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Shear strength theories for beams of variable depth

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Abstract

Flexibly formed reinforced concrete beams usually have varying cross sections along their longitudinal axis, capitalising on the fluidity of concrete to create optimised geometries. According to Orr *et al.* [1], these new shapes have led to challenges for shear design, especially when the depth of the beams is relatively small. It is crucial to be able to accurately determine the shear strength of such beams to maintain structural safety whilst achieving material optimisation.

The effective shear force method is adopted for tapering beams in many design codes. Recent work by Paglietti *et al.* [2] has highlighted concerns over the use of such an approach. In this paper, the theoretical basis for stress distributions in tapered beams built by Timoshenko [3] and Oden [4] in their elastic range is reviewed and then extended to included cracked behaviour.

It is found that the effective shear force method used in design codes does not accurately account for the stress distribution in a section both in elastic and cracked stage of concrete, underestimating the peak shear stress for beams with inclined soffits. This is important for flexibly formed beams, and has implications for designers

As a result of this work, a new calculation and design method for shear reinforcement is proposed.

Keywords: variable depth beam, shear strength, shear stress distribution, flexible formwork.

1. Introduction

Flexible formwork is defined by Orr *et al.* [5] as the formwork using textured fabric to replace rigid formwork (timber or steel) to cast concrete structures. Flexible formwork has been available since the 19th century but new efforts are now being made to re-introduce it to the construction industry as it presents numerous advantages as we look to achieve material efficiency and sustainability targets.

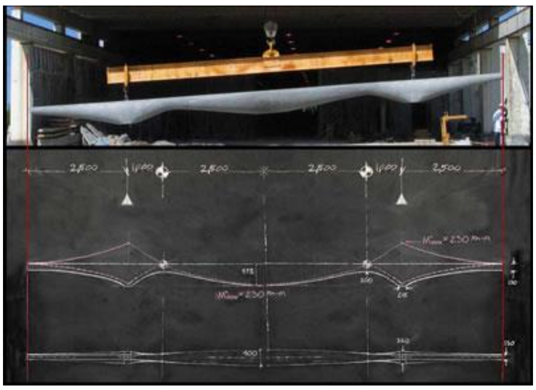

Figure 1: Flexibly formed beam (top) and its bending moment diagram (bottom) by Mark West [6].

The system uses a combination of flexible fabric and fluid concrete to allow the construction of a new range of geometries; with correct design these can be optimised to minimise material use. Such elements usually have a varying cross section along their longitudinal axis (typically following the shape of the bending moment diagram, as shown in Figure 1). These unusual geometries lead to challenges in the prediction of the shear performance of variable depth beams.

Many codes have addressed the shear design of tapered or variable depth beams. The guidance for shear design in BS EN 1992-1-1 [7] and ACI 318 [8] suggest the design shear resistance should be modified to account for the effects of an inclined top or bottom surface on the concrete member. This 'effective shear force' as suggested by Park *et al.* [9] is given as the design shear force minus the vertical components of force in both the compression chord and tension chord of the beam. The reduced shear force is then used to design the shear reinforcement.

Concerns over the use of such an approach have been highlighted in recent work by Paglietti *et al.* [2]. The theoretical basis for stress distributions in variable depth beams in their elastic range can be tracked to Timoshenko [3] and Oden [4]. In this paper, this approach is reviewed and compared with the 'effective shear force' method and then extended to included cracked behaviour. Test data and theoretical calculations are compared. A new calculation and design method for shear reinforcement is then proposed.

A further complexity of variable depth beams is found in their construction. Although steel rods can be bent into standardised shapes, any further complexity can add considerable cost to the construction process. This research is focusing on a novel alternative – replacing the internal steel reinforcement with a knitted composite reinforcement cage made from carbon fibre reinforced polymer (CFRP) tows. By fabricating this reinforcement in exactly the right geometry, this will provide exactly the right strength exactly where it is needed. This will be transformative for concrete construction, and will greatly simplify the reinforcing of complex shapes. Building a feasible calculation method for the shear performance of variable depth beams with steel reinforcement is crucial basis for the future research. The complexities and design challenges of replacing ductile steel with brittle CFRP reinforcement are discussed by Stratford *et al.* [10] and will be considered in later research by the authors.

2. Shear calculations for beams of variable depth

Existing code methods [7, 8] also suggest that the inclined forces can be used to reduce shear forces in the web of a beam section as shown in Figure 2. This 'effective shear force' method is in accordance with force equilibrium. Shear design in such a situation then adopts the equations and specifications used for a beam with constant depth. This indicates that the effective shear force method assumes a parabolic distribution of shear stress on each cross section.

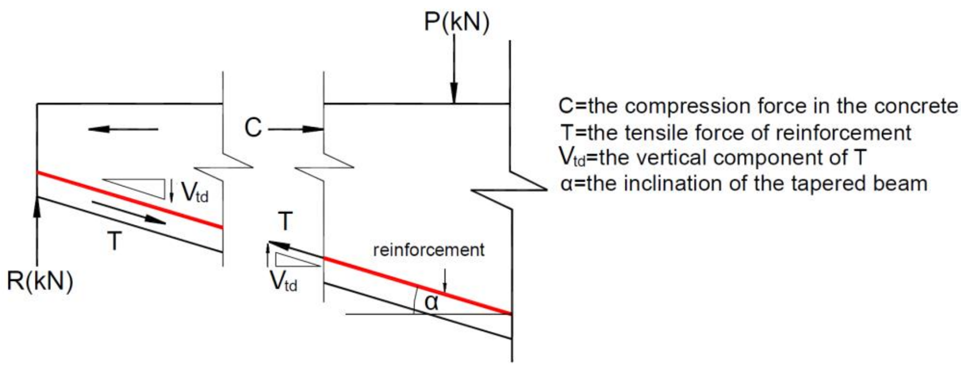

Figure 2: The inclined forces in a tapered beam might provide a contribution to shear capacity

Paglietti *et al.* [2] adopted the theories of Timoshenko [3] and Oden [4] and used these analytical solution for the shear stress of variable depth beams to show that the effective shear force method is often incorrect. Interesting paradoxes are presented by Paglietti *et al.* [2] to demonstrate this. For example, removing material from a constant depth cantilever structure to obtain a variable depth tapered cantilever results in the tapered beam having a higher shear capacity than the constant depth beam if the effective shear force method is adopted, as shown in Figure 3.

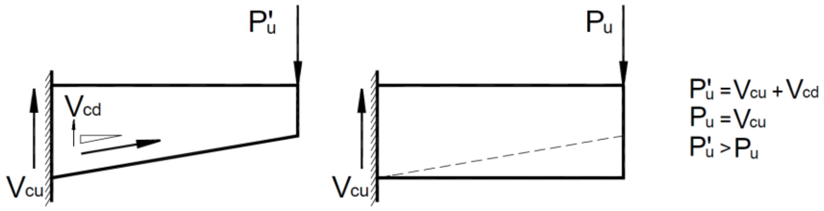

Figure 3: Paglietti's paradox

Paglietti *et al.* [2] suggests that the shear force applied on the cross section should not involve the vertical components of the compression and tension chord. In addition using the theories of Timoshenko [3] and Oden [4], it is seen that the maximum value of shear stress is not located at the centroid of the section, and is larger than is predicted by the effective shear force method.

The effective shear force method is an ultimate-limit-state method, while the work of Paglietti *et al.* [2] considers only elastic theory. It therefore does not prove the effective shear force method is totally incorrect. To consider in more detail the effective shear force method and Paglietti's approach, calculations and comparisons of reinforced concrete beams at the ultimate limit state need to be carried out.

3. Verification at the Ultimate Limit State

To illustrate the ultimate limit state design, a variable cross section beam reinforced using CFRP is considered. A 6m span simply supported beam subject to a uniformly distributed load is to be designed. Figure 4 shows half of the span and the depth of critical sections. Timoshenko's design method [3] and the 'effective shear force' method [2] will be used to illustrate the shear stress distribution of variable depth beams at ULS and comparison of the methods is also presented.

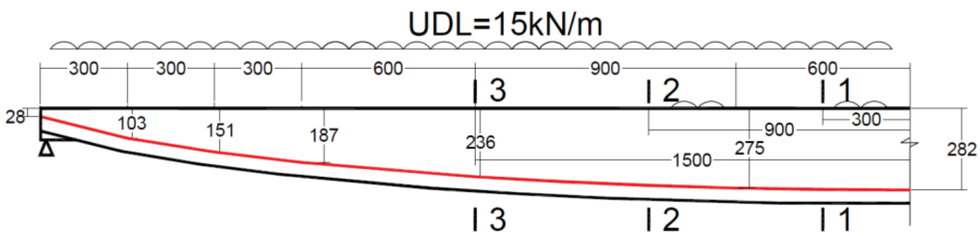

Figure 4: The schematic diagram of the calculation example (showing half the beam span).

The profile of this simply supported variable depth beam is determined initially by matching the bending capacity of the section to the requirements of the bending moment at every section along the beam length using BS EN 1992-1-1 [7]. The resulting beam design is shown in Figure 4. It is assumed in the design that the tensile zone of this beam is fully cracked at the ULS, in which situation the tensile zone of concrete cannot carry any tensile stress. The third assumption is that plane sections remain plane in the process of loading and the strain keeps linearly changing across the section.

3.1 The Timoshenko method

According to Timoshenko [3] the shear stresses on any section of this beam can be calculated as follows.

1. Take two cuts close to the calculation point, and consider the free body in between;

2. Determine the concrete bending stresses σ using moment, force and deformation equilibrium as shown in Figure 5;
3. Choose a series of descending horizontal lines connecting left to right of the free body, and on each of these planes calculate the out-of-balance horizontal shear force acting along this line;
4. Divide this out-of-balance shear force by the breadth and the length of the free body to get the horizontal shear stress, which then (from equilibrium) is also the vertical shear stress at that location.

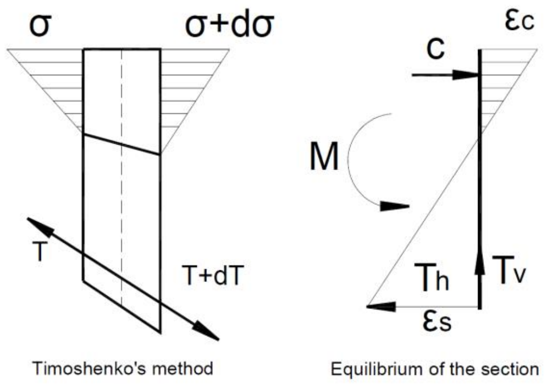

Figure 5: The bending stresses of the Timoshenko method

3.2 The effective shear force method

Following the effective shear force method, the bending stress calculation is different from the Timoshenko method.

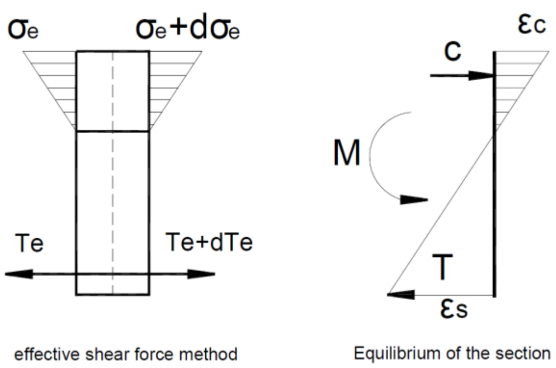

Figure 6: the bending stress of effective shear force method

1. Take two cuts and get the free body for calculation (figure 6).
2. Assume the depth of the two sides of the free body at the calculation point is equal because the effective shear force method adopts the design equations for constant depth beams.
3. Calculate the bending moment on the two sides as the moment at the calculation point plus or minus the effective shear force multiplied by half of the length of the free body as shown in Equation 1.
4. Then following the step 3 and 4 in the Timoshenko method, the shear stress distribution of effective shear force method can be obtained.

$$M_{e1,2} = M \pm V_e \cdot dx / 2 \qquad (1)$$

Where
$Me_{1,2}$ = the calculation moments at the two sides of free body;
M=the moment at the calculation cross section;

V_e =the effective shear force;
dx =width of the free body;

3.3 Comparison between the two methods

The comparison of the shear stress distribution for one cross section of the beam shown in Figure 4, calculated using the two methods (§3.1 and §3.2) is shown in Figure 7 (τ is the true shear stress calculated by Timoshenko's method and τ_e is the shear stress assumed in effective shear force method). The sections are taken from Figure 4.

Figure 7 shows that for both design methods, under the loading shown in Figure 4, the shear stress increases from zero at the top of the beam, before reaching its maximum value which is then maintained until the bottom of the beam. The shear stress distribution diagrams for each section show larger differences between the two design methods for sections closer to the support.

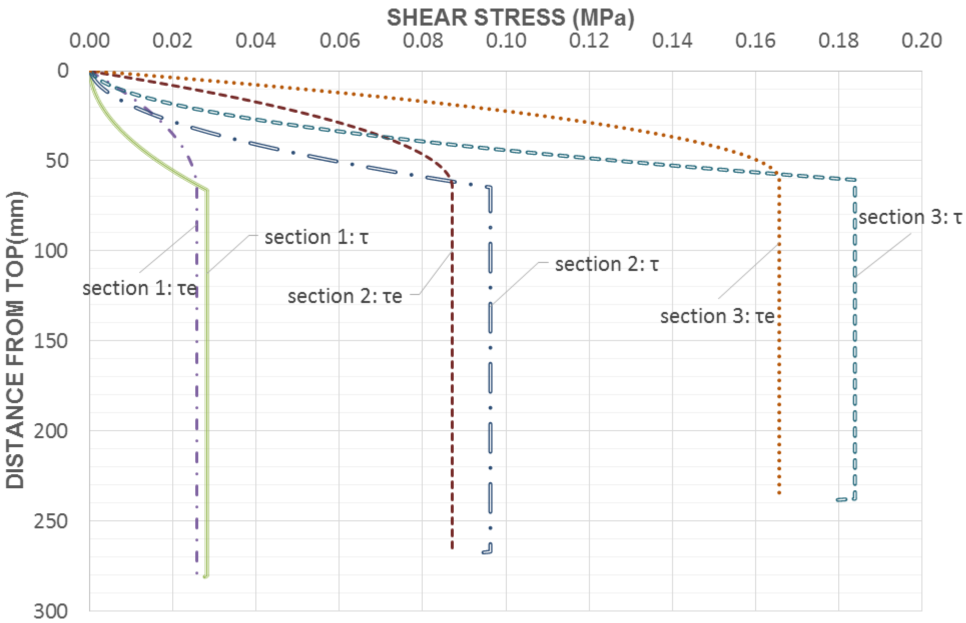

Figure 7: The shear stress distribution of sections

Because of the inclination of variable depth beams, the bending stress of the free body of the Timoshenko method has clear difference from the effective shear force method. As shown in Figure 5, the inclination of the free body leads to a situation that on the left side of the free body (with smaller depth) the bending stress is larger would be found for a beam with constant depth; meanwhile on the right hand side (with greater depth) the bending stress will be relatively smaller. As shown in Figure 8, Timoshenko method has a different '$d\sigma$' distribution on the same calculation section. This leads to the sunken shear stress line for Timoshenko method and the plump shear stress line for effective shear force method as shown in Figure 7. It is also the reason why the Timoshenko method has a higher maximum shear stress than the effective shear force method.

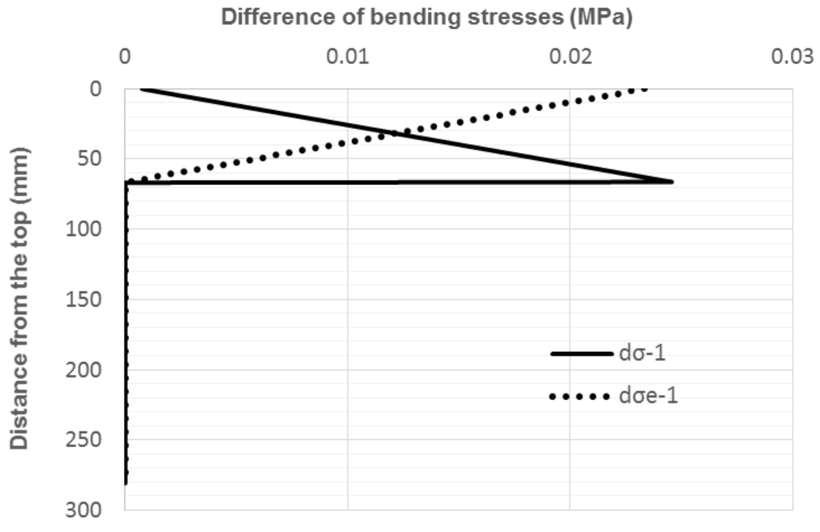

Figure 8: The difference of bending stress of the section 1

By integration of the shear stress diagram, the shear force on each section can be obtained. The results show that the shear force on each section is the effective shear force rather than true shear force mentioned by Paglietti *et al.* [2] as shown in Figure 8. The reason for this result is that the reinforcement takes over what the tensile zone does at elastic state, and carries the tensile force. The shear stress will not continue to grow when it is cracked because there are no tensile stresses in this zone. Hence the inclined reinforcement can carry part of the shear force and the concrete in compression therefore carries only the effective shear force.

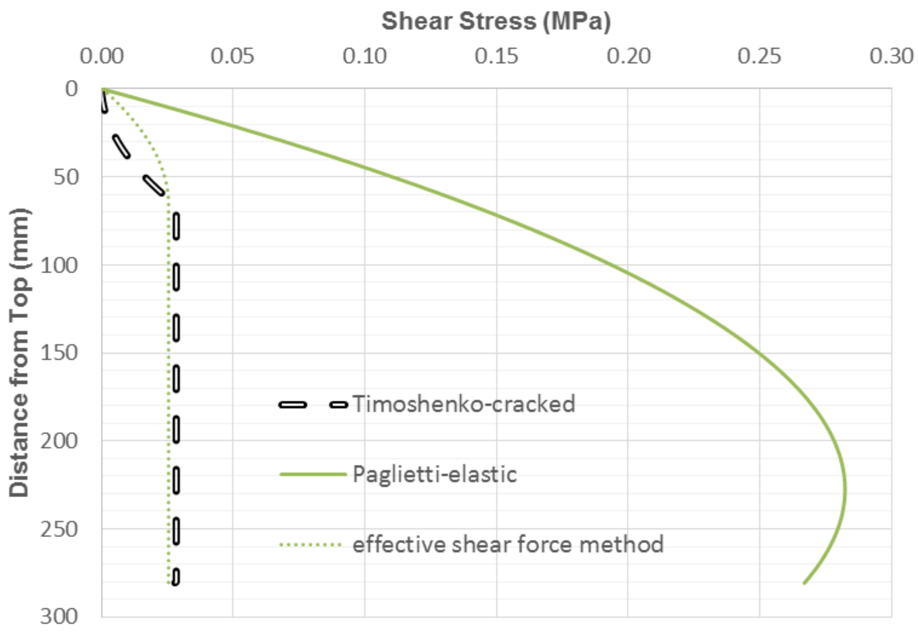

Figure 8: The comparison of shear stress of section 1

Based on this discussion it can be concluded that for cracked concrete beams Timoshenko's method can show the true shear stress distribution of sections, and both the effective shear force method and Paglietti's [2] elastic theory have their own flaws. The effective shear force method adopted by the codes is correct at the first step, using the effective shear force to undertake the section design. However, using a non-inclined (prismatic) cross section to then carry out the shear reinforcement design will underestimate the maximum shear stress, which may lead to unsafe design practice. For the elastic theory, the mismatch of the bending stress in tensile zone results in the continuous increasing of shear stress. However, for reinforced concrete beams, there is no bending stress in the cracked tensile zone and the tensile force is all carried by the inclined reinforcement. The vertical component of this inclined tensile force results in a smaller shear force on the section which is the effective shear force.

4. Comparison with Tests and new recommendations for shear design of variable depth beams

According to the comparison of shear stress of different methods in the previous section, Paglietti's [2] elastic theory and the effective shear force method are verified not valid. On one hand, the calculation results show us that the effective shear force method is right in the first step but it is wrong to adopt the shear design equations for constant depth beams directly. The proportion of the shear force carried by compression zone and tensile zone is changed and the tensile zone will carry more shear force than the constant depth beams. Using the effective shear force method could be unconservative and thus the effective shear force method needs to be modified to account for this.

The variable angle truss model in BS EN 1992-1-1 [7] is composed of an inclined compressive truss of concrete struts, and the tensile truss (the shear reinforcement). Two failure criterion are used for design: concrete crushing and reinforcement yielding. Concrete crushing can be avoided by changing the angle of the truss. However, for most cases, concrete crushing will not occur which leads to the most efficient design: allowing the shallowest truss angle to be adopted to achieve the lowest transverse reinforcement ratio. Whether this design process will be suitable for variable depth beams is not yet clear. Another issue in the variable angle truss model of BS EN 1992-1-1 [7] is that the model neglects all the positive effect of concrete in shear. This is an obstacle to accurately predicting the shear performance of variable depth beams designed using BS EN 1992-1-1 [7].

Based on the assumptions and calculation results shown above, the variable angle truss model could be modified to account for the variable depth section. The compressive concrete carries the total compressive force due to bending and part of the shear force. The cracked zone of concrete, which is composed of a lot of plain concrete cantilevers, provide a mechanism to transfer external loads. The cracked tensile zone does not make contributions to shear capacity. This part of the shear force will be carried by transverse reinforcement which must therefore be designed. It is proposed that the shear capacity of the variable depth beams designed with variable angle truss model could be more accurately determined if the shear force was divided into three parts: 1) the vertical component of inclined bending reinforcement, 2) the compressive concrete; and 3) the shear reinforcement. All these three parts can be calculated but in order to achieve the primary goal of shear design (to make sure shear failure does not happen) it is crucial to determine the angle of the truss model.

4.1 Test data comparison

A series of non-prismatic beam tests and comparison of design methods and test results were carried out by Orr *et al.* [1], using the test set up shown in Figure 10. The members were designed with a truss angle of 22.7 degrees in accordance with BS EN 1992-1-1 [7]. For test member Beam 2_EC2/M and Beam 3_EC2/M, the spans are both 2000mm (see [1] for full details). The shear span a_v for Beam 2-EC2/M is 500mm and for Beam 3-EC2/M it is 300mm. Beam 2-EC2/M (design load 36kN) failed at P = 32.1kN and Beam 3-EC2/m (design load 31.8kN) failed at 18.8kN. The test results showed that the BS EN 1992-1-1 [7] method was unconservative.

In order to compare our theory and the tests, Matlab models were built to simulate the test members including Beam 2_EC2/M and Beam 3_EC2/M. All the models have the same geometry as the test members [1]. The transverse reinforcement design for each beam is shown in Table 1 (see also [1]). The failure load of the test members are applied on the corresponding model and the shear force at four critical sections (each located at a point of changing stirrup spacing) are obtained, as shown in Table 2.

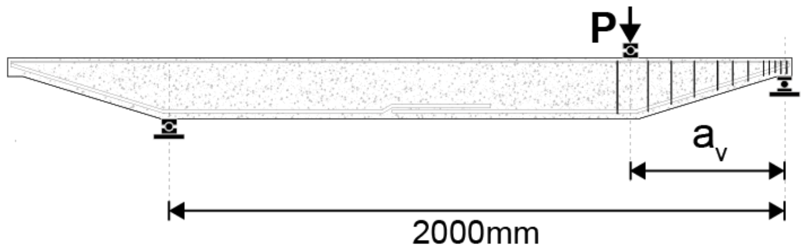

Figure 10: The test arrangement by Orr *et al.* [1]

Table 1: The parameters of tests members

Beam2-EC2/M		Beam3-EC2/M	
Distance from support (mm)	Area of shear reinforcement (mm^2)	Distance from support (mm)	Area of shear reinforcement (mm^2)
25-75	56.52	0-80	56.52
75-225	42.39	80-180	56.52
225-450	42.39	180-300	42.39

Table 2: The tensile zone shear force of the models

Beam2-EC2/M		Beam3-EC2/M	
Distance from support (mm)	Shear force in tensile zone (kN)	Distance from support (mm)	Shear force in tensile zone (kN)
25	9.4	0	7.4
75	7.0	80	4.9
225	2.2	180	2.5
450	0.7	300	1.4

Table 3: Comparison of predicted resistance and loading for Beam2-EC2/M

Distance from support (mm)	Loading	Resistance with variable angle truss model		
	Shear force in tensile zone (kN)	45 degrees (kN)	35 degrees (kN)	22.7 degrees (kN)
25	9.4	9.8	14.0	23.5
75	7.0	6.4	9.1	15.3
225	2.2	8.5	12.2	20.3
450	0.7	11.3	16.1	26.9

Table 4: Comparison of predicted resistance and loading for Beam3-EC2/M

Distance from support (mm)	Loading	Resistance with variable angle truss model		
	Shear force in tensile zone (kN)	45 degrees (kN)	35 degrees (kN)	22.7 degrees (kN)
0	7.4	8.5	12.2	20.4
80	4.9	11.9	17.0	28.5
180	2.5	10.1	14.5	24.3
300	1.4	15.2	21.8	36.4

In order to determine the true angle of the truss model of the test members, the resistances of different truss angles are compared to the applied loading. For Beam2_EC2 and Beam3_EC2, both of the members failed in shear and near the support. By comparing the actual failure load with the prediction made using BS EN 1992-1-1 [7], Eq. (2), the truss model compression strut angle (θ, as shown in Figure 11) can be determined. The comparison of predicted resistance and loading for Beam2-EC2/M and Beam3-EC2/M are shown in Table 3 and 4. The true angles of the truss model in test condition for critical sections in support area are shown in table 5. The results of comparisons are also shown in Figure 12 and Figure 13.

$$V_{Rd,S} = \frac{A_{sw}}{S} \cdot z \cdot f_{ywd} \cdot \cot\theta \qquad (2)$$

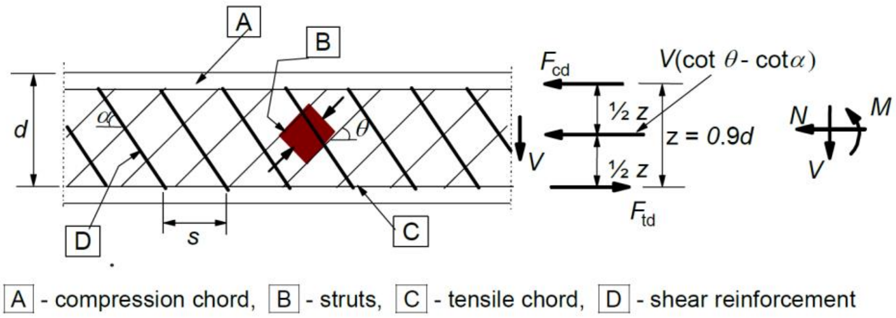

Figure 11: The variable angle truss model in BS EN 1992-1-1

Table 5: The true truss angle at the support area

Beam2-EC2/M			Beam3-EC2/M		
distance from support (mm)	tensile zone shear force (kN)	True truss angle	distance from support	shear force in tensile zone (kN)	True truss angle
25	9.4	46.3	20	7.4	48.9

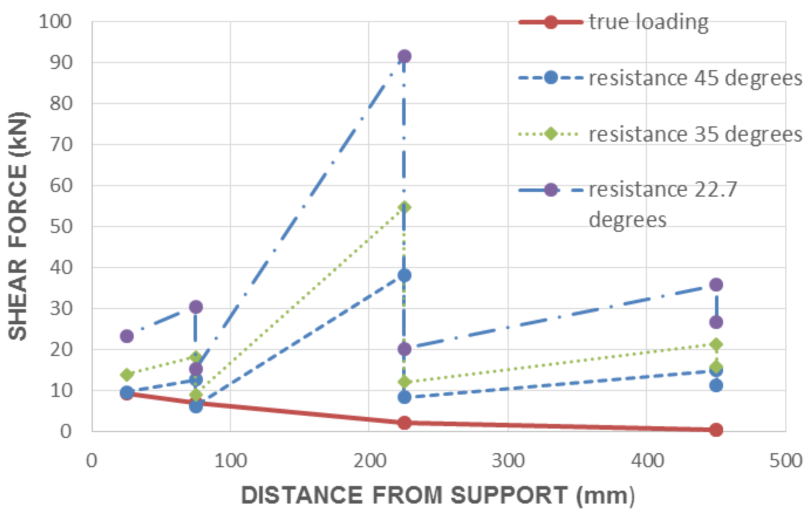

Figure 12: The comparison of Beam2-EC2/M

As shown in Figure 12 and 13, the predicted resistance with different truss angles and the true loading for the two members are compared. The 22.7 degrees and 35 degrees resistances are much higher than the true loading lines. Only the 45 degree lines could match the true loading lines at the support area where the shear failure happened. This also matches the theory built up in the earlier sections: the different distribution of shear stress in variable depth beams leads to a situation that the tensile zone shear force increases and thus it needs more transverse reinforcement to balance the shear force. This situation also implies that the angle of truss model of variable depth beams is not controlled by the concrete crushing and the choice of the angle need to be more conservative for the variable depth beams than the constant depth beams.

The variable angle of truss model of BS EN 1992-1-1 [7] can vary from 22.7 degrees to 45 degrees. The comparison of the resistance and loading, it is feasible to adopt a truss model of forty five degrees to match the test data for corresponding models of two members. With this angle, the shear resistance of transverse reinforcement will have the same value as the shear force in tensile zone of the models when it is at the ultimate limit state. In other words, if the variable depth beams are designed with a forty five degree truss model, shear failure will not happen with applied load much lower than the design load. Relatively accurate shear strength of variable depth beam can be predicted.

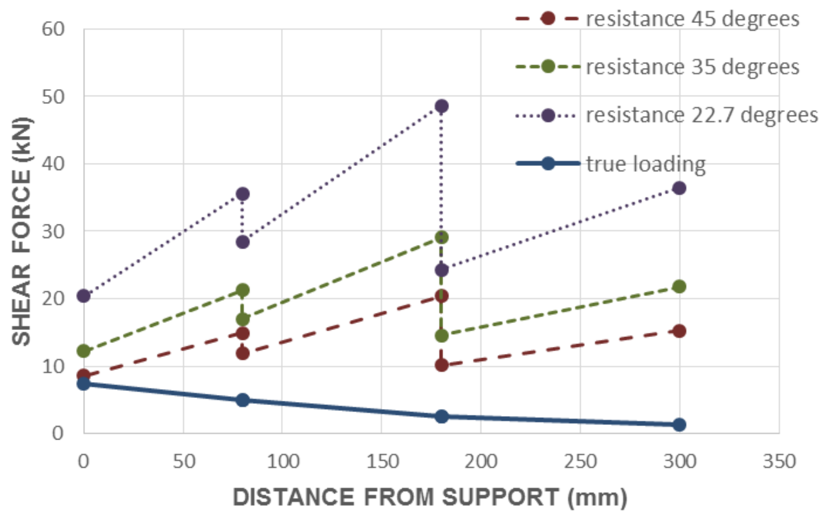

Figure 13: The comparison of Beam3-EC2/M

4.2 Design recommendations

Based on the above suggestions for shear design of variable depth beams can be made. It is proper to consider the positive effect of the inclined bending reinforcement. When the shear stress distribution of a certain section is obtained. The shear stress carried by compressive concrete can be added up as the shear resistance that concrete can supply. This part should be counted in the design. As for the transverse reinforcement, the shear capacity it supplied should in accordance with the shear force in tensile zone of the beams which is calculated by Timoshenko method as shown in Figure 8. Based on BS EN 1992-1-1 [7], the angle of truss model is not controlled by concrete crush and depending on the test data, forty five degree is recommended by the authors.

5. Conclusion

This paper has shown that neither the effective shear force nor the theory of Paglietti [2] are suitable for the shear design of variable depth beams. But the flaws of these two methods are different. For the effective shear force method, it has been proven that it is proper to adopt the effective shear force in the shear design. However, it is not suitable to use the design equations for constant depth beam to design variable depth beams. Paglietti's theory [2] predicts that cracked reinforced concrete variable depth beams may have even more different shear stress distribution than when it is elastic. This prediction is seen to be incorrect. In addition, using the elastic shear stress distribution to undertake shear design is not an option because shear design is based on ultimate limit state and there is a huge shear stress discrepancy between elastic state and fully cracked state.

Comparisons between calculation and test data are made in this paper. They show that there are three parts concerned in the shear design of a variable depth beams: the vertical component of inclined bending reinforcement, the compressive concrete part and the shear reinforcement part. A forty five degree truss model is suggested in the shear design based on BS EN 1992-1-1 [7]. Shear failure is a serious issue for variable depth beams, a good way to predict the shear performance can contribute to more opportunities of application of fabric formwork system and variable depth beams. Further research is now required to determine how variable depth beams reinforced using a brittle material such as FRP can be designed.

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6. Acknowledgement

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7. Data Access Statement

All data created during this research are openly available from the University of Bath data archive at http://dx.doi.org/10.15125/BATH-00086

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