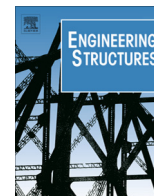


Contents lists available at [ScienceDirect](http://ScienceDirect.com)

# Engineering Structures

journal homepage: [www.elsevier.com/locate/engstruct](http://www.elsevier.com/locate/engstruct)

## The effective length of columns in multi-storey frames

A. Webber<sup>a</sup>, J.J. Orr<sup>a,\*</sup>, P. Shepherd<sup>a</sup>, K. Crothers<sup>b</sup><sup>a</sup> Department of Architecture and Civil Engineering, University of Bath, Bath BA2 7AY, United Kingdom<sup>b</sup> Atkins Design and Engineering, Oxford, United Kingdom

### ARTICLE INFO

#### Article history:

Received 5 January 2015

Revised 24 July 2015

Accepted 27 July 2015

Available online 24 August 2015

#### Keywords:

Steel structures

Multi-storey frames

Effective length method

Buckling

Design methods

Design aids

### ABSTRACT

Codes of practice rely on the effective length method to assess the stability of multi-storey frames. The effective length method involves isolating a critical column within a frame and evaluating the rotational and translational stiffness of its end restraints, so that the critical buckling load may be obtained.

The non-contradictory complementary information (NCCI) document SN008a (Oppe et al., 2005) to BS EN 1993-1 (BSI, 2005) provides erroneous results in certain situations because it omits the contribution made to the rotational stiffness of the end restraints by columns above and below, and to the translational stiffness of end restraints by other columns in the same storey.

Two improvements to the method are proposed in this paper. First, the axial load in adjoining columns is incorporated into the calculation of the effective length. Second, a modification to the effective length ratio is proposed that allows the buckling load of adjacent columns to be considered. The improvements are shown to be effective and consistently provide results within 2% of that computed by structural analysis software, as opposed to the up to 80% discrepancies seen using the NCCI (Oppe et al., 2005).

© 2015 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

### 1. Introduction

Many codes of practice rely on the effective length method to assess the stability of frames. The effective length method allows the buckling capacity of a member in a structural system to be calculated by considering an equivalent pin ended column in Euler buckling. This paper will focus on the non-contradictory complementary information (NCCI) document SN008a [1] to BS EN 1993-1 [2], although many of the findings presented in this paper are also relevant to many other national codes of practice. The NCCI provides a simple method to determine the effective lengths of columns in multi-storey steel frames. Errors in this approach have been identified that arise as the method fails to correctly recognise the contribution made:

1. by adjoining columns, to the rotational stiffness of end restraints; and
2. by other columns in the same storey, to the translational stiffness of end restraints.

Issue (1) concerns both braced and unbraced frames. Using the NCCI [1] it is found that the stiffer an adjoining column, the greater the effective length of the column being analysed, which is

\* Corresponding author. Tel.: +44 1225 385 096.

E-mail address: [j.j.orr@bath.ac.uk](mailto:j.j.orr@bath.ac.uk) (J.J. Orr).

<http://dx.doi.org/10.1016/j.engstruct.2015.07.039>

0141-0296/© 2015 The Authors. Published by Elsevier Ltd.

This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

counter-intuitive. This is demonstrated by considering columns AB and CD in Fig. 1. If AB and CD are stiffened and the loading unchanged, then the rotations at B and C are reduced. The deflected shape shows that the effective length of BC is reduced in this situation, whereas the equations of the NCCI [1] show it to increase, as shown later. A simple improvement to the method is proposed to address this, which incorporates the adjoining columns' axial load into the calculation of the effective length. The improvement is shown to be very effective and consistently provides results within 2% of that computed by structural analysis software.

Issue (2) concerns unbraced frames, and occurs because of the simplifying assumption made in the NCCI [1] that all columns in a storey buckle simultaneously and therefore columns in this storey have end restraints with zero translational stiffness. If the method is applied to unbraced frames where columns of varying stiffness exist in the same storey or columns have different applied loads, then significant errors will be encountered that are potentially unconservative, as seen in Section 3.2.1 below. To address this issue, a modification factor is adopted which is applied to the effective length ratio obtained using the sway design chart, and accounts for columns that will have end restraints with translational stiffnesses between zero (sway case) and infinity (non-sway case) and even negative translational stiffnesses. These are often called partial sway frames. The results obtained from using this factor are shown to be reliable and accurate.

**Nomenclature**

$C$	carry-over factor	$\sum K_{c,I}$	sum of rotational stiffness of the columns converging at node $I$
$E$	Young's modulus	$\sum K_{b,I}$	sum of rotational stiffness of the beams converging at node $I$ modified for axial load and far end support conditions
$\eta_x$	distribution coefficient at node $X$	$L_{E,IJ}$	effective length of column $IJ$
$I$	second moment of area	$L_{IJ}$	physical length of column $IJ$
$I_{IJ}$	second moment of area of column $IJ$	$M_x$	moment at node $X$
$\kappa$	curvature of an elastic member	$P$	applied compression
$K_{IJ}$	nominal rotational stiffness of column $IJ$	$P_{c,IJ}$	critical buckling load for column $IJ$ given by Eq. (3)
$K_{XY}$	nominal rotational stiffness of an adjoining column $XY$	$P_{E,IJ}$	Euler buckling load for column $IJ$ given when $L_{E,IJ} = L_{IJ}$ in Eq. (3)
$K'_{IJ}$	rotational stiffness of column $IJ$ modified for axial load	$P_{IJ}$	applied compression on column $IJ$
$K''_{IJ}$	rotational stiffness of column $IJ$ at node $I$ modified for axial load and support conditions at node $J$	$\theta_x$	rotation at node $X$
$K'''_{IJ}$	rotational stiffness of column $IJ$ at node $J$ modified for axial load and support conditions at node $I$	$S$	stiffness coefficient
$K''''_{XY}$	rotational stiffness of an adjoining column $XY$ modified for axial load and far end support conditions	$v$	deflection

**1.1. Elastic stability**

Buckling is an instability phenomenon in structural systems subjected to compression loads. In columns it is associated with the transition from a straight configuration to a laterally deformed state [3]. The critical load describes the load at which this transition occurs.

Critical loads can be calculated by solving for equilibrium of the laterally deformed column. Assuming deflections and rotations are small, the curvature of a member,  $\kappa$ , can be defined by Eq. (1). If the member is perfectly elastic and the material obeys Hooke's Law, deflection theory [4] states that the bending moment is proportional to the curvature, with the member's flexural stiffness as the constant of proportionality, Eq. (2):

$$\kappa = \frac{d^2 v}{dx^2} \tag{1}$$

$$M = -EI \frac{d^2 v}{dx^2} \tag{2}$$

where  $v$  is the deflection;  $E$  is Young's modulus,  $I$  is the second moment of area.

With the substitution  $k^2 = P/EI$ , the solution for the critical buckling load is given by Eq. (3) where the boundary conditions of the column are used to define the effective length:

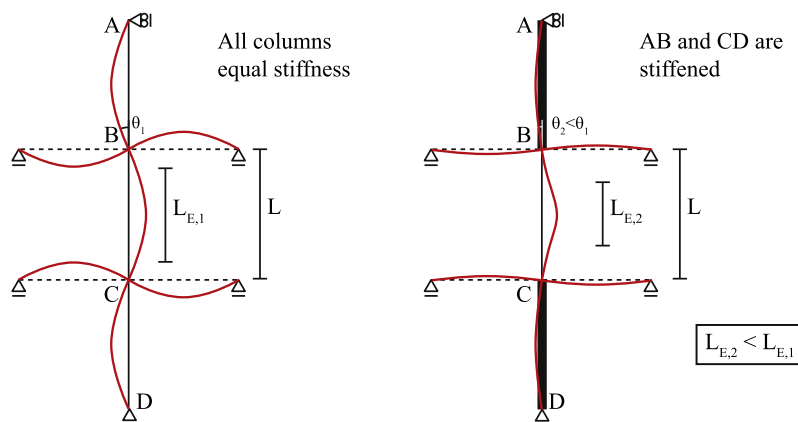
$$P_c = \frac{\pi^2 EI}{L_E^2}, \tag{3}$$

where  $E$  is the Young's modulus,  $I$  is the second moment of area,  $P_c$  is the critical buckling load, and  $L_E$  is the effective length of the column.

**1.2. Effective length**

The effective length,  $L_E$ , depends on the boundary conditions of the column as shown for example in [5]. A pin ended elastic column will have a buckled configuration of a sinusoidal wave. The distance between points of contraflexure, which defines the effective length, is critical in evaluating the stability of the column. Effective lengths given in the codes are generally greater than the theoretical values, as full rigidity at supports is difficult, if not impossible, to achieve.

Theoretical analysis uses idealised end restraints, whose translational and rotational stiffnesses are set to either zero or infinity. In some instances it may be acceptable for the designer to assume a column has these idealised end restraint conditions, especially for preliminary design purposes when a more rigorous analysis is to follow, but care is needed due to the substantial influence that end restraints have on the buckling capacity. In most real structures, the rotational and translational stiffness of the end restraints is somewhere between rigid and free.



**Fig. 1.** The contribution of adjoining columns to rotational stiffness at end restraints.

## 2. Frame stability

Being able to determine the effective length of framed columns is important, as it provides a simple approach to assessing frame stability. To find the effective length of an individual column within a frame, the rotational and translational stiffness of its end restraints must be considered.

A braced frame would usually be categorised as a non-sway frame (lateral displacements are sufficiently small that the secondary forces and moments can be ignored) and the translational stiffness of a column's end restraints are taken as infinity.

In an unbraced frame, the secondary effects caused by lateral displacements are usually significant and consequently the translational stiffness of a column's end restraints is taken as zero.

If all connections between beams and columns are assumed to be fully rigid (i.e. there is no rotation of the beam relative to the column at a connection) it may seem sensible to take an effective length ratio for a column in an unbraced frame of 1.0. However this could be an onerous over simplification because the connecting members will deform when the column buckles. The connecting members restrain the buckling column and provide rotational stiffness. It is also possible that adjoining members provide negative rotational stiffness if they too are subjected to significant axial load and have buckled.

A significant source of inaccuracy in the design of columns using the effective length method is uncertainty in the estimation of rotational boundary conditions for the column. This has been recognised in the literature [6,7]. The effective length method considers columns individually, even though the presence of other members is crucial to buckling behaviour. The contribution of adjoining members is taken into account indirectly through the summation of stiffnesses of the members at the top and bottom of a column. This approach was used by Wood [8], whose work forms the theoretical basis for buckling calculation in BS EN 1993 [2]. However, the work of Wood [8] produces unusual results in some situations, which will be considered later.

The work of Cheong-Siat-Moy [9] provides early insight into the need to consider both individual element and overall system behaviour for accurate buckling analysis, while Bridge and Fraser [10] extended this to consider negative rotational stiffnesses. Hellesland and Bjorhovde [11] also show the importance of fully considering the contributions of adjacent elements to rotational stiffnesses. They propose a 'weighted mean' approach to determining frame buckling from individual element analyses, and importantly applied this method to frames in which column stiffnesses change significantly between storeys. Aristizabal-Ochoa [12] further examined the effect of uniformly distributed axial loads, and the behaviour of frames with partial side-sway [13]. Cheong-Siat-Moy [9] proposed the use of a fictitious lateral load as a method to evaluate the buckling capacity of columns as an alternative to the effective length method, but this has not been adopted by designers.

Ultimately, the critical load of an individual column within a frame cannot be obtained without considering the loads in the rest of the structure as this will affect the stiffness of the column's end restraints and change its effective length. Codes of practice get around this by assuming worst-case scenarios such as that adjoining columns in the storey above and below buckle simultaneously with the column under investigation, and as such reduce the rotational stiffness of its end restraints, and that other columns in the same storey buckle simultaneously. Both of these can lead to over-conservative results.

### 2.1. Stiffness distribution method

The stiffness distribution method, which is used in SN008a [1] employs the stability function 'S' (Eq. (4)) and carry over factor

'C' (Eq. (5)) for non sway cases [14]. For the sway case (see Fig. 2), the stability function is 'n' and the carry over factor 'o', and equations for these coefficients may be found in the literature [14]. They define the end moments of a fixed-pin column  $IJ$  that is rotated by  $\theta$  at its pinned end  $I$ , as displayed in Fig. 2 for sway and non-sway frames.

$$S = \frac{kL_{IJ} [\sin(kL_{IJ}) - kL_{IJ} \cos(kL_{IJ})]}{2 - 2 \cos(kL_{IJ}) - kL_{IJ} \sin(kL_{IJ})} \quad (4)$$

$$C = \frac{kL_{IJ} - \sin(kL_{IJ})}{\sin(kL_{IJ}) - kL_{IJ} \cos(kL_{IJ})} \quad (5)$$

where  $k = \sqrt{\frac{P_{IJ}}{EI_{IJ}}}$ ,  $L_{IJ}$  is the length of column  $IJ$ ,  $P_{IJ}$  is the axial load on column  $IJ$ ,  $I_{IJ}$  is the second moment of area of column  $IJ$ .

Since Eqs. (4) and (5) depend only on  $kL_{IJ}$ , which can alternatively be given by Eq. (6), they are functions of the ratio of the axial load to Euler load ( $P/P_{E,IJ}$ ). Such values have been tabulated extensively in the literature [14,15].

$$kL_{IJ} = L_{IJ} \sqrt{\frac{P_{IJ}}{EI_{IJ}}} = \pi \sqrt{\frac{P_{IJ}}{P_{E,IJ}}} \quad (6)$$

where  $P_{E,IJ}$  is the Euler buckling load of column  $IJ$ , given when  $L_{IJ} = L_{E,IJ}$

When the axial load  $P$  equals zero, the stiffness coefficient  $S$  equals four. Therefore the moment required to rotate the column in Fig. 2(a) by theta degrees is given by Eq. (7):

$$M_I = 4\theta_I \frac{EI}{L} \quad (7)$$

Wood's [8] general definition for rotational stiffness is given by Eq. (8):

$$K_{IJ} = \frac{M_I}{4E\theta_I} \quad (8)$$

The resistance of the fixed-pinned column  $IJ$  (Fig. 3) to rotation at the pinned node  $I$ , when the axial load is zero, is therefore given by Eq. (9):

$$K_{IJ} = \frac{I_{IJ}}{L_{IJ}} \quad (9)$$

This is called the *nominal rotational stiffness* and can be modified to take account of axial load in the column by using the stability function 'S', as shown in Eq. (10):

$$K'_{IJ} = \frac{S}{4} K_{IJ} \quad (10)$$

where  $S$  is given by Eq. (4).

If the end 'J' is in fact not fully rigid, but has a rotational stiffness due to the presence of adjoining beams then this will result in a reduced rotational stiffness at end  $I$ , which is related to the relative stiffness of the adjoining beams at end  $J$ .

To find this reduction in  $K'_{IJ}$  we need to contemplate the following scheme, which is illustrated in Fig. 3:

- Consider the column  $IJ$  where end  $I$  is free and end  $J$  is initially fixed, Fig. 3(a).
- End  $I$  is rotated by  $\theta$ , which requires an applied moment of  $M_I = SEK_{IJ}$ , where  $S$  is the general stiffness coefficient for either sway/non-sway case, and  $K_{IJ}$  is the nominal rotational stiffness from Eq. (9) and  $E$  is the Young's modulus (Fig. 3(a)).
- The moment carried over to end  $J$  is  $CM_I$  where  $C$  is the general carry-over factor.
  - This step is explained by considering the general case of a moment applied at a node 'X' which has three adjoining members (one column and two beams, for example). The

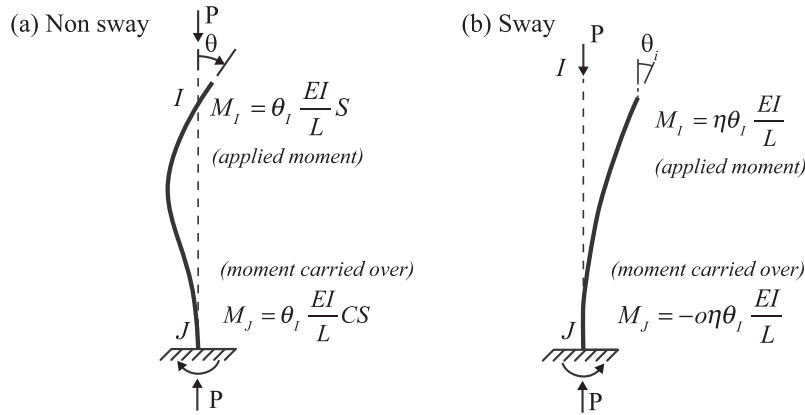


Fig. 2. Demonstration of end moments for fixed-pinned struts in (a) non sway and (b) sway frames.

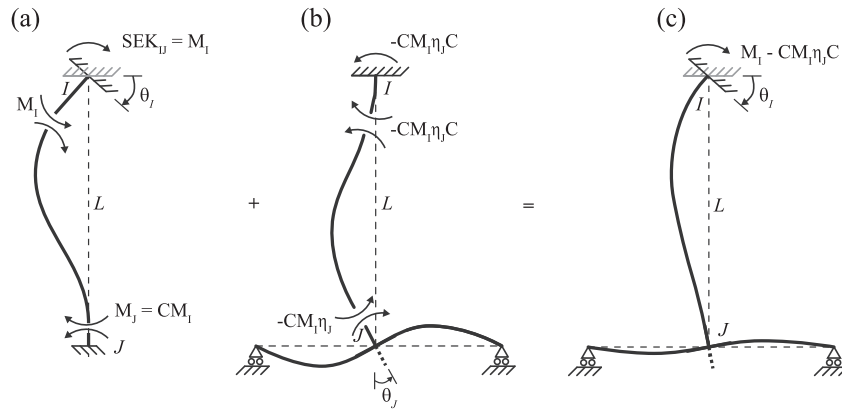


Fig. 3. Moment required to rotate the free end of a column by  $\theta$ , with adjoining beams at far end.

applied moment will be distributed into each adjoining member according to their relative rotational stiffness. The moment distributed into the column in this example is given by Eq. (11):

$$M_{\text{column}} = M_X \eta_X \quad (11)$$

where  $M_X$  is the applied moment at node  $X$ ,  $\eta_X$  is the distribution coefficient (the ratio of the column's rotational stiffness to the total rotational stiffness of the members at the joint), given by Eq. (12):

$$\eta_X = \frac{K_{c,X}}{K_{c,X} + \sum K_{b,X}} \quad (12)$$

where  $K_{c,X}$  is the rotational stiffness of the single column at node  $X$ ; and  $\sum K_{b,X}$  is the sum of the rotational stiffness of each beam at node  $X$ .

- Then, by keeping end  $I$  held at  $\theta$ , and replacing the fixed support at end  $J$  by adjoining beams (Fig. 3(b)) and releasing the moment  $CM_I$ , it follows from Eq. (12) that the moment in the column at end  $J$  is  $-CM_I \eta_J$  and the moment distributed back to end  $I$  is  $-CM_I \eta_J C$  (Fig. 3(b)).
- Therefore the net moment required to rotate end  $I$  by  $\theta$ , for the column which is not fully rigid at joint  $J$ , is:

$$M_I - CM_I \eta_J C \quad (13)$$

Hence, the reduction in  $K'_{IJ}$  is found by combining Eqs. (8) and (13). The resistance of a column  $IJ$  to rotation at node  $I$ , with adjoining beams at node  $J$ , is then given by Eq. (14) and notated as  $K''_{IJ}$ .

$$K''_{IJ} = \frac{M_I - CM_I \eta_J C}{4E\theta_I} = K'_{IJ} (1 - C^2 \eta_J) \quad (14)$$

where  $\eta_J$  is the distribution coefficient for node  $J$  from Eq. (12) and  $K'_{IJ}$  is the rotational stiffness of the column at node  $I$  when node  $J$  is fixed, obtained from Eq. (10).

Combining Eqs. (10), (12) and (14), Eq. (15) is obtained:

$$K''_{IJ} = \frac{K_{IJ} S}{4} \left( 1 - C^2 \left( \frac{K_{IJ} S}{K_{IJ} S + \sum K_{b,J}} \right) \right) \quad (15)$$

Beyond a certain value of  $P_{IJ}/P_{E,IJ}$ , applying the corresponding values of  $S$  and  $C$  to Eq. (15), will result in a negative rotational stiffness.

The criteria for buckling is that the rotational stiffness at a joint is zero. If node  $I$  is pinned then the critical load will be obtained from Eq. (16):

$$K''_{IJ} = 0 \quad (16)$$

However in a frame the rotational restraint provided by any adjoining beams at node  $I$  would also need to be considered. Therefore the critical axial load in the column is reached when Eq. (17) is satisfied.

$$K''_{IJ} + \sum K_{b,I} = 0 \quad (17)$$

where  $\sum K_{b,I}$  is the rotational stiffness of the beams at node  $I$ . This is found by first combining Eqs. (15) and (17) to give Eq. (18):

$$\frac{K_{IJ} S}{4} \left( 1 - C^2 \left( \frac{K_{IJ} S}{K_{IJ} S + \sum K_{b,J}} \right) \right) + \sum K_{b,I} = 0 \quad (18)$$

Rearranging the distribution coefficient (Eq. (12)) to solve for  $\sum K_{b,I}$  and  $\sum K_{b,J}$  gives:

$$\sum K_{b,I} = K_{IJ} \left( \frac{1}{\eta_I} - 1 \right) \quad (19)$$

$$\sum K_{b,J} = K_{JI} \left( \frac{1}{\eta_J} - 1 \right) \quad (20)$$

Substituting these into Eq. (18), noting that the  $K_{IJ}$  terms can now be removed, gives Eq. (21):

$$\frac{S}{4} \left( 1 - C^2 \left( \frac{\frac{S}{4}}{\frac{S}{4} + \left( \frac{1}{\eta_I} - 1 \right)} \right) \right) + \left( \frac{1}{\eta_I} - 1 \right) = 0 \quad (21)$$

Therefore, when  $\eta_I$  and  $\eta_J$  are known, the resulting equation can be satisfied by applying the appropriate stability functions corresponding to the critical value of  $P_{IJ}/P_{E,IJ}$  which causes instability, from which we can get the column's effective length ratio, Eq. (22).

$$\frac{P_{IJ}}{P_{E,IJ}} = \frac{\pi^2 E_{IJ}}{L_{E,IJ}^2} = \left( \frac{L_{IJ}}{L_{E,IJ}} \right)^2 \quad (22)$$

Wood constructed design charts from Eq. (21), which require the designer to know only the top and bottom distribution coefficients ( $\eta_I$  and  $\eta_J$ ) to find the effective length ratio. The charts are symmetrical about their horizontal and vertical axes, which means the designer can apply the top or bottom distribution coefficients ( $\eta_I$  and  $\eta_J$ ) to either the  $x$ - or  $y$ -axis.

### 2.1.1. Multi-storey frames

For the method to be applied to continuous columns in multi-storey frames an adjustment to the distribution coefficient was suggested. Wood [8] recommended the rotational stiffness of an adjoining column is added to both the numerator and denominator of the distribution coefficient as shown in Eq. (23), which can be compared to Eq. (12).

$$\eta_X = \frac{\sum K_{c,X}}{\sum K_{c,X} + \sum K_{b,X}} \quad (23)$$

where  $K_{c,X}$  is the rotational stiffness of each column at node  $X$ ; and  $K_{b,X}$  is the rotational stiffness of each beam at node  $X$ .

The rationale behind this is that both upper and lower columns at any joint are required to be restrained by the beams at that level [8] suggesting that both columns are subjected to sufficient axial load that they both have negative stiffness at the same time. This approach also allows the same design charts as for a single storey frame to be used to find the effective length ratio.

Applying this method to the column  $IJ$  in Fig. 4, Eq. (21) still holds true as the same design charts are being used. Multiplying by  $K_{IJ}$  (see Eqs. (19)–(21)) and substituting values from Eq. (23) into Eq. (21), Eq. (24) is obtained:

$$\left[ \frac{K_{IJ}S}{4} \left( 1 - C^2 \left( \frac{\frac{K_{IJ}S}{4}}{\frac{K_{IJ}S}{4} + \frac{K_{IJ} \sum K_{b,J}}{\sum K_{c,J}}} \right) \right) \right] + \left[ \frac{K_{IJ} \sum K_{b,I}}{\sum K_{c,I}} \right] = 0 \quad (24)$$

$$\text{Since } K_{IJ} \left( \frac{1}{\eta_I} - 1 \right) = K_{IJ} \left( \frac{\sum K_{c,J} + \sum K_{b,J}}{\sum K_{c,J}} - 1 \right) = \frac{K_{IJ} \sum K_{b,J}}{\sum K_{c,J}} \quad (25)$$

The first expression in Eq. (24) represents the effective rotational stiffness of column  $IJ$  at node  $I$  ( $K''_{IJ}$ ), and the second expression represents the rotational stiffness of the adjoining members at node  $I$ .

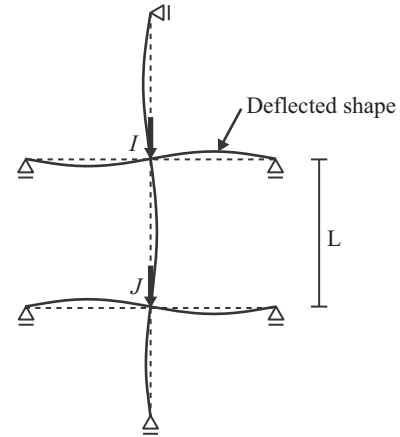


Fig. 4. A simple multi-storey frame.

It can be seen from the second term that the rotational stiffness of the beams at node  $I$  ( $\sum K_{b,I}$ ) has been shared between each column in proportion to their nominal stiffness, i.e. rotational restraint provided to column  $IJ$  at node  $I$  is  $\frac{K_{IJ}}{K_{IJ} + K_{XY,I}} \sum K_{b,I}$  (where  $K_{XY,I}$  is the restraint provided by column  $XY$  at node  $I$ ).

Therefore, the greater the nominal stiffness of an adjoining column relative to that of column  $IJ$ , the lower the rotational restraint provided to column  $IJ$ , and subsequently the lower its buckling load. This is the reason for the anomalies described in Section 1.

This is despite the fact that it is apparent that the greater the nominal stiffness of an adjoining column, the less likely it will require restraining.

Eq. (24) can now alternatively be written as Eq. (26):

$$\frac{K_{IJ}S}{4} \left( 1 - C^2 \left( \frac{\frac{K_{IJ}S}{4}}{\frac{K_{IJ}S}{4} + \sum K_{b,J} - \frac{K_{XY,I}}{\sum K_{c,J}} \sum K_{b,J}} \right) \right) + \sum K_{b,I} - \frac{K_{XY,I}}{\sum K_{c,I}} \sum K_{b,I} = 0 \quad (26)$$

where  $K_{XY,I}$  is the nominal stiffness of the adjoining column  $XY$  at node  $I$ . Therefore the resistance of the adjoining column to rotation at node  $I$  at the critical load can be seen to be estimated from Eq. (27):

$$K''_{XY,I} = - \frac{K_{XY,I}}{\sum K_{c,I}} \sum K_{b,I} \quad (27)$$

There is no consideration given to the adjoining column's axial load or far-end restraint conditions.

The incorporation of an adjoining column's stiffness also has the effect of reducing  $K''_{IJ}$ , therefore reducing the critical load further.

### 2.1.2. Effective rotational stiffness of adjoining members

For the calculation of the distribution coefficients in Eq. (23), the NCCI recommends using the nominal rotational stiffness from Eq. (9) for columns and an effective rotational stiffness for beams [1], modifying the nominal stiffness of a beam to take account of its far end restraint condition and axial load. Wood [8] devised an approach to find a beam's modified rotational stiffness in multi-storey frames, which allows consideration of the rotation at the ends of a beam. Using the slope deflection equation  $M_A = EK(4\theta_A + 2\theta_B)$  [16] the modified rotational stiffness of a beam  $AB$  can be related to the rotation at its ends, as Eq. (28):

$$K'_A = \frac{M_A}{4E\theta_A} = \frac{EK(4\theta_A + 2\theta_B)}{4E\theta_A} = K_A \left( 1 + 0.5 \frac{\theta_B}{\theta_A} \right) \quad (28)$$

where  $\theta_B$  is the rotation at the far end, and  $\theta_A$  is the rotation at the near end.

In the critical buckling mode shape of a sway frame, beams bend in double curvature and the rotations at both ends of a beam are equal  $\theta_A = \theta_B \rightarrow \frac{\theta_B}{\theta_A} = 1$ . Therefore the beams modified rotational stiffness would be:

$$K' = K(1 + 0.5 \times 1) = 1.5K \quad (29)$$

In the critical buckling mode shape of a non-sway frame, beams bend in single curvature and the rotation at both ends of a beam are equal but opposite  $\theta_A = -\theta_B \rightarrow \frac{\theta_B}{\theta_A} = -1$ . Therefore the beams modified rotational stiffness would be:

$$K' = K(1 + 0.5 \times -1) = 0.5K \quad (30)$$

## 2.2. NCCI SN008a

BS EN 1993 [2] refers to the NCCI document SN008a [1]. The method presented in the NCCI for determining the effective length of columns is the same as that presented in Annex E of BS 5950 [17].

The procedure to determine the effective length of steel columns in frames outlined in SN008a [1] is simple to undertake. Distribution coefficients calculated for the top and bottom of the column (notated as  $\eta_1$  and  $\eta_2$  in SN008a [1]) allow the effective length ratio to be extracted from design charts for both sway or non-sway cases. Distribution coefficients can theoretically vary from zero (analogous of a fully rigid support) to one (representing a pure pin).

The distribution coefficients for the top and bottom column nodes are calculated from Eq. (23).

The rotational stiffness is assumed to be linear elastic. The rotational stiffness of a member with a fixed far-end and no axial load can be determined from Eq. (9), which is used in SN008a [1] to calculate column stiffness regardless of their real far end restraint conditions and axial load. The rotational stiffness of adjoining beams can however be modified in SN008a [1]. At present the design charts are only appropriate for frames with fully rigid connections, as the NCCI does not provide any guidance on how to design for semi-rigid connections.

Following the recommendations of BS 5950 Annex E [17], any restraining member required to carry more than 90% of its moment capacity should be assigned a  $K$  value of zero. Similarly, if either end of the column being designed is required to carry more than 90% of its moment-carrying capacity, then the distribution coefficient ( $\eta$ ) should be taken as 1.

In the sway frame design chart, the effective length ratio can vary from one to infinity, with an effective length ratio of infinity corresponding to a sway frame column that has pinned supports at both ends (a mechanism).

## 2.3. The AISC LRFD method

The American Institute for Steel Construction (AISC) Load and Resistance Factor Design (LRFD) manual [18] presents a similar method to the NCCI [1] for the calculation of the effective length of columns in multi storey frames. After evaluating stiffness ratios at the top and bottom of the column, the effective length ratio is read from design nomographs. Like the NCCI, the AISC method assumes adjoining columns buckle simultaneously and therefore adjoining columns provide negative rotational stiffness at restraints. Both methods are examples of linear static analysis of the equilibrium of a column in its deformed state.

In unbraced frames, stronger columns will brace weaker columns in the same storey, and the AISC commentary [18] allows

the column effective length ratio ( $\psi$ ) to be modified to account for this effect by Eq. (31).

$$\psi'_i = \sqrt{\frac{\sum P_u}{P_{ui}} \frac{I_i}{\sum \frac{I}{\psi_0^2}}} \geq \sqrt{\frac{5}{8}} \psi_{i0} \quad (31)$$

where  $\psi'_i$  is the modified column effective length ratio;  $\sum P_u$  is the required axial compressive strength of all columns in a storey;  $P_{ui}$  is the required compressive strength of the column under investigation;  $\sum \frac{I}{\psi_0^2}$  is the ratio of the second moment of area to the effective length factor of each column in the storey based on the sway alignment charts found in [18],  $I_i$  is the second moment of area of the column under investigation;  $\psi_{i0}$  is the unmodified effective length factor for the column.

Eq. (31) allows for column end nodes with translational stiffness' somewhere between zero and infinity, corresponding to the support conditions of a weaker column that is being braced by a stronger one. It also allows for negative translational stiffness, as encountered when the column under investigation is bracing a weaker adjacent column or a leaning column (a leaning column has pinned supports and therefore has no translational stiffness). The application of this modification factor is demonstrated in Section 3.2.1 below.

## 3. Application

The effective length method will be used to evaluate the stability of a variety of framed columns. The results obtained will be compared to that computed using structural analysis software Autodesk Robot Structural Analysis [19] which will be used to perform an eigenvalue buckling analysis. This type of analysis is directly comparable to the Euler formula as it makes all the same assumptions, such as initially straight, perfectly elastic members, and uses a linearised expression for curvature (the one difference being that for compatibility reasons the curvature of members is represented using a cubic form instead of sinusoidal).

### 3.1. The effect of a column from an adjacent storey

The effective length method will be used to assess the stability of the frames in Figs. 5 and 6, considered first as non-sway and then as sway frames. Both frames have regular spaced columns and an even distribution of load (which is common in real building structures). Consequently, it is expected that frame instability will be caused by the buckling of all columns in a single storey as the columns in this storey will have reached their critical load. The effective length method will be used to find the elastic buckling load of a column in the critical storey. The frames are identical, other than the height of the top storey.

All members are  $203 \times 203 \times 60$  UC and all connections are fully rigid. Columns buckle about their weak axis ( $I_{\text{minor}} = 2070 \text{ cm}^4$ ) whilst the beams bend about their strong axis ( $I_{\text{major}} = 6130 \text{ cm}^4$ ), both in the plane of the frame. The beams carry no axial load and do not reach their flexural capacity. The distribution of the design loads are shown in Figs. 5 and 6 along with frame geometry. The frames are braced out of plane. Base nodes of the columns are taken as fully rigid ( $\eta = 0$ ). Frame B might be expected to buckle at a lower load compared to frame A since its top storey column is more slender.

#### 3.1.1. Example 1 – Non sway

Frames A and B are first considered to be non-sway frames. For Frame A (Fig. 5), the NCCI approach suggests that the columns in the middle storey buckle first and cause frame instability. The elastic critical load of column BC therefore defines the load in the rest

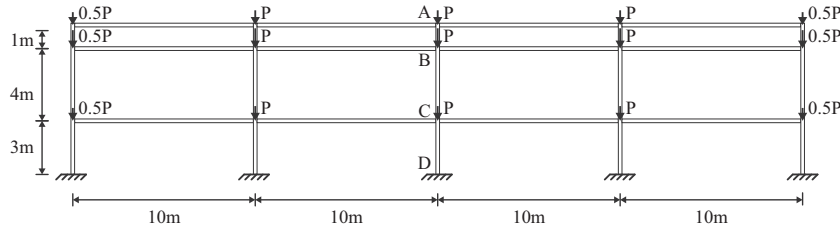


Fig. 5. Frame A: multi-storey fully rigid steel with stiff top storey.

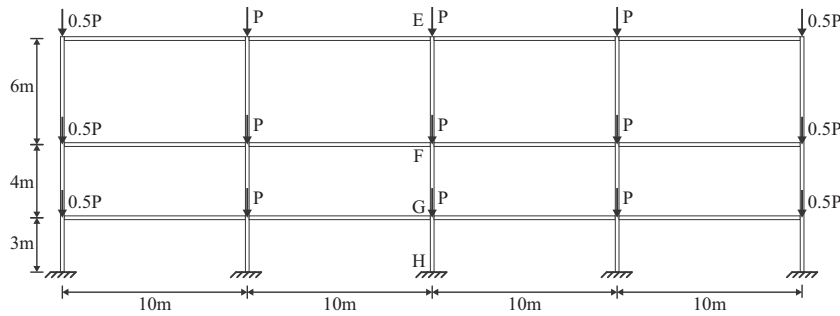


Fig. 6. Frame B: multi-storey fully rigid steel with slender top storey.

of the frame at instability, and is found to be 3846 kN. The axial load in column BC at frame instability, found using an Eigenvalue analysis in Robot [19], is 7054 kN. The NCCI predicts Frame A will buckle at a load 45% lower than that computed by Robot. The calculations for this section are summarised in Table 1. In the non-sway computational analysis supports were added at all nodes which prevented horizontal movement only. These were removed when calculating the sway frames.

Applying the NCCI approach to Frame B (Fig. 6) suggests that column FG will buckle first, at an elastic critical load of 4653 kN. Using Robot, it is found that the axial load in column FG at frame instability is 5622 kN. In this instance the NCCI is closer, but still predicts a collapse load 17% lower than that found using Robot.

Furthermore the NCCI predicts that Frame A buckles at a lower load than Frame B, which from observation does not seem correct. Robot supports the intuitive prediction that Frame B is less stable than Frame A.

The distribution coefficients calculated for nodes B and F using the NCCI are  $\eta_B = 0.808$  and  $\eta_F = 0.585$ . This shows that according to the NCCI, node F has more rotational stiffness than node B, which is evidently incorrect, as the short column AB is stiffer than slender column EF. The distribution coefficients calculated for nodes C and G are the same ( $\eta_C = \eta_G = 0.663$ ) as the members converging on these nodes are identical. So the effective length ratio of column BC and FG, read from the design charts, are  $\frac{L_{E,BC}}{L_{BC}} = 0.825$  and  $\frac{L_{E,FG}}{L_{FG}} = 0.750$ . The calculations for this section are summarised in Table 1.

Table 1  
Calculations for Frame A and Frame B as non-sway frames.

	Frame A	Frame B
Distribution coefficients	$\eta_B = \frac{K_{BA} + K_{BC}}{K_{BA} + K_{BC} + \sum K'_{b,B}} = 0.808$ $\eta_C = \frac{K_{CB} + K_{CD}}{K_{CB} + K_{CD} + \sum K'_{b,C}} = 0.663$ $\eta_D = 0$	$\eta_E = \frac{K_{EF}}{K_{EF} + \sum K'_{b,E}} = 0.360$ $\eta_F = \frac{K_{FE} + K_{FG}}{K_{FE} + K_{FG} + \sum K'_{b,F}} = 0.585$ $\eta_G = \eta_C = 0.663$
Effective length ratios and elastic critical loads using [1]	$\frac{L_{E,BC}}{L_{BC}} = 0.825 \rightarrow P_{C,BC} = 3846 \text{ kN}$ $\frac{L_{E,CD}}{L_{CD}} = 0.825 \rightarrow P_{C,CD} = 11,913 \text{ kN}$	$\frac{L_{E,EF}}{L_{EF}} = 0.675 \rightarrow P_{C,EF} = 2553 \text{ kN}$ $\frac{L_{E,FG}}{L_{FG}} = 0.750 \rightarrow P_{C,FG} = 4653 \text{ kN}$
Conclusion	When column BC buckles the axial load in column CD is: $(3846 \times 1.5 = 5,769 \text{ kN} < 11,913 \text{ kN})$ therefore BC is critical.	When column EF reaches its critical load, the axial load in column FG is $(2553 \times 2 = 5106 \text{ kN} > 4653 \text{ kN})$ therefore column FG buckles first.

### 3.1.2. Example 2 – Sway frames

Frame A and Frame B are now considered to be sway frames.

The NCCI predicts that in both frames, the middle storey columns buckle first. The calculated elastic critical loads of BC and FG are calculated as  $P_{C,BC} = 1211 \text{ kN}$  and  $P_{C,FG} = 1623 \text{ kN}$ , contrary to the expected result that  $P_{C,BC} > P_{C,FG}$ . The analysis using Robot [19] computes the elastic critical loads of columns BC and FG as  $P_{C,BC} = 2230 \text{ kN}$  and  $P_{C,FG} = 1984 \text{ kN}$ .

The error again arises from the calculation of the distribution coefficients for the top nodes, where node B is less stiff than node F. The calculations for this section are summarised in Table 2.

### 3.2. The effect of columns in the same storey

Another limitation of the NCCI approach is that it assumes all columns in a storey buckle simultaneously. One of the consequences is that in unbraced frames the contribution made by adjacent columns in the same storey to the translational stiffness of the column being checked is ignored. In the NCCI, the translational stiffness of the end restraints is assumed to be either zero or infinity for sway or non-sway frames respectively [1].

In unbraced frames, stronger columns will brace weaker columns in the same storey. If the column being analysed is partially braced by a stiffer column then its end restraints will have translational stiffness between zero and infinity. If the column being checked is partially bracing a less stiff column, or fully bracing a pin ended column, the translational stiffness of the restraints will

**Table 2**  
Calculations for Frame A and Frame B as sway frames.

	Frame A	Frame B
Distribution coefficients	$\eta_B = \frac{K_{BA}+K_{BC}}{K_{BA}+K_{BC}+\sum K_{b,B}} = 0.585$ $\eta_C = \frac{K_{CB}+K_{CD}}{K_{CB}+K_{CD}+\sum K_{b,C}} = 0.396$ $\eta_D = 0$	$\eta_E = \frac{K_{EF}}{K_{EF}+\sum K_{b,E}} = 0.158$ $\eta_F = \frac{K_{FE}+K_{FG}}{K_{FE}+K_{FG}+\sum K_{b,F}} = 0.319$ $\eta_G = \eta_C = 0.396$
Effective length ratios and elastic critical loads using [1]	$\frac{L_{EBC}}{L_{BC}} = 1.47 \rightarrow P_{C,BC} = 1211 \text{ kN}$	$\frac{L_{EFF}}{L_{EF}} = 1.16 \rightarrow P_{C,EF} = 865 \text{ kN}$ $\frac{L_{FCG}}{L_{FG}} = 1.27 \rightarrow P_{C,FG} = 1,623 \text{ kN}$
Conclusion	From inspection column BC is the critical column and will buckle first	When column EF reaches its critical load, the axial load in column FG is $(865 \times 2 = 1730 \text{ kN} > 1623 \text{ kN})$ therefore column FG buckles first.

be negative and the NCCI can overestimate the elastic critical load, as shown in Example 3 below.

3.2.1. Example 3

The NCCI approach has been used to find the elastic critical loads of column JK in Frame C, Fig. 7, and column NQ in Frame D, Fig. 8. The results, given in Table 3, highlight the potential errors encountered by ignoring the possibility of restraints having negative translational stiffness. All members are 203x203x60 UC and all connections are fully rigid. Columns buckle about their weak axis ( $I_{minor} = 2070 \text{ cm}^4$ ) whilst the beams bend about their strong axis ( $I_{major} = 6130 \text{ cm}^4$ ), both in the plane of the frame.

The critical load of column JK found using the NCCI is almost double that found using Robot. The critical column for Frame C is LM, so it would be expected that the buckling load of this column will define frame instability. However, using the NCCI, the effective length of column LM would be infinity as both ends are pinned and it is part of a sway frame; therefore it is unable to take any load. The designer may decide that it is fully braced by column JK, and as such read from the non-sway design chart, in which case they would obtain an effective length ratio of one, and an elastic critical load of 2554 kN, which is evidently much too high. The NCCI gives the designer no options in these cases and without careful thought can lead to potentially unconservative errors.

Applying the AISC modification factor (section 2.3) to Frame C, the effective length ratio is given by Eq. (32) and the elastic critical load by Eq. (33), which is in very good agreement with the computer analysis.

Frame C - Leaning column

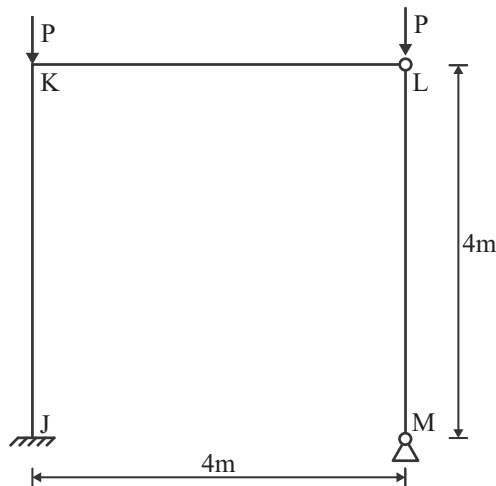


Fig. 7. Frame C: portal frame with a leaning column.

Frame D - No leaning column

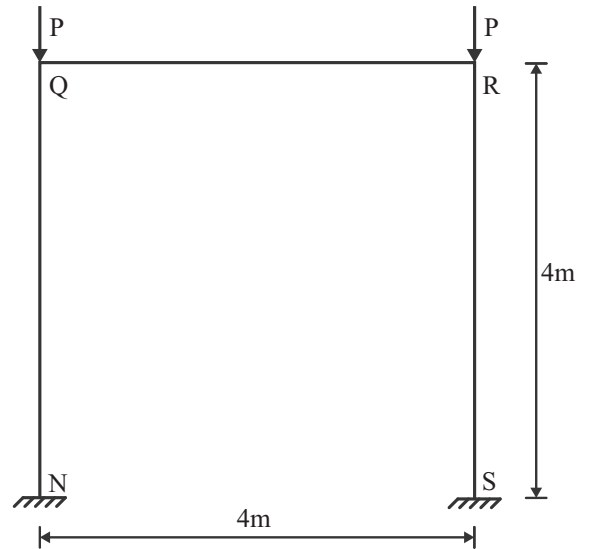


Fig. 8. Frame D: Portal frame without a leaning column.

$$\psi'_{AB} = \sqrt{\frac{\sum P_u I_{AB}}{P_{uAB} \sum \frac{I}{\psi_0^2}}} = \sqrt{2\psi_{0,AB}} = \sqrt{2(1.12)} = 1.50 \tag{32}$$

$$P_C = \frac{\pi^2 EI}{\lambda(L^2)} = \frac{\pi^2 EI}{1.5(4000^2)} = 1163 \text{ kN} \tag{33}$$

3.3. Conclusion

The stiffness distribution method developed by Wood [8] seems logical and effective for single storey frames. For continuous columns in multi-storey frames, the approach adopted by Wood [8] assumes adjoining columns to have negative rotational stiffness. As a result the NCCI approach has been shown to provide unreliable results in certain cases, by incorrectly evaluating the contribution made by adjoining columns to the rotational stiffness of end restraints. This occurs because the method fails to assess the load in the rest of the structure, which evidently has an effect on the stiffness of end restraints.

4. Proposal

There are numerous proposed improvements to the effective length method, but none have replaced the method used in BS 5950 [20] and now the NCCI [1]. The principle reason for this is that



**Table 3**  
Results for Example 3.

	NCCI			Robot	Robot ÷ NCCI	
	Distribution coefficients		Effective length ratio [1]	Elastic critical load (Eq. (3)) (kN)	Elastic critical load (kN)	
Column JK	Node K Node J	$\eta_K = 0.310$ $\eta_J = 0$	1.117	2100	1137	0.52
Column NQ	Node Q Node N	$\eta_Q = 0.184$ $\eta_N = 0$	1.063	2347	2340	1.00

many of the proposed methods are far too complicated and time-consuming to apply by hand, and consequently are unsuitable for preliminary design purposes and impractical to the Design Engineer.

BS 5950 [20] and the NCCI [1] also provide formulae to determine the effective length ratio from distribution coefficients as an alternative to design charts. These formulae are more conservative than the design charts, and more precise formulae have been developed by Smyrell [21] using curve fitting techniques.

Lui [22] developed a method to determine the effective length ratio of framed columns, which explicitly takes into account translational stiffness by applying a fictitious horizontal force to the frame. It also allows for the existence of weaker and stronger columns (or leaning columns) in the same storey. The formula has been shown to provide reliable results when applied to frames where columns in a single storey are of different strengths [23].

#### 4.1. Improved rotational stiffness for adjoining beams

Mageirou and Gantes [7] derived the rotational stiffness' of members using the slope-deflection method, similar to Wood's derivation [8] of the rotational stiffness of adjoining beams. The options for far-end restraint conditions were expanded to include roller supports with various rotational stiffnesses. These options are applicable to the far-end restraint conditions of columns in sway frames and therefore allow an adjoining column's rotational stiffness to be modified appropriately for the calculation of the distribution coefficient. The work of Mageirou and Gantes [7] is also applicable for members with semi-rigid connections, in sway, non-sway and partially sway frames.

Gantes and Mageirou [6] give rotational stiffness values similar to those in the NCCI [1], as shown in Table 4. Gantes and Mageirou [6] define rotational stiffness as  $K = M/\theta$ , so that the stiffness of a fixed end member with no axial load is given by  $4EI/L$ . Wood used Eq. (8), so the nominal stiffness is  $I/L$ . Therefore, the formulae can be made equivalent by dividing those proposed by Gantes and Mageirou [6] by  $4E$  (ultimately the methods from both Wood [8] and Gantes and Mageirou [6] provide the same results).

Mageirou and Gantes [7] have also improved the accuracy to which axial load affects the rotational stiffness and their method is less conservative than the NCCI method, as is shown by the coefficients of  $P/P_E$  used in each equation.

**Table 4**  
Modified rotational stiffness for beams with various far end restraint conditions.

Rotational conditions at far end	Mageirou and Gantes [6]	NCCI [1]
Fixed support (no rotation, no translation)	$4 \frac{EI}{L} \left(1 - 0.33 \frac{P}{P_E}\right)$	$1.0 \frac{I}{L} \left(1 - 0.4 \frac{P}{P_E}\right)$
Pinned support (free rotation, no translation)	$3 \frac{EI}{L} \left(1 - 0.66 \frac{P}{P_E}\right)$	$0.75 \frac{I}{L} \left(1 - 1.0 \frac{P}{P_E}\right)$
Single curvature (rotation equal and opposite to that at near end, no translation)	$2 \frac{EI}{L} \left(1 - 0.82 \frac{P}{P_E}\right)$	$0.50 \frac{I}{L} \left(1 - 1.0 \frac{P}{P_E}\right)$
Roller fixed support (free translation, no rotation)	$\frac{EI}{L} \left(1 - 0.82 \frac{P}{P_E}\right)$	-
Double curvature	$6 \frac{EI}{L} \left(1 - 0.16 \frac{P}{P_E}\right)$	$1.50 \frac{I}{L} \left(1 - 0.2 \frac{P}{P_E}\right)$

#### 4.2. Semi-rigid connections

The assumption of a fully rigid connection implies that no relative rotation of the connection occurs and that the end moment of a beam is completely transferred to the column. On the other hand, a pinned connection implies no rotational restraint is provided and the moment is zero at the connection.

Several non-linear models have been developed that provide a closer approximation to the true moment-rotation behaviour of connections, which employ curve fitting techniques that require the input of connection-dependent parameters that have been tabulated in the literature [7,24].

#### 4.3. New proposal

The main source of discrepancy in the current NCCI method comes from the evaluation of the rotational stiffness of the adjoining columns. An opportunity to modify the NCCI method, so that the rotational stiffness of adjoining columns is appropriately considered, has been identified. Distribution coefficients are proposed for use with the design charts of NCCI [1]:

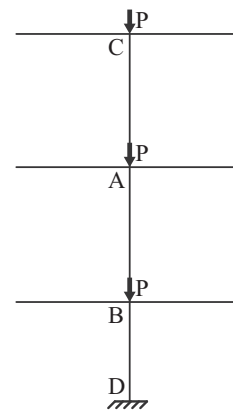
$$\eta_I = \frac{K_{IJ}}{K_{IJ} + K''_{XY,I} + \sum K''_{b,I}} \quad (34)$$

$$\eta_J = \frac{K_{JI}}{K_{JI} + K''_{XY,J} + \sum K''_{b,J}} \quad (35)$$

where  $K_{IJ}$  is the nominal stiffness of the column  $IJ$  which is being analysed;  $K''_{XY,I}$  and  $K''_{XY,J}$  are the effective rotational stiffness of the adjoining columns at nodes  $I$  and  $J$ ; and  $\sum K''_{b,I}$  and  $\sum K''_{b,J}$  are the effective rotational stiffness of the beams converging at nodes  $I$  and  $J$ , evaluated from Table 4.

Rearranging the distribution coefficients gives Eqs. (36) and (37).

$$\left(\frac{1}{\eta_I} - 1\right) = \frac{K''_{XY,I} + \sum K''_{b,I}}{K_{IJ}} \quad (36)$$



**Fig. 9.** Multi-storey frame design loads.

**Table 5**  
Proposed effective rotational stiffness' of adjoining columns.

Support conditions at far end	Effective rotational stiffness of an adjoining column XY, $K''_{XY}$
Fixed support (no rotation, no translation)	$\frac{I_{XY}}{L_{XY}} \left( 1 - 0.33 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right)$
Pinned support (free rotation, no translation)	$0.75 \frac{I_{XY}}{L_{XY}} \left( 1 - 0.66 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right)$
Single curvature (rotation equal and opposite to that at near end, no translation)	$0.5 \frac{I_{XY}}{L_{XY}} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right)$
Roller support (no rotation, free translation)	$0.25 \frac{I_{XY}}{L_{XY}} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{L_{ij}} \right)^2 \right)$

$$\left( \frac{1}{\eta_j} - 1 \right) = \frac{K''_{XY,j} + \sum K''_{b,j}}{K_{ij}} \quad (37)$$

Inputting Eqs. (36) and (37) into the governing design chart Eq. (21) and multiplying by  $K_{ij}$ , the criterion for buckling is given by Eq. (38):

$$\frac{K_{ij}S}{4} \left( 1 - C^2 \left( \frac{\frac{K_{ij}S}{4}}{\frac{K_{ij}S}{4} + K''_{XY,j} + \sum K''_{b,j}} \right) \right) + K''_{XY,j} + \sum K''_{b,j} = 0 \quad (38)$$

The first term of Eq. (38) represents the effective rotational stiffness of the column under consideration,  $K''_{ij}$  (Eq. (39)).

$$K''_{ij} = \frac{K_{ij}S}{4} \left( 1 - C^2 \left( \frac{\frac{K_{ij}S}{4}}{\frac{K_{ij}S}{4} + K''_{XY,j} + \sum K''_{b,j}} \right) \right) \quad (39)$$

**Table 6**  
Test 1 – Non-sway frame results.

	Robot (kN)	NCCI (kN)	Robot/NCCI	Proposed (kN)	Robot/proposed
Frame A Column BC	7054	3846	1.83	7146	0.99
Frame B Column FG	5622	4653	1.21	5654	0.99

**Table 7**  
Calculations for Frame A and Frame B as non-sway frames using the proposed method.

Frame A	Frame B
<p><b>Column BC</b>  <math>K_{BC} = \frac{l}{l} = \frac{20700000}{4000} = 5175</math>  <math>K''_{AB} = 0.5 \frac{l}{l} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = 9809</math>  <math>K''_{CD} = \frac{l}{l} \left( 1 - 0.33 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = 2979</math>  <math>K'_b = 0.5 \frac{l}{l} = 3065</math> (single curvature)  <math>\eta_B = \frac{K_{BC}}{K_{BC} + K_{BA} + \sum K'_{b,B}} = 0.245</math>  <math>\eta_C = \frac{K_{CB}}{K_{CB} + K_{CD} + \sum K'_{b,C}} = 0.362</math>  <math>\frac{L_{E,BC}}{L_{BC}} = 0.650 \rightarrow P_{C,BC} = 7146</math> kN                      At this stage the accuracy can be improved by substituting the found value for <math>L_E</math> into the <math>K'</math> formula for the adjoining column and repeating the analysis.</p>	<p><b>Column EF</b>  <math>K_{EF} = \frac{l}{l} = 3450</math>  <math>K''_{FG} = 0.5 \frac{l}{l} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = -1262</math>  <math>K'_b = 0.5 \frac{l}{l} = 3065</math>  <math>\eta_E = \frac{K_{EF}}{K_{EF} + \sum K'_{b,E}} = 0.360</math>  <math>\eta_F = \frac{K_{FE}}{K_{FE} + K_{FG} + \sum K'_{b,F}} = 0.415</math>  <math>\frac{L_{E,EF}}{L_{EF}} = 0.642 \rightarrow P_{C,EF} = 2827</math> kN</p>
<p><b>Column CD</b>  <math>K_{CD} = \frac{l}{l} = \frac{20700000}{3000} = 6900</math>  <math>K''_{BC} = 0.5 \frac{l}{l} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = -2544</math>  <math>K'_b = 3065</math> (single curvature)  <math>\eta_C = \frac{K_{CD}}{K_{CD} + K_{CB} + \sum K'_{b,C}} = 0.658</math>  <math>\eta_D = 0</math>  <math>\frac{L_{E,CD}}{L_{CD}} = 0.616 \rightarrow P_{C,CD} = 12,266</math> kN</p>	<p><b>Column FG</b>  <math>K_{FG} = \frac{l}{l} = 5175</math>  <math>K''_{EF} = 0.5 \frac{l}{l} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = -1523</math>  <math>K''_{GH} = \frac{l}{l} \left( 1 - 0.33 \frac{P_{d,XY}}{P_{d,ij}} \left( \frac{L_{XY}}{0.7L_{ij}} \right)^2 \right) = 2979</math>  <math>K'_b = 3065</math>  <math>\eta_F = \frac{K_{FG}}{K_{FG} + K_{FE} + \sum K'_{b,F}} = 0.529</math>  <math>\eta_G = \frac{K_{GF}}{K_{GF} + K_{GH} + \sum K'_{b,G}} = 0.362</math>  <math>\frac{L_{E,FG}}{L_{FG}} = 0.670 \rightarrow P_{C,FG} = 5859</math> kN</p>
<p><b>Conclusion:</b> When column BC buckles the axial load in column CD is (7146 × 1.5 = 10,719 kN &lt; 12,266 kN) therefore column BC is critical</p>	<p><b>Conclusion:</b> When column EF buckles, the axial load in column FG is (2827 × 2 = 5654 kN &lt; 5859 kN) therefore EF is critical.</p>

The term  $\frac{K_{ij}S}{4 + K''_{XY,j} + \sum K''_{b,j}}$  in Eq. (39) is the proportion of the carried-over moment transferred back into the column under consideration.

The criterion for buckling can then be given by Eq. (40):

$$K''_{ij} + K''_{XY,j} + \sum K''_{b,j} = 0 \quad (40)$$

To properly evaluate the rotational stiffness of an adjoining column, its far-end restraint conditions and axial load need to be considered. The effective rotational stiffness of a beam with axial load can be determined from Table 4. To use the same equations to obtain the rotational stiffness of adjoining columns, you would require knowledge of its axial load when the critical load of the column under investigation is reached.

Consider the frame in Fig. 9, where a load  $P$  is applied to the column at each floor.

The column CA carries an axial load  $P$ ; column AB carries axial load  $2P$ ; and column BD carries axial load  $3P$ . To determine the effective length of column AB then its load at buckling is  $P_{C,AB} = \frac{\pi^2 EI}{L'^2_{AB}}$  where  $L'$  is a crude approximation of its effective length, which will be taken as its real length for sway frames, or 0.7 times its real length for non-sway frames. This ensures the axial load in adjoining columns is not underestimated.

The load in columns AC and BD can then be estimated as  $P_{AC} = 0.5P_{C,AB}$  and  $P_{BD} = 1.5P_{C,AB}$ . A general equation to estimate the load in an adjoining column (XY) at the point when the critical

**Table 8**  
Test 2 – Sway Frame Results.

	Robot (kN)	NCCI (kN)	Robot/NCCI	Proposed (kN)	Robot/proposed
Frame A Column BC	2230	1211	1.84	2087	1.07
Frame B Column FG	1984	1623	1.20	1958	1.01

**Table 9**  
Calculations for Frame A and Frame B as sway frames using the proposed method.

Frame A	Frame B
<p><b>Column BC</b></p> $K_{BC} = \frac{EI}{L} = 5175$ $K''_{AB} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 5042$ $K''_{CD} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 532$ $K'_b = 1.5 \frac{EI}{L} = 9195 (\text{double curvature})$ $\eta_B = \frac{K_{BC}}{K_{BC} + K_{BA} + \sum K'_{b,B}} = 0.181$ $\eta_C = \frac{K_{CD}}{K_{CD} + K_{CB} + \sum K'_{b,C}} = 0.215$ $\frac{L_{EBC}}{L_{BC}} = 1.12 \rightarrow P_{C,BC} = 2087 \text{ kN}$ <p><b>Column CD</b></p> $K_{CD} = \frac{EI}{L} = \frac{20700000}{3000} = 6900$ $K''_{BC} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 36$ $K'_b = 9195$ $\eta_C = \frac{K_{CD}}{K_{CD} + K_{CB} + \sum K'_{b,C}} = 0.272$ $\eta_D = 0$ $\frac{L_{ECD}}{L_{CD}} = 1.09 \rightarrow P_{C,CD} = 3917 \text{ kN}$ <p><b>Conclusion:</b> When column BC buckles the axial load in column CD is (2087 × 1.5 = 3131 kN &lt; 3817 kN) therefore column BC is critical</p>	<p><b>Column EF</b></p> $K_{EF} = \frac{EI}{L} = 3450$ $K''_{FG} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 351$ $K'_b = 1.5 \frac{EI}{L} = 9195$ $\eta_E = \frac{K_{EF}}{K_{EF} + \sum K'_{b,E}} = 0.158$ $\eta_F = \frac{K_{FE}}{K_{FE} + K_{FG} + \sum K'_{b,F}} = 0.155$ $\frac{L_{EFE}}{L_{EF}} = 1.09 \rightarrow P_{C,EF} = 979 \text{ kN}$ <p><b>Column FG</b></p> $K_{FG} = \frac{EI}{L} = 5175$ $K''_{EF} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 67$ $K''_{GH} = 0.25 \frac{EI}{L} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) = 532$ $K'_b = 9195$ $\eta_F = \frac{K_{FG}}{K_{FG} + K_{FE} + \sum K'_{b,F}} = 0.219$ $\eta_G = \frac{K_{GF}}{K_{GF} + K_{GH} + \sum K'_{b,G}} = 0.215$ $\frac{L_{EFG}}{L_{FG}} = 1.135 \rightarrow P_{C,FG} = 2032 \text{ kN}$ <p><b>Conclusion:</b> When column FG buckles, the axial load in column EF is (979 × 2 = 1958 kN &lt; 2032 kN) therefore EF is critical.</p>

load in the column under consideration ( $IJ$ ) is reached for a frame with any load distribution is given in Eq. (41).

$$P_{XY} = \frac{P_{d,XY}}{P_{d,IJ}} P_{C,IJ} \quad (41)$$

where  $P_{XY}$  is the load in the adjoining column,  $P_{d,IJ}$  is the design load of the column that has reached its critical load  $P_{C,IJ}$ , and  $P_{d,XY}$  is the design load of the adjoining column.

In a sway frame, the rotational stiffness of an adjoining column with a fixed roller support (no rotation, free horizontal translation) at its far-end can therefore be approximated from Eq. (42) (see Table 4).

$$K''_{XY} = 0.25 \frac{I_{XY}}{L_{XY}} \left( 1 - 0.82 \frac{P_{XY}}{P_{E,XY}} \right) \quad (42)$$

Substituting Eq. (41) into Eq. (42), Eq. (43) is obtained:

$$K''_{XY} = 0.25 \frac{I_{XY}}{L_{XY}} \left( 1 - 0.82 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{L_{IJ}} \right)^2 \right) \quad (43)$$

where  $L_{XY}$  is the height of the adjacent storey and  $L_{IJ}$  the height of the critical storey. This equation assumes the critical column and the adjoining column have the same  $EI$  value.

In a similar manner for a non-sway frame, the adjoining column with a fixed far-end can be approximated by Eq. (44).

$$K''_{XY} = \frac{I}{L} \left( 1 - 0.33 \frac{P_{d,XY}}{P_{d,IJ}} \left( \frac{L_{XY}}{0.7L_{IJ}} \right)^2 \right) \quad (44)$$

The proposed effective rotational stiffness of adjoining columns are summarised in Table 5.

The first three far-end support conditions listed in Table 5 correspond to columns in non-sway frames, and as such have the 0.7 effective length ratio included. Only the last rotational condition can be used for columns in sway frames.

#### 4.4. Testing

The proposed method has been used to assess the stability of the frames in Figs. 5 and 6. The results are compared to those found using Robot and the NCCI for both non-sway (Test 1) and sway (Test 2) conditions.

##### 4.4.1. Test 1 – Non-sway frames

For Frame A in Fig. 5 the proposed method predicts that column BC buckles first at a load of 7146 kN, which is within 1.3% of the load found using Robot.

It was shown in section 3.1.1 that using the NCCI approach on Frame B in Fig. 6 suggests column FG buckles first at a load of 4653 kN. The method proposed above predicts column EF buckles first at a load of 2827 kN, at which point column FG would carry a load of 5654 kN. This is within 0.6% of the load found using Robot (5622 kN). The results are summarised in Table 6, with the modified calculations shown in Table 7.

##### 4.4.2. Test 2 – Sway frames

The proposed method predicts column BC in Frame A (Fig. 5) is the critical column, and has an elastic critical load of 2087 kN. The NCCI also predicts BC would buckle first, but at the lower load of 1211 kN. The proposed method is in agreement with Robot, which obtained a load in column BC at frame instability of 2230 kN.

Similarly, the proposed method predicts column *EF* will buckle first in Frame B (Fig. 6) at a load of 979 kN, with a predicted axial load in column *FG* at frame instability of 1958 kN. This is within 1.3% of that computed by Robot, as summarised in Table 8 with the related calculations in Table 9.

The proposed approach correctly recognises the contribution made by an adjoining column to the rotational stiffness. The results obtained from the proposed method have been shown to be in good agreement to that computed from a finite element eigenvalue analysis.

## 5. Conclusion

The current NCCI method consistently underestimates the critical load of columns in multi-storey frames because of the conservative assumption that adjoining columns buckle simultaneously with the column being investigated. Subsequently, adjoining columns only reduce the rotational stiffness of end restraints. A further consequence of this assumption is that the stiffer an adjoining column, the greater the reduction in rotational stiffness of that end restraint. However it is apparent that a stiffer adjoining column is less likely to buckle.

To address these issues, a simple improvement to the calculation of the distribution coefficients has been proposed, which succeeds in accurately assessing the rotational stiffness of adjoining columns by considering the axial load in these columns when the column under investigation buckles. This is shown to significantly improve the accuracy of the results obtained from design charts without overcomplicating the design or even changing the design procedure.

In frames with leaning columns, or where stronger columns brace weaker columns, it is recommended that the NCCI should adopt AISC LRFD's modification factor [18], which has been shown to be a very simple and effective solution. Without this modification, the critical load of frames with leaning columns could be significantly over estimated, which may lead to the under design of frames with potentially disastrous results.

A limitation of the NCCI method is that it can only be applied to multi-storey frames with fully rigid connections. It has been demonstrated that the rotational stiffness of adjoining beams can be modified to account for the rotational stiffness of the beam-to-column connection. However, difficulty comes in assessing the rotational stiffness of the connections. Providing tables matching the different connection types to the corresponding rotational stiffness, in conjunction with limiting criteria, is one possibility. Whilst not a perfectly accurate solution, provided they are less than the real stiffness then the method should be suitable and will improve the accuracy of the results.

It is recognised that simplicity is an advantage in design codes. The methods presented here are intended to clarify and extend the current design guidance, retaining simplicity whilst achieving greater accuracy. It is noted that analytical techniques such as those shown in this paper are very useful to design engineers at preliminary design stages. However, where very high accuracy is required or very complex structures must be analysed, computational methods are widely available and these can also be used.

## 5.1. Future work

More work is required to determine an appropriate analytical method for assessing the rotational stiffness of semi-rigid connections in frames. New full size laboratory tests carried out on elastic frames could be used to observe the buckling modes of multi-storey frames, and measure points of contraflexure on the columns, with the aim of calculating their effective lengths. Additional work is required to extend the approach to three-dimensional frames.

## 6. Data access statement

All data created during this research are openly available from the University of Bath data archive at <<http://dx.doi.org/10.15125/BATH-00131>>.

## References

- [1] Oppe M, Muller C, Iles C. NCCI: buckling lengths of columns: rigorous approach. London: Access Steel; 2005.
- [2] BSI. 1993-1. Design of steel structures general rules and rules for buildings. London: BSI; 2005.
- [3] Simitse GJ. Fundamentals of structural stability. London: Butterworth Heinemann; 2006.
- [4] Timoshenko SP, Gere JM. Mechanics of materials. PWS Publishing Company; 1997.
- [5] Patnaik SN. Strength of materials: a new unified theory for the 21st century. London: Butterworth Heinemann; 2004.
- [6] Gantes CJ, Mageirou GE. Improved stiffness distribution factors for evaluation of effective buckling lengths in multi-storey sway frames. Eng Struct 2005;27:1113–24.
- [7] Mageirou GE, Gantes CJ. Buckling strength of multi-storey sway, non-sway and partially-sway frames with semi-rigid connections. J Construct Steel Res 2006;62.
- [8] Wood RH. Effective lengths of columns in multistorey buildings. Struct Eng 1974;52.
- [9] Cheong-Siat-Moy F. Column design in gravity-loaded frames. J Struct Eng 1991;117.
- [10] Bridge R, Fraser D. Improved G-factor method for evaluating effective lengths of columns. J Struct Eng 1987;113:1341–56.
- [11] Hellesland J, Bjorhovde R. Improved frame stability analysis with effective lengths. J Struct Eng 1996;122:1275–83.
- [12] Aristizabal-Ochoa J. Stability of columns under uniform axial load with semi-rigid connections. J Struct Eng 1994;120:3212–22.
- [13] Aristizabal-Ochoa J. Story stability of braced, partially braced, and unbraced frames: classical approach. J Struct Eng 1997;123.
- [14] Livesley RK, Chandler DB. Stability functions for structural frameworks. Manchester: Manchester University Press; 1956.
- [15] Timoshenko SP, Gere JM. Theory of elastic stability. New York: McGraw Hill; 1961.
- [16] Bazant PZ, Cedolin L. Stability of structures: elastic, inelastic, fracture and damage theories. New York: Oxford University Press; 1991.
- [17] BSI. BS 5950-1. Structural use of steelwork in building Code of practice for design – rolled and welded sections. London: BSI; 2000.
- [18] AISC. AISC manual of steel construction. Load and resistance factor design. Chicago: AISC; 2001.
- [19] Autodesk. Robot structural analysis professional. San Rafael, CA; 2014.
- [20] BSI. BS 5950-1. Structural use of steelwork in building code of practice for design in simple and continuous construction: hot rolled sections. London, UK: BSI; 2000.
- [21] Smyrell AG. Approximate formulae for determining the effective length of steel columns to BS 5950. Struct Eng 1993.
- [22] Lui EM. A novel approach for K-factor design. ASCE; 1992.
- [23] Duan L, Chen WF. Structural engineering handbook. Effective length factor of compression members. London: CRC Press; 1999.
- [24] Chen WF, Goto Y, Liew JYR. Stability design of semi rigid frames. New York: John Wiley and Sons; 1996.