DECOMPOSITIONS OF GRAPHS AND HYPERGRAPHS

by

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Abstract

This thesis contains various new results in the areas of design theory and edge decompositions of graphs and hypergraphs. Most notably, we give a new proof of the existence conjecture, dating back to the 19th century.

For r-graphs F and G, an F-decomposition of G is a collection of edge-disjoint copies of F in G covering all edges of G. In a recent breakthrough, Keevash proved that every sufficiently large quasirandom r-graph G has a $K_f^{(r)}$-decomposition (subject to necessary divisibility conditions), thus proving the existence conjecture.

We strengthen Keevash's result in two major directions: Firstly, our main result applies to decompositions into any r-graph F, which generalises a fundamental theorem of Wilson to hypergraphs. Secondly, our proof framework applies beyond quasirandomness, enabling us e.g. to deduce a minimum degree version.

For graphs, we investigate the minimum degree setting further. In particular, we determine the 'decomposition threshold' of every bipartite graph, and show that the threshold of cliques is equal to its fractional analogue.

We also present theorems concerning optimal path and cycle decompositions of quasirandom graphs.

This thesis is based on joint work with Daniela Kühn and Deryk Osthus [35, 36, 37, 39], Allan Lo [35, 36, 37] and Richard Montgomery [35].

To my wonderful wife Katharina. What is mine is also yours.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Combinatorial designs

> *"Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."*

Nowadays known as 'Kirkman's schoolgirl problem', the above rather innocent-looking problem was proposed by Thomas Kirkman in 1850 in the recreational mathematics journal *The Lady's and Gentleman's Diary*. A solution to this problem, i.e. an arrangement of the ladies with the desired properties, is an example of a *combinatorial design*. The latter term usually refers to a system of finite sets which satisfy some specified balance or symmetry condition, and the study of such systems is called *design theory*. Some well known examples include balanced incomplete block designs, projective planes, Latin squares and Hadamard matrices. These have applications in many areas such as finite geometry, statistics, experiment design, coding theory and cryptography. Even laymen will most likely have encountered combinatorial designs in their leisure time, namely in form of Sudokus.

In this thesis, we consider block designs and Steiner systems. In fact, we study the more general setting of hypergraph decompositions of which block designs and Steiner systems are special cases (see Section 1.2). An (n, f, r, λ)-*design* (or r-(n, f, λ) design) is a set X of f-subsets (called 'blocks') of some n-set V, such that every r-subset of V

belongs to exactly λ elements of X. An $(n, f, r, 1)$-design is also called an (n, f, r)-*Steiner system*, named in the honour of the Swiss mathematician Jakob Steiner, who asked in 1853 for which parameters these systems exist. Steiner systems with $(f, r) = (3, 2)$ are also referred to as *Steiner triple systems of order* n. Note that a solution to Kirkman's schoolgirl problem would yield a Steiner triple system of order 15 (but actually asks for more in that the triples are to be arranged in 'days').

There are some obviously necessary 'divisibility conditions' for the existence of a design: consider some subset S of V of size $i < r$ and assume that X is an (n, f, r, λ)-design. Then the number of elements of X which contain S is $\lambda\binom{n-i}{r-i}/\binom{f-i}{r-i}$. Indeed, there are $\binom{n-i}{r-i}$ r-subsets of V that contain S, and each of these must be contained in exactly λ elements of X. On the other hand, every element of X that contains S contains $\binom{f-i}{r-i}$ r-sets which contain S, proving the claim. We say that the necessary divisibility conditions are satisfied if $\binom{f-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for all $0 \leq i < r$.

In 1846, Kirkman [51] proved that Steiner triple systems exist whenever the necessary divisibility conditions are satisfied (which take on a particularly simple form in this case, namely $n \equiv 1, 3 \mod 6$). Thus Kirkman answered Steiner's question for triple systems even before Steiner asked for it. We note that these triple systems had been considered even earlier by Julius Plücker and Wesley Woolhouse. For more information on the early history, see [83].

In general, it is not true that the necessary divisibility conditions are sufficient for the existence of designs. However, it had been conjectured that there are only few exceptions. More precisely, the 'existence conjecture' states that for given f, r, λ, the necessary divisibility conditions are also sufficient for the existence of an (n, f, r, λ)-design, except for a finite number of exceptional n. It is unclear who first proposed the conjecture in this form, but it might be seen as a speculative answer to Steiner's question.

Over a century later, in a ground-breaking series of papers which transformed the area of design theory, Wilson [84, 85, 86, 87] resolved the case $r = 2$. (In the case when $r = 2$, designs are called 'balanced incomplete block designs'.)

For $r \geq 3$, much less was known until recently. We will revisit the history in Section 2.1. To encapsulate the lack of knowledge at this point, we remark that even the existence of infinitely many Steiner systems with $r \geq 4$ was open and not a single Steiner system with $r \geq 6$ was known to exist.

In a recent breakthrough, Peter Keevash [49] proved the existence conjecture in general. He refers to his proof method as 'randomised algebraic constructions'.

We provide a new proof of the existence conjecture based on the so-called iterative absorption method. Moreover, we are able to strengthen Keevash's result in two major directions. In order to discuss this, we need to introduce some hypergraph terminology first.

1.2 Graphs and hypergraphs

A *hypergraph* G is a pair (V, E), where $V = V(G)$ is the *vertex set* of G and the *edge set* E is a set of subsets of V. We often identify G with E, in particular, we let $|G| := |E|$, and $e \in G$ means $e \in E$. We say that G is an *r-graph* if every edge has size r, and a 2-graph is simply called a *graph*. We let $K_n^{(r)}$ denote the complete r-graph on n vertices, also called a *clique*. As usual, we just write K_n if $r = 2$. (We remark that within Chapter 2 however, we use K_n for the complete *complex* on n vertices instead, see Section 2.2.2.)

We approach the existence conjecture using terminology and methods from *extremal graph theory*. The basic question in this area is: how large or small can a (hyper-)graph be subject to satisfying certain conditions. For example, let G and F be r-graphs. We say that G is *F-free* if it does not contain a subgraph isomorphic to F. A natural question to ask is what is the maximal number of edges an F-free r-graph G on n vertices can have. This number is denoted by $ex(n, F)$, and $\pi(F) := \lim_{n\to\infty} ex(n, F)/\binom{n}{r}$ exists and is called the *Turán density of F*. For graphs, this parameter is well-understood. Turán himself determined the value for cliques. The Erdős-Simonovits-Stone theorem, a cornerstone result in extremal graph theory, generalises this to arbitrary graphs F, showing that $\pi(F) = 1 - 1/(\chi(F) - 1)$, where $\chi(F)$ denotes the *chromatic number* of F. For hypergraphs $r \geq 3$, only few Turán densities are known.

Note that for the Turán problem, it is sufficient to find only one copy of F in G. A more complicated question is the so-called *factor (or tiling) problem*. In this case, the desired object is an *F-factor of G*, i.e. a collection of pairwise vertex-disjoint copies of F is sought in G such that together they cover every vertex of G. Clearly, this is only possible if $|V(F)| \mid |V(G)|$. If F is just a single edge, then this coincides with the *perfect matching problem*. In order to guarantee an F-factor in G, it is no longer enough to assume that G has many edges, as there might still be isolated vertices. Instead, a more suitable question to ask is: if $|V(F)| \mid |V(G)|$ and every vertex is contained in at least $\delta|V(G)|$ edges, does this guarantee an F-factor in G, and what is the smallest such δ? Again, for graphs, this question is satisfyingly answered. The classical Hajnal-Szemerédi theorem provides the solution if F is a clique, and in [4, 53, 54, 59] the problem is solved for arbitrary F. And again, for hypergraphs, much less is known, although some progress has been made using the absorbing method (see Section 1.4). Note however that, even though an F-factor includes all the vertices of G, it uses only a vanishing proportion of the edges of G. Also, if G is complete, then the tiling problem is trivial, even for hypergraphs.

Not so if we move one step further and, instead of 'just' partitioning all the vertices, want to partition the edge set of G into (now edge-disjoint) copies of F. More precisely, an *F-decomposition of G* is a collection $\mathcal{F}$ of copies of F in G such that every edge of G is contained in exactly one of these copies. Note that an (n, f, r)-Steiner system X is equivalent to a $K_f^{(r)}$-decomposition $\mathcal{F}$ of $K_n^{(r)}$. Indeed, the blocks in X, i.e. sets of size f, correspond to the vertex sets of the copies of $K_f^{(r)}$ in $\mathcal{F}$.

The decomposition problem is trivial if F is just a single edge, but NP-complete for all non-trivial graphs F (see [24]). It is thus of interest to find sufficient conditions for the existence of an F-decomposition of a given graph G. As often, it is useful to consider necessary conditions first. Clearly, for an F-decomposition of G to exist, we need to require that the number of edges of G is divisible by the number of edges of F. But there are more such 'divisibility conditions'. For example, suppose that F is a cycle. Then we need to require that every vertex of G has even degree, as every cycle in a decomposition would cover either 0 or 2 edges at every vertex. In the hypergraph case, we also need to consider the 2-degrees, 3-degrees, etc. of F and G. If these divisibility conditions (which we discuss in more detail in Section 2.1.2) are satisfied, we say that G is F*-divisible*.

Hence, F-divisibility of G is necessary for the existence of an F-decomposition of G. On the other hand, it is not sufficient in general. For example, the 6-cycle C_6 is K_3-divisible, but does not have a K_3-decomposition. Our central question is thus:

> *When are the divisibility conditions sufficient for the existence of a decomposition (or design)?*

1.3 Overview of main results

In this section, we briefly outline some of our main results. More details on the history of each problem and previous work as well as further contributions of ourselves can be found in the corresponding chapters of this thesis.

1.3.1 Wilson's theorem for hypergraphs

The following fundamental theorem of Wilson from 1975 gives a positive answer to the above question if the host graph G is complete.

Theorem 1.3.1 (Wilson [87]). *Let F be any graph. For sufficiently large n, K_n has an F-decomposition if it is F-divisible.*

Our results imply the following generalisation of Wilson's theorem to hypergraphs.

Theorem A. *Let F be any r-graph. For sufficiently large n, $K_n^{(r)}$ has an F-decomposition if it is F-divisible.*

This answers a question asked e.g. by Keevash [49] who proved the case when F is a clique, thereby settling the existence conjecture. Previous results in the case when $r \geq 3$

and F is not complete are very sporadic – for instance Hanani [43] settled the problem if F is an octahedron (viewed as a 3-graph). The largest part of this thesis (Chapter 2) is devoted to prove Theorem A.

A natural question is how this can be generalised to non-complete host graphs. Keevash actually proved the existence conjecture in a quasirandom setting, i.e. his result already applies to host graphs which can be far from complete, as long as they are 'typical' (see Section 2.1.2 for the formal definition).

Our Theorem A also follows immediately from a more general result on F-designs of typical r-graphs (Theorem 2.1.1) which we state later. We note that the proof of this theorem does not rely on the concept of typicality, but a more flexible notion of 'supercomplexes' which applies beyond the quasirandom setting.

1.3.2 The decomposition threshold

As discussed above, one way to generalise Wilson's theorem to non-complete host graphs is to consider quasirandom graphs. Another natural way is to consider graphs of large minimum degree. The central conjecture in this area is the triangle decomposition conjecture of Nash-Williams [69] that every sufficiently large K_3-divisible graph G with $\delta(G) \geq 3|V(G)|/4$ has a K_3-decomposition. The bound on the minimum degree here would be best possible. It would be very interesting to have a similar conjecture for hypergraphs. Even for the simplest 'real' hyperclique, the tetrahedron $K_4^{(3)}$, it is unclear what the 'decomposition threshold' should be. Of course, this threshold cannot only be defined for cliques, but for arbitrary r-graphs F.

Definition 1.3.2 (Decomposition threshold)**.** *Given an r-graph F, let δ_F be the infimum of all $\delta \in [0, 1]$ with the following property: There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, every F-divisible r-graph G on n vertices with $\delta(G) \geq \delta n$ has an F-decomposition.*

The result of Keevash [49] implies that if F is complete, then $\delta_F < 1$, because every almost complete r-graph G is still quasirandom. As mentioned before, our methods allow us to obtain results beyond the quasirandom setting. In particular, we obtain a minimum degree version of our decomposition result, which yields the first 'effective' bounds for the decomposition threshold of 'real' hypergraphs (see Section 2.1.3). We remark that Yuster [89] studied the decomposition problem for so-called 'linear' hypertrees, which in their behaviour are very similar to graphs.

For graphs, much more precise bounds on the decomposition threshold are known. Yet the exact value is known only in few cases. We add to this body of work in various ways. For instance, we determine the decomposition threshold for all bipartite graphs F (see Theorem 3.3.1), and show that the threshold of cliques is equal to its fractional analogue (see Corollary 3.1.2). In order to determine the decomposition threshold it is thus sufficient to determine the fractional one. (To appreciate this, note that Wilson's theorem, a landmark result in design theory, becomes trivial in the fractional setting.) We also make progress for general graphs F. Recall that every graph G with $\delta(G) \geq (1-1/(\chi(F)-1)+o(1))|V(G)|$ contains a copy of F by the Erdős-Simonovits-Stone theorem, and every graph G with $|V(F)| \mid |V(G)|$ and $\delta(G) \geq (1-1/\chi(F)+o(1))|V(G)|$ contains an F-factor [4]. We conjecture that every F-divisible graph G with $\delta(G) \geq (1-1/(\chi(F)+1)+o(1))|V(G)|$ has an F-decomposition, or in other words, that $\delta_F \leq 1-1/(\chi(F)+1)$. We again show that it would be enough to obtain the desired bound for the fractional threshold. It is unclear what the precise value of δ_F should be. We prove a 'discretisation result' (see Theorem 3.1.1) that restricts the possible values of δ_F to a small set (where the above values $1-1/(\chi(F)-1), 1-1/\chi(F), 1-1/(\chi(F)+1)$ play a crucial role).

1.3.3 Path and cycle decompositions

So far, we have considered edge decompositions of some host graph G into copies of one given graph F. Clearly, if such a decomposition exists, then the number of copies in the decomposition is $|G|/|F|$. We now consider decomposition problems with a different emphasis. For example, a *path decomposition* is a partition of the edge set of a graph into paths. Obviously, every graph has a path decomposition (e.g. into paths of length one).

The existence question is thus immediately solved, yet the size of a decomposition can vary. A natural question is thus: what is the *minimal number of paths* needed to decompose a given graph? A conjecture of Gallai states that every connected graph on n vertices can be decomposed into $\lceil n/2 \rceil$ paths. There are famous similar conjectures e.g. concerning decompositions into cycles and linear forests. We investigate such decompositions for dense quasirandom graphs and the binomial random graph (see Chapter 4). In particular, we determine the exact minimal number of paths/cycles/linear forests needed to decompose such a graph.

1.4 Iterative absorption

Our results are proven using the iterative absorption method, which we now motivate and briefly sketch. We begin by recalling the 'classical' absorption technique and give some hints why it is not applicable to the edge decomposition setting.

The main idea of the absorbing technique is relatively straightforward. Suppose we want to find some spanning structure in a graph or hypergraph, for instance a perfect matching, a Hamilton cycle, or an F-factor. In many such cases, it is much easier to find an 'almost-spanning' structure, i.e. a matching which covers almost all the vertices, say. Of course, this is not satisfactory for the original problem. The idea of the absorbing technique is to set aside, even before finding the almost-spanning structure, an absorbing structure which is capable of 'absorbing' the leftover vertices into the almost-spanning structure to obtain the desired spanning structure. Such an approach was introduced systematically in the seminal paper by Rödl, Ruciński and Szemerédi [77] to prove an analogue of Dirac's theorem for 3-graphs (but actually goes back further than this, see e.g. the work of Krivelevich [57] on triangle factors in random graphs, and the result of Erdős, Gyárfás and Pyber [31] on vertex coverings with monochromatic cycles). Since then, the absorbing technique has been successfully applied to a wealth of problems concerning spanning structures. Of course, the success of the approach stands and falls with the ability to find this ‘magic’ absorbing structure. One key factor in this is the number of possible leftover configurations. Intuitively, the more possible leftover configurations there are, the more difficult it is to find an absorbing structure which can deal with all of them. Loosely speaking, this makes it much harder (if not impossible) to apply the absorbing technique for edge decomposition problems (see e.g. [9, p. 343] for a back-of-the-envelope calculation).

The ‘iterative absorption’ method tries to overcome this issue by splitting up the absorbing process into many steps, and in each step, the number of possible leftover configurations is drastically reduced using a ‘partial absorbing procedure’, until finally one has enough control over the leftover to absorb it completely. This approach was pioneered by Kühn and Osthus [60] to find Hamilton decompositions of regular robust expanders. The results we present in Chapter 4 are based on this result. The iterative procedure using partial absorbers was also used in [52] to find optimal Hamilton packings in random graphs (yet strictly speaking this is not a decomposition result). In the context of F-decompositions, the method was first applied in [9] to find F-decompositions of graphs of suitably high minimum degree. In particular, this yielded a combinatorial proof of Wilson’s theorem (Theorem 1.3.1). The results from [9] are strengthened in [35]. Even though the overall proof in [35] is more technical, the iterative absorption procedure itself has been simplified therein (see Chapter 3). The method has also been successfully applied to verify the Gyárfás-Lehel tree packing conjecture for bounded degree trees [48], as well as to find decompositions of dense graphs in the partite setting [10].

Here, we develop the iterative absorption method for hypergraphs. We believe that this will pave the way for further applications beyond the graph setting.

CHAPTER 2

WILSON'S THEOREM FOR HYPERGRAPHS

The content of this chapter largely overlaps with the preprints [36] and [37].

2.1 Introduction

In this chapter, we prove Theorem A and various stronger versions thereof.

2.1.1 More Background

Let G and F be r-graphs. Recall from Section 1.2 that an F*-decomposition of* G is a collection $\mathcal{F}$ of copies of F in G such that every edge of G is contained in exactly one of these copies. (Throughout the thesis, we always assume that F is non-empty without mentioning this explicitly.) More generally, an (F,λ)*-design of* G is a collection $\mathcal{F}$ of distinct copies of F in G such that every edge of G is contained in exactly λ of these copies. As discussed in Section 2.1.2, such a design can only exist if G satisfies certain divisibility conditions (e.g. if F is a graph triangle and $\lambda = 1$, then G must have even vertex degrees and the number of edges must be a multiple of three). If F and G are complete, such designs are also referred to as block designs. Recall that an (n,f,r,λ)-*design* (or r-(n,f,λ) design) is a set X of f-subsets of some n-set V, such that every r-subset of V belongs to exactly λ elements of X. The f-subsets are often called 'blocks'. An $(n,f,r,1)$-design is also called an (n,f,r)-*Steiner system*. As noted before, an (n,f,r)-

Steiner system is equivalent to a $K_f^{(r)}$-decomposition of $K_n^{(r)}$. More generally, note that an (n, f, r, λ)-design is equivalent to a $(K_f^{(r)}, \lambda)$-design of $K_n^{(r)}$.

The question of the existence of such designs goes back to the 19th century. For the early history including the works of Plücker, Woolhouse, Kirkman and Steiner, as well as the breakthrough result of Wilson who settled the graph case $r = 2$, we refer to Chapter 1.

For $r \geq 3$, much less was known until very recently. Answering a question of Erdős and Hanani [32], Rödl [75] was able to give an approximate solution to the existence conjecture by constructing near optimal packings of edge-disjoint copies of $K_f^{(r)}$ in $K_n^{(r)}$, i.e. constructing a collection of edge-disjoint copies of $K_f^{(r)}$ which cover almost all the edges of $K_n^{(r)}$. (For this, he introduced his now famous Rödl nibble method, which has since had a major impact in many areas.) His bounds were subsequently improved by increasingly sophisticated randomised techniques (see e.g. [3, 82]). Ferber, Hod, Krivelevich and Sudakov [33] recently observed that this method can be used to obtain an 'almost' Steiner system in the sense that every r-set is covered by either one or two f-sets.

Teirlinck [81] was the first to prove the existence of infinitely many non-trivial (n, f, r, λ)-block designs for arbitrary $r \geq 6$, via an ingenious recursive construction based on the symmetric group (this however requires $f = r+1$ and λ large compared to f). Kuperberg, Lovett and Peled [62] proved a 'localized central limit theorem' for rigid combinatorial structures, which implies the existence of designs for arbitrary f and r, but again for large λ. There are many constructions resulting in sporadic and infinite families of designs (see e.g. the handbook [20]). However, the set of parameters they cover is very restricted. In particular, even the existence of infinitely many Steiner systems with $r \geq 4$ was open until recently, and not a single Steiner system with $r \geq 6$ was known.

In a recent breakthrough, Keevash [49] proved the existence of (n, f, r, λ)-block designs for arbitrary (but fixed) r, f and λ, provided n is sufficiently large. In particular, his result implies the existence of Steiner systems for any admissible range of parameters as long as n is sufficiently large compared to f. The approach in [49] involved 'randomised algebraic constructions' and yielded a far-reaching generalisation to block designs in quasirandom

r-graphs.

Here we develop a non-algebraic approach based on iterative absorption, which additionally yields resilience versions and the existence of block designs in hypergraphs of large minimum degree. Moreover, we are able to go beyond the setting of block designs and show that F-designs also exist for arbitrary r-graphs F whenever the necessary divisibility conditions are satisfied.

2.1.2 F-designs in quasirandom hypergraphs

We now describe the degree conditions which are trivially necessary for the existence of an F-design in an r-graph G. For a set $S \subseteq V(G)$ with $0 \leq |S| \leq r$, the $(r-|S|)$-graph $G(S)$ has vertex set $V(G) \setminus S$ and contains all $(r-|S|)$-subsets of $V(G) \setminus S$ that together with S form an edge in G. ($G(S)$ is often called the *link graph of* S.) Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum $(r-1)$-degree of an r-graph G, respectively, that is, the minimum/maximum value of $|G(S)|$ over all $S \subseteq V(G)$ of size $r-1$. For a (non-empty) r-graph F, we define the *divisibility vector of* F as $Deg(F) := (d_0, \dots, d_{r-1}) \in \mathbb{N}^r$, where $d_i := \gcd\{|F(S)| \,:\, S \in \binom{V(F)}{i}\}$, and we set $Deg(F)_i := d_i$ for $0 \leq i \leq r-1$. Note that $d_0 = |F|$. So if F is a graph triangle K_3, then $Deg(F) = (3, 2)$, and if F is the Fano plane (viewed as a 3-graph), we have $Deg(F) = (7, 3, 1)$.

Given r-graphs F and G, G is called (F, λ)-*divisible* if $Deg(F)_i \mid \lambda |G(S)|$ for all $0 \leq i \leq r-1$ and all $S \in \binom{V(G)}{i}$. Note that G must be (F, λ)-divisible in order to admit an (F, λ)-design. For simplicity, we say that G is F-*divisible* if G is $(F, 1)$-divisible. Thus F-divisibility of G is necessary for the existence of an F-decomposition of G.

As a special case, the following result implies that (F, λ)-divisibility is sufficient to guarantee the existence of an (F, λ)-design when G is complete and λ is not too large. This answers a question asked e.g. by Keevash [49].

In fact, rather than requiring G to be complete, it suffices that G is quasirandom in the following sense. An r-graph G on n vertices is called (c, h, p)-*typical* if for any set A of $(r-1)$-subsets of $V(G)$ with $|A| \leq h$ we have $|\bigcap_{S \in A} G(S)| = (1 \pm c) p^{|A|} n$. Note that this is what one would expect in a random r-graph with edge probability p.

Theorem 2.1.1 (F-designs in typical hypergraphs)**.** *For all $f, r \in \mathbb{N}$ with $f > r$ and all $c, p \in (0, 1]$ with*

$$c \leq 0.9(p/2)^h/(q^r 4^q), \text{ where } q := 2f \cdot f! \text{ and } h := 2^r \binom{q+r}{r},$$

there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds for all $n \geq n_0$. Let F be any r-graph on f vertices and let $\lambda \in \mathbb{N}$ with $\lambda \leq \gamma n$. Suppose that G is a (c, h, p)-typical r-graph on n vertices. Then G has an (F, λ)-design if it is (F, λ)-divisible.

The main result in [49] is also stated in the setting of typical r-graphs, but additionally requires that $c \ll 1/h \ll p, 1/f$ and that $\lambda = \mathcal{O}(1)$ and F is complete.

Previous results in the case when $r \geq 3$ and F is not complete are very sporadic – for instance Hanani [43] settled the problem if F is an octahedron (viewed as a 3-uniform hypergraph) and G is complete.

In Section 2.9, we will deduce Theorem 2.1.1 from a more general result on F-decompositions in supercomplexes G (Theorem 2.4.7). The condition of G being a supercomplex is considerably less restrictive than typicality. Moreover, the F-designs we obtain will have the additional property that $|V(F') \cap V(F'')| \leq r$ for all distinct F', F'' which are included in the design. It is easy to see that with this additional property the bound on λ in Theorem 2.1.1 is best possible up to the value of γ.

We can also deduce the following result which yields 'near-optimal' F-packings in typical r-graphs which are not divisible. (An *F-packing* in G is a collection of edge-disjoint copies of F in G.)

Theorem 2.1.2. *For all $f, r \in \mathbb{N}$ with $f > r$ and all $c, p \in (0, 1]$ with*

$$c \leq 0.9p^h/(q^r 4^q), \text{ where } q := 2f \cdot f! \text{ and } h := 2^r \binom{q+r}{r},$$

there exist $n_0, C \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let F be any r-graph *on f vertices. Suppose that G is a (c,h,p)-typical r-graph on n vertices. Then G has an F-packing $\mathcal{F}$ such that the leftover L consisting of all uncovered edges satisfies $\Delta(L) \leq C$.*

2.1.3 F-designs in hypergraphs of large minimum degree

Once the existence question is settled, a next natural step is to seek F-designs and F-decompositions in r-graphs of large minimum degree. Our next result gives a bound on the minimum degree which ensures an F-decomposition for 'weakly regular' r-graphs F. These are defined as follows.

Definition 2.1.3 (weakly regular)**.** Let F be an r-graph. We say that F is *weakly $(s_0,\dots,s_{r-1})$-regular* if for all $0 \leq i \leq r-1$ and all $S \in \binom{V(F)}{i}$, we have $|F(S)| \in \{0, s_i\}$. We simply say that F is *weakly regular* if it is weakly $(s_0,\dots,s_{r-1})$-regular for suitable s_i's.

So for example, cliques, the Fano plane and the octahedron are all weakly regular but a 3-uniform tight or loose cycle is not.

Theorem 2.1.4 (F-decompositions in hypergraphs of large minimum degree)**.** *Let F be a weakly regular r-graph on f vertices. Let*

$$c_F^{\diamond} := \frac{r!}{3 \cdot 14^r f^{2r}}.$$

There exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that G is an r-graph on n vertices with $\delta(G) \geq (1 - c_F^{\diamond})n$. Then G has an F-decomposition if it is F-divisible.

We will actually deduce Theorem 2.1.4 from a 'resilience version' (Theorem 2.9.3). An analogous (but significantly worse) constant $c_F^{\diamond}$ for r-graphs F which are not weakly regular immediately follows from the case $p = 1$ of Theorem 2.1.1.

Note that Theorem 2.1.4 implies that whenever X is a partial (n,f,r)-Steiner system (i.e. a set of edge-disjoint $K_f^{(r)}$ on n vertices) and $n^* \geq \max\{n_0, n/c^{\diamond}_{K_f^{(r)}}\}$ satisfies the necessary divisibility conditions, then X can be extended to an (n^*, f, r)-Steiner system. For the case of Steiner triple systems (i.e. $f = 3$ and $r = 2$), Bryant and Horsley [17] showed that one can take $n^* = 2n + 1$, which proved a conjecture of Lindner.

Theorem 2.1.4 leads to the concept of the decomposition threshold δ_F of a given r-graph F (see Definition 1.3.2). By Theorem 2.1.4, we have $\delta_F \leq 1 - c_F^\diamond$ whenever F is weakly regular. It is not clear what the correct value should be. We note that for all $r, f, n_0 \in \mathbb{N}$, there exists an r-graph G_n on $n \geq n_0$ vertices with $\delta(G_n) \geq (1 - b_r \frac{\log f}{f^{r-1}})n$ such that G_n does not contain a single copy of $K_f^{(r)}$, where $b_r > 0$ only depends on r. This can be seen by adapting a construction from [56] as follows. Without loss of generality, we may assume that $1/f \ll 1/r$. By a result of [78], for every $r \geq 2$, there exists a constant b_r such that for any large enough f, there exists a partial $(N, r, r-1)$-Steiner system S_N with independence number $\alpha(S_N) < f/(r-1)$ and $1/N \leq b_r \log f / f^{r-1}$. This partial Steiner system can be 'blown up' (cf. [56]) to obtain arbitrarily large r-graphs H_n on n vertices with $\alpha(H_n) < f$ and $\Delta(H_n) \leq n/N \leq b_r n \log f / f^{r-1}$. Then the complement G_n of H_n is $K_f^{(r)}$-free and satisfies $\delta(G_n) \geq (1 - b_r \frac{\log f}{f^{r-1}})n$.

Previously, the only explicit result for the hypergraph case $r \geq 3$ was due to Yuster [89], who showed that if T is a linear r-uniform hypertree, then every T-divisible r-graph G on n vertices with minimum vertex degree at least $(\frac{1}{2^{r-1}} + o(1))\binom{n}{r-1}$ has a T-decomposition. This is asymptotically best possible for nontrivial T. Moreover, the result implies that $\delta_T \leq 1/2^{r-1}$.

For the graph case $r = 2$, much more is known about the decomposition threshold. We refer to Chapter 3 for more details.

2.1.4 Varying block sizes

We now briefly consider a more general notion of block designs, where more than just one block order is admissible. Given $n, r, \lambda \in \mathbb{N}$ as before and $A \subseteq \mathbb{N}$, we say that X is an (n, A, r, λ)-*design* if X consists of subsets of an n-set V such that $|x| \in A$ for every $x \in X$ and such that every r-subset of V is contained in precisely λ elements of X. Similarly, given an r-graph G and a family of r-graphs $\mathcal{K}$, we say that $\mathcal{F}$ is a $\mathcal{K}$-decomposition of G if every edge of G lies in precisely one $F \in \mathcal{F}$ and if $F \in \mathcal{K}$ for each $F \in \mathcal{F}$. For instance, a $\{K_a^{(r)} : a \in A\}$-decomposition of $K_n^{(r)}$ is equivalent to an $(n, A, r, 1)$-design. We say that G is $\mathcal{K}$-*divisible* if $\gcd\{Deg(F)_i : F \in \mathcal{K}\} \mid Deg(G)_i$ for all $0 \leq i \leq r-1$. Clearly, $\mathcal{K}$-divisibility is a necessary condition for the existence of a $\mathcal{K}$-decomposition. Theorem 2.1.1 easily implies the following result (see Section 2.9).

Theorem 2.1.5 (Designs with varying block sizes)**.** *For all $f, r \in \mathbb{N}$ and $p \in (0, 1]$ there exist $c > 0$, $h \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let $\mathcal{K}$ be a family of r-graphs of order at most f each. Suppose that G is a (c, h, p)-typical r-graph on n vertices. Then G has a $\mathcal{K}$-decomposition if it is $\mathcal{K}$-divisible.*

As a very special case, Theorem 2.1.5 resolves a conjecture of Archdeacon on self-dual embeddings of random graphs in orientable surfaces: as proved in [6], a graph has such an embedding if it has a $\{K_4, K_5\}$-decomposition. (In this paragraph, we write K_n for $K_n^{(2)}$.) Note that every graph with an even number of edges is $\{K_4, K_5\}$-divisible. Suppose G is a (c, h, p)-typical graph on n vertices with an even number of edges and $1/n \ll c \ll 1/h \ll p$ (which almost surely holds for the binomial random graph $G_{n,p}$ if we remove at most one edge). Then we can apply Theorem 2.1.5 to obtain a $\{K_4, K_5\}$-decomposition of G. It was also shown in [6] that a graph has a self-dual embedding in a non-orientable surface if it has a $\{K_a : a \geq 4\}$-decomposition. Since every graph is $\{K_4, K_5, K_6\}$-divisible, say, Theorem 2.1.5 implies that almost every graph has a $\{K_4, K_5, K_6\}$-decomposition and thus a self-dual embedding.

2.1.5 Matchings and further results

As another illustration, we now state a consequence of our main result which concerns perfect matchings in hypergraphs that satisfy certain uniformity conditions on their edge distribution. Note that the conditions are much weaker than any standard pseudorandomness notion.

Theorem 2.1.6. *For all $f \geq 2$ and $\xi > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds whenever $n \geq n_0$ and $f \mid n$. Let G be a f-graph on n vertices which satisfies the following properties:*

- *for some $d \geq \xi$, $|G(v)| = (d \pm 0.01\xi)n^{f-1}$ for all $v \in V(G)$;*
- *every vertex is contained in at least ξn^f copies of $K^{(f)}_{f+1}$;*
- *$|G(v) \cap G(w)| \geq \xi n^{f-1}$ for all $v, w \in V(G)$.*

Then G has at least $0.01\xi n^{f-1}$ edge-disjoint perfect matchings.

Note that for $G = K^{(f)}_n$, this is strengthened by Baranyai's theorem [7], which states that $K^{(f)}_n$ has a decomposition into $\binom{n-1}{f-1}$ edge-disjoint perfect matchings. More generally, the interplay between designs and the existence of (almost) perfect matchings in hypergraphs has resulted in major developments over the past decades, e.g. via the Rödl nibble. For more recent progress on results concerning perfect matchings in hypergraphs and related topics, see e.g. the surveys [76, 92, 95].

We discuss further applications of our main result in Section 2.4, e.g. to partite graphs (see Example 2.4.11) and to (n, f, r, λ)-block designs where we allow any $\lambda \leq n^{f-r}/(11 \cdot 7^r f!)$, say (under more restrictive divisibility conditions, see Corollary 2.4.14).

2.1.6 Counting

An approximate F-decomposition of $K^{(r)}_n$ is a set of edge-disjoint copies of F in $K^{(r)}_n$ which together cover almost all edges of $K^{(r)}_n$. Given good bounds on the number of approximate F-decompositions of $K^{(r)}_n$ whose set of leftover edges forms a typical r-graph, one can apply Theorem 2.1.1 to obtain corresponding bounds on the number of F-decompositions in $K^{(r)}_n$ (see [49, 50] for the clique case). Such lower bounds on the number of approximate F-decompositions can be achieved by considering either a random greedy F-removal process or an associated F-nibble removal process. Linial and Luria [64] developed an entropy-based approach which they used to obtain good upper bounds e.g. on the number of

Steiner triple systems. These developments also make it possible to systematically study random designs (see Kwan [63] for an investigation of random Steiner triple systems).

2.1.7 Outline of the chapter

As mentioned earlier, our main result (Theorem 2.4.7) actually concerns F-decompositions in so-called supercomplexes. We will define supercomplexes in Section 2.4 and derive Theorems 2.1.1, 2.1.2, 2.1.4, 2.1.5 and 2.1.6 in Section 2.9. The definition of a supercomplex G involves mainly the distribution of cliques of size f in G (where $f = |V(F)|$). The notion is weaker than usual notions of quasirandomness. This has two main advantages: firstly, our proof is by induction on r, and working with this weaker notion is essential to make the induction proof work. Secondly, this allows us to deduce Theorems 2.1.1, 2.1.2, 2.1.4, 2.1.5 and 2.1.6 from a single statement.

However, Theorem 2.4.7 applies only to F-decompositions of a supercomplex G for weakly regular r-graphs F (which allows us to deduce Theorem 2.1.4 but not Theorem 2.1.1).

To deal with this, in Section 2.9 we first provide an explicit construction which shows that every r-graph F can be 'perfectly' packed into a suitable weakly regular r-graph F^*. In particular, F^* has an F-decomposition. The idea is then to apply Theorem 2.4.7 to find an F^*-decomposition in G. Unfortunately, G may not be F^*-divisible. To overcome this, in Section 2.11 we show that we can remove a small set of copies of F from G to achieve that the leftover G' of G is now F^*-divisible (see Lemma 2.9.4 for the statement). This now implies Theorem 2.1.1 for F-decompositions, i.e. for $\lambda = 1$. However, by repeatedly applying Theorem 2.4.7 in a suitable way, we can actually allow λ to be as large as required in Theorem 2.1.1.

It thus remains to prove Theorem 2.4.7 itself. We achieve this via an approach based on 'iterative absorption'. We give a sketch of the argument in Section 2.3.

As a byproduct of the construction of the weakly regular r-graph F^* outlined above, we prove the existence of resolvable clique decompositions in complete partite r-graphs G

(see Theorem 2.9.1). The construction is explicit and exploits the property that all square submatrices of so-called Cauchy matrices over finite fields are invertible. We believe this construction to be of independent interest. A natural question leading on from the current work would be to obtain such resolvable decompositions also in the general (non-partite) case. For decompositions of $K_n^{(2)}$ into $K_f^{(2)}$, this is due to Ray-Chaudhuri and Wilson [74]. For related results see [28, 66].

2.2 Notation

2.2.1 Basic terminology

We let $[n]$ denote the set $\{1, \dots, n\}$, where $[0] := \emptyset$. Moreover, let $[n]_0 := [n] \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, $\binom{n}{i}$ denotes the binomial coefficient, where we set $\binom{n}{i} := 0$ if $i > n$ or $i < 0$. Moreover, given a set X and $i \in \mathbb{N}_0$, we write $\binom{X}{i}$ for the collection of all i-subsets of X. Hence, $\binom{X}{i} = \emptyset$ if $i > |X|$. If F is a collection of sets, we define $\bigcup F := \bigcup_{f \in F} f$. We write $A \mathbin{\dot\cup} B$ for the union of A and B if we want to emphasise that A and B are disjoint.

We write $X \sim B(n,p)$ if X has binomial distribution with parameters n, p, and we write $bin(n,p,i) := \binom{n}{i} p^i (1-p)^{n-i}$. So by the above convention, $bin(n,p,i) = 0$ if $i > n$ or $i < 0$.

We say that an event holds *with high probability (whp)* if the probability that it holds tends to 1 as $n \to \infty$ (where n usually denotes the number of vertices). We let $\mathcal{H}_r(n,p)$ denote the random binomial r-graph on $[n]$ whose edges appear independently with probability p. If $r = 2$, we write $\mathcal{G}(n,p)$ instead.

We write $x \ll y$ to mean that for any $y \in (0,1]$ there exists an $x_0 \in (0,1)$ such that for all $x \leq x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. We will always assume that the constants in our hierarchies are reals in $(0,1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that x is a natural number. More precisely, $1/x \ll y$ means that for any $y \in (0,1]$ there exists an $x_0 \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ with $x \geq x_0$ the subsequent statement holds.

We write $a = b \pm c$ if $b - c \leq a \leq b + c$. Equations containing $\pm$ are always to be interpreted from left to right, e.g. $b_1 \pm c_1 = b_2 \pm c_2$ means that $b_1 - c_1 \geq b_2 - c_2$ and $b_1 + c_1 \leq b_2 + c_2$. We will often use the fact that for all $0 < x < 1$ and $n \in \mathbb{N}$ we have $(1 \pm x)^n = 1 \pm 2^n x$.

When dealing with multisets, we treat multiple appearances of the same element as distinct elements. In particular, two subsets A, B of a multiset can be disjoint even if they both contain a copy of the same element, and if A and B are disjoint, then the multiplicity of an element in the union $A \cup B$ is obtained by adding the multiplicities of this element in A and B (rather than just taking the maximum).

2.2.2 Hypergraphs and complexes

Let G be an r-graph. Note that $G(\emptyset) = G$. For a set $S \subseteq V(G)$ with $|S| \leq r$ and $L \subseteq G(S)$, let $S \uplus L := \{S \cup e \,:\, e \in L\}$. Clearly, there is a natural bijection between L and $S \uplus L$.

For $i \in [r-1]_0$, we define $\delta_i(G)$ and $\Delta_i(G)$ as the minimum and maximum value of $|G(S)|$ over all i-subsets S of $V(G)$, respectively. As before, we let $\delta(G) := \delta_{r-1}(G)$ and $\Delta(G) := \Delta_{r-1}(G)$. Note that $\delta_0(G) = \Delta_0(G) = |G(\emptyset)| = |G|$.

For two r-graphs G and G', we let $G - G'$ denote the r-graph obtained from G by deleting all edges of G'. We write $G_1 + G_2$ to mean the vertex-disjoint union of G_1 and G_2, and $t \cdot G$ to mean the vertex-disjoint union of t copies of G.

Let F and G be r-graphs. An *F-packing in G* is a set $\mathcal{F}$ of edge-disjoint copies of F in G. We let $\mathcal{F}^{(r)}$ denote the r-graph consisting of all covered edges of G, i.e. $\mathcal{F}^{(r)} = \bigcup_{F' \in \mathcal{F}} F'$.

A *multi-r-graph* G consists of a set of vertices $V(G)$ and a multiset of edges $E(G)$, where each $e \in E(G)$ is a subset of $V(G)$ of size r. We will often identify a multi-r-graph with its edge set. For $S \subseteq V(G)$, let $|G(S)|$ denote the number of edges of G that contain

S (counted with multiplicities). If $|S| = r$, then $|G(S)|$ is called the *multiplicity of S in G*. We say that G is *F-divisible* if $Deg(F)_{|S|}$ divides $|G(S)|$ for all $S \subseteq V(G)$ with $|S| \leq r-1$. An F-decomposition of G is a collection $\mathcal{F}$ of copies of F in G such that every edge $e \in G$ is covered precisely once. (Thus if $S \subseteq V(G)$ has size r, then there are precisely $|G(S)|$ copies of F in $\mathcal{F}$ in which S forms an edge.)

Definition 2.2.1. A *complex* G is a hypergraph which is closed under inclusion, that is, whenever $e' \subseteq e \in G$ we have $e' \in G$. If G is a complex and $i \in \mathbb{N}_0$, we write $G^{(i)}$ for the i-graph on $V(G)$ consisting of all $e \in G$ with $|e| = i$. We say that a complex is empty if $\emptyset \notin G^{(0)}$, that is, if G does not contain any edges.

Suppose G is a complex and $e \subseteq V(G)$. Define $G(e)$ as the complex on vertex set $V(G) \setminus e$ containing all sets $e' \subseteq V(G) \setminus e$ such that $e \cup e' \in G$. Clearly, if $e \notin G$, then $G(e)$ is empty. Observe that if $|e| = i$ and $r \geq i$, then $G^{(r)}(e) = G(e)^{(r-i)}$. We say that G' is a *subcomplex* of G if G' is a complex and a subhypergraph of G.

For a set U, define $G[U]$ as the complex on $U \cap V(G)$ containing all $e \in G$ with $e \subseteq U$. Moreover, for an r-graph H, let $G[H]$ be the complex on $V(G)$ with edge set

$$G[H] := \{e \in G : \binom{e}{r} \subseteq H\},$$

and define $G - H := G[G^{(r)} - H]$. So for $i \in [r-1]$, $G[H]^{(i)} = G^{(i)}$. For $i > r$, we might have $G[H]^{(i)} \subsetneq G^{(i)}$. Moreover, if $H \subseteq G^{(r)}$, then $G[H]^{(r)} = H$. Note that for an r_1-graph H_1 and an r_2-graph H_2, we have $(G[H_1])[H_2] = (G[H_2])[H_1]$. Also, $(G - H_1) - H_2 = (G - H_2) - H_1$, so we may write this as $G - H_1 - H_2$.

If G_1 and G_2 are complexes, we define $G_1 \cap G_2$ as the complex on vertex set $V(G_1) \cap V(G_2)$ containing all sets e with $e \in G_1$ and $e \in G_2$. We say that G_1 and G_2 are *i-disjoint* if $G_1^{(i)} \cap G_2^{(i)}$ is empty.

For any hypergraph H, let $H^{\leq}$ be the complex on $V(H)$ *generated by* H, that is,

$$H^{\leq} := \{e \subseteq V(H) : \exists e' \in H \text{ such that } e \subseteq e'\}.$$

For an r-graph H, we let $H^{\leftrightarrow}$ denote the complex on $V(H)$ that is *induced by* H, that is,

$$H^{\leftrightarrow} := \{e \subseteq V(H) : \binom{e}{r} \subseteq H\}.$$

Note that $H^{\leftrightarrow(r)} = H$ and for each $i \in [r-1]_0$, $H^{\leftrightarrow(i)}$ is the complete i-graph on $V(H)$. Within this chapter, we let K_n denote the complete complex on n vertices (instead of the complete 2-graph).

2.3 Outline of the methods

Rather than an algebraic approach as in [49], we pursue a combinatorial approach based on 'iterative absorption'. In particular, we do not make use of any nontrivial algebraic techniques and results, but rely only on probabilistic tools.

2.3.1 Iterative absorption in vortices

Suppose for simplicity that we aim to find a $K_f^{(r)}$-decomposition of a suitable r-graph G. The Rödl nibble (see e.g. [3, 71, 75, 82]) allows us to obtain an approximate $K_f^{(r)}$-decomposition of G, i.e. a set of edge-disjoint copies of $K_f^{(r)}$ covering almost all edges of G. However, one has little control over the resulting uncovered leftover set of edges. The basic aim of an absorbing approach is to overcome this issue by removing an absorbing structure A right at the beginning and then applying the Rödl nibble to $G - A$, to obtain an approximate decomposition with a very small uncovered remainder R. Ideally, A was chosen in such a way that $A \cup R$ has a $K_f^{(r)}$-decomposition.

In the context of decompositions, the first results based on an absorbing approach were obtained in [52, 60]. In contrast to the construction of spanning subgraphs, the decomposition setting gives rise to the additional challenge that the number of and possible shape of uncovered remainder graphs R is comparatively large. So in general it is much less clear how to construct a structure A which can deal with all such possibilities for R

(to appreciate this issue, note that $V(R) = V(G)$ in this scenario).

The method developed in [52, 60] consisted of an iterative approach: each iteration consists of an approximate decomposition of the previous leftover, together with a partial absorption (or 'cleaning') step, which further restricts the structure of the current leftover. In our context, we carry out this iteration by considering a 'vortex'. Such a vortex is a nested sequence $V(G) = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$, where $|U_i|/|U_{i+1}|$ and $|U_\ell|$ are large but bounded. Crucially, after the ith iteration, all r-edges belonging to the current leftover R_i will be induced by U_i. In the $(i+1)$th iteration, we make use of a suitable r-graph H_i on U_i which we set aside at the start. We first apply the Rödl nibble to R_i to obtain a sparse remainder R_i'. We then apply what we refer to as the 'Cover down lemma' to find a $K_f^{(r)}$-packing $\mathcal{K}_i$ of $H_i \cup R_i'$ so that the remainder R_{i+1} consists entirely of r-edges induced by U_{i+1} (see Lemma 2.7.7). Ultimately, we arrive at a leftover R_ℓ induced by U_ℓ.

Since $|U_\ell|$ is bounded, this means there are only a bounded number of possibilities $S_1, \ldots, S_b$ for R_ℓ. This gives a natural approach to the construction of an absorber A for R_ℓ: it suffices to construct an 'exclusive' absorber A_i for each S_i (in the sense that A_i can absorb S_i but nothing else). More precisely, we aim to construct edge-disjoint r-graphs $A_1, \ldots, A_b$ so that both A_i and $A_i \cup S_i$ have a $K_f^{(r)}$-decomposition, and then let $A := A_1 \cup \cdots \cup A_b$. Then $A \cup R_\ell$ must also have a $K_f^{(r)}$-decomposition.

Iterative absorption based on vortices was introduced in [35], building on a related (but more complicated approach) in [9]. Developing the above approach in the setting of hypergraph decompositions gives rise to two main challenges: constructing the 'exclusive' absorbers and proving the Cover down lemma, which we discuss in the next two subsections, respectively.

One difficulty with the iteration process is that after finishing one iteration, the error terms are too large to carry out the next one. Fortunately, we are able to 'boost' our regularity parameters before each iteration by excluding suitable f-cliques from future consideration (see Lemma 2.6.3). For this, we adopt gadgets introduced in [8]. Moreover, the 'Boost lemma' enables us to obtain explicit bounds e.g. in the minimum degree version

(Theorem 2.1.4).

2.3.2 The Cover down lemma

As indicated above, the goal here is as follows: Given an r-graph G and vertex sets $U_{i+1} \subseteq U_i$ in G, we need to construct H^* in $G[U_i]^{(r)}$ so that for any sparse leftover R on U_i, we can find a $K_f^{(r)}$-packing in $H^* \cup R$ such that any leftover edges lie in U_{i+1}. (In addition, we need to ensure that the distribution of the leftover edges within U_{i+1} is sufficiently well-behaved so that we can continue with the next iteration, but we do not discuss this aspect here.)

We achieve this goal in several stages: given an edge $e \in H^* \cup R$, we refer to the size of its intersection with U_{i+1} as its *type*. Initially, we cover all edges of type 0. This can be done using an appropriate greedy approach, i.e. for each edge e of type 0 in turn, we extend e to a copy of $K_f^{(r)}$ using edges of H^*. In the next stage, we cover all edges of type 1, then all edges of type 2 up to and including type $r-1$. When covering a given set of edges of type j, we will inductively assume that our main decomposition result holds for j-graphs (note that $j < r$). For example, consider the triangle case $f = 3$ and $r = 2$, and suppose $j = 1$. Then for each vertex $v \in U_i \setminus U_{i+1}$, we will inductively find a perfect matching (which can be viewed as a $K_2^{(1)}$-decomposition) on the neighbours of v in U_{i+1}. This yields a triangle packing which covers all (remaining) edges incident to v (note that these edges have type 1). The resulting proof of the Cover down lemma is given in Section 2.10 (which also includes a more detailed sketch of this part of the argument).

2.3.3 Transformers and absorbers

Recall that our remaining goal is to construct an exclusive absorber A_S for a given 'leftover' r-graph S of bounded size. In other words, both $A_S \cup S$ as well as A_S need to have a $K_f^{(r)}$-decomposition. Clearly, we must (and can) assume that S is $K_f^{(r)}$-divisible.

Based on an idea introduced in [9], we will construct A_S as a concatenation of 'transformers': given S, a transformer T_S can be viewed as transforming S into a new leftover L (which has the same number of edges and is still divisible). Formally, we require that $S \cup T_S$ and $T_S \cup L$ both have a $K_f^{(r)}$-decomposition (and will set aside T_S and L at the beginning of the proof). Since transformers act transitively, the idea is to concatenate them in order to transform S into a vertex-disjoint union of $K_f^{(r)}$, i.e. we gradually transform the given leftover S into a graph which is trivially decomposable.

Roughly speaking, we approach this by choosing L to be a suitable 'canonical' graph (i.e. L only depends on $|S|$). Let S' denote the vertex-disjoint union of copies of $K_f^{(r)}$ such that $|S| = |S'|$, and let $T_{S'}$ be the corresponding transformer from S' into L. Then it is easy to see that we could let $A_S := T_S \cup L \cup T_{S'} \cup S'$. The construction of both the canonical graph L as well as that of the transformer T_S is based on an inductive approach, i.e. we assume that our main decomposition result holds for r'-graphs with $1 \leq r' < r$. The above construction is given in Section 2.8.

2.4 Decompositions of supercomplexes

2.4.1 Supercomplexes

We prove our main decomposition theorem for so-called 'supercomplexes'. The crucial property appearing in the definition is that of 'regularity', which means that every r-set of a given complex G is contained in roughly the same number of f-sets (where $f = |V(F)|$). If we view G as a complex which is induced by some r-graph, this means that every edge lies in roughly the same number of cliques of size f. It turns out that this set of conditions is appropriate even when F is not a clique.

A key advantage of the notion of a supercomplex is that the conditions are very flexible, which will enable us to 'boost' their parameters (see Lemma 2.4.4 below).

Definition 2.4.1. Let G be a complex on n vertices, $f \in \mathbb{N}$ and $r \in [f-1]_0$, $0 \leq \varepsilon, d, \xi \leq 1$. We say that G is

(i) (ε, d, f, r)-*regular*, if for all $e \in G^{(r)}$ we have

$$|G^{(f)}(e)| = (d \pm \varepsilon)n^{f-r};$$

(ii) (ξ, f, r)-*dense*, if for all $e \in G^{(r)}$, we have

$$|G^{(f)}(e)| \geq \xi n^{f-r};$$

(iii) (ξ, f, r)-*extendable*, if $G^{(r)}$ is empty or there exists a subset $X \subseteq V(G)$ with $|X| \geq \xi n$ such that for all $e \in \binom{X}{r}$, there are at least ξn^{f-r} $(f-r)$-sets $Q \subseteq V(G) \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G^{(r)}$.

We say that G is a *full* (ε, ξ, f, r)-*complex* if G is

- (ε, d, f, r)-regular for some $d \geq \xi$,
- $(\xi, f + r, r)$-dense,
- (ξ, f, r)-extendable.

We say that G is an (ε, ξ, f, r)-*complex* if there exists an f-graph Y on $V(G)$ such that $G[Y]$ is a full (ε, ξ, f, r)-complex. Note that $G[Y]^{(r)} = G^{(r)}$ (recall that $r < f$).

The additional flexibility offered by considering (ε, ξ, f, r)-complexes rather than full (ε, ξ, f, r)-complexes is key to proving our minimum degree result (via the 'boosting' step discussed below). We also note that for the scope of this thesis, it would be sufficient to define extendability more restrictively, by letting $X := V(G)$. However, for future applications, it might turn out to be useful that we do not require $X = V(G)$.

Fact 2.4.2. *Note that G is an $(\varepsilon, \xi, f, 0)$-complex if and only if G is empty or $|G^{(f)}| \geq \xi n^f$. In particular, every $(\varepsilon, \xi, f, 0)$-complex is a $(0, \xi, f, 0)$-complex.*

Definition 2.4.3. (supercomplex) Let G be a complex. We say that G is an (ε, ξ, f, r)-*supercomplex* if for every $i \in [r]_0$ and every set $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$, we have that $\bigcap_{b \in B} G(b)$ is an $(\varepsilon, \xi, f - i, r - i)$-complex.

In particular, taking $i = 0$ and $B = \{\emptyset\}$ implies that every (ε, ξ, f, r)-supercomplex is also an (ε, ξ, f, r)-complex. Moreover, the above definition ensures that if G is a supercomplex and $b, b' \in G^{(i)}$, then $G(b) \cap G(b')$ is also a supercomplex (cf. Proposition 2.5.5).

In Section 2.4.3, we will give some examples of supercomplexes. As mentioned above, the following lemma allows us to 'boost' the regularity parameters (and thus deduce results with 'effective' bounds). It is an easy consequence of our Boost lemma (Lemma 2.6.3). The key to the proof is that we can (probabilistically) choose some $Y \subseteq G^{(f)}$ so that the parameters of $G[Y]$ in Definition 2.4.1(i) are better than those of G, i.e. the resulting distribution of f-sets is more uniform.

Lemma 2.4.4. *Let $1/n \ll \varepsilon, \xi, 1/f$ and $r \in [f-1]$ with $2(2\sqrt{\mathrm{e}})^r \varepsilon \leq \xi$. Let $\xi' := 0.9(1/4)^{\binom{f+r}{f}}\xi$. If G is an (ε, ξ, f, r)-complex on n vertices, then G is an $(n^{-1/3}, \xi', f, r)$-complex. In particular, if G is an (ε, ξ, f, r)-supercomplex, then it is a $(2n^{-1/3}, \xi', f, r)$-supercomplex.*

2.4.2 The main complex decomposition theorem

The statement of our main complex decomposition theorem involves the concept of 'well separated' decompositions. This is crucial for our inductive proof to work in the context of F-decompositions.

Definition 2.4.5 (well separated)**.** Let F be an r-graph and let $\mathcal{F}$ be an F-packing (in some r-graph G). We say that $\mathcal{F}$ is *κ-well separated* if the following hold:

(WS1) for all distinct $F', F'' \in \mathcal{F}$, we have $|V(F') \cap V(F'')| \leq r$.

(WS2) for every r-set e, the number of $F' \in \mathcal{F}$ with $e \subseteq V(F')$ is at most κ.

We simply say that $\mathcal{F}$ is *well separated* if (WS1) holds.

For instance, any $K_f^{(r)}$-packing is automatically 1-well separated. Moreover, if an F-packing $\mathcal{F}$ is 1-well separated, then for all distinct $F', F'' \in \mathcal{F}$, we have $|V(F') \cap V(F'')| < r$. On the other hand, if F is not complete, we cannot require $|V(F') \cap V(F'')| < r$ in (WS1): this would make it impossible to find an F-decomposition of $K_n^{(r)}$. The notion of being well-separated is a natural relaxation of this requirement, we discuss this in more detail after stating Theorem 2.4.7.

We now define F-divisibility and F-decompositions for complexes G (rather than r-graphs G).

Definition 2.4.6. Let F be an r-graph and $f := |V(F)|$. A complex G is F-*divisible* if $G^{(r)}$ is F-divisible. An F-*packing in* G is an F-packing $\mathcal{F}$ in $G^{(r)}$ such that $V(F') \in G^{(f)}$ for all $F' \in \mathcal{F}$. Similarly, we say that $\mathcal{F}$ is an F-*decomposition of* G if $\mathcal{F}$ is an F-packing in G and $\mathcal{F}^{(r)} = G^{(r)}$.

Note that this implies that every copy F' of F used in an F-packing in G is 'supported' by a clique, i.e. $G^{(r)}[V(F')] \cong K_f^{(r)}$.

We can now state our main complex decomposition theorem.

Theorem 2.4.7 (Main complex decomposition theorem)**.** *For all $r \in \mathbb{N}$, the following is true.*

- $(*)_r$ *Let $1/n \ll 1/\kappa, \varepsilon \ll \xi, 1/f$ and $f > r$. Let F be a weakly regular r-graph on f vertices and let G be an F-divisible (ε, ξ, f, r)-supercomplex on n vertices. Then G has a κ-well separated F-decomposition.*

Note that in light of Lemma 2.4.4, $(*)_r$ already holds if $\varepsilon \leq \frac{\xi}{2(2\sqrt{\mathrm{e}})^r}$. We will prove $(*)_r$ by induction on r in Section 2.9. We do not make any attempt to optimise the values that we obtain for κ.

We now motivate Definitions 2.4.5 and 2.4.6. This involves the following additional concepts, which are also convenient later.

Definition 2.4.8. Let $f := |V(F)|$ and suppose that $\mathcal{F}$ is a well separated F-packing. We let $\mathcal{F}^{\leq}$ denote the complex generated by the f-graph $\{V(F') : F' \in \mathcal{F}\}$. We say that well separated F-packings $\mathcal{F}_1, \mathcal{F}_2$ are *i-disjoint* if $\mathcal{F}_1^{\leq}, \mathcal{F}_2^{\leq}$ are i-disjoint (or equivalently, if $|V(F') \cap V(F'')| < i$ for all $F' \in \mathcal{F}_1$ and $F'' \in \mathcal{F}_2$).

Note that if F is a well-separated F-packing, then the f-graph $\{V(F') : F' \in \mathcal{F}\}$ is simple. Moreover, observe that (WS2) is equivalent to the condition $\Delta_r(\mathcal{F}^{\leq(f)}) \leq \kappa$. Furthermore, if $\mathcal{F}$ is a well separated F-packing in a complex G, then $\mathcal{F}^{\leq}$ is a subcomplex of G by Definition 2.4.6. Clearly, we have $\mathcal{F}^{(r)} \subseteq \mathcal{F}^{\leq(r)}$, but in general equality does not hold. On the other hand, if $\mathcal{F}$ is an F-decomposition of G, then $\mathcal{F}^{(r)} = G^{(r)}$ which implies $\mathcal{F}^{(r)} = \mathcal{F}^{\leq(r)}$.

We now discuss (WS1). During our proof, we will need to find an F-packing which covers a given set of edges. This gives rise to the following task of 'covering down locally'.

($\star$) *Given a set $S \subseteq V(G)$ of size $1 \leq i \leq r-1$, find an F-packing $\mathcal{F}$ which covers all edges of G that contain S.*

(This is crucial in the proof of the Cover down lemma (Lemma 2.7.7). Moreover, a two-sided version of this involving sets S, S' is needed to construct parts of our absorbers, see Section 2.8.1.)

A natural approach to achieve ($\star$) is as follows: Let $T \in \binom{V(F)}{i}$. Suppose that by using the main theorem inductively, we can find an $F(T)$-decomposition $\mathcal{F}'$ of $G(S)$. We now wish to obtain $\mathcal{F}$ by 'extending' $\mathcal{F}'$ as follows: For each copy F' of $F(T)$ in $\mathcal{F}'$, we define a copy $F'_{\triangleleft}$ of F by 'adding S back', that is, $F'_{\triangleleft}$ has vertex set $V(F') \cup S$ and S plays the role of T in $F'_{\triangleleft}$. Then $F'_{\triangleleft}$ covers all edges e with $S \subseteq e$ and $e \setminus S \in F'$. Since $\mathcal{F}'$ is an $F(T)$-decomposition of $G(S)$, the union of all $F'_{\triangleleft}$ would indeed cover all edges of G that contain S, as desired. There are two issues with this 'extension' though. Firstly, it is not clear that $F'_{\triangleleft}$ is a subgraph of G. Secondly, for distinct $F', F'' \in \mathcal{F}'$, it is not clear that $F'_{\triangleleft}$ and $F''_{\triangleleft}$ are edge-disjoint. Definition 2.4.6 (and the succeeding remark) allows us to resolve the first issue. Indeed, if $\mathcal{F}'$ is an $F(T)$-decomposition of the complex $G(S)$, then from $V(F') \in G(S)^{(f-i)}$, we can deduce $V(F'_{\triangleleft}) \in G^{(f)}$ and thus that $F'_{\triangleleft}$ is a subgraph of $G^{(r)}$.

We now consider the second issue. This does not arise if F is a clique. Indeed, in that case $F(T)$ is a copy of $K_{f-i}^{(r-i)}$, and thus for distinct $F', F'' \in \mathcal{F}'$ we have $|V(F') \cap V(F'')| < r-i$. Hence $|V(F'_\triangleleft) \cap V(F''_\triangleleft)| < r-i+|S| = r$, i.e. $F'_\triangleleft$ and $F''_\triangleleft$ are edge-disjoint. If however F is not a clique, then $F', F'' \in \mathcal{F}'$ can overlap in $r-i$ or more vertices (they could in fact have the same vertex set), and the above argument does not work. We will show that under the assumption that $\mathcal{F}'$ is well separated, we can overcome this issue and still carry out the above 'extension'. (Moreover, the resulting F-packing $\mathcal{F}$ will in fact be well separated itself, see Definition 2.7.8 and Proposition 2.7.9). For this it is useful to note that $F(T)$ is an $(r-i)$-graph, and thus we already have $|V(F') \cap V(F'')| \leq r-i$ if $\mathcal{F}'$ is well separated.

The reason why we also include (WS2) in Definition 2.4.5 is as follows. Suppose we have already found a well separated F-packing $\mathcal{F}_1$ in G and now want to find another well separated F-packing $\mathcal{F}_2$ such that we can combine $\mathcal{F}_1$ and $\mathcal{F}_2$. If we find $\mathcal{F}_2$ in $G - \mathcal{F}_1^{(r)}$, then $\mathcal{F}_1^{(r)}$ and $\mathcal{F}_2^{(r)}$ are edge-disjoint and thus $\mathcal{F}_1 \cup \mathcal{F}_2$ will be an F-packing in G, but it is not necessarily well separated. We therefore find $\mathcal{F}_2$ in $G - \mathcal{F}_1^{(r)} - \mathcal{F}_1^{\leq(r+1)}$. This ensures that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(r+1)$-disjoint, which in turn implies that $\mathcal{F}_1 \cup \mathcal{F}_2$ is indeed well separated, as required. But in order to be able to construct $\mathcal{F}_2$, we need to ensure that $G - \mathcal{F}_1^{(r)} - \mathcal{F}_1^{\leq(r+1)}$ is still a supercomplex, which is true if $\Delta(\mathcal{F}_1^{(r)})$ and $\Delta(\mathcal{F}_1^{\leq(r+1)})$ are small (cf. Proposition 2.5.9). The latter in turn is ensured by (WS2) via Fact 2.5.4.

Finally, we discuss why we prove Theorem 2.4.7 for weakly regular r-graphs F. Most importantly, the 'regularity' of the degrees will be crucial for the construction of our absorbers (most notably in Lemma 2.8.25). Beyond that, weakly regular graphs also have useful closure properties (cf. Proposition 2.5.3): they are closed under taking link graphs and divisibility is inherited by link graphs in a natural way.

We prove Theorem 2.4.7 in Sections 2.6–2.8 and 2.9.1. As described in Section 2.1.7, we generalise this to arbitrary F via Lemma 2.9.2 (proved in Section 2.9.2) and Lemma 2.9.4 (proved in Section 2.11): Lemma 2.9.2 shows that for every given r-graph F, there is a weakly regular r-graph F^* which has an F-decomposition. Lemma 2.9.4 then complements this by showing that every F-divisible r-graph G can be transformed into an F^*-divisible

r-graph G' by removing a sparse F-decomposable subgraph of G.

2.4.3 Applications

As the definition of a supercomplex covers a broad range of settings, we give some applications here. We will use Examples 2.4.9, 2.4.10 and 2.4.12 in Section 2.9 to prove Theorems 2.1.1, 2.1.2, 2.1.4, 2.1.5 and 2.1.6. We will also see that random subcomplexes of a supercomplex are again supercomplexes with appropriately adjusted parameters (see Corollary 2.5.19).

Example 2.4.9. Let $1/n \ll 1/f$ and $r \in [f-1]$. It is straightforward to check that the complete complex K_n is a $(0, 0.99/f!, f, r)$-supercomplex.

Recall that (c, h, p)-typicality was defined in Section 2.1.

Example 2.4.10 (Typicality)**.** Suppose that $1/n \ll c, p, 1/f$, that $r \in [f-1]$ and that G is a $(c, 2^r\binom{f+r}{r}, p)$-typical r-graph on n vertices. Then $G^{\leftrightarrow}$ is an (ε, ξ, f, r)-supercomplex, where

$$\varepsilon := 2^{f-r+1}c/(f-r)! \quad \text{and} \quad \xi := (1-2^{f+1}c)p^{2^r\binom{f+r}{r}}/f!.$$

Proof. Let $i \in [r]_0$ and $B \subseteq G^{\leftrightarrow(i)}$ with $1 \leq |B| \leq 2^i$. Let $G_B := \bigcap_{b \in B} G^{\leftrightarrow}(b)$ and $n_B := |V(G) \setminus \bigcup B|$. Let $e \in G_B^{(r-i)}$. To estimate $|G_B^{(f-i)}(e)|$, we let $\mathcal{Q}_e$ be the set of ordered $(f-r)$-tuples $(v_1, \dots, v_{f-r})$ consisting of distinct vertices in $V(G) \setminus (e \cup \bigcup B)$ such that for all $b \in B$, $\binom{b \cup e \cup \{v_1, \dots, v_{f-r}\}}{r} \subseteq G$. Note that $|G_B^{(f-i)}(e)| = |\mathcal{Q}_e|/(f-r)!$. We estimate $|\mathcal{Q}_e|$ by picking $v_1, \dots, v_{f-r}$ sequentially. So let $j \in [f-r]$ and suppose that we have already chosen $v_1, \dots, v_{j-1} \notin e \cup \bigcup B$ such that $\binom{b \cup e \cup \{v_1, \dots, v_{j-1}\}}{r} \subseteq G$ for all $b \in B$. Let $D_j = \bigcup_{b \in B} \binom{b \cup e \cup \{v_1, \dots, v_{j-1}\}}{r-1}$. Thus the possible candidates for v_j are precisely the vertices in $\bigcap_{S \in D_j} G(S)$. Note that $d_j := |D_j| \leq |B|\binom{r+j-1}{r-1}$, and that d_j only depends on the intersection pattern of the $b \in B$, but not on our previous choice of e and $v_1, \dots, v_{j-1}$. Since G is typical, we have $(1 \pm c)p^{d_j}n$ choices for v_j. We conclude that

$$|\mathcal{Q}_e| = (1 \pm c)^{f-r} p^{\sum_{j=1}^{f-r} d_j} n^{f-r} = (1 \pm 2^{f-r+1}c) d_B (f-r)! n_B^{f-r},$$

where $d_B := p^{\sum_{j=1}^{f-r} d_j}/(f-r)!$. Thus, G_B is $(2^{f-r+1}cd_B, d_B, f-i, r-i)$-regular. Since $\sum_{j=1}^{f-r}\binom{r+j-1}{r-1} = \binom{f}{r} - 1$ we have $1/(f-r)! \geq d_B \geq p^{|B|(\binom{f}{r}-1)}/(f-r)! \geq p^{2^r\binom{f}{r}}/(f-r)!$. Similarly, we deduce that G_B is $((1-2^{f-r+1}c)d_B, f-i, r-i)$-extendable. Moreover, we have

$$|G_B^{(f+r-2i)}(e)| \geq \frac{(1-2^{f-i+1}c)p^{2^r\binom{f+r-i}{r}}}{(f-i)!} n_B^{f-i} \geq \xi n_B^{f-i}.$$

Thus, G_B is $(\xi, f+r-2i, r-i)$-dense. We conclude that G_B is an $(\varepsilon, \xi, f-i, r-i)$-complex. □

Example 2.4.11 (Partite graphs)**.** Let $1/N \ll 1/k$ and $2 = r < f \leq k-6$. Let $V_1, \dots, V_k$ be vertex sets of size N each. Let G be the complete k-partite 2-graph on $V_1, \dots, V_k$. It is straightforward to check that $G^{\leftrightarrow}$ is a $(0, k^{-f}, f, 2)$-supercomplex. Thus, using Theorem 2.4.7, we can deduce that G has an F-decomposition if it is F-divisible. To obtain a minimum degree version (and more generally, a resilience version) along the lines of Theorems 2.1.4 and 2.9.3, one can argue similarly as in the proof of Theorem 2.9.3 (cf. Section 2.9).

Results on (fractional) decompositions of dense f-partite 2-graphs into f-cliques are proved in [10, 26, 27, 68]. These have applications to the completion of partial (mutually orthogonal) Latin squares.

Example 2.4.12 (The matching case)**.** Consider $1 = r < f$. Let G be a f-graph on n vertices such that the following conditions hold for some $0 < \varepsilon \leq \xi \leq 1$:

- for some $d \geq \xi - \varepsilon$, $|G(v)| = (d \pm \varepsilon)n^{f-1}$ for all $v \in V(G)$;
- every vertex is contained in at least ξn^f copies of $K_{f+1}^{(f)}$;
- $|G(v) \cap G(w)| \geq \xi n^{f-1}$ for all $v, w \in V(G)$.

Then $G^{\leftrightarrow}$ is an $(\varepsilon, \xi - \varepsilon, f, 1)$-supercomplex.

2.4.4 Disjoint decompositions and designs

Recall that a $K_f^{(r)}$-decomposition of an r-graph is an $(K_f^{(r)}, 1)$-design. We now discuss consequences of our main theorem for general $(K_f^{(r)}, \lambda)$-designs. We can deduce from Theorem 2.4.7 that there are many f-disjoint $K_f^{(r)}$-decompositions, see Corollary 2.4.14. This will easily follow from $(*)_r$ and the next result.

Proposition 2.4.13. *Let $1/n \ll \varepsilon, \xi, 1/f$ and $r \in [f-1]$. Suppose that G is an (ε, ξ, f, r)-supercomplex on n vertices. Let Y_{used} be an f-graph on $V(G)$ with $\Delta_r(Y_{used}) \leq \varepsilon n^{f-r}$. Then $G - Y_{used}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, f, r)$-supercomplex.*

We will apply this when $\mathcal{K}_1, \dots, \mathcal{K}_t$ are $K_f^{(r)}$-packings in some complex G, in which case $Y_{used} := \bigcup_{j\in[t]} \mathcal{K}_j^{(f)}$ satisfies $\Delta_r(Y_{used}) \leq t$.

Proof. Fix $i \in [r]_0$ and $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$. Let $n_B := n - |\bigcup B|$, $G' := \bigcap_{b\in B} G(b)$ and $G'' := \bigcap_{b\in B}(G - Y_{used})(b)$. By assumption, there exists $Y \subseteq G'^{(f-i)}$ such that $G'[Y]$ is a full $(\varepsilon, \xi, f-i, r-i)$-complex. We claim that $G''[Y]$ is a full $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, f-i, r-i)$-complex.

First, there is some $d \geq \xi$ such that $G'[Y]$ is $(\varepsilon, d, f-i, r-i)$-regular. Let $e \in G'^{(r-i)}$. We clearly have $|G''[Y]^{(f-i)}(e)| \leq |G'[Y]^{(f-i)}(e)| \leq (d+\varepsilon)n_B^{f-r}$. Moreover, for each $b \in B$, there are at most εn^{f-r} f-sets in Y_{used} that contain $e \cup b$. Thus, $|G''[Y]^{(f-i)}(e)| \geq (d-\varepsilon)n_B^{f-r} - |B|\varepsilon n^{f-r} \geq (d - \varepsilon - 1.1\cdot 2^i\varepsilon)n_B^{f-r}$. Thus, $G''[Y]$ is $(2^{r+2}\varepsilon, d, f-i, r-i)$-regular.

Next, by assumption we have that $G'[Y]$ is $(\xi, f+r-2i, r-i)$-dense. Let $e \in G'^{(r-i)}$. For each $b \in B$, we claim that the number N_b of $(f+r-i)$-sets in $V(G)$ that contain $e \cup b$ and also contain some f-set from Y_{used} is at most $2^r\varepsilon n^{f-i}$. Indeed, for any $k \in \{i, \dots, r\}$ and any $K \in Y_{used}$ with $|(e\cup b)\cap K| = k$, there are at most n^{k-i} $(f+r-i)$-sets that contain $e \cup b$ and K. Moreover, there are at most $\binom{r}{k}\Delta_k(Y_{used}) \leq \binom{r}{k}n^{r-k}\Delta_r(Y_{used}) \leq \binom{r}{k}\varepsilon n^{f-k}$ f-sets $K \in Y_{used}$ with $|(e \cup b) \cap K| = k$. Hence, $N_b \leq \sum_{k=i}^{r} n^{k-i}\binom{r}{k}\varepsilon n^{f-k} \leq \varepsilon 2^r n^{f-i}$. We then deduce that

$$|G''[Y]^{(f+r-2i)}(e)| \geq \xi n_B^{f-i} - |B|2^r\varepsilon n^{f-i} \geq \xi n_B^{f-i} - \varepsilon 2^{r+i} n^{f-i} \geq (\xi - 2^{2r+1}\varepsilon)n_B^{f-i}.$$

Finally, since $G''[Y]^{(r-i)} = G'[Y]^{(r-i)}$, $G''[Y]$ is $(\xi, f-i, r-i)$-extendable. Thus, $G - Y_{used}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, f, r)$-supercomplex. □

Clearly, any complex G on n vertices can have at most $n^{f-r}/(f-r)!$ f-disjoint $K_f^{(r)}$-decompositions. Moreover, if G has λ f-disjoint $K_f^{(r)}$-decompositions, then $G^{(r)}$ has a $(K_f^{(r)}, \lambda)$-design.

Corollary 2.4.14. *Let $1/n \ll \varepsilon, \xi, 1/f$ and $r \in [f-1]$ with $10 \cdot 7^r \varepsilon \leq \xi$ and assume that $(*)_r$ is true. Suppose that G is a $K_f^{(r)}$-divisible (ε, ξ, f, r)-supercomplex on n vertices. Then G has εn^{f-r} f-disjoint $K_f^{(r)}$-decompositions. In particular, $G^{(r)}$ has a $(K_f^{(r)}, \lambda)$-design for all $1 \leq \lambda \leq \varepsilon n^{f-r}$.*

Proof. Suppose that $\mathcal{K}_1, \dots, \mathcal{K}_t$ are f-disjoint $K_f^{(r)}$-decompositions of G, where $t \leq \varepsilon n^{f-r}$. By Proposition 2.4.13 (and the subsequent remark), $G - \bigcup_{j \in [t]} \mathcal{K}_j^{(f)}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, f, r)$-supercomplex. Since $2(2\sqrt{\mathrm{e}})^r 2^{r+2}\varepsilon \leq \xi - 2^{2r+1}\varepsilon$, $G - \bigcup_{j \in [t]} \mathcal{K}_j^{(f)}$ has a $K_f^{(r)}$-decomposition $\mathcal{K}_{t+1}$ by (the remark after) $(*)_r$, which is f-disjoint from $\mathcal{K}_1, \dots, \mathcal{K}_t$. □

Note that Corollary 2.4.14 together with Example 2.4.9 implies that whenever $1/n \ll 1/f$ and $K_n^{(r)}$ is $K_f^{(r)}$-divisible, then $K_n^{(r)}$ has a $(K_f^{(r)}, \lambda)$-design for all $1 \leq \lambda \leq \frac{1}{11 \cdot 7^r f!} n^{f-r}$, which improves the bound $\lambda / n^{f-r} \ll 1$ in [49].

Using (WS2), we can deduce that there are many f-disjoint F-decompositions of a supercomplex. This will be an important tool in the proof of the Cover down lemma (Lemma 2.7.7), where we will find many candidate F-decompositions and then pick one at random.

Corollary 2.4.15. *Let $1/n \ll \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$ and assume that $(*)_r$ is true. Let F be a weakly regular r-graph on f vertices. Suppose that G is an F-divisible (ε, ξ, f, r)-supercomplex on n vertices. Then the number of pairwise f-disjoint $1/\varepsilon$-well separated F-decompositions of G is at least $\varepsilon^2 n^{f-r}$.*

Proof. Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_t$ are f-disjoint $1/\varepsilon$-well separated F-decompositions of G, where $t \leq \varepsilon^2 n^{f-r}$. Let $Y_{used} := \bigcup_{j \in [t]} \mathcal{F}_j^{\leq (f)}$. By (WS2), we have $\Delta_r(Y_{used}) \leq t/\varepsilon \leq$

εn^{f-r}. Thus, by Proposition 2.4.13, $G - Y_{used}$ is an F-divisible $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, f, r)$-supercomplex and thus has a $1/\varepsilon$-well separated F-decomposition $\mathcal{F}_{t+1}$ by $(*)_r$, which is f-disjoint from $\mathcal{F}_1, \dots, \mathcal{F}_t$. □

2.5 Tools

2.5.1 Basic tools

We will often use the following 'handshaking lemma' for r-graphs: Let G be an r-graph and $0 \le i \le k \le r-1$. Then for every $S \in \binom{V(G)}{i}$ we have

$$|G(S)| = \binom{r-i}{r-k}^{-1} \sum_{T \in \binom{V(G)}{k} :\, S \subseteq T} |G(T)|. \tag{2.5.1}$$

Fact 2.5.1. *Let L be an r-graph on n vertices with $\Delta(L) \le \gamma n$. Then for each $i \in [r-1]_0$, we have $\Delta_i(L) \le \gamma n^{r-i}/(r-i)!$, and for each $S \in \binom{V(L)}{i}$, we have $\Delta(L(S)) \le \gamma n$.*

Proposition 2.5.2. *Let F be an r-graph. Then there exist infinitely many $n \in \mathbb{N}$ such that $K_n^{(r)}$ is F-divisible.*

Proof. Let $p := \prod_{i=0}^{r-1} Deg(F)_i$. We will show that for every $a \in \mathbb{N}$, if we let $n = r!ap + r - 1$ then $K_n^{(r)}$ is F-divisible. Clearly, this implies the claim. In order to see that $K_n^{(r)}$ is F-divisible, it is sufficient to show that $p \mid \binom{n-i}{r-i}$ for all $i \in [r-1]_0$. It is easy to see that this holds for the above choice of n. □

The following proposition shows that the class of weakly regular uniform hypergraphs is closed under taking link graphs.

Proposition 2.5.3. *Let F be a weakly regular r-graph and let $i \in [r-1]$. Suppose that $S \in \binom{V(F)}{i}$ and that $F(S)$ is non-empty. Then $F(S)$ is a weakly regular $(r-i)$-graph and $Deg(F(S))_j = Deg(F)_{i+j}$ for all $j \in [r-i-1]_0$.*

Proof. Let $s_0, \dots, s_{r-1}$ be such that F is weakly $(s_0, \dots, s_{r-1})$-regular. Note that since F is non-empty, we have $s_j > 0$ for all $j \in [r-1]_0$ (and the s_i's are unique). Consider $j \in [r-i-1]_0$. For all $T \in \binom{V(F(S))}{j}$, we have $|F(S)(T)| = |F(S \cup T)| \in \{0, s_{i+j}\}$. Hence, $F(S)$ is weakly $(s_i, \dots, s_{r-1})$-regular. Since F is non-empty, we have $Deg(F) = (s_0, \dots, s_{r-1})$, and since $F(S)$ is non-empty too by assumption, we have $Deg(F(S)) = (s_i, \dots, s_{r-1})$. Therefore, $Deg(F(S))_j = Deg(F)_{i+j}$ for all $j \in [r-i-1]_0$. □

We now list some useful properties of well separated F-packings.

Fact 2.5.4. *Let G be a complex and F an r-graph on $f > r$ vertices. Suppose that $\mathcal{F}$ is a κ-well separated F-packing (in G) and $\mathcal{F}'$ is a κ'-well separated F-packing (in G). Then the following hold.*

(i) $\Delta(\mathcal{F}^{\le(r+1)}) \le \kappa(f-r)$.

(ii) *If $\mathcal{F}^{(r)}$ and $\mathcal{F}'^{(r)}$ are edge-disjoint and $\mathcal{F}$ and $\mathcal{F}'$ are $(r+1)$-disjoint, then $\mathcal{F} \cup \mathcal{F}'$ is a $(\kappa+\kappa')$-well separated F-packing (in G).*

(iii) *If $\mathcal{F}$ and $\mathcal{F}'$ are r-disjoint, then $\mathcal{F} \cup \mathcal{F}'$ is a $\max\{\kappa, \kappa'\}$-well separated F-packing (in G).*

2.5.2 Some properties of supercomplexes

We first state two basic properties of supercomplexes that we will use in Section 2.8 to construct absorbers.

Proposition 2.5.5. *Let G be an (ε, ξ, f, r)-supercomplex and let $B \subseteq G^{(i)}$ with $1 \le |B| \le 2^i$ for some $i \in [r]_0$. Then $\bigcap_{b \in B} G(b)$ is an $(\varepsilon, \xi, f-i, r-i)$-supercomplex.*

Proof. Let $i' \in [r-i]_0$ and $B' \subseteq (\bigcap_{b \in B} G(b))^{(i')}$ with $1 \le |B'| \le 2^{i'}$. Let $B^* := \{b \cup b' : b \in B, b' \in B'\}$. Note that $B^* \subseteq G^{(i+i')}$ and $|B^*| \le 2^{i+i'}$. Thus,

$$\bigcap_{b' \in B'} (\bigcap_{b \in B} G(b))(b') = \bigcap_{b^* \in B^*} G(b^*)$$

is an $(\varepsilon, \xi, f-i-i', r-i-i')$-complex by Definition 2.4.3, as required. □

Fact 2.5.6. *If G is an (ε, ξ, f, r)-supercomplex, then for all distinct $e, e' \in G^{(r)}$, we have $|G^{(f)}(e) \cap G^{(f)}(e')| \geq (\xi - \varepsilon)(n-2r)^{f-r}$.*

In what follows, we gather tools that show that supercomplexes are robust with respect to small perturbations. We first bound the number of f-sets that can affect a given edge e. We provide two bounds, one that we use when optimising our bounds (e.g. in the derivation of Theorem 2.1.4) and a more convenient one that we use when the precise value of the parameters is irrelevant (e.g. in the proof of Proposition 2.5.9).

Proposition 2.5.7. *Let $f, r' \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $f > r$. Let L be an r'-graph on n vertices with $\Delta(L) \leq \gamma n$. Then every $e \in \binom{V(L)}{r}$ that does not contain any edge of L is contained in at most $\min\{2^r, \frac{\binom{f}{r'}}{(f-r)!}\}\gamma n^{f-r}$ f-sets of $V(L)$ that contain an edge of L.*

Proof. Consider any $e \in \binom{V(L)}{r}$ that does not contain any edge of L. For a fixed edge $e' \in L$ with $|e \cup e'| \leq f$ and $|e \cap e'| = i$, there are at most $\binom{n-|e\cup e'|}{f-|e\cup e'|} \leq n^{f-r-r'+i}/(f-r-r'+i)!$ f-sets of $V(L)$ that contain both e and e'. Moreover, since $e' \not\subseteq e$, we have $i < r'$. Hence, by Fact 2.5.1, there are at most $\binom{r}{i}\Delta_i(L) \leq \binom{r}{i}\gamma n^{r'-i}/(r'-i)!$ edges $e' \in L$ with $|e \cap e'| = i$. Let $s := \max\{r + r' - f, 0\}$. Thus, the number of f-sets in $V(L)$ that contain e and an edge of L is at most

$$\sum_{i=s}^{r'-1} \gamma \binom{r}{i} \frac{n^{r'-i}}{(r'-i)!} \frac{n^{f-r-r'+i}}{(f-r-r'+i)!} = \gamma n^{f-r} \sum_{i=s}^{r'-1} \binom{r}{i} \frac{\binom{f-r}{r'-i}}{(f-r)!}.$$

Clearly, $\frac{\binom{f-r}{r'-i}}{(f-r)!} \leq 1$, and we can bound $\sum_{i=s}^{r'-1} \binom{r}{i} \leq 2^r$. Also, using Vandermonde's convolution, we have $\sum_{i=s}^{r'-1} \binom{r}{i} \frac{\binom{f-r}{r'-i}}{(f-r)!} \leq \frac{\binom{f}{r'}}{(f-r)!}$. □

Fact 2.5.8. *Let $0 \leq i \leq r$. For a complex G, an r-graph H and $B \subseteq G^{(i)}$, we have*

$$\bigcap_{b\in B} (G-H)(b) = \bigcap_{b \in B} G(b) - H - \bigcup_{S \in \bigcup B} H(S) - \bigcup_{S \in \bigcup_{b\in B}\binom{b}{2}} H(S) - \cdots - \bigcup_{b \in B} H(b).$$

If $B \not\subseteq (G-H)^{(i)}$, then both sides are empty.

Proposition 2.5.9. *Let $f, r' \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $f > r$ and $r' \geq r$. Let G be a complex on $n \geq r2^{r+1}$ vertices and let H be an r'-graph on $V(G)$ with $\Delta(H) \leq \gamma n$. Then the following hold:*

(i) *If G is (ε, d, f, r)-regular, then $G - H$ is $(\varepsilon + 2^r\gamma, d, f, r)$-regular.*

(ii) *If G is (ξ, f, r)-dense, then $G - H$ is $(\xi - 2^r\gamma, f, r)$-dense.*

(iii) *If G is (ξ, f, r)-extendable, then $G - H$ is $(\xi - 2^r\gamma, f, r)$-extendable.*

(iv) *If G is an (ε, ξ, f, r)-complex, then $G - H$ is an $(\varepsilon + 2^r\gamma, \xi - 2^r\gamma, f, r)$-complex.*

(v) *If G is an (ε, ξ, f, r)-supercomplex, then $G - H$ is an $(\varepsilon + 2^{2r+1}\gamma, \xi - 2^{2r+1}\gamma, f, r)$-supercomplex.*

Proof. (i)–(iii) follow directly from Proposition 2.5.7. (iv) follows from (i)–(iii). To see (v), suppose that $i \in [r]_0$ and $B \subseteq (G-H)^{(i)}$ with $1 \leq |B| \leq 2^i$. By assumption, $\bigcap_{b\in B} G(b)$ is an $(\varepsilon, \xi, f-i, r-i)$-complex. By Fact 2.5.8, we can obtain $\bigcap_{b\in B}(G-H)(b)$ from $\bigcap_{b\in B} G(b)$ by repeatedly deleting an $(r'-|S|)$-graph $H(S)$, where $S \subseteq b \in B$. There are at most $|B|2^i \leq 2^{2i}$ such graphs. Unless $|S| = r'$, we have $\Delta(H(S)) \leq \gamma n \leq 2\gamma(n - |\bigcup B|)$ by Fact 2.5.1. Note that if $|S| = r'$, then $S \in B$ and hence $H(S)$ is empty, in which case we can ignore its removal. Thus, a repeated application of (iv) (with $r' - |S|, r-i$ playing the roles of r', r) shows that $\bigcap_{b\in B}(G-H)(b)$ is an $(\varepsilon + 2^{r+i+1}\gamma, \xi - 2^{r+i+1}\gamma, f-i, r-i)$-complex. □

2.5.3 Probabilistic tools

The following Chernoff-type bounds form the basis of our concentration results that we use for probabilistic arguments.

Lemma 2.5.10 (see [47, Corollary 2.3, Corollary 2.4, Remark 2.5 and Theorem 2.8])**.** *Let X be the sum of n independent Bernoulli random variables. Then the following hold.*

(i) *For all* $t \geq 0$, $\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq 2\mathrm{e}^{-2t^2/n}$.

(ii) *For all* $0 \leq \varepsilon \leq 3/2$, $\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2\mathrm{e}^{-\varepsilon^2 \mathbb{E}X/3}$.

(iii) *If* $t \geq 7\mathbb{E}X$, *then* $\mathbb{P}(X \geq t) \leq \mathrm{e}^{-t}$.

We will also use the following simple result.

Proposition 2.5.11 (Jain, see [73, Lemma 8])**.** *Let* $X_1, \dots, X_n$ *be Bernoulli random variables such that, for any* $i \in [n]$ *and any* $x_1, \dots, x_{i-1} \in \{0, 1\}$,

$$\mathbb{P}(X_i = 1 \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \leq p.$$

Let $B \sim B(n, p)$ *and* $X := X_1 + \cdots + X_n$. *Then* $\mathbb{P}(X \geq a) \leq \mathbb{P}(B \geq a)$ *for any* $a \geq 0$.

Lemma 2.5.12. *Let* $1/n \ll p, \alpha, 1/a, 1/B$. *Let* $\mathcal{I}$ *be a set of size at least* αn^a *and let* $(X_i)_{i \in \mathcal{I}}$ *be a family of Bernoulli random variables with* $\mathbb{P}(X_i = 1) \geq p$. *Suppose that* $\mathcal{I}$ *can be partitioned into at most* Bn^{a-1} *sets* $\mathcal{I}_1, \dots, \mathcal{I}_k$ *such that for each* $j \in [k]$, *the variables* $(X_i)_{i \in \mathcal{I}_j}$ *are independent. Let* $X := \sum_{i \in \mathcal{I}} X_i$. *Then we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geq n^{-1/5}\mathbb{E}X) \leq \mathrm{e}^{-n^{1/6}}.$$

Proof. Let $\mathcal{J}_1 := \{j \in [k] : |\mathcal{I}_j| \geq n^{3/5}\}$ and $\mathcal{J}_2 := [k] \setminus \mathcal{J}_1$. Let $Y_j := \sum_{i \in \mathcal{I}_j} X_i$ and $\varepsilon := n^{-1/5}$. Suppose that $|Y_j - \mathbb{E}Y_j| \leq 0.9\varepsilon \mathbb{E}Y_j$ for all $j \in \mathcal{J}_1$. Then

$$|X - \mathbb{E}X| \leq \sum_{j \in [k]} |Y_j - \mathbb{E}Y_j| \leq n^{3/5} \cdot Bn^{a-1} + \sum_{j \in \mathcal{J}_1} 0.9\varepsilon \mathbb{E}Y_j \leq Bn^{a-2/5} + 0.9\varepsilon \mathbb{E}X \leq \varepsilon \mathbb{E}X.$$

Thus,

$$\begin{aligned}\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) &\leq \sum_{j \in \mathcal{J}_1} \mathbb{P}(|Y_j - \mathbb{E}Y_j| \geq 0.9\varepsilon \mathbb{E}Y_j) \overset{\text{Lemma 2.5.10(ii)}}{\leq} \sum_{j \in \mathcal{J}_1} 2\mathrm{e}^{-0.81\varepsilon^2 \mathbb{E}Y_j/3} \\ &\leq 2Bn^{a-1}\mathrm{e}^{-0.27n^{-2/5}pn^{3/5}} \leq \mathrm{e}^{-n^{1/6}}.\end{aligned}$$

□

Similarly as in [42], Lemma 2.5.12 can be conveniently applied in the following situation: We are given an r-graph H on n vertices and H' is a random subgraph of H, where every edge of H survives with some probability $\geq p$. The following folklore observation allows us to apply Lemma 2.5.12 in order to obtain a concentration result for $|H'|$.

Fact 2.5.13. *Every r-graph on n vertices can be decomposed into rn^{r-1} matchings.*

Corollary 2.5.14. *Let $1/n \ll p, 1/r, \alpha$. Let H be an r-graph on n vertices with $|H| \geq \alpha n^r$. Let H' be a random subgraph of H, where each edge of H survives with some probability $\geq p$. Moreover, suppose that for every matching M in H, the edges of M survive independently. Then we have*

$$\mathbb{P}(||H'| - \mathbb{E}|H'|| \geq n^{-1/5}\mathbb{E}|H'|) \leq \mathrm{e}^{-n^{1/6}}.$$

Whenever we apply Corollary 2.5.14, it will be clear that for every matching M in H, the edges of M survive independently, and we will not discuss this explicitly.

Lemma 2.5.15. *Let $1/n \ll p, 1/r$. Let H be an r-graph on n vertices. Let H' be a random subgraph of H, where each edge of H survives with some probability $\leq p$. Suppose that for every matching M in H, the edges of M survive independently. Then we have*

$$\mathbb{P}(|H'| \geq 7pn^r) \leq rn^{r-1}\mathrm{e}^{-7pn/r}.$$

Proof. Partition H into at most rn^{r-1} matchings $M_1, \dots, M_k$. For each $i \in [k]$, by Lemma 2.5.10(iii) we have $\mathbb{P}(|H' \cap M_i| \geq 7pn/r) \leq \mathrm{e}^{-7pn/r}$ since $\mathbb{E}|H' \cap M_i| \leq pn/r$.

□

2.5.4 Random subsets and subgraphs

In this subsection, we apply the above tools to obtain basic results about random subcomplexes. The first one deals with taking a random subset of the vertex set, and the second one considers the complex obtained by randomly sparsifying $G^{(r)}$.

Proposition 2.5.16. *Let $1/n \ll \varepsilon, \xi, 1/f$ and $1/n \ll \gamma \ll \mu, 1/f$ and $r \in [f-1]_0$. Let G be an (ε, ξ, f, r)-complex on n vertices. Suppose that U is a random subset of $V(G)$ obtained by including every vertex from $V(G)$ independently with probability μ. Then with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, the following holds: for any $W \subseteq V(G)$ with $|W| \leq \gamma n$, $G[U \triangle W]$ is an $(\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}, \xi - n^{-1/5} - \tilde{\gamma}^{2/3}, f, r)$-complex, where $\tilde{\gamma} := \max\{|W|/n, n^{-1/3}\}$.*

Proof. If $G^{(r)}$ is empty, there is nothing to prove, so assume the contrary.

By assumption, there exists $Y \subseteq G^{(f)}$ such that $G[Y]$ is (ε, d, f, r)-regular for some $d \geq \xi$, $(\xi, f+r, r)$-dense and (ξ, f, r)-extendable. The latter implies that there exists $X \subseteq V(G)$ with $|X| \geq \xi n$ such that for all $e \in \binom{X}{r}$, we have $|Ext_e| \geq \xi n^{f-r}$, where Ext_e is the set of all $(f-r)$-sets $Q \subseteq V(G) \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G^{(r)}$.

First, by Lemma 2.5.10(i), with probability at least $1 - 2\mathrm{e}^{-2n^{1/3}}$, we have $|U| = \mu n \pm n^{2/3}$, and with probability at least $1 - 2\mathrm{e}^{-2n^{1/4}}$, $|X \cap U| \geq \mu|X| - |X|^{2/3}$.

Claim 1: For all $e \in G^{(r)}$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, $|G[Y]^{(f)}(e)[U]| = (d \pm (\varepsilon + 2n^{-1/5}))(\mu n)^{f-r}$.

Proof of claim: Fix $e \in G^{(r)}$. Note that $\mathbb{E}|G[Y]^{(f)}(e)[U]| = \mu^{f-r}|G[Y]^{(f)}(e)| = (d \pm \varepsilon)(\mu n)^{f-r}$. Viewing $G[Y]^{(f)}(e)$ as a $(f-r)$-graph and $G[Y]^{(f)}(e)[U]$ as a random subgraph, we deduce with Corollary 2.5.14 that

$$\mathbb{P}(|G[Y]^{(f)}(e)[U]| \neq (1 \pm n^{-1/5})(d \pm \varepsilon)(\mu n)^{f-r}) \leq \mathrm{e}^{-n^{1/6}}.$$

–

Claim 2: For all $e \in G^{(r)}$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, $|G[Y]^{(f+r)}(e)[U]| \geq (\xi - n^{-1/5})(\mu n)^{f}$.

Proof of claim: Note that $\mathbb{E}|G^{(f+r)}(e)[U]| = \mu^f|G^{(f+r)}(e)| \geq \xi(\mu n)^f$. Viewing $G^{(f+r)}(e)$ as a f-graph and $G^{(f+r)}(e)[U]$ as a random subgraph, we deduce with Corollary 2.5.14

that

$$\mathbb{P}(|G^{(f+r)}(e)[U]| \leq (1-n^{-1/5})\xi(\mu n)^f) \leq \mathrm{e}^{-n^{1/6}}.$$

–

For $e \in \binom{X}{r}$, let Ext'_e be the random subgraph of Ext_e containing all $Q \in Ext_e$ with $Q \subseteq U$.

Claim 3: For all $e \in \binom{X}{r}$, with probability at least $1-\mathrm{e}^{-n^{1/6}}$, $|Ext'_e| \geq (\xi - n^{-1/5})(\mu n)^{f-r}$.

Proof of claim: Let $e \in \binom{X}{r}$. Note that $\mathbb{E}|Ext'_e| = \mu^{f-r}|Ext_e| \geq \xi(\mu n)^{f-r}$. Again, Corollary 2.5.14 implies that

$$\mathbb{P}(|Ext'_e| \leq (1-n^{-1/5})\xi(\mu n)^{f-r}) \leq \mathrm{e}^{-n^{1/6}}.$$

–

Hence, a union bound yields that with probability at least $1-\mathrm{e}^{-n^{1/7}}$, we have $|U| = \mu n \pm n^{2/3}$, $|X \cap U| \geq \mu|X| - |X|^{2/3}$ and the above claims hold for all relevant e simultaneously. Assume that this holds for some outcome U. We now deduce the desired result deterministically. Let $W \subseteq V(G)$ with $|W| \leq \gamma n$. Define $G' := G[U \triangle W]$ and $n' := |U \triangle W|$. Note that $\mu n = (1 \pm 4\mu^{-1}\tilde{\gamma})n'$. For all $e \in G'^{(r)}$, we have

$$\begin{aligned}|G'[Y]^{(f)}(e)| = |G[Y]^{(f)}(e)[U]| \pm |W|n^{f-r-1} &= (d \pm (\varepsilon + 2n^{-1/5} + \frac{|W|}{\mu^{f-r}n}))(\mu n)^{f-r} \\ &= (d \pm (\varepsilon + 2n^{-1/5} + \mu^{-(f-r)}\tilde{\gamma}))(1 \pm 2^{f-r}4\mu^{-1}\tilde{\gamma})n'^{f-r} \\ &= (d \pm (\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}))n'^{f-r}\end{aligned}$$

and

$$\begin{aligned}|G'[Y]^{(f+r)}(e)| \geq |G[Y]^{(f+r)}(e)[U]| - |W|n^{f-1} &\geq (\xi - n^{-1/5} - \frac{|W|}{\mu^f n})(\mu n)^f \\ &\geq (\xi - n^{-1/5} - \mu^{-f}\tilde{\gamma})(1 - 2^f 4\mu^{-1}\tilde{\gamma})n'^{f-r} \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'^{f-r},\end{aligned}$$

so $G'[Y]$ is $(\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}, d, f, r)$-regular and $(\xi - n^{-1/5} - \tilde{\gamma}^{2/3}, f + r, r)$-dense.

Finally, let $X' := (X \cap U) \setminus W$. Clearly, $X' \subseteq V(G')$ and $|X'| \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'$. Moreover, for every $e \in \binom{X'}{r}$, there are at least

$$|Ext'_e| - |W|n^{f-r-1} \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'^{f-r}$$

$(f - r)$-sets $Q \subseteq V(G') \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G'^{(r)}$. Thus, G' (and therefore $G'[Y]$) is $(\xi - n^{-1/5} - \tilde{\gamma}^{2/3}, f, r)$-extendable. □

The next result is a straightforward consequence of Proposition 2.5.16 and the definition of a supercomplex.

Corollary 2.5.17. *Let $1/n \ll \gamma \ll \mu \ll \varepsilon \ll \xi, 1/f$ and $r \in [f - 1]$. Let G be an (ε, ξ, f, r)-supercomplex on n vertices. Suppose that U is a random subset of $V(G)$ obtained by including every vertex from $V(G)$ independently with probability μ. Then whp for any $W \subseteq V(G)$ with $|W| \leq \gamma n$, $G[U \triangle W]$ is a $(2\varepsilon, \xi - \varepsilon, f, r)$-supercomplex.*

Next, we investigate the effect on G of inducing to a random subgraph H of $G^{(r)}$. For our applications, we need to be able to choose edges with different probabilities. It turns out that under suitable restrictions on these probabilities, the relevant properties of G are inherited by $G[H]$.

Proposition 2.5.18. *Let $1/n \ll \varepsilon, \gamma, p, \xi, 1/f$ and $r \in [f - 1]$, $i \in [r]_0$. Let*

$$\xi' := 0.95\xi p^{2^r\binom{f+r}{r}} \geq 0.95\xi p^{(8^f)} \text{ and } \gamma' := 1.1 \cdot 2^i \frac{\binom{f+r}{r}}{(f-r)!}\gamma.$$

Let G be a complex on n vertices and $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$. Suppose that

$$G_B := \bigcap_{b \in B} G(b) \text{ is an } (\varepsilon, \xi, f - i, r - i)\text{-complex.}$$

Assume that $\mathcal{P}$ is a partition of $G^{(r)}$ satisfying the following containment conditions:

(I) *For every $b \in B$, there exists a class $\mathcal{E}_b \in \mathcal{P}$ such that $b \cup e \in \mathcal{E}_b$ for all $e \in G_B^{(r-i)}$.*

(II) *For every $\mathcal{E} \in \mathcal{P}$ there exists $D_\mathcal{E} \in \mathbb{N}_0$ such that for all $Q \in G_B^{(f-i)}$, we have that* $|\{e \in \mathcal{E} : \exists b \in B\colon e \subseteq b \cup Q\}| = D_\mathcal{E}$.

Let $\beta\colon \mathcal{P} \to [p,1]$ assign a probability to every class of $\mathcal{P}$. Now, suppose that H is a random subgraph of $G^{(r)}$ obtained by independently including every edge of $\mathcal{E} \in \mathcal{P}$ with probability $\beta(\mathcal{E})$ (for all $\mathcal{E} \in \mathcal{P}$). Then with probability at least $1 - \mathrm{e}^{-n^{1/8}}$, the following holds: for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \gamma n$ and all $(r+1)$-graphs O on $V(G)$ with $\Delta(O) \le f^{-5r}\gamma n$,

$$\bigcap_{b \in B} (G[H \triangle L] - O)(b) \text{ is a } (3\varepsilon + \gamma', \xi' - \gamma', f-i, r-i)\text{-complex.}$$

Note that (I) and (II) certainly hold if $\mathcal{P} = \{G^{(r)}\}$.

Proof. If $G_B^{(r-i)}$ is empty, then the statement is vacuously true. So let us assume that $G_B^{(r-i)}$ is not empty. Let $n_B := |V(G) \setminus \bigcup B| = |V(G_B)|$. By assumption, there exists $Y \subseteq G_B^{(f-i)}$ such that $G_B[Y]$ is $(\varepsilon, d_B, f-i, r-i)$-regular for some $d_B \ge \xi$, $(\xi, f+r-2i, r-i)$-dense and $(\xi, f-i, r-i)$-extendable. Define

$$p_B := \left(\prod_{b \in B} \beta(\mathcal{E}_b)\right)^{-1} \prod_{\mathcal{E} \in \mathcal{P}} (\beta(\mathcal{E}))^{D_\mathcal{E}}.$$

Note that $p_B \ge p^{|B|\binom{f}{r}} \ge p^{2^r\binom{f+r}{r}}$ and thus $p_B d_B \ge \xi'$. For every $e \in G_B^{(r-i)}$, let

$$\mathcal{Q}_e := G_B[Y]^{(f-i)}(e) \quad \text{and} \quad \tilde{\mathcal{Q}}_e := G_B[Y]^{(f+r-2i)}(e).$$

By assumption, we have $|\mathcal{Q}_e| = (d_B \pm \varepsilon) n_B^{f-r}$ and $|\tilde{\mathcal{Q}}_e| \ge \xi n_B^{f-i}$ for all $e \in G_B^{(r-i)}$. Moreover, since $G_B[Y]$ is $(\xi, f-i, r-i)$-extendable, there exists $X \subseteq V(G_B)$ with $|X| \ge \xi n_B$ such that for all $e \in \binom{X}{r-i}$, we have $|Ext_e| \ge \xi n_B^{f-r}$, where Ext_e is the set of all $(f-r)$-sets $Q \subseteq V(G_B) \setminus e$ such that $\binom{Q \cup e}{r-i} \setminus \{e\} \subseteq G_B^{(r-i)} = G_B[Y]^{(r-i)}$.

We consider the following (random) subsets. For every $e \in G_B^{(r-i)}$, let $\mathcal{Q}'_e$ contain all $Q \in \mathcal{Q}_e$ such that for all $b \in B$, we have $\binom{b \cup Q \cup e}{r} \setminus \{b \cup e\} \subseteq H$. Define $\tilde{\mathcal{Q}}'_e$ analogously with $\tilde{\mathcal{Q}}_e$ playing the role of $\mathcal{Q}_e$. For every $e \in \binom{X}{r-i}$, let Ext'_e contain all $Q \in Ext_e$ such that for all $b \in B$ and $e' \in \binom{Q\cup e}{r-i} \setminus \{e\}$, we have $b \cup e' \in H$.

Claim 1: For each $e \in G_B^{(r-i)}$, with probability at least $1 - \mathrm{e}^{-n_B^{1/6}}$, $|\mathcal{Q}'_e| = (p_B d_B \pm 3\varepsilon)n_B^{f-r}$.

Proof of claim: We view $\mathcal{Q}_e$ as a $(f-r)$-graph and $\mathcal{Q}'_e$ as a random subgraph. Note that

$$\mathbb{P}(\forall b \in B\colon b \cup e \in H) = \prod_{b\in B} \mathbb{P}(b \cup e \in H) \overset{\text{(I)}}{=} \prod_{b \in B} \beta(\mathcal{E}_b).$$

Hence, we have for every $Q \in \mathcal{Q}_e$ that

$$\begin{aligned}
\mathbb{P}(Q \in \mathcal{Q}'_e) &= \frac{\mathbb{P}(\forall b \in B\colon \binom{b\cup Q\cup e}{r} \subseteq H)}{\mathbb{P}(\forall b \in B\colon b \cup e \in H)} \\
&= \left(\prod_{b\in B} \beta(\mathcal{E}_b)\right)^{-1} \prod_{e' \in G^{(r)}\colon \exists b \in B\colon e' \subseteq b\cup Q\cup e} \mathbb{P}(e' \in H) \\
&= \left(\prod_{b\in B} \beta(\mathcal{E}_b)\right)^{-1} \prod_{\mathcal{E}\in\mathcal{P}} (\beta(\mathcal{E}))^{|\{e'\in\mathcal{E}\colon \exists b\in B\colon e'\subseteq b\cup Q\cup e\}|} \\
&\overset{\text{(II)}}{=} \left(\prod_{b\in B} \beta(\mathcal{E}_b)\right)^{-1} \prod_{\mathcal{E}\in\mathcal{P}} (\beta(\mathcal{E}))^{D_{\mathcal{E}}} = p_B.
\end{aligned}$$

Thus, $\mathbb{E}|\mathcal{Q}'_e| = p_B|\mathcal{Q}_e|$. Hence, we deduce with Corollary 2.5.14 that with probability at least $1 - \mathrm{e}^{-n_B^{1/6}}$ we have $|\mathcal{Q}'_e| = (1 \pm \varepsilon)\mathbb{E}|\mathcal{Q}'_e| = (p_B d_B \pm 3\varepsilon)n_B^{f-r}$. –

Claim 2: For each $e \in G_B^{(r-i)}$, with probability at least $1 - \mathrm{e}^{-n_B^{1/6}}$, $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_B^{f-i}$.

Proof of claim: We view $\tilde{\mathcal{Q}}_e$ as a $(f-i)$-graph and $\tilde{\mathcal{Q}}'_e$ as a random subgraph. Observe that for every $Q \in \tilde{\mathcal{Q}}_e$, we have

$$\mathbb{P}(Q \in \tilde{\mathcal{Q}}'_e) \geq p^{|B|(\binom{f+r-i}{r}-1)} \geq p^{2^r\binom{f+r}{r}}$$

and thus $\mathbb{E}|\tilde{\mathcal{Q}}'_e| \geq p^{2^r\binom{f+r}{r}}|\tilde{\mathcal{Q}}_e| \geq \xi p^{2^r\binom{f+r}{r}} n_B^{f-i}$. Thus, we deduce with Corollary 2.5.14 that with probability at least $1 - \mathrm{e}^{-n_B^{1/6}}$ we have $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_B^{f-i}$. –

Claim 3: For every $e \in \binom{X}{r-i}$, with probability at least $1 - \mathrm{e}^{-n_B^{1/6}}$, $|Ext'_e| \geq \xi' n_B^{f-r}$.

Proof of claim: We view Ext_e as a $(f-r)$-graph and Ext'_e as a random subgraph. Observe that for every $Q \in Ext_e$, we have

$$\mathbb{P}(Q \in Ext'_e) \geq p^{|B|(\binom{f-i}{r-i}-1)} \geq p^{2^r\binom{f+r}{r}}$$

and thus $\mathbb{E}|Ext'_e| \geq p^{2^r\binom{f+r}{r}}|Ext_e| \geq \xi p^{2^r\binom{f+r}{r}} n_B^{f-r}$. Thus, we deduce with Corollary 2.5.14 that with probability at least $1-\mathrm{e}^{-n_B^{1/6}}$ we have $|Ext'_e| \geq \xi' n_B^{f-r}$. –

Applying a union bound, we can see that with probability at least $1-\mathrm{e}^{-n^{1/8}}$, H satisfies Claims 1–3 simultaneously for all relevant e.

Assume that this applies. We now deduce the desired result deterministically. Let $L \subseteq G^{(r)}$ be any graph with $\Delta(L) \leq \gamma n$ and let O be any $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq f^{-5r}\gamma n$. Let $G' := \bigcap_{b \in B}(G[H \triangle L] - O)(b)$. First, we claim that $G'[Y]$ is $(3\varepsilon + \gamma', p_B d_B, f-i, r-i)$-regular. Consider $e \in G'[Y]^{(r-i)}$. We have that $|\mathcal{Q}'_e| = (p_B d_B \pm 3\varepsilon) n_B^{f-r}$.

Claim 4: If $Q \in G'[Y]^{(f-i)}(e) \triangle \mathcal{Q}'_e$, then there is some $b \in B$ such that $b \cup Q \cup e$ contains some edge from $L - \{b \cup e\}$ or O.

Proof of claim: Clearly, $Q \in G_B[Y]^{(f-i)}(e)$. First, suppose that $Q \in G'[Y]^{(f-i)}(e) - \mathcal{Q}'_e$. Since $Q \notin \mathcal{Q}'_e$, there exists $b \in B$ such that $\binom{b \cup Q \cup e}{r} \setminus \{b \cup e\} \nsubseteq H$, that is, there is $e' \in \binom{b \cup Q \cup e}{r} \setminus \{b \cup e\}$ with $e' \notin H$. But since $Q \in G'[Y]^{(f-i)}(e)$, we have $e' \in H \triangle L$. Thus, $e' \in L$. Next, suppose that $Q \in \mathcal{Q}'_e - G'[Y]^{(f-i)}(e)$. Since $Q \notin G'[Y]^{(f-i)}(e)$, there exists $b \in B$ such that $b \cup Q \cup e \notin G[Y][H \triangle L] - O$. We claim that $b \cup Q \cup e$ contains some edge from $L - \{b \cup e\}$ or O. Since $b \cup Q \cup e \in G[Y]$, there is $e' \in \binom{b \cup Q \cup e}{r}$ with $e' \notin H \triangle L$ or there is $e' \in \binom{b \cup Q \cup e}{r+1}$ with $e' \in O$. In the latter case we are done, so suppose that the first case applies. Since $e \in G'[Y]^{(r-i)}$, we have that $b \cup e \in H \triangle L$, so $e' \neq b \cup e$. Thus, since $Q \in \mathcal{Q}'_e$, we have that $e' \in H$. Therefore, $e' \in L$ and hence $e' \in L - \{b \cup e\}$. –

For fixed $b \in B$, a double application of Proposition 2.5.7 implies that there are at most $\frac{\binom{f}{r}+\binom{f}{r+1}f^{-5r}}{(f-r)!}\gamma n^{f-r}$ f-sets that contain $b \cup e$ and some edge from $L - \{b \cup e\}$ or O.

Thus, we conclude with Claim 4 that $|G'[Y]^{(f-i)}(e) \triangle \mathcal{Q}'_e| \leq |B| \cdot \frac{1.05\binom{f}{r}}{(f-r)!}\gamma n^{f-r}$. Hence,

$$|G'[Y]^{(f-i)}(e)| = |\mathcal{Q}'_e| \pm \gamma' n_B^{f-r} = (p_B d_B \pm (3\varepsilon + \gamma'))n_B^{f-r},$$

meaning that $G'[Y]$ is indeed $(3\varepsilon + \gamma', p_B d_B, f-i, r-i)$-regular.

Next, we claim that $G'[Y]$ is $(\xi' - \gamma', f+r-2i, r-i)$-dense. Consider $e \in G'[Y]^{(r-i)}$. We have that $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_B^{f-i}$. Similarly to Claim 4, for every $Q \in \tilde{\mathcal{Q}}'_e - G'[Y]^{(f+r-2i)}(e)$ there is some $b \in B$ such that $b \cup Q \cup e$ contains some edge from $L - \{b \cup e\}$ or O. Thus, using Proposition 2.5.7 again (with $f+r-i$ playing the role of f), we deduce that

$$|\tilde{\mathcal{Q}}'_e - G'[Y]^{(f+r-2i)}(e)| \leq |B| \cdot \frac{\binom{f+r-i}{r} + \binom{f+r-i}{r+1}f^{-5r}}{(f-i)!}\gamma n^{f-i} \leq 2^i \cdot \frac{1.05\binom{f+r}{r}}{(f-r)!}\gamma n^{f-i}$$

and thus $|G'[Y]^{(f+r-2i)}(e)| \geq (\xi' - \gamma')n_B^{f-i}$.

Finally, we claim that $G'[Y]$ is $(\xi' - \gamma', f-i, r-i)$-extendable. Let $e \in \binom{X}{r-i}$. We have that $|Ext'_e| \geq \xi' n_B^{f-r}$. Let $Ext_{e,G'}$ contain all $Q \in Ext_e$ such that $\binom{Q \cup e}{r-i} \setminus \{e\} \subseteq G'[Y]^{(r-i)}$. Suppose that $Q \in Ext'_e \setminus Ext_{e,G'}$. Then there are $e' \in \binom{Q \cup e}{r-i} \setminus \{e\}$ and $b \in B$ such that $b \cup e' \notin H \triangle L$. On the other hand, we have $b \cup e' \in H$ as $Q \in Ext'_e$. Thus, $b \cup e' \in L$. Thus, for all $Q \in Ext'_e \setminus Ext_{e,G'}$, there is some $b \in B$ such that $b \cup Q \cup e$ contains some edge from $L - \{b \cup e\}$. Proposition 2.5.7 implies that there are at most $|B|\frac{\binom{f}{r}}{(f-r)!}\gamma n^{f-r}$ such Q. Thus,

$$|Ext_{e,G'}| \geq |Ext'_e| - 2^i \frac{\binom{f}{r}}{(f-r)!}\gamma n^{f-r} \geq (\xi' - \gamma')n_B^{f-r}.$$

We conclude that G' is a $(3\varepsilon + \gamma', \xi' - \gamma', f-i, r-i)$-complex, as required. □

In particular, the above proposition implies the following.

Corollary 2.5.19. *Let $1/n \ll \varepsilon, \gamma, \xi, p, 1/f$ and $r \in [f-1]$. Let*

$$\xi' := 0.95\xi p^{2^r\binom{f+r}{r}} \geq 0.95\xi p^{\binom{8^f}{}} \text{ and } \gamma' := 1.1 \cdot 2^r \frac{\binom{f+r}{r}}{(f-r)!}\gamma.$$

Suppose that G is an (ε, ξ, f, r)-supercomplex on n vertices and that $H \subseteq G^{(r)}$ is a random subgraph obtained by including every edge of $G^{(r)}$ independently with probability p. Then whp the following holds: for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \gamma n$, $G[H \triangle L]$ is a $(3\varepsilon + \gamma', \xi' - \gamma', f, r)$-supercomplex.

2.5.5 Rooted Embeddings

We now prove a result (Lemma 2.5.20) which allows us to find edge-disjoint embeddings of graphs with a prescribed 'root embedding'. Let T be an r-graph and suppose that $X \subseteq V(T)$ is such that $T[X]$ is empty. A *root of* (T, X) is a set $S \subseteq X$ with $|S| \in [r-1]$ and $|T(S)| > 0$.

For an r-graph G, we say that $\Lambda \colon X \to V(G)$ is a *G-labelling of* (T, X) if Λ is injective. Our aim is to embed T into G such that the roots of (T, X) are embedded at their assigned position. More precisely, given a G-labelling Λ of (T, X), we say that ϕ is a *Λ-faithful embedding of* (T, X) *into* G if ϕ is an injective homomorphism from T to G with $\phi\restriction_X = \Lambda$. Moreover, for a set $S \subseteq V(G)$ with $|S| \in [r-1]$, we say that Λ *roots* S if $S \subseteq \mathrm{Im}(\Lambda)$ and $|T(\Lambda^{-1}(S))| > 0$, i.e. if $\Lambda^{-1}(S)$ is a root of (T, X).

The *degeneracy of T rooted at X* is the smallest D such that there exists an ordering $v_1, \ldots, v_k$ of the vertices of $V(T) \setminus X$ such that for every $\ell \in [k]$, we have

$$|T[X \cup \{v_1, \ldots, v_\ell\}](v_\ell)| \leq D,$$

i.e. every vertex is contained in at most D edges which lie to the left of that vertex in the ordering.

We need to be able to embed many copies of (T, X) simultaneously (with different labellings) into a given host graph G such that the different embeddings are edge-disjoint. In fact, we need a slightly stronger disjointness criterion. Ideally, we would like to have that two distinct embeddings intersect in less than r vertices. However, this is in general not possible because of the desired rooting. We therefore introduce the following concept of a *hull*. We will ensure that the hulls are edge-disjoint, which will be sufficient for our purposes. Given (T, X) as above, the *hull of* (T, X) is the r-graph T' on $V(T)$ with $e \in T'$ if and only if $e \cap X = \emptyset$ or $e \cap X$ is a root of (T, X). Note that $T \subseteq T' \subseteq K^{(r)}_{V(T)} - K^{(r)}_X$, where $K^{(r)}_Z$ denotes the complete r-graph with vertex set Z. Moreover, the roots of (T', X) are precisely the roots of (T, X).

Lemma 2.5.20. *Let $1/n \ll \gamma \ll \xi, 1/t, 1/D$ and $r \in [t]$. Suppose that $\alpha \in (0, 1]$ is an arbitrary scalar (which might depend on n) and let $m \leq \alpha\gamma n^r$ be an integer. For every $j \in [m]$, let T_j be an r-graph on at most t vertices and $X_j \subseteq V(T_j)$ such that $T_j[X_j]$ is empty and T_j has degeneracy at most D rooted at X_j. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq D$, we have $|\bigcap_{S \in A} G(S)| \geq \xi n$. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. For every $j \in [m]$, let Λ_j be a G-labelling of (T_j, X_j). Suppose that for all $S \subseteq V(G)$ with $|S| \in [r-1]$, we have that*

$$|\{j \in [m] \,:\, \Lambda_j \text{ roots } S\}| \leq \alpha\gamma n^{r-|S|} - 1. \tag{2.5.2}$$

Then for every $j \in [m]$, there exists a Λ_j-faithful embedding ϕ_j of (T_j, X_j) into G such that the following hold:

(i) *for all distinct $j, j' \in [m]$, the hulls of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$ and $(\phi_{j'}(T_{j'}), \mathrm{Im}(\Lambda_{j'}))$ are edge-disjoint;*

(ii) *for all $j \in [m]$ and $e \in O$ with $e \subseteq \mathrm{Im}(\phi_j)$, we have $e \subseteq \mathrm{Im}(\Lambda_j)$;*

(iii) *$\Delta(\bigcup_{j \in [m]} \phi_j(T_j)) \leq \alpha\gamma^{(2^{-r})} n$.*

Note that (i) implies that $\phi_1(T_1), \ldots, \phi_m(T_m)$ are edge-disjoint. We also remark that the T_j do not have to be distinct; in fact, they could all be copies of a single r-graph T.

Proof. For $j \in [m]$ and a set $S \subseteq V(G)$ with $|S| \in [r-1]$, let

$$\mathit{root}(S, j) := |\{j' \in [j] \,:\, \Lambda_{j'} \text{ roots } S\}|.$$

We will define $\phi_1, \ldots, \phi_m$ successively. Once ϕ_j is defined, we let K_j denote the hull of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$. Note that $\phi_j(T_j) \subseteq K_j$ and that K_j is not necessarily a subgraph of G.

Suppose that for some $j \in [m]$, we have already defined $\phi_1, \ldots, \phi_{j-1}$ such that $K_1, \ldots, K_{j-1}$ are edge-disjoint, (ii) holds for all $j' \in [j-1]$, and the following holds for $G_j := \bigcup_{j' \in [j-1]} K_{j'}$, all $i \in [r-1]$ and all $S \in \binom{V(G)}{i}$:

$$|G_j(S)| \leq \alpha\gamma^{(2^{-i})} n^{r-i} + (root(S, j-1) + 1)2^t. \tag{2.5.3}$$

Note that (2.5.3) together with (2.5.2) implies that for all $i \in [r-1]$ and all $S \in \binom{V(G)}{i}$, we have

$$|G_j(S)| \leq 2\alpha\gamma^{(2^{-i})} n^{r-i}. \tag{2.5.4}$$

We will now define a Λ_j-faithful embedding ϕ_j of (T_j, X_j) into G such that K_j is edge-disjoint from G_j, (ii) holds for j, and (2.5.3) holds with j replaced by $j+1$. For $i \in [r-1]$, define $BAD_i := \{S \in \binom{V(G)}{i} : |G_j(S)| \geq \alpha\gamma^{(2^{-i})} n^{r-i}\}$. We view BAD_i as an i-graph. We claim that for all $i \in [r-1]$,

$$\Delta(BAD_i) \leq \gamma^{(2^{-r})} n. \tag{2.5.5}$$

Consider $i \in [r-1]$ and suppose that there exists some $S \in \binom{V(G)}{i-1}$ such that $|BAD_i(S)| > \gamma^{(2^{-r})} n$. We then have that

$$\begin{aligned} |G_j(S)| &= \frac{1}{r-i+1} \sum_{v \in V(G) \setminus S} |G_j(S \cup \{v\})| \geq r^{-1} \sum_{v \in BAD_i(S)} |G_j(S \cup \{v\})| \\ &\geq r^{-1} |BAD_i(S)| \alpha\gamma^{(2^{-i})} n^{r-i} \geq r^{-1} \gamma^{(2^{-r})} n \alpha\gamma^{(2^{-i})} n^{r-i} = r^{-1} \alpha\gamma^{(2^{-r}+2^{-i})} n^{r-(i-1)}. \end{aligned}$$

This contradicts (2.5.4) if $i-1 > 0$ since $2^{-r} + 2^{-i} < 2^{-(i-1)}$. If $i = 1$, then $S = \emptyset$ and we have $|G_j| \geq r^{-1}\alpha\gamma^{(2^{-r}+2^{-1})} n^r$, which is also a contradiction since $|G_j| \leq m\binom{t}{r} \leq \binom{t}{r}\alpha\gamma n^r$ and $2^{-r} + 2^{-1} < 1$ (as $r \geq 2$ if $i \in [r-1]$). This proves (2.5.5).

We now embed the vertices of T_j such that the obtained embedding ϕ_j is Λ_j-faithful. First, embed every vertex from X_j at its assigned position. Since T_j has degeneracy at most D rooted at X_j, there exists an ordering $v_1, \dots, v_k$ of the vertices of $V(T_j) \setminus X_j$ such that for every $\ell \in [k]$, we have

$$|T_j[X_j \cup \{v_1, \dots, v_\ell\}](v_\ell)| \leq D. \tag{2.5.6}$$

Suppose that for some $\ell \in [k]$, we have already embedded $v_1, \dots, v_{\ell-1}$. We now want to define $\phi_j(v_\ell)$. Let $U := \{\phi_j(v) \ : \ v \in X_j \cup \{v_1, \dots, v_{\ell-1}\}\}$ be the set of vertices which have already been used as images for ϕ_j. Let A contain all $(r-1)$-subsets S of U such that $\phi_j^{-1}(S) \cup \{v_\ell\} \in T_j$. We need to choose $\phi_j(v_\ell)$ from the set $(\bigcap_{S \in A} G(S)) \setminus U$ in order to complete ϕ_j to an injective homomorphism from T_j to G. By (2.5.6), we have $|A| \leq D$. Thus, by assumption, $|\bigcap_{S \in A} G(S)| \geq \xi n$.

For $i \in [r-1]$, let O_i consist of all vertices $x \in V(G)$ such that there exists some $S \in \binom{U}{i-1}$ such that $S \cup \{x\} \in BAD_i$ (so $BAD_1 = \binom{O_1}{1}$). We have

$$|O_i| \leq \binom{|U|}{i-1} \Delta(BAD_i) \overset{(2.5.5)}{\leq} \binom{t}{i-1} \gamma^{(2^{-r})} n.$$

Let O_r consist of all vertices $x \in V(G)$ such that $S \cup \{x\} \in G_j$ for some $S \in \binom{U}{r-1}$. By (2.5.4), we have that $|O_r| \leq \binom{|U|}{r-1} \Delta(G_j) \leq \binom{t}{r-1} 2\alpha\gamma^{(2^{-(r-1)})} n \leq \binom{t}{r-1} \gamma^{(2^{-r})} n$. Finally, let O_{r+1} be the set of all vertices $x \in V(G)$ such that there exists some $S \in \binom{U}{r}$ such that $S \cup \{x\} \in O$. By assumption, we have $|O_{r+1}| \leq \binom{|U|}{r} \Delta(O) \leq \binom{t}{r} \gamma n$.

Crucially, we have

$$|\bigcap_{S \in A} G(S)| - |U| - \sum_{i=1}^{r+1} |O_i| \geq \xi n - t - 2^t \gamma^{(2^{-r})} n > 0.$$

Thus, there exists a vertex $x \in V(G)$ such that $x \notin U \cup O_1 \cup \dots \cup O_{r+1}$ and $S \cup \{x\} \in G$ for all $S \in A$. Define $\phi_j(v_\ell) := x$.

Continuing in this way until ϕ_j is defined for every $v \in V(T_j)$ yields an injective homomorphism from T_j to G. By definition of O_{r+1}, (ii) holds for j. Moreover, by definition of O_r, K_j is edge-disjoint from G_j. It remains to show that (2.5.3) holds with j replaced by $j+1$. Let $i \in [r-1]$ and $S \in \binom{V(G)}{i}$. If $S \notin BAD_i$, then we have $|G_{j+1}(S)| \leq |G_j(S)| + \binom{t-i}{r-i} \leq \alpha\gamma^{(2^{-i})}n^{r-i} + 2^t$, so (2.5.3) holds. Now, assume that $S \in BAD_i$. If $S \subseteq \mathrm{Im}(\Lambda_j)$ and $|T_j(\Lambda_j^{-1}(S))| > 0$, then $root(S,j) = root(S,j-1)+1$ and thus $|G_{j+1}(S)| \leq |G_j(S)| + \binom{t-i}{r-i} \leq \alpha\gamma^{(2^{-i})}n^{r-i} + (root(S,j-1)+1)2^t + \binom{t-i}{r-i} \leq \alpha\gamma^{(2^{-i})}n^{r-i} + (root(S,j)+1)2^t$ and (2.5.3) holds. Suppose next that $S \not\subseteq \mathrm{Im}(\Lambda_j)$. We claim that $S \not\subseteq V(\phi_j(T_j))$. Suppose, for a contradiction, that $S \subseteq V(\phi_j(T_j))$. Let $\ell := \max\{\ell' \in [k] : \phi_j(v_{\ell'}) \in S\}$. (Note that the maximum exists since $(S \cap V(\phi_j(T_j))) \setminus \mathrm{Im}(\Lambda_j)$ is not empty.) Hence, $x := \phi_j(v_\ell) \in S$. Recall that when we defined $\phi_j(v_\ell)$, $\phi_j(v)$ had already been defined for all $v \in X_j \cup \{v_1, \dots, v_{\ell-1}\}$ and hence $S \setminus \{x\} \subseteq U$. But since $S \in BAD_i$, we have $x \in O_i$, in contradiction to $x = \phi_j(v_\ell)$. Thus, $S \not\subseteq V(\phi_j(T_j)) = V(K_j)$, which clearly implies that $|G_{j+1}(S)| = |G_j(S)|$ and (2.5.3) holds. The last remaining case is if $S \subseteq \mathrm{Im}(\Lambda_j)$ but $|T_j(\Lambda_j^{-1}(S))| = 0$. But then S is not a root of $(\phi_j(T_j), \mathrm{Im}(\Lambda_j))$ and thus not a root of $(K_j, \mathrm{Im}(\Lambda_j))$. Hence $|K_j(S)| = 0$ and therefore $|G_{j+1}(S)| = |G_j(S)|$ as well.

Finally, if $j = m$, then the fact that (2.5.3) holds with j replaced by $j+1$ together with (2.5.2) implies that $\Delta(\bigcup_{j \in [m]} \phi_j(T_j)) \leq 2\alpha\gamma^{(2^{-(r-1)})}n \leq \alpha\gamma^{(2^{-r})}n$. □

2.6 Nibbles, boosting and greedy covers

2.6.1 The nibble

There are numerous results based on the Rödl nibble which guarantee the existence of an almost perfect matching in a near regular hypergraph with small codegrees. Our application of this is as follows: Let G be a complex. Define the auxiliary $\binom{f}{r}$-graph H with $V(H) = E(G^{(r)})$ and $E(H) = \{\binom{Q}{r} : Q \in G^{(f)}\}$. Note that for every $e \in V(H)$, $|H(e)| = |G^{(f)}(e)|$. Thus, if G is (ε, d, f, r)-regular, then every vertex of H has degree $(d \pm \varepsilon)n^{f-r}$. Moreover, for two vertices $e, e' \in V(H)$, we have $|H(\{e, e'\})| \leq n^{f-r-1}$, thus

$\Delta_2(H) \leq n^{f-r-1}$. Standard nibble theorems would in this setting imply the existence of an almost perfect matching in H, which translates into a $K_f^{(r)}$-packing in G that covers all but $o(n^r)$ r-edges. We need a stronger result in the sense that we want the leftover r-edges to induce an r-graph with small maximum degree. Alon and Yuster [5] observed that one can use a result of Pippenger and Spencer [71] (on the chromatic index of uniform hypergraphs) to show that a near regular hypergraph with small codegrees has an almost perfect matching which is 'well-behaved'. The following is an immediate consequence of Theorem 1.2 in [5] (applied to the auxiliary hypergraph H above).

Theorem 2.6.1 ([5]). *Let $1/n \ll \varepsilon \ll \gamma, d, 1/f$ and $r \in [f-1]$. Suppose that G is an (ε, d, f, r)-regular complex on n vertices. Then G contains a $K_f^{(r)}$-packing $\mathcal{K}$ such that $\Delta(G^{(r)} - \mathcal{K}^{(r)}) \leq \gamma n$.*

2.6.2 The Boost lemma

We will now state and prove the 'Boost lemma', which 'boosts' the regularity of a complex by restricting to a suitable set Y of f-sets. It will help us to keep the error terms under control during the iteration process and also helps us to obtain meaningful resilience and minimum degree bounds.

The proof is based on the following 'edge-gadgets', which were used in [8] to obtain fractional $K_f^{(r)}$-decompositions of r-graphs with high minimum degree. These edge-gadgets allow us to locally adjust a given weighting of f-sets so that this changes the total weight at only one r-set.

Proposition 2.6.2 (see [8, Proposition 3.3]). *Let $f > r \geq 1$ and let e and J be disjoint sets with $|e| = r$ and $|J| = f$. Let G be the complete complex on $e \cup J$. There exists a function $\psi\colon G^{(f)} \to \mathbb{R}$ such that*

(i) *for all $e' \in G^{(r)}$, $\sum_{Q \in G^{(f)}(e')} \psi(Q \cup e') = \begin{cases} 1, & e' = e, \\ 0, & e' \neq e; \end{cases}$*

(ii) *for all* $Q \in G^{(f)}$, $|\psi(Q)| \leq \frac{2^{r-j}(r-j)!}{\binom{f-r+j}{j}}$, *where* $j := |e \cap Q|$.

We use these gadgets as follows. We start off with a complex that is (ε, d, f, r)-regular for some reasonable ε and consider a uniform weighting of all f-sets. We then use the edge-gadgets to shift weights until we have a 'fractional $K_f^{(r)}$-equicovering' in the sense that the weight of each edge is exactly $d'n^{f-r}$ for some suitable d'. We then use this fractional equicovering as an input for a probabilistic argument.

Lemma 2.6.3 (Boost lemma)**.** *Let* $1/n \ll \varepsilon, \xi, 1/f$ *and* $r \in [f-1]$ *such that* $2(2\sqrt{\mathrm{e}})^r\varepsilon \leq \xi$. *Let* $\xi' := 0.9(1/4)^{\binom{f+r}{f}}\xi$. *Suppose that* G *is a complex on* n *vertices and that* G *is* (ε, d, f, r)*-regular for some* $d \geq \xi$ *and* $(\xi, f+r, r)$*-dense. Then there exists* $Y \subseteq G^{(f)}$ *such that* $G[Y]$ *is* $(n^{-(f-r)/2.01}, d/2, f, r)$*-regular and* $(\xi', f+r, r)$*-dense.*

Proof. Let $d' := d/2$. Assume that $\psi \colon G^{(f)} \to [0,1]$ is a function such that for every $e \in G^{(r)}$,

$$\sum_{Q' \in G^{(f)}(e)} \psi(Q' \cup e) = d'n^{f-r},$$

and $1/4 \leq \psi(Q) \leq 1$ for all $Q \in G^{(f)}$. We can then choose $Y \subseteq G^{(f)}$ by including every $Q \in G^{(f)}$ with probability $\psi(Q)$ independently. We then have for every $e \in G^{(r)}$, $\mathbb{E}|G[Y]^{(f)}(e)| = d'n^{f-r}$. By Lemma 2.5.10(ii), we conclude that

$$\mathbb{P}(|G[Y]^{(f)}(e)| \neq (1 \pm n^{-(f-r)/2.01})d'n^{f-r}) \leq 2\mathrm{e}^{-\frac{n^{-2(f-r)/2.01}d'n^{f-r}}{3}} \leq \mathrm{e}^{-n^{0.004}}.$$

Thus, whp $G[Y]$ is $(n^{-(f-r)/2.01}, d', f, r)$-regular. Moreover, for any $e \in G^{(r)}$ and $Q \in G^{(f+r)}(e)$, we have that

$$\mathbb{P}(Q \in G[Y]^{(f+r)}(e)) = \prod_{Q' \in \binom{Q \cup e}{f}} \psi(Q') \geq (1/4)^{\binom{f+r}{f}}.$$

Therefore, $\mathbb{E}|G[Y]^{(f+r)}(e)| \geq (1/4)^{\binom{f+r}{f}}\xi n^f$, and using Corollary 2.5.14 we deduce that

$$\mathbb{P}(|G[Y]^{(f+r)}(e)| \leq 0.9(1/4)^{\binom{f+r}{f}}\xi n^f) \leq \mathrm{e}^{-n^{1/6}}.$$

Thus, whp $G[Y]$ is $(0.9(1/4)^{\binom{f+r}{f}}\xi, f+r, r)$-dense.

It remains to show that ψ exists. For every $e \in G^{(r)}$, define

$$c_e := \frac{d' n^{f-r} - 0.5|G^{(f)}(e)|}{|G^{(f+r)}(e)|}.$$

Observe that $|c_e| \leq \frac{\varepsilon n^{f-r}}{2\xi n^f} = \frac{\varepsilon}{2\xi} n^{-r}$ for all $e \in G^{(r)}$.

By Proposition 2.6.2, for every $e \in G^{(r)}$ and $J \in G^{(f+r)}(e)$, there exists a function $\psi_{e,J} \colon G^{(f)} \to \mathbb{R}$ such that

(i) $\psi_{e,J}(Q) = 0$ for all $Q \not\subseteq e \cup J$;

(ii) for all $e' \in G^{(r)}$, $\sum_{Q' \in G^{(f)}(e')} \psi_{e,J}(Q' \cup e') = \begin{cases} 1, & e' = e, \\ 0, & e' \neq e; \end{cases}$

(iii) for all $Q \in G^{(f)}$, $|\psi_{e,J}(Q)| \leq \frac{2^{r-j}(r-j)!}{\binom{f-r+j}{j}}$, where $j := |e \cap Q|$.

We now define $\psi \colon G^{(f)} \to [0,1]$ as

$$\psi := 1/2 + \sum_{e \in G^{(r)}} c_e \sum_{J \in G^{(f+r)}(e)} \psi_{e,J}.$$

For every $e \in G^{(r)}$, we have

$$\begin{aligned} \sum_{Q' \in G^{(f)}(e)} \psi(Q' \cup e) &= 0.5|G^{(f)}(e)| + \sum_{e' \in G^{(r)}} c_{e'} \sum_{J \in G^{(f+r)}(e')} \sum_{Q' \in G^{(f)}(e)} \psi_{e',J}(Q' \cup e) \\ &\overset{\text{(ii)}}{=} 0.5|G^{(f)}(e)| + c_e |G^{(f+r)}(e)| = d' n^{f-r}, \end{aligned}$$

as desired. Moreover, for every $Q \in G^{(f)}$ and $j \in [r]_0$, there are at most $\binom{n}{r}\binom{f}{j}\binom{r}{r-j}$ pairs

(e,J) for which $e \in G^{(r)}$, $J \in G^{(f+r)}(e)$, $Q \subseteq e \cup J$ and $|Q \cap e| = j$. Hence,

$$\begin{aligned}
|\psi(Q) - 1/2| &= \left| \sum_{e \in G^{(r)}} c_e \sum_{J \in G^{(f+r)}(e)} \psi_{e,J}(Q) \right| \stackrel{\text{(i)}}{\leq} \sum_{e \in G^{(r)}, J \in G^{(f+r)}(e)\colon Q \subseteq e \cup J} |c_e||\psi_{e,J}(Q)| \\
&\stackrel{\text{(iii)}}{\leq} \sum_{j=0}^{r} \binom{n}{r} \binom{f}{j} \binom{r}{r-j} \cdot \frac{\varepsilon}{2\xi} n^{-r} \cdot \frac{2^{r-j}(r-j)!}{\binom{f-r+j}{j}} \\
&\leq \frac{2^{r-1}\varepsilon}{\xi} \sum_{j=0}^{r} \frac{2^{-j}}{j!} \left(\frac{f}{f-r+1} \right)^j \leq \frac{2^{r-1}\varepsilon}{\xi} \sum_{j=0}^{r} \frac{(r/2)^j}{j!} \leq 1/4,
\end{aligned}$$

implying that $1/4 \leq \psi(Q) \leq 3/4$ for all $Q \in G^{(f)}$, as needed. □

Proof of Lemma 2.4.4. Let G be an (ε, ξ, f, r)-complex on n vertices. By definition, there exists $Y \subseteq G^{(f)}$ such that $G[Y]$ is (ε, d, f, r)-regular for some $d \geq \xi$, $(\xi, f+r, r)$-dense and (ξ, f, r)-extendable. We can thus apply the Boost lemma (Lemma 2.6.3) (with $G[Y]$ playing the role of G). This yields $Y' \subseteq Y$ such that $G[Y']$ is $(n^{-1/3}, d/2, f, r)$-regular and $(\xi', f+r, r)$-dense. Since $G[Y']^{(r)} = G[Y]^{(r)}$, $G[Y']$ is also (ξ, f, r)-extendable. Thus, G is an $(n^{-1/3}, \xi', f, r)$-complex.

Suppose now that G is an (ε, ξ, f, r)-supercomplex. Let $i \in [r]_0$ and $B \subseteq G^{(i)}$ with $1 \leq |B| \leq 2^i$. We have that $G_B := \bigcap_{b \in B} G(b)$ is an $(\varepsilon, \xi, f-i, r-i)$-complex. If $i < r$, we deduce by the above that G_B is an $(n_B^{-1/3}, \xi', f-i, r-i)$-complex. If $i = r$, this also holds by Fact 2.4.2. □

Lemma 2.6.3 together with Theorem 2.6.1 immediately implies the following 'Boosted nibble lemma'. In contrast to Theorem 2.6.1, we do not need to require $\varepsilon \ll \gamma$ here.

Lemma 2.6.4 (Boosted nibble lemma)**.** *Let $1/n \ll \gamma, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let G be a complex on n vertices such that G is (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense for some $d \geq \xi$. Then G contains a $K_f^{(r)}$-packing $\mathcal{K}$ such that $\Delta(G^{(r)} - \mathcal{K}^{(r)}) \leq \gamma n$.*

2.6.3 Approximate F-decompositions

We now prove an F-nibble lemma which allows us to find κ-well separated approximate F-decompositions in supercomplexes. Whenever we need an approximate decomposition in the proof of Theorem 2.4.7, we will obtain it via Lemma 2.6.5.

Lemma 2.6.5 (F-nibble lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f vertices. Let G be a complex on n vertices such that G is (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense for some $d \geq \xi$. Then G contains a κ-well separated F-packing $\mathcal{F}$ such that $\Delta(G^{(r)} - \mathcal{F}^{(r)}) \leq \gamma n$.*

Let F be an r-graph on f vertices. Given a collection $\mathcal{K}$ of edge-disjoint copies of $K_f^{(r)}$, we define the *$\mathcal{K}$-random F-packing* $\mathcal{F}$ as follows: For every $K \in \mathcal{K}$, choose a random bijection from $V(F)$ to $V(K)$ and let F_K be a copy of F on $V(K)$ embedded by this bijection. Let $\mathcal{F} := \{F_K \,:\, K \in \mathcal{K}\}$.

Clearly, if $\mathcal{K}$ is a $K_f^{(r)}$-decomposition of a complex G, then the $\mathcal{K}$-random F-packing $\mathcal{F}$ is a 1-well separated F-packing in G. Moreover, writing $p := 1 - |F|/\binom{f}{r}$, we have $|\mathcal{F}^{(r)}| = |F||\mathcal{K}| = |F||G^{(r)}|/\binom{f}{r} = (1-p)|G^{(r)}|$, and for every $e \in G^{(r)}$, we have $\mathbb{P}(e \in G^{(r)} - \mathcal{F}^{(r)}) = p$. As turns out, the leftover $G^{(r)} - \mathcal{F}^{(r)}$ behaves essentially like a p-random subgraph of $G^{(r)}$ (cf. Lemma 2.6.6). Our strategy to prove Lemma 2.6.5 is thus as follows: We apply Lemma 2.6.4 to G to obtain a $K_f^{(r)}$-packing $\mathcal{K}_1$ such that $\Delta(G^{(r)} - \mathcal{K}_1^{(r)}) \leq \gamma n$. The leftover here is negligible, so assume for the moment that $\mathcal{K}_1$ is a $K_f^{(r)}$-decomposition. We then choose a $\mathcal{K}_1$-random F-packing $\mathcal{F}_1$ in G and continue the process with $G - \mathcal{F}_1^{(r)}$. In each step, the leftover decreases by a factor of p. Thus after $\log_p \gamma$ steps, the leftover will have maximum degree at most γn.

Lemma 2.6.6. *Let $1/n \ll \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f-vertices with $p := 1 - |F|/\binom{f}{r} \in (0,1)$. Let G be an (ε, d, f, r)-regular and $(\xi, f+r, r)$-dense complex on n vertices for some $d \geq \xi$. Suppose that $\mathcal{K}$ is a $K_f^{(r)}$-decomposition of G. Let $\mathcal{F}$ be the $\mathcal{K}$-random F-packing in G. Then whp the following hold for $G' := G - \mathcal{K}^{\leq(r+1)} - \mathcal{F}^{(r)}$.*

(i) *G' is $(2\varepsilon, p^{\binom{f}{r}-1} d, f, r)$-regular;*

(ii) *G' is $(0.9p^{\binom{f+r}{r}-1}\xi, f+r, r)$-dense;*

(iii) $\Delta(G'^{(r)}) \leq 1.1p\Delta(G^{(r)})$.

Proof. For $e \in G^{(r)}$, we let K_e be the unique element of $\mathcal{K}^{\leq(f)}$ with $e \subseteq K_e$. Let $G_{ind} := G - \mathcal{K}^{\leq(r+1)}$. $G'^{(r)}$ is a random subgraph of $G^{(r)}_{ind}$, where for any $\mathcal{I} \subseteq G^{(r)}$, the events $\{e \in G'^{(r)}\}_{e\in\mathcal{I}}$ are independent if the sets $\{K_e\}_{e\in\mathcal{I}}$ are distinct. Since $\Delta(\mathcal{K}^{\leq(r+1)}) \leq f-r$, Proposition 2.5.9 implies that G_{ind} is $(1.1\varepsilon, d, f, r)$-regular and $(\xi - \varepsilon, f+r, r)$-dense.

For $e \in G^{(r)}$, let $\mathcal{Q}_e := G^{(f)}_{ind}(e)$ and $\tilde{\mathcal{Q}}_e := G^{(f+r)}_{ind}(e)$. Thus, $|\mathcal{Q}_e| = (d \pm 1.1\varepsilon)n^{f-r}$ and $|\tilde{\mathcal{Q}}_e| \geq 0.95\xi n^f$. Let $\mathcal{Q}'_e$ be the random subgraph of $\mathcal{Q}_e$ consisting of all $Q \in \mathcal{Q}_e$ with $\binom{Q\cup e}{r} \setminus \{e\} \subseteq G'^{(r)}$. Similarly, let $\tilde{\mathcal{Q}}'_e$ be the random subgraph of $\tilde{\mathcal{Q}}_e$ consisting of all $Q \in \tilde{\mathcal{Q}}_e$ with $\binom{Q\cup e}{r} \setminus \{e\} \subseteq G'^{(r)}$. Note that if $e \in G'^{(r)}$, then $\mathcal{Q}'_e = G'^{(f)}(e)$. Moreover, note that by definition of G_{ind}, we have

$$|(e \cup Q) \cap K| \leq r \text{ for all } Q \in \mathcal{Q}_e, K \in \mathcal{K}. \tag{2.6.1}$$

Consider $Q \in \mathcal{Q}_e$. By (2.6.1), the $K_{e'}$ with $e' \in \binom{Q\cup e}{r} \setminus \{e\}$ are all distinct, hence we have $\mathbb{P}(Q \in \mathcal{Q}'_e) = p^{\binom{f}{r}-1}$. Thus, $\mathbb{E}|\mathcal{Q}'_e| = p^{\binom{f}{r}-1}|\mathcal{Q}_e|$.

Define an auxiliary graph A_e on vertex set $\mathcal{Q}_e$ where $QQ' \in A_e$ if and only if there exists $K \in \mathcal{K}^{\leq(f)} \setminus \{K_e\}$ such that $|(e\cup Q)\cap K| = r$ and $|(e\cup Q')\cap K| = r$. Using (2.6.1), it is easy to see that if Y is an independent set in A_e, then the events $\{Q \in \mathcal{Q}'_e\}_{Q\in Y}$ are independent.

Claim 1: $\mathcal{Q}_e$ can be partitioned into $2\binom{f}{r}^2 n^{f-r-1}$ independent sets in A_e.

Proof of claim: It is sufficent to prove that $\Delta(A_e) \leq \binom{f}{r}^2 n^{f-r-1}$. Fix $Q \in V(A_e)$. There are $\binom{f}{r} - 1$ r-subsets e' of $e \cup Q$ other than e. For each of these, $K_{e'}$ is the unique $K \in \mathcal{K}^{\leq(f)} \setminus \{K_e\}$ which contains e'. Each choice of $K_{e'}$ has $\binom{f}{r}$ r-subsets e''. If we want $e\cup Q'$ to contain e'', then since $e'' \neq e$, we have $|e\cup e''| \geq r+1$ and thus there are at most n^{f-r-1} possibilities for Q'. –

By Lemma 2.5.12, we thus have $\mathbb{P}(|\mathcal{Q}'_e| \neq (1 \pm n^{-1/5})\mathbb{E}|\mathcal{Q}'_e|) \leq \mathrm{e}^{-n^{1/6}}$. We conclude that with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have $|\mathcal{Q}'_e| = (p^{\binom{f}{r}-1}d \pm 2\varepsilon)n^{f-r}$. Together with a union bound, this implies that whp G' is $(2\varepsilon, p^{\binom{f}{r}-1}d, f, r)$-regular, which proves (i).

A similar argument shows that whp G' is $(0.9p^{\binom{f+r}{r}-1}\xi, f+r, r)$-dense.

To prove (iii), let $S \in \binom{V(G)}{r-1}$. Clearly, we have $\mathbb{E}|G'^{(r)}(S)| = p|G^{(r)}(S)|$. If $|G^{(r)}(S)| = 0$, then we clearly have $|G^{(r)}(S)| \le 1.1p\Delta(G^{(r)})$, so assume that $S \subseteq e \in G^{(r)}$. Since e is contained in at least $0.5\xi n^{f-r}$ f-sets in G, and every r-set $e' \neq e$ is contained in a most $n^{f-(r+1)}$ of these, we can deduce that $|G^{(r)}(S)| \ge 0.5\xi n$. Define the auxiliary graph A_S with vertex set $G^{(r)}(S)$ such that $e_1e_2 \in A_S$ if and only if $K_{S\cup e_1} = K_{S\cup e_2}$. Again, we have $\Delta(A_S) \le f-r$ and thus $G^{(r)}(S)$ can be partitioned into $f-r+1$ sets which are independent in A_S. By Lemma 2.5.12, we thus have $\mathbb{P}(|G'^{(r)}(S)| \neq (1\pm n^{-1/5})p|G^{(r)}(S)|) \le \mathrm{e}^{-n^{1/6}}$. Using a union bound, we conclude that whp $\Delta(G'^{(r)}) \le 1.1p\Delta(G^{(r)})$. $\square$

Proof of Lemma 2.6.5. Let $p := 1 - |F|/\binom{f}{r}$. If $F = K_f^{(r)}$, then we are done by Lemma 2.6.4. We may thus assume that $p \in (0,1)$. Choose $\varepsilon' > 0$ such that $1/n \ll \varepsilon' \ll 1/\kappa \ll \gamma, \varepsilon \ll p, 1-p, \xi, 1/f$. We will now repeatedly apply Lemma 2.6.4. More precisely, let $\xi_0 := 0.9(1/4)^{\binom{f+r}{f}}\xi$ and define $\xi_j := (0.5p)^{j\binom{f+r}{r}}\xi_0$ for $j \ge 1$. For every $j \in [\kappa]_0$, we will find $\mathcal{F}_j$ and G_j such that the following hold:

(a)$_j$ $\mathcal{F}_j$ is a j-well separated F-packing in G and $G_j \subseteq G - \mathcal{F}_j^{(r)}$;

(b)$_j$ $\Delta(L_j) \le j\varepsilon' n$, where $L_j := G^{(r)} - \mathcal{F}_j^{(r)} - G_j^{(r)}$;

(c)$_j$ G_j is $(2^{(r+1)j}\varepsilon', d_j, f, r)$-regular and $(\xi_j, f+r, r)$-dense for some $d_j \ge \xi_j$;

(d)$_j$ $\mathcal{F}_j^{\le}$ and G_j are $(r+1)$-disjoint;

(e)$_j$ $\Delta(G_j^{(r)}) \le (1.1p)^j n$.

First, apply Lemma 2.6.3 to G in order to find $Y \subseteq G^{(f)}$ such that $G_0 := G[Y]$ is $(\varepsilon', d/2, f, r)$-regular and $(\xi_0, f+r, r)$-dense. Hence, (a)$_0$–(e)$_0$ hold with $\mathcal{F}_0 := \emptyset$. Also note that $\mathcal{F}_\kappa$ will be a κ-well separated F-packing in G and $\Delta(G^{(r)} - \mathcal{F}_\kappa^{(r)}) \le \Delta(L_\kappa) + \Delta(G_\kappa^{(r)}) \le \kappa\varepsilon' n + (1.1p)^\kappa n \le \gamma n$, so we can take $\mathcal{F} := \mathcal{F}_\kappa$.

Now, assume that for some $j \in [\kappa]$, we have found $\mathcal{F}_{j-1}$ and G_{j-1} and now need to find $\mathcal{F}_j$ and G_j. By (c)$_{j-1}$, G_{j-1} is $(\sqrt{\varepsilon'}, d_{j-1}, f, r)$-regular and $(\xi_{j-1}, f+r, r)$-dense for some $d_{j-1} \geq \xi_{j-1}$. Thus, we can apply Lemma 2.6.4 to obtain a $K_f^{(r)}$-packing $\mathcal{K}_j$ in G_{j-1} such that $\Delta(L'_j) \leq \varepsilon' n$, where $L'_j := G_{j-1}^{(r)} - \mathcal{K}_j^{(r)}$. Let $G'_j := G_{j-1} - L'_j$. Clearly, $\mathcal{K}_j$ is a $K_f^{(r)}$-decomposition of G'_j. Moreover, by (c)$_{j-1}$ and Proposition 2.5.9 we have that G'_j is $(2^{(r+1)(j-1)+r}\varepsilon', d_{j-1}, f, r)$-regular and $(0.9\xi_{j-1}, f+r, r)$-dense. By Lemma 2.6.6, there exists a 1-well separated F-packing $\mathcal{F}'_j$ in G'_j such that the following hold for $G_j := G'_j - \mathcal{F}'^{(r)}_j - \mathcal{K}_j^{\leq(r+1)} = G'_j - \mathcal{F}'^{(r)}_j - \mathcal{F}'^{\leq(r+1)}_j$:

(i) G_j is $(2^{(r+1)(j-1)+r+1}\varepsilon', p^{\binom{f}{r}-1} d_{j-1}, f, r)$-regular;

(ii) G_j is $(0.81 p^{\binom{f+r}{r}-1}\xi_{j-1}, f+r, r)$-dense;

(iii) $\Delta(G_j^{(r)}) \leq 1.1 p \Delta(G'^{(r)}_j)$.

Let $\mathcal{F}_j := \mathcal{F}_{j-1} \cup \mathcal{F}'_j$ and $L_j := G^{(r)} - \mathcal{F}_j^{(r)} - G_j^{(r)}$. Note that $\mathcal{F}_{j-1}^{(r)} \cap \mathcal{F}'^{(r)}_j = \emptyset$ by (a)$_{j-1}$. Moreover, $\mathcal{F}_{j-1}$ and $\mathcal{F}'_j$ are $(r+1)$-disjoint by (d)$_{j-1}$. Thus, $\mathcal{F}_j$ is $(j-1+1)$-well separated by Fact 2.5.4(ii). Moreover, using (a)$_{j-1}$, we have

$$G_j \subseteq G_{j-1} - \mathcal{F}'^{(r)}_j \subseteq G - \mathcal{F}_{j-1}^{(r)} - \mathcal{F}'^{(r)}_j,$$

thus (a)$_j$ holds. Observe that $L_j \setminus L_{j-1} \subseteq L'_j$. Thus, we clearly have $\Delta(L_j) \leq \Delta(L_{j-1}) + \Delta(L'_j) \leq j\varepsilon' n$, so (b)$_j$ holds. Moreover, (c)$_j$ follows directly from (i) and (ii), and (e)$_j$ follows from (e)$_{j-1}$ and (iii). To see (d)$_j$, observe that $\mathcal{F}_{j-1}^{\leq}$ and G_j are $(r+1)$-disjoint by (d)$_{j-1}$ and since $G_j \subseteq G_{j-1}$, and $\mathcal{F}'^{\leq}_j$ and G_j are $(r+1)$-disjoint by definition of G_j. Thus, (a)$_j$–(e)$_j$ hold and the proof is completed. □

2.6.4 Greedy coverings and divisibility

The following lemma allows us to extend a given collection of r-sets into suitable r-disjoint f-cliques (see Corollary 2.6.9). The full strength of Lemma 2.6.7 will only be needed in Section 2.8. The proof consists of a sequential random greedy algorithm.

Lemma 2.6.7. *Let $1/n \ll \gamma \ll \alpha, 1/s, 1/f$ and $r \in [f-1]$. Let G be a complex on n vertices and let $L \subseteq G^{(r)}$ satisfy $\Delta(L) \leq \gamma n$. Suppose that L decomposes into $L_1, \dots, L_m$ with $1 \leq |L_j| \leq s$. Suppose that for every $j \in [m]$, we are given some candidate set $\mathcal{Q}_j \subseteq \bigcap_{e \in L_j} G^{(f)}(e)$ with $|\mathcal{Q}_j| \geq \alpha n^{f-r}$. Then there exists $Q_j \in \mathcal{Q}_j$ for each $j \in [m]$ such that, writing $K_j := (Q_j \uplus L_j)^{\leq}$, we have that K_j and $K_{j'}$ are r-disjoint for all distinct $j, j' \in [m]$, and $\Delta(\bigcup_{j \in [m]} K_j^{(r)}) \leq \sqrt{\gamma} n$.*

Proof. Let $t := 0.5\alpha n^{f-r}$ and consider Algorithm 2.6.8. We claim that with positive

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Algorithm 2.6.8

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for j from 1 to m **do**
  define the r-graph $T_j := \bigcup_{j'=1}^{j-1} K_{j'}^{(r)}$ and let $\mathcal{Q}'_j$ contain all $Q \in \mathcal{Q}_j$ such that $(Q \uplus L_j)^{\leq}$ does not contain any edge from T_j or $L - L_j$.
  if $|\mathcal{Q}'_j| \geq t$ **then**
    pick $Q \in \mathcal{Q}'_j$ uniformly at random and let $K_j := (Q \uplus L_j)^{\leq}$
  else
    return 'unsuccessful'
  end if
end for

---

probability, Algorithm 2.6.8 outputs $K_1, \dots, K_m$ as desired.

It is enough to ensure that with positive probability, $\Delta(T_j) \leq sfr\gamma^{2/3}n$ for all $j \in [m]$. Indeed, note that we have $L_j \cap T_j = \emptyset$ by construction. Hence, if $\Delta(T_j) \leq sfr\gamma^{2/3}n$, then Proposition 2.5.7 implies that every $e \in L_j$ is contained in at most $(\gamma + sfr\gamma^{2/3})2^r n^{f-r}$ f-sets of $V(G)$ that also contain an edge of $T_j \cup (L - L_j)$. Thus, there are at most $s(\gamma + sfr\gamma^{2/3})2^r n^{f-r} \leq 0.5\alpha n^{f-r}$ candidates $Q \in \mathcal{Q}_j$ such that $(Q \uplus L_j)^{\leq}$ contains some edge from $T_j \cup (L - L_j)$. Hence, $|\mathcal{Q}'_j| \geq |\mathcal{Q}_j| - 0.5\alpha n^{f-r} \geq t$, so the algorithm succeeds in round j.

For every $(r-1)$-set $S \subseteq V(G)$ and $j \in [m]$, let Y_j^S be the indicator variable of the event that S is covered by K_j.

For every $(r-1)$-set $S \subseteq V(G)$ and $k \in [r-1]_0$, define $\mathcal{J}_{S,k} := \{j \in [m] : \max_{e \in L_j} |S \cap e| = k\}$. Observe that if $Y_j^S = 1$, then K_j covers at most sf r-edges that contain S.

Therefore, we have

$$|T_j(S)| \leq sf \sum_{j'=1}^{j-1} Y_{j'}^S = sf \sum_{k=0}^{r-1} \sum_{j' \in \mathcal{J}_{S,k} \cap [j-1]} Y_{j'}^S.$$

The following claim thus implies the lemma.

Claim 1: With positive probability, we have $\sum_{j' \in \mathcal{J}_{S,k} \cap [j-1]} Y_{j'}^S \leq \gamma^{2/3} n$ *for all* $(r-1)$*-sets* S, $k \in [r-1]_0$ *and* $j \in [m]$.

Fix an $(r-1)$-set S, $k \in [r-1]_0$ and $j \in [m]$. For $j' \in \mathcal{J}_{S,k}$, there are at most

$$\sum_{e \in L_{j'}} n^{f-|S \cup e|} \leq sn^{\max_{e \in L_{j'}}(f-|S \cup e|)} = sn^{f-2r+1+k}$$

f-sets that contain S and some edge of $L_{j'}$.

In order to apply Proposition 2.5.11, let $j_1, \dots, j_b$ be an enumeration of $\mathcal{J}_{S,k} \cap [j-1]$. We then have for all $a \in [b]$ and all $y_1, \dots, y_{a-1} \in \{0,1\}$ that

$$\mathbb{P}(Y_{j_a}^S = 1 \mid Y_{j_1}^S = y_1, \dots, Y_{j_{a-1}}^S = y_{a-1}) \leq \frac{sn^{f-2r+1+k}}{t} = 2s\alpha^{-1} n^{-r+k+1}.$$

Let $p := \min\{2s\alpha^{-1} n^{-r+k+1}, 1\}$ and let $B \sim Bin(|\mathcal{J}_{S,k} \cap [j-1]|, p)$.

Note that $|\mathcal{J}_{S,k}| \leq \binom{|S|}{k} \Delta_k(L) \leq \binom{r-1}{k} \gamma n^{r-k}$ by Fact 2.5.1. Thus,

$$7\mathbb{E}B = 7|\mathcal{J}_{S,k} \cap [j-1]| \cdot p \leq 7 \cdot \binom{r-1}{k} \gamma n^{r-k} \cdot 2s\alpha^{-1} n^{-r+k+1} \leq \gamma^{2/3} n.$$

Therefore,

$$\mathbb{P}\Big(\sum_{j' \in \mathcal{J}_{S,k} \cap [j-1]} Y_{j'}^S \geq \gamma^{2/3} n\Big) \overset{\text{Proposition 2.5.11}}{\leq} \mathbb{P}(B \geq \gamma^{2/3} n) \overset{\text{Lemma 2.5.10(iii)}}{\leq} \mathrm{e}^{-\gamma^{2/3} n}.$$

A union bound now easily proves the claim. □

Corollary 2.6.9. *Let* $1/n \ll \gamma \ll \alpha, 1/f$ *and* $r \in [f-1]$*. Suppose that* F *is an* r*-graph* *on f vertices. Let G be a complex on n vertices and let $H \subseteq G^{(r)}$ with $\Delta(H) \leq \gamma n$ and $|G^{(f)}(e)| \geq \alpha n^{f-r}$ for all $e \in H$. Then there is a 1-well separated F-packing $\mathcal{F}$ in G that covers all edges of H and such that $\Delta(\mathcal{F}^{(r)}) \leq \sqrt{\gamma}n$.*

Proof. Let $e_1, \dots, e_m$ be an enumeration of H. For $j \in [m]$, define $L_j := \{e_j\}$ and $\mathcal{Q}_j := G^{(f)}(e)$. Apply Lemma 2.6.7 to obtain $K_1, \dots, K_m$. For each $j \in [m]$, let F_j be a copy of F with $V(F_j) = K_j$ and such that $e_j \in F_j$. Then $\mathcal{F} := \{F_1, \dots, F_m\}$ is as desired. □

We can conveniently combine Lemma 2.6.5 and Corollary 2.6.9 to deduce the following result. It allows us to make an r-graph divisible by deleting a small fraction of edges (even if we are forbidden to delete a certain set of edges H). We will prove a similar result (Corollary 2.9.5) in Section 2.11 under different assumptions.

Corollary 2.6.10. *Let $1/n \ll \gamma, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f vertices. Suppose that G is a complex on n vertices which is (ε, d, f, r)-regular for some $d \geq \xi$ and $(\xi, f+r, r)$-dense. Let $H \subseteq G^{(r)}$ satisfy $\Delta(H) \leq \varepsilon n$. Then there exists $L \subseteq G^{(r)} - H$ such that $\Delta(L) \leq \gamma n$ and $G^{(r)} - L$ is F-divisible.*

Proof. We clearly have $|G^{(f)}(e)| \geq 0.5\xi n^{f-r}$ for all $e \in H$. Thus, by Corollary 2.6.9, there exists an F-packing $\mathcal{F}_0$ in G which covers all edges of H and satisfies $\Delta(\mathcal{F}_0^{(r)}) \leq \sqrt{\varepsilon}n$. By Proposition 2.5.9(i) and (ii), $G' := G - \mathcal{F}_0^{(r)}$ is still $(2^{r+1}\sqrt{\varepsilon}, d, f, r)$-regular and $(\xi/2, f+r, r)$-dense. Thus, by Lemma 2.6.5, there exists an F-packing $\mathcal{F}_{nibble}$ in G' such that $\Delta(L) \leq \gamma n$, where $L := G'^{(r)} - \mathcal{F}_{nibble}^{(r)} = G^{(r)} - \mathcal{F}_0^{(r)} - \mathcal{F}_{nibble}^{(r)} \subseteq G^{(r)} - H$. Clearly, $G^{(r)} - L$ is F-divisible (in fact, F-decomposable). □

2.7 Vortices

A vortex is best thought of as a sequence of nested ‘random-like’ subsets of the vertex set of a supercomplex G. In our approach, the final set of the vortex has bounded size.

The main results of this section are Lemmas 2.7.4 and 2.7.5, where the first one shows that vortices exist, and the latter one shows that given a vortex, we can find an F-packing covering all edges which do not lie inside the final vortex set. We now give the formal definition of what it means to be a 'random-like' subset.

Definition 2.7.1. Let G be a complex on n vertices. We say that U is $(\varepsilon, \mu, \xi, f, r)$-*random in* G if there exists an f-graph Y on $V(G)$ such that the following hold:

(R1) $U \subseteq V(G)$ with $|U| = \mu n \pm n^{2/3}$;

(R2) there exists $d \geq \xi$ such that for all $x \in [f-r]_0$ and all $e \in G^{(r)}$, we have that

$$|\{Q \in G[Y]^{(f)}(e) \,:\, |Q \cap U| = x\}| = (1 \pm \varepsilon) bin(f-r, \mu, x) dn^{f-r};$$

(R3) for all $e \in G^{(r)}$ we have $|G[Y]^{(f+r)}(e)[U]| \geq \xi(\mu n)^f$;

(R4) for all $h \in [r]_0$ and all $B \subseteq G^{(h)}$ with $1 \leq |B| \leq 2^h$ we have that $\bigcap_{b \in B} G(b)[U]$ is an $(\varepsilon, \xi, f-h, r-h)$-complex.

We record the following easy consequences for later use.

Fact 2.7.2. *The following hold.*

(i) *If G is an (ε, ξ, f, r)-supercomplex, then $V(G)$ is $(\varepsilon/\xi, 1, \xi, f, r)$-random in G.*

(ii) *If U is $(\varepsilon, \mu, \xi, f, r)$-random in G, then $G[U]$ is an (ε, ξ, f, r)-supercomplex.*

Here, (ii) follows immediately from (R4). Note that (R4) is stronger in the sense that B is not restricted to U. Having defined what it means to be a 'random-like' subset, we can now define what a vortex is.

Definition 2.7.3 (Vortex)**.** Let G be a complex. An $(\varepsilon, \mu, \xi, f, r, m)$-*vortex in* G is a sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ such that

(V1) $U_0 = V(G)$;

(V2) $|U_i| = \lfloor \mu |U_{i-1}| \rfloor$ for all $i \in [\ell]$;

(V3) $|U_\ell| = m$;

(V4) for all $i \in [\ell]$, U_i is $(\varepsilon, \mu, \xi, f, r)$-random in $G[U_{i-1}]$;

(V5) for all $i \in [\ell - 1]$, $U_i \setminus U_{i+1}$ is $(\varepsilon, \mu(1-\mu), \xi, f, r)$-random in $G[U_{i-1}]$.

We will show in Section 2.7.2 that a vortex can be found in a supercomplex by repeatedly taking random subsets.

Lemma 2.7.4. *Let $1/m' \ll \varepsilon \ll \mu, \xi, 1/f$ such that $\mu \leq 1/2$ and $r \in [f-1]$. Let G be an (ε, ξ, f, r)-supercomplex on $n \geq m'$ vertices. Then there exists a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, f, r, m)$-vortex in G for some $\mu m' \leq m \leq m'$.*

The following is the main lemma of this section. Given a vortex in a supercomplex G, it allows us to cover all edges of $G^{(r)}$ except possibly some from inside the final vortex set. We will prove Lemma 2.7.5 in Section 2.7.4.

Lemma 2.7.5. *Let $1/m \ll 1/\kappa \ll \varepsilon \ll \mu \ll \xi, 1/f$ and $r \in [f-1]$. Assume that $(*)_k$ is true for all $k \in [r-1]$. Let F be a weakly regular r-graph on f vertices. Let G be an F-divisible (ε, ξ, f, r)-supercomplex and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ an $(\varepsilon, \mu, \xi, f, r, m)$-vortex in G. Then there exists a 4κ-well separated F-packing $\mathcal{F}$ in G which covers all edges of $G^{(r)}$ except possibly some inside U_ℓ.*

The proof of Lemma 2.7.5 consists of an 'iterative absorption' procedure, where the key ingredient is the Cover down lemma (Lemma 2.7.7). Roughly speaking, given a supercomplex G and a 'random-like' subset $U \subseteq V(G)$, the Cover down lemma allows us to find a 'partial absorber' $H \subseteq G^{(r)}$ such that for any sparse $L \subseteq G^{(r)}$, $H \cup L$ has an F-packing which covers all edges of $H \cup L$ except possibly some inside U. Together with the F-nibble lemma (Lemma 2.6.5), this allows us to cover all edges of G except possibly some inside U whilst using only few edges inside U. Indeed, set aside H as above, which is reasonably sparse. Then apply the Lemma 2.6.5 to $G - G^{(r)}[U] - H$ to obtain an F-packing $\mathcal{F}_{nibble}$ with a very sparse leftover L. Combine H and L to find an F-packing $\mathcal{F}_{clean}$ whose leftover lies inside U.

Now, if $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ is a vortex, then U_1 is 'random-like' in G and thus we can cover all edges which are not inside U_1 by using only few edges inside U_1 (and in this step we forbid edges inside U_2 from being used.) Then U_2 is still 'random-like' in the remainder of $G[U_1]$, and hence we can iterate until we have covered all edges of G except possibly some inside U_ℓ.

2.7.1 The Cover down lemma

We now provide the formal statement of the Cover down lemma. We will prove it in Section 2.10.

Definition 2.7.6. Let G be a complex on n vertices and $H \subseteq G^{(r)}$. We say that G is (ξ, f, r)*-dense with respect to* H if for all $e \in G^{(r)}$, we have $|G[H \cup \{e\}]^{(f)}(e)| \geq \xi n^{f-r}$.

Lemma 2.7.7 (Cover down lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/f$ and $r \in [f-1]$ with $\mu \leq 1/2$. Assume that $(*)_i$ is true for all $i \in [r-1]$ and that F is a weakly regular r-graph on f vertices. Let G be a complex on n vertices and suppose that U is $(\varepsilon, \mu, \xi, f, r)$-random in G. Let $\tilde{G}$ be a complex on $V(G)$ with $G \subseteq \tilde{G}$ such that $\tilde{G}$ is (ε, f, r)-dense with respect to $G^{(r)} - G^{(r)}[\bar{U}]$, where $\bar{U} := V(G) \setminus U$.*

Then there exists a subgraph $H^* \subseteq G^{(r)} - G^{(r)}[\bar{U}]$ with $\Delta(H^*) \leq \nu n$ such that for any $L \subseteq \tilde{G}^{(r)}$ with $\Delta(L) \leq \gamma n$ and $H^* \cup L$ being F-divisible and any $(r+1)$-graph O on $V(G)$ with $\Delta(O) \leq \gamma n$, there exists a κ-well separated F-packing in $\tilde{G}[H^* \cup L] - O$ which covers all edges of $H^* \cup L$ except possibly some inside U.

Roughly speaking, the proof of the Cover down lemma proceeds as follows. Suppose that we have already chosen H^* and that L is any sparse (leftover) r-graph. For an edge $e \in H^* \cup L$, we refer to $|e \cap U|$ as its type. Since L is very sparse, we can greedily cover all edges of L using edges of H^* in a first step. In particular, this covers all type-0-edges.

We will now continue and cover all type-1-edges. Note that every type-1-edge contains a unique $S \in \binom{V(G)\setminus U}{r-1}$. For a given set $S \in \binom{V(G)\setminus U}{r-1}$, we would like to cover all remaining edges of H^* that contain S simultaneously. Assuming a suitable choice of H^*, this can be achieved as follows. Let L_S be the link graph of S after the first step. Let $T \in \binom{V(F)}{r-1}$ be such that $F(T)$ is non-empty. By Proposition 2.5.3, L_S will be $F(T)$-divisible. Thus, by $(*)_1$, L_S has a κ-well separated $F(T)$-decomposition $\mathcal{F}'_S$. Proposition 2.7.9 below implies that we can 'extend' $\mathcal{F}'_S$ to a κ-well separated F-packing $\mathcal{F}_S$ which covers all edges that contain S.

However, in order to cover all type-1-edges, we need to obtain such a packing $\mathcal{F}_S$ for every $S \in \binom{V(G)\setminus U}{r-1}$, and these packings are to be r-disjoint for their union to be a κ-well separated F-packing again. The real difficulty thus lies in choosing H^* in such a way that the link graphs L_S do not interfere too much with each other, and then to choose the decompositions $\mathcal{F}'_S$ sequentially (see the discussion in the beginning of Section 2.10). We would then continue to cover all type-2-edges using $(*)_2$, etc., until we finally cover all type-$(r-1)$-edges using $(*)_{r-1}$. The only remaining edges are then type-r-edges, which are contained in U, as desired.

We now show how the notion of well separated F-packings allows us to 'extend' a decomposition of a link complex to a packing which covers all edges that contain a given set S (cf. the discussion in Section 2.4.2).

Definition 2.7.8. Let F be an r-graph, $i \in [r-1]$ and assume that $T \in \binom{V(F)}{i}$ is such that $F(T)$ is non-empty. Let G be a complex and $S \in \binom{V(G)}{i}$. Suppose that $\mathcal{F}'$ is a well separated $F(T)$-packing in $G(S)$. We then define $S \triangleleft \mathcal{F}'$ as follows: For each $F' \in \mathcal{F}'$, let $F'_{\triangleleft}$ be an (arbitrary) copy of F on vertex set $S \cup V(F')$ such that $F'_{\triangleleft}(S) = F'$. Let

$$S \triangleleft \mathcal{F}' := \{F'_{\triangleleft} : F' \in \mathcal{F}'\}.$$

The following proposition is crucial and guarantees that the above extension yields a packing which covers the desired set of edges. It is also used in the construction of so-called 'transformers' (see Section 2.8.1).

Proposition 2.7.9. *Let F, r, i, T, G, S be as in Definition 2.7.8. Let $L \subseteq G(S)^{(r-i)}$. Suppose that $\mathcal{F}'$ is a κ-well separated $F(T)$-decomposition of $G(S)[L]$. Then $\mathcal{F} := S \triangleleft \mathcal{F}'$ is a κ-well separated F-packing in G and $\{e \in \mathcal{F}^{(r)} : S \subseteq e\} = S \uplus L$.*

In particular, if $L = G(S)^{(r-i)}$, i.e. if $\mathcal{F}'$ is a κ-well separated $F(T)$-decomposition of $G(S)$, then $\mathcal{F}$ is a κ-well separated F-packing in G which covers all r-edges of G that contain S.

Proof. We first check that $\mathcal{F}$ is an F-packing in G. Let $f := |V(F)|$. For each $F' \in \mathcal{F}'$, we have $V(F') \in G(S)[L]^{(f-i)} \subseteq G(S)^{(f-i)}$. Hence, $V(F'_{\triangleleft}) \in G^{(f)}$. In particular, $G^{(r)}[V(F'_{\triangleleft})]$ is a clique and thus $F'_{\triangleleft}$ is a subgraph of $G^{(r)}$. Suppose, for a contradiction, that for distinct $F', F'' \in \mathcal{F}'$, $F'_{\triangleleft}$ and $F''_{\triangleleft}$ both contain $e \in G^{(r)}$. By (WS1) we have that $|V(F') \cap V(F'')| \leq r - i$, and thus we must have $e = S \cup (V(F') \cap V(F''))$. Since $V(F') \cap V(F'') \in G(S)[L]$, we have $e \setminus S \in G(S)[L]^{(r-i)}$, and thus $e \setminus S$ belongs to at most one of F' and F''. Without loss of generality, assume that $e \setminus S \notin F'$. Then we have $e \setminus S \notin F'_{\triangleleft}(S)$ and thus $e \notin F'_{\triangleleft}$, a contradiction. Thus, $\mathcal{F}$ is an F-packing in G.

We next show that $\mathcal{F}$ is κ-well separated. Clearly, for distinct $F', F'' \in \mathcal{F}'$, we have $|V(F'_{\triangleleft}) \cap V(F''_{\triangleleft})| \leq r - i + |S| = r$, so (WS1) holds. To check (WS2), consider $e \in \binom{V(G)}{r}$. Let e' be an $(r-i)$-subset of $e \setminus S$. By definition of $\mathcal{F}$, we have that the number of $F'_{\triangleleft} \in \mathcal{F}$ with $e \subseteq V(F'_{\triangleleft})$ is at most the number of $F' \in \mathcal{F}'$ with $e' \subseteq V(F')$, where the latter is at most κ since $\mathcal{F}'$ is κ-well separated.

Finally, we check that $\{e \in \mathcal{F}^{(r)} : S \subseteq e\} = S \uplus L$. Let e be any r-set with $S \subseteq e$. By Definition 2.7.8, we have $e \in \mathcal{F}^{(r)}$ if and only if $e \setminus S \in \mathcal{F}'^{(r-i)}$. Since $\mathcal{F}'$ is an $F(T)$-decomposition of $G(S)[L]^{(r-i)} = L$, we have $e \setminus S \in \mathcal{F}'^{(r-i)}$ if and only if $e \setminus S \in L$. Thus, $e \in \mathcal{F}^{(r)}$ if and only if $e \in S \uplus L$. □

2.7.2 Existence of vortices

The goal of this subsection is to prove Lemma 2.7.4, which guarantees the existence of a vortex in a supercomplex.

Fact 2.7.10. *For all $p_1, p_2 \in [0,1]$ and $i, n \in \mathbb{N}_0$, we have*

$$\sum_{j=i}^{n} bin(n, p_1, j) bin(j, p_2, i) = bin(n, p_1 p_2, i). \tag{2.7.1}$$

Proposition 2.7.11. *Let $1/n \ll \varepsilon \ll \mu_1, \mu_2, 1-\mu_2, \xi, 1/f$ and $r \in [f-1]$. Let G be a complex on n vertices and suppose that U is $(\varepsilon, \mu_1, \xi, f, r)$-random in G. Let U' be a random subset of U obtained by including every vertex from U independently with probability μ_2. Then whp for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$, $U' \triangle W$ is $(\varepsilon + 0.5|U|^{-1/6}, \mu_1\mu_2, \xi - 0.5|U|^{-1/6}, f, r)$-random in G.*

Proof. Let $Y \subseteq G^{(f)}$ and $d \geq \xi$ be such that (R1)–(R4) hold for U. By Lemma 2.5.10(i) we have that whp $|U'| = \mu_2|U| \pm |U|^{3/5}$. So for any admissible W, we have that $|U' \triangle W| = \mu_2|U| \pm 2|U|^{3/5} = \mu_1\mu_2 n \pm (\mu_2 n^{2/3} + 2n^{3/5}) = \mu_1\mu_2 n \pm n^{2/3}$, implying (R1).

We next check (R2). For all $x \in [f-r]_0$ and $e \in G^{(r)}$, we have that $|\mathcal{Q}_{e,x}| = (1 \pm \varepsilon) bin(f-r, \mu_1, x) dn^{f-r}$, where $\mathcal{Q}_{e,x} := \{Q \in G[Y]^{(f)}(e) : |Q \cap U| = x\}$. Consider $e \in G^{(r)}$ and $x, y \in [f-r]_0$. We view $\mathcal{Q}_{e,x}$ as a $(f-r)$-graph and consider the random subgraph $\mathcal{Q}_{e,x,y}$ containing all $Q \in \mathcal{Q}_{e,x}$ such that $|Q \cap U'| = y$.

By the random choice of U', for all $e \in G^{(r)}$ and $x, y \in [f-r]_0$, we have

$$\mathbb{E}|\mathcal{Q}_{e,x,y}| = bin(x, \mu_2, y)|\mathcal{Q}_{e,x}|.$$

Thus, by Corollary 2.5.14 whp we have for all $e \in G^{(r)}$ and $x, y \in [f-r]_0$ that

$$\begin{aligned} |\mathcal{Q}_{e,x,y}| &= (1 \pm n^{-1/5}) bin(x, \mu_2, y)|\mathcal{Q}_{e,x}| \\ &= (1 \pm n^{-1/5}) bin(x, \mu_2, y)(1 \pm \varepsilon) bin(f-r, \mu_1, x) dn^{f-r} \\ &= (1 \pm (\varepsilon + 2n^{-1/5})) bin(f-r, \mu_1, x) bin(x, \mu_2, y) dn^{f-r}. \end{aligned}$$

Assuming that the above holds for U', we have for all $y \in [f-r]_0$, $e \in G^{(r)}$ and $W \subseteq U$ of size $|W| \leq |U|^{3/5}$ that

$$
\begin{aligned}
&|\{Q \in G[Y]^{(f)}(e) \,:\, |Q \cap (U' \triangle W)| = y\}| = \sum_{x=y}^{f-r} |\mathcal{Q}_{e,x,y}| \pm |W| n^{f-r-1} \\
= \quad & \sum_{x=y}^{f-r} (1 \pm (\varepsilon + 2n^{-1/5})) bin(f-r, \mu_1, x) bin(x, \mu_2, y) dn^{f-r} \pm n^{-2/5} n^{f-r} \\
\overset{(2.7.1)}{=} \quad & (1 \pm (\varepsilon + 3n^{-1/5})) bin(f-r, \mu_1\mu_2, y) dn^{f-r}.
\end{aligned}
$$

We now check (R3). Consider $e \in G^{(r)}$ and let $\tilde{\mathcal{Q}}_e := G[Y]^{(f+r)}(e)[U]$. We have $|\tilde{\mathcal{Q}}_e| \geq \xi(\mu_1 n)^f$. Consider the random subgraph of $\tilde{\mathcal{Q}}'_e$ consisting of all f-sets $Q \in \tilde{\mathcal{Q}}_e$ satisfying $Q \subseteq U'$. For every $Q \in \tilde{\mathcal{Q}}_e$, we have $\mathbb{P}(Q \subseteq U') = \mu_2^f$. Hence, $\mathbb{E}|\tilde{\mathcal{Q}}'_e| = \mu_2^f |\tilde{\mathcal{Q}}_e| \geq \xi(\mu_1\mu_2 n)^f$. Thus, using Corollary 2.5.14 and a union bound, we deduce that whp for all $e \in G^{(r)}$, we have $|G[Y]^{(f+r)}(e)[U']| \geq (1 - |U|^{-1/5})\xi(\mu_1\mu_2 n)^f$. Assuming that this holds for U', it is easy to see that for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$, we have $|G[Y]^{(f+r)}(e)[U' \triangle W]| \geq (1 - |U|^{-1/5})\xi(\mu_1\mu_2 n)^f - |W| n^{f-1} \geq (\xi - 2|U|^{-1/5})(\mu_1\mu_2 n)^f$.

Finally, we check (R4). Let $h \in [r]_0$ and $B \subseteq G^{(h)}$ with $1 \leq |B| \leq 2^h$. Since U is $(\varepsilon, \mu_1, \xi, f, r)$-random in G, we have that $\bigcap_{b \in B} G(b)[U]$ is an $(\varepsilon, \xi, f-h, r-h)$-complex. Then, by Proposition 2.5.16, with probability at least $1 - \mathrm{e}^{-|U|/8}$, $\bigcap_{b \in B} G(b)[U' \triangle W]$ is an $(\varepsilon + 4|U|^{-1/5}, \xi - 3|U|^{-1/5}, f-h, r-h)$-complex for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$. Thus, a union bound yields the desired result. □

Proposition 2.7.12. *Let $1/n \ll \varepsilon \ll \mu_1, \mu_2, 1-\mu_2, \xi, 1/f$ and $r \in [f-1]$. Let G be a complex on n vertices and let $U \subseteq V(G)$ be of size $\lfloor \mu_1 n \rfloor$ and $(\varepsilon, \mu_1, \xi, f, r)$-random in G. Then there exists $\tilde{U} \subseteq U$ of size $\lfloor \mu_2 |U| \rfloor$ such that*

(i) *$\tilde{U}$ is $(\varepsilon + |U|^{-1/6}, \mu_2, \xi - |U|^{1/6}, f, r)$-random in $G[U]$ and*

(ii) *$U \setminus \tilde{U}$ is $(\varepsilon + |U|^{-1/6}, \mu_1(1-\mu_2), \xi - |U|^{1/6}, f, r)$-random in G.*

Proof. Pick $U' \subseteq U$ randomly by including every vertex from U independently with probability μ_2. Clearly, by Lemma 2.5.10(i), we have with probability at least $1 - 2\mathrm{e}^{-2|U|^{1/7}}$

that $|U'| = \mu_2|U| \pm |U|^{4/7}$.

It is easy to see that U is $(\varepsilon+0.5|U|^{-1/6}, 1, \xi-0.5|U|^{-1/6}, f, r)$-random in $G[U]$. Hence, by Proposition 2.7.11, whp $U' \triangle W$ is $(\varepsilon+|U|^{-1/6}, \mu_2, \xi-|U|^{1/6}, f, r)$-random in $G[U]$ for all $W \subseteq U$ of size $|W| \le |U|^{3/5}$. Moreover, since $U'' := U \setminus U'$ is a random subset obtained by including every vertex from U independently with probability $1-\mu_2$, Proposition 2.7.11 implies that whp $U'' \triangle W$ is $(\varepsilon+0.5|U|^{-1/6}, \mu_1(1-\mu_2), \xi-0.5|U|^{1/6}, f, r)$-random in G for all $W \subseteq U$ of size $|W| \le |U|^{3/5}$.

Let U' be a set that has the above properties. Let $W \subseteq V(G)$ be a set with $|W| \le |U|^{3/5}$ such that $|U' \triangle W| = \lfloor \mu_2|U| \rfloor$ and let $\tilde{U} := U' \triangle W$. By the above, $\tilde{U}$ satisfies (i) and (ii). □

We can now obtain a vortex by inductively applying Proposition 2.7.12.

Proof of Lemma 2.7.4. Recursively define $n_0 := n$ and $n_i := \lfloor \mu n_{i-1} \rfloor$. Observe that $\mu^i n \ge n_i \ge \mu^i n - 1/(1-\mu)$. Further, for $i \in \mathbb{N}$, let $a_i := 2n^{-1/6}\sum_{j\in[i]} \mu^{-(j-1)/6}$. Let $\ell := 1 + \max\{i \ge 0 : n_i \ge m'\}$ and let $m := n_\ell$. Note that $\lfloor \mu m' \rfloor \le m \le m'$. Moreover, we have that

$$a_\ell = 2n^{-1/6}\frac{\mu^{-\ell/6}-1}{\mu^{-1/6}-1} \le 2\frac{(\mu^{\ell-1}n)^{-1/6}}{1-\mu^{1/6}} \le 2\frac{m'^{-1/6}}{1-\mu^{1/6}} \le \varepsilon$$

since $\mu^{\ell-1}n \ge n_{\ell-1} \ge m'$.

By Fact 2.7.2, $U_0 := V(G)$ is $(\varepsilon/\xi, 1, \xi, f, r)$-random in G. Hence, by Proposition 2.7.12, there exists a set $U_1 \subseteq U_0$ of size n_1 such that U_1 is $(\sqrt{\varepsilon}+a_1, \mu, \xi-a_1, f, r)$-random in $G[U_0]$. If $\ell = 1$, this completes the proof, so assume that $\ell \ge 2$.

Now, suppose that for some $i \in [\ell-1]$, we have already found a $(\sqrt{\varepsilon}+a_i, \mu, \xi - a_i, f, r, n_i)$-vortex $U_0, \dots, U_i$ in G. Note that this is true for $i = 1$. In particular, U_i is $(\sqrt{\varepsilon}+a_i, \mu, \xi-a_i, f, r)$-random in $G[U_{i-1}]$ by (V4). By Proposition 2.7.12, there exists a subset U_{i+1} of U_i of size n_{i+1} such that U_{i+1} is $(\sqrt{\varepsilon}+a_i+n_i^{-1/6}, \mu, \xi-a_i-n_i^{-1/6}, f, r)$-random in $G[U_i]$ and $U_i \setminus U_{i+1}$ is $(\sqrt{\varepsilon}+a_i+n_i^{-1/6}, \mu(1-\mu), \xi-a_i-n_i^{-1/6}, f, r)$-random in $G[U_{i-1}]$. Thus, $U_0, \dots, U_{i+1}$ is a $(\sqrt{\varepsilon}+a_{i+1}, \mu, \xi-a_{i+1}, f, r, n_{i+1})$-vortex in G.

Finally, $U_0, \dots, U_\ell$ is an $(\sqrt{\varepsilon} + a_\ell, \mu, \xi - a_\ell, f, r, m)$-vortex in G. □

Proposition 2.7.13. *Let $1/n \ll \varepsilon \ll \mu, \xi, 1/f$ such that $\mu \leq 1/2$ and $r \in [f-1]$. Suppose that G is a complex on n vertices and U is $(\varepsilon, \mu, \xi, f, r)$-random in G. Suppose that $L \subseteq G^{(r)}$ and $O \subseteq G^{(r+1)}$ satisfy $\Delta(L) \leq \varepsilon n$ and $\Delta(O) \leq \varepsilon n$. Then U is still $(\sqrt{\varepsilon}, \mu, \xi - \sqrt{\varepsilon}, f, r)$-random in $G - L - O$.*

Proof. Clearly, (R1) still holds. Moreover, using Proposition 2.5.7 it is easy to see that (R2) and (R3) are preserved. To see (R4), let $h \in [r]_0$ and $B \subseteq (G - L - O)^{(h)}$ with $1 \leq |B| \leq 2^h$. By assumption, we have that $\bigcap_{b \in B} G(b)[U]$ is an $(\varepsilon, \xi, f-h, r-h)$-complex. By Fact 2.5.8, we can obtain $\bigcap_{b \in B}(G - L - O)(b)[U]$ from $\bigcap_{b \in B} G(b)[U]$ by successively deleting $(r-|S|)$-graphs $L(S)$ and $(r+1-|S|)$-graphs $O(S)$, where $S \subseteq b \in B$. There are at most $2|B|2^h \leq 2^{2h+1}$ such graphs. By Fact 2.5.1, we have $\Delta(L(S)) \leq \varepsilon n \leq \varepsilon^{2/3}|U - \bigcup B|$ if $|S| < r$. If $|S| = r$, we have $S \in B$ and thus $L(S)$ is empty, in which case we can ignore its removal. Moreover, again by Fact 2.5.1, we have $\Delta(O(S)) \leq \varepsilon n \leq \varepsilon^{2/3}|U - \bigcup B|$ for all $S \subseteq b \in B$. Thus, a repeated application of Proposition 2.5.9(iv) (with $r - |S|, r - h, f - h, L(S), \varepsilon^{2/3}$ playing the roles of r', r, f, H, γ or with $r + 1 - |S|, r - h, f - h, O(S), \varepsilon^{2/3}$ playing the roles of r', r, f, H, γ, respectively) shows that $\bigcap_{b \in B}(G - L - O)(b)[U]$ is a $(\sqrt{\varepsilon}, \xi - \sqrt{\varepsilon}, f - h, r - h)$-complex, as needed. □

2.7.3 Existence of cleaners

Recall that the Cover down down lemma guarantees the existence of a suitable 'cleaning graph' or 'partial absorber' which allows us to 'clean' the leftover of an application of the F-nibble lemma in the sense that the new leftover is guaranteed to lie in the next vortex set. For technical reasons, we will in fact find all cleaning graphs first (one for each vortex set) and set them aside even before the first nibble.

The aim of this subsection is to apply the Cover down lemma to each 'level' i of the vortex to obtain a 'cleaning graph' H_i (playing the role of H^*) for each $i \in [\ell]$ (see

Lemma 2.7.15). Let G be a complex and $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ a vortex in G. We say that $H_1, \dots, H_\ell$ is a (γ, ν, κ, F)*-cleaner (for the said vortex)* if the following hold for all $i \in [\ell]$:

(C1) $H_i \subseteq G^{(r)}[U_{i-1}] - G^{(r)}[U_{i+1}]$, where $U_{\ell+1} := \emptyset$;

(C2) $\Delta(H_i) \leq \nu |U_{i-1}|$;

(C3) H_i and H_{i+1} are edge-disjoint, where $H_{\ell+1} := \emptyset$;

(C4) whenever $L \subseteq G^{(r)}[U_{i-1}]$ is such that $\Delta(L) \leq \gamma |U_{i-1}|$ and $H_i \cup L$ is F-divisible and O is an $(r+1)$-graph on U_{i-1} with $\Delta(O) \leq \gamma |U_{i-1}|$, there exists a κ-well separated F-packing $\mathcal{F}$ in $G[H_i \cup L][U_{i-1}] - O$ which covers all edges of $H_i \cup L$ except possibly some inside U_i.

Note that (C1) and (C3) together imply that $H_1, \dots, H_\ell$ are edge-disjoint. The following proposition will be used to ensure (C3).

Proposition 2.7.14. *Let $1/n \ll \varepsilon \ll \mu, \xi, 1/f$ and $r \in [f-1]$. Let $\xi' := \xi (1/2)^{(8^f+1)}$. Let G be a complex on n vertices and let $U \subseteq V(G)$ of size μn and $(\varepsilon, \mu, \xi, f, r)$-random in G. Suppose that H is a random subgraph of $G^{(r)}$ obtained by including every edge of $G^{(r)}$ independently with probability $1/2$. Then with probability at least $1 - \mathrm{e}^{-n^{1/10}}$,*

(i) *U is $(\sqrt{\varepsilon}, \mu, \xi', f, r)$-random in $G[H]$ and*

(ii) *G is $(\sqrt{\varepsilon}, f, r)$-dense with respect to $H - G^{(r)}[\bar{U}]$, where $\bar{U} := V(G) \setminus U$.*

Proof. Let $Y \subseteq G^{(f)}$ and $d \geq \xi$ be such that (R1)–(R4) hold for U and G. We first consider (i). Clearly, (R1) holds. We next check (R2). For $e \in G^{(r)}$ and $x \in [f-r]_0$, let $\mathcal{Q}_{e,x} := \{Q \in G[Y]^{(f)}(e) : |Q \cap U| = x\}$. Thus, $|\mathcal{Q}_{e,x}| = (1 \pm \varepsilon) bin(f-r, \mu, x) d n^{f-r}$.

Consider $e \in G^{(r)}$ and $x \in [f-r]_0$. We view $\mathcal{Q}_{e,x}$ as a $(f-r)$-graph and consider the random subgraph $\mathcal{Q}'_{e,x}$ containing all $Q \in \mathcal{Q}_{e,x}$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_{e,x}$, we have $\mathbb{P}(Q \in \mathcal{Q}'_{e,x}) = (1/2)^{\binom{f}{r}-1}$. Thus, using Corollary 2.5.14 we deduce that with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have

$$|\mathcal{Q}'_{e,x}| = (1\pm\varepsilon)\mathbb{E}|\mathcal{Q}'_{e,x}| = (1\pm\varepsilon)(1/2)^{\binom{f}{r}-1}(1\pm\varepsilon)bin(f-r,\mu,x)dn^{f-r}$$
$$= (1\pm\sqrt{\varepsilon})d'bin(f-r,\mu,x)dn^{f-r},$$

where $d' := d(1/2)^{\binom{f}{r}-1} \geq \xi'$. Thus, a union bound yields that with probability at least $1-\mathrm{e}^{-n^{1/7}}$, (R2) holds.

Next, we check (R3). By assumption, we have $|G[Y]^{(f+r)}(e)[U]| \geq \xi(\mu n)^f$ for all $e\in G^{(r)}$. Let $Q_e := G[Y]^{(f+r)}(e)[U]$ and consider the random subgraph $\mathcal{Q}'_e$ containing all $Q\in\mathcal{Q}_e$ such that $\binom{Q\cup e}{r}\setminus\{e\}\subseteq H$. For each $Q\in\mathcal{Q}_e$, we have $\mathbb{P}(Q\in\mathcal{Q}'_e) = (1/2)^{\binom{f+r}{r}-1}$. Thus, using Corollary 2.5.14 we deduce that with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have

$$|\mathcal{Q}'_e| = (1\pm\varepsilon)\mathbb{E}|\mathcal{Q}'_e| \geq (1-\varepsilon)(1/2)^{\binom{f+r}{r}-1}\xi(\mu n)^f \geq \xi'(\mu n)^f,$$

and a union bound implies that this is true for all $e\in G^{(r)}$ with probability at least $1-\mathrm{e}^{-n^{1/7}}$.

Next, we check (R4). Let $h\in[r]_0$ and $B\subseteq G^{(h)}$ with $1\leq |B|\leq 2^h$. We know that $\bigcap_{b\in B} G(b)[U]$ is an $(\varepsilon,\xi,f-h,r-h)$-complex. By Proposition 2.5.18 (applied with $G[U\cup\bigcup B], \{G[U\cup\bigcup B]^{(r)}\}$ playing the roles of $G,\mathcal{P}$), with probability at least $1-\mathrm{e}^{-|U|^{1/8}}$, $\bigcap_{b\in B} G[H](b)[U]$ is a $(\sqrt{\varepsilon},\xi',f-h,r-h)$-complex. Thus, a union bound over all $h\in[r]_0$ and $B\subseteq G^{(h)}$ with $1\leq|B|\leq 2^h$ yields that with probability at least $1-\mathrm{e}^{-n^{1/9}}$, (R4) holds.

Finally, we check (ii). Consider $e\in G^{(r)}$ and let $Q_e := G[(G^{(r)}-G^{(r)}[\bar{U}])\cup e]^{(f)}(e)$. Note by (R2), we have $|G[Y]^{(f)}(e)[U]| = (1\pm\varepsilon)bin(f-r,\mu,f-r)dn^{f-r}$, so $|\mathcal{Q}_e| \geq |G[Y]^{(f)}(e)[U]| \geq (1-\varepsilon)\xi\mu^{f-r}n^{f-r}$. We view Q_e as a $(f-r)$-graph and consider the random subgraph Q'_e containing all $Q\in\mathcal{Q}_e$ such that $\binom{Q\cup e}{r}\setminus\{e\}\subseteq H$. For each $Q\in\mathcal{Q}_e$, we have $\mathbb{P}(Q\in Q'_e) = (1/2)^{\binom{f}{r}-1}$. Thus, using Corollary 2.5.14 we deduce that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have

$$|\mathcal{Q}'_e| \geq 0.9\mathbb{E}|\mathcal{Q}'_e| \geq 0.9(1/2)^{\binom{f}{r}-1}(1-\varepsilon)\xi\mu^{f-r}n^{f-r} \geq \sqrt{\varepsilon}n^{f-r}.$$

A union bound easily implies that with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, this holds for all $e \in G^{(r)}$. □

The following lemma shows that cleaners exist.

Lemma 2.7.15. *Let $1/m \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/f$ be such that $\mu \leq 1/2$ and $r \in [f-1]$. Assume that $(*)_i$ is true for all $i \in [r-1]$ and that F is a weakly regular r-graph on f vertices. Let G be a complex and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ an $(\varepsilon, \mu, \xi, f, r, m)$-vortex in G. Then there exists a (γ, ν, κ, F)-cleaner.*

Proof. For $i \in [\ell]$, define $U'_i := U_i \setminus U_{i+1}$, where $U_{\ell+1} := \emptyset$. For $i \in [\ell-1]$, let $\mu_i := \mu(1-\mu)$, and let $\mu_\ell := \mu$. By (V4) and (V5), we have for all $i \in [\ell]$ that U'_i is $(\varepsilon, \mu_i, \xi, f, r)$-random in $G[U_{i-1}]$.

Split $G^{(r)}$ randomly into G_0 and G_1, that is, independently for every edge $e \in G^{(r)}$, put e into G_0 with probability $1/2$ and into G_1 otherwise. We claim that with positive probability, the following hold for every $i \in [\ell]$:

(i) U'_i is $(\sqrt{\varepsilon}, \mu_i, \xi(1/2)^{(8^f+1)}, f, r)$-random in $G[G_{i \bmod 2}][U_{i-1}]$;

(ii) $G[U_{i-1}]$ is $(\sqrt{\varepsilon}, f, r)$-dense with respect to $G_{i \bmod 2}[U_{i-1}] - G^{(r)}[U_{i-1} \setminus U'_i]$.

By Proposition 2.7.14, the probability that (i) or (ii) do not hold for $i \in [\ell]$ is at most $\mathrm{e}^{-|U_{i-1}|^{1/10}} \leq |U_{i-1}|^{-2}$. Since $\sum_{i=1}^{\ell} |U_{i-1}|^{-2} < 1$, we deduce that with positive probability, (i) and (ii) hold for all $i \in [\ell]$.

Therefore, there exist G_0, G_1 satisfying the above properties. For every $i \in [\ell]$, we will find H_i using the Cover down lemma (Lemma 2.7.7). Let $i \in [\ell]$. Apply Lemma 2.7.7 with the following objects/parameters:

object/parameter	$G[G_{i \bmod 2}][U_{i-1}]$	U'_i	$G[U_{i-1}]$	F	$\lvert U_{i-1}\rvert$	κ	γ	$\sqrt{\varepsilon}$	ν	μ_i	$\xi(1/2)^{(8^f+1)}$	f	r
playing the role of	G	U	$\tilde{G}$	F	n	κ	γ	ε	ν	μ	ξ	f	r

Hence, there exists

$$H_i \subseteq G_{i \bmod 2}[U_{i-1}] - G_{i \bmod 2}[U_{i-1} \setminus U_i'] \subseteq G_{i \bmod 2}[U_{i-1}] - G^{(r)}[U_{i+1}]$$

with $\Delta(H_i) \leq \nu|U_{i-1}|$ and the following 'cleaning' property: for all $L \subseteq G^{(r)}[U_{i-1}]$ with $\Delta(L) \leq \gamma|U_{i-1}|$ such that $H_i \cup L$ is F-divisible and all $(r+1)$-graphs O on U_{i-1} with $\Delta(O) \leq \gamma|U_{i-1}|$, there exists a κ-well separated F-packing $\mathcal{F}$ in $G[H_i \cup L][U_{i-1}] - O$ which covers all edges of $H_i \cup L$ except possibly some inside $U_i' \subseteq U_i$. Thus, (C1), (C2) and (C4) hold.

Since G_0 and G_1 are edge-disjoint, (C3) holds as well. Thus, $H_1, \dots, H_\ell$ is a (γ, ν, κ, F)-cleaner. □

2.7.4 Obtaining a near-optimal packing

Recall that Lemma 2.7.5 guarantees an F-packing covering all edges except those in the final set U_ℓ of a vortex. We prove this by applying successively the F-nibble lemma (Lemma 2.6.5) and the definition of a cleaner to each set U_i in the vortex.

Proof of Lemma 2.7.5. Choose new constants $\gamma, \nu > 0$ such that

$$1/m \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu \ll \mu \ll \xi, 1/f.$$

Apply Lemma 2.7.15 to obtain a (γ, ν, κ, F)-cleaner $H_1, \dots, H_\ell$. Note that by (V4) and Fact 2.7.2(ii), $G[U_i]$ is an (ε, ξ, f, r)-supercomplex for all $i \in [\ell]$, and the same holds for $i = 0$ by assumption. Let $H_{\ell+1} := \emptyset$ and $U_{\ell+1} := \emptyset$.

For $i \in [\ell]_0$ and $\mathcal{F}_i^*$, define the following conditions:

(FP1*)$_i$ $\mathcal{F}_i^*$ is a 4κ-well separated F-packing in $G - H_{i+1} - G^{(r)}[U_{i+1}]$;

(FP2*)$_i$ $\mathcal{F}_i^*$ covers all edges of $G^{(r)}$ that are not inside U_i;

(FP3*)$_i$ for all $e \in G^{(r)}[U_i]$, $|\mathcal{F}_i^{*\leq(f)}(e)| \leq 2\kappa$;

(FP4*)$_i$ $\Delta(\mathcal{F}_i^{*(r)}[U_i]) \leq \mu|U_i|$.

Note that (FP1*)$_0$–(FP4*)$_0$ hold trivially with $\mathcal{F}_0^* := \emptyset$. We will now proceed inductively until we obtain $\mathcal{F}_\ell^*$ satisfying (FP1*)$_\ell$–(FP4*)$_\ell$. Clearly, taking $\mathcal{F} := \mathcal{F}_\ell^*$ completes the proof (using (FP1*)$_\ell$ and (FP2*)$_\ell$).

Suppose that for some $i \in [\ell]$, we have found $\mathcal{F}_{i-1}^*$ such that (FP1*)$_{i-1}$–(FP4*)$_{i-1}$ hold. Let

$$G_i := G[U_{i-1}] - (\mathcal{F}_{i-1}^{*(r)} \cup H_{i+1} \cup G^{(r)}[U_{i+1}]) - \mathcal{F}_{i-1}^{*\leq(r+1)}.$$

We now intend to find $\mathcal{F}_i$ such that:

(FP1) $\mathcal{F}_i$ is a 2κ-well separated F-packing in G_i;

(FP2) $\mathcal{F}_i$ covers all edges from $G^{(r)}[U_{i-1}] - \mathcal{F}_{i-1}^{*(r)}$ that are not inside U_i;

(FP3) $\Delta(\mathcal{F}_i^{(r)}[U_i]) \leq \mu|U_i|$.

We first observe that this is sufficient for $\mathcal{F}_i^* := \mathcal{F}_{i-1}^* \cup \mathcal{F}_i$ to satisfy (FP1*)$_i$–(FP4*)$_i$. Note that $\mathcal{F}_i^{(r)}$ and $\mathcal{F}_{i-1}^{*(r)}$ are edge-disjoint, and $\mathcal{F}_i$ and $\mathcal{F}_{i-1}^*$ are $(r+1)$-disjoint by definition of G_i. Together with (FP1*)$_{i-1}$ this implies that $\mathcal{F}_i^*$ is a well separated F-packing in $G - H_{i+1} - G^{(r)}[U_{i+1}]$. Let $e \in G^{(r)}$. If $e \not\subseteq U_{i-1}$, then $|\mathcal{F}_i^{\leq(f)}(e)| = 0$ and hence $|\mathcal{F}_i^{*\leq(f)}(e)| = |\mathcal{F}_{i-1}^{*\leq(f)}(e)| \leq 4\kappa$. If $e \subseteq U_{i-1}$, then we have $|\mathcal{F}_i^{*\leq(f)}(e)| = |\mathcal{F}_{i-1}^{*\leq(f)}(e)| + |\mathcal{F}_i^{\leq(f)}(e)| \leq 4\kappa$ by (FP3*)$_{i-1}$ and (FP1). Thus, $\mathcal{F}_i^*$ is 4κ-well separated and (FP1*)$_i$ holds.

Clearly, (FP2*)$_{i-1}$ and (FP2) imply (FP2*)$_i$. Moreover, observe that $\mathcal{F}_{i-1}^{*\leq(r)}[U_i]$ is empty by (FP1*)$_{i-1}$. Thus, (FP3*)$_i$ holds since $\mathcal{F}_i$ is 2κ-well separated, and (FP3) implies (FP4*)$_i$.

It thus remains to show that $\mathcal{F}_i$ satisfying (FP1)–(FP3) exists. We will obtain $\mathcal{F}_i$ as the union of two packings, one obtained from the F-nibble lemma (Lemma 2.6.5) and one using (C4). Let $G_{i,nibble} := G[U_{i-1}] - (\mathcal{F}^{*(r)}_{i-1} \cup H_i \cup G^{(r)}[U_i]) - \mathcal{F}^{*\leq(r+1)}_{i-1}$. Recall that $G[U_{i-1}]$ is an (ε, ξ, f, r)-supercomplex. In particular, it is (ε, d, f, r)-regular for some $d \geq \xi$, and $(\xi, f + r, r)$-dense. Note that by (FP4*)$_{i-1}$, (C2) and (V2) we have

$$\Delta(\mathcal{F}^{*(r)}_{i-1}[U_{i-1}] \cup H_i \cup G^{(r)}[U_i]) \leq \mu|U_{i-1}| + \nu|U_{i-1}| + \mu|U_{i-1}| \leq 3\mu|U_{i-1}|.$$

Moreover, $\Delta(\mathcal{F}^{*\leq(r+1)}_{i-1}) \leq 4\kappa(f - r) \leq \mu|U_{i-1}|$ by Fact 2.5.4(i). Thus, Proposition 2.5.9(i) and (ii) imply that $G_{i,nibble}$ is still $(2^{r+3}\mu, d, f, r)$-regular and $(\xi/2, f + r, r)$-dense. Since $\mu \ll \xi$, we can apply Lemma 2.6.5 to obtain a κ-well separated F-packing $\mathcal{F}_{i,nibble}$ in $G_{i,nibble}$ such that $\Delta(L_{i,nibble}) \leq \frac{1}{2}\gamma|U_{i-1}|$, where $L_{i,nibble} := G^{(r)}_{i,nibble} - \mathcal{F}^{(r)}_{i,nibble}$. Since by (FP2*)$_{i-1}$,

$$\begin{aligned} G^{(r)} - \mathcal{F}^{*(r)}_{i-1} - \mathcal{F}^{(r)}_{i,nibble} &= G^{(r)}[U_{i-1}] - \mathcal{F}^{*(r)}_{i-1} - \mathcal{F}^{(r)}_{i,nibble} \\ &= (G^{(r)}_{i,nibble} \cup H_i \cup G^{(r)}[U_i]) - \mathcal{F}^{(r)}_{i,nibble} \\ &= H_i \cup G^{(r)}[U_i] \cup L_{i,nibble}, \end{aligned}$$

we know that $H_i \cup G^{(r)}[U_i] \cup L_{i,nibble}$ is F-divisible. By (C1) and (C3), we know that $H_{i+1} \cup G^{(r)}[U_{i+1}] \subseteq G^{(r)}[U_i] - H_i$. Moreover, by (C2) and Proposition 2.5.9(v) we have that $G[U_i] - H_i$ is a $(2\mu, \xi/2, f, r)$-supercomplex. We can thus apply Corollary 2.6.10 (with $G[U_i] - H_i$, $H_{i+1} \cup G^{(r)}[U_{i+1}]$, 2μ playing the roles of G, H, ε) to find an F-divisible subgraph R_i of $G^{(r)}[U_i] - H_i$ containing $H_{i+1} \cup G^{(r)}[U_{i+1}]$ such that $\Delta(L_{i,res}) \leq \frac{1}{2}\gamma|U_i|$, where $L_{i,res} := G^{(r)}[U_i] - H_i - R_i$.

Let $L_i := L_{i,nibble} \cup L_{i,res}$. Clearly, $L_i \subseteq G^{(r)}[U_{i-1}]$ and $\Delta(L_i) \leq \gamma|U_{i-1}|$. Note that

$$H_i \cup L_i = (H_i \,\dot\cup\, (G^{(r)}[U_i] - H_i) \,\dot\cup\, L_{i,nibble}) - R_i = G^{(r)} - \mathcal{F}^{*(r)}_{i-1} - \mathcal{F}^{(r)}_{i,nibble} - R_i \quad (2.7.2)$$

is F-divisible. Moreover, $\Delta(\mathcal{F}^{*\leq(r+1)}_{i-1} \cup \mathcal{F}^{\leq(r+1)}_{i,nibble}) \leq 5\kappa(f - r)$ by Fact 2.5.4(i). Thus, by

(C4) there exists a κ-well separated F-packing $\mathcal{F}_{i,clean}$ in

$$G_{i,clean} := G[H_i \cup L_i][U_{i-1}] - \mathcal{F}_{i-1}^{*\leq(r+1)} - \mathcal{F}_{i,nibble}^{\leq(r+1)}$$

which covers all edges of $H_i \cup L_i$ except possibly some inside U_i.

We claim that $\mathcal{F}_i := \mathcal{F}_{i,nibble} \cup \mathcal{F}_{i,clean}$ is the desired packing. Since $\mathcal{F}_{i,nibble}^{(r)}$ and $\mathcal{F}_{i,clean}^{(r)}$ are edge-disjoint and $\mathcal{F}_{i,nibble}$ and $\mathcal{F}_{i,clean}$ are $(r+1)$-disjoint, we have that $\mathcal{F}_i$ is a 2κ-well separated F-packing by Fact 2.5.4(ii). Moreover, it is easy to see from (C1) that $G_{i,nibble} \subseteq G_i$. Crucially, since R_i was chosen to contain $H_{i+1} \cup G^{(r)}[U_{i+1}]$, we have from (FP2*)$_{i-1}$ that

$$H_i \cup L_i \overset{(2.7.2)}{\subseteq} G^{(r)}[U_{i-1}] - R_i - \mathcal{F}_{i-1}^{*(r)} \subseteq G^{(r)}[U_{i-1}] - (\mathcal{F}_{i-1}^{*(r)} \cup H_{i+1} \cup G^{(r)}[U_{i+1}])$$

and thus $G_{i,clean} \subseteq G_i$ as well. Hence, (FP1) holds.

Clearly, $\mathcal{F}_i$ covers all edges of $G^{(r)}[U_{i-1}] - \mathcal{F}_{i-1}^{*(r)}$ that are not inside U_i, thus (FP2) holds. Finally, since $\mathcal{F}_{i,nibble}^{(r)}[U_i]$ is empty, we have $\Delta(\mathcal{F}_i^{(r)}[U_i]) \leq \Delta(H_i \cup L_i) \leq \nu|U_{i-1}| + \gamma|U_{i-1}| \leq \mu|U_i|$, as needed for (FP3). □

2.8 Absorbers

In this section we show that for any (divisible) r-graph H in a supercomplex G, we can find an 'exclusive' absorber r-graph A (as discussed in Section 2.1.7, one may think of H as a potential leftover from an approximate F-decomposition and A will be set aside earlier to absorb H into an F-decomposition). The following definition makes this precise. The main result of this section is Lemma 2.8.2, which constructs an absorber provided that F is weakly regular. Building on [9], we will construct absorbers as a concatenation of 'transformers' and special 'canonical graphs'. The goal is to transform an arbitrary divisible r-graph H into a canonical graph. In the following subsection, we will construct transformers. In Section 2.8.2, we will prove the existence of suitable canonical graphs. We will prove Lemma 2.8.2 in Section 2.8.3.

Definition 2.8.1 (Absorber)**.** Let F, H and A be r-graphs. We say that A is an *F-absorber for H* if A and H are edge-disjoint and both A and $A \cup H$ have an F-decomposition. More generally, if G is a complex and $H \subseteq G^{(r)}$, then $A \subseteq G^{(r)}$ is a *κ-well separated F-absorber for H in G* if A and H are edge-disjoint and there exist κ-well separated F-packings $\mathcal{F}_\circ$ and $\mathcal{F}_\bullet$ in G such that $\mathcal{F}_\circ^{(r)} = A$ and $\mathcal{F}_\bullet^{(r)} = A \cup H$.

Lemma 2.8.2 (Absorbing lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, 1/h, \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph on f vertices, let G be an (ε, ξ, f, r)-supercomplex on n vertices and let H be an F-divisible subgraph of $G^{(r)}$ with $|H| \leq h$. Then there exists a κ-well separated F-absorber A for H in G with $\Delta(A) \leq \gamma n$.*

We now briefly discuss the case $r = 1$. We write $V(F) = \{x_1, \dots, x_f\}$ and can assume that $F = \{\{x_1\}, \dots, \{x_t\}\}$ for some $t \in [f]$.

Assume first that $H = \{e_1, \dots, e_t\}$. Choose any f-set $Q_0 \in G^{(f)}$ and write $Q_0 = \{v_1, \dots, v_f\}$. Let F_0 be a copy of F with vertex set Q_0 such that $F_0 = \{\{v_1\}, \dots, \{v_t\}\}$. Now, for every $i \in [t]$, choose a $Q_i \in G^{(f)}(e_i) \cap G^{(f)}(\{v_i\})$ (cf. Fact 2.5.6). Choose these sets such that $\bigcup H, Q_0, \dots, Q_t$ are pairwise disjoint. For every $i \in [t]$, let F_i and F'_i be copies of F such that $V(F_i) = Q_i \cup e_i$, $V(F'_i) = Q_i \cup \{v_i\}$ and $F_i \bigtriangleup F'_i = \{e_i, \{v_i\}\}$.

Now, let $A := \bigcup_{i \in [t]} F'_i$. Then $\mathcal{F}_\circ := \{F'_1, \dots, F'_t\}$ is a 1-well separated F-packing in G with $\mathcal{F}_\circ^{(1)} = A$, and $\mathcal{F}_\bullet := \{F_0, F_1, \dots, F_t\}$ is a 1-well separated F-packing in G with $\mathcal{F}_\bullet^{(1)} = A \cup H$. Thus, A is a 1-well separated F-absorber for H in G. More generally, if H is any F-divisible 1-graph, then $t \mid |H|$, so we can partition the edges of H into $|H|/t$ subgraphs of size t and then find an absorber for each of these subgraphs (successively so that they are appropriately disjoint.) Thus, for the remainder of this section, we will assume that $r \geq 2$.

2.8.1 Transformers

Roughly speaking, a transformer T can be viewed as transforming a given leftover graph H into a new leftover H' (where we set aside T and H' earlier).

Definition 2.8.3 (Transformer)**.** Let F be an r-graph, G a complex and assume that $H, H' \subseteq G^{(r)}$. A subgraph $T \subseteq G^{(r)}$ is a *κ-well separated $(H, H'; F)$-transformer in G* if T is edge-disjoint from both H and H' and there exist κ-well separated F-packings $\mathcal{F}$ and $\mathcal{F}'$ in G such that $\mathcal{F}^{(r)} = T \cup H$ and $\mathcal{F}'^{(r)} = T \cup H'$.

Our 'Transforming lemma' (Lemma 2.8.5) guarantees the existence of a transformer for H and H' if H' is *obtained from H by identifying vertices* (modulo deleting some isolated vertices from H'). To make this more precise, given a multi-r-graph H and $x, x' \in V(H)$, we say that x and x' are *identifiable* if $|H(\{x, x'\})| = 0$, that is, if identifying x and x' does not create an edge of size less than r. For multi-r-graphs H and H', we write $H \rightsquigarrow\!\!\!\rightsquigarrow H'$ if there is a sequence $H_0, \dots, H_t$ of multi-r-graphs such that $H_0 \cong H$, H_t is obtained from H' by deleting isolated vertices, and for every $i \in [t]$, there are two identifiable vertices $x, x' \in V(H_{i-1})$ such that H_i is obtained from H_{i-1} by identifying x and x'.

If H and H' are (simple) r-graphs and $H \rightsquigarrow\!\!\!\rightsquigarrow H'$, we just write $H \rightsquigarrow H'$ to indicate the fact that during the identification steps, only vertices $x, x' \in V(H_{i-1})$ with $H_{i-1}(\{x\}) \cap H_{i-1}(\{x'\}) = \emptyset$ were identified (i.e. if we did not create multiple edges).

Clearly, $\rightsquigarrow\!\!\!\rightsquigarrow$ is a reflexive and transitive relation on the class of multi-r-graphs, and $\rightsquigarrow$ is a reflexive and transitive relation on the class of r-graphs.

It is easy to see that $H \rightsquigarrow H'$ if and only if there is an *edge-bijective homomorphism* from H to H' (see Proposition 2.8.4(i)). Given r-graphs H, H', a *homomorphism from H to H'* is a map $\phi \colon V(H) \to V(H')$ such that $\phi(e) \in H'$ for all $e \in H$. Note that this implies that $\phi\restriction_e$ is injective for all $e \in H$. We let $\phi(H)$ denote the subgraph of H' with vertex set $\phi(V(H))$ and edge set $\{\phi(e) : e \in H\}$. We say that ϕ is *edge-bijective* if $|H| = |\phi(H)| = |H'|$. For two r-graphs H and H', we write $H \overset{\phi}{\rightsquigarrow} H'$ if ϕ is an edge-bijective homomorphism from H to H'.

We now record a few simple observations about the relation $\rightsquigarrow$ for future reference.

Proposition 2.8.4. *The following hold.*

(i) *$H \rightsquigarrow H'$ if and only if there exists ϕ such that $H \overset{\phi}{\rightsquigarrow} H'$.*

(ii) *Let $H_1, H_1', \dots, H_t, H_t'$ be r-graphs such that $H_1, \dots, H_t$ are vertex-disjoint and $H_1', \dots, H_t'$ are edge-disjoint and $H_i \cong H_i'$ for all $i \in [t]$. Then*

$$H_1 + \dots + H_t \rightsquigarrow H_1' \mathbin{\dot\cup} \dots \mathbin{\dot\cup} H_t'.$$

(iii) *If $H \rightsquigarrow H'$ and H is F-divisible, then H' is F-divisible.*

The following lemma guarantees the existence of a transformer from H to H' if F is weakly regular and $H \rightsquigarrow H'$. The proof relies inductively on the assertion of the main complex decomposition theorem (Theorem 2.4.7).

Lemma 2.8.5 (Transforming lemma)**.** *Let $1/n \ll 1/\kappa \ll \gamma, 1/h, \varepsilon \ll \xi, 1/f$ and $2 \le r < f$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph on f vertices, let G be an (ε, ξ, f, r)-supercomplex on n vertices and let H, H' be vertex-disjoint F-divisible subgraphs of $G^{(r)}$ of order at most h and such that $H \rightsquigarrow H'$. Then there exists a κ-well separated $(H, H'; F)$-transformer T in G with $\Delta(T) \le \gamma n$.*

A key operation in the proof of Lemma 2.8.5 is the ability to find 'localised transformers'. Let $i \in [r-1]$ and let $S \subseteq V(H)$, $S' \subseteq V(H')$ and $S^* \subseteq V(F)$ be sets of size i. For an $(r-i)$-graph L in the link graph of both S and S', we can view an $F(S^*)$-decomposition $\mathcal{F}_L$ of L (which exists by $(*)_{r-i}$) as a localised transformer between $S \uplus L$ and $S' \uplus L$. Indeed, similarly to the situation described in Sections 2.4.2 and 2.7.1, we can extend $\mathcal{F}_L$ 'by adding S back' to obtain an F-packing $\mathcal{F}$ which covers all edges of $S \uplus L$. By 'mirroring' this extension, we can also obtain an F-packing $\mathcal{F}'$ which covers all edges of $S' \uplus L$ (see Definition 2.8.8 and Proposition 2.8.9). To make this more precise, we introduce the following notation.

Definition 2.8.6. Let V be a set and let V_1, V_2 be disjoint subsets of V having equal size. Let $\phi\colon V_1 \to V_2$ be a bijection. For a set $S \subseteq V \setminus V_2$, define $\phi(S) := (S \setminus V_1) \cup \phi(S \cap V_1)$. Moreover, for an r-graph R with $V(R) \subseteq V \setminus V_2$, we let $\phi(R)$ be the r-graph on $\phi(V(R))$ with edge set $\{\phi(e) : e \in R\}$.

The following facts are easy to see.

Fact 2.8.7. *Suppose that V, V_1, V_2 and ϕ are as above. Then the following hold for every r-graph R with $V(R) \subseteq V \setminus V_2$:*

(i) *$\phi(R) \cong R$;*

(ii) *if $R = R_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} R_k$, then $\phi(R) = \phi(R_1) \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \phi(R_k)$ and thus $\phi(R_1) = \phi(R) - \phi(R_2) - \cdots - \phi(R_k)$.*

The following definition is a two-sided version of Definition 2.7.8.

Definition 2.8.8. Let F be an r-graph, $i \in [r-1]$ and assume that $S^* \in \binom{V(F)}{i}$ is such that $F(S^*)$ is non-empty. Let G be a complex and assume that $S_1, S_2 \in \binom{V(G)}{i}$ are disjoint and that a bijection $\phi\colon S_1 \to S_2$ is given. Suppose that $\mathcal{F}'$ is a well separated $F(S^*)$-packing in $G(S_1) \cap G(S_2)$. We then define $S_1 \triangleleft \mathcal{F}' \triangleright S_2$ as follows: For each $F' \in \mathcal{F}'$ and $j \in \{1,2\}$, let F'_j be a copy of F on vertex set $S_j \cup V(F')$ such that $F'_j(S_j) = F'$ and such that $\phi(F'_1) = F'_2$. Let

$$\mathcal{F}_1 := \{F'_1 : F' \in \mathcal{F}'\};$$
$$\mathcal{F}_2 := \{F'_2 : F' \in \mathcal{F}'\};$$
$$S_1 \triangleleft \mathcal{F}' \triangleright S_2 := (\mathcal{F}_1, \mathcal{F}_2).$$

The next proposition is proved using its one-sided counterpart, Proposition 2.7.9. As in Proposition 2.7.9, the notion of well separatedness (Definition 2.4.5) is crucial here.

Proposition 2.8.9. *Let F, r, i, S^*, G, S_1, S_2 and ϕ be as in Definition 2.8.8. Suppose that $L \subseteq G(S_1)^{(r-i)} \cap G(S_2)^{(r-i)}$ and that $\mathcal{F}'$ is a κ-well separated $F(S^*)$-decomposition of $(G(S_1) \cap G(S_2))[L]$. Then the following holds for $(\mathcal{F}_1, \mathcal{F}_2) = S_1 \triangleleft \mathcal{F}' \triangleright S_2$:*

(i) *for* $j \in [2]$, $\mathcal{F}_j$ *is a* κ*-well separated* F*-packing in* G *with* $\{e \in \mathcal{F}_j^{(r)} : S_j \subseteq e\} = S_j \uplus L$*;*

(ii) $V(\mathcal{F}_1^{(r)}) \subseteq V(G) \setminus S_2$ *and* $\phi(\mathcal{F}_1^{(r)}) = \mathcal{F}_2^{(r)}$*.*

Proof. Let $j \in [2]$. Since $(G(S_1) \cap G(S_2))[L] \subseteq G(S_j)$, we can view $\mathcal{F}_j$ as $S_j \triangleleft \mathcal{F}'$ (cf. Definition 2.7.8). Moreover, since $(G(S_1) \cap G(S_2))[L]^{(r-i)} = L = G(S_j)[L]^{(r-i)}$, we can conclude that $\mathcal{F}'$ is a κ-well separated $F(S^*)$-decomposition of $G(S_j)[L]$. Thus, by Proposition 2.7.9, $\mathcal{F}_j$ is a κ-well separated F-packing in G with $\{e \in \mathcal{F}_j^{(r)} : S_j \subseteq e\} = S_j \uplus L$.

Moreover, we have $V(\mathcal{F}_1^{(r)}) \subseteq \bigcup_{F' \in \mathcal{F}'} V(F_1') \subseteq V(G) \setminus S_2$ and by Fact 2.8.7(ii)

$$\phi(\mathcal{F}_1^{(r)}) = \phi(\dot{\bigcup_{F' \in \mathcal{F}'}} F_1') = \dot{\bigcup_{F' \in \mathcal{F}'}} \phi(F_1') = \dot{\bigcup_{F' \in \mathcal{F}'}} F_2' = \mathcal{F}_2^{(r)}.$$

□

We now sketch the proof of Lemma 2.8.5. Suppose for simplicity that H' is simply a copy of H, i.e. $H' = \phi(H)$ where ϕ is an isomorphism from H to H'. We aim to construct an $(H, H'; F)$-transformer. In a first step, for every edge $e \in H$, we introduce a set X_e of $|V(F)| - r$ new vertices and let F_e be a copy of F such that $V(F_e) = e \cup X_e$ and $e \in F_e$. Let $T_1 := \bigcup_{e \in H} F_e[X_e]$ and $R_1 := \bigcup_{e \in H} F_e - T_1 - H$. Clearly, $\{F_e : e \in H\}$ is an F-decomposition of $H \cup R_1 \cup T_1$. By Fact 2.8.7(ii), we also have that $\{\phi(F_e) : e \in H\}$ is an F-decomposition of $H' \cup \phi(R_1) \cup T_1$. Hence, T_1 is an $(H \cup R_1, H' \cup \phi(R_1); F)$-transformer. Note that at this stage, it would suffice to find an $(R_1, \phi(R_1); F)$-transformer T_1', as then $T_1 \cup T_1' \cup R_1 \cup \phi(R_1)$ would be an $(H, H'; F)$-transformer. The crucial difference now to the original problem is that every edge of R_1 contains at most $r-1$ vertices from $V(H)$. On the other hand, every edge in R_1 contains at least one vertex in $V(H)$ as otherwise it would belong to T_1. We view this as Step 1 and will now proceed inductively. After Step i, we will have an r-graph R_i and an $(H \cup R_i, H' \cup \phi(R_i); F)$-transformer T_i such that every edge $e \in R_i$ satisfies $1 \leq |e \cap V(H)| \leq r - i$. Thus, after Step r we can terminate the process as R_r must be empty and thus T_r is an $(H, H'; F)$-transformer.

In Step $i+1$, where $i \in [r-1]$, we use $(*)_i$ inductively as follows. Let R'_i consist of all edges of R_i which intersect $V(H)$ in $r-i$ vertices. We decompose R'_i into 'local' parts. For every edge $e \in R'_i$, there exists a unique set $S \in \binom{V(H)}{r-i}$ such that $S \subseteq e$. For each $S \in \binom{V(H)}{r-i}$, let $L_S := R'_i(S)$. Note that the 'local' parts $S \uplus L_S$ form a decomposition of R'_i. The problem of finding R_{i+1} and T_{i+1} can be reduced to finding a 'localised transformer' between $S \uplus L_S$ and $\phi(S) \uplus L_S$ for every S, as described above. At this stage, by Proposition 2.5.3, L_S will automatically be $F(S^*)$-divisible, where $S^* \in \binom{V(F)}{r-i}$ is such that $F(S^*)$ is non-empty. If we were given an $F(S^*)$-decomposition $\mathcal{F}'_S$ of L_S, we could use Proposition 2.8.9 to extend $\mathcal{F}'_S$ to an F-packing $\mathcal{F}_S$ which covers all edges of $S \uplus L_S$, and all new edges created by this extension intersect S (and $V(H)$) in at most $r-i-1$ vertices, as desired. It is possible to combine these localised transformers with T_i and R_i in such a way that we obtain T_{i+1} and R_{i+1}.

Unfortunately, $(G(S) \cap G(\phi(S)))[L_S]$ might not be a supercomplex (one can think of L_S as some leftover from previous steps) and so $\mathcal{F}'_S$ may not exist. However, by Proposition 2.5.5, we have that $G(S) \cap G(\phi(S))$ is a supercomplex. Thus we can (randomly) choose a suitable i-subgraph A_S of $(G(S) \cap G(\phi(S)))^{(i)}$ such that A_S is $F(S^*)$-divisible and edge-disjoint from L_S. Instead of building a localised transformer for L_S directly, we will now build one for A_S and one for $A_S \cup L_S$, using $(*)_i$ both times to find the desired $F(S^*)$-decomposition. These can then be combined into a localised transformer for L_S.

Lemma 2.8.10. *Let $1/n \ll \gamma' \ll \gamma, 1/\kappa, \varepsilon \ll \xi, 1/f$ and $1 \le i < r < f$. Assume that $(*)_{r-i}$ is true. Let F be a weakly regular r-graph on f vertices and assume that $S^* \in \binom{V(F)}{i}$ is such that $F(S^*)$ is non-empty. Let G be an (ε, ξ, f, r)-supercomplex on n vertices, let $S_1, S_2 \in G^{(i)}$ with $S_1 \cap S_2 = \emptyset$, and let $\phi \colon S_1 \to S_2$ be a bijection. Moreover, suppose that L is an $F(S^*)$-divisible subgraph of $G(S_1)^{(r-i)} \cap G(S_2)^{(r-i)}$ with $|V(L)| \le \gamma' n$.*

Then there exist $T, R \subseteq G^{(r)}$ such that the following hold:

(TR1) *$V(R) \subseteq V(G) \setminus S_2$ and $|e \cap S_1| \in [i-1]$ for all $e \in R$ (so if $i = 1$, then R must be empty since $[0] = \emptyset$);*

(TR2) *T is a $(\kappa+1)$-well separated $((S_1 \uplus L) \cup \phi(R), (S_2 \uplus L) \cup R; F)$-transformer in G;*

(TR3) *$|V(T \cup R)| \leq \gamma n$.*

Proof. We may assume that $\gamma' \ll \gamma \ll 1/\kappa, \varepsilon$. Choose $\mu > 0$ with $\gamma' \ll \mu \ll \gamma \ll 1/\kappa, \varepsilon$. We split the argument into two parts. First, we will establish the following claim, which is the essential part and relies on $(*)_{r-i}$.

Claim 1: There exist $\hat{T}, R_{1,A}, R_{1,A\cup L} \subseteq G^{(r)}$ and κ-well separated F-packings $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2$ in G such that the following hold:

(tr1) *$V(R_{1,A} \cup R_{1,A\cup L}) \subseteq V(G) \setminus S_2$ and $|e \cap S_1| \in [i-1]$ for all $e \in R_{1,A} \cup R_{1,A\cup L}$;*

(tr2) *$\hat{T}$, $S_1 \uplus L$, $S_2 \uplus L$, $R_{1,A}$, $\phi(R_{1,A})$, $R_{1,A\cup L}$, $\phi(R_{1,A\cup L})$ are pairwise edge-disjoint subgraphs of $G^{(r)}$;*

(tr3) *$\hat{\mathcal{F}}_1^{(r)} = \hat{T} \cup (S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A})$ and $\hat{\mathcal{F}}_2^{(r)} = \hat{T} \cup (S_2 \uplus L) \cup R_{1,A} \cup \phi(R_{1,A\cup L})$;*

(tr4) *$|V(\hat{T} \cup R_{1,A} \cup R_{1,A\cup L})| \leq 2\mu n$.*

Proof of claim: By Corollary 2.5.17 and Lemma 2.5.10(i), there exists a subset $U \subseteq V(G)$ with $0.9\mu n \leq |U| \leq 1.1\mu n$ such that $G' := G[U \cup S_1 \cup S_2 \cup V(L)]$ is a $(2\varepsilon, \xi - \varepsilon, f, r)$-supercomplex. By Proposition 2.5.5, $G'' := G'(S_1) \cap G'(S_2)$ is a $(2\varepsilon, \xi - \varepsilon, f - i, r - i)$-supercomplex. Clearly, $L \subseteq G''^{(r-i)}$ and $\Delta(L) \leq \gamma' n \leq \sqrt{\gamma'}|U|$. Thus, by Proposition 2.5.9(v), $G'' - L$ is a $(3\varepsilon, \xi - 2\varepsilon, f - i, r - i)$-supercomplex. By Corollary 2.6.10, there exists $H \subseteq G''^{(r-i)} - L$ such that $A := G''^{(r-i)} - L - H$ is $F(S^*)$-divisible and $\Delta(H) \leq \gamma' n$. In particular, by Proposition 2.5.9(v) we have that

(i) $G''[A]$ is an $F(S^*)$-divisible $(3\varepsilon, \xi/2, f - i, r - i)$-supercomplex;

(ii) $G''[A \cup L]$ is an $F(S^*)$-divisible $(3\varepsilon, \xi/2, f - i, r - i)$-supercomplex.

Recall that F being weakly regular implies that $F(S^*)$ is weakly regular as well (see Proposition 2.5.3). By (i) and $(*)_{r-i}$, there exists a κ-well separated $F(S^*)$-decomposition $\mathcal{F}_A$ of $G''[A]$. By Fact 2.5.4(i), $\Delta(\mathcal{F}_A^{\leq(r-i+1)}) \leq \kappa f$. Thus, by (ii), Proposition 2.5.9(v) and $(*)_{r-i}$, there also exists a κ-well separated $F(S^*)$-decomposition $\mathcal{F}_{A\cup L}$ of $G''[A\cup L] - \mathcal{F}_A^{\leq(r-i+1)}$. In particular, $\mathcal{F}_A$ and $\mathcal{F}_{A\cup L}$ are $(r-i+1)$-disjoint.

We define

$$(\mathcal{F}_{1,A}, \mathcal{F}_{2,A}) := S_1 \triangleleft \mathcal{F}_A \triangleright S_2,$$
$$(\mathcal{F}_{1,A\cup L}, \mathcal{F}_{2,A\cup L}) := S_1 \triangleleft \mathcal{F}_{A\cup L} \triangleright S_2.$$

By Proposition 2.8.9(i), for $j \in [2]$, $\mathcal{F}_{j,A}$ is a κ-well separated F-packing in $G' \subseteq G$ with $\{e \in \mathcal{F}_{j,A}^{(r)} : S_j \subseteq e\} = S_j \uplus A$ and $\mathcal{F}_{j,A\cup L}$ is a κ-well separated F-packing in $G' \subseteq G$ with $\{e \in \mathcal{F}_{j,A\cup L}^{(r)} : S_j \subseteq e\} = S_j \uplus (A \cup L)$.

For $j \in [2]$, let

$$T_{j,A} := \{e \in \mathcal{F}_{j,A}^{(r)} : |e \cap S_j| = 0\},$$
$$T_{j,A\cup L} := \{e \in \mathcal{F}_{j,A\cup L}^{(r)} : |e \cap S_j| = 0\},$$
$$R_{j,A} := \{e \in \mathcal{F}_{j,A}^{(r)} : |e \cap S_j| \in [i-1]\},$$
$$R_{j,A\cup L} := \{e \in \mathcal{F}_{j,A\cup L}^{(r)} : |e \cap S_j| \in [i-1]\}.$$

By Definition 2.8.8, we have that $T_{1,A} = T_{2,A}$ and $T_{1,A\cup L} = T_{2,A\cup L}$. We thus set

$$T_A := T_{1,A} = T_{2,A} \quad \text{and} \quad T_{A\cup L} := T_{1,A\cup L} = T_{2,A\cup L}.$$

Moreover, we have

$$\phi(R_{1,A}) = R_{2,A} \quad \text{and} \quad \phi(R_{1,A\cup L}) = R_{2,A\cup L}. \tag{2.8.1}$$

Note that $R_{1,A}, R_{2,A}, R_{1,A\cup L}, R_{2,A\cup L}$ are empty if $i = 1$. Crucially, since $\mathcal{F}_A$ and $\mathcal{F}_{A\cup L}$ are $(r-i+1)$-disjoint, it is easy to see (by contradiction) that T_A and $T_{A\cup L}$ are edge-disjoint, and that for $j \in [2]$, $R_{j,A}$ and $R_{j,A\cup L}$ are edge-disjoint. Further, since A and L are edge-disjoint, we clearly have for $j \in [2]$ that $S_j \uplus L$ and $S_j \uplus A$ are edge-disjoint.

Using this, it is straightforward to see that

(†) $S_1 \uplus L$, $S_2 \uplus L$, $S_1 \uplus A$, $S_2 \uplus A$, T_A, $T_{A\cup L}$, $R_{1,A}$, $R_{2,A}$, $R_{1,A\cup L}$, $R_{2,A\cup L}$ are pairwise edge-disjoint subgraphs of $G^{(r)}$.

Observe that for $j \in [2]$, we have

$$\mathcal{F}^{(r)}_{j,A} = (S_j \uplus A) \mathbin{\dot\cup} R_{j,A} \mathbin{\dot\cup} T_A; \tag{2.8.2}$$

$$\mathcal{F}^{(r)}_{j,A\cup L} = (S_j \uplus (A \cup L)) \mathbin{\dot\cup} R_{j,A\cup L} \mathbin{\dot\cup} T_{A\cup L}. \tag{2.8.3}$$

Define

$$\hat{T} := (S_1 \uplus A) \cup (S_2 \uplus A) \cup T_A \cup T_{A\cup L};$$
$$\hat{\mathcal{F}}_1 := \mathcal{F}_{1,A\cup L} \cup \mathcal{F}_{2,A};$$
$$\hat{\mathcal{F}}_2 := \mathcal{F}_{1,A} \cup \mathcal{F}_{2,A\cup L}.$$

We now check that (tr1)–(tr4) hold. First note that by (†) we clearly have $\hat{T}, R_{1,A}, R_{1,A\cup L} \subseteq G^{(r)}$. Moreover, since $\mathcal{F}_A$ and $\mathcal{F}_{A\cup L}$ are $(r-i+1)$-disjoint, we have that $\mathcal{F}_{1,A\cup L}$ and $\mathcal{F}_{2,A}$ are r-disjoint and thus $\hat{\mathcal{F}}_1$ is a κ-well separated F-packing in G by Fact 2.5.4(iii). Similarly, $\hat{\mathcal{F}}_2$ is a κ-well separated F-packing in G.

To check (tr1), note that $V(R_{1,A}) \subseteq V(\mathcal{F}^{(r)}_{1,A}) \subseteq V(G)\setminus S_2$ and $V(R_{1,A\cup L}) \subseteq V(\mathcal{F}^{(r)}_{1,A\cup L}) \subseteq V(G) \setminus S_2$ by Proposition 2.8.9(ii). Moreover, for all $e \in R_{1,A} \cup R_{1,A\cup L}$, we have $|e \cap S_1| \in [i-1]$ by definition. Hence, (tr1) holds. Clearly, (2.8.1) and (†) imply (tr2). Crucially, by (2.8.1)–(2.8.3) we have that

$$\hat{\mathcal{F}}^{(r)}_1 = \mathcal{F}^{(r)}_{1,A\cup L} \mathbin{\dot\cup} \mathcal{F}^{(r)}_{2,A} = \hat{T} \cup (S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A});$$
$$\hat{\mathcal{F}}^{(r)}_2 = \mathcal{F}^{(r)}_{1,A} \mathbin{\dot\cup} \mathcal{F}^{(r)}_{2,A\cup L} = \hat{T} \cup (S_2 \uplus L) \cup R_{1,A} \cup \phi(R_{1,A\cup L}).$$

Thus, (tr3) is satisfied. Finally, $|V(\hat{T} \cup R_{1,A} \cup R_{1,A\cup L})| \leq |V(G')| \leq 2\mu n$, proving the claim. $-$

The transformer $\hat{T}$ almost has the required properties, except that to satisfy (TR2) we would have needed $R_{1,A\cup L}$ and $\phi(R_{1,A\cup L})$ to be on the 'other side' of the transformation. In order to resolve this, we carry out an additional transformation step. (Since $R_{1,A}$ and $R_{1,A\cup L}$ are empty if $i=1$, this additional step is vacuous in this case.)

Claim 2: There exist $T', R' \subseteq G^{(r)}$ and 1-well separated F-packings $\mathcal{F}'_1, \mathcal{F}'_2$ in $G - \hat{\mathcal{F}}_1^{\leq(r+1)} - \hat{\mathcal{F}}_2^{\leq(r+1)}$ such that the following hold:

(tr1′) $V(R') \subseteq V(G) \setminus S_2$ *and* $|e \cap S_1| \in [i-1]$ *for all* $e \in R'$;

(tr2′) T', R', $\phi(R')$, $\hat{T}$, $S_1 \uplus L$, $S_2 \uplus L$, $R_{1,A}$, $\phi(R_{1,A})$, $R_{1,A\cup L}$, $\phi(R_{1,A\cup L})$ *are pairwise edge-disjoint r-graphs;*

(tr3′) $\mathcal{F}_1'^{(r)} = T' \cup R_{1,A\cup L} \cup R'$ *and* $\mathcal{F}_2'^{(r)} = T' \cup \phi(R_{1,A\cup L}) \cup \phi(R')$;

(tr4′) $|V(T' \cup R')| \leq 0.7\gamma n$.

Proof of claim: Let $H' := \hat{T} \cup R_{1,A} \cup \phi(R_{1,A}) \cup (S_1 \uplus L) \cup (S_2 \uplus L)$. Clearly, $\Delta(H') \leq 5\mu n$.

Let $W := V(R_{1,A\cup L}) \cup V(\phi(R_{1,A\cup L}))$. By (tr4), we have that $|W| \leq 4\mu n$. Similarly to the beginning of the proof of Claim 1, by Corollary 2.5.17 and Lemma 2.5.10(i), there exists a subset $U' \subseteq V(G)$ with $0.4\gamma n \leq |U'| \leq 0.6\gamma n$ such that $G''' := G[U' \cup W]$ is a $(2\varepsilon, \xi - \varepsilon, f, r)$-supercomplex. Let $\tilde{n} := |U' \cup W|$. Note that

$$\Delta(H') \leq 5\mu n \leq \sqrt{\mu}\tilde{n} \quad \text{and} \quad \Delta(\hat{\mathcal{F}}_j^{\leq(r+1)}) \leq \kappa(f-r)$$

for $j \in [2]$ by Fact 2.5.4(i). Thus, by Proposition 2.5.9(v),

$$\tilde{G} := G''' - H' - \hat{\mathcal{F}}_1^{\leq(r+1)} - \hat{\mathcal{F}}_2^{\leq(r+1)}$$

is still a $(3\varepsilon, \xi - 2\varepsilon, f, r)$-supercomplex. For every $e \in R_{1,A\cup L}$, let

$$\mathcal{Q}_e := \{Q \in \tilde{G}^{(f)}(e) \cap \tilde{G}^{(f)}(\phi(e)) \,:\, Q \cap (S_1 \cup S_2) = \emptyset\}.$$

By Fact 2.5.6, for every $e \in R_{1,A\cup L} \subseteq \tilde{G}^{(r)}$, we have that $|\tilde{G}^{(f)}(e) \cap \tilde{G}^{(f)}(\phi(e))| \geq 0.5\xi\tilde{n}^{f-r}$.

Thus, we have that $|\mathcal{Q}_e| \geq 0.4\xi\tilde{n}^{f-r}$. Since $\Delta(R_{1,A\cup L} \cup \phi(R_{1,A\cup L})) \leq 4\mu n \leq \sqrt{\mu}\tilde{n}$, we can apply Lemma 2.6.7 (with $|R_{1,A\cup L}|, 2, \{e, \phi(e)\}, \mathcal{Q}_e$ playing the roles of $m, s, L_j, \mathcal{Q}_j$) to find for every $e \in R_{1,A\cup L}$ some $Q_e \in \mathcal{Q}_e$ such that, writing $K_e := (Q_e \uplus \{e, \phi(e)\})^{\leq}$, we have that

$$K_e \text{ and } K_{e'} \text{ are } r\text{-disjoint for distinct } e, e' \in R_{1,A\cup L}. \tag{2.8.4}$$

For each $e \in R_{1,A\cup L}$, let $\tilde{F}_{e,1}$ and $\tilde{F}_{e,2}$ be copies of F with $V(\tilde{F}_{e,1}) = e \cup Q_e$ and $V(\tilde{F}_{e,2}) = \phi(e) \cup Q_e$ and such that $e \in \tilde{F}_{e,1}$ and $\phi(\tilde{F}_{e,1}) = \tilde{F}_{e,2}$. Clearly, we have that $\phi(e) \in \tilde{F}_{e,2}$. Moreover, since $e \subseteq V(R_{1,A\cup L}) \subseteq V(G) \setminus S_2$ by (tr1) and $Q_e \cap (S_1 \cup S_2) = \emptyset$, we have $V(\tilde{F}_{e,1}) \subseteq V(G) \setminus S_2$. Let

$$\mathcal{F}_1' := \{\tilde{F}_{e,1} \,:\, e \in R_{1,A\cup L}\}; \tag{2.8.5}$$

$$\mathcal{F}_2' := \{\tilde{F}_{e,2} \,:\, e \in R_{1,A\cup L}\}. \tag{2.8.6}$$

By (2.8.4), $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are both 1-well separated F-packings in $\tilde{G} \subseteq G - \hat{\mathcal{F}}_1^{\leq(r+1)} - \hat{\mathcal{F}}_2^{\leq(r+1)}$. Moreover, $V(\mathcal{F}_1'^{(r)}) \subseteq V(G) \setminus S_2$ and $\phi(\mathcal{F}_1'^{(r)}) = \mathcal{F}_2'^{(r)}$. Let

$$T' := \mathcal{F}_1'^{(r)} \cap \mathcal{F}_2'^{(r)}; \tag{2.8.7}$$

$$R' := \mathcal{F}_1'^{(r)} - T' - R_{1,A\cup L}. \tag{2.8.8}$$

We clearly have $T', R' \subseteq G^{(r)}$ and now check (tr1′)–(tr4′). Note that no edge of T' intersects $S_1 \cup S_2$. For (tr1′), we first have that $V(R') \subseteq V(\mathcal{F}_1'^{(r)}) \subseteq V(G) \setminus S_2$. Now, consider $e' \in R'$. There exists $e \in R_{1,A\cup L}$ with $e' \in \tilde{F}_{e,1}$ and thus $e' \subseteq e \cup Q_e$. If we had $e' \cap S_1 = \emptyset$, then $e' \subseteq (e \setminus S_1) \cup Q_e$. Since $\phi(\tilde{F}_{e,1}) = \tilde{F}_{e,2}$, it follows that $e' \in T'$, a contradiction to (2.8.8). Hence, $|e' \cap S_1| > 0$. Moreover, by (tr1) we have $|e' \cap S_1| \leq |(e \cup Q_e) \cap S_1| = |e \cap S_1| \leq i-1$. Therefore, $|e' \cap S_1| \in [i-1]$ and (tr1′) holds.

In order to check (tr3′), observe first that by (2.8.8) and (2.8.5), we have $\mathcal{F}_1'^{(r)} =$

$T' \cupdot R_{1,A\cup L} \cupdot R'$. Hence, by Fact 2.8.7(ii), we have

$$\mathcal{F}_2'^{(r)} = \phi(\mathcal{F}_1'^{(r)}) = \phi(T') \cupdot \phi(R_{1,A\cup L}) \cupdot \phi(R') = T' \cupdot \phi(R_{1,A\cup L}) \cupdot \phi(R'), \tag{2.8.9}$$

so (tr3′) is satisfied.

We now check (tr2′). Note that $T', R', \phi(R') \subseteq \tilde{G}^{(r)} \subseteq G^{(r)} - H'$. Thus, by (tr2), it is enough to check that $T', R', \phi(R'), R_{1,A\cup L}, \phi(R_{1,A\cup L})$ are pairwise edge-disjoint. Recall that no edge of T' intersects $S_1 \cup S_2$. Moreover, for every $e \in R' \cup R_{1,A\cup L}$, we have $|e \cap S_1| \in [i-1]$ and $e \cap S_2 = \emptyset$, and for every $e \in \phi(R') \cup \phi(R_{1,A\cup L})$, we have $|e \cap S_2| \in [i-1]$ and $e \cap S_1 = \emptyset$. Since R' and $R_{1,A\cup L}$ are edge-disjoint by (2.8.8) and $\phi(R')$ and $\phi(R_{1,A\cup L})$ are edge-disjoint by (2.8.9), this implies that $T', R', \phi(R'), R_{1,A\cup L}, \phi(R_{1,A\cup L})$ are indeed pairwise edge-disjoint, proving (tr2′).

Finally, we can easily check that $|V(T' \cup R')| \leq \tilde{n} \leq 0.7\gamma n$. —

We now combine the results of Claims 1 and 2. Let

$$\begin{aligned} T &:= \hat{T} \cup R_{1,A\cup L} \cup \phi(R_{1,A\cup L}) \cup T'; \\ R &:= R_{1,A} \cup R'; \\ \mathcal{F}_1 &:= \hat{\mathcal{F}}_1 \cup \mathcal{F}_2'; \\ \mathcal{F}_2 &:= \hat{\mathcal{F}}_2 \cup \mathcal{F}_1'. \end{aligned}$$

Clearly, (tr1) and (tr1′) imply that (TR1) holds. Moreover, (tr2′) implies that T is edge-disjoint from both $(S_1 \uplus L) \cup \phi(R)$ and $(S_2 \uplus L) \cup R$. Using (tr3) and (tr3′), observe that

$$\begin{aligned} T \cup (S_1 \uplus L) \cup \phi(R) &= \hat{T} \cup R_{1,A\cup L} \cup \phi(R_{1,A\cup L}) \cup T' \cup (S_1 \uplus L) \cup \phi(R_{1,A}) \cup \phi(R') \\ &= (\hat{T} \cup (S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A})) \cupdot (T' \cup \phi(R_{1,A\cup L}) \cup \phi(R')) \\ &= \hat{\mathcal{F}}_1^{(r)} \cupdot \mathcal{F}_2'^{(r)} = \mathcal{F}_1^{(r)}. \end{aligned}$$

Similarly, $\mathcal{F}_2^{(r)} = \hat{\mathcal{F}}_2^{(r)} \cupdot \mathcal{F}_1'^{(r)} = T \cup (S_2 \uplus L) \cup R$. In particular, by Fact 2.5.4(ii) we can see that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(\kappa+1)$-well separated F-packings in G. Thus, T is a $(\kappa+1)$-well separated $((S_1 \uplus L) \cup \phi(R), (S_2 \uplus L) \cup R; F)$-transformer in G, so (TR2) holds. Finally, we have $|V(T \cup R)| \leq 4\mu n + 0.7\gamma n \leq \gamma n$ by (tr4) and (tr4′). □

So far, our maps $\phi\colon S_1 \to S_2$ were bijections. When ϕ is an edge-bijective homomorphism from H to H', ϕ is in general not injective. In order to still have a meaningful notion of 'mirroring' as before, we introduce the following notation.

Definition 2.8.11. Let V be a set and let V_1, V_2 be disjoint subsets of V, and let $\phi\colon V_1 \to V_2$ be a map. For a set $S \subseteq V \setminus V_2$, define $\phi(S) := (S \setminus V_1) \cup \phi(S \cap V_1)$. Let $r \in \mathbb{N}$ and suppose that R is an r-graph with $V(R) \subseteq V$ and $i \in [r]_0$. We say that R is (ϕ, V, V_1, V_2, i)-*projectable* if the following hold:

(Y1) for every $e \in R$, we have that $e \cap V_2 = \emptyset$ and $|e \cap V_1| \in [i]$ (so if $i = 0$, then R must be empty since $[0] = \emptyset$);

(Y2) for every $e \in R$, we have $|\phi(e)| = r$;

(Y3) for every two distinct edges $e, e' \in R$, we have $\phi(e) \neq \phi(e')$.

Note that if ϕ is injective and $e \cap V_2 = \emptyset$ for all $e \in R$, then (Y2) and (Y3) always hold. If R is (ϕ, V, V_1, V_2, i)-projectable, then let $\phi(R)$ be the r-graph on $\phi(V(R) \setminus V_2)$ with edge set $\{\phi(e) : e \in R\}$. For an r-graph P with $V(P) \subseteq V \setminus V_2$ that satisfies (Y2), let P^ϕ be the r-graph on $V(P) \cup V_1$ that consists of all $e \in \binom{V \setminus V_2}{r}$ such that $\phi(e) = \phi(e')$ for some $e' \in P$.

The following facts are easy to see.

Proposition 2.8.12. *Let $V, V_1, V_2, \phi, R, r, i$ be as above and assume that R is (ϕ, V, V_1, V_2, i)-projectable. Then the following hold:*

(i) *$R \rightsquigarrow \phi(R)$;*

(ii) *every subgraph of R is (ϕ, V, V_1, V_2, i)-projectable;*

(iii) *for all* $e' \in \phi(R)$*, we have* $e' \cap V_1 = \emptyset$ *and* $|e' \cap V_2| \in [i]$*;*

(iv) *assume that for all* $e \in R$*, we have* $|e \cap V_1| = i$*, and let* $\mathcal{S}$ *contain all* $S \in \binom{V_1}{i}$ *such that* S *is contained in some edge of* R*, then*

$$R = \dot{\bigcup_{S \in \mathcal{S}}}(S \uplus R(S)) \quad \text{and} \quad \phi(R) = \dot{\bigcup_{S \in \mathcal{S}}}(\phi(S) \uplus R(S)).$$

We can now prove the Transforming lemma by combining many localised transformers.

Proof of Lemma 2.8.5. We can assume that $1/\kappa \ll \gamma \ll 1/h, \varepsilon$. Choose new constants $\kappa' \in \mathbb{N}$ and $\gamma_2, \ldots, \gamma_r, \gamma'_2 \ldots, \gamma'_r > 0$ such that

$$1/n \ll 1/\kappa \ll \gamma_r \ll \gamma'_r \ll \gamma_{r-1} \ll \gamma'_{r-1} \ll \cdots \ll \gamma_2 \ll \gamma'_2 \ll \gamma \ll 1/\kappa', 1/h, \varepsilon \ll \xi, 1/f.$$

Let $\phi\colon V(H) \to V(H')$ be an edge-bijective homomorphism from H to H'. Extend ϕ as in Definition 2.8.11 with $V(H), V(H')$ playing the roles of V_1, V_2. Since ϕ is edge-bijective, we have that

$$\phi\restriction_S \text{ is injective whenever } S \subseteq e \text{ for some } e \in H. \tag{2.8.10}$$

For every $e \in H$, we have $|G^{(f)}(e) \cap G^{(f)}(\phi(e))| \geq 0.5\xi n^{f-r}$ by Fact 2.5.6. It is thus easy to find for each $e \in H$ some $Q_e \in G^{(f)}(e) \cap G^{(f)}(\phi(e))$ with $Q_e \cap (V(H) \cup V(H')) = \emptyset$ such that $Q_e \cap Q_{e'} = \emptyset$ for all distinct $e, e' \in H$. For each $e \in H$, let $\tilde{F}_{e,1}$ and $\tilde{F}_{e,2}$ be copies of F with $V(\tilde{F}_{e,1}) = e \cup Q_e$ and $V(\tilde{F}_{e,2}) = \phi(e) \cup Q_e$ and such that $e \in \tilde{F}_{e,1}$ and $\phi(\tilde{F}_{e,1}) = \tilde{F}_{e,2}$. Clearly, we have that $\phi(e) \in \tilde{F}_{e,2}$. For $j \in [2]$, define $\mathcal{F}^*_{r,j} := \{\tilde{F}_{e,j} \,:\, e \in H\}$. Clearly, $\mathcal{F}^*_{r,1}$ and $\mathcal{F}^*_{r,2}$ are both 1-well separated F-packings in G. Define

$$\begin{aligned} T^*_r &:= \mathcal{F}^{*(r)}_{r,1} \cap \mathcal{F}^{*(r)}_{r,2}, \\ R^*_r &:= \mathcal{F}^{*(r)}_{r,1} - T^*_r - H. \end{aligned} \tag{2.8.11}$$

Let $\gamma_1 := \gamma$. Furthermore, let $\kappa_r := 1$ and recursively define $\kappa_i := \kappa_{i+1} + \binom{h}{i}\kappa'$ for all $i \in [r-1]$.

Given $i \in [r-1]_0$ and $T^*_{i+1}, R^*_{i+1}, \mathcal{F}^*_{i+1,1}, \mathcal{F}^*_{i+1,2}$, we define the following conditions:

(TR1*)$_i$ R^*_{i+1} is $(\phi, V(G), V(H), V(H'), i)$-projectable;

(TR2*)$_i$ $T^*_{i+1}, R^*_{i+1}, \phi(R^*_{i+1}), H, H'$ are edge-disjoint subgraphs of $G^{(r)}$;

(TR3*)$_i$ $\mathcal{F}^*_{i+1,1}$ and $\mathcal{F}^*_{i+1,2}$ are κ_{i+1}-well separated F-packings in G with $\mathcal{F}^{*(r)}_{i+1,1} = T^*_{i+1} \cup H \cup R^*_{i+1}$ and $\mathcal{F}^{*(r)}_{i+1,2} = T^*_{i+1} \cup H' \cup \phi(R^*_{i+1})$;

(TR4*)$_i$ $|V(T^*_{i+1} \cup R^*_{i+1})| \leq \gamma_{i+1} n$.

We will first show that the above choices of $T^*_r, R^*_r, \mathcal{F}^*_{r,1}, \mathcal{F}^*_{r,2}$ satisfy (TR1*)$_{r-1}$–(TR4*)$_{r-1}$. We will then proceed inductively until we obtain $T^*_1, R^*_1, \mathcal{F}^*_{1,1}, \mathcal{F}^*_{1,2}$ satisfying (TR1*)$_0$–(TR4*)$_0$, which will then easily complete the proof.

Claim 1: $T^*_r, R^*_r, \mathcal{F}^*_{r,1}, \mathcal{F}^*_{r,2}$ satisfy (TR1*)$_{r-1}$–(TR4*)$_{r-1}$.

Proof of claim: (TR4*)$_{r-1}$ clearly holds. To see (TR1*)$_{r-1}$, consider any $e' \in R^*_r$. There exists $e \in H$ such that $e' \in \tilde{F}_{e,1}$. In particular, $e' \subseteq e \cup Q_e$. If $e' \subseteq V(H)$, then $e' = e \in H$, and if $e' \cap V(H) = \emptyset$, then $e' \in \tilde{F}_{e,2}$ since $\phi(\tilde{F}_{e,1}) = \tilde{F}_{e,2}$ and thus $e' \in T^*_r$. Hence, by definition of R^*_r, we must have $|e' \cap V(H)| \in [r-1]$. Clearly, $e' \cap V(H') \subseteq (e \cup Q_e) \cap V(H') = \emptyset$, so (Y1) holds. Moreover, $e' \cap V(H) \subseteq e$, so $\phi\restriction_{e' \cap V(H)}$ is injective by (2.8.10), and (Y2) holds. Let $e', e'' \in R^*_r$ and suppose that $\phi(e') = \phi(e'')$. We thus have $e' \setminus V(H) = e'' \setminus V(H) \neq \emptyset$. Since the Q_e's were chosen to be vertex-disjoint, we must have $e', e'' \subseteq e \cup Q_e$ for some $e \in H$. Hence, $(e' \cup e'') \cap V(H) \subseteq e$ and so $\phi\restriction_{(e' \cup e'') \cap V(H)}$ is injective by (2.8.10). Since $\phi(e' \cap V(H)) = \phi(e'' \cap V(H))$ by assumption, we have $e' \cap V(H) = e'' \cap V(H)$, and thus $e' = e''$. Altogether, (Y3) holds, so (TR1*)$_{r-1}$ is satisfied. In particular, $\phi(R^*_r)$ is well-defined. Observe that

$$\phi(R^*_r) = \mathcal{F}^{*(r)}_{r,2} - T^*_r - H'.$$

Clearly, $T_r^*, R_r^*, \phi(R_r^*), H, H'$ are subgraphs of $G^{(r)}$. Using Proposition 2.8.12(iii), it is easy to see that they are indeed edge-disjoint, so (TR2*) holds. Moreover, note that $\mathcal{F}_{r,1}^*$ and $\mathcal{F}_{r,2}^*$ are 1-well separated F-packings in G with $\mathcal{F}_{r,1}^{*(r)} = T_r^* \cup H \cup R_r^*$ and $\mathcal{F}_{r,2}^{*(r)} = T_r^* \cup H' \cup \phi(R_r^*)$, so T_r^* satisfies (TR3*)$_{r-1}$. $-$

Suppose that for some $i \in [r-1]$, we have already found $T_{i+1}^*, R_{i+1}^*, \mathcal{F}_{i+1,1}^*, \mathcal{F}_{i+1,2}^*$ such that (TR1*)$_i$–(TR4*)$_i$ hold. We will now find $T_i^*, R_i^*, \mathcal{F}_{i,1}^*, \mathcal{F}_{i,2}^*$ such that (TR1*)$_{i-1}$–(TR4*)$_{i-1}$ hold. To this end, let

$$R_i := \{e \in R_{i+1}^* : |e \cap V(H)| = i\}.$$

By Proposition 2.8.12(ii), R_i is $(\phi, V(G), V(H), V(H'), i)$-projectable. Let $\mathcal{S}_i$ be the set of all $S \in \binom{V(H)}{i}$ such that S is contained in some edge of R_i. For each $S \in \mathcal{S}_i$, let $L_S := R_i(S)$. By Proposition 2.8.12(iv), we have that

$$R_i = \dot{\bigcup_{S \in \mathcal{S}_i}} (S \uplus L_S) \quad \text{and} \quad \phi(R_i) = \dot{\bigcup_{S \in \mathcal{S}_i}} (\phi(S) \uplus L_S). \tag{2.8.12}$$

We intend to apply Lemma 2.8.10 to each pair $S, \phi(S)$ with $S \in \mathcal{S}_i$ individually. For each $S \in \mathcal{S}_i$, define

$$V_S := (V(G) \setminus (V(H) \cup V(H'))) \cup S \cup \phi(S).$$

Claim 2: For every $S \in \mathcal{S}_i$, $L_S \subseteq G[V_S](S)^{(r-i)} \cap G[V_S](\phi(S))^{(r-i)}$ and $|V(L_S)| \leq 1.1\gamma_{i+1}|V_S|$.

Proof of claim: The second assertion clearly holds by (TR4*)$_i$. To see the first one, let $e' \in L_S = R_i(S)$. Since $R_i \subseteq R_{i+1}^* \subseteq G^{(r)}$, we have $e' \in G(S)^{(r-i)}$. Moreover, $\phi(S) \cup e' \in \phi(R_i) \subseteq \phi(R_{i+1}^*) \subseteq G^{(r)}$ by (2.8.12). Since R_{i+1}^* is $(\phi, V(G), V(H), V(H'), i)$-projectable, we have that $e' \cap (V(H) \cup V(H')) = \emptyset$. Thus, $S \cup e' \subseteq V_S$ and $\phi(S) \cup e' \subseteq V_S$. $-$

Let $S^* \in \binom{V(F)}{i}$ be such that $F(S^*)$ is non-empty.

Claim 3: For every $S \in \mathcal{S}_i$, L_S is $F(S^*)$-divisible.

Proof of claim: Consider $b \subseteq V(L_S)$ with $|b| < r-i$. We have to check that $Deg(F(S^*))_{|b|} \mid |L_S(b)|$. By (TR3*)$_i$, both $T^*_{i+1} \cup H \cup R^*_{i+1}$ and $T^*_{i+1} \cup H' \cup \phi(R^*_{i+1})$ are necessarily F-divisible. Clearly, H' does not contain an edge that contains S. Note that by (TR1*)$_i$ and Proposition 2.8.12(iii), $\phi(R^*_{i+1})$ does not contain an edge that contains S either, hence $|T^*_{i+1}(S\cup b)| = |(T^*_{i+1}\cup H'\cup\phi(R^*_{i+1}))(S\cup b)| \equiv 0 \mod Deg(F)_{|S\cup b|}$. Moreover, since H is F-divisible, we have $|(T^*_{i+1}\cup R^*_{i+1})(S\cup b)| \equiv |(T^*_{i+1}\cup H\cup R^*_{i+1})(S\cup b)| \equiv 0 \mod Deg(F)_{|S\cup b|}$. Thus, we have $Deg(F)_{|S\cup b|} \mid |R^*_{i+1}(S\cup b)|$. Moreover, $|R^*_{i+1}(S\cup b)| = |R_i(S\cup b)| = |L_S(b)|$. Hence, $Deg(F)_{|S\cup b|} \mid |L_S(b)|$, which proves the claim as $Deg(F)_{|S\cup b|} = Deg(F(S^*))_{|b|}$ by Proposition 2.5.3. $-$

We now intend to apply Lemma 2.8.10 for every $S \in \mathcal{S}_i$ in order to define $T_S, R_S \subseteq G^{(r)}$ and κ'-well separated F-packings $\mathcal{F}_{S,1}, \mathcal{F}_{S,2}$ in G such that the following hold:

(TR1$'$) R_S is $(\phi, V(G), V(H), V(H'), i-1)$-projectable;

(TR2$'$) $T_S, R_S, \phi(R_S), S \uplus L_S, \phi(S) \uplus L_S$ are edge-disjoint;

(TR3$'$) $\mathcal{F}^{(r)}_{S,1} = T_S \cup (S \uplus L_S) \cup \phi(R_S)$ and $\mathcal{F}^{(r)}_{S,2} = T_S \cup (\phi(S) \uplus L_S) \cup R_S$;

(TR4$'$) $|V(T_S \cup R_S)| \le \gamma'_{i+1} n$.

We also need to ensure that all these graphs and packings satisfy several 'disjointness properties' (see (a)–(c)), and we will therefore choose them successively. Recall that P^ϕ (for a given r-graph P) was defined in Definition 2.8.11. Let $\mathcal{S}' \subseteq \mathcal{S}_i$ be the set of all $S' \in \mathcal{S}_i$ for which $T_{S'}, R_{S'}$ and $\mathcal{F}_{S',1}, \mathcal{F}_{S',2}$ have already been defined such that (TR1$'$)–(TR4$'$) hold. Suppose that next we want to find T_S, R_S, $\mathcal{F}_{S,1}$ and $\mathcal{F}_{S,2}$. Let

$$\begin{aligned}
P_S &:= R^*_{i+1} \cup \bigcup_{S'\in\mathcal{S}'} R_{S'},\\
M_S &:= T^*_{i+1} \cup R^*_{i+1} \cup \phi(R^*_{i+1}) \cup \bigcup_{S'\in\mathcal{S}'} (T_{S'} \cup R_{S'} \cup \phi(R_{S'})),\\
O_S &:= \mathcal{F}^{*\le(r+1)}_{i+1,1} \cup \mathcal{F}^{*\le(r+1)}_{i+1,2} \cup \bigcup_{S'\in\mathcal{S}'} \mathcal{F}^{\le(r+1)}_{S',1} \cup \mathcal{F}^{\le(r+1)}_{S',2},\\
G_S &:= G[V_S] - ((M_S \cup P^\phi_S) - ((S \uplus L_S) \cup (\phi(S) \uplus L_S))) - O_S.
\end{aligned}$$

Observe that (TR4*)$_i$ and (TR4′) imply that

$$|V(M_S \cup P_S)| \leq |V(T^*_{i+1} \cup R^*_{i+1} \cup \phi(R^*_{i+1}))| + \sum_{S' \in \mathcal{S}'} |V(T_{S'} \cup R_{S'} \cup \phi(R_{S'}))|$$
$$\leq 2\gamma_{i+1} n + 2\binom{h}{i}\gamma'_{i+1} n \leq \gamma_i n.$$

In particular, $|V(P^\phi_S)| \leq |V(P_S) \cup V(H)| \leq \gamma_i n + h$. Moreover, by Fact 2.5.4(i), (TR3*)$_i$ and (TR3′), we have that $\Delta(O_S) \leq (2\kappa_{i+1} + 2\binom{h}{i}\kappa')(f-r)$. Thus, by Proposition 2.5.9(v) G_S is still a $(2\varepsilon, \xi/2, f, r)$-supercomplex. Moreover, note that $L_S \subseteq G_S(S)^{(r-i)} \cap G_S(\phi(S))^{(r-i)}$ and $|V(L_S)| \leq 1.1\gamma_{i+1}|V_S|$ by Claim 2 and that L_S is $F(S^*)$-divisible by Claim 3.

Finally, by definition of $\mathcal{S}_i$, S is contained in some $e \in R_i$. Since R_i satisfies (Y2) by (TR1*)$_i$, we know that $\phi\restriction_e$ is injective. Thus, $\phi\restriction_S : S \to \phi(S)$ is a bijection. We can thus apply Lemma 2.8.10 with the following objects/parameters:

object/parameter	G_S	i	S	$\phi(S)$	$\phi\restriction_S$	L_S	$1.1\gamma_{i+1}$	γ'_{i+1}	2ε	$\lvert V_S\rvert$	$\xi/2$	f	r	F	S^*	$\kappa'/2$
playing the role of	G	i	S_1	S_2	ϕ	L	γ'	γ	ε	n	ξ	f	r	F	S^*	κ

This yields $T_S, R_S \subseteq G_S^{(r)}$ and $\kappa'/2$-well separated F-packings $\mathcal{F}_{S,1}, \mathcal{F}_{S,2}$ such that (TR2′)–(TR4′) hold, $V(R_S) \subseteq V(G_S) \setminus \phi(S)$ and $|e \cap S| \in [i-1]$ for all $e \in R_S$. Note that the latter implies that R_S is $(\phi, V(G), V(H), V(H'), i-1)$-projectable as $V(H) \cap V(G_S) = S$ and $V(H') \cap V(G_S) = \phi(S)$, so (TR1′) holds as well. Moreover, using (TR2*)$_i$ and (TR2′) it is easy to see that our construction ensures that

(a) $H, H', T^*_{i+1}, R^*_{i+1}, \phi(R^*_{i+1}), (T_S)_{S \in \mathcal{S}_i}, (R_S)_{S \in \mathcal{S}_i}, (\phi(R_S))_{S \in \mathcal{S}_i}$ are pairwise edge-disjoint;

(b) for all distinct $S, S' \in \mathcal{S}_i$ and all $e \in R_S$, $e' \in R_{S'}$, $e'' \in R^*_{i+1} - R_i$ we have that $\phi(e)$, $\phi(e')$ and $\phi(e'')$ are pairwise distinct;

(c) for any $j, j' \in [2]$ and all distinct $S, S' \in \mathcal{S}_i$, $\mathcal{F}_{S,j}$ is $(r+1)$-disjoint from $\mathcal{F}^*_{i+1,j'}$ and from $\mathcal{F}_{S',j'}$.

Indeed, (a) holds by the choice of M_S, (b) holds by definition of P^ϕ_S, and (c) holds by definition of O_S. Let

$$T_i^* := T_{i+1}^* \cup R_i \cup \phi(R_i) \cup \bigcup_{S \in \mathcal{S}_i} T_S;$$
$$R_i^* := (R_{i+1}^* - R_i) \cup \bigcup_{S \in \mathcal{S}_i} R_S;$$
$$\mathcal{F}_{i,1}^* := \mathcal{F}_{i+1,1}^* \cup \bigcup_{S \in \mathcal{S}_i} \mathcal{F}_{S,2};$$
$$\mathcal{F}_{i,2}^* := \mathcal{F}_{i+1,2}^* \cup \bigcup_{S \in \mathcal{S}_i} \mathcal{F}_{S,1}.$$

Using (TR3*)$_i$, (TR3′), (a) and (2.8.12), it is easy to check that both $\mathcal{F}_{i,1}^*$ and $\mathcal{F}_{i,2}^*$ are F-packings in G. We check that (TR1*)$_{i-1}$–(TR4*)$_{i-1}$ hold. Using (TR4*)$_i$ and (TR4′), we can confirm that

$$\begin{aligned} |V(T_i^* \cup R_i^*)| &\leq |V(T_{i+1}^* \cup R_{i+1}^* \cup \phi(R_{i+1}^*))| + \sum_{S \in \mathcal{S}_i} |V(T_S \cup R_S)| \\ &\leq 2\gamma_{i+1} n + \binom{h}{i} \gamma'_{i+1} n \leq \gamma_i n, \end{aligned}$$

so (TR4*)$_{i-1}$ holds.

In order to check (TR1*)$_{i-1}$, i.e. that R_i^* is $(\phi, V(G), V(H), V(H'), i-1)$-projectable, note that (Y1) and (Y2) hold by (TR1*)$_i$, the definition of R_i and (TR1′). Moreover, (Y3) is implied by (TR1*)$_i$, (TR1′) and (b).

Moreover, (TR2*)$_{i-1}$ follows from (a). Finally, we check (TR3*)$_{i-1}$. Observe that

$$\begin{aligned} T_i^* \cup H \cup R_i^* &= T_{i+1}^* \cup R_i \cup \phi(R_i) \cup \bigcup_{S \in \mathcal{S}_i} T_S \cup H \cup (R_{i+1}^* - R_i) \cup \bigcup_{S \in \mathcal{S}_i} R_S \\ &\overset{(2.8.12)}{=} (T_{i+1}^* \cup H \cup R_{i+1}^*) \cup \bigcup_{S \in \mathcal{S}_i} (T_S \cup (\phi(S) \uplus L_S) \cup R_S), \\ T_i^* \cup H' \cup \phi(R_i^*) &= T_{i+1}^* \cup R_i \cup \phi(R_i) \cup \bigcup_{S \in \mathcal{S}_i} T_S \cup H' \cup (\phi(R_{i+1}^*) - \phi(R_i)) \cup \bigcup_{S \in \mathcal{S}_i} \phi(R_S) \\ &\overset{(2.8.12)}{=} (T_{i+1}^* \cup H' \cup \phi(R_{i+1}^*)) \cup \bigcup_{S \in \mathcal{S}_i} (T_S \cup (S \uplus L_S) \cup \phi(R_S)). \end{aligned}$$

Thus, by (TR3*)$_i$ and (TR3$'$), $\mathcal{F}^*_{i,1}$ is an F-decomposition of $T^*_i \cup H \cup R^*_i$ and $\mathcal{F}^*_{i,2}$ is an F-decomposition of $T^*_i \cup H' \cup \phi(R^*_i)$. Moreover, by (c) and Fact 2.5.4(ii), $\mathcal{F}^*_{i,1}$ and $\mathcal{F}^*_{i,2}$ are both $(\kappa_{i+1} + \binom{h}{i}\kappa')$-well separated in G. Since $\kappa_{i+1} + \binom{h}{i}\kappa' = \kappa_i$, this establishes (TR3*)$_{i-1}$.

Finally, let $T^*_1, R^*_1, \mathcal{F}^*_{1,1}, \mathcal{F}^*_{1,2}$ satisfy (TR1*)$_0$–(TR4*)$_0$. Note that R^*_1 is empty by (TR1*)$_0$ and (Y1). Moreover, $T^*_1 \subseteq G^{(r)}$ is edge-disjoint from H and H' by (TR2*)$_0$ and $\Delta(T^*_1) \leq \gamma_1 n$ by (TR4*)$_0$. Most importantly, $\mathcal{F}^*_{1,1}$ and $\mathcal{F}^*_{1,2}$ are κ_1-well separated F-packings in G with $\mathcal{F}^{*(r)}_{1,1} = T^*_1 \cup H$ and $\mathcal{F}^{*(r)}_{1,2} = T^*_1 \cup H'$ by (TR3*)$_0$. Therefore, T^*_1 is a κ_1-well separated $(H, H'; F)$-transformer in G with $\Delta(T^*_1) \leq \gamma_1 n$. Recall that $\gamma_1 = \gamma$ and note that $\kappa_1 \leq 2^h\kappa' \leq \kappa$. Thus, T^*_1 is the desired transformer. □

2.8.2 Canonical multi-r-graphs

Roughly speaking, the aim of this section is to show that any F-divisible r-graph H can be transformed into a canonical multigraph M_h which does not depend on the structure of H. However, it turns out that for this we need to move to a 'dual' setting, where we consider ∇H which is obtained from H by applying an F-extension operator ∇. This operator allows us to switch between multi-r-graphs (which arise naturally in the construction but are not present in the complex G we are decomposing) and (simple) r-graphs (see e.g. Fact 2.8.18).

Given a multi-r-graph H and a set X of size r, we say that ψ is an *X-orientation of H* if ψ is a collection of bijective maps $\psi_e \colon X \to e$, one for each $e \in H$. (For $r = 2$ and $X = \{1, 2\}$, say, this coincides with the notion of an oriented multigraph, e.g. by viewing $\psi_e(1)$ as the tail and $\psi_e(2)$ as the head of e, where parallel edges can be oriented in opposite directions.)

Given an r-graph F and a distinguished edge $e_0 \in F$, we introduce the following 'extension' operators $\tilde{\nabla}_{(F,e_0)}$ and $\nabla_{(F,e_0)}$.

Definition 2.8.13 (Extension operators $\tilde{\nabla}$ and ∇)**.** Given a (multi-)r-graph H with an

e_0-orientation ψ, let $\tilde{\nabla}_{(F,e_0)}(H,\psi)$ be obtained from H by extending every edge of H into a copy of F, with e_0 being the rooted edge. More precisely, let Z_e be vertex sets of size $|V(F) \setminus e_0|$ such that $Z_e \cap Z_{e'} = \emptyset$ for all distinct (but possibly parallel) $e, e' \in H$ and $V(H) \cap Z_e = \emptyset$ for all $e \in H$. For each $e \in H$, let F_e be a copy of F on vertex set $e \cup Z_e$ such that $\psi_e(v)$ plays the role of v for all $v \in e_0$ and Z_e plays the role of $V(F) \setminus e_0$. Then $\tilde{\nabla}_{(F,e_0)}(H,\psi) := \bigcup_{e \in H} F_e$. Let $\nabla_{(F,e_0)}(H,\psi) := \tilde{\nabla}_{(F,e_0)}(H,\psi) - H$.

Note that $\nabla_{(F,e_0)}(H,\psi)$ is a (simple) r-graph even if H is a multi-r-graph. If F, e_0 and ψ are clear from the context, or if we only want to motivate an argument before giving the formal proof, we just write $\tilde{\nabla}H$ and ∇H.

Fact 2.8.14. *Let F be an r-graph and $e_0 \in F$. Let H be a multi-r-graph and let ψ be any e_0-orientation of H. Then the following hold:*

(i) $\tilde{\nabla}_{(F,e_0)}(H,\psi)$ *is F-decomposable;*

(ii) $\nabla_{(F,e_0)}(H,\psi)$ *is F-divisible if and only if H is F-divisible.*

The goal of this subsection is to show that for every $h \in \mathbb{N}$, there is a multi-r-graph M_h such that for any F-divisible r-graph H on at most h vertices, we have

$$\nabla(\nabla(H + t \cdot F) + s \cdot F) \rightsquigarrow \nabla M_h \tag{2.8.13}$$

for suitable $s, t \in \mathbb{N}$. The multigraph M_h is *canonical* in the sense that it does not depend on H, but only on h. The benefit is, very roughly speaking, that it allows us to transform any given leftover r-graph H into the empty r-graph, which is trivially decomposable, and this will enable us to construct an absorber for H. Indeed, to see that (2.8.13) allows us to transform H into the empty r-graph, let

$$H' := \nabla(\nabla(H + t \cdot F) + s \cdot F) = \nabla\nabla H + t \cdot \nabla\nabla F + s \cdot \nabla F$$

and observe that the r-graph $T := \nabla H + t \cdot \tilde{\nabla} F + s \cdot F$ 'between' H and H' can be chosen in such a way that

$$T \cup H = \tilde{\nabla} H + t \cdot \tilde{\nabla} F + s \cdot F,$$
$$T \cup H' = \tilde{\nabla}(\nabla H) + t \cdot (\tilde{\nabla}(\nabla F) \cupdot F) + s \cdot \tilde{\nabla} F,$$

i.e. T is an $(H, H'; F)$-transformer (cf. Fact 2.8.14(i)). Hence, together with (2.8.13) and Lemma 2.8.5, this means that we can transform H into ∇M_h. Since M_h does not depend on H, we can also transform the empty r-graph into ∇M_h, and by transitivity we can transform H into the empty graph, which amounts to an absorber for H (the detailed proof of this can be found in Section 2.8.3).

We now give the rigorous statement of (2.8.13), which is the main lemma of this subsection.

Lemma 2.8.15. *Let $r \geq 2$ and assume that $(*)_i$ is true for all $i \in [r-1]$. Let F be a weakly regular r-graph and $e_0 \in F$. Then for all $h \in \mathbb{N}$, there exists a multi-r-graph M_h such that for any F-divisible r-graph H on at most h vertices, we have*

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F, \psi_3) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi_2)$$

for suitable $s, t \in \mathbb{N}$, where ψ_1 and ψ_2 can be arbitrary e_0-orientations of $H + t \cdot F$ and M_h, respectively, and ψ_3 is an e_0-orientation depending on these.

The above graphs $\nabla(\nabla(H+t\cdot F)+s\cdot F)$ and ∇M_h will be part of our F-absorber for H. We therefore need to make sure that we can actually find them in a supercomplex G. This requirement is formalised by the following definition.

Definition 2.8.16. Let G be a complex, $X \subseteq V(G)$, F an r-graph with $f := |V(F)|$ and $e_0 \in F$. Suppose that $H \subseteq G^{(r)}$ and that ψ is an e_0-orientation of H. By *extending H with a copy of $\nabla_{(F,e_0)}(H, \psi)$ in G (whilst avoiding X)* we mean the following: for each $e \in H$, let $Z_e \in G^{(f)}(e)$ be such that $Z_e \cap (V(H) \cup X) = \emptyset$ for every $e \in H$ and $Z_e \cap Z_{e'} = \emptyset$ for all distinct $e, e' \in H$. For each $e \in H$, let F_e be a copy of F on vertex set $e \cup Z_e$

(so $F_e \subseteq G^{(r)}$) such that $\psi_e(v)$ plays the role of v for all $v \in e_0$ and Z_e plays the role of $V(F) \setminus e_0$. Let $H^\nabla := \bigcup_{e \in H} F_e - H$ and $\mathcal{F} := \{F_e : e \in H\}$ be the output of this.

For our purposes, the set $|V(H) \cup X|$ will have a small bounded size compared to $|V(G)|$. Thus, if the $G^{(f)}(e)$ are large enough (which is the case e.g. in an (ε, ξ, f, r)-supercomplex), then the above extension can be carried out simply by picking the sets Z_e one by one.

Fact 2.8.17. *Let $(H^\nabla, \mathcal{F})$ be obtained by extending $H \subseteq G^{(r)}$ with a copy of $\nabla_{(F,e_0)}(H, \psi)$ in G. Then $H^\nabla \subseteq G^{(r)}$ is a copy of $\nabla_{(F,e_0)}(H, \psi)$ and $\mathcal{F}$ is a 1-well separated F-packing in G with $\mathcal{F}^{(r)} = H \cup H^\nabla$ such that for all $F' \in \mathcal{F}$, $|V(F') \cap V(H)| \leq r$.*

For a partition $\mathcal{P} = \{V_x\}_{x \in X}$ whose classes are indexed by a set X, we define $V_Y := \bigcup_{x \in Y} V_x$ for every subset $Y \subseteq X$. Recall that for a multi-r-graph H and $e \in \binom{V(H)}{r}$, $|H(e)|$ denotes the multiplicity of e in H. For multi-r-graphs H, H', we write $H \overset{\mathcal{P}}{\rightsquigarrow} H'$ if $\mathcal{P} = \{V_{x'}\}_{x' \in V(H')}$ is a partition of $V(H)$ such that

(I1) for all $x' \in V(H')$ and $e \in H$, $|V_{x'} \cap e| \leq 1$;

(I2) for all $e' \in \binom{V(H')}{r}$, $\sum_{e \in \binom{V_{e'}}{r}} |H(e)| = |H'(e')|$.

Given $\mathcal{P}$, define $\phi_{\mathcal{P}} \colon V(H) \to V(H')$ as $\phi_{\mathcal{P}}(x) := x'$ where x' is the unique $x' \in V(H')$ such that $x \in V_{x'}$. Note that by (I1), we have $|\{\phi_{\mathcal{P}}(x) : x \in e\}| = r$ for all $e \in H$. Further, by (I2), there exists a bijection $\Phi_{\mathcal{P}} \colon H \to H'$ between the multi-edge-sets of H and H' such that for every edge $e \in H$, the image $\Phi_{\mathcal{P}}(e)$ is an edge consisting of the vertices $\phi_{\mathcal{P}}(x)$ for all $x \in e$. It is easy to see that $H \rightsquigarrow H'$ if and only if there is some $\mathcal{P}$ such that $H \overset{\mathcal{P}}{\rightsquigarrow} H'$.

The extension operator ∇ is well behaved with respect to the identification relation $\rightsquigarrow$ in the following sense: if $H \rightsquigarrow H'$, then $\nabla H \rightsquigarrow \nabla H'$. More precisely, let H and H' be multi-r-graphs and suppose that $H \overset{\mathcal{P}}{\rightsquigarrow} H'$. Let $\phi_{\mathcal{P}}$ and $\Phi_{\mathcal{P}}$ be defined as above. Let F be an r-graph and $e_0 \in F$. For any e_0-orientation ψ' of H', we define an e_0-orientation ψ of H *induced by* ψ' as follows: for every $e \in H$, let $e' := \Phi_{\mathcal{P}}(e)$ be the image of e with respect to $\overset{\mathcal{P}}{\rightsquigarrow}$. We have that $\phi_{\mathcal{P}}\restriction_e : e \to e'$ is a bijection. We now define the bijection $\psi_e : e_0 \to e$ as $\psi_e := \phi_{\mathcal{P}}\restriction_e^{-1} \circ \psi'_{e'}$, where $\psi'_{e'} : e_0 \to e'$. Thus, the collection ψ of all ψ_e, $e \in H$, is an e_0-orientation of H. It is easy to see that ψ satisfies the following.

Fact 2.8.18. *Let F be an r-graph and $e_0 \in F$. Let H, H' be multi-r-graphs and suppose that $H \rightsquigarrow H'$. Then for any e_0-orientation ψ' of H', we have $\nabla_{(F,e_0)}(H,\psi) \rightsquigarrow \nabla_{(F,e_0)}(H',\psi')$, where ψ is induced by ψ'.*

We now define the multi-r-graphs which will serve as the canonical multi-r-graphs M_h in (2.8.13). For $r \in \mathbb{N}$, let $\mathcal{M}_r$ contain all pairs $(k,m) \in \mathbb{N}_0^2$ such that $\frac{m}{r-i}\binom{k-i}{r-1-i}$ is an integer for all $i \in [r-1]_0$.

Definition 2.8.19 (Canonical multi-r-graph)**.** Let F^* be an r-graph and $e^* \in F^*$. Let $V' := V(F^*) \setminus e^*$. If $(k,m) \in \mathcal{M}_r$, define the multi-r-graph $M_{k,m}^{(F^*,e^*)}$ on vertex set $[k] \cup V'$ such that for every $e \in \binom{[k]\cup V'}{r}$, the multiplicity of e is

$$|M_{k,m}^{(F^*,e^*)}(e)| = \begin{cases} 0 & \text{if } e \subseteq [k]; \\ \frac{m}{r-|e\cap[k]|}\binom{k-|e\cap[k]|}{r-1-|e\cap[k]|} & \text{if } |e\cap[k]|>0, |e\cap V'|>0; \\ 0 & \text{if } e \subseteq V', e \notin F^*; \\ \frac{m}{r}\binom{k}{r-1} & \text{if } e \subseteq V', e \in F^*. \end{cases}$$

We will require the graph F^* in Definition 2.8.19 to have a certain symmetry property with respect to e^*, which we now define. We will prove the existence of a suitable (F-decomposable) symmetric r-extender in Lemma 2.8.26.

Definition 2.8.20 (symmetric r-extender)**.** We say that (F^*, e^*) is a *symmetric r-extender* if F^* is an r-graph, $e^* \in F^*$ and the following holds:

(SE) for all $e' \in \binom{V(F^*)}{r}$ with $e' \cap e^* \neq \emptyset$, we have $e' \in F^*$.

Note that if (F^*, e^*) is a symmetric r-extender, then the operators $\tilde{\nabla}_{(F^*,e^*)}, \nabla_{(F^*,e^*)}$ are labelling-invariant, i.e. $\tilde{\nabla}_{(F^*,e^*)}(H,\psi_1) \cong \tilde{\nabla}_{(F^*,e^*)}(H,\psi_2)$ and $\nabla_{(F^*,e^*)}(H,\psi_1) \cong \nabla_{(F^*,e^*)}(H,\psi_2)$

for all e^*-orientations ψ_1, ψ_2 of a multi-r-graph H. We therefore simply write $\tilde{\nabla}_{(F^*,e^*)}H$ and $\nabla_{(F^*,e^*)}H$ in this case.

To prove Lemma 2.8.15 we introduce so called strong colourings. Let H be an r-graph and C a set. A map $c\colon V(H) \to C$ is a *strong C-colouring of H* if for all distinct $x, y \in V(H)$ with $|H(\{x,y\})| > 0$, we have $c(x) \neq c(y)$, that is, no colour appears twice in one edge. For $\alpha \in C$, we let $c^{-1}(\alpha)$ denote the set of all vertices coloured α. For a set $C' \subseteq C$, we let $c^{\subseteq}(C') := \{e \in H : C' \subseteq c(e)\}$. We say that c is *m-regular* if $|c^{\subseteq}(C')| = m$ for all $C' \in \binom{C}{r-1}$. For example, an r-partite r-graph H trivially has a strong $|H|$-regular $[r]$-colouring.

Fact 2.8.21. *Let H be an r-graph and let c be a strong m-regular $[k]$-colouring of H. Then $|c^{\subseteq}(C')| = \frac{m}{r-i}\binom{k-i}{r-1-i}$ for all $i \in [r-1]_0$ and all $C' \in \binom{[k]}{i}$.*

Lemma 2.8.22. *Let (F^*, e^*) be a symmetric r-extender. Suppose that H is an r-graph and suppose that c is a strong m-regular $[k]$-colouring of H. Then $(k, m) \in \mathcal{M}_r$ and*

$$\nabla_{(F^*,e^*)}H \rightsquigarrow M_{k,m}^{(F^*,e^*)}.$$

Proof. By Fact 2.8.21, $(k, m) \in \mathcal{M}_r$, thus $M_{k,m}^{(F^*,e^*)}$ is defined. Recall that $M_{k,m}^{(F^*,e^*)}$ has vertex set $[k] \cup V'$, where $V' := V(F^*) \setminus e^*$. Let $V(H) \cup \bigcup_{e \in H} Z_e$ be the vertex set of $\nabla_{(F^*,e^*)}H$ as in Definition 2.8.13, with $Z_e = \{z_{e,v} : v \in V'\}$. We define a partition $\mathcal{P}$ of $V(H) \cup \bigcup_{e \in H} Z_e$ as follows: for all $i \in [k]$, let $V_i := c^{-1}(i)$. For all $v \in V'$, let $V_v := \{z_{e,v} : e \in H\}$. We now claim that $\nabla_{(F^*,e^*)}H \overset{\mathcal{P}}{\rightsquigarrow} M_{k,m}^{(F^*,e^*)}$.

Clearly, $\mathcal{P}$ satisfies (I1) because c is a strong colouring of H. For a set $e' \in \binom{[k] \cup V'}{r}$, define

$$S_{e'} := \{e'' \in \nabla_{(F^*,e^*)}H : e'' \subseteq V_{e'}\}.$$

Since $\nabla_{(F^*,e^*)}H$ is simple, in order to check (I2), it is enough to show that for all $e' \in \binom{[k] \cup V'}{r}$, we have $|S_{e'}| = |M_{k,m}^{(F^*,e^*)}(e')|$. We distinguish three cases.

Case 1: $e' \subseteq [k]$

In this case, $|M_{k,m}^{(F^*,e^*)}(e')| = 0$. Since $V_{e'} \subseteq V(H)$ and $(\nabla_{(F^*,e^*)}H)[V(H)]$ is empty, we have $S_{e'} = \emptyset$, as desired.

Case 2: $e' \subseteq V'$

In this case, $S_{e'}$ consists of all edges of $\nabla_{(F^*,e^*)}H$ which play the role of e' in F^*_e for some $e \in H$. Hence, if $e' \notin F^*$, then $|S_{e'}| = 0$, and if $e' \in F^*$, then $|S_{e'}| = |H|$. Fact 2.8.21 applied with $i = 0$ yields $|H| = \frac{m}{r}\binom{k}{r-1}$, as desired.

Case 3: $|e' \cap [k]| > 0$ and $|e' \cap V'| > 0$

We claim that $|S_{e'}| = |c^{\subseteq}(e' \cap [k])|$. In order to see this, we define a bijection $\pi \colon c^{\subseteq}(e' \cap [k]) \to S_{e'}$ as follows: for every $e \in H$ with $e' \cap [k] \subseteq c(e)$, define

$$\pi(e) := (e \cap c^{-1}(e' \cap [k])) \cup \{z_{e,v} \ : \ v \in e' \cap V'\}.$$

We first show that $\pi(e) \in S_{e'}$. Note that $e \cap c^{-1}(e' \cap [k])$ is a subset of e of size $|e' \cap [k]|$ and $\{z_{e,v} \ : \ v \in e' \cap V'\}$ is a subset of Z_e of size $|e' \cap V'|$. Hence, $\pi(e) \in \binom{V(F^*_e)}{r}$ and $|\pi(e) \cap e| = |e' \cap [k]| > 0$. Thus, by (SE), we have $\pi(e) \in F^*_e \subseteq \nabla_{(F^*,e^*)}H$. (This is in fact the crucial point where we need (SE).) Moreover,

$$\pi(e) \subseteq c^{-1}(e' \cap [k]) \cup \{z_{e,v} \ : \ v \in e' \cap V'\} \subseteq V_{e' \cap [k]} \cup V_{e' \cap V'} = V_{e'}.$$

Therefore, $\pi(e) \in S_{e'}$. It is straightforward to see that π is injective. Finally, for every $e'' \in S_{e'}$, we have $e'' = \pi(e)$, where $e \in H$ is the unique edge of H with $e'' \in F^*_e$. This establishes our claim that π is bijective and hence $|S_{e'}| = |c^{\subseteq}(e' \cap [k])|$. Since $1 \leq |e' \cap [k]| \leq r-1$, Fact 2.8.21 implies that

$$|S_{e'}| = |c^{\subseteq}(e' \cap [k])| = \frac{m}{r - |e' \cap [k]|}\binom{k - |e' \cap [k]|}{r - 1 - |e' \cap [k]|} = |M_{k,m}^{(F^*,e^*)}(e')|,$$

as required. □

Next, we establish the existence of suitable strong regular colourings. As a tool we need the following result about decompositions of very dense multi-r-graphs (which we will apply with $r-1$ playing the role of r).

Lemma 2.8.23. *Let $r \in \mathbb{N}$ and assume that $(*)_r$ is true. Let $1/n \ll 1/h, 1/f$ with $f > r$, let F be a weakly regular r-graph on f vertices and assume that $K_n^{(r)}$ is F-divisible. Let $m \in \mathbb{N}$. Suppose that H is an F-divisible multi-r-graph on $[h]$ with multiplicity at most $m-1$ and let K be the complete multi-r-graph on $[n]$ with multiplicity m. Then $K - H$ has an F-decomposition.*

Proof. Choose $\varepsilon > 0$ such that $1/n \ll \varepsilon \ll 1/h, 1/f$. Fix an edge $e_0 \in F$. Let ψ be any e_0-orientation of H. We may assume that $\tilde{H} := \tilde{\nabla}_{(F,e_0)}(H, \psi)$ is a multi-r-graph on $[n]$. Let $\tilde{\psi}$ be any e_0-orientation of $H^* := \tilde{H} - H$. We may also assume that $\hat{H} := \tilde{\nabla}_{(F,e_0)}(H^*, \tilde{\psi})$ is an r-graph on $[n]$. Let $H^\dagger := \hat{H} - H^*$. Using Fact 2.8.14, observe that the following are true:

(a) $\tilde{H}$ can be decomposed into $m-1$ (possibly empty) F-decomposable (simple) r-graphs $H'_1, \dots, H'_{m-1}$;

(b) $\hat{H}$ is an F-decomposable (simple) r-graph;

(c) $H^\dagger$ is an F-divisible (simple) r-graph;

(d) $H \cup \hat{H} = \tilde{H} \cup H^\dagger$.

By (d), we have that

$$K - H = (K - H - \hat{H}) \cup \hat{H} = \hat{H} \cup (K - \tilde{H} - H^\dagger).$$

Let K' be the complete (simple) r-graph on $[n]$. For each $i \in [m-1]$, define $H_i := K' - H'_i$, and let $H_m := K' - H^\dagger$. We thus have $K - \tilde{H} - H^\dagger = \bigcup_{i \in [m]} H_i$ by (a).

Recall that $K'^{\leftrightarrow}$ is a $(0, 0.99/f!, f, r)$-supercomplex (cf. Example 2.4.9). We conclude with Proposition 2.5.9(v) that $H_i^{\leftrightarrow} = K'^{\leftrightarrow} - H'_i$ is an $(\varepsilon, 0.5/f!, f, r)$-supercomplex for every $i \in [m]$. Recall that K' is F-divisible by assumption. Thus, by (a) and (c), each H_i

is F-divisible. Hence, by $(*)_r$, H_i is F-decomposable for every $i \in [m]$. Thus,

$$K - H = \hat{H} \cup (K - \tilde{H} - H^\dagger) = \hat{H} \cup \bigcup_{i \in [m]} H_i$$

has an F-decomposition by (b). $\square$

The next lemma guarantees the existence of a suitable strong regular colouring. For this, we apply Lemma 2.8.23 to the shadow of F. For an r-graph F, define the *shadow* F^{sh} *of* F to be the $(r-1)$-graph on $V(F)$ where an $(r-1)$-set S is an edge if and only if $|F(S)| > 0$. We need the following fact.

Fact 2.8.24. *If F is a weakly $(s_0, \dots, s_{r-1})$-regular r-graph, then F^{sh} is a weakly $(s'_0, \dots, s'_{r-2})$-regular $(r-1)$-graph, where $s'_i := \frac{r-i}{s_{r-1}} s_i$ for all $i \in [r-2]_0$.*

Proof. Let $i \in [r-2]_0$. For every $T \in \binom{V(F)}{i}$, we have $|F^{sh}(T)| = \frac{r-i}{s_{r-1}} |F(T)|$ since every edge of F which contains T contains $r - i$ edges of F^{sh} which contain T, but each such edge of F^{sh} is contained in s_{r-1} such edges of F. This implies the claim. $\square$

Lemma 2.8.25. *Let $r \geq 2$ and assume that $(*)_{r-1}$ holds. Let F be a weakly regular r-graph. Then for all $h \in \mathbb{N}$, there exist $k, m \in \mathbb{N}$ such that for any F-divisible r-graph H on at most h vertices, there exists $t \in \mathbb{N}$ such that $H + t \cdot F$ has a strong m-regular $[k]$-colouring.*

Proof. Let $f := |V(F)|$ and suppose that F is weakly $(s_0, \dots, s_{r-1})$-regular. Thus, for every $S \in \binom{V(F)}{r-1}$, we have

$$|F(S)| = \begin{cases} s_{r-1} & \text{if } S \in F^{sh}; \\ 0 & \text{otherwise.} \end{cases} \tag{2.8.14}$$

By Proposition 2.5.2, we can choose $k \in \mathbb{N}$ such that $1/k \ll 1/h, 1/f$ and such that $K_k^{(r-1)}$ is F^{sh}-divisible. Let G be the complete multi-$(r-1)$-graph on $[k]$ with multiplicity $m' := h + 1$ and let $m := s_{r-1} m'$.

Let H be any F-divisible r-graph on at most h vertices. By adding isolated vertices to H if necessary, we may assume that $V(H) = [h]$. We first define a multi-$(r-1)$-graph H' on $[h]$ as follows: For each $S \in \binom{[h]}{r-1}$, let the multiplicity of S in H' be $|H'(S)| := |H(S)|$. Clearly, H' has multiplicity at most h. Observe that for each $S \subseteq [h]$ with $|S| \leq r-1$, we have

$$|H'(S)| = (r - |S|)|H(S)|. \tag{2.8.15}$$

Note that since H is F-divisible, we have that $s_{r-1} \mid |H(S)|$ for all $S \in \binom{[h]}{r-1}$. Thus, the multiplicity of each $S \in \binom{[h]}{r-1}$ in H' is divisible by s_{r-1}. Let H'' be the multi-$(r-1)$-graph on $[h]$ obtained from H' by dividing the multiplicity of each $S \in \binom{[h]}{r-1}$ by s_{r-1}. Hence, by (2.8.15), for all $S \subseteq [h]$ with $|S| \leq r-1$, we have

$$|H''(S)| = \frac{|H'(S)|}{s_{r-1}} = \frac{r-|S|}{s_{r-1}}|H(S)|. \tag{2.8.16}$$

For each $S \in \binom{[k]}{r-1}$ with $S \not\subseteq [h]$, we set $|H''(S)| := |H(S)| := 0$. Then (2.8.16) still holds.

We claim that H'' is F^{sh}-divisible. Recall that by Fact 2.8.24,

$$F^{sh} \text{ is weakly } (\frac{r}{s_{r-1}}s_0, \ldots, \frac{r-i}{s_{r-1}}s_i, \ldots, \frac{2}{s_{r-1}}s_{r-2})\text{-regular.}$$

Let $i \in [r-2]_0$ and let $S \in \binom{[h]}{i}$. We need to show that $|H''(S)| \equiv 0 \mod Deg(F^{sh})_i$, where $Deg(F^{sh})_i = \frac{r-i}{s_{r-1}}s_i$. Since H is F-divisible, we have $|H(S)| \equiv 0 \mod s_i$. Together with (2.8.16), we deduce that $|H''(S)| \equiv 0 \mod \frac{r-i}{s_{r-1}}s_i$. Hence, H'' is F^{sh}-divisible. Therefore, by Lemma 2.8.23 (with $k, m', r-1, F^{sh}$ playing the roles of n, m, r, F) and our choice of k, $G - H''$ has an F^{sh}-decomposition $\mathcal{F}$ into t edge-disjoint copies $F'_1, \ldots, F'_t$ of F^{sh}.

We will show that t is as required in Lemma 2.8.25. To do this, let $F_1, \ldots, F_t$ be vertex-disjoint copies of F which are also vertex-disjoint from H. We will now define a strong m-regular $[k]$-colouring c of

$$H^+ := H \cup \bigcup_{j\in[t]} F_j.$$

Let c_0 be the identity map on $V(H) = [h]$, and for each $j \in [t]$, let

$$c_j \colon V(F_j) \to V(F'_j) \text{ be an isomorphism from } F_j^{sh} \text{ to } F'_j \tag{2.8.17}$$

(recall that $V(F_j^{sh}) = V(F_j)$). Since $H, F_1, \dots, F_t$ are vertex-disjoint and $V(H) \cup \bigcup_{j\in[t]} V(F'_j) \subseteq [k]$, we can combine $c_0, c_1, \dots, c_t$ to a map

$$c\colon V(H^+) \to [k],$$

i.e. for $x \in V(H^+)$, we let $c(x) := c_j(x)$, where either j is the unique index for which $x \in V(F_j)$ or $j = 0$ if $x \in V(H)$. For every edge $e \in H^+$, we have $e \subseteq V(H)$ or $e \subseteq V(F_j)$ for some $j \in [t]$, thus $c\restriction_e$ is injective. Therefore, c is a strong $[k]$-colouring of H^+.

It remains to check that c is m-regular. Let $C \in \binom{[k]}{r-1}$. Clearly, $|c^{\subseteq}(C)| = \sum_{j=0}^{t} |c_j^{\subseteq}(C)|$. Since every c_j is a bijection, we have

$$|c_0^{\subseteq}(C)| = |\{e \in H \,:\, c_0^{-1}(C) \subseteq e\}| = |H(c_0^{-1}(C))| = |H(C)| \quad \text{and}$$

$$|c_j^{\subseteq}(C)| = |F_j(c_j^{-1}(C))| \overset{(2.8.14)}{=} \begin{cases} s_{r-1} & \text{if } c_j^{-1}(C) \in F_j^{sh} \overset{(2.8.17)}{\Leftrightarrow} C \in F'_j; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have $|c^{\subseteq}(C)| = |H(C)| + s_{r-1}|J(C)|$, where

$$J(C) := \{j \in [t] \,:\, C \in F'_j\}.$$

Now crucially, since $\mathcal{F}$ is an F^{sh}-decomposition of $G - H''$, we have that $|J(C)|$ is equal to the multiplicity of C in $G - H''$, i.e. $|J(C)| = m' - |H''(C)|$. Thus,

$$|c^{\subseteq}(C)| = |H(C)| + s_{r-1}|J(C)| \stackrel{(2.8.16)}{=} s_{r-1}(|H''(C)| + |J(C)|) = s_{r-1}m' = m,$$

completing the proof. □

Before we can prove Lemma 2.8.15, we need to show the existence of a symmetric r-extender F^* which is F-decomposable. For some F we could actually take $F^* = F$ (e.g. if F is a clique). For general (weakly regular) r-graphs F, we will use the Cover down lemma (Lemma 2.7.7) to find F^*. At first sight, appealing to the Cover down lemma may seem rather heavy handed, but a direct construction seems to be quite difficult.

Lemma 2.8.26. *Let F be a weakly regular r-graph, $e_0 \in F$ and assume that $(*)_i$ is true for all $i \in [r-1]$. There exists a symmetric r-extender (F^*, e^*) such that F^* has an F-decomposition $\mathcal{F}$ with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'.*

Proof. Let $f := |V(F)|$. By Proposition 2.5.2, we can choose $n \in \mathbb{N}$ and $\gamma, \varepsilon, \nu, \mu > 0$ such that $1/n \ll \gamma \ll \varepsilon \ll \nu \ll \mu \ll 1/f$ and such that $K_n^{(r)}$ is F-divisible. By Example 2.4.9, K_n is a $(0, 0.99/f!, f, r)$-supercomplex. By Fact 2.7.2(i) and Proposition 2.7.12, there exists $U \subseteq V(K_n)$ of size $\lfloor \mu n \rfloor$ which is $(\varepsilon, \mu, 0.9/f!, f, r)$-random in K_n. Let $\bar{U} := V(K_n) \setminus U$. Using (R2) of Definition 2.7.1, it is easy to see that K_n is (ε, f, r)-dense with respect to $K_n^{(r)} - K_n^{(r)}[\bar{U}]$ (see Definition 2.7.6). Thus, by the Cover down lemma (Lemma 2.7.7), there exists a subgraph H^* of $K_n^{(r)} - K_n^{(r)}[\bar{U}]$ with $\Delta(H^*) \leq \nu n$ and the following property: for all $L \subseteq K_n^{(r)}$ such that $\Delta(L) \leq \gamma n$ and $H^* \cup L$ is F-divisible, $H^* \cup L$ has an F-packing which covers all edges except possibly some inside U.

Let F' be a copy of F with $V(F') \subseteq \bar{U}$. Let $G_{nibble} := K_n - H^* - F'$. By Proposition 2.5.9(v), G_{nibble} is a $(2^{2r+2}\nu, 0.8/f!, f, r)$-supercomplex. Thus, by Lemma 2.6.5, there exists an F-packing $\mathcal{F}_{nibble}$ in $G_{nibble}^{(r)}$ such that $\Delta(L) \leq \gamma n$, where $L := G_{nibble}^{(r)} - \mathcal{F}_{nibble}^{(r)}$. Clearly, $H^* \cup L = K_n^{(r)} - \mathcal{F}_{nibble}^{(r)} - F'$ is F-divisible. Thus, there exists an F-packing $\mathcal{F}^*$ in $H^* \cup L$ which covers all edges of $H^* \cup L$ except possibly some inside U. Let $\mathcal{F} := \{F'\} \cup \mathcal{F}_{nibble} \cup \mathcal{F}^*$. Let $F^* := \mathcal{F}^{(r)}$ and let e^* be the edge in F' which plays the role of e_0.

Clearly, $\mathcal{F}$ is an F-decomposition of F^* with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'. It remains to check (SE). Let $e' \in \binom{V(K_n^{(r)})}{r}$ with $e' \cap e^* \neq \emptyset$. Since $e^* \subseteq \bar{U}$, e' cannot be inside U. Thus, e' is covered by $\mathcal{F}$ and we have $e' \in F^*$. □

Note that $|V(F^*)|$ is quite large here, in particular $1/|V(F^*)| \ll 1/f$ for $f = |V(F)|$. This means that G being an (ε, ξ, f, r)-supercomplex does not necessarily allow us to extend a given subgraph H of $G^{(r)}$ to a copy of $\nabla_{(F^*,e^*)}H$ as described in Definition 2.8.16. Fortunately, this will in fact not be necessary, as F^* will only serve as an abstract auxiliary graph and will not appear as a subgraph of the absorber. (This is crucial since otherwise we would not be able to prove our main theorems with explicit bounds, let alone the bounds given in Theorem 2.1.4.)

We are now ready to prove Lemma 2.8.15.

Proof of Lemma 2.8.15. Given F and e_0, we first apply Lemma 2.8.26 to obtain a symmetric r-extender (F^*, e^*) such that F^* has an F-decomposition $\mathcal{F}$ with $e^* \in F' \in \mathcal{F}$ and e^* plays the role of e_0 in F'. For given $h \in \mathbb{N}$, let $k, m \in \mathbb{N}$ be as in Lemma 2.8.25. Clearly, we may assume that there exists an F-divisible r-graph on at most h vertices. Together with Lemma 2.8.22, this implies that $(k, m) \in \mathcal{M}_r$. Define

$$M_h := M_{k,m}^{(F^*,e^*)}.$$

Now, let H be any F-divisible r-graph on at most h vertices. By Lemma 2.8.25, there exists $t \in \mathbb{N}$ such that $H + t \cdot F$ has a strong m-regular $[k]$-colouring. By Lemma 2.8.22, we have

$$\nabla_{(F^*,e^*)}(H + t \cdot F) \rightsquigarrow M_h.$$

Let ψ_1 be any e_0-orientation of $H + t \cdot F$. Observe that since e^* plays the role of e_0 in F', $\nabla_{(F^*,e^*)}(H + t \cdot F)$ can be decomposed into a copy of $\nabla_{(F,e_0)}(H + t \cdot F, \psi_1)$ and s copies of

F (where $s = |H + t \cdot F| \cdot |\mathcal{F} \setminus \{F'\}|$). Hence, we have

$$\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F \rightsquigarrow \nabla_{(F^*,e^*)}(H + t \cdot F)$$

by Proposition 2.8.4(ii). Thus, $\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F \leftrightsquigarrow M_h$ by transitivity of $\leftrightsquigarrow$. Finally, let ψ_2 be any e_0-orientation of M_h. By Fact 2.8.18, there exists an e_0-orientation ψ_3 of $\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F$ such that

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t \cdot F, \psi_1) + s \cdot F, \psi_3) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi_2).$$

□

2.8.3 Proof of the Absorbing lemma

As discussed at the beginning of Section 2.8.2, we can now combine Lemma 2.8.5 and Lemma 2.8.15 to construct the desired absorber by concatenating transformers between certain auxiliary r-graphs, in particular the extension ∇M_h of the canonical multi-r-graph M_h. It is relatively straightforward to find these auxiliary r-graphs within a given supercomplex G. The step when we need to find ∇M_h is the reason why the definition of a supercomplex includes the notion of extendability.

Proof of Lemma 2.8.2. If H is empty, then we can take A to be empty, so let us assume that H is not empty. In particular, $G^{(r)}$ is not empty. Recall also that we assume $r \geq 2$. Let $e_0 \in F$ and let M_h be as in Lemma 2.8.15. Fix any e_0-orientation ψ of M_h. By Lemma 2.8.15, there exist $t_1, t_2, s_1, s_2, \psi_1, \psi_2, \psi'_1, \psi'_2$ such that

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H + t_1 \cdot F, \psi_1) + s_1 \cdot F, \psi'_1) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi); \quad (2.8.18)$$

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(t_2 \cdot F, \psi_2) + s_2 \cdot F, \psi'_2) \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi). \quad (2.8.19)$$

We can assume that $1/n \ll 1/\ell$ where $\ell := \max\{|V(M_h)|, t_1, t_2, s_1, s_2\}$.

Since G is $(\xi, f+r, r)$-dense, there exist disjoint $Q_{1,1}, \dots, Q_{1,t_1}, Q_{2,1}, \dots, Q_{2,t_2} \in G^{(f)}$ which are also disjoint from $V(H)$. For $i \in [2]$ and $j \in [t_i]$, let $F_{i,j}$ be a copy of F with $V(F_{i,j}) = Q_{i,j}$. Let $H_1 := H \cup \bigcup_{j \in [t_1]} F_{1,j}$ and $H_2 := \bigcup_{j \in [t_2]} F_{2,j}$ and for $i \in [2]$, define

$$\mathcal{F}_i := \{F_{i,j} \ : \ j \in [t_i]\}.$$

So H_1 is a copy of $H + t_1 \cdot F$ and H_2 is a copy of $t_2 \cdot F$. In fact, we will from now on assume (by redefining ψ_i and ψ_i') that for $i \in [2]$, we have

$$\nabla_{(F,e_0)}(\nabla_{(F,e_0)}(H_i, \psi_i) + s_i \cdot F, \psi_i') \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi). \tag{2.8.20}$$

For $i \in [2]$, let $(H_i', \mathcal{F}_i')$ be obtained by extending H_i with a copy of $\nabla_{(F,e_0)}(H_i, \psi_i)$ in G (cf. Definition 2.8.16). We can assume that H_1' and H_2' are vertex-disjoint by first choosing H_1' whilst avoiding $V(H_2)$ and subsequently choosing H_2' whilst avoiding $V(H_1')$. (To see that this is possible we can e.g. use the fact that G is (ε, d, f, r)-regular for some $d \geq \xi$.)

There exist disjoint $Q_{1,1}', \dots, Q_{1,s_1}', Q_{2,1}', \dots, Q_{2,s_2}' \in G^{(f)}$ which are also disjoint from $V(H_1') \cup V(H_2')$. For $i \in [2]$ and $j \in [s_i]$, let $F_{i,j}'$ be a copy of F with $V(F_{i,j}') = Q_{i,j}'$. For $i \in [2]$, let

$$H_i'' := H_i' \cup \bigcup_{j \in [s_i]} F_{i,j}';$$
$$\mathcal{F}_i'' := \{F_{i,j}' \ : \ j \in [s_i]\}.$$

Since H_i'' is a copy of $\nabla_{(F,e_0)}(H_i, \psi_i) + s_i \cdot F$, we can assume (by redefining ψ_i') that

$$\nabla_{(F,e_0)}(H_i'', \psi_i') \rightsquigarrow \nabla_{(F,e_0)}(M_h, \psi). \tag{2.8.21}$$

For $i \in [2]$, let $(H_i''', \mathcal{F}_i''')$ be obtained by extending H_i'' with a copy of $\nabla_{(F,e_0)}(H_i'', \psi_i')$ in G (cf. Definition 2.8.16). We can assume that H_1''' and H_2''' are vertex-disjoint.

Since G is (ξ, f, r)-extendable, it is straightforward to find a copy M' of $\nabla_{(F,e_0)}(M_h, \psi)$

in $G^{(r)}$ which is vertex-disjoint from H_1''' and H_2'''.

Since H_i''' is a copy of $\nabla_{(F,e_0)}(H_i'', \psi_i')$, by (2.8.21) we have $H_i''' \rightsquigarrow M'$ for $i \in [2]$. Using Fact 2.8.14(ii) repeatedly, we can see that both H_1''' and H_2''' are F-divisible. Together with Proposition 2.8.4(iii), this implies that M' is F-divisible as well.

Let $T_1 := (H_1 - H) \cup H_1''$ and $T_2 := H_2 \cup H_2''$. For $i \in [2]$, let

$$\mathcal{F}_{i,1} := \mathcal{F}_i' \cup \mathcal{F}_i'' \text{ and } \mathcal{F}_{i,2} := \mathcal{F}_i \cup \mathcal{F}_i'''.$$

We claim that $\mathcal{F}_{1,1}, \mathcal{F}_{1,2}, \mathcal{F}_{2,1}, \mathcal{F}_{2,2}$ are 2-well separated F-packings in G such that

$$\mathcal{F}_{1,1}^{(r)} = T_1 \cup H, \quad \mathcal{F}_{1,2}^{(r)} = T_1 \cup H_1''', \quad \mathcal{F}_{2,2}^{(r)} = T_2 \cup H_2''' \quad \text{and} \quad \mathcal{F}_{2,1}^{(r)} = T_2. \tag{2.8.22}$$

(In particular, T_1 is a 2-well separated $(H, H_1'''; F)$-transformer in G and T_2 is a 2-well separated $(H_2''', \emptyset; F)$-transformer in G.) Indeed, we clearly have that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_1'', \mathcal{F}_2''$ are 1-well separated F-packings in G, where $\mathcal{F}_1^{(r)} = H_1 - H$, $\mathcal{F}_2^{(r)} = H_2$, and for $i \in [2]$, $\mathcal{F}_i''^{(r)} = H_i'' - H_i'$. Moreover, by Fact 2.8.17, for $i \in [2]$, $\mathcal{F}_i'$ and $\mathcal{F}_i'''$ are 1-well separated F-packings in G with $\mathcal{F}_i'^{(r)} = H_i \cup H_i'$ and $\mathcal{F}_i'''^{(r)} = H_i'' \cup H_i'''$. Note that

$$\begin{aligned}
T_1 \cup H = H_1 \cup H_1'' = (H_1 \cup H_1') \mathbin{\dot\cup} (H_1'' - H_1') = \mathcal{F}_1'^{(r)} \mathbin{\dot\cup} \mathcal{F}_1''^{(r)} = \mathcal{F}_{1,1}^{(r)};\\
T_1 \cup H_1''' = (H_1 - H) \mathbin{\dot\cup} (H_1'' \cup H_1''') = \mathcal{F}_1^{(r)} \mathbin{\dot\cup} \mathcal{F}_1'''^{(r)} = \mathcal{F}_{1,2}^{(r)};\\
T_2 \cup H_2''' = H_2 \mathbin{\dot\cup} (H_2'' \cup H_2''') = \mathcal{F}_2^{(r)} \mathbin{\dot\cup} \mathcal{F}_2'''^{(r)} = \mathcal{F}_{2,2}^{(r)};\\
T_2 = H_2 \cup H_2'' = (H_2 \cup H_2') \mathbin{\dot\cup} (H_2'' - H_2') = \mathcal{F}_2'^{(r)} \mathbin{\dot\cup} \mathcal{F}_2''^{(r)} = \mathcal{F}_{2,1}^{(r)}.
\end{aligned}$$

To check that $\mathcal{F}_{1,1}$, $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,1}$ and $\mathcal{F}_{2,2}$ are 2-well separated F-packings, by Fact 2.5.4(ii) it is now enough to show for $i \in [2]$ that $\mathcal{F}_i'$ and $\mathcal{F}_i''$ are $(r+1)$-disjoint and that $\mathcal{F}_i$ and $\mathcal{F}_i'''$ are $(r+1)$-disjoint. Note that for all $F' \in \mathcal{F}_i'$ and $F'' \in \mathcal{F}_i''$, we have $V(F') \subseteq V(H_i')$ and $V(F'') \cap V(H_i') = \emptyset$, thus $V(F') \cap V(F'') = \emptyset$. For all $F' \in \mathcal{F}_i$ and $F'' \in \mathcal{F}_i'''$, we have $V(F') \subseteq V(H_i)$ and $|V(F'') \cap V(H_i)| \leq |V(F'') \cap V(H_i'')| \leq r$ by Fact 2.8.17, thus $|V(F') \cap V(F'')| \leq r$. This completes the proof of (2.8.22).

Let

$$O_r := H_1 \cup H_1'' \cup H_2 \cup H_2'';$$

$$O_{r+1,3} := \mathcal{F}_{1,1}^{\leq(r+1)} \cup \mathcal{F}_{1,2}^{\leq(r+1)} \cup \mathcal{F}_{2,1}^{\leq(r+1)} \cup \mathcal{F}_{2,2}^{\leq(r+1)}.$$

By Fact 2.5.4(i), $\Delta(O_{r+1,3}) \leq 8(f-r)$. Note that $H_1''', M' \subseteq G^{(r)} - (O_r \cup H_2''')$. Thus, by Proposition 2.5.9(v) and Lemma 2.8.5, there exists a $(\kappa/3)$-well separated $(H_1''', M'; F)$-transformer T_3 in $G - (O_r \cup H_2''') - O_{r+1,3}$ with $\Delta(T_3) \leq \gamma n/3$. Let $\mathcal{F}_{3,1}$ and $\mathcal{F}_{3,2}$ be $(\kappa/3)$-well separated F-packings in $G - (O_r \cup H_2''') - O_{r+1,3}$ such that $\mathcal{F}_{3,1}^{(r)} = T_3 \cup H_1'''$ and $\mathcal{F}_{3,2}^{(r)} = T_3 \cup M'$.

Similarly, let $O_{r+1,4} := O_{r+1,3} \cup \mathcal{F}_{3,1}^{\leq(r+1)} \cup \mathcal{F}_{3,2}^{\leq(r+1)}$. By Fact 2.5.4(i), $\Delta(O_{r+1,4}) \leq (8+2\kappa/3)(f-r)$. Note that $H_2''', M' \subseteq G^{(r)} - (O_r \cup H_1''' \cup T_3)$. Using Proposition 2.5.9(v) and Lemma 2.8.5 again, we can find a $(\kappa/3)$-well separated $(H_2''', M'; F)$-transformer T_4 in $G - (O_r \cup H_1''' \cup T_3) - O_{r+1,4}$ with $\Delta(T_4) \leq \gamma n/3$. Let $\mathcal{F}_{4,1}$ and $\mathcal{F}_{4,2}$ be $(\kappa/3)$-well separated F-packings in $G - (O_r \cup H_1''' \cup T_3) - O_{r+1,4}$ such that of $\mathcal{F}_{4,1}^{(r)} = T_4 \cup H_2'''$ and $\mathcal{F}_{4,2}^{(r)} = T_4 \cup M'$.

Let

$$A := T_1 \mathbin{\dot\cup} H_1''' \mathbin{\dot\cup} T_3 \mathbin{\dot\cup} M' \mathbin{\dot\cup} T_4 \mathbin{\dot\cup} H_2''' \mathbin{\dot\cup} T_2;$$

$$\mathcal{F}_\circ := \mathcal{F}_{1,2} \cup \mathcal{F}_{3,2} \cup \mathcal{F}_{4,1} \cup \mathcal{F}_{2,1};$$

$$\mathcal{F}_\bullet := \mathcal{F}_{1,1} \cup \mathcal{F}_{3,1} \cup \mathcal{F}_{4,2} \cup \mathcal{F}_{2,2}.$$

Clearly, $A \subseteq G^{(r)}$, and $\Delta(A) \leq \gamma n$. Moreover, A and H are edge-disjoint. Using (2.8.22), we can check that

$$\mathcal{F}_\circ^{(r)} = \mathcal{F}_{1,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{3,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{4,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{2,1}^{(r)} = (T_1 \cup H_1''') \mathbin{\dot\cup} (T_3 \cup M') \mathbin{\dot\cup} (T_4 \cup H_2''') \mathbin{\dot\cup} T_2 = A;$$

$$\mathcal{F}_\bullet^{(r)} = \mathcal{F}_{1,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{3,1}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{4,2}^{(r)} \mathbin{\dot\cup} \mathcal{F}_{2,2}^{(r)} = (H \cup T_1) \mathbin{\dot\cup} (H_1''' \cup T_3) \mathbin{\dot\cup} (M' \cup T_4) \mathbin{\dot\cup} (H_2''' \cup T_2) = A \cup H.$$

By definition of $O_{r+1,3}$ and $O_{r+1,4}$, we have that $\mathcal{F}_{1,2}, \mathcal{F}_{3,2}, \mathcal{F}_{4,1}, \mathcal{F}_{2,1}$ are $(r+1)$-disjoint.

Thus, $\mathcal{F}_\circ$ is a $(2 \cdot \kappa/3 + 4)$-well separated F-packing in G by Fact 2.5.4(ii). Similarly, $\mathcal{F}_\bullet$ is a $(2 \cdot \kappa/3 + 4)$-well separated F-packing in G. So A is indeed a κ-well separated F-absorber for H in G. □

2.9 Proof of the main theorems

2.9.1 Main complex decomposition theorem

We can now deduce our main decomposition result for supercomplexes (modulo the proof of the Cover down lemma). The main ingredients for the proof of Theorem 2.4.7 are Lemma 2.7.4 (to find a vortex), Lemma 2.8.2 (to find absorbers for the possible leftovers in the final vortex set), and Lemma 2.7.5 (to cover all edges outside the final vortex set).

Proof of Theorem 2.4.7. We proceed by induction on r. The case $r = 1$ forms the base case of the induction and in this case we do not rely on any inductive assumption. Suppose that $r \in \mathbb{N}$ and that $(*)_i$ is true for all $i \in [r-1]$.

We may assume that $1/n \ll 1/\kappa \ll \varepsilon$. Choose new constants $\kappa', m' \in \mathbb{N}$ and $\gamma, \mu > 0$ such that

$$1/n \ll 1/\kappa \ll \gamma \ll 1/m' \ll 1/\kappa' \ll \varepsilon \ll \mu \ll \xi, 1/f$$

and suppose that F is a weakly regular r-graph on $f > r$ vertices.

Let G be an F-divisible (ε, ξ, f, r)-supercomplex on n vertices. We are to show the existence of a κ-well separated F-decomposition of G. By Lemma 2.7.4, there exists a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, f, r, m)$-vortex $U_0, U_1, \dots, U_\ell$ in G for some $\mu m' \le m \le m'$. Let $H_1, \dots, H_s$ be an enumeration of all spanning F-divisible subgraphs of $G[U_\ell]^{(r)}$. Clearly, $s \le 2^{\binom{m}{r}}$. We will now find edge-disjoint subgraphs $A_1, \dots, A_s$ of $G^{(r)}$ and $\sqrt{\kappa}$-well separated F-packings $\mathcal{F}_{1,\circ}, \mathcal{F}_{1,\bullet}, \dots, \mathcal{F}_{s,\circ}, \mathcal{F}_{s,\bullet}$ in G such that for all $i \in [s]$ we have that

(A1) $\mathcal{F}_{i,\circ}^{(r)} = A_i$ and $\mathcal{F}_{i,\bullet}^{(r)} = A_i \cup H_i$;

(A2) $\Delta(A_i) \le \gamma n$;

(A3) $A_i[U_1]$ is empty;

(A4) $\mathcal{F}^{\leq}_{i,\bullet}, G[U_1], \mathcal{F}^{\leq}_{1,\circ}, \dots, \mathcal{F}^{\leq}_{i-1,\circ}, \mathcal{F}^{\leq}_{i+1,\circ}, \dots, \mathcal{F}^{\leq}_{s,\circ}$ are $(r+1)$-disjoint.

Suppose that for some $t \in [s]$, we have already found edge-disjoint $A_1, \dots, A_{t-1}$ together with $\mathcal{F}_{1,\circ}, \mathcal{F}_{1,\bullet}, \dots, \mathcal{F}_{t-1,\circ}, \mathcal{F}_{t-1,\bullet}$ that satisfy (A1)–(A4) (with $t-1$ playing the role of s). Let

$$T_t := (G^{(r)}[U_1] - H_t) \cup \bigcup_{i \in [t-1]} A_i;$$
$$T'_t := G^{(r+1)}[U_1] \cup \bigcup_{i \in [t-1]} (\mathcal{F}^{\leq (r+1)}_{i,\circ} \cup \mathcal{F}^{\leq (r+1)}_{i,\bullet}).$$

Clearly, $\Delta(T_t) \leq \mu n + s\gamma n \leq 2\mu n$ by (V2) and (A2). Also, $\Delta(T'_t) \leq \mu n + 2s\sqrt{\kappa}(f-r) \leq 2\mu n$ by (V2) and Fact 2.5.4(i). Thus, applying Proposition 2.5.9(v) twice we see that $G_{abs,t} := G - T_t - T'_t$ is still a $(\sqrt{\mu}, \xi/2, f, r)$-supercomplex. Moreover, $H_t \subseteq G^{(r)}_{abs,t}$ by (A3). Hence, by Lemma 2.8.2, there exists a $\sqrt{\kappa}$-well separated F-absorber A_t for H_t in $G_{abs,t}$ with $\Delta(A_t) \leq \gamma n$. Let $\mathcal{F}_{t,\circ}$ and $\mathcal{F}_{t,\bullet}$ be $\sqrt{\kappa}$-well separated F-packings in $G_{abs,t} \subseteq G$ such that $\mathcal{F}^{(r)}_{t,\circ} = A_t$ and $\mathcal{F}^{(r)}_{t,\bullet} = A_t \cup H_t$. Clearly, A_t is edge-disjoint from $A_1, \dots, A_{t-1}$. Moreover, (A3) holds since $G^{(r)}_{abs,t}[U_1] = H_t$ and A_t is edge-disjoint from H_t, and (A4) holds with t playing the role of s due to the definition of T'_t.

Let $A^* := A_1 \cup \dots \cup A_s$ and $T^* := \bigcup_{i \in [s]} (\mathcal{F}^{\leq (r+1)}_{i,\circ} \cup \mathcal{F}^{\leq (r+1)}_{i,\bullet})$. We claim that the following hold:

(A1′) for every F-divisible subgraph H^* of $G[U_\ell]^{(r)}$, $A^* \cup H^*$ has an $s\sqrt{\kappa}$-well separated F-decomposition $\mathcal{F}^*$ with $\mathcal{F}^{*\leq} \subseteq G[T^*]$;

(A2′) $\Delta(A^*) \leq \varepsilon n$ and $\Delta(T^*) \leq 2s\sqrt{\kappa}(f-r) \leq \varepsilon n$;

(A3′) $A^*[U_1]$ and $T^*[U_1]$ are empty.

For (A1′), we have that $H^* = H_t$ for some $t \in [s]$. Then $\mathcal{F}^* := \mathcal{F}_{t,\bullet} \cup \bigcup_{i \in [s] \setminus \{t\}} \mathcal{F}_{i,\circ}$ is an F-decomposition of $A^* \cup H^* = (A_t \cup H_t) \cup \bigcup_{i \in [s] \setminus \{t\}} A_i$ by (A1) and since $H_t, A_1, \dots, A_s$ are pairwise edge-disjoint. By (A4) and Fact 2.5.4(ii), $\mathcal{F}^*$ is $s\sqrt{\kappa}$-well separated. We clearly have $\mathcal{F}^{*\leq} \subseteq G$ and $\mathcal{F}^{*\leq(r+1)} \subseteq T^*$. Thus $\mathcal{F}^{*\leq} \subseteq G[T^*]$ and so (A1′) holds. It is straightforward to check that (A2′) follows from (A2) and Fact 2.5.4(i), and that (A3′) follows from (A3) and (A4).

Let $G_{almost} := G - A^* - T^*$. By (A2′) and Proposition 2.5.9(v), G_{almost} is an $(\sqrt{\varepsilon}, \xi/2, f, r)$-supercomplex. Moreover, since A^* must be F-divisible, we have that G_{almost} is F-divisible. By (A3′), $U_1, \dots, U_\ell$ is a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, f, r, m)$-vortex in $G_{almost}[U_1]$. Moreover, (A2′) and Proposition 2.7.13 imply that U_1 is $(\varepsilon^{1/5}, \mu, \xi/2, f, r)$-random in G_{almost} and $U_1 \setminus U_2$ is $(\varepsilon^{1/5}, \mu(1-\mu), \xi/2, f, r)$-random in G_{almost}. Hence, $U_0, U_1, \dots, U_\ell$ is still an $(\varepsilon^{1/5}, \mu, \xi/2, f, r, m)$-vortex in G_{almost}. Thus, by Lemma 2.7.5, there exists a $4\kappa'$-well separated F-packing $\mathcal{F}_{almost}$ in G_{almost} which covers all edges of $G^{(r)}_{almost}$ except possibly some inside U_ℓ. Let $H^* := (G^{(r)}_{almost} - \mathcal{F}^{(r)}_{almost})[U_\ell]$. Since H^* is F-divisible, $A^* \cup H^*$ has an $s\sqrt{\kappa}$-well separated F-decomposition $\mathcal{F}^*$ with $\mathcal{F}^{*\leq} \subseteq G[T^*]$ by (A1′). Clearly,

$$G^{(r)} = G^{(r)}_{almost} \mathbin{\dot\cup} A^* = \mathcal{F}^{(r)}_{almost} \mathbin{\dot\cup} H^* \mathbin{\dot\cup} A^* = \mathcal{F}^{(r)}_{almost} \mathbin{\dot\cup} \mathcal{F}^{*(r)},$$

and $\mathcal{F}_{almost}$ and $\mathcal{F}^*$ are $(r+1)$-disjoint. Thus, by Fact 2.5.4(ii), $\mathcal{F}_{almost} \cup \mathcal{F}^*$ is a $(4\kappa' + s\sqrt{\kappa})$-well separated F-decomposition of G, completing the proof. □

2.9.2 Resolvable partite designs

Perhaps surprisingly, it is much easier to obtain decompositions of complete partite r-graphs than of complete (non-partite) r-graphs. In fact, we can obtain (explicit) resolvable decompositions (sometimes referred to as *Kirkman systems*) in the partite setting using basic linear algebra. We believe that this result and the corresponding construction are of independent interest. Here, we will use this result to show that for every r-graph F, there is a weakly regular r-graph F^* which is F-decomposable (see Lemma 2.9.2).

Let G be a complex. We say that a $K^{(r)}_f$-decomposition $\mathcal{K}$ of G is *resolvable* if $\mathcal{K}$ can be partitioned into $K^{(r-1)}_f$-decompositions of G, that is, $\mathcal{K}^{\leq(f)}$ can be partitioned into sets $Y_1, \dots, Y_t$ such that for each $i \in [t]$, $\mathcal{K}_i := \{G^{(r-1)}[Q] : Q \in Y_i\}$ is a $K^{(r-1)}_f$-decomposition of G. Clearly, $\mathcal{K}_1, \dots, \mathcal{K}_t$ are r-disjoint.

Let $K_{n\times k}$ be the complete k-partite complex with each vertex class having size n. More precisely, $K_{n\times k}$ has vertex set $V_1 \,\dot\cup\, \dots \,\dot\cup\, V_k$ such that $|V_i| = n$ for all $i \in [k]$ and $e \in K_{n\times k}$ if and only if e is *crossing*, that is, intersects with each V_i in at most one vertex. Since every subset of a crossing set is crossing, this defines a complex.

Theorem 2.9.1. *Let q be a prime power and $2f \le q$. Then for every $r \in [f-1]$, $K_{q\times f}$ has a resolvable $K_f^{(r)}$-decomposition.*

Let us first motivate the proof of Theorem 2.9.1. Let $\mathbb{F}$ be the finite field of order q. Assume that each class of $K_{q\times f}$ is a copy of $\mathbb{F}$. Suppose further that we are given a matrix $A \in \mathbb{F}^{(f-r)\times f}$ with the property that every $(f-r)\times(f-r)$-submatrix is invertible. Identifying $K^{(f)}_{q\times f}$ with $\mathbb{F}^f$ in the obvious way, we let $\mathcal{K}$ be the set of all $Q \in K^{(f)}_{q\times f}$ with $AQ = 0$. Fixing the entries of r coordinates of Q (which can be viewed as fixing an r-set) transforms this into an equation $A'Q' = b'$, where A' is an $(f-r)\times(f-r)$-submatrix of A. Thus, there exists a unique solution, which will translate into the fact that every r-set of $K_{q\times f}$ is contained in exactly one f-set of $\mathcal{K}$, i.e. we have a $K_f^{(r)}$-decomposition.

There are several known classes of matrices over finite fields which have the desired property that every square submatrix is invertible. We use so-called Cauchy matrices, introduced by Cauchy [18], which are very convenient for our purposes. For an application of Cauchy matrices to coding theory, see e.g. [11].

Let $\mathbb{F}$ be a field and let $x_1, \dots, x_m, y_1, \dots, y_n$ be distinct elements of $\mathbb{F}$. The *Cauchy matrix generated by $(x_i)_{i\in[m]}$ and $(y_j)_{j\in[n]}$* is the $m \times n$-matrix $A \in \mathbb{F}^{m\times n}$ defined by $a_{i,j} := (x_i - y_j)^{-1}$. Obviously, every submatrix of a Cauchy matrix is itself a Cauchy matrix. For $m = n$, it is well known that the Cauchy determinant is given by the following formula (cf. [79]):

$$\det(A) = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_i - y_j)}{\prod_{1\le i,j\le n}(x_i - y_j)}.$$

In particular, every square Cauchy matrix is invertible.

Proof of Theorem 2.9.1. Let $\mathbb{F}$ be the finite field of order q. Since $2f \leq q$, there exists a Cauchy matrix $A \in \mathbb{F}^{(f-r+1)\times f}$. Let $\hat{\mathbf{a}}$ be the final row of A and let $A' \in \mathbb{F}^{(f-r)\times f}$ be obtained from A by deleting $\hat{\mathbf{a}}$.

We assume that the vertex set of $K_{q\times f}$ is $\mathbb{F} \times [f]$. Hence, for every $e \in K_{q\times f}$, there are unique $1 \leq i_1 < \cdots < i_{|e|} \leq f$ and $x_1, \ldots, x_{|e|} \in \mathbb{F}$ such that $e = \{(x_j, i_j) : j \in [|e|]\}$. Let

$$I_e := \{i_1, \ldots, i_{|e|}\} \subseteq [f] \quad \text{and} \quad \mathbf{x_e} := \begin{pmatrix} x_1 \\ \vdots \\ x_{|e|} \end{pmatrix} \in \mathbb{F}^{|e|}.$$

Clearly, $Q \in K^{(f)}_{q\times f}$ is uniquely determined by $\mathbf{x_Q}$.

Define $Y \subseteq K^{(f)}_{q\times f}$ as the set of all $Q \in K^{(f)}_{q\times f}$ which satisfy $A' \cdot \mathbf{x_Q} = \mathbf{0}$. Moreover, for each $x^* \in \mathbb{F}$, define $Y_{x^*} \subseteq Y$ as the set of all $Q \in Y$ which satisfy $\hat{\mathbf{a}} \cdot \mathbf{x_Q} = x^*$. Clearly, $\{Y_{x^*} : x^* \in \mathbb{F}\}$ is a partition of Y. Let $\mathcal{K} := \{K^{(r)}_{q\times f}[Q] : Q \in Y\}$ and $\mathcal{K}_{x^*} := \{K^{(r-1)}_{q\times f}[Q] : Q \in Y_{x^*}\}$ for each $x^* \in \mathbb{F}$. We claim that $\mathcal{K}$ is a $K^{(r)}_f$-decomposition of $K_{q\times f}$ and that $\mathcal{K}_{x^*}$ is a $K^{(r-1)}_f$-decomposition of $K_{q\times f}$ for each $x^* \in \mathbb{F}$.

For $I \subseteq [f]$, let A_I be the $(f-r+1)\times |I|$-submatrix of A obtained by deleting the columns which are indexed by $[f] \setminus I$. Similarly, for $I \subseteq [f]$, let A'_I be the $(f-r)\times|I|$-submatrix of A' obtained by deleting the columns which are indexed by $[f] \setminus I$. Finally, for a vector $\mathbf{x} \in \mathbb{F}^f$ and $I \subseteq [f]$, let $\mathbf{x}_I \in \mathbb{F}^{|I|}$ be the vector obtained from $\mathbf{x}$ by deleting the coordinates not in I.

Observe that for all $e \in K_{q\times f}$ and $Q \in K^{(f)}_{q\times f}$, we have

$$e \subseteq Q \text{ if and only if } \mathbf{x_Q}_{I_e} = \mathbf{x_e}. \tag{2.9.1}$$

Consider $e \in K^{(r)}_{q\times f}$. By (2.9.1), the number of $Q \in Y$ containing e is equal to the number of $\mathbf{x} \in \mathbb{F}^f$ such that $A' \cdot \mathbf{x} = \mathbf{0}$ and $\mathbf{x}_{I_e} = \mathbf{x_e}$, or equivalently, the number of $\mathbf{x}' \in \mathbb{F}^{f-r}$ satisfying $A'_{I_e} \cdot \mathbf{x_e} + A'_{[f]\setminus I_e} \cdot \mathbf{x}' = \mathbf{0}$. Since $A'_{[f]\setminus I_e}$ is an $(f-r)\times(f-r)$-Cauchy matrix, the equation $A'_{[f]\setminus I_e} \cdot \mathbf{x}' = -A'_{I_e} \cdot \mathbf{x_e}$ has a unique solution $\mathbf{x}' \in \mathbb{F}^{f-r}$, i.e. there is exactly one $Q \in Y$ which contains e. Thus, $\mathcal{K}$ is a $K_f^{(r)}$-decomposition of $K_{q\times f}$.

Now, fix $x^* \in \mathbb{F}$ and $e \in K_{q\times f}^{(r-1)}$. By (2.9.1), the number of $Q \in Y_{x^*}$ containing e is equal to the number of $\mathbf{x} \in \mathbb{F}^f$ such that $A' \cdot \mathbf{x} = \mathbf{0}$, $\hat{\mathbf{a}} \cdot \mathbf{x} = x^*$ and $\mathbf{x}_{I_e} = \mathbf{x_e}$, or equivalently, the number of $\mathbf{x}' \in \mathbb{F}^{f-(r-1)}$ satisfying $A_{I_e} \cdot \mathbf{x_e} + A_{[f]\setminus I_e} \cdot \mathbf{x}' = \begin{pmatrix} \mathbf{0} \\ x^* \end{pmatrix}$. Since $A_{[f]\setminus I_e}$ is an $(f-r+1)\times(f-r+1)$-Cauchy matrix, this equation has a unique solution $\mathbf{x}' \in \mathbb{F}^{f-r+1}$, i.e. there is exactly one $Q \in Y_{x^*}$ which contains e. Hence, $\mathcal{K}_{x^*}$ is a $K_f^{(r-1)}$-decomposition of $K_{q\times f}$. $\square$

Our application of Theorem 2.9.1 is as follows.

Lemma 2.9.2. *Let $2 \le r < f$. Let F be any r-graph on f vertices. There exists a weakly regular r-graph F^* on at most $2f\cdot f!$ vertices which has a 1-well separated F-decomposition.*

Proof. Choose a prime power q with $f! \le q \le 2f!$. Let $V(F) = \{v_1, \dots, v_f\}$. By Theorem 2.9.1, there exists a resolvable $K_f^{(r)}$-decomposition $\mathcal{K}$ of $K_{q\times f}$. Let the vertex classes of $K_{q\times f}$ be $V_1, \dots, V_f$. Let $\mathcal{K}_1, \dots, \mathcal{K}_q$ be a partition of $\mathcal{K}$ into $K_f^{(r-1)}$-decompositions of $K_{q\times f}$. (We will only need $\mathcal{K}_1, \dots, \mathcal{K}_{f!}$.) We now construct F^* with vertex set $V(K_{q\times f})$ as follows: Let $\pi_1, \dots, \pi_{f!}$ be an enumeration of all permutations on $[f]$. For every $i \in [f!]$ and $Q \in \mathcal{K}_i^{\le(f)}$, let $F_{i,Q}$ be a copy of F with $V(F) = Q$ such that for every $j \in [f]$, the unique vertex in $Q \cap V_{\pi_i(j)}$ plays the role of v_j. Let

$$F^* := \bigcup_{i\in[f!],Q\in\mathcal{K}_i^{\le(f)}} F_{i,Q};$$

$$\mathcal{F} := \{F_{i,Q} \,:\, i \in [f!], Q \in \mathcal{K}_i^{\le(f)}\}.$$

Since $\mathcal{K}_1, \dots, \mathcal{K}_{f!}$ are r-disjoint, we have $|V(F') \cap V(F'')| < r$ for all distinct $F', F'' \in \mathcal{F}$. Thus, $\mathcal{F}$ is a 1-well separated F-decomposition of F^*.

We now show that F^* is weakly regular. Let $i \in [r-1]_0$ and $S \in \binom{V(F^*)}{i}$. If S is not crossing, then $|F^*(S)| = 0$, so assume that S is crossing. If $i = r-1$, then S plays the role of every $(r-1)$-subset of $V(F)$ exactly k times, where k is the number of permutations on $[f]$ that map $[r-1]$ to $[r-1]$. Hence,

$$|F^*(S)| = |F|rk = |F| \cdot r!(f-r+1)! =: s_{r-1}.$$

If $i < r-1$, then S is contained in exactly $c_i := \binom{f-i}{r-1-i}q^{r-1-i}$ crossing $(r-1)$-sets. Thus,

$$|F^*(S)| = \frac{s_{r-1}c_i}{r-i} =: s_i.$$

Therefore, F^* is weakly $(s_0, \dots, s_{r-1})$-regular. $\square$

2.9.3 Proofs of Theorems 2.1.1, 2.1.2, 2.1.4, 2.1.5 and 2.1.6

We now prove our main theorems which guarantee F-decompositions in r-graphs of high minimum degree (for weakly regular r-graphs F, see Theorem 2.1.4), and F-designs in typical r-graphs (for arbitrary r-graphs F, see Theorem 2.1.1). We will also derive Theorems 2.1.2, 2.1.5 and 2.1.6.

We first prove the minimum degree version (for weakly regular r-graphs F). Instead of directly proving Theorem 2.1.4 we actually prove a stronger 'local resilience version'. Recall that $\mathcal{H}_r(n,p)$ denotes the random binomial r-graph on $[n]$ whose edges appear independently with probability p.

Theorem 2.9.3 (Resilience version)**.** *Let $p \in (0,1]$ and $f, r \in \mathbb{N}$ with $f > r$ and let*

$$c(f,r,p) := \frac{r!p^{2^r\binom{f+r}{r}}}{3 \cdot 14^r f^{2r}}.$$

Then the following holds whp for $H \sim \mathcal{H}_r(n,p)$. For every weakly regular r-graph F on f vertices and any r-graph L on $[n]$ with $\Delta(L) \leq c(f,r,p)n$, $H \triangle L$ has an F-decomposition whenever it is F-divisible.

The case $p = 1$ immediately implies Theorem 2.1.4.

Proof. Choose $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $1/n_0 \ll \varepsilon \ll p, 1/f$ and let $n \geq n_0$,

$$c' := \frac{1.1 \cdot 2^r \binom{f+r}{r}}{(f-r)!} c(f,r,p), \quad \xi := 0.99/f!, \quad \xi' := 0.95 \xi p^{2^r \binom{f+r}{r}}, \quad \xi'' := 0.9(1/4)^{\binom{f+r}{f}}(\xi' - c').$$

Recall that the complete complex K_n is an (ε, ξ, f, r)-supercomplex (cf. Example 2.4.9). Let $H \sim \mathcal{H}_r(n,p)$. We can view H as a random subgraph of $K_n^{(r)}$. By Corollary 2.5.19, the following holds whp for all $L \subseteq K_n^{(r)}$ with $\Delta(L) \leq c(f,r,p)n$:

$$K_n[H \bigtriangleup L] \text{ is a } (3\varepsilon + c', \xi' - c', f, r)\text{-supercomplex.}$$

Note that $c' \leq \frac{p^{2^r \binom{f+r}{r}}}{2.7(2\sqrt{e})^r f!}$. Thus, $2(2\sqrt{e})^r \cdot (3\varepsilon + c') \leq \xi' - c'$. Lemma 2.4.4 now implies that $K_n[H \bigtriangleup L]$ is an $(\varepsilon, \xi'', f, r)$-supercomplex. Hence, if $H \bigtriangleup L$ is F-divisible, it has an F-decomposition by Theorem 2.4.7. □

Next, we derive Theorem 2.1.1. As indicated previously, we cannot apply Theorem 2.4.7 directly, but have to carry out two reductions. As shown in Lemma 2.9.2, we can 'perfectly' pack any given r-graph F into a weakly regular r-graph F^*. We also need the following lemma, which we will prove later in Section 2.11. It allows us to remove a sparse F-decomposable subgraph L from an F-divisible r-graph G to achieve that $G - L$ is F^*-divisible. Note that we do not need to assume that F^* is weakly regular.

Lemma 2.9.4. *Let $1/n \ll \gamma \ll \xi, 1/f^*$ and $r \in [f^* - 1]$. Let F be an r-graph. Let F^* be an r-graph on f^* vertices which has a 1-well separated F-decomposition. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq \binom{f^*-1}{r-1}$, we have $|\bigcap_{S \in A} G(S)| \geq \xi n$. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. Then there exists an F-divisible subgraph $D \subseteq G$ with $\Delta(D) \leq \gamma^{-2}$ such that the following holds: for every F-divisible r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F^*-divisible and $D - D^*$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.*

In particular, we will apply this lemma when G is F-divisible and thus $H := G - D$

is F-divisible. Then $L := D - D^*$ is a subgraph of G with $\Delta(L) \leq \gamma^{-2}$ and has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint. Moreover, $G - L = H \cup D^*$ is F^*-divisible.

We can deduce the following corollary from the case $F = K_r^{(r)}$ of Lemma 2.9.4.

Corollary 2.9.5. *Let $1/n \ll \gamma \ll \xi, 1/f$ and $r \in [f-1]$. Let F be an r-graph on f vertices. Let G be an r-graph on n vertices such that for all $A \subseteq \binom{V(G)}{r-1}$ with $|A| \leq \binom{f-1}{r-1}$, we have $|\bigcap_{S \in A} G(S)| \geq \xi n$. Then there exists a subgraph $D \subseteq G$ with $\Delta(D) \leq \gamma^{-2}$ such that the following holds: for any r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F-divisible.*

In particular, using $H := G - D$, there exists a subgraph $L := D - D^* \subseteq G$ with $\Delta(L) \leq \gamma^{-2}$ such that $G - L = H \cup D^*$ is F-divisible.

Proof. Apply Lemma 2.9.4 with $F, K_r^{(r)}$ playing the roles of F^*, F. □

We now prove the following theorem, which immediately implies the case $\lambda = 1$ of Theorem 2.1.1.

Theorem 2.9.6. *Let $1/n \ll \gamma, 1/\kappa \ll c, p, 1/f$ and $r \in [f-1]$, and*

$$c \leq p^h/(q^r 4^q), \text{ where } h := 2^r \binom{q+r}{r} \text{ and } q := 2f \cdot f!. \tag{2.9.2}$$

Let F be any r-graph on f vertices. Suppose that G is a (c, h, p)-typical F-divisible r-graph on n vertices. Let O be an $(r+1)$-graph on $V(G)$ with $\Delta(O) \leq \gamma n$. Then G has a κ-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.

Proof. By Lemma 2.9.2, there exists a weakly regular r-graph F^* on $f^* \leq q$ vertices which has a 1-well separated F-decomposition.

By Lemma 2.9.4 (with $0.5p^{\binom{f^*-1}{r-1}}$ playing the role of ξ), there exists a subgraph $L \subseteq G$ with $\Delta(L) \leq \gamma^{-2}$ such that $G - L$ is F^*-divisible and L has a 1-well separated F-decomposition $\mathcal{F}_{div}$ such that $\mathcal{F}_{div}^{\leq(r+1)}$ and O are edge-disjoint. By Fact 2.5.4(i),

$\Delta(\mathcal{F}_{div}^{\leq(r+1)}) \leq f - r$. Let

$$G' := G^{\leftrightarrow} - L - \mathcal{F}_{div}^{\leq(r+1)} - O.$$

By Example 2.4.10, $G^{\leftrightarrow}$ is an $(\varepsilon, \xi, f^*, r)$-supercomplex, where $\varepsilon := 2^{f^*-r+1}c/(f^*-r)!$ and $\xi := (1-2^{f^*+1}c)p^{2^r\binom{f^*+r}{r}}/f^*!$. Observe that assumption (2.9.2) now guarantees that $2(2\sqrt{\mathrm{e}})^r\varepsilon \leq \xi$. Thus, by Lemma 2.4.4, $G^{\leftrightarrow}$ is a (γ, ξ', f^*, r)-supercomplex, where $\xi' := 0.9(1/4)^{\binom{f^*+r}{r}}\xi$. By Proposition 2.5.9(v), we have that G' is a $(\sqrt{\gamma}, \xi'/2, f^*, r)$-supercomplex. Moreover, G' is F^*-divisible. Thus, by Theorem 2.4.7, G' has a $(\kappa-1)$-well separated F^*-decomposition $\mathcal{F}^*$. Since F^* has a 1-well separated F-decomposition, we can conclude that G' has a $(\kappa-1)$-well separated F-decomposition $\mathcal{F}_{complex}$. Let $\mathcal{F} := \mathcal{F}_{div} \cup \mathcal{F}_{complex}$. By Fact 2.5.4(ii), $\mathcal{F}$ is a κ-well separated F-decomposition of G. Moreover, $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint. □

We next derive Theorem 2.1.1 from Theorem 2.9.6 and Corollary 2.9.5.

Proof of Theorem 2.1.1. Choose a new constant $\kappa \in \mathbb{N}$ such that

$$1/n \ll \gamma \ll 1/\kappa \ll c, p, 1/f.$$

Suppose that G is a (c, h, p)-typical (F, λ)-divisible r-graph on n vertices. Split G into two subgraphs G_1' and G_2' which are both $(c+\gamma, h, p/2)$-typical (a standard Chernoff-type bound shows that whp a random splitting of G yields the desired property).

By Corollary 2.9.5 (applied with $G_2', 0.5(p/2)^{\binom{f-1}{r-1}}$ playing the roles of G, ξ), there exists a subgraph $L^* \subseteq G_2'$ with $\Delta(L^*) \leq \kappa$ such that $G_2 := G_2' - L^*$ is F-divisible. Let $G_1 := G_1' \cup L^* = G - G_2$. Clearly, G_1 is still (F, λ)-divisible. By repeated applications of Corollary 2.9.5, we can find edge-disjoint subgraphs $L_1, \dots, L_\lambda$ of G_1 such that $R_i := G_1 - L_i$ is F-divisible and $\Delta(L_i) \leq \kappa$ for all $i \in [\lambda]$. Indeed, suppose that we have already found $L_1, \dots, L_{i-1}$. Then $\Delta(L_1 \cup \dots \cup L_{i-1}) \leq \lambda\kappa \leq \gamma^{1/2}n$ (recall that $\lambda \leq \gamma n$). Thus, by Corollary 2.9.5, there exists a subgraph $L_i \subseteq G_1' - (L_1 \cup \dots \cup L_{i-1})$ with $\Delta(L_i) \leq \kappa$ such that $G_1 - L_i$ is F-divisible.

Let $G_2'' := G_2 \cup L_1 \cup \cdots \cup L_\lambda$. We claim that G_2'' is F-divisible. Indeed, let $S \subseteq V(G)$ with $|S| \leq r-1$. We then have that $|G_2''(S)| = |G_2(S)| + \sum_{i\in[\lambda]} |(G_1 - R_i)(S)| = |G_2(S)| + \lambda|G_1(S)| - \sum_{i\in[\lambda]} |R_i(S)| \equiv 0 \mod Deg(F)_{|S|}$.

Since G_1' and G_2' are both $(c+\gamma, h, p/2)$-typical and $\Delta(L^* \cup L_1 \cup \cdots \cup L_\lambda) \leq 2\gamma^{1/2}n$, we have that each of G_2, G_2'', $R_1, \ldots, R_\lambda$ is $(c+\gamma^{1/3}, h, p/2)$-typical (and they are F-divisible by construction).

Using Theorem 2.9.6 repeatedly, we can thus find κ-well separated F-decompositions $\mathcal{F}_1, \ldots, \mathcal{F}_{\lambda-1}$ of G_2, a κ-well separated F-decomposition $\mathcal{F}^*$ of G_2'', and for each $i \in [\lambda]$, a κ-well separated F-decomposition $\mathcal{F}_i'$ of R_i. Moreover, we can assume that all these decompositions are pairwise $(r+1)$-disjoint. Indeed, this can be achieved by choosing them successively: Let O consist of the $(r+1)$-sets which are covered by the decompositions we have already found. Then by Fact 2.5.4(i) we have that $\Delta(O) \leq 2\lambda \cdot \kappa(f-r) \leq \gamma^{1/2}n$. Hence, using Theorem 2.9.6, we can find the next κ-well separated F-decomposition which is $(r+1)$-disjoint from the previously chosen ones.

Then $\mathcal{F} := \mathcal{F}^* \cup \bigcup_{i\in[\lambda-1]} \mathcal{F}_i \cup \bigcup_{i\in[\lambda]} \mathcal{F}_i'$ is the desired (F, λ)-design. Indeed, every edge of $G_1 - (L_1 \cup \cdots \cup L_\lambda)$ is covered by each of $\mathcal{F}_1', \ldots, \mathcal{F}_\lambda'$. For each $i \in [\lambda]$, every edge of L_i is covered by $\mathcal{F}^*$ and each of $\mathcal{F}_1', \ldots, \mathcal{F}_{i-1}', \mathcal{F}_{i+1}', \ldots, \mathcal{F}_\lambda'$. Finally, every edge of G_2 is covered by each of $\mathcal{F}_1, \ldots, \mathcal{F}_{\lambda-1}$ and $\mathcal{F}^*$. □

Using the same strategy, a similar result which holds in the more general setting of supercomplexes can be obtained by using Corollary 2.6.10 instead of Corollary 2.9.5.

Theorem 2.1.2 is an immediate consequence of Theorem 2.9.6 and Corollary 2.9.5.

Proof of Theorem 2.1.2. Apply Corollary 2.9.5 (with $G, 0.5p^{\binom{f-1}{r-1}}$ playing the roles of G, ξ) to find a subgraph $L \subseteq G$ with $\Delta(L) \leq C$ such that $G - L$ is F-divisible. It is easy to see that $G - L$ is $(1.1c, h, p)$-typical. Thus, we can apply Theorem 2.9.6 to obtain an F-decomposition $\mathcal{F}$ of $G - L$. □

Proof of Theorem 2.1.6. By Example 2.4.12, we have that $G^{\leftrightarrow}$ is an $(0.01\xi, 0.99\xi, f, 1)$-supercomplex. Moreover, since $f \mid n$, $G^{\leftrightarrow}$ is $K_f^{(1)}$-divisible. Thus, by Corollary 2.4.14, $G^{\leftrightarrow}$

has $0.01\xi n^{f-1}$ f-disjoint $K_f^{(1)}$-decompositions, i.e. G has $0.01\xi n^{f-1}$ edge-disjoint perfect matchings. □

Finally, we also prove Theorem 2.1.5, which is an easy corollary of Theorem 2.1.1.

Proof of Theorem 2.1.5. Choose c, h, n_0 such that $1/n_0 \ll c \ll 1/h \ll p, 1/f$. Let $\mathcal{K} = \{F_1, \dots, F_t\}$. Thus $t \leq 2^{\binom{f}{r}}$. Let $F^* := F_1 + \dots + F_t$ and let $a_1, \dots, a_t$ be integers such that $e := \gcd\{|F_1|, \dots, |F_t|\} = a_1|F_1| + \dots + a_t|F_t|$.

Now, assume that G is (c, h, p)-typical and $\mathcal{K}$-divisible. In particular, $e \mid |G|$. Since $e \mid |F^*|$, we have $|G| \equiv xe \mod |F^*|$ for some $x \in \mathbb{Z}$. With the above, $|G| \equiv \sum_{i\in[t]} a'_i|F_i| \mod |F^*|$ for some integers a'_i. Clearly, we may assume that $0 \leq a'_i < |F^*|$. Let $\mathcal{F}_0$ be a set of a'_i copies of F_i in G for all $i \in [t]$, all edge-disjoint. Let $G' := G - \mathcal{F}_0^{(r)}$. It is easy to check that G' is F^*-divisible. Thus, since G' is still $(2c, h, p)$-typical, Theorem 2.1.1 implies that G' has an F^*-decomposition. In particular, G' has a $\mathcal{K}$-decomposition $\mathcal{F}_1$. Finally, $\mathcal{F}_0 \cup \mathcal{F}_1$ is a $\mathcal{K}$-decomposition of G. □

2.10 Covering down

The aim of this section is to prove the Cover down lemma (Lemma 2.7.7). Suppose that G is a supercomplex and U is a 'random-like' subset of $V(G)$. The Cover down lemma shows the existence of a 'cleaning graph' H^* so that for any sparse leftover graph L^*, $G[H^* \cup L^*]$ has an F-packing covering all edges of $H^* \cup L^*$ except possibly some inside U.

We now briefly sketch how one can attempt to construct such a graph H^*. As in Section 2.7.1, for an edge e, we refer to $|e \cap U|$ as its *type*. For the moment, suppose that H^* and L^* are given. A natural way (for divisibility reasons) to try to cover all edges of $H^* \cup L^*$ which are not inside U is to first cover all type-0-edges, then all type-1-edges, etc. and finally all type-$(r-1)$-edges. It is comparatively easy to cover all type-0-edges. The reason for this is that a type-0-edge can be covered by a copy of F that contains no other type-0-edge. Thus, if H^* is a random subgraph of $G^{(r)} - G^{(r)}[V(G) \setminus U]$, then every type-0-edge (from L^*) is contained in many copies of F. Since $\Delta(L^*)$ is very small, this allows us to apply Corollary 2.6.9 in order to cover all type-0-edges with edge-disjoint copies of F.

The situation is very different for edges of higher types. Suppose that for some $i \in [r-1]$, we have already covered all edges of types $0, \dots, r-i-1$ and now want to cover all edges of type $r-i$. Every such edge contains a unique $S \in \binom{V(G)\setminus U}{i}$. As indicated in Section 2.7.1, we seek to cover all edges containing a fixed $S \in \binom{V(G)\setminus U}{i}$ simultaneously using Proposition 2.7.9 as follows: Let $T \in \binom{V(F)}{i}$. Roughly speaking, for every $S \in \binom{V(G)\setminus U}{i}$, we reserve a random subgraph H_S of $G(S)[U]^{(r-i)}$ and protect all the H_S's when applying the nibble. Let L be the leftover resulting from this application and let $L_S := L(S)$. Assuming that there are no more leftover edges of types $0, \dots, r-i-1$ implies that $L_S \subseteq G(S)[U]^{(r-i)}$ and that $H_S \cup L_S$ is $F(T)$-divisible. We want to use $(*)_{r-i}$ inductively to find a well separated $F(T)$-decomposition $\mathcal{F}_S$ of $H_S \cup L_S$ (provided that $H_S \cup L_S$ is quasirandom). Using Proposition 2.7.9, $\mathcal{F}_S$ can then be 'extended' to an F-packing $S \triangleleft \mathcal{F}_S$ which covers all edges that contain S. The hope is that the H_S's do not intersect too much, so that it is possible to find an $F(T)$-decomposition $\mathcal{F}_S$ for each S such that the extended F-packings $S \triangleleft \mathcal{F}_S$ are r-disjoint. Their union would then yield an F-packing covering all edges of type $r-i$.

There are two natural candidates for selecting H_S:

(A) Choose H_S by including every edge of $G(S)[U]^{(r-i)}$ with probability ν.

(B) Choose a random subset U_S of U of size $\rho|U|$ and let $H_S := G(S)^{(r-i)}[U_S]$.

The advantage of Strategy (A) is that $H_S \cup L_S$ is quasirandom if L_S is sparse. This is not the case for (B): even if the maximum degree of L_S is sublinear, its edges might be spread out over the whole of U (while H_S is restricted to U_S). Unfortunately, when pursuing Strategy (A), the H_S intersect too much, so it is not clear how to find the desired decompositions due to the interference between different H_S. However, it turns out that under the additional assumption that $V(L_S) \subseteq U_S$, Strategy (B) does work. We call the corresponding result the 'Localised cover down lemma' (Lemma 2.10.8).

We will combine both strategies as follows: For each S, we will choose H_S as in (A) and U_S as in (B) and let $J_S := G(S)^{(r-i)}[U_S]$. In a first step we use H_S to find an $F(T)$-packing covering all edges $e \in H_S \cup L_S$ with $e \not\subseteq U_S$, and then afterwards we apply the Localised cover down lemma to cover all remaining edges. Note that the first step resembles the original problem: We are given a graph $H_S \cup L_S$ on U and want to cover all edges that are not inside $U_S \subseteq U$. But the resulting types are now more restricted. This enables us to prove a more general Cover down lemma, the 'Cover down lemma for setups' (Lemma 2.10.24), by induction on $r - i$, which will allow us to perform the first step in the above combined strategy for all S simultaneously.

2.10.1 Systems and focuses

In this subsection, we prove the Localised cover down lemma, which shows that Strategy (B) works under the assumption that each L_S is 'localised'.

Definition 2.10.1. Given $i \in \mathbb{N}_0$, an *i-system in a set V* is a collection $\mathcal{S}$ of distinct subsets of V of size i. A subset of V is called *$\mathcal{S}$-important* if it contains some $S \in \mathcal{S}$, otherwise we call it *$\mathcal{S}$-unimportant*. We say that $\mathcal{U} = (U_S)_{S \in \mathcal{S}}$ is a *focus for $\mathcal{S}$* if for each $S \in \mathcal{S}$, U_S is a subset of $V \setminus S$.

Definition 2.10.2. Let G be a complex and $\mathcal{S}$ an i-system in $V(G)$. We call G *r-exclusive with respect to $\mathcal{S}$* if every $e \in G$ with $|e| \geq r$ contains at most one element of $\mathcal{S}$. Let $\mathcal{U}$ be a focus for $\mathcal{S}$. If G is r-exclusive with respect to $\mathcal{S}$, the following functions are well-defined: For $r' \geq r$, let $\mathcal{E}_{r'}$ denote the set of $\mathcal{S}$-important r'-sets in G. Define $\tau_{r'} \colon \mathcal{E}_{r'} \to [r' - i]_0$ as $\tau_{r'}(e) := |e \cap U_S|$, where S is the unique $S \in \mathcal{S}$ contained in e. We call $\tau_{r'}$ the *type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$*.

Fact 2.10.3. *Let $r \in \mathbb{N}$ and $i \in [r-1]_0$. Let G be a complex and $\mathcal{S}$ an i-system in $V(G)$. Let $\mathcal{U}$ be a focus for $\mathcal{S}$ and suppose that G is r-exclusive with respect to $\mathcal{S}$. For $r' \geq r$,* *let* $\tau_{r'}\colon \mathcal{E}_{r'} \to [r'-i]_0$ *denote the type function of* $G^{(r')}, \mathcal{S}, \mathcal{U}$*. Let* $e \in G$ *with* $|e| \geq r$ *be* $\mathcal{S}$*-important and let* $\mathcal{E}' := \mathcal{E}_r \cap \binom{e}{r}$*. Then we have*

(i) $\max_{e' \in \mathcal{E}'} \tau_r(e') \leq \tau_{|e|}(e) \leq |e| - r + \min_{e' \in \mathcal{E}'} \tau_r(e')$,

(ii) $\min_{e' \in \mathcal{E}'} \tau_r(e') = \max\{r + \tau_{|e|}(e) - |e|, 0\}$.

Proof. Let $S \subseteq e$ with $S \in \mathcal{S}$. Clearly, for every $\mathcal{S}$-important r-subset e' of e, S is the unique element from $\mathcal{S}$ that e' contains. For any such e', we have $\tau_{|e|}(e) = |e \cap U_S| \geq |e' \cap U_S| = \tau_r(e')$, implying the first inequality of (i). Also, $|e| - \tau_{|e|}(e) = |e \setminus U_S| \geq |e' \setminus U_S| = r - \tau_r(e')$, implying the second inequality of (i).

This also implies that $\min_{e' \in \mathcal{E}'} \tau_r(e') \geq \max\{r + \tau_{|e|}(e) - |e|, 0\}$. To see the converse, note that $|e \setminus U_S| = |e| - \tau_{|e|}(e)$. Hence, we can choose an r-set $e' \subseteq e$ with $S \subseteq e'$ and $|e' \setminus U_S| = \min\{|e| - \tau_{|e|}(e), r\}$. Note that $e' \in \mathcal{E}'$ and $\tau_r(e') = r - |e' \setminus U_S| = r - \min\{|e| - \tau_{|e|}(e), r\} = \max\{r + \tau_{|e|}(e) - |e|, 0\}$. This completes the proof of (ii). $\square$

Definition 2.10.4. Let G be a complex and $\mathcal{S}$ an i-system in $V(G)$. Let $\mathcal{U}$ be a focus for $\mathcal{S}$ and suppose that G is r-exclusive with respect to $\mathcal{S}$. For $i' \in \{i+1, \dots, r-1\}$, we define $\mathcal{T}$ as the set of all i'-subsets T of $V(G)$ which satisfy $S \subseteq T \subseteq e \setminus U_S$ for some $S \in \mathcal{S}$ and $e \in G^{(r)}$. We call $\mathcal{T}$ the *i'-extension of $\mathcal{S}$ in G around $\mathcal{U}$*.

Clearly, $\mathcal{T}$ is an i'-system in $V(G)$. Moreover, note that for every $T \in \mathcal{T}$, there is a unique $S \in \mathcal{S}$ with $S \subseteq T$ because G is r-exclusive with respect to $\mathcal{S}$. We let $T{\restriction_{\mathcal{S}}} := S$ denote this element. (On the other hand, we may have $|\mathcal{T}| < |\mathcal{S}|$.) Note that $\mathcal{U}' := \{U_{T\restriction_{\mathcal{S}}} : T \in \mathcal{T}\}$ is a focus for $\mathcal{T}$ as $T \cap U_{T\restriction_{\mathcal{S}}} = \emptyset$ for all $T \in \mathcal{T}$.

The following proposition contains some basic properties of i'-extensions.

Proposition 2.10.5. *Let* $0 \leq i < i' < r$*. Let* G *be a complex and* $\mathcal{S}$ *an* i*-system in* $V(G)$*. Let* $\mathcal{U}$ *be a focus for* $\mathcal{S}$ *and suppose that* G *is* r*-exclusive with respect to* $\mathcal{S}$*. Let* $\mathcal{T}$ *be the* i'*-extension of* $\mathcal{S}$ *in* G *around* $\mathcal{U}$*. For* $r' \geq r$*, let* $\tau_{r'}$ *be the type function of* $G^{(r')}$*,* $\mathcal{S}$*,* $\mathcal{U}$*. Then the following hold for*

$$G' := G - \{e \in G^{(r)} : e \text{ is } \mathcal{S}\text{-important and } \tau_r(e) < r - i'\} :$$

(i) *G' is r-exclusive with respect to $\mathcal{T}$;*

(ii) *for all $e \in G$ with $|e| \geq r$, we have*

$$e \notin G' \quad \Leftrightarrow \quad e \text{ is } \mathcal{S}\text{-important and } \tau_{|e|}(e) < |e| - i';$$

(iii) *for $r' \geq r$, the $\mathcal{T}$-important elements of $G'^{(r')}$ are precisely the elements of $\tau_{r'}^{-1}(r'-i')$.*

Proof. To see (i), suppose, for a contradiction, that there is some $e' \in G'$ with $|e'| \geq r$ and distinct $T, T' \in \mathcal{T}$ such that e' contains both T and T'. Let $S := T\lceil_{\mathcal{S}}$ and $S' := T'\lceil_{\mathcal{S}}$. Clearly, $S, S' \subseteq e' \in G$. Since G is r-exclusive with respect to $\mathcal{S}$, we must have $S = S'$ and thus $U_S = U_{S'}$. Since T and T' are distinct, we have that $|T \cup T'| > i'$. Let e be a subset of e' of size r containing S and at least $i'+1$ vertices from $T \cup T'$. Since $e \subseteq e' \in G'$, we must have $e \in G'^{(r)}$. On the other hand, since $S \subseteq e$, e is $\mathcal{S}$-important. However, as $T \cup T' \subseteq V(G) \setminus U_S$, we have $\tau_r(e) = |e \cap U_S| < r - i'$, contradicting the definition of G'.

For (ii), let $\mathcal{E}_e$ be the set of $\mathcal{S}$-important r-sets in e. By definition of G', we have $e \notin G'$ if and only if e is $\mathcal{S}$-important, $\mathcal{E}_e \neq \emptyset$ and $\min_{e' \in \mathcal{E}_e} \tau_r(e') < r - i'$. Then Fact 2.10.3(ii) implies the claim.

Finally, we prove (iii). Suppose first that $e \in G'^{(r')}$ is $\mathcal{T}$-important. Clearly, we have $\tau_{r'}(e) \leq r' - i'$. Also, since e must also be $\mathcal{S}$-important, but $e \in G'$, (ii) implies that $\tau_{r'}(e) \geq r' - i'$. Hence, $e \in \tau_{r'}^{-1}(r'-i')$. Now, suppose that $e \in \tau_{r'}^{-1}(r'-i')$. By (ii), we have $e \in G'$ and it remains to show that e is $\mathcal{T}$-important. Since e is $\mathcal{S}$-important, there is a unique $S \in \mathcal{S}$ such that $S \subseteq e$. Let $T := e \setminus U_S$. Clearly, $S \subseteq T \subseteq e \setminus U_S$. Moreover, $|T| = |e| - |e \cap U_S| = r' - \tau_{r'}(e) = i'$. Thus, $T \in \mathcal{T}$, implying that e is $\mathcal{T}$-important. □

Let $\mathcal{Z}_{r,i}$ be the set of all quadruples $(z_0, z_1, z_2, z_3) \in \mathbb{N}_0^4$ such that $z_0 + z_1 < i$, $z_0 + z_3 < i$ and $z_0 + z_1 + z_2 + z_3 = r$. Clearly, $|\mathcal{Z}_{r,i}| \leq (r+1)^3$, and $\mathcal{Z}_{r,i} = \emptyset$ if $i = 0$.

Definition 2.10.6. Let V be a set of size n, let $\mathcal{S}$ be an i-system in V and let $\mathcal{U}$ be a focus for $\mathcal{S}$. We say that $\mathcal{U}$ is a *μ-focus for $\mathcal{S}$* if each $U_S \in \mathcal{U}$ has size $\mu n \pm n^{2/3}$. For all

$S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and all $(z_1 + z_2 - 1)$-sets $b \subseteq V \setminus S$, define

$$\begin{aligned}\mathcal{J}^b_{S,z} &:= \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, b \subseteq S' \cup U_{S'}, |U_{S'} \cap S| \geq z_3\},\\ \mathcal{J}^b_{S,z,1} &:= \{S' \in \mathcal{J}^b_{S,z} \,:\, |b \cap S'| = z_1\},\\ \mathcal{J}^b_{S,z,2} &:= \{S' \in \mathcal{J}^b_{S,z} \,:\, |b \cap S'| = z_1 - 1, |U_S \cap (S' \setminus b)| \geq 1\}.\end{aligned}$$

We say that $\mathcal{U}$ is a (ρ_{size}, ρ, r)-*focus for* $\mathcal{S}$ if

(F1) each U_S has size $\rho_{size}\rho n \pm n^{2/3}$;

(F2) $|U_S \cap U_{S'}| \leq 2\rho^2 n$ for distinct $S, S' \in \mathcal{S}$;

(F3) for all $S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $b \subseteq V \setminus S$, we have

$$\begin{aligned}|\mathcal{J}^b_{S,z,1}| &\leq 2^{6r} \rho^{z_2+z_3-1} n^{i-z_0-z_1},\\ |\mathcal{J}^b_{S,z,2}| &\leq 2^{9r} \rho^{z_2+z_3+1} n^{i-z_0-z_1+1}.\end{aligned}$$

The sets S' in $\mathcal{J}^b_{S,z,1}$ and $\mathcal{J}^b_{S,z,2}$ are those which may give rise to interference when covering the edges containing S. (F3) ensures that there are not too many of them. The next lemma states that a suitable random choice of the U_S yields a (ρ_{size}, ρ, r)-focus.

Lemma 2.10.7. *Let $1/n \ll \rho \ll \rho_{size}, 1/r$ and $i \in [r-1]$. Let V be a set of size n, let $\mathcal{S}$ be an i-system in V and let $\mathcal{U}' = (U'_S)_{S \in \mathcal{S}}$ be a ρ_{size}-focus for $\mathcal{S}$. Let $\mathcal{U} = (U_S)_{S \in \mathcal{S}}$ be a random focus obtained as follows: independently for all pairs $S \in \mathcal{S}$ and $x \in U'_S$, retain x in U_S with probability ρ. Then whp $\mathcal{U}$ is a (ρ_{size}, ρ, r)-focus for $\mathcal{S}$.*

Proof. Clearly, $U_S \subseteq V \setminus S$ for all $S \in \mathcal{S}$.

Step 1: Probability estimates for (F1) and (F2)

For $S \in \mathcal{S}$, Lemma 2.5.10(i) implies that with probability at least $1 - 2\mathrm{e}^{-0.5|U'_S|^{1/3}}$, we have $|U_S| = \mathbb{E}(|U_S|) \pm 0.5|U'_S|^{2/3} = \rho\rho_{size} n \pm (\rho n^{2/3} + 0.5|U'_S|^{2/3})$. Thus, with probability at least $1 - \mathrm{e}^{-n^{1/4}}$, (F1) holds.

Let $S, S' \in \mathcal{S}$ be distinct. If $|U'_S \cap U'_{S'}| \leq \rho^2 n$, then we surely have $|U_S \cap U_{S'}| \leq \rho^2 n$, so assume that $|U'_S \cap U'_{S'}| \geq \rho^2 n$. Lemma 2.5.10(i) implies that with probability at least $1 - 2\mathrm{e}^{-2\rho^4 |U'_S \cap U'_{S'}|}$, we have $|U_S \cap U_{S'}| \leq \mathbb{E}(|U_S \cap U_{S'}|) + \rho^2 |U'_S \cap U'_{S'}| \leq 2\rho^2 n$. Thus, with probability at least $1 - \mathrm{e}^{-n^{1/2}}$, (F2) holds.

Step 2: Probability estimates for (F3)

Now, fix $S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and an $(z_1 + z_2 - 1)$-set $b \subseteq V \setminus S$. In order to estimate $|\mathcal{J}^b_{S,z,1}|$ and $|\mathcal{J}^b_{S,z,2}|$, define

$$\mathcal{J}' := \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, |b \cap S'| = z_1\},$$
$$\mathcal{J}'' := \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, |b \cap S'| = z_1 - 1\}.$$

Clearly, $\mathcal{J}^b_{S,z,1} \subseteq \mathcal{J}'$ and $\mathcal{J}^b_{S,z,2} \subseteq \mathcal{J}''$. Moreover, since $b \cap S = \emptyset$, we have that

$$|\mathcal{J}'| \leq \binom{i}{z_0}\binom{z_1 + z_2 - 1}{z_1} n^{i - z_0 - z_1} \leq 2^{2r} n^{i - z_0 - z_1},$$
$$|\mathcal{J}''| \leq \binom{i}{z_0}\binom{z_1 + z_2 - 1}{z_1 - 1} n^{i - z_0 - z_1 + 1} \leq 2^{2r} n^{i - z_0 - z_1 + 1}.$$

Consider $S' \in \mathcal{J}'$. By the random choice of $U_{S'}$ and since $b \cap S = \emptyset$, we have that

$$\mathbb{P}(S' \in \mathcal{J}^b_{S,z,1}) = \mathbb{P}(b \setminus S' \subseteq U_{S'}, |U_{S'} \cap S| \geq z_3) = \mathbb{P}(b \setminus S' \subseteq U_{S'}) \cdot \mathbb{P}(|U_{S'} \cap S| \geq z_3).$$

Note that $\mathbb{P}(b \setminus S' \subseteq U_{S'}) \leq \rho^{z_2 - 1}$ since $|b \setminus S'| = z_2 - 1$. Moreover, $\mathbb{P}(|U_{S'} \cap S| \geq z_3) \leq \binom{i}{z_3}\rho^{z_3} \leq 2^i \rho^{z_3}$.

Hence, $7\mathbb{E}|\mathcal{J}^b_{S,z,1}| \leq 2^3 2^i \rho^{z_2 + z_3 - 1} 2^{2r} n^{i - z_0 - z_1}$. Since $i - z_0 - z_1 \geq 1$ and $U_{S'}$ and $U_{S''}$ are chosen independently for any two distinct $S', S'' \in \mathcal{J}'$, Lemma 2.5.10(iii) implies that

$$\mathbb{P}(|\mathcal{J}^b_{S,z,1}| \geq 2^{6r} \rho^{z_2 + z_3 - 1} n^{i - z_0 - z_1}) \leq \mathrm{e}^{-2^{6r} \rho^{z_2 + z_3 - 1} n^{i - z_0 - z_1}} \leq \mathrm{e}^{-\sqrt{n}}. \tag{2.10.1}$$

Now, consider $S' \in \mathcal{J}''$. By the random choice of U_S and $U_{S'}$, we have that

$$\begin{aligned}\mathbb{P}(S' \in \mathcal{J}^b_{S,z,2}) &= \mathbb{P}(b \setminus S' \subseteq U_{S'}, |U_{S'} \cap S| \geq z_3, |U_S \cap (S' \setminus b)| \geq 1)\\ &= \mathbb{P}(b \setminus S' \subseteq U_{S'}) \cdot \mathbb{P}(|U_{S'} \cap S| \geq z_3) \cdot \mathbb{P}(|U_S \cap (S' \setminus b)| \geq 1)\\ &\leq \rho^{z_2} \cdot \binom{i}{z_3} \rho^{z_3} \cdot (i - z_1 + 1)\rho \leq r2^r \rho^{z_2+z_3+1}.\end{aligned}$$

However, note that the events $S' \in \mathcal{J}^b_{S,z,2}$ and $S'' \in \mathcal{J}^b_{S,z,2}$ are not necessarily independent. To deal with this, define the auxiliary $(i - z_0 - z_1 + 1)$-graph A on V with edge set $\{S' \setminus (S \cup b) : S' \in \mathcal{J}''\}$ and let A' be the (random) subgraph with edge set $\{S' \setminus (S \cup b) : S' \in \mathcal{J}^b_{S,z,2}\}$. Note that for every edge $e \in A$, there are at most $\binom{i}{z_0}\binom{z_1+z_2-1}{z_1-1} \leq 2^{2r}$ elements $S' \in \mathcal{J}''$ with $e = S' \setminus (S \cup b)$. Hence, $|\mathcal{J}^b_{S,z,2}| \leq 2^{2r}|A'|$. Moreover, every edge of A survives (i.e. lies in A') with probability at most $2^{2r} \cdot r2^r \rho^{z_2+z_3+1}$, and for every matching M in A, the edges of M survive independently. Thus, by Lemma 2.5.15, we have that

$$\mathbb{P}(|A'| \geq 7r2^{3r}\rho^{z_2+z_3+1}n^{i-z_0-z_1+1}) \leq (i - z_0 - z_1 + 1)n^{i-z_0-z_1}\mathrm{e}^{-7 \cdot 2^{3r}\rho^{z_2+z_3+1}n}$$

and thus

$$\mathbb{P}(|\mathcal{J}^b_{S,z,2}| \geq 7r2^{5r}\rho^{z_2+z_3+1}n^{i-z_0-z_1+1}) \leq rn^r\mathrm{e}^{-7 \cdot 2^{3r}\rho^{z_2+z_3+1}n} \leq \mathrm{e}^{-\sqrt{n}}. \tag{2.10.2}$$

Since $|\mathcal{S}| \leq n^i$, a union bound applied to (2.10.1) and (2.10.2) shows that with probability at least $1 - \mathrm{e}^{-n^{1/3}}$, (F3) holds. □

The following 'Localised cover down lemma' allows us to simultaneously cover all $\mathcal{S}$-important edges of an i-system $\mathcal{S}$ provided that the associated focus $\mathcal{U}$ satisfies (F1)–(F3) and all $\mathcal{S}$-important edges are 'localised' in the sense that their links are contained in the respective focus set (or, equivalently, their type is maximal).

Lemma 2.10.8 (Localised cover down lemma)**.** *Let $1/n \ll \rho \ll \rho_{size}, \xi, 1/f$ and $1 \leq$*

$i<r<f$. Assume that $(*)_{r-i}$ is true. Let F be a weakly regular r-graph on f vertices and $S^* \in \binom{V(F)}{i}$ such that $F(S^*)$ is non-empty. Let G be a complex on n vertices and let $\mathcal{S}=\{S_1,\dots,S_p\}$ be an i-system in G such that G is r-exclusive with respect to $\mathcal{S}$. Let $\mathcal{U}=\{U_1,\dots,U_p\}$ be a (ρ_{size},ρ,r)-focus for $\mathcal{S}$. Suppose further that whenever $S_j \subseteq e \in G^{(r)}$, we have $e \setminus S_j \subseteq U_j$. Finally, assume that $G(S_j)[U_j]$ is an $F(S^*)$-divisible $(\rho,\xi,f-i,r-i)$-supercomplex for all $j\in[p]$.

Then there exists a $\rho^{-1/12}$-well separated F-packing $\mathcal{F}$ in G covering all $\mathcal{S}$-important r-edges.

Proof. Recall that by Proposition 2.5.3, $F(S^*)$ is a weakly regular $(r-i)$-graph. We will use $(*)_{r-i}$ together with Corollary 2.4.15 in order to find many $F(S^*)$-decompositions of $G(S_j)[U_j]$ and then pick one of these at random. Let $t:=\rho^{1/6}(0.5\rho\rho_{size}n)^{f-r}$ and $\kappa:=\rho^{-1/12}$. For all $j\in[p]$, define $G_j:=G(S_j)[U_j]$. Consider Algorithm 2.10.9 which, if successful, outputs a κ-well separated $F(S^*)$-decomposition $\mathcal{F}_j$ of G_j for every $j\in[p]$.

Algorithm 2.10.9

> **for** j from 1 to p **do**
> for all $z=(z_0,z_1,z_2,z_3)\in\mathcal{Z}_{r,i}$, define T^j_z as the (z_1+z_2)-graph on U_j containing all $Z_1 \mathbin{\dot\cup} Z_2 \subseteq U_j$ with $|Z_1|=z_1, |Z_2|=z_2$ such that for some $j'\in[j-1]$ with $|S_j\cap S_{j'}|=z_0$ and some $K'\in\mathcal{F}_{j'}^{\le(f-i)}$, we have $Z_1\subseteq S_{j'}$, $Z_2\subseteq K'$ and $|K'\cap S_j|=z_3$
> **if** there exist κ-well separated $F(S^*)$-decompositions $\mathcal{F}_{j,1},\dots,\mathcal{F}_{j,t}$ of $G_j-\bigcup_{z\in\mathcal{Z}_{r,i}}T^j_z$ which are pairwise $(f-i)$-disjoint **then**
> pick $s\in[t]$ uniformly at random and let $\mathcal{F}_j:=\mathcal{F}_{j,s}$
> **else**
> **return** 'unsuccessful'
> **end if**
> **end for**

Claim 1: If Algorithm 2.10.9 outputs $\mathcal{F}_1,\dots,\mathcal{F}_p$, then $\mathcal{F}:=\bigcup_{j\in[p]}\tilde{\mathcal{F}}_j$ is a packing as desired, where $\tilde{\mathcal{F}}_j:=S_j\triangleleft\mathcal{F}_j$.

Proof of claim: Since $z_1+z_2>r-i$, we have $G_j^{(r-i)}=(G_j-\bigcup_{z\in\mathcal{Z}_{r,i}}T^j_z)^{(r-i)}$. Hence, $\mathcal{F}_j$ is indeed an $F(S^*)$-decomposition of G_j. Thus, by Proposition 2.7.9, $\tilde{\mathcal{F}}_j$ is a κ-well separated F-packing in G covering all r-edges containing S_j. Therefore, $\mathcal{F}$ covers all $\mathcal{S}$-important r-edges of G. By Fact 2.5.4(iii) it suffices to show that $\tilde{\mathcal{F}}_1,\dots,\tilde{\mathcal{F}}_p$ are r-disjoint.

To this end, let $j' < j$ and suppose, for a contradiction, that there exist $\tilde{K} \in \tilde{\mathcal{F}}_j^{\leq(f)}$ and $\tilde{K}' \in \tilde{\mathcal{F}}_{j'}^{\leq(f)}$ such that $|\tilde{K} \cap \tilde{K}'| \geq r$. Let $K := \tilde{K} \setminus S_j$ and $K' := \tilde{K}' \setminus S_{j'}$. Then $K \in \mathcal{F}_j^{\leq(f-i)}$ and $K' \in \mathcal{F}_{j'}^{\leq(f-i)}$ and $|(S_j \cup K) \cap (S_{j'} \cup K')| \geq r$. Let $z_0 := |S_j \cap S_{j'}|$ and $z_3 := |S_j \cap K'|$. Hence, we have $|K \cap (S_{j'} \cup K')| \geq r - z_0 - z_3$. Choose $X \subseteq K$ such that $|X \cap (S_{j'} \cup K')| = r - z_0 - z_3$ and let $Z_1 := X \cap S_{j'}$ and $Z_2 := X \cap K'$. We claim that $z := (z_0, |Z_1|, |Z_2|, z_3) \in \mathcal{Z}_{r,i}$. Clearly, we have $z_0 + |Z_1| + |Z_2| + z_3 = r$. Furthermore, note that $z_0 + z_3 < i$. Indeed, we clearly have $z_0 + z_3 = |S_j \cap (S_{j'} \cup K')| \leq |S_j| = i$, and equality can only hold if $S_j \subseteq S_{j'} \cup K' = \tilde{K}'$, which is impossible since G is r-exclusive. Similarly, we have $z_0 + |Z_1| < i$. Thus, $z \in \mathcal{Z}_{r,i}$. But this implies that $Z_1 \cup Z_2 \in T_z^j$, in contradiction to $Z_1 \cup Z_2 \subseteq K$. $-$

In order to prove the lemma, it is thus sufficient to prove that with positive probability, $\Delta(T_z^j) \leq 2^{2r} f \kappa \rho^{1/2} |U_j|$ for all $j \in [p]$ and $z \in \mathcal{Z}_{r,i}$. Indeed, this would imply that $\Delta(\bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j) \leq (r+1)^3 2^{2r} f \rho^{1/2-1/12} |U_j|$, and by Proposition 2.5.9(v), $G_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j$ would be a $(\rho^{1/12}, \xi/2, f-i, r-i)$-supercomplex. By Corollary 2.4.15 and since $|U_j| \geq 0.5\rho\rho_{size}n$, the number of pairwise $(f-i)$-disjoint κ-well separated $F(S^*)$-decompositions in $G_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j$ is at least $\rho^{2/12}|U_j|^{(f-i)-(r-i)} \geq t$, so the algorithm would succeed.

In order to analyse $\Delta(T_z^j)$, we define the following variables. Suppose that $1 \leq j' < j \leq p$, that $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $b \subseteq U_j$ is a $(z_1 + z_2 - 1)$-set. Let $Y_{j,z}^{b,j'}$ denote the random indicator variable of the event that each of the following holds:

(a) there exists some $K' \in \mathcal{F}_{j'}^{\leq(f-i)}$ with $|K' \cap S_j| = z_3$;

(b) there exist $Z_1 \subseteq S_{j'}$, $Z_2 \subseteq K'$ with $|Z_1| = z_1$, $|Z_2| = z_2$ such that $b \subsetneq Z_1 \cup Z_2 \subseteq U_j$;

(c) $|S_j \cap S_{j'}| = z_0$.

We say that $v \in \binom{U_j \setminus b}{1}$ is a *witness for* j' if (a)–(c) hold with $Z_1 \cup Z_2 = b \cup v$. For all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $b \subseteq U_j$, let $X_{j,z}^b := \sum_{j'=1}^{j-1} Y_{j,z}^{b,j'}$.

Claim 2: For all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $b \subseteq U_j$, we have $|T_z^j(b)| \leq 2^{2r} f \kappa X_{j,z}^b$.

Proof of claim: Let j, z and b be fixed. Clearly, if $v \in T_z^j(b)$, then by Algorithm 2.10.9, v is a witness for some $j' \in [j-1]$. Conversely, we claim that for each $j' \in [j-1]$, there are at most $2^{2r} f\kappa$ witnesses for j'. Clearly, this would imply that $|T_z^j(b)| \leq 2^{2r} f\kappa |\{j' \in [j-1] : Y_{j,z}^{b,j'} = 1\}| = 2^{2r} f\kappa X_{j,z}^b$.

Fix $j' \in [j-1]$. If v is a witness for j', then there exists $K_v \in \mathcal{F}_{j'}^{\leq(f-i)}$ such that (a)–(c) hold with $Z_1 \cupdot Z_2 = b \cupdot v$ and K_v playing the role of K'. By (b) we must have $v \subseteq Z_1 \cup Z_2 \subseteq S_{j'} \cup K_v$. Since $|S_{j'} \cup K_v| = f$, there are at most f witnesses v' for j' such that K_v can play the role of $K_{v'}$. It is thus sufficient to show that there are at most $2^{2r}\kappa$ $K' \in \mathcal{F}_{j'}^{\leq(f-i)}$ such that (a)–(c) hold.

Note that for any possible choice of Z_1, Z_2, K', we must have $|b \cap Z_2| \in \{z_2, z_2 - 1\}$ and $b \cap Z_2 \subseteq Z_2 \subseteq K'$ by (b). For any $Z_2' \subseteq b$ with $|Z_2'| \in \{z_2, z_2-1\}$ and any $Z_3 \in \binom{S_j}{z_3}$, there can be at most κ $K' \in \mathcal{F}_{j'}^{\leq(f-i)}$ with $Z_2' \subseteq K'$ and $K' \cap S_j = Z_3$. This is because $\mathcal{F}_{j'}$ is a κ-well separated $F(S^*)$-decomposition and $|Z_2' \cup Z_3| \geq z_2 - 1 + z_3 \geq r - i$. Hence, there can be at most $2^{|b|}\binom{i}{z_3}\kappa \leq 2^{2r}\kappa$ possible choices for K'. –

The following claim thus implies the lemma.

Claim 3: With positive probability, we have $X_{j,z}^b \leq \rho^{1/2}|U_j|$ for all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $b \subseteq U_j$.

Proof of claim: Fix j, z, b as above. We split $X_{j,z}^b$ into two sums. For this, let

$$\begin{aligned} \mathcal{J}_{j,z}^b &:= \{j' \in [j-1] : |S_j \cap S_{j'}| = z_0, b \setminus S_{j'} \subseteq U_{j'}, |U_{j'} \cap S_j| \geq z_3\}, \\ \mathcal{J}_{j,z,1}^b &:= \{j' \in \mathcal{J}_{j,z}^b : |b \cap S_{j'}| = z_1\}, \\ \mathcal{J}_{j,z,2}^b &:= \{j' \in \mathcal{J}_{j,z}^b : |b \cap S_{j'}| = z_1 - 1, |U_j \cap (S_{j'} \setminus b)| \geq 1\}. \end{aligned}$$

Since $\mathcal{U}$ is a (ρ_{size}, ρ, r)-focus for $\mathcal{S}$, (F3) implies that

$$|\mathcal{J}_{j,z,1}^b| \leq 2^{6r} \rho^{z_2+z_3-1} n^{i-z_0-z_1}, \tag{2.10.3}$$

$$|\mathcal{J}_{j,z,2}^b| \leq 2^{9r} \rho^{z_2+z_3+1} n^{i-z_0-z_1+1}. \tag{2.10.4}$$

Note that if $Y_{j,z}^{b,j'} = 1$, then $j' \in \mathcal{J}_{j,z,1}^b \cup \mathcal{J}_{j,z,2}^b$. Hence, we have $X_{j,z}^b = X_{j,z,1}^b + X_{j,z,2}^b$, where $X_{j,z,1}^b := \sum_{j' \in \mathcal{J}_{j,z,1}^b} Y_{j,z}^{b,j'}$ and $X_{j,z,2}^b := \sum_{j' \in \mathcal{J}_{j,z,2}^b} Y_{j,z}^{b,j'}$. We will bound $X_{j,z,1}^b$ and $X_{j,z,2}^b$ separately.

For $j' \in \mathcal{J}_{j,z,1}^b \cup \mathcal{J}_{j,z,2}^b$, define

$$\mathcal{K}_{j,z}^{b,j'} := \{K' \in \binom{U_{j'}}{f-i} : b \subseteq S_{j'} \cup K', |K' \cap U_j| \geq z_2, |K' \cap S_j| = z_3\}. \tag{2.10.5}$$

Note that if $Y_{j,z}^{b,j'} = 1$, then $\mathcal{F}_{j',k}^{\leq (f-i)} \cap \mathcal{K}_{j,z}^{b,j'} \neq \emptyset$. Recall that the candidates $\mathcal{F}_{j',1}, \dots, \mathcal{F}_{j',t}$ in Algorithm 2.10.9 from which $\mathcal{F}_{j'}$ was chosen at random are $(f-i)$-disjoint. We thus have

$$\mathbb{P}(Y_{j,z}^{b,j'} = 1) \leq \frac{|\{k \in [t] : \mathcal{F}_{j',k}^{\leq (f-i)} \cap \mathcal{K}_{j,z}^{b,j'} \neq \emptyset\}|}{t} \leq \frac{|\mathcal{K}_{j,z}^{b,j'}|}{t}.$$

This upper bound still holds if we condition on variables $Y_{j,z}^{b,j''}$, $j'' \neq j'$. We thus need to bound $|\mathcal{K}_{j,z}^{b,j'}|$ in order to bound $X_{j,z,1}^b$ and $X_{j,z,2}^b$.

Step 1: Estimating $X_{j,z,1}^b$

Consider $j' \in \mathcal{J}_{j,z,1}^b$. For all $K' \in \mathcal{K}_{j,z}^{b,j'}$, we have $b \backslash S_{j'} \subseteq K'$ and $|b \cap K'| = |b| - |b \cap S_{j'}| = z_2 - 1$, and the sets $b \cap K'$, $K' \cap S_j$, $(K' \setminus b) \cap (U_j \cap U_{j'})$ are disjoint. Moreover, we have $|(K' \setminus b) \cap (U_j \cap U_{j'})| = |(K' \setminus b) \cap U_j| \geq |K' \cap U_j| - |b \cap K'| \geq 1$. We can thus count

$$|\mathcal{K}_{j,z}^{b,j'}| \leq \binom{|S_j|}{z_3} \cdot |U_j \cap U_{j'}| \cdot |U_{j'}|^{f-i-(z_2-1)-1-z_3} \leq 2^i \cdot 2\rho^2 n \cdot (2\rho\rho_{size} n)^{f-i-z_2-z_3}.$$

Let $\tilde{\rho}_1 := \rho^{z_0+z_1-i+5/3} \rho_{size} n^{1+z_0+z_1-i} \in [0,1]$. In order to apply Proposition 2.5.11, let $j_1, \dots, j_m$ be an enumeration of $\mathcal{J}_{j,z,1}^b$. We then have for all $k \in [m]$ and all $y_1, \dots, y_{k-1} \in \{0,1\}$ that

$$\begin{aligned}\mathbb{P}(Y_{j,z}^{b,j_k} = 1 \mid Y_{j,z}^{b,j_1} = y_1, \dots, Y_{j,z}^{b,j_{k-1}} = y_{k-1}) &\leq \frac{|\mathcal{K}_{j,z}^{b,j_k}|}{t} \leq \frac{2^i \cdot 2\rho^2 n \cdot (2\rho\rho_{size} n)^{f-i-z_2-z_3}}{\rho^{1/6}(0.5\rho\rho_{size} n)^{f-r}} \\ &= 2^{2f-r+1-z_2-z_3} \rho^{11/6} (\rho\rho_{size})^{z_0+z_1-i} n^{1+z_0+z_1-i} \\ &\leq \tilde{\rho}_1.\end{aligned}$$

Let $B_1 \sim Bin(|\mathcal{J}^b_{j,z,1}|, \tilde{\rho}_1)$ and observe that

$$7\mathbb{E}B_1 = 7|\mathcal{J}^b_{j,z,1}|\tilde{\rho}_1 \overset{(2.10.3)}{\leq} 7 \cdot 2^{6r} \rho^{z_2+z_3-1} n^{i-z_0-z_1} \cdot \rho^{z_0+z_1-i+5/3} \rho_{size} n^{1+z_0+z_1-i}$$
$$= 7 \cdot 2^{6r} \rho^{r-i+2/3} \rho_{size} n \leq 0.5\rho^{1/2}|U_j|.$$

Thus,

$$\mathbb{P}(X^b_{j,z,1} \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Proposition 2.5.11}}{\leq} \mathbb{P}(B_1 \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Lemma 2.5.10(iii)}}{\leq} \mathrm{e}^{-0.5\rho^{1/2}|U_j|}.$$

Step 2: Estimating $X^b_{j,z,2}$

Consider $j' \in \mathcal{J}^b_{j,z,2}$. This time, since $|b \cap S_{j'}| = z_1 - 1$, we have $|K' \cap b| = |b \setminus S_{j'}| = z_2$ for all $K' \in \mathcal{K}^{b,j'}_{j,z}$. Thus, we count

$$|\mathcal{K}^{b,j'}_{j,z}| \leq \binom{|S_j|}{z_3} \cdot |U_{j'}|^{f-i-z_2-z_3} \leq 2^i \cdot (2\rho\rho_{size}n)^{f-i-z_2-z_3}.$$

Let $\tilde{\rho}_2 := \rho^{z_0+z_1-i-1/5}\rho_{size}n^{z_0+z_1-i} \in [0,1]$. In order to apply Proposition 2.5.11, let $j_1, \ldots, j_m$ be an enumeration of $\mathcal{J}^f_{j,z,2}$. We then have for all $k \in [m]$ and all $y_1, \ldots, y_{k-1} \in \{0,1\}$ that

$$\mathbb{P}(Y^{b,j_k}_{j,z} = 1 \mid Y^{b,j_1}_{j,z} = y_1, \ldots, Y^{b,j_{k-1}}_{j,z} = y_{k-1}) \leq \frac{|\mathcal{K}^{b,j_k}_{j,z}|}{t} \leq \frac{2^i \cdot (2\rho\rho_{size}n)^{f-i-z_2-z_3}}{\rho^{1/6}(0.5\rho\rho_{size}n)^{f-r}}$$
$$= 2^{2f-r-z_2-z_3}\rho^{-1/6}(\rho\rho_{size}n)^{z_0+z_1-i}$$
$$\leq \tilde{\rho}_2.$$

Let $B_2 \sim Bin(|\mathcal{J}^b_{j,z,2}|, \tilde{\rho}_2)$ and observe that

$$7\mathbb{E}B_2 = 7|\mathcal{J}^b_{j,z,2}|\tilde{\rho}_2 \overset{(2.10.4)}{\leq} 7 \cdot 2^{9r} \rho^{z_2+z_3+1} n^{i-z_0-z_1+1} \cdot \rho^{z_0+z_1-i-1/5} \rho_{size} n^{z_0+z_1-i}$$
$$= 7 \cdot 2^{9r} \rho^{r-i+4/5} \rho_{size} n \leq 0.5\rho^{1/2}|U_j|.$$

Thus,

$$\mathbb{P}(X^b_{j,z,2} \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Proposition 2.5.11}}{\leq} \mathbb{P}(B_2 \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Lemma 2.5.10(iii)}}{\leq} \mathrm{e}^{-0.5\rho^{1/2}|U_j|}.$$

Hence,

$$\mathbb{P}(X^b_{j,z} \geq \rho^{1/2}|U_j|) \leq \mathbb{P}(X^b_{j,z,1} \geq 0.5\rho^{1/2}|U_j|) + \mathbb{P}(X^b_{j,z,2} \geq 0.5\rho^{1/2}|U_j|) \leq 2\mathrm{e}^{-0.5\rho^{1/2}|U_j|}.$$

Since $p = |\mathcal{S}| \leq n^i$, a union bound easily implies Claim 3. —

This completes the proof of Lemma 2.10.8. □

2.10.2 Partition pairs

We now develop the appropriate framework to be able to state the Cover down lemma for setups (Lemma 2.10.24). Recall that we will consider (and cover) r-sets separately according to their type. The type of an r-set e naturally imposes constraints on the type of an f-set which covers e. We will need to track and adjust the densities of r-sets with respect to f-sets for each pair of types separately. This gives rise to the following concepts of partition pairs and partition regularity (see Section 2.10.3). We will sometimes refer to r-sets as 'edges' and to f-sets as 'cliques'.

Let X be a set. We say that $\mathcal{P} = (X_1, \dots, X_a)$ is an ordered partition of X if the X_i are disjoint subsets of X whose union is X. We let $\mathcal{P}(i) := X_i$ and $\mathcal{P}([i]) := (X_1, \dots, X_i)$. If $\mathcal{P} = (X_1, \dots, X_a)$ is an ordered partition of X and $X' \subseteq X$, we let $\mathcal{P}[X']$ denote the ordered partition $(X_1 \cap X', \dots, X_a \cap X')$ of X'. If $\{X', X''\}$ is a partition of X, $\mathcal{P}' = (X'_1, \dots, X'_a)$ is an ordered partition of X' and $\mathcal{P}'' = (X''_1, \dots, X''_b)$ is an ordered partition of X'', we let

$$\mathcal{P}' \sqcup \mathcal{P}'' := (X'_1, \dots, X'_a, X''_1, \dots, X''_b).$$

Definition 2.10.10. Let G be a complex and let $f > r \geq 1$. An (r, f)*-partition pair of* G is a pair $(\mathcal{P}_r, \mathcal{P}_f)$, where $\mathcal{P}_r$ is an ordered partition of $G^{(r)}$ and $\mathcal{P}_f$ is an ordered partition of $G^{(f)}$, such that for all $\mathcal{E} \in \mathcal{P}_r$ and $\mathcal{Q} \in \mathcal{P}_f$, every $Q \in \mathcal{Q}$ contains the same number $C(\mathcal{E}, \mathcal{Q})$ of elements from $\mathcal{E}$. We call $C\colon \mathcal{P}_r \times \mathcal{P}_f \to [\binom{f}{r}]_0$ the *containment function* of the partition pair. We say that $(\mathcal{P}_r, \mathcal{P}_f)$ is *upper-triangular* if $C(\mathcal{P}_r(\ell), \mathcal{P}_f(k)) = 0$ whenever $\ell > k$.

Clearly, for every $\mathcal{Q} \in \mathcal{P}_f$, $\sum_{\mathcal{E}\in\mathcal{P}_r} C(\mathcal{E}, \mathcal{Q}) = \binom{f}{r}$. If $(\mathcal{P}_r, \mathcal{P}_f)$ is an (r, f)-partition pair of G and $G' \subseteq G$ is a subcomplex, we define

$$(\mathcal{P}_r, \mathcal{P}_f)[G'] := (\mathcal{P}_r[G'^{(r)}], \mathcal{P}_f[G'^{(f)}]).$$

Clearly, $(\mathcal{P}_r, \mathcal{P}_f)[G']$ is an (r, f)-partition pair of G'.

Example 2.10.11. Suppose that G is a complex and $U \subseteq V(G)$. For $\ell \in [r]_0$, define $\mathcal{E}_\ell := \{e \in G^{(r)} : |e \cap U| = \ell\}$. For $k \in [f]_0$, define $\mathcal{Q}_k := \{Q \in G^{(f)} : |Q \cap U| = k\}$. Let $\mathcal{P}_r := (\mathcal{E}_0, \dots, \mathcal{E}_r)$ and $\mathcal{P}_f := (\mathcal{Q}_0, \dots, \mathcal{Q}_f)$. Then clearly $(\mathcal{P}_r, \mathcal{P}_f)$ is an (r, f)-partition pair of G, where the containment function is given by $C(\mathcal{E}_\ell, \mathcal{Q}_k) = \binom{k}{\ell}\binom{f-k}{r-\ell}$. In particular, $C(\mathcal{E}_\ell, \mathcal{Q}_k) = 0$ whenever $\ell > k$ or $k > f - r + \ell$. We say that $(\mathcal{P}_r, \mathcal{P}_f)$ is the (r, f)*-partition pair of* G, U.

The partition pairs we use are generalisations of the above example. More precisely, suppose that G is a complex, $\mathcal{S}$ is an i-system in $V(G)$ and $\mathcal{U}$ is a focus for $\mathcal{S}$. Moreover, assume that G is r-exclusive with respect to $\mathcal{S}$. For $r' \geq r$, let $\tau_{r'}$ denote the type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$. As in the above example, if $\mathcal{E}_\ell := \tau_r^{-1}(\ell)$ for all $\ell \in [r - i]_0$ and $\mathcal{Q}_k := \tau_f^{-1}(k)$ for all $k \in [f - i]_0$, then every $Q \in \mathcal{Q}_k$ contains exactly $\binom{k}{\ell}\binom{f-i-k}{r-i-\ell}$ elements from $\mathcal{E}_\ell$. However, we also have to consider $\mathcal{S}$-unimportant edges and cliques. It turns out that it is useful to assume that the unimportant edges and cliques are partitioned into i parts each, in an upper-triangular fashion.

More formally, for $r' \geq r$, let $\mathcal{D}_{r'}$ denote the set of $\mathcal{S}$-unimportant r'-sets of G and assume that $\mathcal{P}^*_r$ is an ordered partition of $\mathcal{D}_r$ and $\mathcal{P}^*_f$ is an ordered partition of $\mathcal{D}_f$. We say that $(\mathcal{P}_r^*, \mathcal{P}_f^*)$ is *admissible with respect to* G, $\mathcal{S}$, $\mathcal{U}$ if the following hold:

(P1) $|\mathcal{P}_r^*| = |\mathcal{P}_f^*| = i$;

(P2) for all $S \in \mathcal{S}$, $h \in [r-i]_0$ and $B \subseteq G(S)^{(h)}$ with $1 \leq |B| \leq 2^h$ and all $\ell \in [i]$, there exists $D(S, B, \ell) \in \mathbb{N}_0$ such that for all $Q \in \bigcap_{b \in B} G(S \cup b)[U_S]^{(f-i-h)}$, we have that

$$|\{e \in \mathcal{P}_r^*(\ell) \,:\, \exists b \in B \colon e \subseteq S \cup b \cup Q\}| = D(S, B, \ell);$$

(P3) $(\mathcal{P}_r^* \sqcup \{G^{(r)} \setminus \mathcal{D}_r\}, \mathcal{P}_f^* \sqcup \{G^{(f)} \setminus \mathcal{D}_f\})$ is an upper-triangular (r,f)-partition pair of G.

Note that for $i = 0$, $\mathcal{S} = \{\emptyset\}$ and $\mathcal{U} = \{U\}$ for some $U \subseteq V(G)$, the pair $(\emptyset, \emptyset)$ trivially satisfies these conditions. Also note that (P2) can be viewed as an analogue of the containment function (from Definition 2.10.10) which is suitable for dealing with supercomplexes.

Assume that $(\mathcal{P}_r^*, \mathcal{P}_f^*)$ is admissible with respect to G, $\mathcal{S}$, $\mathcal{U}$. Define

$$\mathcal{P}_r := \mathcal{P}_r^* \sqcup (\tau_r^{-1}(0), \ldots, \tau_r^{-1}(r-i)),$$
$$\mathcal{P}_f := \mathcal{P}_f^* \sqcup (\tau_f^{-1}(0), \ldots, \tau_f^{-1}(f-i)).$$

It is not too hard to see that $(\mathcal{P}_r, \mathcal{P}_f)$ is an (r,f)-partition pair of G. Indeed, $\mathcal{P}_r$ clearly is a partition of $G^{(r)}$ and $\mathcal{P}_f$ is a partition of $G^{(f)}$. Suppose that C is the containment function of $(\mathcal{P}_r^* \sqcup \{G^{(r)} \setminus \mathcal{D}_r\}, \mathcal{P}_f^* \sqcup \{G^{(f)} \setminus \mathcal{D}_f\})$. Then C' as defined below is the containment function of $(\mathcal{P}_r, \mathcal{P}_f)$:

- For all $\mathcal{E} \in \mathcal{P}_r^*$ and $\mathcal{Q} \in \mathcal{P}_f^*$, let $C'(\mathcal{E}, \mathcal{Q}) := C(\mathcal{E}, \mathcal{Q})$.
- For all $\ell \in [r-i]_0$ and $\mathcal{Q} \in \mathcal{P}_f^*$, let $C'(\tau_r^{-1}(\ell), \mathcal{Q}) := 0$.
- For all $\mathcal{E} \in \mathcal{P}_r^*$ and $k \in [f-i]_0$, define $C'(\mathcal{E}, \tau_f^{-1}(k)) := C(\mathcal{E}, \{G^{(f)} \setminus \mathcal{D}_f\})$.

	$\mathcal{P}_f^*(1)$	$\dots$	$\mathcal{P}_f^*(i)$	$\tau_f^{-1}(0)$	$\tau_f^{-1}(1)$	$\dots$	$\dots$	$\tau_f^{-1}(f-r)$	$\dots$	$\dots$	$\tau_f^{-1}(f-i)$
$\mathcal{P}_r^*(1)$	*										
$\dots$	0	*									
$\mathcal{P}_r^*(i)$	0	0	*								
$\tau_r^{-1}(0)$	0	0	0	*				*	0	0	0
$\dots$	0	0	0	0	*				*	0	0
$\dots$	0	0	0	0	0	*				*	0
$\tau_r^{-1}(r-i)$	0	0	0	0	0	0	*				*

Figure 2.1: The above table sketches the containment function of an (r, f)-partition pair induced by $(\mathcal{P}_r^*, \mathcal{P}_f^*)$ and $\mathcal{U}$. The cells marked with $*$ and the shaded subtable will play an important role later on.

- For all $\ell \in [r-i]_0$, $k \in [f-i]_0$, let

$$C'(\tau_r^{-1}(\ell), \tau_f^{-1}(k)) := \binom{k}{\ell}\binom{f-i-k}{r-i-\ell}. \tag{2.10.6}$$

We say that $(\mathcal{P}_r, \mathcal{P}_f)$ as defined above is *induced by* $(\mathcal{P}_r^*, \mathcal{P}_f^*)$ *and* $\mathcal{U}$.

Finally, we say that $(\mathcal{P}_r, \mathcal{P}_f)$ is an *(r, f)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$*, if

- $(\mathcal{P}_r([i]), \mathcal{P}_f([i]))$ is admissible with respect to G, $\mathcal{S}$, $\mathcal{U}$;
- $(\mathcal{P}_r, \mathcal{P}_f)$ is induced by $(\mathcal{P}_r([i]), \mathcal{P}_f([i]))$ and $\mathcal{U}$.

The next proposition summarises basic properties of an (r, f)-partition pair of $G, \mathcal{S}, \mathcal{U}$.

Proposition 2.10.12. *Let $0 \le i < r < f$ and suppose that G is a complex, $\mathcal{S}$ is an i-system in $V(G)$ and $\mathcal{U}$ is a focus for $\mathcal{S}$. Moreover, assume that G is r-exclusive with respect to $\mathcal{S}$. Let $(\mathcal{P}_r, \mathcal{P}_f)$ be an (r, f)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$ with containment function C. Then the following hold:*

(P1′) $|\mathcal{P}_r| = r+1$ *and* $|\mathcal{P}_f| = f+1$;

(P2′) *for* $i < \ell \le r+1$, $\mathcal{P}_r(\ell) = \tau_r^{-1}(\ell-i-1)$, *and for* $i < k \le f+1$, $\mathcal{P}_f(k) = \tau_f^{-1}(k-i-1)$;

(P3′) $(\mathcal{P}_r, \mathcal{P}_f)$ *is upper-triangular;*

(P4′) $C(\mathcal{P}_r(\ell), \mathcal{P}_f(k)) = 0$ *whenever both* $\ell > i$ *and* $k > f-r+\ell$;

(P5′) (P2) *holds for all* $\ell \in [r+1]$, *with* $\mathcal{P}_r$ *playing the role of* $\mathcal{P}_r^*$.

(P6′) *if* $i = 0$, $\mathcal{S} = \{\emptyset\}$ *and* $\mathcal{U} = \{U\}$ *for some* $U \subseteq V(G)$, *then the (unique)* (r, f)*-partition pair of* $G, \mathcal{S}, \mathcal{U}$ *is the* (r, f)*-partition pair of* G, U *(cf. Example 2.10.11);*

(P7′) *for every subcomplex* $G' \subseteq G$, $(\mathcal{P}_r, \mathcal{P}_f)[G']$ *is an* (r, f)*-partition pair of* $G', \mathcal{S}, \mathcal{U}$.

Proof. Clearly, (P1′), (P2′) and (P6′) hold, and it is also straightforward to check (P7′). Moreover, (P3′) holds because of (P3) and (2.10.6). The latter also implies (P4′).

Finally, consider (P5′). For $\ell \in [i]$, this holds since $(\mathcal{P}_r([i]), \mathcal{P}_f([i]))$ is admissible, so assume that $\ell > i$. We have $\mathcal{P}_r(\ell) = \tau_r^{-1}(\ell - i - 1)$. Let $S \in \mathcal{S}$, $h \in [r - i]_0$ and $B \subseteq G(S)^{(h)}$ with $1 \leq |B| \leq 2^h$.

For $Q \in \bigcap_{b \in B} G(S \cup b)[U_S]^{(f-i-h)}$, let

$$\mathcal{D}_Q := \{e \in G^{(r)} \,:\, S \subseteq e, |e \cap U_S| = \ell - i - 1, \exists b \in B \colon e \setminus S \subseteq b \cup Q\}.$$

It is easy to see that

$$\{e \in \mathcal{P}_r(\ell) \,:\, \exists b \in B \colon e \subseteq S \cup b \cup Q\} = \mathcal{D}_Q.$$

Note that for every $e \in \mathcal{D}_Q$, we have $e = S \cup (\bigcup B \cap e) \cup (Q \cap e)$.

It remains to show that for all $Q, Q' \in \bigcap_{b \in B} G(S \cup b)[U_S]^{(f-i-h)}$, we have $|\mathcal{D}_Q| = |\mathcal{D}_{Q'}|$. Let $\pi \colon Q \to Q'$ be any bijection. For each $e \in \mathcal{D}_Q$, define $\pi'(e) := S \cup (\bigcup B \cap e) \cup \pi(Q \cap e)$. It is straightforward to check that $\pi' \colon \mathcal{D}_Q \to \mathcal{D}_{Q'}$ is a bijection. □

2.10.3 Partition regularity

Definition 2.10.13. Let G be a complex on n vertices and $(\mathcal{P}_r, \mathcal{P}_f)$ an (r, f)-partition pair of G with $a := |\mathcal{P}_r|$ and $b := |\mathcal{P}_f|$. Let $A = (a_{\ell,k}) \in [0, 1]^{a \times b}$. We say that G is *(ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f)$* if for all $\ell \in [a]$, $k \in [b]$ and $e \in \mathcal{P}_r(\ell)$, we have

$$|(\mathcal{P}_f(k))(e)| = (a_{\ell,k} \pm \varepsilon) n^{f-r}, \tag{2.10.7}$$

where we view $\mathcal{P}_f(k)$ as a subgraph of $G^{(f)}$. If $\mathcal{E} \subseteq \mathcal{P}_r(\ell)$ and $\mathcal{Q} \subseteq \mathcal{P}_f(k)$, we will often write $A(\mathcal{E}, \mathcal{Q})$ instead of $a_{\ell,k}$.

For $A \in [0,1]^{a\times b}$ with $1 \leq t \leq a \leq b$, we define

- $\min^{\backslash}(A) := \min\{a_{j,j} : j \in [a]\}$ as the minimum value on the diagonal,
- $\min^{\backslash t}(A) := \min\{a_{j,j+b-a} : j \in \{a-t+1, \ldots, a\}\}$ and
- $\min^{\backslash\backslash t}(A) := \min\{\min^{\backslash}(A), \min^{\backslash t}(A)\}$.

Note that $\min^{\backslash\backslash r-i+1}(A)$ is the minimum value of the entries in A that correspond to the entries marked with $*$ in Figure 2.1.

Example 2.10.14. Suppose that G is a complex and that $U \subseteq V(G)$ is $(\varepsilon, \mu, \xi, f, r)$-random in G (see Definition 2.7.1). Let $(\mathcal{P}_r, \mathcal{P}_f)$ be the (r, f)-partition pair of G, U (cf. Example 2.10.11). Let $Y \subseteq G^{(f)}$ and $d \geq \xi$ be such that (R2) holds. Define the matrix $A \in [0,1]^{(r+1)\times(f+1)}$ as follows: for all $\ell \in [r+1]$ and $k \in [f+1]$, let

$$a_{\ell,k} := bin(f-r, \mu, k-\ell)d.$$

For all $\ell \in [r+1]$, $k \in [f+1]$ and $e \in \mathcal{P}_r(\ell) = \{e' \in G^{(r)} : |e' \cap U| = \ell - 1\}$, we have that

$$\begin{aligned} |(\mathcal{P}_f[Y](k))(e)| &= |\{Q \in G[Y]^{(f)}(e) : |(e \cup Q) \cap U| = k-1\}| \\ &= |\{Q \in G[Y]^{(f)}(e) : |Q \cap U| = k-\ell\}| \\ &\overset{(R2)}{=} (1 \pm \varepsilon) bin(f-r, \mu, k-\ell) d n^{f-r} = (a_{\ell,k} \pm \varepsilon) n^{f-r}. \end{aligned}$$

In other words, $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f[Y])$. Note also that $\min^{\backslash\backslash r+1}(A) = \min\{bin(f-r, \mu, 0), bin(f-r, \mu, f-r)\}d \geq (\min\{\mu, 1-\mu\})^{f-r}\xi$.

In the proof of the Cover down lemma for setups, we face (amongst others) the following two challenges: (i) given an (ε, A, f, r)-regular complex G for some suitable A, we need to find an efficient F-packing in G; (ii) if A is not suitable for (i), we need to find a 'representative' subcomplex G' of G which is (ε, A', f, r)-regular for some A' that is suitable for (i). The strategy to implement (i) is similar to that of the Boost lemma (Lemma 2.6.3): We randomly sparsify $G^{(f)}$ according to a suitably chosen (non-uniform) probability distribution in order to find $Y^* \subseteq G^{(f)}$ such that $G[Y^*]$ is (ε, d, f, r)-regular. We can then apply the Boosted nibble lemma (Lemma 2.6.4). The desired probability distribution arises from a non-negative solution to the equation $Ax = \mathbb{1}$. The following condition on A allows us to find such a solution (cf. Proposition 2.10.16).

Definition 2.10.15. We say that $A \in [0,1]^{a\times b}$ is *diagonal-dominant* if $a_{\ell,k} \leq a_{k,k}/2(a-\ell)$ for all $1 \leq \ell < k \leq \min\{a,b\}$.

Definition 2.10.15 also allows us to achieve (ii). Given some A, we can find a 'representative' subcomplex G' of G which is (ε, A', f, r)-regular for some A' that is diagonal-dominant (cf. Lemma 2.10.20).

Proposition 2.10.16. *Let $A \in [0,1]^{a\times b}$ be upper-triangular and diagonal-dominant with $a \leq b$. Then there exists $x \in [0,1]^b$ such that $x \geq \min^{\backslash}(A)/4b$ and $Ax = \min^{\backslash}(A)\mathbb{1}$.*

Proof. If $\min^{\backslash}(A) = 0$, we can take $x = 0$, so assume that $\min^{\backslash}(A) > 0$. For $k > a$, let $y_k := 1/4b$. For k from a down to 1, let $y_k := a_{k,k}^{-1}(1 - \sum_{j=k+1}^{b} a_{k,j}y_j)$. Since A is upper-triangular, we have $Ay = \mathbb{1}$. We claim that $1/4b \leq y_k \leq a_{k,k}^{-1}$ for all $k \in [b]$. This clearly holds for all $k > a$. Suppose that for some $k \in [a]$, we have already checked that $1/4b \leq y_j \leq a_{j,j}^{-1}$ for all $j > k$. We now check that

$$1 \geq 1 - \sum_{j=k+1}^{b} a_{k,j}y_j \geq 1 - \sum_{j=k+1}^{a} \frac{a_{j,j}}{2(a-k)}y_j - \frac{b-a}{4b} \geq \frac{3}{4} - \frac{a-k}{2(a-k)} = \frac{1}{4}$$

and so $1/4b \leq y_k \leq a_{k,k}^{-1}$. Thus we can take $x := \min^{\backslash}(A)y$. □

Lemma 2.10.17. *Let $1/n \ll \varepsilon \ll \xi, 1/f$ and $r \in [f-1]$. Suppose that G is a complex on n vertices and $(\mathcal{P}_r, \mathcal{P}_f)$ is an upper-triangular (r,f)-partition pair of G with $|\mathcal{P}_r| \leq |\mathcal{P}_f| \leq f+1$. Let $A \in [0,1]^{|\mathcal{P}_r|\times|\mathcal{P}_f|}$ be diagonal-dominant with $d := \min^{\backslash}(A) \geq \xi$. Suppose that*

G is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f)$ and $(\xi, f+r, r)$-dense. Then there exists $Y^* \subseteq G^{(f)}$ such that $G[Y^*]$ is $(2f\varepsilon, d, f, r)$-regular and $(0.9\xi(\xi/4(f+1))^{\binom{f+r}{f}}, f+r, r)$-dense.

Proof. Since $(\mathcal{P}_r, \mathcal{P}_f)$ is upper-triangular, we may assume that A is upper-triangular too. By Proposition 2.10.16, there exists a vector $x \in [0,1]^{|\mathcal{P}_f|}$ with $x \geq \min^{\setminus}(A)/4(f+1) \geq \xi/4(f+1)$ and $Ax = d\mathbb{1}$.

Obtain $Y^* \subseteq G^{(f)}$ randomly by including every $Q \in G^{(f)}$ that belongs to $\mathcal{P}_f(k)$ with probability x_k, all independently. Let $e \in \mathcal{P}_r(\ell)$ for any $\ell \in [|\mathcal{P}_r|]$. We have

$$\mathbb{E}|G[Y^*]^{(f)}(e)| = \sum_{k=1}^{|\mathcal{P}_f|} x_k(a_{\ell,k} \pm \varepsilon)n^{f-r} = (d \pm (f+1)\varepsilon)n^{f-r}.$$

Then, combining Lemma 2.5.10(ii) with a union bound, we conclude that whp $G[Y^*]$ is $(2f\varepsilon, d, f, r)$-regular.

Let $e \in G^{(r)}$. Since $|G^{(f+r)}(e)| \geq \xi n^f$ and every $Q \in G^{(f+r)}(e)$ belongs to $G[Y^*]^{(f+r)}(e)$ with probability at least $(\xi/4(f+1))^{\binom{f+r}{f}}$, we conclude with Corollary 2.5.14 that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, we have

$$|G[Y^*]^{(f+r)}(e)| \geq 0.9(\xi/4(f+1))^{\binom{f+r}{f}}|G^{(f+r)}(e)| \geq 0.9\xi(\xi/4(f+1))^{\binom{f+r}{f}}n^f.$$

Applying a union bound shows that whp $G[Y^*]$ is $(0.9\xi(\xi/4(f+1))^{\binom{f+r}{f}}, f+r, r)$-dense.

□

The following concept of a setup turns out to be the appropriate generalisation of Definition 2.7.1 to i-systems and partition pairs.

Definition 2.10.18 (Setup)**.** Let G be a complex on n vertices and $0 \leq i < r < f$. We say that $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ form an $(\varepsilon, \mu, \xi, f, r, i)$-*setup* for G if there exists an f-graph Y on $V(G)$ such that the following hold:

(S1) $\mathcal{S}$ is an i-system in $V(G)$ such that G is r-exclusive with respect to $\mathcal{S}$; $\mathcal{U}$ is a μ-focus for $\mathcal{S}$ and $(\mathcal{P}_r, \mathcal{P}_f)$ is an (r, f)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$;

(S2) there exists a matrix $A \in [0,1]^{(r+1)\times(f+1)}$ with $\min^{\backslash\backslash r-i+1}(A) \geq \xi$ such that $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f)[G[Y]] = (\mathcal{P}_r, \mathcal{P}_f[Y])$;

(S3) every $\mathcal{S}$-unimportant $e \in G^{(r)}$ is contained in at least $\xi(\mu n)^f$ $\mathcal{S}$-unimportant $Q \in G[Y]^{(f+r)}$, and for every $\mathcal{S}$-important $e \in G^{(r)}$ with $e \supseteq S \in \mathcal{S}$, we have $|G[Y]^{(f+r)}(e)[U_S]| \geq \xi(\mu n)^f$;

(S4) for all $S \in \mathcal{S}$, $h \in [r-i]_0$ and all $B \subseteq G(S)^{(h)}$ with $1 \leq |B| \leq 2^h$ we have that $\bigcap_{b\in B} G(S \cup b)[U_S]$ is an $(\varepsilon, \xi, f-i-h, r-i-h)$-complex.

Moreover, if (S1)–(S4) are true and A is diagonal-dominant, then we say that $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ form a *diagonal-dominant* $(\varepsilon, \mu, \xi, f, r, i)$*-setup* for G.

Note that (S4) implies that $G(S)[U_S]$ is an $(\varepsilon, \xi, f-i, r-i)$-supercomplex for every $S \in \mathcal{S}$, but is stronger in the sense that B is not restricted to U_S. The following observation shows that Definition 2.10.18 does indeed generalise Definition 2.7.1. (Recall that the partition pair of G, U was defined in Example 2.10.11.) We will use it to derive the Cover down lemma from the more general Cover down lemma for setups.

Proposition 2.10.19. *Let G be a complex on n vertices and suppose that $U \subseteq V(G)$ is $(\varepsilon, \mu, \xi, f, r)$-random in G. Let $(\mathcal{P}_r, \mathcal{P}_f)$ be the (r, f)-partition pair of G, U. Then $\{\emptyset\}, \{U\}, (\mathcal{P}_r, \mathcal{P}_f)$ form an $(\varepsilon, \mu, \tilde{\mu}\xi, f, r, 0)$-setup for G, where $\tilde{\mu} := (\min\{\mu, 1-\mu\})^{f-r}$.*

Proof. We first check (S1). Clearly, $\mathcal{S}$ is a 0-system in $V(G)$. Moreover, G is trivially r-exclusive with respect to $\mathcal{S}$ since $|\mathcal{S}| < 2$. Moreover, by (R1), $\mathcal{U}$ is a μ-focus for $\mathcal{S}$, and $(\mathcal{P}_r, \mathcal{P}_f)$ is an (r, f)-partition pair of $G, \mathcal{S}, \mathcal{U}$ by (P6′) in Proposition 2.10.12. Note that (S4) follows immediately from (R4). In order to check (S2) and (S3), assume that $Y \subseteq G^{(f)}$ and $d \geq \xi$ are such that (R2) and (R3) hold. Clearly, all $e \in G^{(r)}$ are $\mathcal{S}$-important, and by (R3), we have for all $e \in G^{(r)}$ that $|G[Y]^{(f+r)}(e)[U]| \geq \xi(\mu n)^f$, so (S3) holds. Finally, we have seen in Example 2.10.14 that there exists a matrix $A \in [0,1]^{(r+1)\times(f+1)}$ with $\min^{\backslash\backslash r-i+1}(A) \geq \tilde{\mu}\xi$ such that $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f[Y])$.

□

The following lemma shows that we can (probabilistically) sparsify a given setup so that the resulting setup is diagonal-dominant.

Lemma 2.10.20. *Let $1/n \ll \varepsilon \ll \nu \ll \mu, \xi, 1/f$ and $0 \le i < r < f$. Let $\xi' := \nu^{8^f \cdot f+1}$. Let G be a complex on n vertices and suppose that*

$$\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f) \text{ form an } (\varepsilon, \mu, \xi, f, r, i)\text{-setup for } G.$$

Then there exists a subgraph $H \subseteq G^{(r)}$ with $\Delta(H) \le 1.1\nu n$ and the following property: for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \varepsilon n$ and all $(r+1)$-graphs O on $V(G)$ with $\Delta(O) \le \varepsilon n$, the following holds for $G' := G[H \triangle L] - O$:

$$\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)[G'] \text{ form a diagonal-dominant } (\sqrt{\varepsilon}, \mu, \xi', f, r, i)\text{-setup for } G'.$$

Proof. Let $Y \subseteq G^{(f)}$ and $A \in [0,1]^{(r+1)\times(f+1)}$ be such that (S1)–(S4) hold for G. Let $C \colon \mathcal{P}_r \times \mathcal{P}_f \to [\binom{f}{r}]_0$ be the containment function of $(\mathcal{P}_r, \mathcal{P}_f)$. We will write $c_{\ell,k} := C(\mathcal{P}_r(\ell), \mathcal{P}_f(k))$ for all $\ell \in [r+1]$ and $k \in [f+1]$. We may assume that $a_{\ell,k} = 0$ whenever $c_{\ell,k} = 0$ (and $\min^{\backslash\backslash r-i+1}(A) \ge \xi$ still holds).

Define the matrix $A' \in [0,1]^{(r+1)\times(f+1)}$ by letting $a'_{\ell,k} := a_{\ell,k}\nu^{-\ell} \prod_{\ell' \in [r+1]} \nu^{\ell' c_{\ell',k}}$. Note that we always have $a'_{\ell,k} \le a_{\ell,k}$.

Claim 1: A' is diagonal-dominant and $\min^{\backslash\backslash r-i+1}(A') \ge \xi'$.

Proof of claim: For $1 \le \ell < k \le r+1$,

$$\frac{a'_{\ell,k}}{a'_{k,k}} = \frac{a_{\ell,k}\nu^{-\ell}}{a_{k,k}\nu^{-k}} \le \frac{\nu^{k-\ell}}{\xi} \le \frac{1}{2(r+1-\ell)}.$$

Moreover, we have $\min^{\backslash\backslash r-i+1}(A') \ge \xi \nu^{(r+1)\binom{f}{r}-1} \ge \xi'$. –

We choose H randomly by including independently each $e \in \mathcal{P}_r(\ell)$ with probability ν^ℓ, for all $\ell \in [r+1]$. A standard application of Lemma 2.5.10 shows that whp $\Delta(H) \le 1.1\nu n$.

We now check (S1)–(S4) for $G', \mathcal{S}, \mathcal{U}$ and $(\mathcal{P}_r, \mathcal{P}_f)[G']$. For any L and O, G' is r-exclusive with respect to $\mathcal{S}$, and $(\mathcal{P}_r, \mathcal{P}_f)[G']$ is an (r,f)-partition pair of G', $\mathcal{S}$, $\mathcal{U}$ by

(P7′) in Proposition 2.10.12. Thus, (S1) holds.

We now consider (S2). Let $\ell \in [r+1]$, $k \in [f+1]$ and $e \in \mathcal{P}_r(\ell)$. Define

$$\mathcal{Q}_{e,k} := (\mathcal{P}_f[Y](k))(e).$$

By (2.10.7) and (S2) for $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$, we have that $|\mathcal{Q}_{e,k}| = (a_{\ell,k} \pm \varepsilon)n^{f-r}$. We view $\mathcal{Q}_{e,k}$ as a $(f-r)$-graph and consider the random subgraph $\mathcal{Q}'_{e,k}$ that contains all $Q \in \mathcal{Q}_{e,k}$ with $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. If $a_{\ell,k} \neq 0$, then for all $Q \in \mathcal{Q}_{e,k}$, we have

$$\mathbb{P}(Q \in \mathcal{Q}'_{e,k}) = \nu^{-\ell} \prod_{\ell' \in [r+1]} \nu^{\ell' c_{\ell',k}} = \frac{a'_{\ell,k}}{a_{\ell,k}}.$$

Thus, $\mathbb{E}|\mathcal{Q}'_{e,k}| = (a'_{\ell,k} \pm \varepsilon)n^{f-r}$. This also holds if $a_{\ell,k} = 0$ (and thus $a'_{\ell,k} = 0$). Using Corollary 2.5.14 and a union bound, we thus conclude that with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, we have $|\mathcal{Q}'_{e,k}| = (a'_{\ell,k} \pm \varepsilon^{2/3})n^{f-r}$ for all $\ell \in [r+1]$, $k \in [f+1]$ and $e \in \mathcal{P}_r(\ell)$. (Technically, we can only apply Corollary 2.5.14 if $|\mathcal{Q}_{e,k}| \geq 2\varepsilon n^{f-r}$, say. Note that the result holds trivially if $|\mathcal{Q}_{e,k}| \leq 2\varepsilon n^{f-r}$.) Assuming that this holds for H, a double application of Proposition 2.5.7 shows that any $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n$ and any $(r+1)$-graph O on $V(G)$ with $\Delta(O) \leq \varepsilon n$ results in $G'[Y]$ being $(\sqrt{\varepsilon}, A', f, r)$-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f)[G'[Y]]$.

We now check (S3). Let $e \in G^{(r)}$. If e is $\mathcal{S}$-unimportant then let $\mathcal{Q}_e$ be the set of all $Q \in G[Y]^{(f+r)}(e)$ such that $Q \cup e$ is $\mathcal{S}$-unimportant, otherwise let $\mathcal{Q}_e := G[Y]^{(f+r)}(e)[U_S]$. By (S3) for $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$, we have that $|\mathcal{Q}_e| \geq \xi(\mu n)^f$. We view $\mathcal{Q}_e$ as a f-graph and consider the random subgraph $\mathcal{Q}'_e$ containing all $Q \in \mathcal{Q}_e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_e$, we have

$$\mathbb{P}(Q \in \mathcal{Q}'_e) \geq \nu^{(r+1)\binom{f+r}{r}-1} \geq \nu^{f(4^f)},$$

thus $\mathbb{E}|\mathcal{Q}'_e| \geq \nu^{f(4^f)}\xi(\mu n)^f$. Using Corollary 2.5.14 and a union bound, we conclude that whp $|\mathcal{Q}'_e| \geq 2\xi'(\mu n)^f$ for all $e \in G^{(r)}$. Assuming that this holds for H, Proposition 2.5.7

implies that for any admissible choices of L and O, (S3) still holds.

Finally, we check (S4). Let $S \in \mathcal{S}$, $h \in [r-i]_0$ and $B \subseteq G(S)^{(h)}$ with $1 \leq |B| \leq 2^h$. By assumption, $G_{S,B} := \bigcap_{b \in B} G(S \cup b)[U_S]$ is an $(\varepsilon, \xi, f-i-h, r-i-h)$-complex. We intend to apply Proposition 2.5.18 with $i+h$, $G[U_S \cup S \cup \bigcup B]$, $\mathcal{P}_r[G^{(r)}[U_S \cup S \cup \bigcup B]]$, $\{b \cup S \ : \ b \in B\}$, ν^{r+1}, $\varepsilon^{2/3}$ playing the roles of $i, G, \mathcal{P}, B, p, \gamma$. Note that for every $b \in B$ and all $e \in G_{S,B}^{(r-i-h)}$, $S \cup b \cup e$ is $\mathcal{S}$-important and $\tau_r(S \cup b \cup e) = |(S \cup b \cup e) \cap U_S| = |b \cap U_S| + r - i - h$. Hence, $S \cup b \cup e \in \mathcal{P}_r(|b \cap U_S| + r - h + 1)$. Thus, condition (I) in Proposition 2.5.18 is satisfied. Moreover, (II) is also satisfied because of (P5$'$) in Proposition 2.10.12. Therefore, by Proposition 2.5.18, with probability at least $1 - \mathrm{e}^{-|U_S|^{1/8}}$, for any $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n \leq 2\varepsilon |U_S|/\mu \leq \varepsilon^{2/3}|U_S|$ and any $(r+1)$-graph O on $V(G)$ with $\Delta(O) \leq \varepsilon n \leq f^{-5r}\varepsilon^{2/3}|U_S|$, we have that $\bigcap_{b \in B} G'(S \cup b)[U_S]$ is a $(\sqrt{\varepsilon}, \xi', f-i-h, r-i-h)$-complex. A union bound now shows that with probability at least $1 - \mathrm{e}^{-n^{1/10}}$, (S4) holds.

Thus, there exists an H with the desired properties. $\square$

We also need a similar result which 'sparsifies' the neighbourhood complexes of an i-system.

Lemma 2.10.21. *Let $1/n \ll \varepsilon \ll \mu, \beta, \xi, 1/f$ and $1 \leq i < r < f$. Let $\xi' := 0.9\xi\beta^{(8^f)}$. Let G be a complex on n vertices and let $\mathcal{S}$ be an i-system in G such that G is r-exclusive with respect to $\mathcal{S}$. Let $\mathcal{U}$ be a μ-focus for $\mathcal{S}$. Suppose that*

$$G(S)[U_S] \text{ is an } (\varepsilon, \xi, f-i, r-i)\text{-supercomplex for every } S \in \mathcal{S}.$$

Then there exists a subgraph $H \subseteq G^{(r)}$ with $\Delta(H) \leq 1.1\beta n$ and the following property: for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n$ and all $(r+1)$-graphs O on $V(G)$ with $\Delta(O) \leq \varepsilon n$, the following holds for $G' := G[H \triangle L] - O$:

$$G'(S)[U_S] \text{ is a } (\sqrt{\varepsilon}, \xi', f-i, r-i)\text{-supercomplex for every } S \in \mathcal{S}.$$

Proof. Choose H randomly by including each $e \in G^{(r)}$ independently with probability β.

Clearly, whp $\Delta(H) \leq 1.1\beta n$. Now, consider $S \in \mathcal{S}$. Let $h \in [r-i]_0$ and $B \subseteq G(S)[U_S]^{(h)}$ with $1 \leq |B| \leq 2^h$. By assumption, $G_{S,B} := \bigcap_{b \in B} G(S)[U_S](b) = \bigcap_{b \in B} G(S \cup b)[U_S]$ is an $(\varepsilon, \xi, f-i-h, r-i-h)$-complex. Proposition 2.5.18 (applied with $G[U_S \cup S \cup \bigcup B] =: G_1, \{b \cup S : b \in B\}, i+h, \{G_1^{(r)}\}, \beta, \varepsilon^{2/3}$ playing the roles of $G, B, i, \mathcal{P}, p, \gamma$) implies that with probability at least $1 - \mathrm{e}^{-|U_S|^{1/8}}$, H has the property that for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n \leq \varepsilon^{2/3}|U_S|$ and all $(r+1)$-graphs O on $V(G)$ with $\Delta(O) \leq \varepsilon n \leq f^{-5r}\varepsilon^{2/3}|U_S|$, $\bigcap_{b \in B} G'(S \cup b)[U_S] = \bigcap_{b \in B} G'(S)[U_S](b)$ is a $(\sqrt{\varepsilon}, \xi', f-i-h, r-i-h)$-complex.

Therefore, applying a union bound to all $S \in \mathcal{S}$, $h \in [r-i]_0$ and $B \subseteq G(S)[U_S]^{(h)}$ with $1 \leq |B| \leq 2^h$, we conclude that whp H has the property that for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n$ and all $(r+1)$-graphs O on $V(G)$ with $\Delta(O) \leq \varepsilon n$, $G'(S)[U_S]$ is a $(\sqrt{\varepsilon}, \xi', f-i, r-i)$-supercomplex for every $S \in \mathcal{S}$. Thus, there exists an H with the desired properties. $\square$

The final tool that we need is the following lemma. Given a setup in a supercomplex G and an i'-extension $\mathcal{T}$ of the respective i-system $\mathcal{S}$, it allows us to find a new focus $\mathcal{U}'$ for $\mathcal{T}$ and a suitable partition pair which together form a new setup in the complex G' (which is the complex we look at after all edges with type less than $r-i'$ have been covered).

Lemma 2.10.22. *Let $1/n \ll \varepsilon \ll \rho \ll \mu, \xi, 1/f$ and $0 \leq i < i' < r < f$. Let G be a complex on n vertices and suppose that $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ form an $(\varepsilon, \mu, \xi, f, r, i)$-setup for G. For $r' \geq r$, let $\tau_{r'}$ be the type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$. Let $\mathcal{T}$ be the i'-extension of $\mathcal{S}$ in G around $\mathcal{U}$, and let*

$$G' := G - \{e \in G^{(r)} : e \text{ is } \mathcal{S}\text{-important and } \tau_r(e) < r - i'\}.$$

Then there exist $\mathcal{U}', \mathcal{P}'_r, \mathcal{P}'_f$ with the following properties:

(i) *$\mathcal{U}'$ is a (μ, ρ, r)-focus for $\mathcal{T}$ such that $U_T \subseteq U_{T\restriction_{\mathcal{S}}}$ for all $T \in \mathcal{T}$;*

(ii) *$\mathcal{T}, \mathcal{U}', (\mathcal{P}'_r, \mathcal{P}'_f)$ form a $(1.1\varepsilon, \rho\mu, \rho^{f-r}\xi, f, r, i')$-setup for G';*

(iii) $G'(T)[U_T]$ *is a* $(1.1\varepsilon, 0.9\xi, f-i', r-i')$*-supercomplex for every* $T \in \mathcal{T}$*.*

Proof. Let $\ell := r - i'$. Let $Y \subseteq G^{(f)}$ and $A \in [0,1]^{(r+1)\times(f+1)}$ be such that (S1)–(S4) hold for $G, \mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$. We choose $\mathcal{U}'$ randomly as follows: for every $T \in \mathcal{T}$ we let U_T be a random subset of $U_{T\restriction_\mathcal{S}}$, obtained by including every $x \in U_{T\restriction_\mathcal{S}}$ with probability ρ, and all these choices are made independently. Let $\mathcal{U}' := (U_T)_{T\in\mathcal{T}}$. Clearly, $\mathcal{U}'$ is a focus for $\mathcal{T}$ and $U_T \subseteq U_{T\restriction_\mathcal{S}}$ for all $T \in \mathcal{T}$. We will prove that (i)–(iii) hold whp.

By Proposition 2.10.5, the following hold:

(a) G' is r-exclusive with respect to $\mathcal{T}$;

(b) for all $e \in G$ with $|e| \geq r$, we have

$$e \notin G' \quad \Leftrightarrow \quad e \text{ is } \mathcal{S}\text{-important and } \tau_{|e|}(e) < |e| - i';$$

(c) for $r' \geq r$, the $\mathcal{T}$-important elements of $G'^{(r')}$ are precisely the elements of $\tau_{r'}^{-1}(r'-i')$.

For $r' \geq r$, property (a) allows us to consider the type function $\tau'_{r'}$ of $G'^{(r')}, \mathcal{T}, \mathcal{U}'$. As a consequence of (b), we have for each $r' \geq r$ that

$$G'^{(r')} = G^{(r')} \setminus \bigcup_{k=0}^{r'-i'-1} \tau_{r'}^{-1}(k). \tag{2.10.8}$$

In what follows, we define a suitable (r,f)-partition pair $(\mathcal{P}'_r, \mathcal{P}'_f)$ of G'. Recall that every element of a class from $\mathcal{P}_r([i])$ and $\mathcal{P}_f([i])$ is $\mathcal{S}$-unimportant, and thus $\mathcal{T}$-unimportant as well. By (2.10.8) and (c), the $\mathcal{T}$-unimportant r-sets of G' that are $\mathcal{S}$-important are precisely the elements of $\tau_r^{-1}(\ell+1), \dots, \tau_r^{-1}(r-i)$, and the $\mathcal{T}$-unimportant f-sets of G' that are $\mathcal{S}$-important are precisely the elements of $\tau_f^{-1}(f-r+\ell+1), \dots, \tau_f^{-1}(f-i)$. Thus, we aim to attach these classes to $\mathcal{P}_r([i])$ and $\mathcal{P}_f([i])$, respectively, in order to obtain partitions of the $\mathcal{T}$-unimportant r-sets and f-sets of G'. When doing so, we reverse their

	$\mathcal{P}_f([i])$	$\tau_f^{-1}(f-i)$	$\dots$	$\tau_f^{-1}(f-i'+1)$	$\tau_f^{-1}(f-i')$
$\mathcal{P}_r([i])$	$*$ 0 $*$ 0 0 $*$				
$\tau_r^{-1}(r-i)$	0	$*$			
$\dots$	0	0	$*$		
$\tau_r^{-1}(\ell+1)$	0	0	0	$*$	
$\tau_r^{-1}(\ell)$	0	0	0	0	$*$

Figure 2.2: The above table sketches the containment function of $(\mathcal{P}_r'^* \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}_f'^* \sqcup \{\tau_f^{-1}(f-r+\ell)\})$. Note that the shaded subtable corresponds to the shaded subtable in Figure 2.1, but has been flipped to make it upper-triangular instead of lower-triangular.

order. This will ensure that the new partition pair is again upper-triangular (cf. Figure 2.2).

Define

$$\mathcal{P}_r'^* := \mathcal{P}_r([i]) \sqcup (\tau_r^{-1}(r-i), \dots, \tau_r^{-1}(\ell+1)), \tag{2.10.9}$$

$$\mathcal{P}_f'^* := \mathcal{P}_f([i]) \sqcup (\tau_f^{-1}(f-i), \dots, \tau_f^{-1}(f-r+\ell+1)). \tag{2.10.10}$$

Claim 1: $(\mathcal{P}_r'^*, \mathcal{P}_f'^*)$ is admissible with respect to G', $\mathcal{T}$, $\mathcal{U}'$.

Proof of claim: By (2.10.8) and (c), we have that $\mathcal{P}_r'^*$ is a partition of the $\mathcal{T}$-unimportant elements of $G'^{(r)}$ and $\mathcal{P}_f'^*$ is a partition of the $\mathcal{T}$-unimportant elements of $G'^{(f)}$. Moreover, note that $|\mathcal{P}_r'^*| = i + (r-i-\ell) = i'$ and $|\mathcal{P}_f'^*| = i + (f-i) - (f-r+\ell) = i'$, so (P1) holds.

We proceed with checking (P3). By (c), $\tau_r^{-1}(\ell)$ consists of all $\mathcal{T}$-important edges of $G'^{(r)}$, and $\tau_f^{-1}(f-r+\ell)$ consists of all $\mathcal{T}$-important f-sets of $G'^{(f)}$. Thus, $(\mathcal{P}_r'^* \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}_f'^* \sqcup \{\tau_f^{-1}(f-r+\ell)\})$ clearly is an (r,f)-partition pair of G'. If $0 \le k' < \ell' \le i'-i$, then no $Q \in \tau_f^{-1}(f-i-k')$ contains any element from $\tau_r^{-1}(r-i-\ell')$ by (2.10.6), so $(\mathcal{P}_r'^* \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}_f'^* \sqcup \{\tau_f^{-1}(f-r+\ell)\})$ is upper-triangular (cf. Figure 2.2).

It remains to check (P2). Let $T \in \mathcal{T}$, $h' \in [r-i']_0$ and $B' \subseteq G'(T)^{(h')}$ with $1 \le |B'| \le 2^{h'}$. Let $S := T\restriction_{\mathcal{S}}$, let $h := h' + i' - i \in [r-i]_0$ and $B := \{(T \setminus S) \cup b' : b' \in B'\}$. Clearly, $B \subseteq G(S)^{(h)}$ with $1 \le |B| \le 2^h$. Thus, by (P5′) in Proposition 2.10.12, we have for all

$\mathcal{E} \in \mathcal{P}_r$ that there exists $D(S, B, \mathcal{E}) \in \mathbb{N}_0$ such that for all $Q \in \bigcap_{b\in B} G(S \cup b)[U_S]^{(f-i-h)}$, we have that

$$|\{e \in \mathcal{E} \,:\, \exists b \in B \colon e \subseteq S \cup b \cup Q\}| = D(S, B, \mathcal{E}).$$

For each $\mathcal{E} \in \mathcal{P}'^{*}_r$, define $D'(T, B', \mathcal{E}) := D(S, B, \mathcal{E})$. Thus, since $U_T \subseteq U_S$, we have for all $Q \in \bigcap_{b'\in B'} G'(T \cup b')[U_T]^{(f-i'-h')}$ that

$$|\{e \in \mathcal{E} \,:\, \exists b' \in B' \colon e \subseteq T \cup b' \cup Q\}| = D'(T, B', \mathcal{E}).$$

–

Let $(\mathcal{P}'_r, \mathcal{P}'_f)$ be the (r, f)-partition pair of G' induced by $(\mathcal{P}'^{*}_r, \mathcal{P}'^{*}_f)$ and $\mathcal{U}'$. Recall that $\tau'_{r'}$ denotes the type function of $G'^{(r')}$, $\mathcal{T}$, $\mathcal{U}'$ (for any $r' \geq r$). Define the matrix $A' \in [0, 1]^{(r+1)\times(f+1)}$ such that the following hold:

- For all $\mathcal{E} \in \mathcal{P}'^{*}_r$ and $\mathcal{Q} \in \mathcal{P}'^{*}_f$, let $A'(\mathcal{E}, \mathcal{Q}) := A(\mathcal{E}, \mathcal{Q})$.
- For all $\ell' \in [r - i']_0$ and $\mathcal{Q} \in \mathcal{P}'^{*}_f$, let $A'(\tau'^{-1}_r(\ell'), \mathcal{Q}) := 0$.
- For all $\mathcal{E} \in \mathcal{P}'^{*}_r$ and $k' \in [f - i']_0$, define

$$A'(\mathcal{E}, \tau'^{-1}_f(k')) := bin(f - i', \rho, k')A(\mathcal{E}, \tau^{-1}_f(f - r + \ell)).$$

- For all $\ell' \in [r - i']_0$, $k' \in [f - i']_0$, let

$$A'(\tau'^{-1}_r(\ell'), \tau'^{-1}_f(k')) := bin(f - r, \rho, k' - \ell')A(\tau^{-1}_r(\ell), \tau^{-1}_f(f - r + \ell)).$$

Claim 2: $\min^{\backslash\backslash r-i'+1}(A') \geq \rho^{f-r}\xi$.

Proof of claim: Let

$$a'_1 := \min_{\ell'\in[r-i']_0} A'(\tau'^{-1}_r(\ell'), \tau'^{-1}_f(\ell')) \quad \text{and} \quad a'_2 := \min_{\ell'\in[r-i']_0} A'(\tau'^{-1}_r(\ell'), \tau'^{-1}_f(f - r + \ell')).$$

Observe that $\min^{\backslash\backslash r-i'+1}(A') \geq \min\{\min^{\backslash\backslash r-i+1}(A), a'_1, a'_2\}$. Since $\min^{\backslash\backslash r-i+1}(A) \geq \xi$, $a'_1 \geq (1-\rho)^{f-r}\xi$ and $a'_2 \geq \rho^{f-r}\xi$, the claim follows. –

We now prove in a series of claims that (i)–(iii) hold whp. By Lemma 2.10.7 (applied with $\mathcal{T}$, $\{U_{T\restriction_{\mathcal{S}}} : T \in \mathcal{T}\}$ playing the roles of $\mathcal{S}$,$\mathcal{U}$), whp $\mathcal{U}'$ is a (μ, ρ, r)-focus for $\mathcal{T}$, so (i) holds. In particular, whp $\mathcal{U}'$ is a $\rho\mu$-focus for $\mathcal{T}$, implying that (S1) holds for G' with $\mathcal{T}, \mathcal{U}'$ and $(\mathcal{P}'_r, \mathcal{P}'_f)$. We now check (S2)–(S4) and (iii).

Claim 3: Whp $G'[Y]$ is $(1.1\varepsilon, A', f, r)$-regular with respect to $(\mathcal{P}'_r, \mathcal{P}'_f[Y])$ (cf. (S2)*).*

Proof of claim: By definition of $(\mathcal{P}'^*_r, \mathcal{P}'^*_f)$, we have for all $\mathcal{E} \in \mathcal{P}'^*_r \sqcup \{\tau_r^{-1}(\ell)\}$ and $\mathcal{Q} \in (\mathcal{P}'^*_f \sqcup \{\tau_f^{-1}(f-r+\ell)\})[Y]$ that $\mathcal{E} \in \mathcal{P}_r$ and $\mathcal{Q} \in \mathcal{P}_f[Y]$. Since $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f[Y])$, we have thus for all $e \in \mathcal{E}$ that

$$|\mathcal{Q}(e)| = (A(\mathcal{E}, \mathcal{Q}) \pm \varepsilon)n^{f-r}. \tag{2.10.11}$$

We have to show that for all $\mathcal{E} \in \mathcal{P}^\ell_r$, $\mathcal{Q} \in \mathcal{P}^\ell_f[Y]$ and $e \in \mathcal{E}$, we have $|\mathcal{Q}(e)| = (A'(\mathcal{E}, \mathcal{Q}) \pm 1.1\varepsilon)n^{f-r}$. We distinguish four cases as in the definition of A'.

Firstly, for all $\mathcal{E} \in \mathcal{P}'^*_r$, $\mathcal{Q} \in \mathcal{P}'^*_f[Y]$ and $e \in \mathcal{E}$, we have by (2.10.11) that $|\mathcal{Q}(e)| = (A(\mathcal{E}, \mathcal{Q}) \pm \varepsilon)n^{f-r} = (A'(\mathcal{E}, \mathcal{Q}) \pm \varepsilon)n^{f-r}$ with probability 1.

Also, for all $\ell' \in [r-i']_0$, $\mathcal{Q} \in \mathcal{P}'^*_f[Y]$ and $e \in \tau'^{-1}_r(\ell')$, we have $|\mathcal{Q}(e)| = 0 = A'(\tau'^{-1}_r(\ell'), \mathcal{Q})n^{f-r}$ with probability 1.

Let $\mathcal{E} \in \mathcal{P}'^*_r \sqcup \{\tau_r^{-1}(\ell)\}$ and consider $e \in \mathcal{E}$. Let $\mathcal{Q}_e := (Y \cap \tau_f^{-1}(f-r+\ell))(e)$. By (2.10.11), we have that $|\mathcal{Q}_e| = (A(\mathcal{E}, \tau_f^{-1}(f-r+\ell)) \pm \varepsilon)n^{f-r}$.

First, assume that $e \in \mathcal{E} \in \mathcal{P}'^*_r$. For each $k' \in [f-i']_0$, we consider the random subgraph $\mathcal{Q}^{k'}_e$ of $\mathcal{Q}_e$ that contains all $Q \in \mathcal{Q}_e$ with $Q \cup e \in \tau'^{-1}_f(k')$. Hence, $\mathcal{Q}^{k'}_e = (Y \cap \tau'^{-1}_f(k'))(e)$. For each $Q \in \mathcal{Q}_e$, there are unique $T_Q \in \mathcal{T}$ and $S_Q \in \mathcal{S}$ with $S_Q \subseteq T_Q \subseteq Q \cup e$ and $(Q \cup e) \setminus T_Q \subseteq U_{S_Q}$.

For each $Q \in \mathcal{Q}_e$, we then have

$$\mathbb{P}(Q \in \mathcal{Q}^{k'}_e) = \mathbb{P}(\tau'_f(Q \cup e) = k') = \mathbb{P}(|(Q \cup e) \cap U_{T_Q}| = k') = bin(f-i', \rho, k').$$

Thus, $\mathbb{E}|\mathcal{Q}_e^{k'}| = bin(f-i',\rho,k')|\mathcal{Q}_e|$. For each $T \in \mathcal{T}$, let $\mathcal{Q}_T$ be the set of all those $Q \in \mathcal{Q}_e$ for which $T_Q = T$. Since e is $\mathcal{T}$-unimportant, we have $|T \setminus e| > 0$ and thus $|\mathcal{Q}_T| \leq n^{f-r-1}$ for all $T \in \mathcal{T}$. Thus we can partition $\mathcal{Q}_e$ into n^{f-r-1} subgraphs such that each of them intersects each $\mathcal{Q}_T$ in at most one element. For all Q lying in the same subgraph, the events $Q \in \mathcal{Q}_e^{k'}$ are now independent. Hence, by Lemma 2.5.12, we conclude that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have that

$$
\begin{aligned}
|\mathcal{Q}_e^{k'}| &= (1 \pm \varepsilon^2)\mathbb{E}|\mathcal{Q}_e^{k'}| = (1 \pm \varepsilon^2) bin(f-i',\rho,k')|\mathcal{Q}_e| \\
&= (1 \pm \varepsilon^2) bin(f-i',\rho,k')(A(\mathcal{E},\tau_f^{-1}(f-r+\ell)) \pm \varepsilon)n^{f-r} \\
&= (A'(\mathcal{E},\tau_f'^{-1}(k')) \pm 1.1\varepsilon)n^{f-r}.
\end{aligned} \tag{2.10.12}
$$

(Technically, we can only apply Lemma 2.5.12 if $|\mathcal{Q}_e| \geq 0.1\varepsilon n^{f-r}$, say. Note that (2.10.12) holds trivially if $|\mathcal{Q}_e| \leq 0.1\varepsilon n^{f-r}$.)

Finally, consider the case $e \in \mathcal{E} = \tau_r^{-1}(\ell)$. By (c), e is $\mathcal{T}$-important, so let $T \in \mathcal{T}$ be such that $T \subseteq e$. Note that for every $Q \in \mathcal{Q}_e$, we have $(e \setminus T) \cup Q \subseteq U_S$, where $S := T\restriction_{\mathcal{S}}$. For every $x \in [f-r]_0$, let $\mathcal{Q}_e^x$ be the random subgraph of $\mathcal{Q}_e$ that contains all $Q \in \mathcal{Q}_e$ with $|Q \cap U_T| = x$. By the random choice of U_T, for each $Q \in \mathcal{Q}$ and $x \in [f-r]_0$, we have

$$
\mathbb{P}(Q \in \mathcal{Q}_e^x) = bin(f-r,\rho,x).
$$

Using Corollary 2.5.14 we conclude that for $x \in [f-r]_0$, with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have that

$$
\begin{aligned}
|\mathcal{Q}_e^x| &= (1 \pm \varepsilon^2)\mathbb{E}|\mathcal{Q}_e^x| = (1 \pm \varepsilon^2) bin(f-r,\rho,x)|\mathcal{Q}_e| \\
&= (1 \pm \varepsilon^2) bin(f-r,\rho,x)(A(\tau_r^{-1}(\ell),\tau_f^{-1}(f-r+\ell)) \pm \varepsilon)n^{f-r} \\
&= (bin(f-r,\rho,x)A(\tau_r^{-1}(\ell),\tau_f^{-1}(f-r+\ell)) \pm 1.1\varepsilon)n^{f-r}.
\end{aligned}
$$

Thus for all $\ell' \in [r-i']_0$, $k' \in [f-i']_0$ and $e \in \tau_r'^{-1}(\ell')$ with $k' \geq \ell'$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have

$$|(Y \cap \tau_f'^{-1}(k'))(e)| = |\mathcal{Q}_e^{k'-\ell'}| = (A'(\tau_r'^{-1}(\ell'), \tau_f'^{-1}(k')) \pm 1.1\varepsilon)n^{f-r},$$

and if $\ell' > k'$ then trivially $|(Y \cap \tau_f'^{-1}(k'))(e)| = 0 = A'(\tau_r'^{-1}(\ell'), \tau_f'^{-1}(k'))n^{f-r}$. Thus, a union bound implies the claim. $-$

Claim 4: Whp every $\mathcal{T}$-unimportant $e \in G'^{(r)}$ is contained in at least $0.9\xi(\rho\mu n)^f$ $\mathcal{T}$-unimportant $Q \in G'[Y]^{(f+r)}$, and for every $\mathcal{T}$-important $e \in G'^{(r)}$ with $e \supseteq T \in \mathcal{T}$, we have $|G'[Y]^{(f+r)}(e)[U_T]| \geq 0.9\xi(\rho\mu n)^f$ (cf. (S3)*).*

Proof of claim: Let $e \in G'^{(r)}$ be $\mathcal{T}$-unimportant. By (b) and (c), we thus have that e is $\mathcal{S}$-unimportant or $\tau_r(e) > \ell$. In the first case, we have that e is contained in at least $\xi(\mu n)^f$ $\mathcal{S}$-unimportant $Q \in G[Y]^{(f+r)}$ by (S3) for $\mathcal{U}, G, \mathcal{S}$. But each such Q is clearly $\mathcal{T}$-unimportant as well and contained in $G'[Y]$. If the second case applies, assume that e contains $S \in \mathcal{S}$. By (S3) for $G, \mathcal{S}, \mathcal{U}$, we have that $|G[Y]^{(f+r)}(e)[U_S]| \geq \xi(\mu n)^f$. For every $Q \in G[Y]^{(f+r)}(e)[U_S]$, we have that $\tau_{f+r}(Q \cup e) = |(Q \cup e) \cap U_S| = f + \tau_r(e) > f + \ell$. Thus, (b) implies that $Q \cup e \in G'[Y]$, and by (c) we have that $Q \cup e$ is $\mathcal{T}$-unimportant. Altogether, every $\mathcal{T}$-unimportant edge $e \in G'^{(r)}$ is contained in at least $\xi(\mu n)^f \geq 0.9\xi(\rho\mu n)^f$ $\mathcal{T}$-unimportant $Q \in G'[Y]^{(f+r)}$.

Let $e \in G'^{(r)}$ be $\mathcal{T}$-important. Assume that e contains $T \in \mathcal{T}$ and let $S := T{\restriction_{\mathcal{S}}}$. By (S3) for $G, \mathcal{S}, \mathcal{U}$, we have that $|G[Y]^{(f+r)}(e)[U_S]| \geq \xi(\mu n)^f$. As before, for every $Q \in G[Y]^{(f+r)}(e)[U_S]$, we have $Q \cup e \in G'[Y]$. Moreover, $\mathbb{P}(Q \subseteq U_T) = \rho^f$. Thus, by Corollary 2.5.14, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have that $|G'[Y]^{(f+r)}(e)[U_T]| \geq 0.9\xi(\rho\mu n)^f$. A union bound hence implies the claim. $-$

Claim 5: Whp for all $T \in \mathcal{T}$, $h' \in [r - i']_0$ and $B' \subseteq G'(T)^{(h')}$ with $1 \leq |B'| \leq 2^{h'}$ we have that $\bigcap_{b' \in B'} G'(T \cup b')[U_T]$ is an $(1.1\varepsilon, 0.9\xi, f - i' - h', r - i' - h')$-complex (cf. (S4) *and* (iii)*).*

Proof of claim: Let $T \in \mathcal{T}$, $h' \in [r - i']_0$ and $B' \subseteq G'(T)^{(h')}$ with $1 \leq |B'| \leq 2^{h'}$. Let

$S := T\restriction_{\mathcal{S}}$. We claim that

$$\bigcap_{b' \in B'} G'(T \cup b')[U_S] \text{ is an } (\varepsilon, \xi, f - i' - h', r - i' - h')\text{-complex.} \tag{2.10.13}$$

If $\bigcap_{b' \in B'} G'(T \cup b')[U_S]^{(r-i'-h')}$ is empty, then there is nothing to prove, thus assume the contrary. We claim that we must have $b' \subseteq U_S$ for all $b' \in B'$. Indeed, let $b' \in B'$ and $g_0 \in G'(T \cup b')[U_S]^{(r-i'-h')}$. Hence, $g_0 \cup T \cup b' \in G'^{(r)}$. By (b), we must have $|(g_0 \cup T \cup b') \cap U_S| \geq |g_0 \cup T \cup b'| - i'$. But since $T \cap U_S = \emptyset$, we must have $b' \subseteq U_S$.

Let $h := h' + i' - i \in [r - i]_0$ and $B := \{(T \setminus S) \cup b' : b' \in B'\} \subseteq G(S)^{(h)}$. (S4) for $\mathcal{U}, G, \mathcal{S}$ implies that $\bigcap_{b \in B} G(S \cup b)[U_S]$ is an $(\varepsilon, \xi, f - i - h, r - i - h)$-complex. To prove (2.10.13), it thus suffices to show that $G(T \cup b')[U_S]^{(r')} = G'(T \cup b')[U_S]^{(r')}$ for all $r' \geq r - i - h$ and $b' \in B'$. To this end, let $b' \in B'$, $r' \geq r - i - h$ and suppose that $g \in G(T \cup b')[U_S]^{(r')}$. Observe that $|(g \cup T \cup b') \cap U_S| = |g \cup T \cup b'| - i'$, so (b) implies that $g \cup T \cup b' \in G'$ and thus $g \in G'(T \cup b')[U_S]^{(r')}$. This proves (2.10.13).

By Proposition 2.5.16, with probability at least $1 - \mathrm{e}^{-|U_S|/8}$, $\bigcap_{b' \in B'} G'(T \cup b')[U_T]$ is an $(1.1\varepsilon, 0.9\xi, f - i' - h', r - i' - h')$-complex.

Applying a union bound to all $T \in \mathcal{T}$, $h' \in [r - i']_0$ and $B' \subseteq G'(T)^{(h')}$ with $1 \leq |B'| \leq 2^{h'}$ then establishes the claim. $-$

By the above claims, $\mathcal{U}'$ satisfies (S2)–(S4) whp and thus (ii). Moreover, Claim 5 implies that whp (iii) holds. Thus, the random choice $\mathcal{U}'$ satisfies (i)–(iii) whp. □

2.10.4 Proof of the Cover down lemma

In this subsection, we state and prove the Cover down lemma for setups and deduce the Cover down lemma (Lemma 2.7.7).

Definition 2.10.23. Let F and G be r-graphs, let $\mathcal{S}$ be an i-system in $V(G)$, and let $\mathcal{U}$ be a focus for $\mathcal{S}$. We say that G is *F-divisible with respect to $\mathcal{S}$,$\mathcal{U}$*, if for all $S \in \mathcal{S}$ and all $T \subseteq V(G) \setminus S$ with $|T| \leq r - i - 1$ and $|T \setminus U_S| \geq 1$, we have $Deg(F)_{i+|T|} \mid |G(S \cup T)|$.

Note that if G is F-divisible, then it is F-divisible with respect to any i-system and any associated focus.

Recall that a setup for G was defined in Definition 2.10.18, and G being (ξ, f, r)-dense with respect to $H \subseteq G^{(r)}$ in Definition 2.7.6. We will prove the Cover down lemma for setups by induction on $r - i$. We will deduce the Cover down lemma by applying this lemma with $i = 0$.

Lemma 2.10.24 (Cover down lemma for setups)**.** *Let $1/n \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/f$ and $0 \leq i < r < f$. Let F be a weakly regular r-graph on f vertices. Assume that $(*)_\ell$ is true for all $\ell \in [r-i-1]$. Let G be a complex on n vertices and suppose that $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ form an $(\varepsilon, \mu, \xi, f, r, i)$-setup for G. For $r' \geq r$, let $\tau_{r'}$ denote the type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$. Then the following hold.*

(i) *Let $\tilde{G}$ be a complex on $V(G)$ with $G \subseteq \tilde{G}$ such that $\tilde{G}$ is (ε, f, r)-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$. Then there exists a subgraph $H^* \subseteq G^{(r)} - \tau_r^{-1}(0)$ with $\Delta(H^*) \leq \nu n$ such that for any $L^* \subseteq \tilde{G}^{(r)}$ with $\Delta(L^*) \leq \gamma n$ and $H^* \cup L^*$ being F-divisible with respect to $\mathcal{S}, \mathcal{U}$ and any $(r+1)$-graph O^* on $V(G)$ with $\Delta(O^*) \leq \gamma n$, there exists a κ-well separated F-packing in $\tilde{G}[H^* \cup L^*] - O^*$ which covers all edges of L^*, and all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$.*

(ii) *If $G^{(r)}$ is F-divisible with respect to $\mathcal{S}, \mathcal{U}$ and the setup is diagonal-dominant, then there exists a 2κ-well separated F-packing in G which covers all $\mathcal{S}$-important r-edges except possibly some from $\tau_r^{-1}(r-i)$.*

Before proving Lemma 2.10.24, we show how it implies the Cover down lemma (Lemma 2.7.7). Note that we only need part (i) of Lemma 2.10.24 to prove Lemma 2.7.7. (ii) is used in the inductive proof of Lemma 2.10.24 itself.

Proof of Lemma 2.7.7. Let $\mathcal{S} := \{\emptyset\}, \mathcal{U} := \{U\}$ and let $(\mathcal{P}_r, \mathcal{P}_f)$ be the (r, f)-partition pair of G, U. By Proposition 2.10.19, $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ form a $(\varepsilon, \mu, \mu^{f-r}\xi, f, r, 0)$-setup for G. We can thus apply Lemma 2.10.24(i) with $\mu^{f-r}\xi$ playing the role of ξ. Recall that all

r-edges of G are $\mathcal{S}$-important. Moreover, let τ_r denote the type function of $G^{(r)}$, $\mathcal{S}$, $\mathcal{U}$. We then have $\tau_r^{-1}(0) = G^{(r)}[\bar{U}]$ and $\tau_r^{-1}(r) = G^{(r)}[U]$, where $\bar{U} := V(G) \setminus U$. $\square$

Proof of Lemma 2.10.24. The proof is by induction on $r - i$. For $i = r - 1$, we will prove the statement directly. For $i < r - 1$, we assume that the statement is true for all $i' \in \{i + 1, \dots, r - 1\}$. We will first prove (i) using (ii) inductively, and then derive (ii) from (i) (for the same value of $r - i$).

Proof of (i).

If $i < r - 1$, choose new constants $\nu_1, \rho_1, \beta_1, \dots, \nu_{r-i-1}, \rho_{r-i-1}, \beta_{r-i-1}$ such that

$$1/n \ll 1/\kappa \ll \gamma \ll \varepsilon \ll \nu_1 \ll \rho_1 \ll \beta_1 \ll \dots \ll \nu_{r-i-1} \ll \rho_{r-i-1} \ll \beta_{r-i-1} \ll \nu \ll \mu, \xi, 1/f.$$

For every $\ell \in [r - i - 1]$, let

$$G_\ell := G - \{e \in G^{(r)} : e \text{ is } \mathcal{S}\text{-important and } \tau_r(e) < \ell\}. \tag{2.10.14}$$

For every $i' \in \{i + 1, \dots, r - 1\}$, let $\mathcal{T}^{i'}$ be the i'-extension of $\mathcal{S}$ in G around $\mathcal{U}$. By Proposition 2.10.5, the following hold for all $i' \in \{i + 1, \dots, r - 1\}$:

(I) $G_{r-i'}$ is r-exclusive with respect to $\mathcal{T}^{i'}$;

(II) the elements of $\tau_r^{-1}(r - i')$ are precisely the $\mathcal{T}^{i'}$-important elements of $G^{(r)}_{r-i'}$.

By Lemma 2.10.22, for every $i' \in \{i + 1, \dots, r - 1\}$, there exist $\mathcal{U}^{i'}$, $\mathcal{P}^{i'}_r$, $\mathcal{P}^{i'}_f$ such that the following hold:

(a) $\mathcal{U}^{i'}$ is a $(\mu, \rho_{r-i'}, r)$-focus for $\mathcal{T}^{i'}$ such that $U_T \subseteq U_{T\restriction_\mathcal{S}}$ for all $T \in \mathcal{T}^{i'}$;

(b) $\mathcal{T}^{i'}, \mathcal{U}^{i'}, (\mathcal{P}^{i'}_r, \mathcal{P}^{i'}_f)$ form a $(1.1\varepsilon, \rho_{r-i'}\mu, \rho_{r-i'}^{f-r}\xi, f, r, i')$-setup for $G_{r-i'}$;

(c) $G_{r-i'}(T)[U_T]$ is a $(1.1\varepsilon, 0.9\xi, f - i', r - i')$-supercomplex for every $T \in \mathcal{T}^{i'}$.

(I) allows us to consider the type function $\tau_{r-i',r}$ of $G^{(r)}_{r-i'}, \mathcal{T}^{i'}, \mathcal{U}^{i'}$.

Step 1: Reserving subgraphs

In this step, we will find a number of subgraphs of $G^{(r)} - \tau_r^{-1}(0)$ whose union will be the r-graph H^* we seek in (i). Let $\tilde{G}$ be a complex as specified in (i). Let $\beta_0 := \varepsilon$. Let H_0 be a subgraph of $G^{(r)} - \tau_r^{-1}(0)$ with $\Delta(H_0) \leq 1.1\beta_0 n$ such that for all $e \in \tilde{G}^{(r)}$, we have

$$|\tilde{G}[H_0 \cup \{e\}]^{(f)}(e)| \geq 0.9\beta_0^{\binom{f}{r}} n^{f-r}. \tag{2.10.15}$$

(H_0 will be used to greedily cover L^*.) That such a subgraph exists can be seen by a probabilistic argument: let H_0 be obtained by including every edge of $G^{(r)} - \tau_r^{-1}(0)$ with probability β_0. Clearly, whp $\Delta(H_0) \leq 1.1\beta_0 n$. Also, since $\tilde{G}$ is (ε, f, r)-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$ by assumption, we have for all $e \in \tilde{G}^{(r)}$ that

$$\mathbb{E}|\tilde{G}[H_0 \cup \{e\}]^{(f)}(e)| = \beta_0^{\binom{f}{r}-1}|\tilde{G}[(G^{(r)} - \tau_r^{-1}(0)) \cup \{e\}]^{(f)}(e)| \geq \beta_0^{\binom{f}{r}-1}\varepsilon n^{f-r}.$$

Using Corollary 2.5.14 and a union bound, it is then easy to see that whp H_0 satisfies (2.10.15) for all $e \in \tilde{G}^{(r)}$.

Step 1.1: Defining 'sparse' induction graphs H_ℓ.

Consider $\ell \in [r-i-1]$ and let $i' := r - \ell$. Let $\xi_\ell := \nu_\ell^{8^f \cdot f+1}$. By (b) and Lemma 2.10.20 (with $G_\ell, 3\beta_{\ell-1}, \nu_\ell, \rho_\ell\mu, \rho_\ell^{f-r}\xi, i'$ playing the roles of $G, \varepsilon, \nu, \mu, \xi, i$), there exists a subgraph $H_\ell \subseteq G_\ell^{(r)}$ with $\Delta(H_\ell) \leq 1.1\nu_\ell n$ and the following property: for all $L \subseteq G_\ell^{(r)}$ with $\Delta(L) \leq 3\beta_{\ell-1}n$ and every $(r+1)$-graph O on $V(G_\ell)$ with $\Delta(O) \leq 3\beta_{\ell-1}n$, the following holds for $G' := G_\ell[H_\ell \triangle L] - O$:

$$\begin{aligned}&\mathcal{T}^{i'}, \mathcal{U}^{i'}, (\mathcal{P}_r^{i'}, \mathcal{P}_f^{i'})[G'] \text{ form a diagonal-dominant} \\ &(\sqrt{3\beta_{\ell-1}}, \rho_\ell\mu, \xi_\ell, f, r, i')\text{-setup for } G'.\end{aligned} \tag{2.10.16}$$

Step 1.2: Defining 'localised' cleaning graphs J_ℓ.

Again, consider $\ell \in [r-i-1]$ and let $i' := r - \ell$. Let

$$G_\ell^* := G_\ell - \{e \in G_\ell^{(r)} : e \text{ is } \mathcal{T}^{i'}\text{-important and } \tau_{\ell,r}(e) < \ell\}. \tag{2.10.17}$$

We claim that $G_\ell^*(T)[U_T] = G_\ell(T)[U_T]$ for every $T \in \mathcal{T}^{i'}$. Indeed, consider any $T \in \mathcal{T}^{i'}$ and $e \in G_\ell(T)[U_T]$. Hence, $e \subseteq U_T$ and $e \cup T \in G_\ell$. We need to show that $e \cup T \in G_\ell^*$, i.e. that there is no $\mathcal{T}^{i'}$-important r-subset e' of $e \cup T$ with $\tau_{\ell,r}(e') < \ell$. However, if $e' \in \binom{e \cup T}{r}$ is $\mathcal{T}^{i'}$-important, then $|e \cup T| \geq |e'| = r$ and since G_ℓ is r-exclusive with respect to $\mathcal{T}^{i'}$ by (I), we must have $T \subseteq e'$. As $e' \setminus T \subseteq e \subseteq U_T$, we deduce that $\tau_{\ell,r}(e') = |e' \cap U_T| = |e' \setminus T| = r - i' = \ell$.

Hence, by (c), for every $T \in \mathcal{T}^{i'}$, $G_\ell^*(T)[U_T]$ is a $(1.1\varepsilon, 0.9\xi, f-i', r-i')$-supercomplex. Thus, by Lemma 2.10.21 (with $G_\ell^*, 3\nu_\ell, \rho_\ell\mu, \beta_\ell, 0.9\xi$ playing the roles of $G, \varepsilon, \mu, \beta, \xi$), there exists a subgraph $J_\ell \subseteq G_\ell^{*(r)}$ with $\Delta(J_\ell) \leq 1.1\beta_\ell n$ and the following property: for all $L \subseteq G_\ell^{*(r)}$ with $\Delta(L) \leq 3\nu_\ell n$ and every $(r+1)$-graph O on $V(G_\ell^*)$ with $\Delta(O) \leq 3\nu_\ell n$, the following holds for $G^* := G_\ell^*[J_\ell \triangle L] - O$:

$$G^*(T)[U_T] \text{ is a } (\sqrt{3\nu_\ell}, 0.81\xi\beta_\ell^{(8^f)}, f-i', r-i')\text{-supercomplex for every } T \in \mathcal{T}^{i'}. \tag{2.10.18}$$

We have defined subgraphs $H_0, H_1, \ldots, H_{r-i-1}, J_1, \ldots, J_{r-i-1}$ of $G^{(r)} - \tau_r^{-1}(0)$. Note that they are not necessarily edge-disjoint. Let $H_0^* := H_0$ and for all $\ell \in [r-i-1]$ define inductively

$$H_\ell' := H_{\ell-1}^* \cup H_\ell,$$
$$H_\ell^* := H_{\ell-1}^* \cup H_\ell \cup J_\ell = H_\ell' \cup J_\ell,$$
$$H^* := H_{r-i-1}^*.$$

Clearly, $\Delta(H_\ell^*) \leq 2\beta_\ell n$ for all $\ell \in [r-i-1]_0$ and $\Delta(H_\ell') \leq 2\nu_\ell n$ for all $\ell \in [r-i-1]$. In particular, $\Delta(H^*) \leq 2\beta_{r-i-1} n \leq \nu n$, as desired.

Step 2: Covering down

Let L^* be any subgraph of $\tilde{G}^{(r)}$ with $\Delta(L^*) \leq \gamma n$ such that $H^* \cup L^*$ is F-divisible with respect to $\mathcal{S}$,$\mathcal{U}$, and let $O^* \subseteq \tilde{G}^{(r+1)}$ with $\Delta(O^*) \leq \gamma n$. We need to find a κ-well separated F-packing $\mathcal{F}$ in $\tilde{G}[H^* \cup L^*] - O^*$ which covers all edges of L^*, and covers all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$. We will do so by inductively showing that the following holds for all $\ell \in [r-i]$.

$(\#)_\ell$ There exists a $(3\ell\sqrt{\kappa})$-well separated F-packing $\mathcal{F}^*_{\ell-1}$ in $\tilde{G}[H^*_{\ell-1} \cup L^*] - O^*$ covering all edges of L^*, and all $\mathcal{S}$-important $e \in H^*_{\ell-1}$ with $\tau_r(e) < \ell$.

Clearly, $(\#)_{r-i}$ establishes (i).

Claim 1: $(\#)_1$ is true.

Proof of claim: Let $H'_0 := H_0 \cup L^* = H^*_0 \cup L^*$. By (2.10.15) and Proposition 2.5.7, for all $e \in L^*$ we have that

$$|(\tilde{G}[H'_0] - O^*)^{(f)}(e)| \geq |\tilde{G}[H_0 \cup e]^{(f)}(e)| - 2^r \gamma n^{f-r} \geq 0.8\beta_0^{\binom{f}{r}} n^{f-r}.$$

By Corollary 2.6.9, there is a 1-well separated F-packing $\mathcal{F}^*_0$ in $\tilde{G}[H'_0] - O^*$ covering all edges of L^*. Since H^*_0 does not contain any edges from $\tau_r^{-1}(0)$, $\mathcal{F}^*_0$ satisfies $(\#)_1$. $-$

If $i = r-1$, we can take $\mathcal{F}^*_0$ and complete the proof of (i). So assume that $i < r-1$ and that Lemma 2.10.24 holds for larger values of i.

Suppose that for some $\ell \in [r-i-1]$, $\mathcal{F}^*_{\ell-1}$ satisfies $(\#)_\ell$. Let $i' := r - \ell > i$. We will now find a $3\sqrt{\kappa}$-well separated F-packing $\mathcal{F}_\ell$ in $G[H^*_\ell] - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{*\leq(r+1)}_{\ell-1} - O^*$ such that $\mathcal{F}_\ell$ covers all edges of $H^*_\ell - \mathcal{F}^{*(r)}_{\ell-1}$ that belong to $\tau_r^{-1}(\ell)$.

Then $\mathcal{F}^*_\ell := \mathcal{F}^*_{\ell-1} \cup \mathcal{F}_\ell$ covers all edges of L^* and all $\mathcal{S}$-important $e \in H^*_\ell$ with $\tau_r(e) < \ell+1$. By Fact 2.5.4(ii), $\mathcal{F}^*_\ell$ is $(3\ell\sqrt{\kappa} + 3\sqrt{\kappa})$-well separated, implying that $(\#)_{\ell+1}$ is true.

Crucially, by (II), all the edges of $\tau_r^{-1}(\ell)$ that we seek to cover in this step are $\mathcal{T}^{i'}$-important. We will obtain $\mathcal{F}_\ell$ as the union of $\mathcal{F}^\circ_\ell$ and $\mathcal{F}^\dagger_\ell$, where

(COV1) $\mathcal{F}_\ell^\circ$ is $2\sqrt{\kappa}$-well separated F-packing in $G[H_\ell^*] - \mathcal{F}_{\ell-1}^{*(r)} - \mathcal{F}_{\ell-1}^{*\leq(r+1)} - O^*$ which covers all $\mathcal{T}^{i'}$-important edges of $H_\ell^* - \mathcal{F}_{\ell-1}^{*(r)}$ except possibly some from $\tau_{\ell,r}^{-1}(\ell)$;

(COV2) $\mathcal{F}_\ell^\dagger$ is a $\sqrt{\kappa}$-well separated F-packing in $G[H_\ell^*]-\mathcal{F}_{\ell-1}^{*(r)}-\mathcal{F}_\ell^{\circ(r)}-\mathcal{F}_{\ell-1}^{*\leq(r+1)}-\mathcal{F}_\ell^{\circ\leq(r+1)}-O^*$ which covers all $\mathcal{T}^{i'}$-important edges of $H_\ell^* - \mathcal{F}_{\ell-1}^{*(r)} - \mathcal{F}_\ell^{\circ(r)}$.

Since $\mathcal{F}_\ell^\dagger$ and $\mathcal{F}_\ell^\circ$ are $(r+1)$-disjoint, $\mathcal{F}_\ell := \mathcal{F}_\ell^\circ \cup \mathcal{F}_\ell^\dagger$ is $3\sqrt{\kappa}$-well separated by Fact 2.5.4(ii). Clearly, $\mathcal{F}_\ell$ covers all $\mathcal{T}^{i'}$-important edges of $H_\ell^* - \mathcal{F}_{\ell-1}^{*(r)}$, as required. We will obtain $\mathcal{F}_\ell^\circ$ by using (ii) of this lemma inductively, and $\mathcal{F}_\ell^\dagger$ by an application of the Localised cover down lemma (Lemma 2.10.8).

Recall that F-divisibility with respect to $\mathcal{T}^{i'},\mathcal{U}^{i'}$ was defined in Definition 2.10.23. Let $H_\ell'' := H_\ell' - \mathcal{F}_{\ell-1}^{*(r)}$.

Claim 2: H_ℓ'' is F-divisible with respect to $\mathcal{T}^{i'},\mathcal{U}^{i'}$.

Proof of claim: Let $T \in \mathcal{T}^{i'}$ and $b' \subseteq V(G) \setminus T$ with $|b'| \leq r - i' - 1$ and $|b' \setminus U_T| \geq 1$. We have to show that $Deg(F)_{i'+|b'|} \mid |H_\ell''(T \cup b')|$. Let $S := T\restriction_{\mathcal{S}}$ and $b := b' \cup (T \setminus S)$. Hence, $|b| = |b'| + i' - i$. Clearly, $b \subseteq V(G) \setminus S$, $|b| \leq r - i - 1$ and $|b \setminus U_S| \geq |T \setminus S| \geq 1$. Hence, since $H^* \cup L^*$ is F-divisible with respect to $\mathcal{S},\mathcal{U}$ by assumption, we have $Deg(F)_{i+|b|} \mid |(H^* \cup L^*)(S \cup b)|$, and this implies that $Deg(F)_{i+|b|} \mid |((H^* \cup L^*) - \mathcal{F}_{\ell-1}^{*(r)})(S \cup b)|$. It is thus sufficient to show that

$$H_\ell''(T \cup b') = ((H^* \cup L^*) - \mathcal{F}_{\ell-1}^{*(r)})(S \cup b).$$

Clearly, we have $T \cup b' = S \cup b$ and $H_\ell'' \subseteq H^* - \mathcal{F}_{\ell-1}^{*(r)}$. Conversely, observe that every $e \in H^* \cup L^*$ that contains $T \cup b'$ and is not covered by $\mathcal{F}_{\ell-1}^*$ must belong to H_ℓ''. Indeed, since e contains T, we have that $\tau_r(e) \leq r - i' = \ell$, so $e \in H_\ell^*$. Moreover, by $(\#)_\ell$ we must have $\tau_r(e) \geq \ell$. Hence, $\tau_r(e) = \ell$. But since $|b' \setminus U_T| \geq 1$, we have $\tau_{\ell,r}(e) < \ell$. By (2.10.17), $e \notin J_\ell$. Thus, $e \in H_\ell' - \mathcal{F}_{\ell-1}^{*(r)} = H_\ell''$. Hence, $H_\ell''(T \cup b') = ((H^* \cup L^*) - \mathcal{F}_{\ell-1}^{*(r)})(S \cup b)$. This implies the claim. —

Let $L_\ell' := H_\ell'' \triangle H_\ell$. So $H_\ell'' = H_\ell \triangle L_\ell'$.

Claim 3: $L'_\ell \subseteq G_\ell^{(r)}$ and $\Delta(L'_\ell) \le 3\beta_{\ell-1}n$.

Proof of claim: Suppose, for a contradiction, that there is $e \in H''_\ell \triangle H_\ell$ with $e \notin G_\ell^{(r)}$. Since $H_\ell \subseteq G_\ell^{(r)}$, we must have $e \in H''_\ell = H'_\ell - \mathcal{F}_{\ell-1}^{*(r)}$. Thus, since e is not covered by $\mathcal{F}^*_{\ell-1}$, $(\#)_\ell$ implies that e is $\mathcal{S}$-unimportant or $\tau_r(e) \ge \ell$, both contradicting $e \notin G_\ell^{(r)}$.

In order to see the second part, observe that $L'_\ell = ((H^*_{\ell-1} \cup H_\ell) - \mathcal{F}_{\ell-1}^{*(r)}) \triangle H_\ell \subseteq H^*_{\ell-1} \cup L^*$ since $\mathcal{F}_{\ell-1}^{*(r)} \subseteq L^* \cup H^*_{\ell-1}$. Thus, $\Delta(L'_\ell) \le \Delta(H^*_{\ell-1}) + \Delta(L^*) \le 3\beta_{\ell-1}n$. −

Note that Claim 3 implies that $H''_\ell \subseteq G_\ell^{(r)}$. Let $G_{\ell,ind} := G_\ell[H''_\ell] - \mathcal{F}_{\ell-1}^{*\le(r+1)} - O^*$. By Fact 2.5.4(i) and $(\#)_\ell$, we have that $\Delta(\mathcal{F}_{\ell-1}^{*\le(r+1)} \cup O^*) \le (3\ell\sqrt{\kappa})(f-r) + \gamma n \le 2\gamma n$. Thus, by (2.10.16) and Claim 3, $\mathcal{T}^{i'}$,$\mathcal{U}^{i'}$, $(\mathcal{P}_r^{i'}, \mathcal{P}_f^{i'})[G_{\ell,ind}]$ form a diagonal-dominant $(\sqrt{3\beta_{\ell-1}}, \rho_\ell\mu, \xi_\ell, f, r, i')$-setup for $G_{\ell,ind}$. We can thus apply Lemma 2.10.24(ii) inductively with the following objects/parameters.

object/parameter	$G_{\ell,ind}$	n	$\sqrt{3\beta_{\ell-1}}$	$\rho_\ell\mu$	ξ_ℓ	i'	$\mathcal{T}^{i'}$	$\mathcal{U}^{i'}$	$(\mathcal{P}_r^{i'}, \mathcal{P}_f^{i'})[G_{\ell,ind}]$	$\sqrt{\kappa}$	f	r	F
playing the role of	G	n	ε	μ	ξ	i	$\mathcal{S}$	$\mathcal{U}$	$(\mathcal{P}_r, \mathcal{P}_f)$	κ	f	r	F

Since $G_{\ell,ind}^{(r)} = H''_\ell$ is F-divisible with respect to $\mathcal{T}^{i'}$,$\mathcal{U}^{i'}$ by Claim 2, there exists a $2\sqrt{\kappa}$-well separated F-packing $\mathcal{F}^\circ_\ell$ in $G_{\ell,ind}$ covering all $\mathcal{T}^{i'}$-important edges of H''_ℓ except possibly some from $\tau_{\ell,r}^{-1}(r-i') = \tau_{\ell,r}^{-1}(\ell)$. Note that $H^*_\ell - H'_\ell \subseteq J_\ell$ and that every $\mathcal{T}^{i'}$-important edge of J_ℓ lies in $\tau_{\ell,r}^{-1}(\ell)$. Thus $\mathcal{F}^\circ_\ell$ does indeed cover all $\mathcal{T}^{i'}$-important edges of $H^*_\ell - \mathcal{F}_{\ell-1}^{*(r)}$ except possibly some from $\tau_{\ell,r}^{-1}(\ell)$, as required for (COV1).

We will now use J_ℓ to cover the remaining $\mathcal{T}^{i'}$-important edges of H^*_ℓ. Let $J'_\ell := H^*_\ell - \mathcal{F}_{\ell-1}^{*(r)} - \mathcal{F}_\ell^{\circ(r)}$. Let $S^*_{i'} \in \binom{V(F)}{i'}$ be such that $F(S^*_{i'})$ is non-empty.

Claim 4: $J'_\ell(T)[U_T]$ is $F(S^*_{i'})$-divisible for every $T \in \mathcal{T}^{i'}$.

Proof of claim: Let $T \in \mathcal{T}^{i'}$ and $b' \subseteq U_T$ with $|b'| \le r - i' - 1$. We have to show that $Deg(F(S^*_{i'}))_{|b'|} \mid |J'_\ell(T)[U_T](b')|$. Note that for every $e \in J'_\ell \subseteq G_\ell^{*(r)}$ containing T, we have $\tau_{\ell,r}(e) = r - i'$. Thus, $J'_\ell(T)[U_T]$ is identical with $J'_\ell(T)$ except for the different vertex sets. It is thus sufficient to show that $Deg(F(S^*_{i'}))_{|b'|} \mid |J'_\ell(T \cup b')|$. By Proposition 2.5.3, we have that $Deg(F(S^*_{i'}))_{|b'|} = Deg(F)_{i'+|b'|}$. Let $S := T\lceil_{\mathcal{S}}$ and $b := b' \cup (T \setminus S)$. By assumption, $H^* \cup L^*$ is F-divisible with respect to $\mathcal{S}$,$\mathcal{U}$. Thus, since $S \in \mathcal{S}$, $|b| \le r - i - 1$

and $|b \setminus U_S| \geq |T \setminus S| \geq 1$, we have that $Deg(F)_{i+|b|} \mid |(H^* \cup L^*)(S \cup b)|$. This implies that $Deg(F)_{i+|b|} \mid |((H^* \cup L^*) - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell)(S \cup b)|$. It is thus sufficient to prove that $J'_\ell(T \cup b') = ((H^* \cup L^*) - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell)(S \cup b)$. Clearly, $J'_\ell \subseteq H^* - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell$ by definition. Conversely, observe that every $e \in (H^* \cup L^*) - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell$ that contains $T \cup b'$ must belong to J'_ℓ. Indeed, since $L^* \subseteq \mathcal{F}^{*(r)}_{\ell-1}$, we have $e \in H^*$, and since e contains T, we have $\tau_r(e) \leq \ell$. Hence, $e \in H^*_\ell$ and thus $e \in J'_\ell$. This implies the claim. –

Let $L''_\ell := J'_\ell \triangle J_\ell$. So $J'_\ell = J_\ell \triangle L''_\ell$.

Claim 5: $L''_\ell \subseteq G^{*(r)}_\ell$ and $\Delta(L''_\ell) \leq 3\nu_\ell n$.

Proof of claim: Suppose, for a contradiction, that there is $e \in J'_\ell \triangle J_\ell$ with $e \notin G^{*(r)}_\ell$. By (2.10.14) and (2.10.17), the latter implies that e is $\mathcal{S}$-important with $\tau_r(e) < \ell$ or $\mathcal{T}^{i'}$-important with $\tau_{\ell,r}(e) < \ell$. However, since $J_\ell \subseteq G^{*(r)}_\ell$, we must have $e \in J'_\ell - J_\ell$ and thus $e \in H'_\ell$ and $e \notin \mathcal{F}^{*(r)}_{\ell-1} \cup \mathcal{F}^{\circ(r)}_\ell$. In particular, $e \in H''_\ell$. Now, if e was $\mathcal{S}$-important with $\tau_r(e) < \ell$, then $e \in H'_\ell - H_\ell \subseteq H^*_{\ell-1}$. But then e would be covered by $\mathcal{F}^*_{\ell-1}$, a contradiction. So e must be $\mathcal{T}^{i'}$-important with $\tau_{\ell,r}(e) < \ell$. But since $e \in H''_\ell$, e would be covered by $\mathcal{F}^\circ_\ell$ unless $\tau_{\ell,r}(e) = \ell$, a contradiction.

In order to see the second part, observe that

$$L''_\ell = ((H'_\ell \cup J_\ell) - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell) \triangle J_\ell \subseteq H'_\ell \cup L^*$$

since $\mathcal{F}^{*(r)}_{\ell-1} \cup \mathcal{F}^{\circ(r)}_\ell \subseteq H'_\ell \cup L^*$. Thus, $\Delta(L''_\ell) \leq \Delta(H'_\ell) + \Delta(L^*) \leq 3\nu_\ell n$. –

Note that Claim 5 implies that $J'_\ell \subseteq G^{*(r)}_\ell$. Let

$$G_{\ell,clean} := G^*_\ell[J'_\ell] - \mathcal{F}^{*\leq(r+1)}_{\ell-1} - \mathcal{F}^{\circ\leq(r+1)}_\ell - O^*.$$

By $(\#)_\ell$, (COV1) and Fact 2.5.4(i), we have that

$$\Delta(\mathcal{F}^{*\leq(r+1)}_{\ell-1} \cup \mathcal{F}^{\circ\leq(r+1)}_\ell \cup O^*) \leq (3\ell\sqrt{\kappa})(f-r) + (2\sqrt{\kappa})(f-r) + \gamma n \leq 2\gamma n.$$

Thus, by (2.10.18), Claim 4 and Claim 5, $G_{\ell,clean}(T)[U_T]$ is an $F(S^*_{i'})$-divisible $(\rho_\ell, \beta_\ell^{(8^f)+1}, f-i', r-i')$-supercomplex for every $T \in \mathcal{T}^{i'}$. Moreover, whenever there are $T \in \mathcal{T}^{(i')}$ and $e \in G^{(r)}_{\ell,clean} \subseteq G^{*(r)}_\ell$ with $T \subseteq e$, then $|(e \setminus T) \cap U_T| = \tau_{\ell,r}(e) = \ell = |e \setminus T|$ and thus $e \setminus T \subseteq U_T$. By (I), $G_{\ell,clean} \subseteq G_\ell$ is r-exclusive with respect to $\mathcal{T}^{i'}$, and by (a), $\mathcal{U}^{i'}$ is a (μ, ρ_ℓ, r)-focus for $\mathcal{T}^{i'}$. We can therefore apply the Localised cover down lemma (Lemma 2.10.8) with the following objects/parameters.

object/parameter	n	ρ_ℓ	μ	$\beta_\ell^{(8^f)+1}$	i'	$G_{\ell,clean}$	$\mathcal{T}^{i'}$	$\mathcal{U}^{i'}$	r	f	F	$S^*_{i'}$
playing the role of	n	ρ	ρ_{size}	ξ	i	G	$\mathcal{S}$	$\mathcal{U}$	r	f	F	S^*

This yields a $\rho_\ell^{-1/12}$-well separated F-packing $\mathcal{F}^\dagger_\ell$ in $G_{\ell,clean}$ covering all $\mathcal{T}^{i'}$-important edges of $G^{(r)}_{\ell,clean} = J'_\ell = H^*_\ell - \mathcal{F}^{*(r)}_{\ell-1} - \mathcal{F}^{\circ(r)}_\ell$. Thus $\mathcal{F}^\dagger_\ell$ is as required in (COV2). As observed before, this completes the proof of $(\#)_{\ell+1}$ and thus the proof of (i). □

Proof of (ii).

Let $Y \subseteq G^{(f)}$ and $A \in [0,1]^{(r+1)\times(f+1)}$ be such that (S1)–(S4) hold. We assume that $G^{(r)}$ is F-divisible with respect to $\mathcal{S},\mathcal{U}$ and that A is diagonal-dominant.

Claim 6: G is $(\xi - \varepsilon, f, r)$-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$.

Proof of claim: Let $e \in G^{(r)}$ and let $\ell' \in [r+1]$ be such that $e \in \mathcal{P}_r(\ell')$. Suppose first that $\ell' \leq i$. Then no f-set from $\mathcal{P}_f(\ell')$ contains any edge from $\tau_r^{-1}(0)$ (as such an f-set is $\mathcal{S}$-unimportant). Recall from (S2) for $\mathcal{S}, \mathcal{U}, (\mathcal{P}_r, \mathcal{P}_f)$ that $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f[Y])$ and $\min^{\backslash\backslash r-i+1}(A) \geq \xi$. Thus,

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(f)}(e)| \geq |(Y \cap \mathcal{P}_f(\ell'))(e)| \geq (a_{\ell',\ell'} - \varepsilon)n^{f-r} \geq (\xi - \varepsilon)n^{f-r}.$$

If $\ell' > i+1$, then by (P2′) in Proposition 2.10.12, no f-set from $\mathcal{P}_f(f-r+\ell')$ contains any edge from $\tau_r^{-1}(0)$. Thus, we have

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(f)}(e)| \geq (a_{\ell',f-r+\ell'} - \varepsilon)n^{f-r} \geq (\xi - \varepsilon)n^{f-r}.$$

If $\ell' = i+1$, then $\mathcal{P}_r(\ell') = \tau_r^{-1}(0)$ by (P2′). However, every f-set from $\tau_f^{-1}(f-r) = \mathcal{P}_f(f-r+\ell')$ that contains e contains no other edge from $\tau_r^{-1}(0)$. Thus,

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(f)}(e)| \geq (a_{\ell', f-r+\ell'} - \varepsilon)n^{f-r} \geq (\xi - \varepsilon)n^{f-r}.$$

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By Claim 6, we can choose $H^* \subseteq G^{(r)} - \tau_r^{-1}(0)$ such that (i) holds with G playing the role of $\tilde{G}$. Let

$$H_{nibble} := G^{(r)} - H^*.$$

Recall that by (S2), $G[Y]$ is (ε, A, f, r)-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f[Y])$, and (S3) implies that $G[Y]$ is $(\mu^f \xi, f+r, r)$-dense. Let

$$G_{nibble} := (G[Y])[H_{nibble}].$$

Using Proposition 2.5.7, it is easy to see that G_{nibble} is $(2^{r+1}\nu, A, f, r)$-regular with respect to $(\mathcal{P}_r, \mathcal{P}_f)[G_{nibble}]$. Moreover, by Proposition 2.5.9(ii), G_{nibble} is $(\mu^f \xi/2, f+r, r)$-dense. Thus, by Lemma 2.10.17, there exists $Y^* \subseteq G^{(f)}_{nibble}$ such that $G_{nibble}[Y^*]$ is $(\sqrt{\nu}, d, f, r)$-regular for $d := \min^{\setminus}(A) \geq \xi$ and $(0.45\mu^f \xi(\mu^f \xi/8(f+1))^{\binom{f+r}{f}}, f+r, r)$-dense. Thus, by Lemma 2.6.5 there is a κ-well separated F-packing $\mathcal{F}_{nibble}$ in $G_{nibble}[Y^*]$ such that $\Delta(L_{nibble}) \leq \gamma n$, where $L_{nibble} := G_{nibble}[Y^*]^{(r)} - \mathcal{F}^{(r)}_{nibble} = H_{nibble} - \mathcal{F}^{(r)}_{nibble}$. Since $G^{(r)}$ is F-divisible with respect to $\mathcal{S}, \mathcal{U}$, we clearly have that $H^* \cup L_{nibble} = G^{(r)} - \mathcal{F}^{(r)}_{nibble}$ is F-divisible with respect to $\mathcal{S}, \mathcal{U}$. By Fact 2.5.4(i), we have that $\Delta(\mathcal{F}^{\leq(r+1)}_{nibble}) \leq \kappa(f-r) \leq \gamma n$. Thus, by (i), there exists a κ-well separated F-packing $\mathcal{F}^*$ in $G[H^* \cup L_{nibble}] - \mathcal{F}^{\leq(r+1)}_{nibble}$ which covers all edges of L_{nibble}, and all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$. But then, by Fact 2.5.4(ii), $\mathcal{F}_{nibble} \cup \mathcal{F}^*$ is a 2κ-well separated F-packing in G which covers all $\mathcal{S}$-important r-edges except possibly some from $\tau_r^{-1}(r-i)$, completing the proof. □

This completes the proof of Lemma 2.10.24. □

2.11 Achieving divisibility

It remains to show that we can turn every F-divisible r-graph G into an F^*-divisible r-graph G' by removing a sparse F-decomposable subgraph of G, that is, to prove Lemma 2.9.4. Note that in Lemma 2.9.4, we do not need to assume that F^* is weakly regular. On the other hand, our argument heavily relies on the assumption that F^* is F-decomposable.

We first sketch the argument. Let F^* be F-decomposable, let $b_k := Deg(F^*)_k$ and $h_k := Deg(F)_k$. Clearly, we have $h_k \mid b_k$. First, consider the case $k = 0$. Then $b_0 = |F^*|$ and $h_0 = |F|$. We know that $|G|$ is divisible by h_0. Let $0 \leq x < b_0$ be such that $|G| \equiv x \mod b_0$. Since h_0 divides $|G|$ and b_0, it follows that $x = ah_0$ for some $0 \leq a < b_0/h_0$. Thus, removing a edge-disjoint copies of F from G yields an r-graph G' such that $|G'| = |G| - ah_0 \equiv 0 \mod b_0$, as desired. This will in fact be the first step of our argument.

We then proceed by achieving $Deg(G')_1 \equiv 0 \mod b_1$. Suppose that the vertices of G' are ordered $v_1, \dots, v_n$. We will construct a *degree shifter* which will fix the degree of v_1 by allowing the degree of v_2 to change, whereas all other degrees are unaffected (modulo b_1). Step by step, we will fix all the degrees from $v_1, \dots, v_{n-1}$. Fortunately, the degree of v_n will then automatically be divisible by b_1. For $k > 1$, we will proceed similarly, but the procedure becomes more intricate. It is in general impossible to shift degree from one k-set to another one without affecting the degrees of any other k-set. Roughly speaking, the degree shifter will contain a set of $2k$ special 'root vertices', and the degrees of precisely 2^k k-subsets of this root set change, whereas all other k-degrees are unaffected (modulo b_k). This will allow us to fix all the degrees of k-sets in G' except the ones inside some final $(2k-1)$-set, where we use induction on k as well. Fortunately, the remaining k-sets will again automatically satisfy the desired divisibility condition (cf. Lemma 2.11.5).

The proof of Lemma 2.9.4 divides into three parts. In the first subsection, we will construct the degree shifters. In the second subsection, we show on a very abstract level (without considering a particular host graph) how the shifting has to proceed in order to achieve overall divisibility. Finally, we will prove Lemma 2.9.4 by embedding our constructed shifters (using Lemma 2.5.20) according to the given shifting procedure.

2.11.1 Degree shifters

The aim of this subsection is to show the existence of certain r-graphs which we call degree shifters. They allow us to locally 'shift' degree among the k-sets of some host graph G.

Definition 2.11.1 (**x**-shifter)**.** Let $1 \leq k < r$ and let F, F^* be r-graphs. Given an r-graph T_k and distinct vertices $x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1$ of T_k, we say that T_k is an $(x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1)$-*shifter with respect to* F, F^* if the following hold:

(SH1) T_k has a 1-well separated F-decomposition $\mathcal{F}$ such that for all $F' \in \mathcal{F}$ and all $i \in [k]$, $|V(F') \cap \{x_i^0, x_i^1\}| \leq 1$;

(SH2) $|T_k(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k)$ with $|S| < k$;

(SH3) for all $S \in \binom{V(T_k)}{k}$,

$$|T_k(S)| \equiv \begin{cases} (-1)^{\sum_{i\in[k]} z_i} Deg(F)_k \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

We will now show that such shifters exist. Ultimately, we seek to find them as rooted subgraphs in some host graph G. Therefore, we impose additional conditions which will allow us to apply Lemma 2.5.20.

Lemma 2.11.2. *Let $1 \leq k < r$, let F, F^* be r-graphs and suppose that F^* has a 1-well separated F-decomposition $\mathcal{F}$. Let $f^* := |V(F^*)|$. There exists an $(x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1)$-shifter T_k with respect to F, F^* such that $T_k[X]$ is empty and T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X, where $X := \{x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1\}$.*

In order to prove Lemma 2.11.2, we will first prove a multigraph version (Lemma 2.11.4), which is more convenient for our construction. We will then recover the desired (simple)

r-graph by applying an operation similar to the extension operator $\nabla_{(F,e_0)}$ defined in Section 2.8.2. The difference is that instead of extending every edge to a copy of F, we will consider an F-decomposition of the multigraph shifter and then extend every copy of F in this decomposition to a copy of F^* (and then delete the original multigraph).

For a word $w = w_1 \dots w_k \in \{0,1\}^k$, let $|w|_0$ denote the number of 0's in w and let $|w|_1$ denote the number of 1's in w. Let $W_e(k)$ be the set of words $w \in \{0,1\}^k$ with $|w|_1$ being even, and let $W_o(k)$ be the set of words $w \in \{0,1\}^k$ with $|w|_1$ being odd.

Fact 2.11.3. *For every $k \geq 1$, $|W_e(k)| = |W_o(k)| = 2^{k-1}$.*

Lemma 2.11.4. *Let $1 \leq k < r$ and let F, F^* be r-graphs such that F^* is F-decomposable. Let $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1$ be distinct vertices. There exists a multi-r-graph T_k^* which satisfies (SH1)–(SH3), except that $\mathcal{F}$ does not need to be 1-well separated.*

Proof. Let $\mathcal{S}_k := \binom{V(F)}{k}$. For every $S^* \in \mathcal{S}_k$, we will construct a multi-r-graph T_{k,S^*} such that $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1 \in V(T_{k,S^*})$ and

(sh1) T_{k,S^*} has an F-decomposition $\mathcal{F}$ such that for all $F' \in \mathcal{F}$ and all $i \in [k]$, $|V(F') \cap \{x_i^0, x_i^1\}| \leq 1$;

(sh2) $|T_{k,S^*}(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_{k,S^*})$ with $|S| < k$;

(sh3) for all $S \in \binom{V(T_{k,S^*})}{k}$,

$$|T_{k,S^*}(S)| \equiv \begin{cases} (-1)^{\sum_{i\in[k]} z_i}|F(S^*)| \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

Following from this, it easy to construct T_k^* by overlaying the above multi-r-graphs T_{k,S^*}. Indeed, there are integers $(a'_{S^*})_{S^*\in\mathcal{S}_k}$ such that $\sum_{S^*\in\mathcal{S}_k} a'_{S^*}|F(S^*)| = Deg(F)_k$. Hence, there are positive integers $(a_{S^*})_{S^*\in\mathcal{S}_k}$ such that

$$\sum_{S^*\in\mathcal{S}_k} a_{S^*}|F(S^*)| \equiv Deg(F)_k \mod Deg(F^*)_k. \tag{2.11.1}$$

Therefore, we take T_k^* to be the union of a_{S^*} copies of T_{k,S^*} for each $S^* \in \mathcal{S}_k$. Then T_k^* has the desired properties.

Let $S^* \in \mathcal{S}_k$. It remains to construct T_{k,S^*}. Let $X_0 := \{x_1^0, \ldots, x_k^0\}$ and $X_1 := \{x_1^1, \ldots, x_k^1\}$. We may assume that $V(F^*) \cap (X_0 \cup X_1) = \emptyset$. Let $\mathcal{F}^*$ be an F-decomposition of F^* and $F' \in \mathcal{F}^*$. Let $X = \{x_1, \ldots, x_k\} \subseteq V(F')$ be the k-set which plays the role of S^* in F', in particular $|F'(X)| = |F(S^*)|$. We first define an auxiliary r-graph T_{1,x_k} as follows: Let F'' be obtained from F' by replacing x_k with a new vertex $\hat{x}_k$. Then let

$$T_{1,x_k} := (F^* - F') \mathbin{\dot\cup} F''.$$

Clearly, $(\mathcal{F}^* \setminus \{F'\}) \cup \{F''\}$ is an F-decomposition of T_{1,x_k}. Moreover, observe that for every set $S \subseteq V(T_{1,x_k})$ with $|S| < r$, we have

$$|T_{1,x_k}(S)| = \begin{cases} 0 & \text{if } \{x_k, \hat{x}_k\} \subseteq S; \\ |F^*(S)| & \text{if } \{x_k, \hat{x}_k\} \cap S = \emptyset; \\ |F^*(S)| - |F'(S)| & \text{if } x_k \in S, \hat{x}_k \notin S; \\ |F''(S)| = |F'((S \setminus \{\hat{x}_k\}) \cup \{x_k\})| & \text{if } x_k \notin S, \hat{x}_k \in S. \end{cases} \tag{2.11.2}$$

We now overlay copies of T_{1,x_k} in a suitable way in order to obtain the multi-r-graph T_{k,S^*}. The vertex set of T_{k,S^*} will be

$$V(T_{k,S^*}) = (V(F^*) \setminus X) \mathbin{\dot\cup} X_0 \mathbin{\dot\cup} X_1.$$

For every word $w = w_1 \ldots w_{k-1} \in \{0,1\}^{k-1}$, let T_w be a copy of T_{1,x_k}, where

(a) for each $i \in [k-1]$, $x_i^{w_i}$ plays the role of x_i (and $x_i^{1-w_i} \notin V(T_w)$);

(b) if $|w|_1$ is odd, then x_k^0 plays the role of x_k and x_k^1 plays the role of $\hat{x}_k$, whereas if $|w|_1$ is even, then x_k^0 plays the role of $\hat{x}_k$ and x_k^1 plays the role of x_k;

(c) the vertices in $V(T_{1,x_k}) \setminus \{x_1, \ldots, x_{k-1}, x_k, \hat{x}_k\}$ keep their role.

Let

$$T_{k,S^*} := \bigcup_{w\in\{0,1\}^{k-1}} T_w.$$

(Note that if $k = 1$, then T_{k,S^*} is just a copy of T_{1,x_k}, where x_1^0 plays the role of $\hat{x}_1$ and x_1^1 plays the role of x_1.) We claim that T_{k,S^*} satisfies (sh1)–(sh3). Clearly, (sh1) is satisfied because each T_w is a copy of T_{1,x_k} which is F-decomposable, and for all $w \in \{0,1\}^{k-1}$ and all $i \in [k-1]$, $|V(T_w) \cap \{x_i^0, x_i^1\}| = 1$, and since $x_k \notin V(F'')$.

We will now use (2.11.2) in order to determine an expression for $|T_{k,S^*}(S)|$ (see (2.11.3)) which will imply (sh2) and (sh3). Call $S \subseteq V(T_{k,S^*})$ *degenerate* if $\{x_i^0, x_i^1\} \subseteq S$ for some $i \in [k]$. Clearly, if S is degenerate, then $|T_w(S)| = 0$ for all $w \in \{0,1\}^{k-1}$. If $S \subseteq V(T_{k,S^*})$ is non-degenerate, define $I(S)$ as the set of all indices $i \in [k]$ such that $|S \cap \{x_i^0, x_i^1\}| = 1$, and define the 'projection'

$$\pi(S) := (S \setminus (X_0 \cup X_1)) \cup \{x_i \,:\, i \in I(S)\}.$$

Clearly, $\pi(S) \subseteq V(F^*)$ and $|\pi(S)| = |S|$. Note that if $S \subseteq V(T_w)$ and $k \notin I(S)$, then S plays the role of $\pi(S) \subseteq V(T_{1,x_k})$ in T_w by (a). For $i \in I(S)$, let $z_i(S) \in \{0,1\}$ be such that $S \cap \{x_i^0, x_i^1\} = \{x_i^{z_i(S)}\}$, and let $z(S) := \sum_{i\in I(S)} z_i(S)$. We claim that the following holds:

$$|T_{k,S^*}(S)| \equiv \begin{cases} (-1)^{z(S)}|F'(\pi(S))| \mod Deg(F^*)_{|S|} & \text{if } S \text{ is non-degenerate and } |I(S)| = k; \\ 0 \mod Deg(F^*)_{|S|} & \text{otherwise.} \end{cases} \tag{2.11.3}$$

As seen above, if S is degenerate, then we have $|T_{k,S^*}(S)| = 0$. From now on, we assume that S is non-degenerate. Let $W(S)$ be the set of words $w = w_1 \dots w_{k-1} \in \{0,1\}^{k-1}$ such that $w_i = z_i(S)$ for all $i \in I(S)\setminus\{k\}$. Clearly, if $w \in \{0,1\}^{k-1}\setminus W(S)$, then $|T_w(S)| = 0$ by (a). Suppose that $w \in W(S)$. If $k \notin I(S)$, then S plays the role of $\pi(S)$ in T_w and hence we have $|T_w(S)| = |T_{1,x_k}(\pi(S))| = |F^*(\pi(S))|$ by (2.11.2). It follows that $|T_{k,S^*}(S)| \equiv 0$

mod $Deg(F^*)_{|S|}$, as required.

From now on, suppose that $k \in I(S)$. Let

$$W_e(S) := \{w \in W(S) \,:\, |w|_1 + z_k(S) \text{ is even}\};$$
$$W_o(S) := \{w \in W(S) \,:\, |w|_1 + z_k(S) \text{ is odd}\}.$$

By (b), we know that $x_k^{z_k(S)}$ plays the role of x_k in T_w if $w \in W_o(S)$ and the role of $\hat{x}_k$ if $w \in W_e(S)$. Hence, if $w \in W_o(S)$ then S plays the role of $\pi(S)$ in T_w, and if $w \in W_e(S)$, then S plays the role of $(\pi(S) \setminus \{x_k\}) \cup \{\hat{x}_k\}$ in T_w. Thus, we have

$$|T_w(S)| = \begin{cases} |T_{1,x_k}(\pi(S))| \overset{(2.11.2)}{=} |F^*(\pi(S))| - |F'(\pi(S))| & \text{if } w \in W_o(S); \\ |T_{1,x_k}((\pi(S) \setminus \{x_k\}) \cup \{\hat{x}_k\})| \overset{(2.11.2)}{=} |F'(\pi(S))| & \text{if } w \in W_e(S); \\ 0 & \text{if } w \notin W(S). \end{cases}$$

It follows that

$$|T_{k,S^*}(S)| = \sum_{w \in \{0,1\}^{k-1}} |T_w(S)| \equiv (|W_e(S)| - |W_o(S)|)|F'(\pi(S))| \mod Deg(F^*)_{|S|}.$$

Observe that

$$|W_e(S)| = |\{w' \in \{0,1\}^{k-|I(S)|} \,:\, |w'|_1 + z(S) \text{ is even}\}|;$$
$$|W_o(S)| = |\{w' \in \{0,1\}^{k-|I(S)|} \,:\, |w'|_1 + z(S) \text{ is odd}\}|.$$

Hence, if $|I(S)| < k$, then by Fact 2.11.3 we have $|W_e(S)| = |W_o(S)| = 2^{k-|I(S)|-1}$. If $|I(S)| = k$, then $|W_e(S)| = 1$ if $z(S)$ is even and $|W_e(S)| = 0$ if $z(S)$ is odd, and for $W_o(S)$, the reverse holds. Altogether, this implies (2.11.3).

It remains to show that (2.11.3) implies (sh2) and (sh3). Clearly, (sh2) holds. Indeed, if $|S| < k$, then S is degenerate or we have $|I(S)| < k$, and (2.11.3) implies that $|T_{k,S^*}(S)| \equiv 0 \mod Deg(F^*)_{|S|}$.

Finally, consider $S \in \binom{V(T_{k,S^*})}{k}$. If S does not have the form $\{x_i^{z_i} : i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0,1\}$, then S is degenerate or $|I(S)| < k$ and (2.11.3) implies that $|T_{k,S^*}(S)| \equiv 0 \mod Deg(F^*)_k$, as required. Assume now that $S = \{x_i^{z_i} : i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0,1\}$. Then S is not degenerate, $I(S) = [k]$, $z(S) = \sum_{i\in[k]} z_i$ and $\pi(S) = \{x_1, \dots, x_k\} = X$, in which case (2.11.3) implies that

$$|T_{k,S^*}(S)| \equiv (-1)^{z(S)}|F'(X)| = (-1)^{z(S)}|F(S^*)| \mod Deg(F^*)_k,$$

as required for (sh3). □

Proof of Lemma 2.11.2. By applying Lemma 2.11.4 (with x_k^0 and x_k^1 swapping their roles), we can see that there exists a multi-r-graph T_k^* with $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1 \in V(T_k^*)$ such that the following properties hold:

- T_k^* has an F-decomposition $\{F_1, \dots, F_m\}$ such that for all $j \in [m]$ and all $i \in [k]$, we have $|V(F_j) \cap \{x_i^0, x_i^1\}| \leq 1$;
- $|T_k^*(S)| \equiv 0 \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k^*)$ with $|S| < k$;
- for all $S \in \binom{V(T_k^*)}{k}$,

$$|T_k^*(S)| \equiv \begin{cases} (-1)^{\sum_{i\in[k-1]} z_i + (1-z_k)} Deg(F)_k \mod Deg(F^*)_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod Deg(F^*)_k & \text{otherwise.} \end{cases}$$

Let $f := |V(F)|$. For every $j \in [m]$, let Z_j be a set of $f^* - f$ new vertices, such that $Z_j \cap Z_{j'} = \emptyset$ for all distinct $j, j' \in [m]$ and $Z_j \cap V(T_k^*) = \emptyset$ for all $j \in [m]$. Now, for every $j \in [m]$, let F_j^* be a copy of F^* on vertex set $V(F_j) \cup Z_j$ such that $\mathcal{F}_j \cup \{F_j\}$ is a 1-well separated F-decomposition of F_j^*. In particular, we have that

(a) $(F_j^* - F_j)[V(F_j)]$ is empty;

(b) $\mathcal{F}_j$ is a 1-well separated F-decomposition of $F_j^* - F_j$ such that for all $F' \in \mathcal{F}_j$, $|V(F') \cap V(F_j)| \leq r-1$.

Let

$$T_k := \dot{\bigcup_{j\in[m]}} (F_j^* - F_j).$$

We claim that T_k is the desired shifter. First, observe that T_k is a (simple) r-graph since $(F_j^* - F_j)[V(F_j)]$ is empty for every $j \in [m]$ by (a). Moreover, since $\mathcal{F}_1, \dots, \mathcal{F}_m$ are r-disjoint by (b), Fact 2.5.4(iii) implies that $\mathcal{F} := \mathcal{F}_1 \cup \dots \cup \mathcal{F}_m$ is a 1-well separated F-decomposition of T_k, and for each $j \in [m]$, all $F' \in \mathcal{F}_j$ and all $i \in [k]$, we have $|V(F') \cap \{x_i^0, x_i^1\}| \leq |V(F_j) \cap \{x_i^0, x_i^1\}| \leq 1$. Thus, (SH1) holds.

Moreover, note that for every $j \in [m]$, we have $|(F_j^* - F_j)(S)| \equiv -|F_j(S)| \mod Deg(F^*)_{|S|}$ for all $S \subseteq V(T_k)$ with $|S| \leq r-1$. Thus,

$$|T_k(S)| \equiv \sum_{j\in[m]} -|F_j(S)| = -|T_k^*(S)| \mod Deg(F^*)_{|S|}$$

for all $S \subseteq V(T_k)$ with $|S| \leq r-1$. Hence, (SH2) clearly holds. If $S = \{x_i^{z_i} : i \in [k]\}$ for suitable $z_1, \dots, z_k \in \{0, 1\}$, then

$$|T_k(S)| \equiv -|T_k^*(S)| \equiv (-1)^{\sum_{i\in[k]} z_i} Deg(F)_k \mod Deg(F^*)_k$$

and (SH3) holds. Thus, T_k is indeed an $(x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1)$-shifter with respect to F, F^*.

Finally, to see that T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X, consider the vertices of $V(T_k) \setminus X$ in an ordering where the vertices of $V(T_k^*) \setminus X$ precede all the vertices in sets Z_j, for $j \in [m]$. Note that $T_k[V(T_k^*)]$ is empty by (a), i.e. a vertex in $V(T_k^*) \setminus X$ has no 'backward' edges. Moreover, if $z \in Z_j$ for some $j \in [m]$, then $|T_k(\{z\})| = |F_j^*(\{z\})| \leq \binom{f^*-1}{r-1}$. □

2.11.2 Shifting procedure

In the previous section, we constructed degree shifters which allow us to locally change the degrees of k-sets in some host graph. We will now show how to combine these local shifts in order to transform any given F-divisible r-graph G into an F^*-divisible r-graph. It turns out to be more convenient to consider the shifting for 'r-set functions' rather than r-graphs. We will then recover the graph theoretical statement by considering a graph as an indicator set function (see below).

Let $\phi : \binom{V}{r} \to \mathbb{Z}$. (Think of ϕ as the multiplicity function of a multi-r-graph.) We extend ϕ to $\phi : \bigcup_{k \in [r]_0} \binom{V}{k} \to \mathbb{Z}$ by defining for all $S \subseteq V$ with $|S| = k \leq r$,

$$\phi(S) := \sum_{S' \in \binom{V}{r} : S \subseteq S'} \phi(S'). \tag{2.11.4}$$

Thus for all $0 \leq i \leq k \leq r$ and all $S \in \binom{V}{i}$,

$$\binom{r-i}{k-i} \phi(S) = \sum_{S' \in \binom{V}{k} : S \subseteq S'} \phi(S'). \tag{2.11.5}$$

For $k \in [r-1]_0$ and $b_0, \dots, b_k \in \mathbb{N}$, we say that ϕ is $(b_0, \dots, b_k)$-*divisible* if $b_{|S|} \mid \phi(S)$ for all $S \subseteq V$ with $|S| \leq k$.

If G is an r-graph with $V(G) \subseteq V$, we define $\mathbb{1}_G \colon \binom{V}{r} \to \mathbb{Z}$ as

$$\mathbb{1}_G(S) := \begin{cases} 1 & \text{if } S \in G; \\ 0 & \text{if } S \notin G. \end{cases}$$

and extend $\mathbb{1}_G$ as in (2.11.4). Hence, for a set $S \subseteq V$ with $|S| < r$, we have $\mathbb{1}_G(S) = |G(S)|$. Thus, (2.11.5) corresponds to the handshaking lemma for r-graphs (cf. (2.5.1)). Clearly, if G and G' are edge-disjoint, then we have $\mathbb{1}_G + \mathbb{1}_{G'} = \mathbb{1}_{G \cup G'}$. Moreover, for an r-graph F, G is F-divisible if and only if $\mathbb{1}_G$ is $(Deg(F)_0, \dots, Deg(F)_{r-1})$-divisible.

As mentioned before, our strategy is to successively fix the degrees of k-sets until we have fixed the degrees of all k-sets except possibly the degrees of those k-sets contained in some final vertex set K which is too small as to continue with the shifting. However, as the following lemma shows, divisibility is then automatically satisfied for all the k-sets lying inside K. For this to work it is essential that the degrees of all i-sets for $i < k$ are already fixed.

Lemma 2.11.5. *Let $1 \leq k < r$ and $b_0, \dots, b_k \in \mathbb{N}$ be such that $\binom{r-i}{k-i} b_i \equiv 0 \mod b_k$ for all $i \in [k]_0$. Let $\phi : \binom{V}{r} \to \mathbb{Z}$ be a $(b_0, \dots, b_{k-1})$-divisible function. Suppose that there exists a subset $K \subseteq V$ of size $2k-1$ such that if $S \in \binom{V}{k}$ with $\phi(S) \not\equiv 0 \mod b_k$, then $S \subseteq K$. Then ϕ is $(b_0, \dots, b_k)$-divisible.*

Proof. Let $\mathcal{K}$ be the set of all subsets T'' of K of size less than k. We first claim that for all $T'' \in \mathcal{K}$, we have

$$\sum_{T' \in \binom{K}{k} :\, T'' \subseteq T'} \phi(T') \equiv 0 \mod b_k. \tag{2.11.6}$$

Indeed, suppose that $|T''| = i < k$, then we have

$$\sum_{T' \in \binom{K}{k} :\, T'' \subseteq T'} \phi(T') \equiv \sum_{T' \in \binom{V}{k} :\, T'' \subseteq T'} \phi(T') \overset{(2.11.5)}{=} \binom{r-i}{k-i} \phi(T'') \mod b_k.$$

Since ϕ is $(b_0, \dots, b_{k-1})$-divisible, we have $\phi(T'') \equiv 0 \mod b_i$, and since $\binom{r-i}{k-i} b_i \equiv 0 \mod b_k$, the claim follows.

Let $T \in \binom{K}{k}$. We need to show that $\phi(T) \equiv 0 \mod b_k$. To this end, define the function $f \colon \mathcal{K} \to \mathbb{Z}$ as

$$f(T'') := \begin{cases} (-1)^{|T''|} & \text{if } T'' \subseteq K \setminus T; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for all $T' \in \binom{K}{k}$, we have

$$\sum_{T'' \subsetneq T'} f(T'') = \begin{cases} 1 & \text{if } T' = T; \\ 0 & \text{otherwise.} \end{cases} \tag{2.11.7}$$

Indeed, let $T' \in \binom{K}{k}$, and set $t := |T' \setminus T|$. We then check that (using $|K| < 2k$ in the first equality)

$$\sum_{T'' \subsetneq T'} f(T'') = \sum_{T'' \subseteq (K \setminus T) \cap T'} (-1)^{|T''|} = \sum_{j=0}^{t} (-1)^j \binom{t}{j} = \begin{cases} 1 & \text{if } t = 0; \\ 0 & \text{if } t > 0. \end{cases}$$

We can now conclude that

$$\phi(T) \overset{(2.11.7)}{=} \sum_{T' \in \binom{K}{k}} \phi(T') \sum_{T'' \subsetneq T'} f(T'') = \sum_{T'' \in \mathcal{K}} f(T'') \left(\sum_{T' \in \binom{K}{k} \colon T'' \subseteq T'} \phi(T') \right) \overset{(2.11.6)}{\equiv} 0 \mod b_k,$$

as desired. □

We now define a more abstract version of degree shifters, which we call adapters. They represent the effect of shifters and will finally be replaced by shifters again.

Definition 2.11.6 (**x**-adapter)**.** Let V be a vertex set and $k, r, b_0, \dots, b_k, h_k \in \mathbb{N}$ be such that $k < r$ and $h_k \mid b_k$. For distinct vertices $x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1$ in V, we say that $\tau \colon \binom{V}{r} \to \mathbb{Z}$ is an $(x_1^0, \dots, x_k^0, x_1^1, \dots, x_k^1)$*-adapter with respect to* $(b_0, \dots, b_k; h_k)$ if τ is $(b_0, \dots, b_{k-1})$-divisible and for all $S \in \binom{V}{k}$,

$$\tau(S) \equiv \begin{cases} (-1)^{\sum_{i \in [k]} z_i} h_k \mod b_k & \text{if } S = \{x_i^{z_i} : i \in [k]\}, \\ 0 \mod b_k & \text{otherwise.} \end{cases}$$

Note that such an adapter τ is $(b_0, \dots, b_{k-1}, h_k)$-divisible.

Fact 2.11.7. *If T is an $\mathbf{x}$-shifter with respect to F, F^*, then $\mathbb{1}_T$ is an $\mathbf{x}$-adapter with respect to $(Deg(F^*)_0, \ldots, Deg(F^*)_k; Deg(F)_k)$.*

The following definition is crucial for the shifting procedure. Given some function ϕ, we intend to add adapters in order to obtain a divisible function. Every adapter is characterised by a tuple $\mathbf{x}$ consisting of $2k$ distinct vertices, which tells us where to apply the adapter. All these tuples are contained within a multiset Ω, which we call a balancer. Ω is capable of dealing with any input function ϕ in the sense that there is a multisubset of Ω which tells us where to apply the adapters in order to make ϕ divisible. Moreover, as we finally want to replace the adapters by shifters (and thus embed them into some host graph), there must not be too many of them.

Definition 2.11.8 (balancer)**.** Let $r, k, b_0, \ldots, b_k \in \mathbb{N}$ with $k < r$ and let U, V be sets with $U \subseteq V$. Let Ω_k be a multiset containing ordered tuples $\mathbf{x} = (x_1, \ldots, x_{2k})$, where $x_1, \ldots, x_{2k} \in U$ are distinct. We say that Ω_k is a *$(b_0, \ldots, b_k)$-balancer for V with uniformity r acting on U* if for any $h_k \in \mathbb{N}$ with $h_k \mid b_k$, the following holds: let $\phi \colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \ldots, b_{k-1}, h_k)$-divisible function such that $S \subseteq U$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. There exists a multisubset Ω' of Ω_k such that $\phi + \tau_{\Omega'}$ is $(b_0, \ldots, b_k)$-divisible, where $\tau_{\Omega'} := \sum_{\mathbf{x} \in \Omega'} \tau_{\mathbf{x}}$ and $\tau_{\mathbf{x}}$ is any $\mathbf{x}$-adapter with respect to $(b_0, \ldots, b_k; h_k)$.

For a set $S \in \binom{V}{k}$, let $\deg_{\Omega_k}(S)$ be the number of $\mathbf{x} = (x_1, \ldots, x_{2k}) \in \Omega_k$ such that $|S \cap \{x_i, x_{i+k}\}| = 1$ for all $i \in [k]$. Furthermore, we denote $\Delta(\Omega_k)$ to be the maximum value of $\deg_{\Omega_k}(S)$ over all $S \in \binom{V}{k}$.

The following lemma shows that these balancers exist, i.e. that the local shifts performed by the degree shifters guaranteed by Lemma 2.11.2 are sufficient to obtain global divisibility (for which we apply Lemma 2.11.5).

Lemma 2.11.9. *Let $1 \leq k < r$. Let $b_0, \ldots, b_k \in \mathbb{N}$ be such that $\binom{r-s}{k-s} b_s \equiv 0 \mod b_k$ for all $s \in [k]_0$. Let U be a set of $n \geq 2k$ vertices and $U \subseteq V$. Then there exists a $(b_0, \ldots, b_k)$-balancer Ω_k for V with uniformity r acting on U such that $\Delta(\Omega_k) \leq 2^k (k!)^2 b_k$.*

Proof. We will proceed by induction on k. First, consider the case when $k = 1$. Write $U = \{v_1, \ldots, v_n\}$. Define Ω_1 to be the multiset containing precisely $b_1 - 1$ copies of (v_j, v_{j+1}) for all $j \in [n-1]$. Note that $\Delta(\Omega_1) \leq 2b_1$.

We now show that Ω_1 is a (b_0, b_1)-balancer for V with uniformity r acting on U. Let $\phi : \binom{V}{r} \to \mathbb{Z}$ be (b_0, h_1)-divisible for some $h_1 \in \mathbb{N}$ with $h_1 \mid b_1$, such that $v \in U$ whenever $v \in V$ and $\phi(\{v\}) \not\equiv 0 \mod b_1$. Let $m_0 := 0$. For each $j \in [n-1]$, let $0 \leq m_j < b_1$ be such that $(m_{j-1} - m_j)h_1 \equiv \phi(\{v_j\}) \mod b_1$. Let $\Omega' \subseteq \Omega_1$ consist of precisely m_j copies of (v_j, v_{j+1}) for all $j \in [n-1]$. Let $\tau := \sum_{\mathbf{x} \in \Omega'} \tau_{\mathbf{x}}$, where $\tau_{\mathbf{x}}$ is an $\mathbf{x}$-adapter with respect to $(b_0, b_1; h_1)$, and let $\phi' := \phi + \tau$. Clearly, ϕ' is (b_0)-divisible. Note that, for all $j \in [n-1]$,

$$\begin{aligned} \tau(\{v_j\}) &\equiv m_{j-1}\tau_{(v_{j-1}, v_j)}(\{v_j\}) + m_j\tau_{(v_j, v_{j+1})}(\{v_j\}) \mod b_1 \\ &\equiv (-m_{j-1} + m_j)h_1 \equiv -\phi(\{v_j\}) \mod b_1, \end{aligned} \tag{2.11.8}$$

implying that $\phi'(\{v_j\}) \equiv 0 \mod b_1$ for all $j \in [n-1]$. Moreover, for all $v \in V \setminus U$, we have $\phi(\{v\}) \equiv 0 \mod b_1$ by assumption and $\tau(\{v\}) \equiv 0 \mod b_1$ since no element of Ω_1 contains v. Thus, by Lemma 2.11.5 (with $\{v_n\}$ playing the role of K), ϕ' is (b_0, b_1)-divisible, as required.

We now assume that $k > 1$ and that the statement holds for smaller k. Again, write $U = \{v_1, \ldots, v_n\}$. For every $\ell \in [n]$, let $U_\ell := \{v_j : j \in [\ell]\}$. We construct Ω_k inductively. For each $\ell \in \{2k, \ldots, n\}$, we define a multiset $\Omega_{k,\ell}$ as follows. Let $\Omega_{k-1,\ell-1}$ be a $(b_1, \ldots, b_k)$-balancer for $V \setminus \{v_\ell\}$ with uniformity $r - 1$ acting on $U_{\ell-1}$ and

$$\Delta(\Omega_{k-1,\ell-1}) \leq 2^{k-1}(k-1)!^2 b_k.$$

(Indeed, $\Omega_{k-1,\ell-1}$ exists by our induction hypothesis with $r-1, k-1, b_1, \ldots, b_k, U_{\ell-1}, V \setminus \{v_\ell\}$ playing the roles of $r, k, b_0, \ldots, b_k, U, V$.) For each $\mathbf{v} = (v_{j_1}, \ldots, v_{j_{2k-2}}) \in \Omega_{k-1,\ell-1}$, let

$$\mathbf{v}' := (v_\ell, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_{\mathbf{v}}}, v_{j_k}, \ldots, v_{j_{2k-2}}) \in U_\ell \times U_{\ell-1}^{2k-1}, \tag{2.11.9}$$

such that $j_{\mathbf{v}} \in \{\ell - 2k + 1, \ldots, \ell\} \setminus \{\ell, j_1, \ldots, j_{2k-2}\}$ (which exists since $\ell \geq 2k$). We let $\Omega_{k,\ell} := \{\mathbf{v}' : \mathbf{v} \in \Omega_{k-1,\ell-1}\}$. Now, define

$$\Omega_k := \bigcup_{\ell=2k}^{n} \Omega_{k,\ell}.$$

Claim 1: $\Delta(\Omega_k) \leq 2^k (k!)^2 b_k$

Proof of claim: Consider any $S \in \binom{V}{k}$. Clearly, if $S \not\subseteq U$, then $\deg_{\Omega_k}(S) = 0$, so assume that $S \subseteq U$. Let i_0 be the largest $i \in [n]$ such that $v_i \in S$.

First note that for all $\ell \in \{2k, \ldots, n\}$, we have

$$\deg_{\Omega_{k,\ell}}(S) \leq \sum_{v \in S} \deg_{\Omega_{k-1,\ell-1}}(S \setminus \{v\}) \leq k\Delta(\Omega_{k-1,\ell-1}).$$

On the other hand, we claim that if $\ell < i_0$ or $\ell \geq i_0 + 2k$, then $\deg_{\Omega_{k,\ell}}(S) = 0$. Indeed, in the first case, we have $S \not\subseteq U_\ell$ which clearly implies that $\deg_{\Omega_{k,\ell}}(S) = 0$. In the latter case, for any $\mathbf{v} \in \Omega_{k-1,\ell-1}$, we have $j_{\mathbf{v}} \geq \ell - 2k + 1 > i_0$ and thus $|S \cap \{v_\ell, v_{j_{\mathbf{v}}}\}| = 0$, which also implies $\deg_{\Omega_{k,\ell}}(S) = 0$. Therefore,

$$\deg_{\Omega_k}(S) = \sum_{\ell=2k}^{n} \deg_{\Omega_{k,\ell}}(S) \leq 2k^2 \Delta(\Omega_{k-1,\ell-1}) \leq 2^k (k!)^2 b_k,$$

as required. –

We now show that Ω_k is indeed a $(b_0, \ldots, b_k)$-balancer on V with uniformity r acting on U. The key to this is the following claim, which we will apply repeatedly.

Claim 2: Let $2k \leq \ell \leq n$. Let $\phi_\ell \colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \ldots, b_{k-1}, h_k)$-divisible function for some $h_k \in \mathbb{N}$ with $h_k \mid b_k$. Suppose that if $\phi_\ell(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_\ell$. Then there exists $\Omega'_{k,\ell} \subseteq \Omega_{k,\ell}$ such that $\phi_{\ell-1} := \phi_\ell + \tau_{\Omega'_{k,\ell}}$ is $(b_0, \ldots, b_{k-1}, h_k)$-divisible and if $\phi_{\ell-1}(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_{\ell-1}$.

(Here, $\tau_{\Omega'_{k,\ell}}$ is as in Definition 2.11.8, i.e. $\tau_{\Omega'_{k,\ell}} := \sum_{\mathbf{v}' \in \Omega'_{k,\ell}} \tau_{\mathbf{v}'}$ and $\tau_{\mathbf{v}'}$ is an arbitrary $\mathbf{v}'$-adapter with respect to $(b_0, \ldots, b_k; h_k)$.)

Proof of claim: Define $\rho : \binom{V\setminus\{v_\ell\}}{r-1} \to \mathbb{Z}$ such that for all $S \in \binom{V\setminus\{v_\ell\}}{r-1}$,

$$\rho(S) := \phi_\ell(S \cup \{v_\ell\}).$$

It is easy to check that this identity transfers to smaller sets S, that is, for all $S \subseteq V\setminus\{v_\ell\}$, with $|S| \le r-1$, we have $\rho(S) = \phi_\ell(S\cup\{v_\ell\})$, where $\rho(S)$ and $\phi_\ell(S\cup\{v_\ell\})$ are as defined in (2.11.4).

Hence, since ϕ_ℓ is $(b_0,\dots,b_{k-1},h_k)$-divisible, ρ is $(b_1,\dots,b_{k-1},h_k)$-divisible. Moreover, for all $S \in \binom{V\setminus\{v_\ell\}}{k-1}$ with $\rho(S) \not\equiv 0 \mod b_k$, we have $S \subseteq U_{\ell-1}$.

Recall that $\Omega_{k-1,\ell-1}$ is a $(b_1,\dots,b_k)$-balancer for $V\setminus\{v_\ell\}$ with uniformity $r-1$ acting on $U_{\ell-1}$. Thus, there exists a multiset $\Omega' \subseteq \Omega_{k-1,\ell-1}$ such that

$$\rho + \tau_{\Omega'} \text{ is } (b_1,\dots,b_k)\text{-divisible.} \tag{2.11.10}$$

Let $\Omega'_{k,\ell} \subseteq \Omega_{k,\ell}$ be induced by Ω', that is, $\Omega'_{k,\ell} := \{\mathbf{v}' : \mathbf{v} \in \Omega'\}$ (see (2.11.9)). Let $\mathbf{v}' \in \Omega'_{k,\ell}$ and let $\tau_{\mathbf{v}'}$ be any $\mathbf{v}'$-adapter with respect to $(b_0,\dots,b_k;h_k)$. As noted after Definition 2.11.6, $\tau_{\mathbf{v}'}$ is $(b_0,\dots,b_{k-1},h_k)$-divisible. Crucially, if $S \in \binom{V}{k}$ and $v_\ell \in S$, then $\tau_{\mathbf{v}'}(S) \equiv \tau_{\mathbf{v}}(S\setminus\{v_\ell\}) \mod b_k$. Indeed, let $x_1^0,\dots,x_{k-1}^0,x_1^1,\dots,x_{k-1}^1$ be such that $\mathbf{v} = (x_1^0,\dots,x_{k-1}^0,x_1^1,\dots,x_{k-1}^1)$ and thus $\mathbf{v}' = (v_\ell,x_1^0,\dots,x_{k-1}^0,v_{j_\mathbf{v}},x_1^1,\dots,x_{k-1}^1)$. Then by Definition 2.11.6, as $v_\ell \in S$, we have

$$\begin{aligned}\tau_{\mathbf{v}'}(S) &\equiv \begin{cases}(-1)^{0+\sum_{i\in[k-1]} z_i} h_k \mod b_k & \text{if } S\setminus\{v_\ell\} = \{x_i^{z_i} : i \in [k-1]\},\\ 0 \mod b_k & \text{otherwise,}\end{cases}\\ &\equiv \tau_{\mathbf{v}}(S\setminus\{v_\ell\}) \mod b_k.\end{aligned}$$

Let $\tau_{\Omega'_{k,\ell}} := \sum_{\mathbf{v}'\in\Omega'_{k,\ell}} \tau_{\mathbf{v}'}$ and $\phi_{\ell-1} := \phi_\ell + \tau_{\Omega'_{k,\ell}}$. Note that for all $S \not\subseteq U_\ell$, we have $\tau_{\Omega'_{k,\ell}}(S) = 0$ by (2.11.9). Moreover, if $S \in \binom{V}{k}$ and $v_\ell \in S$, then $\tau_{\Omega'_{k,\ell}}(S) \equiv \tau_{\Omega'}(S\setminus\{v_\ell\}) \mod b_k$ by the above.

Clearly, $\phi_{\ell-1}$ is $(b_0,\dots,b_{k-1},h_k)$-divisible. Now, consider any $S \in \binom{V}{k}$ with $S \not\subseteq U_{\ell-1}$.

If $S \not\subseteq U_\ell$, then

$$\phi_{\ell-1}(S) = \phi_\ell(S) + \tau_{\Omega'_{k,\ell}}(S) \equiv 0 + 0 \equiv 0 \mod b_k.$$

If $S \subseteq U_\ell$, then since $S \not\subseteq U_{\ell-1}$ we must have $v_\ell \in S$, and so

$$\phi_{\ell-1}(S) = \phi_\ell(S) + \tau_{\Omega'_{k,\ell}}(S) \equiv \rho(S \setminus \{v_\ell\}) + \tau_{\Omega'}(S \setminus \{v_\ell\}) \overset{(2.11.10)}{\equiv} 0 \mod b_k.$$

This completes the proof of the claim. —

Now, let $h_k \in \mathbb{N}$ with $h_k \mid b_k$ and let $\phi\colon \binom{V}{r} \to \mathbb{Z}$ be any $(b_0, \ldots, b_{k-1}, h_k)$-divisible function such that $S \subseteq U$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. Let $\phi_n := \phi$ and note that $U = U_n$. Thus, by Claim 2, there exists $\Omega'_{k,n} \subseteq \Omega_{k,n}$ such that $\phi_{n-1} := \phi_n + \tau_{\Omega'_{k,n}}$ is $(b_0, \ldots, b_{k-1}, h_k)$-divisible and if $\phi_{n-1}(S) \not\equiv 0 \mod b_k$ for some $S \in \binom{V}{k}$, then $S \subseteq U_{n-1}$. Repeating this step finally yields some $\Omega'_k \subseteq \Omega_k$ such that $\phi^* := \phi + \tau_{\Omega'_k}$ is $(b_0, \ldots, b_{k-1}, h_k)$-divisible and such that $S \subseteq U_{2k-1}$ whenever $S \in \binom{V}{k}$ and $\phi(S) \not\equiv 0 \mod b_k$. By Lemma 2.11.5 (with U_{2k-1} playing the role of K), ϕ^* is then $(b_0, \ldots, b_k)$-divisible. Thus Ω_k is indeed a $(b_0, \ldots, b_k)$-balancer. □

2.11.3 Proof of Lemma 2.9.4

We now prove Lemma 2.9.4. For this, we consider the balancers Ω_k guaranteed by Lemma 2.11.9. Recall that these consist of suitable adapters, and that Lemma 2.11.2 guarantees the existence of shifters corresponding to these adapters. It remains to embed these shifters in a suitable way, which is achieved via Lemma 2.5.20. The following fact will help us to verify the conditions of Lemma 2.11.9.

Fact 2.11.10. *Let F be an r-graph. Then for all $0 \le i \le k < r$, we have $\binom{r-i}{k-i} Deg(F)_i \equiv 0 \mod Deg(F)_k$.*

Proof. Let S be any i-set in $V(F)$. By (2.5.1), we have that

$$\binom{r-i}{k-i}|F(S)| = \sum_{T\in\binom{V(F)}{k}:\, S\subseteq T} |F(T)| \equiv 0 \mod \mathit{Deg}(F)_k,$$

and this implies the claim. □

Proof of Lemma 2.9.4. Let $x^0_1,\dots,x^0_{r-1},x^1_1,\dots,x^1_{r-1}$ be distinct vertices (not in $V(G)$). For $k\in[r-1]$, let $X_k := \{x^0_1,\dots,x^0_k,x^1_1,\dots,x^1_k\}$. By Lemma 2.11.2, for every $k\in[r-1]$, there exists an $(x^0_1,\dots,x^0_k,x^1_1,\dots,x^1_k)$-shifter T_k with respect to F,F^* such that $T_k[X_k]$ is empty and T_k has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X_k. Note that (SH1) implies that

$$|T_k(\{x^0_i,x^1_i\})| = 0 \text{ for all } i\in[k]. \tag{2.11.11}$$

We may assume that there exists $t\geq \max_{k\in[r-1]}|V(T_k)|$ such that $1/n \ll \gamma \ll 1/t \ll \xi, 1/f^*$. Let $\mathit{Deg}(F)=(h_0,h_1,\dots,h_{r-1})$ and let $\mathit{Deg}(F^*)=(b_0,b_1,\dots,b_{r-1})$. Since F^* is F-decomposable and thus F-divisible, we have $h_k \mid b_k$ for all $k\in[r-1]_0$.

By Fact 2.11.10, we have $\binom{r-i}{k-i}b_i\equiv 0 \mod b_k$ for all $0\leq i\leq k<r$. For each $k\in[r-1]$ with $h_k<b_k$, we apply Lemma 2.11.9 to obtain a $(b_0,\dots,b_k)$-balancer Ω_k for $V(G)$ with uniformity r acting on $V(G)$ such that $\Delta(\Omega_k)\leq 2^k(k!)^2b_k$. For values of k for which we have $h_k=b_k$, we let $\Omega_k:=\emptyset$. For every $k\in[r-1]$ and every $\mathbf{v}=(v_1,\dots,v_{2k})\in\Omega_k$, define the labelling $\Lambda_{\mathbf{v}}\colon X_k\to V(G)$ by setting $\Lambda_{\mathbf{v}}(x^0_i):=v_i$ and $\Lambda_{\mathbf{v}}(x^1_i):=v_{i+k}$ for all $i\in[k]$.

For technical reasons, let T_0 be a copy of F and let $X_0:=\emptyset$. Let Ω_0 be the multiset containing b_0/h_0 copies of $\emptyset$, and for every $\mathbf{v}\in\Omega_0$, let $\Lambda_{\mathbf{v}}\colon X_0\to V(G)$ be the trivial G-labelling of (T_0,X_0). Note that T_0 has degeneracy at most $\binom{f^*-1}{r-1}$ rooted at X_0. Note also that $\Lambda_{\mathbf{v}}$ does not root any set $S\subseteq V(G)$ with $|S|\in[r-1]$.

We will apply Lemma 2.5.20 in order to find faithful embeddings of the T_k into G. Let $\Omega:=\bigcup_{k=0}^{r-1}\Omega_k$. Let $\alpha:=\gamma^{-2}/n$.

Claim 1: For every $k \in [r-1]$ and every $S \subseteq V(G)$ with $|S| \in [r-1]$, we have $|\{\mathbf{v} \in \Omega_k : \Lambda_{\mathbf{v}} \text{ roots } S\}| \leq r^{-1}\alpha\gamma n^{r-|S|}$. Moreover, $|\Omega_k| \leq r^{-1}\alpha\gamma n^r$.

Proof of claim: Let $k \in [r-1]$ and $S \subseteq V(G)$ with $|S| \in [r-1]$. Consider any $\mathbf{v} = (v_1, \dots, v_{2k}) \in \Omega_k$ and suppose that $\Lambda_{\mathbf{v}}$ roots S, i.e. $S \subseteq \{v_1, \dots, v_{2k}\}$ and $|T_k(\Lambda_{\mathbf{v}}^{-1}(S))| > 0$. Note that if we had $\{x_i^0, x_i^1\} \subseteq \Lambda_{\mathbf{v}}^{-1}(S)$ for some $i \in [k]$ then $|T_k(\Lambda_{\mathbf{v}}^{-1}(S))| = 0$ by (2.11.11), a contradiction. We deduce that $|S \cap \{v_i, v_{i+k}\}| \leq 1$ for all $i \in [k]$, in particular $|S| \leq k$. Thus there exists $S' \supseteq S$ with $|S'| = k$ and such that $|S' \cap \{v_i, v_{i+k}\}| = 1$ for all $i \in [k]$. However, there are at most $n^{k-|S|}$ sets S' with $|S'| = k$ and $S' \supseteq S$, and for each such S', the number of $\mathbf{v} = (v_1, \dots, v_{2k}) \in \Omega_k$ with $|S' \cap \{v_i, v_{i+k}\}| = 1$ for all $i \in [k]$ is at most $\Delta(\Omega_k)$. Thus, $|\{\mathbf{v} \in \Omega_k : \Lambda_{\mathbf{v}} \text{ roots } S\}| \leq n^{k-|S|}\Delta(\Omega_k) \leq n^{r-1-|S|}2^k(k!)^2 b_k \leq r^{-1}\alpha\gamma n^{r-|S|}$. Similarly, we have $|\Omega_k| \leq n^k\Delta(\Omega_k) \leq r^{-1}\alpha\gamma n^r$. −

Claim 1 implies that for every $S \subseteq V(G)$ with $|S| \in [r-1]$, we have

$$|\{\mathbf{v} \in \Omega : \Lambda_{\mathbf{v}} \text{ roots } S\}| \leq \alpha\gamma n^{r-|S|} - 1,$$

and we have $|\Omega| \leq b_0/h_0 + \sum_{k=1}^{r-1} |\Omega_k| \leq \alpha\gamma n^r$. Therefore, by Lemma 2.5.20, for every $k \in [r-1]_0$ and every $\mathbf{v} \in \Omega_k$, there exists a $\Lambda_{\mathbf{v}}$-faithful embedding $\phi_{\mathbf{v}}$ of (T_k, X_k) into G, such that, letting $T_{\mathbf{v}} := \phi_{\mathbf{v}}(T_k)$, the following hold:

(a) for all distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$, the hulls of $(T_{\mathbf{v}_1}, \mathrm{Im}(\Lambda_{\mathbf{v}_1}))$ and $(T_{\mathbf{v}_2}, \mathrm{Im}(\Lambda_{\mathbf{v}_2}))$ are edge-disjoint;

(b) for all $\mathbf{v} \in \Omega$ and $e \in O$ with $e \subseteq V(T_{\mathbf{v}})$, we have $e \subseteq \mathrm{Im}(\Lambda_{\mathbf{v}})$;

(c) $\Delta(\bigcup_{\mathbf{v} \in \Omega} T_{\mathbf{v}}) \leq \alpha\gamma^{(2^{-r})}n$.

Note that by (a), all the graphs $T_{\mathbf{v}}$ are edge-disjoint. Let

$$D := \bigcup_{\mathbf{v} \in \Omega} T_{\mathbf{v}}.$$

By (c), we have $\Delta(D) \leq \gamma^{-2}$. We will now show that D is as desired.

For every $k \in [r-1]$ and $\mathbf{v} \in \Omega_k$, we have that $T_{\mathbf{v}}$ is a $\mathbf{v}$-shifter with respect to F, F^* by definition of $\Lambda_{\mathbf{v}}$ and since $\phi_{\mathbf{v}}$ is $\Lambda_{\mathbf{v}}$-faithful. Thus, by Fact 2.11.7,

$$\mathbb{1}_{T_{\mathbf{v}}} \text{ is a } \mathbf{v}\text{-adapter with respect to } (b_0, \ldots, b_k; h_k). \tag{2.11.12}$$

Claim 2: For every $\Omega' \subseteq \Omega$, $\bigcup_{\mathbf{v}\in\Omega'} T_{\mathbf{v}}$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.

Proof of claim: Clearly, for every $\mathbf{v} \in \Omega_0$, $T_{\mathbf{v}}$ is a copy of F and thus has a 1-well separated F-decomposition $\mathcal{F}_{\mathbf{v}} = \{T_{\mathbf{v}}\}$. Moreover, for each $k \in [r-1]$ and all $\mathbf{v} = (v_1, \ldots, v_{2k}) \in \Omega_k$, $T_{\mathbf{v}}$ has a 1-well separated F-decomposition $\mathcal{F}_{\mathbf{v}}$ by (SH1) such that for all $F' \in \mathcal{F}_{\mathbf{v}}$ and all $i \in [k]$, $|V(F') \cap \{v_i, v_{i+k}\}| \leq 1$.

In order to prove the claim, it is thus sufficient to show that for all distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$, $\mathcal{F}_{\mathbf{v}_1}$ and $\mathcal{F}_{\mathbf{v}_2}$ are r-disjoint (implying that $\mathcal{F} := \bigcup_{\mathbf{v}\in\Omega'} \mathcal{F}_{\mathbf{v}}$ is 1-well separated by Fact 2.5.4(iii)) and that for every $\mathbf{v} \in \Omega$, $\mathcal{F}_{\mathbf{v}}^{\leq(r+1)}$ and O are edge-disjoint.

To this end, we first show that for every $\mathbf{v} \in \Omega$ and $F' \in \mathcal{F}_{\mathbf{v}}$, we have that $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| < r$ and every $e \in \binom{V(F')}{r}$ belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$. If $\mathbf{v} \in \Omega_0$, this is clear since $\mathrm{Im}(\Lambda_{\mathbf{v}}) = \emptyset$ and $F' = T_{\mathbf{v}}$, so suppose that $\mathbf{v} = (v_1, \ldots, v_{2k}) \in \Omega_k$ for some $k \in [r-1]$. (In particular, $h_k < b_k$.) By the above, we have $|V(F') \cap \{v_i, v_{i+k}\}| \leq 1$ for all $i \in [k]$. In particular, $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| \leq k < r$, as desired. Moreover, suppose that $e \in \binom{V(F')}{r}$. If $e \cap \mathrm{Im}(\Lambda_{\mathbf{v}}) = \emptyset$, then e belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$, so suppose further that $S := e \cap \mathrm{Im}(\Lambda_{\mathbf{v}})$ is not empty. Clearly, $|S \cap \{v_i, v_{i+k}\}| \leq |V(F') \cap \{v_i, v_{i+k}\}| \leq 1$ for all $i \in [k]$. Thus, there exists $S' \supseteq S$ with $|S'| = k$ and $|S' \cap \{v_i, v_{i+k}\}| = 1$ for all $i \in [k]$. By (SH3) (and since $h_k < b_k$), we have that $|T_{\mathbf{v}}(S')| > 0$, which clearly implies that $|T_{\mathbf{v}}(S)| > 0$. Thus, $e \cap \mathrm{Im}(\Lambda_{\mathbf{v}}) = S$ is a root of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$ and therefore e belongs to the hull of $(T_{\mathbf{v}}, \mathrm{Im}(\Lambda_{\mathbf{v}}))$.

Now, consider distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$ and suppose, for a contradiction, that there is $e \in \binom{V(G)}{r}$ such that $e \subseteq V(F') \cap V(F'')$ for some $F' \in \mathcal{F}_{\mathbf{v}_1}$ and $F'' \in \mathcal{F}_{\mathbf{v}_2}$. But by the above, e belongs to the hulls of both $(T_{\mathbf{v}_1}, \mathrm{Im}(\Lambda_{\mathbf{v}_1}))$ and $(T_{\mathbf{v}_2}, \mathrm{Im}(\Lambda_{\mathbf{v}_2}))$, a contradiction to (a).

Finally, consider $\mathbf{v} \in \Omega$ and $e \in O$. We claim that $e \notin \mathcal{F}_{\mathbf{v}}^{\leq(r+1)}$. Let $F' \in \mathcal{F}_{\mathbf{v}}$ and suppose, for a contradiction, that $e \subseteq V(F')$. By (b), we have $e \subseteq \mathrm{Im}(\Lambda_{\mathbf{v}})$. On the other hand, by the above, we have $|V(F') \cap \mathrm{Im}(\Lambda_{\mathbf{v}})| < r$, a contradiction. $-$

Clearly, D is F-divisible by Claim 2. We will now show that for every F-divisible r-graph H on $V(G)$ which is edge-disjoint from D, there exists a subgraph $D^* \subseteq D$ such that $H \cup D^*$ is F^*-divisible and $D - D^*$ has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq(r+1)}$ and O are edge-disjoint.

Let H be any F-divisible r-graph on $V(G)$ which is edge-disjoint from D. We will inductively prove that the following holds for all $k \in [r-1]_0$:

SHIFT$_k$ there exists $\Omega_k^* \subseteq \Omega_0 \cup \dots \cup \Omega_k$ such that $\mathbb{1}_{H \cup D_k^*}$ is $(b_0, \dots, b_k)$-divisible, where $D_k^* := \bigcup_{\mathbf{v} \in \Omega_k^*} T_{\mathbf{v}}$.

We first establish SHIFT$_0$. Since H is F-divisible, we have $|H| \equiv 0 \mod h_0$. Since $h_0 \mid b_0$, there exists some $0 \leq a < b_0/h_0$ such that $|H| \equiv a h_0 \mod b_0$. Let Ω_0^* be the multisubset of Ω_0 consisting of $b_0/h_0 - a$ copies of $\emptyset$. Let $D_0^* := \bigcup_{\mathbf{v} \in \Omega_0^*} T_{\mathbf{v}}$. Hence, D_0^* is the edge-disjoint union of $b_0/h_0 - a$ copies of F. We thus have $|H \cup D_0^*| \equiv a h_0 + |F|(b_0/h_0 - a) \equiv a h_0 + b_0 - a h_0 \equiv 0 \mod b_0$. Therefore, $\mathbb{1}_{H \cup D_0^*}$ is (b_0)-divisible, as required.

Suppose now that SHIFT$_{k-1}$ holds for some $k \in [r-1]$, that is, there is $\Omega_{k-1}^* \subseteq \Omega_0 \cup \dots \cup \Omega_{k-1}$ such that $\mathbb{1}_{H \cup D_{k-1}^*}$ is $(b_0, \dots, b_{k-1})$-divisible, where $D_{k-1}^* := \bigcup_{\mathbf{v} \in \Omega_{k-1}^*} T_{\mathbf{v}}$. Note that D_{k-1}^* is F-divisible by Claim 2. Thus, since both H and D_{k-1}^* are F-divisible, we have $\mathbb{1}_{H \cup D_{k-1}^*}(S) = |(H \cup D_{k-1}^*)(S)| \equiv 0 \mod h_k$ for all $S \in \binom{V(G)}{k}$. Hence, $\mathbb{1}_{H \cup D_{k-1}^*}$ is in fact $(b_0, \dots, b_{k-1}, h_k)$-divisible. Thus, if $h_k = b_k$, then $\mathbb{1}_{H \cup D_{k-1}^*}$ is $(b_0, \dots, b_k)$-divisible and we let $\Omega_k' := \emptyset$. Now, assume that $h_k < b_k$. Recall that Ω_k is a $(b_0, \dots, b_k)$-balancer and that $h_k \mid b_k$. Thus, there exists a multisubset Ω_k' of Ω_k such that the function $\mathbb{1}_{H \cup D_{k-1}^*} + \sum_{\mathbf{v} \in \Omega_k'} \tau_{\mathbf{v}}$ is $(b_0, \dots, b_k)$-divisible, where $\tau_{\mathbf{v}}$ is any $\mathbf{v}$-adapter with respect to

$(b_0, \ldots, b_k; h_k)$. Recall that by (2.11.12) we can take $\tau_{\mathbf{v}} = \mathbb{1}_{T_{\mathbf{v}}}$. In both cases, let

$$\Omega_k^* := \Omega_{k-1}^* \cup \Omega_k' \subseteq \Omega_0 \cup \cdots \cup \Omega_k;$$
$$D_k' := \bigcup_{\mathbf{v} \in \Omega_k'} T_{\mathbf{v}};$$
$$D_k^* := \bigcup_{\mathbf{v} \in \Omega_k^*} T_{\mathbf{v}} = D_{k-1}^* \cup D_k'.$$

Thus, $\sum_{\mathbf{v} \in \Omega_k'} \tau_{\mathbf{v}} = \mathbb{1}_{D_k'}$ and hence $\mathbb{1}_{H \cup D_k^*} = \mathbb{1}_{H \cup D_{k-1}^*} + \mathbb{1}_{D_k'}$ is $(b_0, \ldots, b_k)$-divisible, as required.

Finally, SHIFT_{r-1} implies that there exists $\Omega_{r-1}^* \subseteq \Omega$ such that $\mathbb{1}_{H \cup D^*}$ is $(b_0, \ldots, b_{r-1})$-divisible, where $D^* := \bigcup_{\mathbf{v} \in \Omega_{r-1}^*} T_{\mathbf{v}}$. Clearly, $D^* \subseteq D$, and we have that $H \cup D^*$ is F^*-divisible. Finally, by Claim 2,

$$D - D^* = \bigcup_{\mathbf{v} \in \Omega \setminus \Omega_{r-1}^*} T_{\mathbf{v}}$$

has a 1-well separated F-decomposition $\mathcal{F}$ such that $\mathcal{F}^{\leq (r+1)}$ and O are edge-disjoint, completing the proof. □

CHAPTER 3

THE DECOMPOSITION THRESHOLD OF A GIVEN GRAPH

This chapter contains an overview of the results proved in [35]. The proofs themselves are omitted in the thesis because of space constraints.

In this chapter, we investigate the F-decomposition threshold δ_F in the graph setting. In particular, we determine δ_F for all bipartite graphs, improve existing bounds for general F and present a 'discretisation' result for the possible values of δ_F. We write $gcd(F) := Deg(F)_1$ for the greatest common divisor of the vertex degrees of F. Also, we use standard graph theory notation and write $e(G)$ for the number of edges of G, and $d_G(x)$ for the degree of x in G. Thus, a graph G is F-divisible if $e(F) \mid e(G)$ and $gcd(F) \mid d_G(x)$ for all $x \in V(G)$.

Recall that the main achievement of an absorption approach is to turn an approximate decomposition into a full decomposition. In the quasirandom setting (and more generally that of supercomplexes as in Chapter 2), approximate decompositions can be obtained 'on the spot' by using a nibble approach. In the minimum degree setting, we pursue a different approach. We assume the ability to get approximate decompositions above a certain minimum degree threshold (via blackbox results) and investigate under which conditions such approximate decompositions can be completed to real decompositions. More precisely, given a graph F, we define an approximate decomposition threshold δ_F^{0+} and then aim to determine δ_F up to the unknown δ_F^{0+}. In order to determine δ_F, it would then suffice to investigate δ_F^{0+}, which is a much simpler task.

3.1 A discretisation result

Our first main result (Theorem 3.1.1) bounds the decomposition threshold δ_F in terms of the approximate decomposition threshold δ_F^{0+}, the fractional decomposition threshold δ_F^*, and the threshold δ_F^e for covering a given edge. We now introduce these formally.

Let F be a fixed graph. For $\eta \geq 0$, an *η-approximate F-decomposition* of an n-vertex graph G is a collection of edge-disjoint copies of F contained in G which together cover all but at most ηn^2 edges of G. Let δ_F^η be the infimum of all $\delta \geq 0$ with the following property: there exists an $n_0 \in \mathbb{N}$ such that whenever G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$, then G has an η-approximate F-decomposition. Clearly, $\delta_F^{\eta'} \geq \delta_F^\eta$ whenever $\eta' \leq \eta$. We let $\delta_F^{0+} := \sup_{\eta>0} \delta_F^\eta$.

Let G^F be the set of copies of F in G. A *fractional F-decomposition of* G is a function $\omega\colon G^F \to [0,1]$ such that, for each $e \in E(G)$,

$$\sum_{F' \in G^F\,:\, e \in E(F')} \omega(F') = 1. \tag{3.1.1}$$

Note that every F-decomposition is a fractional F-decomposition where $\omega(F) \in \{0,1\}$.

Let δ_F^* be the infimum of all $\delta \geq 0$ with the following property: there exists an $n_0 \in \mathbb{N}$ such that whenever G is an F-divisible graph on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$, then G has a fractional F-decomposition. Usually the definition considers all graphs G (and not only those which are F-divisible) but it is convenient for us to make this additional restriction here as δ_F^* is exactly the relevant parameter when investigating δ_F (in particular, we trivially have $\delta_F^* \leq \delta_F$). Haxell and Rödl [44] used Szemerédi's regularity lemma to show that a fractional F-decomposition of a graph G can be turned into an approximate F-decomposition of G. This can be used to show that $\delta_F^{0+} \leq \delta_F^*$.

Let δ_F^e be the infimum of all $\delta \geq 0$ with the following property: there exists an $n_0 \in \mathbb{N}$

such that whenever G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$, and e' is an edge in G, then G contains a copy of F which contains e'.

Our first result bounds δ_F in terms of the approximate decomposition threshold δ_F^{0+} and the chromatic number of F. Parts (ii) and (iii) give much more precise information if $\chi \geq 5$. We obtain a 'discretisation result' in terms of the parameters introduced above. We do not believe that this result extends to $\chi = 3, 4$. On the other hand, we do have $\delta_F \in \{0, 1/2, 2/3\}$ if $\chi(F) = 2$ (see Section 3.3). We also believe that none of the terms in the discretisation statement can be omitted.

Theorem 3.1.1. *Let F be a graph with $\chi := \chi(F)$.*

(i) Then $\delta_F \leq \max\{\delta_F^{0+}, 1 - 1/(\chi + 1)\}$.

(ii) If $\chi \geq 5$, then $\delta_F \in \{\max\{\delta_F^{0+}, \delta_F^{e}\}, 1 - 1/\chi, 1 - 1/(\chi + 1)\}$.

(iii) If $\chi \geq 5$, then $\delta_F \in \{\delta_F^{*}, 1 - 1/\chi, 1 - 1/(\chi + 1)\}$.

Theorem 3.1.1(i) improves a bound of $\delta_F \leq \max\{\delta_F^{0+}, 1 - 1/3r\}$ proved in [9] for r-regular graphs F. Also, the cases where $F = K_3$ or C_4 of (i) were already proved in [9].

Since it is known that $\delta_{K_r}^{0+} \geq 1 - 1/(r + 1)$ (see e.g. [91]), Theorem 3.1.1 implies that the decomposition threshold for cliques equals its fractional relaxation.

Corollary 3.1.2. *For all $r \geq 3$, $\delta_{K_r} = \delta_{K_r}^{*} = \delta_{K_r}^{0+}$.*

3.2 Explicit bounds

Theorem 3.1.1 involves several 'auxiliary thresholds' and parameters that play a role in the construction of an F-decomposition. Bounds on these of course lead to better 'explicit' bounds on δ_F which we now discuss.

The central conjecture in the area is due to Nash-Williams [69] (for the triangle case) and Gustavsson [40] (for the general case).

Conjecture 3.2.1 (Gustavsson [40], Nash-Williams [69])**.** For every $r \geq 3$, there exists an $n_0 = n_0(r)$ such that every K_r-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/(r+1))n$ has a K_r-decomposition.

For general F, the following conjecture provides a natural upper bound for δ_F which would be best possible for the case of cliques. It is not clear to us what a formula for general F might look like.

Conjecture 3.2.2. For all graphs F, $\delta_F \leq 1 - 1/(\chi(F) + 1)$.

Note that by Theorem 3.1.1 in order to prove Conjecture 3.2.2 it suffices to show $\delta_F^{0+} \leq 1 - 1/(\chi(F) + 1)$. This in turn implies that Conjecture 3.2.2 is actually a special case of Conjecture 3.2.1. Indeed, it follows from a result of Yuster [93] that for every graph F, $\delta_F^{0+} \leq \delta_{K_{\chi(F)}}^{0+}$, and thus $\delta_F^{0+} \leq \delta^*_{K_{\chi(F)}} \leq \delta_{K_{\chi(F)}}$.

In view of this, bounds on $\delta^*_{K_r}$ are of considerable interest. The following result gives the best bound for general r (see [8]) and triangles (see [25]).

Theorem 3.2.3 ([8], [25])**.**

(i) For every $r \geq 3$, we have $\delta^*_{K_r} \leq 1 - 10^{-4} r^{-3/2}$.

(ii) $\delta^*_{K_3} \leq 9/10$.

This improved earlier bounds by Yuster [91] and Dukes [26, 27]. Together with the results in [9], part (ii) implies $\delta_{K_3} \leq 9/10$. More generally, combining Theorem 3.2.3 and Theorem 3.1.1(i) with the fact that $\delta_F^{0+} \leq \delta_{K_{\chi(F)}}^{0+} \leq \delta^*_{K_{\chi(F)}}$, one obtains the following explicit upper bound on the decomposition threshold.

Corollary 3.2.4.

(i) For every graph F, $\delta_F \leq 1 - 10^{-4} \chi(F)^{-3/2}$.

(ii) If $\chi(F) = 3$, then $\delta_F \leq 9/10$.

Here, (i) improves a bound of $1 - 1/\max\{10^4\chi(F)^{3/2}, 6e(F)\}$ obtained by combining the results of [8] and [9] (see [8]). It also improves earlier bounds by Gustavsson [40] and Yuster [91, 94]. A bound of $1 - \varepsilon$ also follows from the results of Keevash [49].

In the r-partite setting an analogue of Corollary 3.1.2 was proved in [10], an analogue of Theorem 3.2.3(i) (with weaker bounds) in [68] and an analogue of Theorem 3.2.3(ii) (again with weaker bounds) in [14]. These bounds can be combined to give results on the completion of (mutually orthogonal) partially filled in Latin squares. Moreover, it turns out that if $\delta_F > \delta_F^*$ (in the non-partite setting), then there exist extremal graphs that are extremely close to large complete partite graphs, which adds further relevance to results on the r-partite setting.

3.3 Decompositions into bipartite graphs

Let F be a bipartite graph. Yuster [90] showed that $\delta_F = 1/2$ if F is connected and contains a vertex of degree one. Moreover, Barber, Kühn, Lo and Osthus [9] showed that $\delta_{C_4} = 2/3$ and $\delta_{C_\ell} = 1/2$ for all even $\ell \geq 6$ (which improved a bound of $\delta_{C_4} \leq 31/32$ by Bryant and Cavenagh [16]). Here we generalise these results to arbitrary bipartite graphs.

Note that if F is bipartite, then $\delta_F^{0+} = 0$. This is a consequence of the fact that bipartite graphs have vanishing Turán density. This allows us to determine δ_F for any bipartite graph F. It would be interesting to see if this can be generalised to r-partite r-graphs.

To state our result, we need the following definitions. A set $X \subseteq V(F)$ is called C_4-*supporting in* F if there exist distinct $a, b \in X$ and $c, d \in V(F) \setminus X$ such that $ac, bd, cd \in E(F)$. We define

$$\tau(F) := gcd\{e(F[X]) \,:\, X \subseteq V(F) \text{ is not } C_4\text{-supporting in } F\},$$

$$\tilde{\tau}(F) := gcd\{e(C) \,:\, C \text{ is a component of } F\}.$$

So for example $\tau(F) = 1$ if there exists an edge in F that is not contained in any cycle of length 4, and $\tilde{\tau}(F) > 1$ if F is connected (and $e(F) \geq 2$). The definition of τ can be motivated by considering the following graph G: Let A, B, C be sets of size $n/3$ with $G[A], G[C]$ complete, B independent and $G[A, B]$ and $G[B, C]$ complete bipartite. Note that $\delta(G) \sim 2n/3$. It turns out that the extremal examples which we construct showing $\delta_F \geq 2/3$ for certain bipartite graphs F are all similar to G. Moreover, $\tau(F) = 1$ if for any large c there is a set of copies of F in G whose number of edges in $G[A]$ add up to c.

We note that $\tau(F) \mid gcd(F)$ and $gcd(F) \mid \tilde{\tau}(F)$. The following theorem determines δ_F for every bipartite graph F.

Theorem 3.3.1. *Let F be a bipartite graph. Then*

$$\delta_F = \begin{cases} 2/3 & \text{if } \tau(F) > 1; \\ 0 & \text{if } \tilde{\tau}(F) = 1 \text{ and } F \text{ has a bridge}; \\ 1/2 & \text{otherwise.} \end{cases}$$

The next corollary translates Theorem 3.3.1 into explicit results for important classes of bipartite graphs.

Corollary 3.3.2. *The following hold.*

(i) Let $s, t \in \mathbb{N}$ with $s + t > 2$. Then $\delta_{K_{s,t}} = 1/2$ if s and t are coprime and $\delta_{K_{s,t}} = 2/3$ otherwise.

(ii) If $gcd(F) = 1$ and F is connected, then $\delta_F = 1/2$.

(iii) If F is connected and has an edge that is not contained in any cycle of length 4, then $\delta_F = 1/2$.

(For (ii) and (iii) recall that we always assume $e(F) \geq 2$.) Note that $\tau(K_{s,t}) = gcd(s, t)$. Then (i)–(iii) follow from the definitions of τ and $\tilde{\tau}$.

3.4 Near-optimal decompositions

Along the way to proving Theorem 3.1.1 we obtain the following bound guaranteeing a 'near-optimal' decomposition. For this, let δ_F^{vx} be the infimum of all $\delta \geq 0$ with the following property: there exists an $n_0 \in \mathbb{N}$ such that whenever G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$, and x is a vertex of G with $gcd(F) \mid d_G(x)$, then G contains a collection $\mathcal{F}$ of edge-disjoint copies of F such that $\{xy \,:\, y \in N_G(x)\} \subseteq \bigcup \mathcal{F}$. Loosely speaking, δ_F^{vx} is the threshold that allows us to cover all edges at one vertex. For example, if F is a triangle, then δ_F^{vx} is essentially the threshold that $N_G(x)$ contains a perfect matching whenever $d_G(x)$ is even. Note that $\delta_F^{vx} \geq \delta_F^{e}$.

The following theorem roughly says that if we do not require to cover all edges of G with edge-disjoint copies of F, but accept a bounded number of uncovered edges, then the minimum degree required can be less than if we need to cover all edges.

Theorem 3.4.1. *For any graph F and $\mu > 0$ there exists a constant $C = C(F, \mu)$ such that whenever G is an F-divisible graph on n vertices satisfying*

$$\delta(G) \geq (\max\{\delta_F^{0+}, \delta_F^{vx}\} + \mu)n$$

then G contains a collection of edge-disjoint copies of F covering all but at most C edges.

It can be shown that $\delta_F^{vx} \leq 1 - 1/\chi(F)$. For many bipartite graphs F, e.g. trees and complete balanced bipartite graphs, our results imply that $\max\{\delta_F^{0+}, \delta_F^{vx}\} < \delta_F$. It seems plausible to believe that there also exist graphs F with $\chi(F) \geq 3$ such that $\max\{\delta_F^{0+}, \delta_F^{vx}\} < \delta_F$. However, the current bounds on δ_F^{0+} do not suffice to verify this.

3.5 Overview of the proofs

One key ingredient in the proofs of Theorems 3.1.1, 3.3.1 and 3.4.1 is the iterative absorption method. As in Chapter 2, we carry out this iteration inside a vortex until we have a 'near-optimal decomposition' which covers all but a bounded number of edges. The corresponding 'Cover down lemma' is much easier than in the hypergraph setting. Roughly speaking, we show that if G is a graph with $\delta(G) \geq (\max\{\delta_F^{0+}, \delta_F^{vx}\} + o(1))|V(G)|$, then we can cover down into a 'random-like' subset $U \subseteq V(G)$. Here, δ_F^{0+} is needed to obtain an approximate decomposition, and the definition of δ_F^{vx} is used to 'clean' the remaining edges at vertices which lie outside U. Intuitively, it is also clear that δ_F^{0+} and δ_F^{vx} should be lower bounds for δ_F and thus that the Cover down lemma performs optimally for our purposes (see Corollary 11.4 in [35]). The iterative application of the Cover down lemma yields a 'near-optimal decomposition'. Theorem 3.4.1 is a byproduct of this.

As in Chapter 2, the idea to deal with the final leftover is to use 'exclusive absorbers', and each absorber is constructed as a concatenation of transformers and certain canonical structures between them. This approach was first introduced in [9]. For more details on this part of the argument, we refer to Section 2.3.3.

The difficulty here is to construct transformers with 'low degeneracy' which can be embedded once the minimum degree of the host graph is large enough. The crucial feature in proving our results here, which allows us to go significantly beyond the results in [9], is to break down the construction of transformers into even smaller pieces. We construct them from building blocks called 'switchers'. These switchers are transformers with more limited capabilities. The most important switchers are C_6-switchers and $K_{2,r}$-switchers. A C_6-switcher S transforms the perfect matching $E^+ := \{u_1u_2, u_3u_4, u_5u_6\}$ into its 'complement' $E^- := \{u_2u_3, u_4u_5, u_6u_1\}$ along a 6-cycle. (The formal requirement is that both $S \cup E^+$ and $S \cup E^-$ have an F-decomposition.) A $K_{2,r}$-switcher transforms a star with r leaves centred at x into a star with the same leaves centred at x'. Surprisingly, it turns out that these building blocks suffice to build the desired transformers.

Apart from proving the existence of switchers, we also need to be able to find them in G. This is where we may need the condition that $\delta(G) \geq (1 - 1/(\chi + 1) + o(1))|V(G)|$. To achieve this, we will apply Szemerédi's regularity lemma to G to obtain its reduced graph R. We will then find a 'compressed' version (i.e. a suitable homomorphism) of the switcher in R. This then translates to the existence of the desired switcher in G via standard regularity techniques.

The switchers are also key to our discretisation results in Theorem 3.1.1(ii) and (iii). We show that if $\delta_F < 1 - 1/(\chi + 1)$, then to find the relevant switchers (and hence, as described above, the relevant absorbers) we need the graph G only to have minimum degree $(1 - 1/\chi + o(1))|V(G)|$. Roughly speaking, the idea is that if $\delta_F < 1 - 1/(\chi + 1)$, then the minimum degree of an F-divisible graph which is close to a sufficiently large complete $(\chi + 1)$-partite graph is large enough to guarantee an F-decomposition. In particular, we can find S such that $S \cup \{u_1u_2, u_3u_4\}$ is such a graph. Moreover, the divisibility of $S \cup \{u_2u_3, u_1u_4\}$ follows automatically. Thus, by the definition of δ_F, both have an F-decomposition, i.e. S is a C_4-switcher (see Lemma 10.1 in [35]). The switcher S may be quite large indeed, but the fact that it is $(\chi+1)$-partite will allow us to embed it in a graph G with $(1-1/\chi+o(1))|V(G)|$ using regularity methods. Recall that to build transformers, we need C_6-switchers and $K_{2,r}$-switchers, whilst our implicit construction above yields C_4-switchers. An important part of the proof of the discretisation results in Theorem 3.1.1(ii) and (iii) are several 'reductions'. For example, we can build a C_6-switcher by combining C_4-switchers in a suitable way. These reductions are also the reason why we need the assumption $\chi \geq 5$.

Similarly, if $\delta_F < 1 - 1/\chi$, the minimum degree we require is only $(1 - 1/(\chi - 1) + o(1))|V(G)|$. As discussed earlier we require the minimum degree to be at least $(\max\{\delta_F^{0+}, \delta_F^{vx}\} + o(1))|V(G)|$ in order to iteratively cover all but a constant number of edges in G (see Theorem 3.4.1). This may not be sufficiently high to construct our absorbers, but this discretisation argument allows us to conclude that if δ_F exceeds $\max\{\delta_F^{0+}, \delta_F^{vx}\}$ then it can take at most two other values, $1 - 1/(\chi + 1)$ or $1 - 1/\chi$.

Note that the parameter δ_F^{vx} does not appear in Theorem 3.1.1. We investigate δ_F^{vx} separately. Note that if $F = K_r$, then the problem of covering all edges at a vertex x reduces to finding a K_{r-1}-factor on the neighbours of x. As discussed in Section 1.2, factor problems are much easier than decomposition problems. The Hajnal-Szemerédi theorem implies here that $\delta^{vx}_{K_r} \le 1 - 1/r$. For general F, the determination of δ^{vx}_F does not reduce to a 'pure' factor problem. We use a theorem of Komlós [53] on approximate F-factors to reduce δ^{vx}_F to δ^e_F.

Most of the above steps are common to the proof of Theorems 3.1.1 and 3.3.1, i.e. we can prove them in a unified way. The key additional difficulty in the bipartite case is proving the existence of a C_6-switcher for those F with $\delta_F = 1/2$.

CHAPTER 4

OPTIMAL PATH AND CYCLE DECOMPOSITIONS

This chapter contains an overview of the results proved in [39]. The proofs themselves are omitted in the thesis because of space constraints. Section 4.3 is based on [38].

There are several longstanding and beautiful conjectures on decompositions of graphs into cycles and/or paths. In this chapter, we consider four of the most well-known in the setting of dense quasirandom and random graphs: the Erdős-Gallai conjecture, Gallai's conjecture on path decompositions, the linear arboricity conjecture as well as the overfull subgraph conjecture.

4.1 Decompositions of random graphs

A classical result of Lovász [65] on decompositions of graphs states that the edges of any graph on n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ cycles and paths. Erdős and Gallai [29, 30] made the related conjecture that the edges of every graph G on n vertices can be decomposed into $\mathcal{O}(n)$ cycles and edges. Conlon, Fox and Sudakov [21] recently showed that $\mathcal{O}(n \log\log n)$ cycles and edges suffice and that the conjecture holds for graphs with linear minimum degree. They also proved that the conjecture holds whp for the binomial random graph $G \sim \mathcal{G}(n,p)$. Korándi, Krivelevich and Sudakov [55] carried out a more systematic study of the problem for $\mathcal{G}(n,p)$: for a large range of p, whp $\mathcal{G}(n,p)$

can be decomposed into $n/4+np/2+o(n)$ cycles and edges, which is asymptotically best possible. They also asked for improved error terms. For constant p, we give an exact formula.

A further related conjecture of Gallai (see [65]) states that every connected graph on n vertices can be decomposed into $\lceil n/2 \rceil$ paths. The result of Lovász mentioned above implies that for every (not necessarily connected) graph, $n-1$ paths suffice. This has been improved to $\lfloor 2n/3 \rfloor$ paths [23, 88]. Here we determine the number of paths in an optimal path decomposition of $\mathcal{G}(n,p)$ for constant p. In particular this implies that Gallai's conjecture holds (with room to spare) for almost all graphs.

Next, recall that an edge colouring of a graph is a partition of its edge set into matchings. A matching can be viewed as a forest whose connected components are edges. As a relaxation of this, a *linear forest* is a forest whose components are paths, and the least possible number of linear forests needed to partition the edge set of a graph G is called the *linear arboricity* of G, denoted by $la(G)$. Clearly, in order to cover all edges at any vertex of maximum degree, we need at least $\lceil \Delta(G)/2 \rceil$ linear forests. However, for some graphs (e.g. complete graphs on an odd number of vertices) we need at least $\lceil (\Delta(G)+1)/2 \rceil$ linear forests. The following conjecture is known as the linear arboricity conjecture and can be viewed as an analogue to Vizing's theorem.

Conjecture 4.1.1 (Akiyama, Exoo, Harary [1])**.** For every graph G, $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$.

This is equivalent to the statement that for all d-regular graphs G, $la(G) = \lceil (d+1)/2 \rceil$. Alon [2] proved an approximate version of the conjecture for sufficiently large values of $\Delta(G)$. Using his approach, McDiarmid and Reed [67] confirmed the conjecture for random regular graphs with fixed degree. We show that, for a large range of p, whp the random graph $G \sim \mathcal{G}(n,p)$ can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests. Moreover, we use the recent confirmation [22] of the so-called 'Hamilton decomposition conjecture' to deduce that the linear arboricity conjecture holds for large and sufficiently dense regular graphs (see Corollary 6.4 in [39]).

The following theorem summarises our optimal decomposition results for dense random graphs. We denote by $odd(G)$ the number of odd degree vertices in a graph G.

Theorem 4.1.2. *Let $0 < p < 1$ be constant and let $G \sim \mathcal{G}(n,p)$. Then whp the following hold:*

(i) G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.

(ii) G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.

(iii) G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests, i.e. $la(G) = \lceil \Delta(G)/2 \rceil$.

Clearly, each of the given bounds is best possible. Moreover, as observed e.g. in [55], for a large range of p, whp $odd(\mathcal{G}(n,p)) = (1+o(1))n/2$. This means that for fixed $p < 1/2$, the size of an optimal path decomposition of $\mathcal{G}(n,p)$ is determined by the number of odd degree vertices, whereas for $p > 1/2$, the maximum degree is the crucial parameter.

A related result of Gao, Pérez-Giménez and Sato [34] determines the arboricity and spanning tree packing number of $\mathcal{G}(n,p)$. Optimal results on packing Hamilton cycles in $\mathcal{G}(n,p)$ which together cover essentially the whole range of p were proven in [52, 58].

One can extend Theorem 4.1.2(iii) to the range $\frac{\log^{117} n}{n} \leq p = o(1)$ by applying a recent result in [45] on covering $\mathcal{G}(n,p)$ by Hamilton cycles (see Corollary 6.2 in [39]). It would be interesting to obtain corresponding exact results also for (i) and (ii). In particular we believe that the following should hold.

Conjecture 4.1.3. Suppose $p = o(1)$ and $\frac{pn}{\log n} \to \infty$. Then whp $G \sim \mathcal{G}(n,p)$ can be decomposed into $odd(G)/2$ paths.

By tracking the number of cycles in the decomposition constructed in [55] and by splitting every such cycle into two paths, one immediately obtains an approximate version of Conjecture 4.1.3. Note that this argument does not yield an approximate version of Theorem 4.1.2(ii) in the case when p is constant.

4.2 Dense quasirandom graphs

We actually deduce Theorem 4.1.2 from quasirandom versions of the corresponding results. As our notion of quasirandomness, we will consider the following one-sided version of ε-regularity. Let $0 < \varepsilon, p < 1$. A graph G on n vertices is called *lower-(p, ε)-regular* if we have $e_G(S,T) \geq (p-\varepsilon)|S||T|$ for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$. In order to deduce Theorem 4.1.2 from its quasirandom version, we use the following well-known facts about random graphs.

Lemma 4.2.1. *Let $0 < \varepsilon, p < 1$ be constant. The following holds whp for the random graph $G \sim \mathcal{G}(n,p)$:*

(i) $\Delta(G) - \delta(G) \leq 4\sqrt{n \log n}$,

(ii) *G is lower-(p, ε)-regular,*

(iii) *G has a unique vertex of maximum degree.*

Indeed, using Lemma 2.5.10, it is easy to establish (i) and (ii). For (iii), we refer to Theorem 3.15 in [12]. We also need to prove another important property of G, which is that whp there is a perfect matching on the vertices of odd degree (see Lemma 3.7 in [39]).

The next theorem is a quasirandom version of Theorem 4.1.2(i). Indeed, Theorem 4.1.2(i) can be deduced from Theorem 4.2.2 as follows: Let $G \sim \mathcal{G}(n,p)$. In a first step, find a perfect matching M on the vertices of G which have odd degree. Then $G-M$ is Eulerian and, using Lemma 4.2.1, we can apply Theorem 4.2.2 to $G-M$. Since $\Delta(G-M) = 2\lfloor \Delta(G)/2 \rfloor$, $G-M$ can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles, as desired.

Theorem 4.2.2. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$ and that G is Eulerian. Then G can be decomposed into $\Delta(G)/2$ cycles.*

This confirms the following conjecture of Hajós (see [65]) for quasirandom graphs (with room to spare): Every Eulerian graph on n vertices has a decomposition into $\lfloor n/2 \rfloor$ cycles. (It is easy to see that this conjecture implies the Erdős-Gallai conjecture.)

Similarly, the following theorem immediately implies parts (ii) and (iii) of Theorem 4.1.2 via Lemma 4.2.1.

Theorem 4.2.3. *Let $1/n \ll \eta, \varepsilon \ll p < 1$. Suppose G is a lower-(p, ε)-regular graph on n vertices such that $\Delta(G) - \delta(G) \leq \eta n$. Then the following hold.*

(i) G can be decomposed into $\max\{odd(G)/2, \lceil (\Delta(G)+1)/2 \rceil\}$ paths. If G has a unique vertex of maximum degree, then G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.

(ii) G can be decomposed into $\lceil (\Delta(G) + 1)/2 \rceil$ linear forests. If G has a unique vertex of maximum degree, then G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.

We also apply our approach to edge colourings of dense quasirandom graphs. Recall that in general it is NP-complete to decide whether a graph G has chromatic index $\Delta(G)$ or $\Delta(G) + 1$ (see for example [46]). We will show that for dense quasirandom graphs of even order this decision problem can be solved in quadratic time without being trivial. For this, call a subgraph H of G *overfull* if $e(H) > \Delta(G) \lfloor |V(H)|/2 \rfloor$. Clearly, if G contains any overfull subgraph, then $\chi'(G) = \Delta(G) + 1$. The following conjecture is known as the overfull subgraph conjecture and dates back to 1986.

Conjecture 4.2.4 (Chetwynd, Hilton [19])**.** A graph G on n vertices with $\Delta(G) > n/3$ satisfies $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph.

This conjecture implies the 1-factorization conjecture, that every regular graph of sufficiently high degree and even order can be decomposed into perfect matchings, which was recently proved for large graphs in [22]. Minimum degree conditions under which the overfull subgraph conjecture is true were first investigated in [13, 72]. (We refer to [80] for a more thorough discussion of the area.) We prove the overfull subgraph conjecture for quasirandom graphs of even order, even if the maximum degree is smaller than stated in the conjecture, as long as it is linear.

Theorem 4.2.5. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices and n is even. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph. Further, there is a polynomial time algorithm which finds an optimal colouring.*

At first glance, the overfull subgraph criterion seems not very helpful in terms of time complexity, as it involves all subgraphs of G. (On the other hand, Niessen [70] proved that in the case when $\Delta(G) \geq |V(G)|/2$ there is a polynomial time algorithm which finds all overfull subgraphs.) Our proof of Theorem 4.2.5 will actually yield a simple criterion whether G is class 1 or class 2. Moreover, the proof is constructive, thus using appropriate running time statements for our tools, this yields a polynomial time algorithm which finds an optimal colouring.

The condition of n being even is essential for our proof as we colour Hamilton cycles with two colours each. It would be interesting to obtain a similar result for graphs of odd order.

Conjecture 4.2.6. For every $0 < p < 1$ there exist $\varepsilon, \eta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Whenever G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices, where n is odd, and $\Delta(G) - \delta(G) \leq \eta n$, then $\chi'(G) = \Delta(G)$ if and only if $\sum_{x \in V(G)}(\Delta(G) - d_G(x)) \geq \Delta(G)$.

Note that the condition $\sum_{x \in V(G)}(\Delta(G) - d_G(x)) \geq \Delta(G)$ in Conjecture 4.2.6 is equivalent to the requirement that G itself is not overfull. Also note that the corresponding question for $\mathcal{G}(n, p)$ is easily solved if p does not tend to 0 or 1 too quickly: It is well-known that in this case whp $G \sim \mathcal{G}(n, p)$ satisfies $\chi'(G) = \Delta(G)$, which follows from the fact that whp G has a unique vertex of maximum degree.

4.3 Proof overviews

Our main tool is a result on Hamilton decompositions of regular robust expanders by Kühn and Osthus [60, 61]. Robust expansion is another variant of quasirandomness, which we do not introduce formally here. It is enough to note that it is implied by lower-ε-regularity (see Proposition 3.10 in [39]).

Note that our main results concern almost regular graphs. So the key step is to partially decompose a given graph (into paths, cycles or appropriate linear forests) optimally such that the remaining graph is regular. We sketch the proofs of Theorems 4.2.2 and 4.2.5. Theorem 4.2.3 is proved using a few tricks which obtain the desired path or linear forest decomposition from a cycle decomposition of a suitably defined auxiliary graph.

4.3.1 Proof sketch of Theorem 4.2.2

If an Eulerian graph G has a decomposition into $\Delta(G)/2$ cycles, then any vertex of maximum degree must be contained in any cycle of the decomposition. Let Z contain the vertices of maximum degree in G. We want to find a cycle C that contains Z. A cycle on Z would be desirable, yet too much to hope for. However, suppose we are given a set of vertices S (not necessarily disjoint from Z) such that $G[S \cup Z]$ is lower-ε-regular and has linear minimum degree. Then we can find a Hamilton cycle C in $G[S \cup Z]$. Let G' be obtained from G by removing the edges of C. Hence, when going from G to G', the maximum degree decreases by two. Let Z' contain the vertices of maximum degree in G'. Again, we aim at finding a cycle C' that contains Z'. In addition, if $\delta(G') < \delta(G)$, then we want to make sure that C' does not contain any vertex of degree $\delta(G')$. We achieve this as follows. We find another set S' such that $G[S' \cup Z']$ is lower-ε-regular and has linear minimum degree, and critically, S' is disjoint from S. Then we can take C' to be a Hamilton cycle in $G[S' \cup Z']$. In this way we have reduced the maximum degree by 4 and the minimum degree by at most 2 by removing the edges of two cycles. By repeating this 2-step procedure, we will eventually obtain a dense regular graph which can be decomposed into Hamilton cycles.

4.3.2 Proof sketch of Theorem 4.2.5

Roughly speaking, instead of inductively removing cycles, we aim to remove paths in order to make our graph regular and then decompose the regular remainder into Hamilton cycles. We can then simply colour each path with two colours and, since our graph has even order, each Hamilton cycle with two colours. We can translate the condition that G does not contain any overfull subgraph into a simple condition on the degree sequence of G. Together with a classic result on multigraphic degree sequences by Hakimi [41], we find an auxiliary multigraph A on $V(G)$ such that $d_A(x) = \Delta(G) - d_G(x)$ for all $x \in V(G)$. If we removed the edges of a Hamilton path from G joining a and b for every edge $ab \in E(A)$, then the leftover would be a regular graph. However, too many iterations would be needed and we could not ensure that the regular remainder is still dense enough to apply the Hamilton decomposition result in [61]. Therefore, we split $E(A)$ into matchings, and for every such matching M we remove a linear forest from G whose leaves are the vertices covered by M. In order to actually find these linear forests, we observe that lower-(p, ε)-regular graphs contain 'spanning linkages' for arbitrary pairs of vertices.

CHAPTER 5

CONCLUSION

We gave a new proof of the existence conjecture based on the iterative absorption method, which we developed in the hypergraph setting. This opens the door for further applications of this method beyond the graph setting. Of particular interest would be to explore the possibility of an existence theory for *q-analogs of Steiner systems*. There, instead of finding f-sets in an n-set which cover every r-set exactly once, the aim is to find a set of f-dimensional subspaces of an n-dimensional vector space (over $GF(q)$) such that every r-dimensional subspace is covered exactly once. The current state of knowledge for this problem is sobering: for $r \geq 2$, the only set of parameters for which the existence of such a structure is known is $(n, f, r, q) = (13, 3, 2, 2)$ [15]. Yet Keevash's proof of the existence conjecture and our alternative proof using iterative absorption give some hope that this problem is not totally out of reach.

We also generalised Wilson's fundamental theorem on F-decompositions to hypergraphs (Theorem A), and our methods made it possible to study the decomposition problem even beyond the quasirandom setting. In particular, we initiated the systematic study of the decomposition threshold for hypergraphs. As demonstrated in the graph case, the iterative absorption method is capable of delivering exact results for this problem, but significant new ideas will be needed in order to extend this to hypergraphs.

For graphs, we determined the decomposition threshold of every bipartite graph, and showed that the threshold of a clique equals its fractional counterpart. It would be interesting to study the problem for general F further, i.e. to determine δ_F up to δ_F^*. Yet perhaps the more important problem is to improve the bounds for the fractional decomposition threshold of cliques.

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