^2(v)\zeta_{m,n}dyds-\int_Q\eta_\delta'(k-v)\hat{h}(v)\zeta_{m,n}dydw(s),
\end{aligned}$$

where we used the fact that for any fixed $(t,x)\in Q$, $\zeta_{m,n}(t,x,s,y)\in\mathcal{D}(Q)$.

Multiplying the above inequality by $\varrho_l(u-k)$ and integrating in k over $\mathbb{R}$, in (t,x) over Q and in α over $(0,1)$,respectively, and taking expectation, we have

$$
\begin{aligned}
0 \quad \le \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(k-v)\zeta_{m,n}(t,x,0,y)\varrho_l(u-k)d\alpha dkdydxdt \\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-v)(\partial_s\varphi\varrho_m+\varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_l(u-k)d\alpha dkdxdt \\
& -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-v)\Delta_y v\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \hat{F}^{\breve{\eta}_\delta}(v,k)\cdot\nabla_y\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-v)\left[\sum_{i=1}^N\frac{\partial\hat{f}_i}{\partial x_i}(t,x,v)-\hat{g}(t,x,v)\right] \\
& \times\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \zeta_{m,n}\left(\int_k^v\eta'_\delta(k-\sigma)\sum_{i=1}^N\frac{\partial^2\hat{f}_i}{\partial x_i\partial\sigma}(t,x,\sigma)d\sigma\right) \\
& \times\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
& +\frac{1}{2}\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta''_\delta(k-v)\hat{h}^2(v)\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-v)\hat{h}(v)\zeta_{m,n}dydw(s)\varrho_l(u-k)d\alpha dkdxdt \\
=: \quad & J_1+J_2+\cdots+J_8.
\end{aligned}
$$

Noting that $\varrho_m(t)=0$, $t\in[0,T]$, we have

$$
\begin{aligned}
I_1+J_1 \quad &= \quad \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u-k)\zeta_{m,n}(0,x,s,y)\varrho_l(k-v)d\alpha dkdydxds \\
&= \quad \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u-k)\varphi\phi\rho_n\varrho_m(-s)\varrho_l(k-v)d\alpha dkdydxds \\
&\longrightarrow_{m,l} \quad \int_D\int_D \eta_\delta(u_0-v_0)\varphi\phi\rho_n dydx
\end{aligned}
$$

Due to $u\in N_w^2(0,T,L^2(D))$, u_0, $v_0\in L^2(D)$ and the compact support of $\zeta_{m,n}$, we know that the convergences in above inequality hold, see [2] for the similar proof.

By using the fact $\partial_t \varrho_m(t-s)+\partial_s \varrho_m(t-s)=0$ and changing variable technique, we get

$$\begin{aligned}
I_2+J_2 &= \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u-k)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(k-v)dkdxdtdyds \\
&\quad +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-v)(\partial_s\varphi\varrho_m+\varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_l(u-k)d\alpha dkdxdt \\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-v)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_l(u-k)d\alpha dkdxdt \\
&\quad +\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1 \eta_\delta(u-v-\tau)\varphi\phi\rho_n\partial_t\varrho_m(t-s)d\alpha\varrho_l(\tau)d\tau dxdtdyds \\
&\quad +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(u-v-\tau)\varphi\phi\rho_n\partial_s\varrho_m(t-s)dyds\varrho_l(\tau)d\alpha d\tau dxdt \\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-v)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_l(u-k)d\alpha dkdxdt \\
&\to_{l,m} \mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta(u-v)\partial_t\varphi\phi\rho_n dyd\alpha dxdt.
\end{aligned}$$

For the term J_3, we have

$$\begin{aligned}
J_3 &= -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'_\delta(k-v)\Delta_y v\zeta_{m,n}dyds\varrho_l(u-k)d\alpha dkdxdt \\
&\to_{l,m} -\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \eta'_\delta(u-v)\Delta_y v\varphi\phi\rho_n dyd\alpha dxdt.
\end{aligned}$$

Similar to the case I_2+J_2, and noting that $\nabla_x\rho_m(y-x)=-\nabla_y\rho_m(y-x)$, we have

$$I_3+J_4 \to_{m,l} \quad I^{f,\hat{f}}(\varphi\phi)-I^{\hat{f}}(\varphi\phi).$$

By the definition of stochastic entropy solution and the compact support of the test function, we know that the following limit holds

$$\begin{aligned}
&I_4+J_5+I_5+J_6 \to_{l,m} J^{f,\hat{f}}(\varphi\phi)-J^{g,\hat{g}}(\varphi\phi); \\
&I_6+J_7 \to_{l,m} \frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u-v)\left(h^2(u)+\hat{h}^2(v)\right)\varphi(t,y)\phi(y)\rho_n(y-x)dyd\alpha dxdt.
\end{aligned}$$

Now, we come to the estimate of most interesting part, the stochastic integrals. Since $\alpha(t)=\int_0^1\varrho_l(u(t,x,\tau)-k)d\tau$ is predictable and if one denotes

$$\beta(s):=\int_D \eta'_\delta(k-v)\hat{h}(v)\zeta_{m,n}dy,$$

we have

$$\mathbb{E}\left[\alpha(t)\int_t^T\beta(s)dw(s)\right]=\mathbb{E}\left[\alpha(t)\int_0^T\beta(s)dw(s)\right]-\mathbb{E}\left[\alpha(t)\int_0^t\beta(s)dw(s)\right]=0$$

due to that

$$\mathbb{E}\left[\alpha(t)\int_0^T\beta(s)dw(s)\right]=\mathbb{E}\left[\alpha(t)\mathbb{E}\left(\int_0^T\beta(s)dw(s)|\mathcal{F}_t\right)\right]=\mathbb{E}\left[\alpha(t)\int_0^t\beta(s)dw(s)\right].$$

Similarly, let $\alpha\left(s-\frac{2}{m}\right)=\varrho_l(k-v)$ and

$$\beta(t)=\int_D\int_0^1\eta'_\delta(u-k)h(u)\zeta_{m,n}dxd\alpha,$$

then we get that

$$\mathbb{E}\int_Q\int_{\mathbb{R}}\alpha\left(s-\frac{2}{m}\right)\int_0^T\beta(t)dw(t)dkdyds=\int_Q\int_{\mathbb{R}}\mathbb{E}\alpha\left(s-\frac{2}{m}\right)\int_{\left(s-\frac{2}{m}\right)^+}^s\beta(t)dw(t)dkdyds=0.$$

Thus, we have

$$\begin{aligned}
I_7+J_8 &= \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\int_0^1\eta'_\delta(u-k)h(u)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k-v)dkdyds\\
&\quad -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_t^T\int_0^1\eta'_\delta(k-v)\hat{h}(v)\zeta_{m,n}dydw(s)\varrho_l(u-k)d\alpha dkdxdt\\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\int_0^1\eta'_\delta(u-k)h(u)\zeta_{m,n}dxdw(t)d\alpha\varrho_l(k-v)dkdyds\\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\int_0^1\eta'_\delta(u-k)h(u)\zeta_{m,n}dxdw(t)d\alpha\\
&\quad\times\left[\varrho_l(k-v(s,y))-\varrho_l\left(k-v\left(s-\frac{2}{m},y\right)\right)\right]dkdyds
\end{aligned}$$

As $dv=[\varepsilon\Delta v+div(\hat{f}(t,y,v))-\hat{g}(t,y,v)]dt+\hat{h}(v)dw(t):=A_\varepsilon dt+\hat{h}(v)dw(t)$, by Itô's formula, we arrive that

$$\begin{aligned}
&\varrho_l(k-v(s,y))-\varrho_l\left(k-v\left(s-\frac{2}{m},y\right)\right)\\
&= -\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_l(k-v(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\\
&\quad -\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_l(k-v(\sigma,y))h(v(\sigma,y))dw(\sigma)\\
&\quad +\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho''_l(k-v(\sigma,y))h^2(v(\sigma,y))d\sigma\\
&= -\frac{\partial}{\partial k}\left\{\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho_l(k-v(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\right.\\
&\quad +\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho_l(k-v(\sigma,y))\hat{h}(v(\sigma,y))dw(\sigma)\\
&\quad \left.-\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_l(k-v(\sigma,y))\hat{h}^2(v(\sigma,y))d\sigma\right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_7+J_8 &= -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\int_0^1\eta''_\delta(u-k)h(u)\zeta_{m,n}dxdw(t)d\alpha\\
&\quad\times\{\cdots\}dkdyds\\
&=: L_1+L_2+L_3.
\end{aligned}$$

As in [2, 3], one can prove that

$$|L_1|\to_m 0,\qquad |L_3|\to_m 0.$$

Thanks to Fubini's theorem and the properties of Itô integral, we have

$$
\begin{aligned}
& \lim_m (L_1 + L_2 + L_3) \\
= \; & -\lim_m \mathbb{E} \int_Q \int_{\mathbb{R}} \int_{\left(s-\frac{2}{m}\right)^+}^{s} \int_D \int_0^1 \eta_\delta''(u-k) h(u) \zeta_{m,n} dx dw(t) d\alpha \\
& \times \int_{\left(s-\frac{2}{m}\right)^+}^{s} \varrho_l(k - v(\sigma, y)) \hat{h}(v(\sigma, y)) dw(\sigma) dk dy ds \\
= \; & -\lim_m \mathbb{E} \int_Q \int_{\mathbb{R}} \int_{\left(s-\frac{2}{m}\right)^+}^{s} \int_D \int_0^1 \eta_\delta''(u-k) h(u) \zeta_{m,n} d\alpha \\
& \times \varrho_l(k - v(t,y)) \hat{h}(v(t,y)) dt dx dk dy ds \\
\to_l \; & -\mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta''(u - v(t,y)) h(u) \varphi(t,y) \phi(x) \rho_n(y-x) \hat{h}(v(t,y)) d\alpha dt dx dy
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\lim_{m,l} (I_6 + J_7 + I_7 + J_8) \; = \; & \frac{1}{2} \mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta''(u - v) \\
& \times \left(h^2(u) - 2h(u)\hat{h}(v) + \hat{h}^2(v) \right) \varphi(t,y) \phi(y) \rho_n(y-x) dy d\alpha dx dt \\
= \; & \frac{1}{2} \mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta''(u - v) \\
& \times \left(h(u) - \hat{h}(v) \right)^2 \varphi(t,y) \phi(y) \rho_n(y-x) dy d\alpha dx dt.
\end{aligned}
$$

Lastly, we consider I_8. From the assumptions for a_2^ε, we have

$$
\begin{aligned}
I_8 \; = \; & \mathbb{E} \int_Q \int_{\mathbb{R}} \int_\Sigma \eta_\delta'(a-k) \zeta_{m,n} \omega^+(x,k,a) dS dt \varrho_l(k-v) dk dy ds \\
\to_{m,l} \; & \int_\Sigma \int_D \eta_\delta'(a - \hat{a}) \varphi \phi \rho_n(y-x) \omega^+(x, \hat{a}, a).
\end{aligned}
$$

Combining all estimates above then yields

$$
\begin{aligned}
0 \; \leq \; & \mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta(u(t,x,\alpha) - v(t,y)) \rho_n(y-x) \partial_t \varphi \phi d\alpha dy dx dt \\
& -\varepsilon \int_Q \int_D \int_0^1 \eta_\delta'(u(t,x,\alpha) - v(t,y)) \rho_n(y-x) \varphi \phi \Delta_y v d\alpha dy dx dt \\
& + \frac{1}{2} \mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta''(u(t,x,\alpha) - v(t,y)) \rho_n(y-x) (h(u) - \hat{h}(v))^2 \varphi \phi d\alpha dy dx dt \\
& + I^{f,\hat{f}}(\varphi\phi) - I^{\hat{f}}(\varphi\phi) + J^{f,\hat{f}}(\varphi\phi) - J^{g,\hat{g}}(\varphi\phi) + \int_\Sigma \int_D \eta_\delta'(a - \hat{a}) \varphi \phi \rho_n(y-x) \omega^+(x, \hat{a}, a) \\
& + \int_D \int_D \eta_\delta(u_0 - v_0) \varphi(0) \phi \rho_n(y-x) dx dy,
\end{aligned}
$$

Summing over $i = 0, 1, \cdots, m_r$, taking into account the local inequality (4.3) for $i = 0$, we obtain the desired inequality (4.2). This completes the proof. $\square$

Proof of Theorem 4.1 Let $f = \hat{f}$, $g = \hat{g}$ and $h = \hat{h}$ in inequality (4.2). It is easy to see that $I^{f,\hat{f}}(\varphi) = 0$. Next, we will show that

$$
-\lim_{\varepsilon,\delta \to 0} \varepsilon \mathbb{E} \int_Q \int_D \int_0^1 \eta_\delta'(u(t,x,\alpha) - v(t,y)) \rho_n(y-x) \varphi \Delta_y v d\alpha dy dx dt \leq 0.
$$

By using the fact that $\eta'' \geq 0$ and

$$\Delta_y \eta_\delta(u-v) = \eta''_\delta(u-v)|\nabla_y v|^2 - \eta'_\delta(u-v)\Delta_y v,$$

we have

$$\begin{aligned}
& -\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \eta'_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dy dx dt \\
= \;& \varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \Delta_y\eta_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dy dx dt \\
& -\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u(t,x,\alpha)-v(t,y))|\nabla_y v|^2\rho_n(y-x)\varphi\Delta_y v d\alpha dy dx dt \\
\leq \;& \varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \Delta_y\eta_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi d\alpha dy dx dt \\
= \;& \varepsilon\mathbb{E}\int_Q\int_0^1\int_{\partial D} \nabla_y\eta_\delta(u-v)\cdot\nu\rho_n(y-x)\varphi d\alpha dS dx dt \\
& -\varepsilon\mathbb{E}\int_Q\int_0^1\int_{\partial D} \eta_\delta(u-v)\nabla_y(\rho_n(y-x)\varphi)\cdot\nu d\alpha dS dx dt \\
& +\varepsilon\mathbb{E}\int_Q\int_0^1\int_D \eta_\delta(u-v)\Delta_y(\rho_n(y-x)\varphi) d\alpha dy dx dt \\
=: \;& J_{31}+J_{32}+J_{33}.
\end{aligned}$$

Using the bound of $\nabla_y v$ and v on $\bar{D}$, we get $\lim\limits_{\varepsilon\to 0}(J_{31}+J_{32}) = 0$.

$$\begin{aligned}
\lim_{\delta\to 0} J_{33} \;=\;& \varepsilon\mathbb{E}\int_Q\int_0^1\int_D |u-v|\Delta_y(\rho_n(y-x)\varphi) d\alpha dy dx dt \\
\leq\;& \varepsilon\mathbb{E}\int_Q\int_0^1\int_D |u|\Delta_y(\rho_n(y-x)\varphi) d\alpha dy dx dt \\
& +\varepsilon C\mathbb{E}\left(\int_D |v|^2 dy\right)^{\frac{1}{2}}\left(\int_D [\Delta_y(\rho_n(y-x)\varphi)]^2 dy\right)^{\frac{1}{2}} \\
\to\;& 0, \quad as\ \varepsilon\to 0,
\end{aligned}$$

where we have used the fact that $\|v\|_{L^2(D)}$ is uniform bounded for $\varepsilon > 0$. Thus, we get the desired result. Noting that $\lim\limits_{\delta\to 0}\eta''(u) = \delta_0(u)$, where $\delta_0(x) = 1$ if $x = 0$ and $\delta_0(x) = 0$ otherwise, we have

$$\lim_{\delta\to 0}\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \eta''_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)(h(u)-h(v))^2\varphi\phi d\alpha dy dx dt = 0.$$

Then taking limits in (4.2), we have

$$\begin{aligned}
0 \;\leq\; & \lim_{m\to\infty}\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \text{RHS of (4.2)} \\
= \;& \mathbb{E}\int_Q\int_0^1\int_0^1 (u(t,x,\alpha)-v(t,y,\beta))^+\partial_t\varphi d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(u(t,x,\alpha),v(t,x,\beta))\nabla\varphi(t,x)d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0^+(u-v)[g(t,x,u(t,x,\alpha))-g(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0^+(u-v)\sum_{i=1}^N[\frac{\partial f_i}{\partial x_i}(t,x,u(t,x,\alpha))+\frac{\partial f_i}{\partial x_i}(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& +\int_\Sigma \varphi\omega^+(x,\hat{a},a)dSdt+\int_D (u_0-v_0)^+\varphi(0)dx.
\end{aligned}$$

Next, we consider the second half. Similarly, as u is an entropy solution, using the other half of Definition 2.2, and applying Itô's formula to $\int_D \eta_\delta(v-k)$, we have the following

$$\begin{aligned}
0 \;\leq\; & \mathbb{E}\int_Q\int_0^1\int_0^1 (v(t,y,\beta)-u(t,x,\alpha))^+\partial_t\varphi d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 F^+(v(t,x,\beta),u(t,x,\alpha))\nabla\varphi(t,x)d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0^+(v-u)[g(t,x,u(t,x,\alpha))-g(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0^+(v-u)\sum_{i=1}^N[\frac{\partial f_i}{\partial x_i}(t,x,u(t,x,\alpha))+\frac{\partial f_i}{\partial x_i}(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& +\int_\Sigma \varphi\omega^-(x,\hat{a},a)dSdt+\int_D (v_0-u_0)^+\varphi(0)dx.
\end{aligned}$$

Summing the above two inequalities, and using the fact $|a-b|=(a-b)^+ + (b-a)^+$, we have

$$\begin{aligned}
0 \;\leq\; & \mathbb{E}\int_Q\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,y,\beta)|\partial_t\varphi d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 F(u(t,x,\alpha),v(t,x,\beta))\nabla\varphi(t,x)d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(u-v)[g(t,x,u(t,x,\alpha))-g(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(u-v)\sum_{i=1}^N[\frac{\partial f_i}{\partial x_i}(t,x,u(t,x,\alpha))+\frac{\partial f_i}{\partial x_i}(t,x,v(t,x,\beta))]d\alpha d\beta dx dt \\
& +\int_\Sigma \varphi \max_{\min(a,\hat{a})\leq r,s\leq \max(a,\hat{a})} |(f(t,x,r)-f(t,x,s))\cdot\vec{n}(x)|dSdt+\int_D |u_0-v_0|\varphi(0)dx.
\end{aligned}$$

where we used the fact that

$$\omega^-(x,\hat{a},a)+\omega^+(x,\hat{a},a_1)=\max_{\min(a,\hat{a})\leq r,s\leq \max(a,\hat{a})} |(f(t,x,r)-f(t,x,s))\cdot\vec{n}(x)|. \tag{4.4}$$

We see that it is different from the case that $f = f(u)$ and $g = 0$. Clearly, the following two terms

$$-\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(u-v)[g(t,x,u(t,x,\alpha))+g(t,x,v(t,x,\beta))]d\alpha d\beta dxdt$$
$$-\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(u-v)\sum_{i=1}^N[\frac{\partial f_i}{\partial x_i}(t,x,u(t,x,\alpha))+\frac{\partial f_i}{\partial x_i}(t,x,v(t,x,\beta))]d\alpha d\beta dxdt$$

would not vanish. Fortunately, we remark that $|u-v| = |v-u|$, so we can obtain the following inequality exactly as in the previous proof

$$\begin{aligned}
0 \ \leq \ & \mathbb{E}\int_Q\int_0^1\int_0^1 |v(t,y,\beta)-u(t,x,\alpha)|\partial_t\varphi d\alpha d\beta dxdt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 F(v(t,x,\beta),u(t,x,\alpha))\nabla\varphi(t,x)d\alpha d\beta dxdt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(v-u)[g(t,x,v(t,x,\beta))-g(t,x,u(t,x,\alpha))]d\alpha d\beta dxdt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 sgn_0(v-u)\sum_{i=1}^N[\frac{\partial f_i}{\partial x_i}(t,x,u(t,x,\alpha))+\frac{\partial f_i}{\partial x_i}(t,x,v(t,x,\beta))]d\alpha d\beta dxdt \\
& +\int_\Sigma\varphi\max_{\min(a,\hat{a})\leq r,s\leq\max(a,\hat{a})}|(f(t,x,r)-f(t,x,s))\cdot\vec{n}(x)|dSdt+\int_D|v_0-u_0|\varphi(0)dx.
\end{aligned}$$

Notice that $F(v(t,x,\beta),u(t,x,\alpha)) = F(u(t,x,\alpha),v(t,x,\beta))$, adding up the two inequalities then yields

$$\begin{aligned}
0 \ \leq \ & \mathbb{E}\int_Q\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,y,\beta)|\partial_t\varphi d\alpha d\beta dxdt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 F(u(t,x,\alpha),v(t,x,\beta))\nabla\varphi(t,x)d\alpha d\beta dxdt \\
& -\mathbb{E}\int_Q\int_0^1\int_0^1 |g(t,x,v(t,x,\beta))-g(t,x,u(t,x,\alpha))|d\alpha d\beta dxdt \\
& +\int_\Sigma\varphi\max_{\min(a,\hat{a})\leq r,s\leq\max(a,\hat{a})}|(f(t,x,r)-f(t,x,s))\cdot\vec{n}(x)|dSdt \\
& +\int_D|u_0-v_0|\varphi(0)dx.
\end{aligned}$$

The rest of the proof of Theorem 4.1 is routine (cf the proof of Theorem 3.8 in [4] for details). We omit it here. This completes the proof. □

Remark 4.2 *We remark that there is a significant difference in the proof of the uniqueness from that in [30]. To be more precise, there is a big difference between the deterministic case and the stochastic case. The reason is that we can not add the two inequalities, i.e.,* $\mu_{\eta,\varphi}\geq 0$ *and* $\mu_{\breve{\eta},\varphi}\geq 0$*. But in deterministic case, one can add the two inequalities in the definition, see Lemma 16 in [30]. Therefore, for stochastic case, it becomes more difficult.*

Another difference from [30] is that here we did not assume that the flux function fulfils the Lipschitz condition. Moreover, the boundary data a satisfies a different condition from that in [30] (cf Definition 1 of [30]). Our definition is a natural extension from those in [1, 31, 32].

5 Continuous Dependence Estimates

The aim of this section is to prove the second part of Theorem 2.2, that is, we will show the following

Theorem 5.1 *(Continuous dependence estimats) Suppose* $(H_1), (H_2), (H_3'), (H_4)$ *hold for the two data sets* (u_0, a, f, g, h) *and* $(v_0, \hat{a}, \hat{f}, \hat{g}, \hat{h})$. *Let* u *be a solution of (1.1)-(1.3) in the sense of Definition 2.2. Let* v *be the solution to the stochastic parabolic problem (4.1). In addition, we assume that either*

$$u, v \in L^\infty(\Omega \times Q) \quad \text{for any } T > 0,$$

or

$$\frac{\partial^2 f}{\partial u^2}, \frac{\partial f}{\partial u} - \frac{\partial \hat{f}}{\partial u}, \frac{\partial f_i}{\partial x_i} - \frac{\partial \hat{f}_i}{\partial x_i}, g - \hat{g}, h - \hat{h} \in L^\infty.$$

Then, there is a constant $C_T > 0$ *such that*

$$\begin{aligned}
&\mathbb{E}\left[\int_D \int_0^1 \int_0^1 |u(t,x,\alpha) - v(t,x,\beta)|\psi(x) d\alpha d\beta dx\right] \\
\leq\ & C_T \left(\int_D |u_0(x) - v_0(x)|\psi(x)dx + \sqrt{t}\|h - \hat{h}\|_{L^\infty} + \|a - \hat{a}\|_{L^\infty}\right. \\
&\left. + \|g - \hat{g}\|_{L^\infty} + |v_0|_{BV(D)} \|\frac{\partial f}{\partial u} - \frac{\partial \hat{f}}{\partial u}\|_{L^\infty} + \|h - \hat{h}\|_{L^\infty}\right),
\end{aligned}$$

where the constant $C_T > 0$ *is independent of* $|v_0|_{BV(D)}$ *and* $\psi(x) \in \mathcal{D}^+(\mathbb{R}^N)$ *is any function satisfying* $|\psi| \leq C_0$ *and* $|\nabla\psi| \leq C_0\psi$, *which includes* $\psi(x) = 1$ *when* $|x| \leq R$ *and* $\psi(x) = 0$ *when* $|x| \geq 2R$. *In particular, we have*

$$\begin{aligned}
&\mathbb{E}\left[\int_D \int_0^1 \int_0^1 |u(t,x,\alpha) - v(t,x,\beta)| dx\right] \\
\leq\ & C_T \left(\int_D |u_0(x) - v_0(x)|dx + \sqrt{t}\|h - \hat{h}\|_{L^\infty} + \|a - \hat{a}\|_{L^\infty}\right. \\
&\left. + \|g - \hat{g}\|_{L^\infty} + |v_0|_{BV(D)} \|\frac{\partial f}{\partial u} - \frac{\partial \hat{f}}{\partial u}\|_{L^\infty} + \|h - \hat{h}\|_{L^\infty}\right).
\end{aligned}$$

Proof Denote $\tilde{\eta}_\delta(x) := \eta_\delta(x) + \check{\eta}_\delta(x)$. Then, $\tilde{\eta}_\delta$ satisfies $\tilde{\eta}_\delta((0) = 0$, $\tilde{\eta}_\delta((x) = \tilde{\eta}_\delta((-x)$. From the Notation in section, we can assume that

$$|r| - M_1\delta \leq \tilde{\eta}_\delta((r) \leq |r|, \qquad 0 \leq \tilde{\eta}_\delta''((r) \leq \frac{M_2}{\delta} 1_{|r|<\delta},$$

where $M_i > 0$, $i = 1, 2$. Such function can be easily given, for example the function in [9] will be satisfied the above assumptions.

It follows from Lemma 4.1 and Remark 4.1 that

$$\begin{aligned}
0 \;\leq\; & \mathbb{E}\int_Q\int_D\int_0^1 \tilde{\eta}_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\partial_t\varphi d\alpha dydxdt \\
& -\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \tilde{\eta}_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dydxdt \\
& +\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \tilde{\eta}_\delta''(u(t,x,\alpha)-v(t,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\varphi d\alpha dydxdt \\
& +\mathbb{E}\int_Q\int_D\int_0^1 \left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\varphi\nabla_y\rho_n(y-x)d\alpha dydxdt \\
& -\mathbb{E}\int_Q\int_D\int_0^1 \hat{F}^{\tilde{\eta}_\delta}(v,u)\rho_n(y-x)\nabla_y\varphi d\alpha dydxdt \\
& +\mathbb{E}\int_Q\int_D\int_0^1 \varphi\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(\sigma-v)\frac{\partial f_i}{\partial x_i}(t,x,\sigma)d\alpha\rho_n(y-x)dydxdt \\
& +\mathbb{E}\int_Q\int_D\int_0^1 \varphi\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(u-\sigma)\frac{\partial \hat{f}_i}{\partial x_i}(t,y,\sigma)d\alpha\rho_n(y-x)dydxdt \\
& -\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi[g(t,x,u)-\hat{g}(t,y,v)]d\alpha dydxdt \\
& +\int_\Sigma\int_D \varphi\rho_n(y-x)\omega(x,\hat{a},a)+\int_D\int_D \tilde{\eta}_\delta(u_0-v_0)\varphi(0)\rho_n(y-x)dxdy, \qquad (5.1)
\end{aligned}$$

where we have used (4.4). Here and after, denote

$$\omega(x,\hat{a},a) := \max_{min(a,\hat{a})\leq r,s\leq max(a,\hat{a})} |(f(t,x,r)-f(t,x,s))\cdot\vec{n}(x)|.$$

For each $h>0$ and $0\leq t<T$, define

$$\phi_h(s) = \begin{cases} 1 & \text{if } s\leq t, \\ 1-\frac{s-t}{h} & \text{if } t<s\leq t+h, \\ 0 & \text{if } s>t+h. \end{cases}$$

Then, by standard approximation, truncation and mollification argument, (5.1) holds with

$$\varphi(s,x) = \phi_h(s)\psi(x),$$

where ψ satisfies the assumptions in Theorem 5.1.

Define

$$A(s) := \mathbb{E}\left[\int_D\int_0^1 \tilde{\eta}_\delta(u(s,x,\alpha)-v(s,x))d\alpha dx\right],$$

then $A\in L^1_{loc}(0,T)$. It is easy to check that any right Lebesgue point of $A(s)$ is also a right Lebesgue point of

$$A_\psi(s) = \mathbb{E}\left[\int_D\int_D\int_0^1 \tilde{\eta}_\delta(u(s,x,\alpha)-v(s,x))\rho_n(y-x)d\alpha\psi(y)dxdy\right].$$

Let t be a right Lebesgue point of A. We choose this t in the definition of $\phi_h(s)$. Then the inequality

(5.1) implies that

$$
\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\mathbb{E}\int_D\int_D\int_0^1 \tilde{\eta}_\delta(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\psi(y)d\alpha dydxds\\
\leq\ & -\varepsilon\mathbb{E}\int_Q\int_D\int_0^1 \tilde{\eta}_\delta'(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\phi_h(s)\psi(y)\Delta_y v d\alpha dydxds\\
&+\frac{1}{2}\mathbb{E}\int_Q\int_D\int_0^1 \tilde{\eta}_\delta''(u(s,x,\alpha)-v(s,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\phi_h(s)\psi(y)d\alpha dydxds\\
&+\mathbb{E}\int_Q\int_D\int_0^1 \left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\phi_h(s)\psi(y)\nabla_y\rho_n(y-x)d\alpha dydxds\\
&-\mathbb{E}\int_Q\int_D\int_0^1 \hat{F}^{\tilde{\eta}_\delta}(v,u)\rho_n(y-x)\nabla_y\phi_h(s)\psi(y)d\alpha dydxds\\
&+\mathbb{E}\int_Q\int_D\int_0^1 \phi_h(s)\psi(y)\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(\sigma-v)\frac{\partial f_i}{\partial x_i}(s,x,\sigma)d\alpha\rho_n(y-x)dydxds\\
&+\mathbb{E}\int_Q\int_D\int_0^1 \phi_h(s)\psi(y)\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(u-\sigma)\frac{\partial \hat{f}_i}{\partial x_i}(s,y,\sigma)d\alpha\rho_n(y-x)dydxds\\
&-\mathbb{E}\int_Q\int_D\int_0^1 \eta_\delta'(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\phi_h(s)\psi(y)[g(s,x,u)-\hat{g}(s,y,v)]d\alpha dydxds\\
&+\int_\Sigma\int_D \psi(y)\rho_n(y-x)\omega(x,\hat{a},a)+\int_D\int_D \tilde{\eta}_\delta(u_0-v_0)\psi(y)\rho_n(y-x)dxdy. \qquad (5.2)
\end{aligned}
$$

Letting $h\to 0$ in (5.2), we get

$$
\begin{aligned}
&\mathbb{E}\int_D\int_D\int_0^1 \tilde{\eta}_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\psi(y)d\alpha dydx\\
\leq\ & -\varepsilon\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \tilde{\eta}_\delta'(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\psi(y)\Delta_y v d\alpha dydxds\\
&+\frac{1}{2}\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \tilde{\eta}_\delta''(u(s,x,\alpha)-v(s,y))\rho_n(y-x)(h(u)-\hat{h}(v))^2\psi(y)d\alpha dydxds\\
&+\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\psi(y)\nabla_y\rho_n(y-x)d\alpha dydxds\\
&-\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \hat{F}^{\tilde{\eta}_\delta}(v,u)\rho_n(y-x)\nabla_y\psi(y)d\alpha dydxds\\
&+\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \psi(y)\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(\sigma-v)\frac{\partial f_i}{\partial x_i}(s,x,\sigma)d\alpha\rho_n(y-x)dydxds\\
&+\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \psi(y)\sum_{i=1}^N\int_v^u \tilde{\eta}_\delta''(u-\sigma)\frac{\partial \hat{f}_i}{\partial x_i}(s,y,\sigma)d\alpha\rho_n(y-x)dydxds\\
&-\mathbb{E}\int_0^t\int_D\int_D\int_0^1 \eta_\delta'(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\psi(y)[g(s,x,u)-\hat{g}(s,y,v)]d\alpha dydxds\\
&+\int_\Sigma\int_D \psi(y)\rho_n(y-x)\omega(x,\hat{a},a)+\int_D\int_D \tilde{\eta}_\delta(u_0-v_0)\psi(y)\rho_n(y-x)dxdy. \qquad (5.3)
\end{aligned}
$$

From the above assumptions on $\tilde{\eta}$, we know that the function

$$F^{\tilde{\eta}_\delta}(u,v):=\int_v^u \tilde{\eta}_\delta'(\sigma-v)\frac{\partial f}{\partial\sigma}(t,x,\sigma)d\sigma$$

satisfies

$$\Big|\partial_u\left(F^{\tilde{\eta}_\delta}(u,v)-F^{\tilde{\eta}_\delta}(v,u)\right)\Big|\leq M_2\|\frac{\partial^2 f}{\partial u^2}\|_{L^\infty(Q\times\mathbb{R})}\delta.$$

Notice that

$$\begin{aligned}&\nabla_y\left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\\ =\ &\nabla_y v\cdot\partial_v\left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)|_{(u,v)=(u(t,x),v(t,y))},\end{aligned}$$

thus

$$\begin{aligned}&\Big|\partial_v\left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\Big|\\ =\ &\Big|\partial_v\left(F^{\tilde{\eta}_\delta}(v,u)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)+\partial_v\left(F^{\tilde{\eta}_\delta}(u,v)-F^{\tilde{\eta}_\delta}(v,u)\right)\Big|\\ \leq\ &|\frac{\partial f}{\partial v}(t,x,v)-\frac{\partial\hat{f}}{\partial v}(t,x,v)|+M_2\|\frac{\partial^2 f}{\partial u^2}\|_{L^\infty(Q\times\mathbb{R})}\delta.\end{aligned}$$

Hence, after an integration by part, we get

$$\begin{aligned}&\Big|\mathbb{E}\int_0^t\int_D\int_D\int_0^1\left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\psi(y)\nabla_y\rho_n(y-x)d\alpha dydxds\\ &-\mathbb{E}\int_0^t\int_D\int_D\int_0^1\hat{F}^{\tilde{\eta}_\delta}(v,u)\rho_n(y-x)\nabla_y\psi(y)d\alpha dydxds\Big|\\ =\ &\Big|\mathbb{E}\int_0^t\int_D\int_D\int_0^1\nabla_y\left(F^{\tilde{\eta}_\delta}(u,v)-\hat{F}^{\tilde{\eta}_\delta}(v,u)\right)\psi(y)\rho_n(y-x)d\alpha dydxds\\ &-\mathbb{E}\int_0^t\int_D\int_D\int_0^1F^{\tilde{\eta}_\delta}(u,v)\rho_n(y-x)\nabla_y\psi(y)d\alpha dydxds\Big|\\ \leq\ &t\mathbb{E}[|v_0|_{BV(D)}]\|\psi\|_{L^\infty}\left(\|\frac{\partial f}{\partial u}-\frac{\partial\hat{f}}{\partial u}\|_{L^\infty}+M_2\|\frac{\partial^2 f}{\partial u^2}\|_{L^\infty}\delta\right)\\ &+C\|\frac{\partial f}{\partial u}\|_{L^\infty}\int_0^t\mathbb{E}\left[\int_D\int_D\int_0^1\tilde{\eta}_\delta(u(s,x,\alpha)-v(s,y))\rho_n(y-x)\psi(y)d\alpha dydx\right]ds,\end{aligned}$$

where we have used the properties of ψ.

Similar to the previous section, one can prove that

$$\Big|\varepsilon\int_Q\int_D\int_0^1\tilde{\eta}'_\delta(u(t,x,\alpha)-v(t,y))\rho_n(y-x)\varphi\Delta_y v d\alpha dydxdt\Big|\to 0,\quad\text{as }\varepsilon\to 0.$$

Consequently, using the properties of $\tilde{\eta}_\delta$ and the above discussion and letting $n\to\infty$ and $\varepsilon\to 0$, we can write (5.3) as

$$
\begin{aligned}
&\mathbb{E}\int_D\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,x,\beta)|\psi(x)d\alpha dx\\
\leq\ & \frac{1}{2}\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \tilde{\eta}''_\delta(u(s,x,\alpha)-v(s,x,\beta))(h(u)-\hat{h}(v))^2\psi(x)d\alpha d\beta dxds\\
&+\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(x)\sum_{i=1}^N\int_v^u \tilde{\eta}''_\delta(\sigma-v)\frac{\partial f_i}{\partial x_i}(s,x,\sigma)d\alpha d\beta dxdt\\
&+\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(y)\sum_{i=1}^N\int_v^u \tilde{\eta}''_\delta(u-\sigma)\frac{\partial \hat{f}_i}{\partial x_i}(s,x,\sigma)d\alpha d\beta dxdt\\
&-\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \eta'_\delta(u(s,x,\alpha)-v(s,x,\beta))\psi(x)[g(s,x,u)-\hat{g}(s,x,v)]d\alpha d\beta dxds\\
&+t\mathbb{E}[|v_0|_{BV(D)}]\|\psi\|_{L^\infty}\left(\|\frac{\partial f}{\partial u}-\frac{\partial \hat{f}}{\partial u}\|_{L^\infty}+C\delta\right)+C\|\frac{\partial f}{\partial u}\|_{L^\infty}\\
&\times\int_0^t\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u(s,x,\alpha)-v(s,x,\beta)|\psi(x)d\alpha d\beta dx\right]ds\\
&+\int_\Sigma \psi(x)\omega(x,\hat{a},a)+\int_D |u_0(x)-v_0(x)|\psi(x)dx.
\end{aligned}
$$

By the assumptions of $\tilde{\eta}$, we have

$$
\begin{aligned}
&\Big|\frac{1}{2}\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \tilde{\eta}''_\delta(u(s,x,\alpha)-v(s,x,\beta))(h(u)-\hat{h}(v))^2\psi(x)d\alpha d\beta dxds\Big|\\
\leq\ & \mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \frac{M_2}{\delta}1_{|u(s,x,\alpha)-v(s,x,\beta)|<\delta}(h(u)-\hat{h}(u))^2\psi(x)d\alpha d\beta dxds\\
&+\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \frac{M_2}{\delta}1_{|u(s,x,\alpha)-v(s,x,\beta)|<\delta}(\hat{h}(u)-\hat{h}(v))^2\psi(x)d\alpha d\beta dxdt\\
=:\ & A_1+A_2.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
|A_1| &\leq C\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \frac{(h(u)-\hat{h}(u))^2}{\delta}\psi(x)d\alpha d\beta dxds\\
&\leq C\|\psi\|_{L^1(D)}\frac{t\|h-\hat{h}\|^2_{L^\infty}}{\delta}
\end{aligned}
$$

and in view of H_2,

$$
|A_2|\leq CL_{\hat{h}}\int_0^t\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u(s,x,\alpha)-v(s,x,\beta)|\psi(x)d\alpha d\beta dx\right]ds.
$$

By the assumption (H_4) and using integration by part, we get

$$
\begin{aligned}
& \Big|\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(x)\sum_{i=1}^N\int_v^u \tilde{\eta}''_\delta(\sigma-v)\frac{\partial f_i}{\partial x_i}(s,x,\sigma)d\alpha d\beta dxds \\
& +\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(x)\sum_{i=1}^N\int_v^u \tilde{\eta}''_\delta(u-\sigma)\frac{\partial \hat{f}_i}{\partial x_i}(s,x,\sigma)d\alpha d\beta dxdt\Big| \\
\le\ & CL_{\frac{\partial f}{\partial x}}\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(x)\left(\int_v^u [\tilde{\eta}''_\delta(\sigma-v)+\tilde{\eta}''_\delta(u-\sigma)]|\sigma-v|d\sigma\right)d\alpha d\beta dxds \\
& +\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \psi(x)\tilde{\eta}'_\delta(u-v)\sum_{i=1}^N\left(\frac{\partial f_i}{\partial x_i}(s,x,v)-\frac{\partial \hat{f}_i}{\partial x_i}(s,x,u)\right)d\alpha d\beta dxds \\
\le\ & Ct\|\psi\|_{L^\infty}\|\frac{\partial f_i}{\partial x_i}-\frac{\partial \hat{f}_i}{\partial x_i}\|_{L^\infty} \\
& +CL_{\frac{\partial f}{\partial x}}\int_0^t\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u(s,x,\alpha)-v(s,x,\beta)|\psi(x)d\alpha d\beta dx\right]ds,
\end{aligned}
$$

where $L_{\frac{\partial f}{\partial x}} := \max_{i=1,\cdots,N} L_{\frac{\partial f_i}{\partial x_i}}$. In view of (H_2), we obtain

$$
\begin{aligned}
& \Big|\mathbb{E}\int_0^t\int_D\int_0^1\int_0^1 \eta'_\delta(u(s,x,\alpha)-v(s,x,\beta))\psi(x)[g(s,x,u)-\hat{g}(s,x,v)]d\alpha d\beta dxds\Big| \\
\le\ & C\|g-\hat{g}\|_{L^\infty}.
\end{aligned}
$$

Notice that $\omega(x,\hat{a},a)\le C\|a-\hat{a}\|_{L^\infty}$, we have then arrived at

$$
\begin{aligned}
& \mathbb{E}\int_D\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,x,\beta)|\psi(x)d\alpha dx \\
\le\ & \int_D |u_0(x)-v_0(x)|\psi(x)dx + C\left(\|\frac{\partial f}{\partial u}\|_{L^\infty}+L_{\frac{\partial f}{\partial x}}+L_{\hat{h}}\right) \\
& \times\int_0^t\mathbb{E}\left[\int_D\int_0^1\int_0^1 |u(s,x,\alpha)-v(s,x,\beta)|\psi(x)d\alpha d\beta dx\right]ds \\
& +C\Bigg(\|a-\hat{a}\|_{L^\infty}+\frac{t\|h-\hat{h}\|^2_{L^\infty}}{\delta}+\|\frac{\partial f_i}{\partial x_i}-\frac{\partial \hat{f}_i}{\partial x_i}\|_{L^\infty}+\|g-\hat{g}\|_{L^\infty} \\
& +t|v_0|_{BV(D)}\|\psi\|_{L^\infty}\|\frac{\partial f}{\partial u}-\frac{\partial \hat{f}}{\partial u}\|_{L^\infty}+\delta\Bigg),
\end{aligned}
$$

which implies via the Gronwall inequality that, for any $t>0$,

$$
\begin{aligned}
& \mathbb{E}\int_D\int_0^1\int_0^1 |u(t,x,\alpha)-v(t,x,\beta)|\psi(x)d\alpha dx \\
\le\ & e^{\mathcal{B}t}\int_D |u_0(x)-v_0(x)|\psi(x)dx + Ce^{\mathcal{B}t}\Bigg(\|a-\hat{a}\|_{L^\infty}+\frac{t\|h-\hat{h}\|^2_{L^\infty}}{\delta} \\
& +\|\frac{\partial f_i}{\partial x_i}-\frac{\partial \hat{f}_i}{\partial x_i}\|_{L^\infty}+\|g-\hat{g}\|_{L^\infty}+t|v_0|_{BV(D)}\|\psi\|_{L^\infty}\|\frac{\partial f}{\partial u}-\frac{\partial \hat{f}}{\partial u}\|_{L^\infty}+\delta\Bigg),
\end{aligned}
$$

where $\mathcal{B} := C\left(\|\frac{\partial f}{\partial u}\|_{L^\infty}+L_{\frac{\partial f}{\partial x}}+L_{\hat{h}}\right)$. The desired inequality is then obtained by choosing $\delta=\sqrt{t}\|h-\hat{h}\|_{L^\infty}$. We are done. □

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