

Valuing Infrastructure Investments as Portfolios of Interdependent Real Options

Sebastian Maier

Department of Civil and Environmental Engineering

Imperial College London

A thesis submitted as fulfilment of the requirements for the degree of
Doctor of Philosophy of Imperial College London

November 2017

Declaration

I hereby declare that I have personally carried out the work described in this thesis and all material in this thesis which is not my own work has been appropriately referenced.

Sebastian Maier

The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives licence. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the licence terms of this work.

Abstract

The value of infrastructure investments is frequently influenced by enormous uncertainty surrounding both exogenous and endogenous factors. At the same time, however, their value is generally driven by much flexibility – i.e. options – with respect to design, financing, construction and operation. Real options analysis aims to pro-actively manage risks by valuing the flexibilities inherent in uncertain investments. Although real options generally occur within portfolios whose value is affected by both exogenous and endogenous uncertainty, most existing valuation approaches focus on single (i.e. individual) options and consider only exogenous uncertainty.

In this thesis, we introduce an approach for modelling and approximating the value of portfolios of interdependent real options under exogenous uncertainty, using both influence diagrams and simulation-and-regression. The key features of this approach are that it translates the interdependencies between real options into linear constraints and then integrates these in a portfolio optimisation problem, formulated as a multi-stage stochastic integer programme. To approximate the value of this optimisation problem we present a transparent valuation algorithm based on simulation and parametric regression that explicitly takes into account the state variable’s multidimensional resource component.

We operationalise this approach using three numerical examples of increasing complexity: an American put option in a simple single-factor setting; a natural resource investment with a switching option in a one-factor setting; and the same investment in a three-factor setting. Subsequently, we demonstrate the ability of the proposed approach to evaluate a complex natural resource investment that features both a large portfolio of interdependent real options and four underlying uncertainties. We show how our approach can be used to investigate the way in which the value of that portfolio and its individual real options are affected by the underlying operating margin and the degrees of different uncertainties.

Lastly, we extend this approach to include endogenous, decision- and state-dependent uncertainties. We present an efficient valuation algorithm that is more transparent than those used in existing approaches; by exploiting the problem structure it explicitly accounts for the path dependencies of the state variables. The applicability of the extended approach to complex investment projects is illustrated by valuing an urban infrastructure investment. We show the way in which the optimal value of the portfolio and its single, well-defined options are affected by the initial operating revenues, and by the degrees of exogenous and endogenous uncertainty.

Acknowledgements

It would not have been possible to conduct the research for this PhD thesis – and be in a position to conduct research in the first place – without the support of a large number of people including my family, friends and colleagues. While I can only acknowledge some of them here, I would like to thank all of them for their support and contribution.

First and foremost, my sincere thanks go to my two supervisors at Imperial College London. I would therefore like to express my gratitude to Professor John Polak, my primary supervisor, for giving me the opportunity to undertake this research and to develop as a researcher as well as for his invaluable support, guidance and advice. I am also grateful to my second supervisor, Professor David Gann, for his guidance throughout my studies, and especially for providing me with opportunities to interact with colleagues both within and outside the College. I would also like to thank my PhD examiners, Professors Afzal Siddiqui and Nilay Shah, for their insightful comments and invaluable feedback.

I also wish to thank colleagues at Imperial and at other organisations for their encouragement, inspiration and support. This includes colleagues within Imperial’s Centres for Transport Studies and Process Systems Engineering as well as participants of the Young Scientists Summer Program at the International Institute for Applied Systems Analysis (IIASA), where I initiated part of this research with financial support from the Austrian National Member Organization. I am grateful to my main supervisor at IIASA, Professor Georg Pflug, for his guidance and inspiration. In addition, this work was supported by the Grantham Institute at Imperial College London and the European Institute of Innovation & Technology’s Climate-KIC. This work has also benefited from various presentations at national and international conferences and seminars and I am thankful for the comments received at these events.

Finally, and most importantly, I would like to thank my family and my friends whose support has been crucial throughout this chapter of my life. In particular, I am indebted to my parents for nurturing (and tolerating) my curiosity and for providing me with opportunities without which this would not have been possible; and to Stefania for being by my side and for supporting and encouraging me when I needed it most.

Contents

1	Introduction	12
1.1	Background and Motivation	12
1.2	Research Aim and Objectives	16
1.3	Structure of the Thesis	16
2	Literature Review	18
2.1	Real Options Analysis in Infrastructure Investments	18
2.2	Simulation-and-Regression Methods	21
2.3	Portfolios of Real Options Approaches	26
2.4	Uncertainty Classification and Modelling	28
2.5	Summary	30
3	Modelling and Valuing Portfolios of Interdependent Real Options	32
3.1	Conceptual Framework	32
3.2	Problem Formulation	33
3.2.1	Modelling Flexibilities with Influence Diagrams	33
3.2.2	Portfolio Optimisation Problem	35
3.2.3	Curses of Dimensionality	36
3.3	The Valuation Algorithm	37
3.3.1	Approximating the Continuation Function by Parametric Regression	37
3.3.2	The Simulation-and-Regression-based Valuation Algorithm	37
3.3.3	Computational Efficiency and Numerical Accuracy	40
3.4	Summary	41
4	Numerical Examples	43
4.1	Valuing an American Put Option	43
4.1.1	Problem Setting	43
4.1.2	Modelling	44
4.1.3	Valuation	46
4.1.4	Results and Discussion	49
4.2	Re-evaluating Natural Resource Investments	50
4.2.1	Problem Setting	50
4.2.2	Modelling	51

4.2.3	Valuation	54
4.2.4	Results and Discussion	55
4.3	Re-evaluating Natural Resource Investments under Different Model Dynamics	62
4.3.1	Problem Setting	62
4.3.2	Modelling	63
4.3.3	Valuation	64
4.3.4	Results and Discussion	65
4.4	Discussion	73
4.5	Summary	75
5	Evaluating Complex Natural Resource Investments	77
5.1	The Investment Problem	77
5.1.1	Portfolio of Interdependent Real Options	78
5.1.2	Characterisation of Uncertainties	79
5.2	Modelling	80
5.3	Valuation	83
5.4	Results and Discussion	85
5.5	Summary	90
6	Extending the Framework to Endogenous Uncertainties	92
6.1	Extended Conceptual Framework	92
6.2	The Investment Problem	94
6.2.1	Portfolio of Interdependent Real Options	94
6.2.2	Characterisation of Uncertainties	95
6.3	Problem Formulation	97
6.3.1	Modelling Flexibilities with Influence Diagrams	97
6.3.2	Portfolio Optimisation Problem	100
6.4	The Valuation Algorithm	101
6.4.1	The Simulation-and-Regression-based Valuation Algorithm	101
6.4.2	Solution Procedure	103
6.4.3	Computational Efficiency and Numerical Accuracy	105
6.5	Case Study	106
6.5.1	Expansion of District Heating Network	106
6.5.2	Generated State Space and Utilised Basis Functions	106
6.6	Results and Discussion	109
6.7	Summary	113
7	Conclusions	115
7.1	Summary of Contributions	116
7.2	Limitations and Future Work	118

Bibliography	121
Appendices	136
A Nomenclature	137
B Additional Analyses of Chapter 4	139
B.1 Effect of Copper Price and Uncertainty on Mine Value	139
B.2 Effect of Interest Rate Uncertainty and Correlation on Mine Value	139
C Additional Analyses of Chapter 5	142
C.1 Effect of Parametric Model Choices on Portfolio Value	142
C.2 Effect of Production Cost and Copper Price Uncertainty on Mine Value . .	144
D Additional Analyses of Chapter 6	145
D.1 Effect of Revenue and Technical Uncertainty on Investment Value	145
D.2 Effect of Pay-out Ratios and Salvage Value Uncertainty on Investment Value	146
D.3 Effect of Depreciation Rate on Portfolio Value	146
E List of Publications	149
E.1 Papers Currently under Review	149
E.2 Accepted Papers in Conference Proceedings	149
E.3 Research Reports	149

List of Tables

4.1	Sample realisations of stock prices $X_t(\omega)$ over four time periods.	46
4.2	Steps performed at resource state $R_3 = (3, 1)$ at time $t = 3$	47
4.3	Steps performed at resource state $R_2 = (2, 1)$ at time $t = 2$	48
4.4	Steps performed at resource state $R_1 = (1, 1)$ at time $t = 1$	49
4.5	Input data for hypothetical copper mine of Brennan and Schwartz (1985). .	54
4.6	Value (in US\$ millions) of mine for different copper prices according to alternative numerical methods.	56
4.7	Value (in US\$ millions) of copper mine for different copper price levels. . .	57
4.8	Value (in US\$ millions) of copper mine for different copper price levels according to Brennan and Schwartz (1985).	60
4.9	Parameters of convenience yield process for different specifications as of Tsekrekos et al. (2012).	64
4.10	Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), under the two-factor model for the three specifications of Table 4.9 according to different numerical methods.	66
4.11	Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), under the three-factor model for the three specifications of Table 4.9 according to different numer- ical methods.	66
4.12	Results of equilibrium analysis for both the two- and three-factor model with the three specifications of Table 4.9.	71
5.1	Summary of stochastic factors considered in this example.	79
5.2	Input data for complex mine development project adapted from Brennan and Schwartz (1985); Tsekrekos et al. (2012) and own estimates.	84
5.3	Value of complex mine development project (in US\$ millions) for different levels of initial copper prices.	86
5.4	Modelling complexity of different (real) option problems.	91
6.1	Summary of stochastic factors considered in this investment problem. . . .	96
6.2	Input data for district heating network expansion adapted from Grainger and Etherington (2014); HM Treasury (2011) and own estimates.	107
6.3	Value of investment project (in £millions) for different levels of initial an- nual revenues.	110

C.1	Parameters of convenience yield process for different specifications of Tsekrekos et al. (2012).	142
C.2	Value of investment project with portfolio of options (in US\$ millions) for the specifications of Table C.1 and different parametric models.	143

List of Figures

4.1	Influence diagram for American option.	44
4.2	State transition diagram with nodes and arcs representing states $S_t = (t, N_t, X_t)$ and transitions $h \in \mathcal{H}$, respectively.	47
4.3	Influence diagram for the mine development project.	51
4.4	Selection of 5 equally likely paths for the evolution of the copper price, X_t	55
4.5	Influence diagrams for different copper mine settings.	59
4.6	Selection of 5 equally likely paths for the evolution of the three stochastic factors with specification # 11 and $\rho_{x,\delta} = 0.60$	65
4.7	Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), and volatility in two-factor model ($\sigma_{M_2}^2$) as a function of correlation between convenience yield and copper price process ($\rho_{x,\delta}$).	69
4.8	Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), and volatility in three-factor model ($\sigma_{M_3}^2$) as a function of correlation between convenience yield and copper price process ($\rho_{x,\delta}$).	70
4.9	Volatility in three-factor model ($\sigma_{M_3}^2$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.15$	72
5.1	Influence diagram for the complex mine development project.	81
5.2	Selection of 5 equally likely paths for the evolution of X_t , δ_t , r_t and A_t	87
5.3	Value of investment project, $\bar{G}_0(S_0)$ (in US\$ millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of production cost (σ_a) and copper price (σ_x) uncertainty.	89
6.1	Flexibilities provided by portfolio of interdependent real options.	98
6.2	Main steps of interleaved solution approach given we are in state S_t at time t	104
6.3	Selection of 5 equally likely paths for the evolution of K_t , V_t , μ_t and X_t	108
6.4	Results of reachability analysis.	109

6.5	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty.	111
6.6	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x	113
B.1	Value of (opened) copper mine with portfolio of options and without options as well as added value of portfolio as a function of initial copper price (X_0) and copper price uncertainty (σ_x) – all values are in US\$ millions.	140
B.2	Volatility in three-factor model ($\sigma_{M_3}^2$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.12$	141
C.1	Value of investment project, $\bar{G}_0(S_0)$ (in US\$ millions), with portfolio of real options and without options as well as portfolio's least valuable individual options (filled circles), as a function of degrees of production cost (σ_a) and copper price (σ_x) uncertainty.	144
D.1	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's least valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty.	145
D.2	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's least valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x	146
D.3	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty, for $\xi = 0.80\%$ and $\gamma = \delta = 0.50$	147
D.4	Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x , for $\xi = 0.80\%$	148

Chapter 1

Introduction

1.1 Background and Motivation

Although investments in modern and efficient infrastructures are widely recognized to play a key role in managing the world's transitions towards a resilient, low-carbon future, insufficient spending as well as inefficient prioritization and delivery in the last few decades have led to a substantial global infrastructure investment gap (World Economic Forum, 2012, 2013). Addressing this gap, which is estimated at US\$ 1-1.5 trillion per year (World Bank Group, 2015), and meeting the estimated future demand will require annual investments of US\$ 5-6 trillion in infrastructure assets between 2015 and 2030, totalling around US\$ 90 trillion over this 15-year period (Bhattacharya et al., 2015). While the specific development strategies may vary between countries – emerging and developing countries often have to build new infrastructure, whereas countries with already existing yet often ageing infrastructure need to either rebuild or maintain their assets (World Economic Forum, 2014) –, the public financing of infrastructure is becoming increasingly challenging for most countries due to ever more constrained budgets, unsatisfactory experiences with public expenditures and inefficient management of infrastructure (Della Croce and Gatti, 2014).

Whatever sources of finance governments access, further infrastructure investments will have to be made in the context of enormous uncertainties¹. Indeed, as noted by Arrow and Lind (1970); Flyvbjerg et al. (2003, 2009), there are various sources of uncertainty that may affect – both adversely and beneficially – the performance of real investment projects. Uncertainty sources include (geo)political, regulatory, technical, and economic conditions; direct and indirect environmental impacts including climate change; future technological innovation and behavioural adaptation/change; future demographic trends; and the true effects and impacts of the new investment projects themselves. A recent example that highlights some of these challenges can be found in Edinburgh, the capital of Scotland. Having started operation in May 2014 more than six years after construction began in

¹Even though some researchers treat the concepts of “risk” and “uncertainty” as if they were equivalent, even interchangeable, we acknowledge that there may be reasons for having significant distinction between risk and uncertainty, as discussed, for example, in the seminal work of Knight (1964).

2007, the Edinburgh tram route, which cost £776 millions and covers around 14km from Edinburgh’s city centre to Edinburgh Airport, BBC (2014) noted that

“In the decade since the first money was allocated to the project, the price has doubled, the network has halved and it has taken twice as long to build as originally planned.”

Making sound investment decisions in the context of uncertainty and irreversibility is a challenging problem for decision makers in both the public and private sectors. It is essential for decision makers to understand both the value of and the risks inherent in investment projects (Borgonovo et al., 2010; Borgonovo and Gatti, 2013), whether it is a government investing in public infrastructure while facing budget constraints or a corporation launching a new service under uncertain demand. The value of an investment project and its exposure to risks are generally affected directly by both a sequence of decisions, which may be strategic, operational or tactical (Vidal and Goetschalckx, 1997; Chevalier-Roignant et al., 2011; Azevedo and Paxson, 2014), and their interactions (e.g. between financing and investment decisions (Myers, 1974)). In addition, decisions makers often have to make such sequential decisions simultaneously with respect to several interacting projects, thus have to manage portfolios of investment projects, rather than individual ones. Classifying the risks of an investment into non-diversifiable (or systematic, market) and diversifiable (or unsystematic, specific) risks, the former can be managed through applying a risk-adjusted discount rate and the latter through using a portfolio approach, thereby making use of diversification (Ben-Horim and Levy, 1980). In the light of these risks, Hirshleifer (1961) stressed the importance of applying appropriate techniques to evaluate risky investment propositions.

While there exist a range of capital budgeting techniques that can be used by decision makers to appraise and analyse investment projects, the valuation methods based on simple temporal discounting – also referred to as discounting cash flow (DCF) methods – such as net present value (NPV) have become standard and widely accepted among practitioners (Ryan and Ryan, 2002; Hermes et al., 2007; Bennouna et al., 2010). For example, “Cost-Benefit Analysis” (CBA), which is one of the most widely applied appraisal frameworks in the public-sector, determines the NPV as part of its decision rule if applied correctly (Pearce et al., 2006). According to Garvin and Cheah (2004), the NPV metric is the preferred method to value infrastructure investment projects. Given their popularity and prevalence, DCF methods have been extended considerably in order to allow for more accurate investment analyses. Popular extensions include the Weighted Average Cost of Capital, or WACC (Miles and Ezzell, 1980); the Capital Asset Pricing Model, or CAPM (Ben-Horim and Levy, 1980); Monte Carlo simulation (Borgonovo and Gatti, 2013); and sensitivity analysis (Borgonovo and Plischke, 2016). Despite all these extensions, however, the inherent limitation of DCF methods remains: they do not correctly take into account

the value of managerial flexibility by assuming investment decisions are “now-or-never” propositions, thereby undervalue uncertain investment projects.

Overcoming the limitation of DCF methods by valuing the managerial flexibility inherent in investment projects, real options analysis (ROA), or real options valuation, has received widespread attention over the last four decades. As a consequence, the umbrella of ROA comprises nowadays a range of different valuation approaches and techniques that can be applied to a wide range of risky investment projects with inherent managerial flexibility (De Reyck et al., 2008). Popular textbook introductions to ROA include Trigeorgis (1996); Dixit and Pindyck (1994). For an overview of applications see Trigeorgis (2005). However, recent surveys indicate that ROA is mostly used by companies and industries where more sophisticated analyses are generally being conducted anyway (Block, 2007) and when financial uncertainty is high (Verbeeten, 2006). Lander and Pinches (1998) offered some challenges for the practical application of ROA using option pricing techniques: corporate decision makers and practitioners often do not have the knowledge to apply option pricing theory; modelling assumptions of ROA approaches are in many practical applications violated; and assumptions made to allow for mathematical tractability constrain the applicability of ROA.

Widening and enhancing its applicability, ROA has extended its focus beyond the consideration of single real options alone to portfolios of real options. However, while the consideration of portfolios of interdependent real options has been considered to have great potential to widen the applicability of ROA to many practical situations (Trigeorgis, 1995), a decade later Trigeorgis (2005) stated that the development of a “more credible general portfolio theory for (possibly interdependent) options” remains a research challenge to be addressed. Zapata and Reklaitis (2010) noted there are several limitations inherent to ROA when used in a portfolio context which include path-dependency of options, curse of dimensionality and combinatorial burden. Despite some recent advances, however, the modelling and valuation of portfolios of interdependent real options remains a challenging problem. This is because ROA generally aims at valuing single, well-defined options yet fails to find the portfolio of options (Wallace, 2010) and traditional option valuation techniques (e.g. binomial/lattice and finite difference) become impractical (Longstaff and Schwartz, 2001; Gamba, 2003).

A fundamental issue in ROA and decision-making under uncertainty is how to account correctly and adequately for the multiple sources of uncertainty occurring in most practical real-life situations. In these situations it is generally assumed that the effective sources of uncertainty are purely exogenous and, as such, are independent of both the actions taken by the decision maker and the state of the underlying system affected by these decisions. For example, in the case of investment in a new wind farm, while the wind farm’s performance depends on factors such as location, time of day and the wind turbines’ height, parameters such as the wind speed to which the turbines will be exposed to, and

consequently the amount of power generated, are independent of the investor’s decision of whether to build the wind farm or not. Likewise, if the amount of power generated by such wind farm is sufficiently small and/or the relevant wholesale electricity market to which the power is sold is comparatively large, then the underlying wholesale price of electricity, and consequently the investor’s revenues are also independent of the investor’s decision.

There are, however, many practical situations in which the relevant sources of uncertainty are endogenous, i.e. dependent on the decision maker’s actions or the underlying system’s state, or both. In the case of the wind farm example, if the above-mentioned conditions are violated, i.e. if the new wind farm is sufficiently large and/or the electricity market relatively small, then the introduction of a new wind farm will affect the wholesale price of electricity and hence the investor’s future revenues. Similarly, although the “off-the-shelf” cost of new wind turbines may be known and a feasibility study may provide a construction cost estimate, the actual cost of building a new wind farm will not be known until the investor actually builds it. During the building process, the investor reveals and learns its true capital cost. If the investor wants to sell the wind farm at the end of its lifetime, in the absence of a second hand market, the resale value will depend on its “state”, which may include such factors as its lifetime, asset value, wear and tear, and decommissioning cost.

Despite the ubiquity of exogenous and endogenous uncertainties in many real-life situations, there remains a need for a unified approach that accounts for both when real options analysis is used to evaluate practical investment problems. Including both types of uncertainty in a real options approach has rarely been studied in the related literature (Ahsan and Musteen, 2011). Although portfolio of real options approaches have been applied when there is only exogenous uncertainty, there is a need to include both types because that enables decision-makers to manage the two uncertainty types simultaneously (Otim and Grover, 2012). Some authors have therefore suggested that future work should examine the relationship and interactions between different sources of uncertainty and the portfolio’s individual options. For example, Tiwana et al. (2006) stated that future research should investigate how the comparative performance of individual real options is affected by multiple sources of uncertainty, and Li et al. (2007) called for studies to investigate how investment decisions are affected individually and interactively by multiple uncertainty sources. More recently, the critical review of Trigeorgis and Reuer (2017) has suggested four extensions, three of which are addressed in this thesis: portfolios of interdependent real options, multiple sources of uncertainty, and endogenous resolution of uncertainty through learning.

1.2 Research Aim and Objectives

Investment decisions in infrastructure systems such as in transport, water, waste, energy and ICT are frequently made in the context of enormous uncertainty surrounding both the investments intrinsic risks and the highly volatile supply and demand patterns. Traditional investment appraisal techniques such as those based on simple temporal discounting are widely regarded as inadequate since they do not correctly take into account the value of flexibilities (i.e. real options), therefore undervalue infrastructure investments under uncertainty. Given that these investments are not only being made in the context of significant uncertainty but also contain many flexibilities, a portfolio of real options approach is needed to correctly value such infrastructure investments whilst pro-actively managing the many risks involved.

The overall aim of this thesis is to develop a real options-based framework for the valuation of infrastructure investments as portfolios of interdependent real options under both exogenous and endogenous sources of uncertainty, and to illustrate its application. The specific core objectives, which guide the development process of this research, consist of:

1. To develop an approach for the modelling and valuation of a portfolio of interdependent real options under exogenous uncertainty that is capable of accounting for multiple, possibly interdependent real options and various, possibly interlinked, sources of uncertainty.
2. To operationalise the approach using practical, relevant examples of increasing complexity in terms of both the portfolio of real options and the uncertainties considered, and to comprehensively evaluate the comparative performance of the conventional and new approach.
3. To demonstrate the ability of the approach to evaluate a complex natural resource investment project that features both a large portfolio of interdependent real options and multiple underlying uncertainties.
4. To extend the portfolio-based real options approach to include endogenous, decision and state dependent uncertainties, and to illustrate the applicability of the extended approach by valuing a district heating network expansion investment.

1.3 Structure of the Thesis

This thesis is divided into seven chapters. Chapter 2 contains a review of the relevant literature, that is the one related to both the application area of this research in general and relevant ROA approaches in particular, covering simulation-and-regression and portfolio

of real options approaches, as well as uncertainty classification and modelling. Chapter 3 presents the conceptual framework, describes the approach that a decision maker can apply to model inherent flexibilities and formulates the corresponding portfolio optimisation problem, and then presents a valuation algorithm that approximates the optimal value of portfolios of interdependent real options. Chapter 4 delivers the operationalisation of the new approach by considering the numerical examples of valuing a simple American put option and evaluating a natural resource investment (i.e. a copper mine), firstly, under copper price uncertainty alone, and subsequently under three sources of uncertainty. Chapter 5 demonstrates how the approach can be applied to the example of evaluating a complex natural resource investment, which represents a substantial extension of the third example of Chapter 4 in terms of both options portfolio and uncertainties considered. Chapter 6 contains the further development of the framework presented in Chapter 3 to include endogenous sources of uncertainty and its application to the real-case of a district heating network expansion investment in the London borough of Islington. Finally, Chapter 7 concludes this thesis by providing a summary of the contributions of this research, a discussion of its limitations and suggestions for future research.

Chapter 2

Literature Review

The purpose of this chapter is to review the relevant literature and provide the necessary background for this thesis. The rest of this chapter is organised as follows: Section 2.1 briefly reviews the state of the practice with regard to ROA applied in the context of infrastructure investments. Section 2.2 reviews simulation-and-regression methods proposed to solve real option problems. Section 2.3 reviews the relevant literature on portfolios of real options. Section 2.4 reviews the classification of uncertainties into exogenous and endogenous with an emphasis on the operational research as well as on the finance and management literature. Lastly, Section 2.5 concludes this chapter by summarising the gaps in the literature relevant to this research.

2.1 Real Options Analysis in Infrastructure Investments

It is widely acknowledged that Real Options Analysis (ROA) has substantial potential as a framework for the adequate appraisal of infrastructure investments given that these investments are not only being made in the context of significant uncertainty, but also – naturally or intentionally – “ripe with flexibility” (Cheah and Garvin, 2009). Common uncertainties are exogenous volatility in supply and demand conditions as well as intrinsic technical and other endogenous risks. Flexibilities are commonly represented through real options and generally exist in the form of managerial flexibility, but also in other forms like design flexibility. The former includes options that, for example, allow decision makers to alter an infrastructure’s scale of operation – see Trigeorgis (1996) for a list of traditional types of real options –, whereas the latter relates to integrating flexibility into the design of an infrastructure, for example, through product design modularity (de Neufville, 2003; Gil, 2007; de Neufville and Scholtes, 2011).

Even some governments recently realised the potential benefits of ROA. For example, appraisal guidelines of the United Kingdom’s Treasury Department state that “it is important to incorporate the value of flexibility” (HM Treasury, 2015) and that ROA may be appropriate “if an activity has uncertainty, flexibility and learning potential” (HM Treasury, 2009). The Office of Gas and Electricity Markets (Ofgem), which is the UK’s independent national regulatory authority, is considering the incorporation of ROA within

their investment and policy appraisal framework (Grayburn, 2012) stating ROA “should help decision making where the investment environment is characterised by uncertainty and management flexibility in responding to investment needs”. Furthermore, even though solely presenting the Black-Scholes-Merton approach to options pricing, the European Investment Bank (2013) recognised in “The Economic Appraisal of Investment Projects at the EIB” that managerial flexibility becomes valuable when facing both high uncertainty and irreversible investments. The authors justified their choice of valuation method by succinctly noting that “it is the simplest to apply”.

Most of the recent infrastructure-related applications of ROA fall into one of two categories: physical and digital infrastructure investment projects. With regard to the former, Zhao and Tseng (2003) appraised flexible design alternatives for the construction of public parking garages. Arguing with the inappropriateness of complex option valuation techniques, de Neufville et al. (2006) proposed a simple spreadsheet approach for the valuation of the flexibility incorporated in the design of a parking garage. Another early study (Gil, 2007) on infrastructure design investigated the effects of modularization – that is product design modularity – in airport expansions programmes. Garvin and Cheah (2004) applied options pricing in a case study of a toll road project to comparatively evaluate the project’s economic viability under the NPV and options approach (deferment option). A few years earlier, Rose (1998) valued complex interacting real options that represent contractual agreements using Monte Carlo simulation. Investments into urban transportation infrastructure have been considered by Saphores and Boarnet (2004), whose modelling approach took into account the impact of the variation of a city’s population on land rents and prices as well as on transportation costs. More recently, Munoz et al. (2014) applied a two-stage stochastic programming approach to transmission planning under uncertainty, where the added value of the first-stage transmission investment quantifies the real option value.

ROA has also been applied in the context of digital infrastructures such as information technology (IT) infrastructures. Benaroch (2002) mentioned that real options generally must be intentionally planned in an IT investment project, rather than being “inherently” embedded like in physical infrastructure. One of the first works in this area was presented by Kambil et al. (1991), who recognised the growth options often embedded in such investments. Panayi and Trigeorgis (1998) applied a multi-stage (compound) real options approach to the case of an IT infrastructure investment faced by CYTA, the state telecommunications authority of Cyprus. Benaroch and Kauffman (1999) argued that investments in IT infrastructures generally do not result in immediate expected paybacks, but rather can provide the basis for profitable future investment opportunities. Miller et al. (2004) applied ROA to evaluate the “Korean Information superhighway infrastructure” investment project. Subsequently, Benaroch (2002) claimed that there exists a number of gaps between real options theory and what is required to adequately model and appraise

real-world IT investments. One of these gaps has been tackled by Kumar (2004), who developed a novel evaluation framework based on the “asset valuation” literature.

In addition to the above applications, a number of papers have dealt with issues related to the provision and ownership of infrastructure systems. Considering different private sector participation arrangements available under the umbrella of Public Private Partnership (PPP), Cheah and Garvin (2009) discussed the potential application of ROA in infrastructure projects, noting typical options are call, put, switching, timing, compound, and learning options. Ho and Liu (2002) proposed a quantitative model based on real options theory to evaluate the economic viability of privatised (build-operate-transfer, BOT) infrastructure projects from the perspective of both the government and the project promoter. Cheah and Liu (2006) investigated the case of the Malaysia-Singapore Second Crossing and developed a methodology to value governmental support in BOT infrastructure projects by modelling the government guarantee as a put option and the potential repayment – i.e. a cap on the return – from the private sector participant to the government as a call option. In contrast, Chiara et al. (2007) argued that a revenue guarantee in a BOT infrastructure project should be modelled as an American-style real option rather than the European-style option used by Cheah and Liu (2006). Alonso-Conde et al. (2007) applied ROA to analyse the contractual terms associated with the case of the PPP of the Melbourne CityLink Project, whereas Krüger (2012) analysed the implications of PPP agreements on the execution of expansion options in road infrastructure.

A few researchers even extended ROA approaches in the context of infrastructure investments by considering either game theoretic interactions or multiple objectives. As noted by Smit and Trigeorgis (2009), option games support strategic capital investments through actively accounting for multiple decision makers, thereby improving strategic planning in a competitive environment. Considering the case of the Amsterdam Airport Schiphol, Smit (2003) applied a real options game approach to value airport expansion, i.e. its growth option. Suttinon et al. (2012) illustrated their methodology through a game setting where the public sector (Government of Thailand) may invest in tap and industrial water supply, whereas the private sector firm may invest into recycled-water development. Despite the potential usefulness of game theory for risk management in infrastructures (Cox Jr, 2009), its combination with real options theory has not been widely used yet to strategically assess and analyse investments into both technology (Smit and Trigeorgis, 2007) and infrastructure systems (Smit and Trigeorgis, 2006a). On the other hand, considering multiple objectives when evaluating electricity infrastructure investments, Cesena and Davalos (2011) extended the real options approach and illustrated the proposed methodology by investigating the case of a distribution network investment in Mexico. Marques et al. (2015) developed a real options-based approach for water distribution network investments that takes into account both design and cost uncertainty.

However, despite its great potential and evidence (Martins et al., 2015) of the large

growth in the past 15 years in the number of publications advocating the use of ROA in infrastructure projects, the absolute number is still comparatively low when compared with most other areas of applications. Indeed, Garvin and Ford (2012) noted that it is not widely applied in practice and, as Gil and Beckman (2009) pointed out, applying ROA to infrastructure design “is still in its infancy”. There remains a need for a practical and powerful investment appraisal technique that is capable of valuing the portfolio of possibly interdependent real options generally available in complex infrastructure projects whilst, at the same time, takes into account multiple exogenous and endogenous sources of uncertainty. Such an approach would have to overcome several limitations inherent to ROA when used in a portfolio context, particularly path-dependency of options, curse of dimensionality, and combinatorial burden (Zapata and Reklaitis, 2010).

2.2 Simulation-and-Regression Methods

Simulation-based option pricing methods have received considerable attention over the past four decades. Introduced to the pricing of European call options by Boyle (1977), subsequent works¹ have presented simulation-based approaches to value American (Tilley, 1993; Barraquand and Martineau, 1995; Broadie and Glasserman, 1997) and Asian options (Broadie and Glasserman, 1996; Grant et al., 1997). Despite adding computational complexity, simulation techniques have significant advantages over more traditional option pricing techniques such as analytical and numerical methods – including lattice/tree and PDE/finite difference – since they are flexible and relatively easy to apply, which makes them suitable in (complex) situations where traditional methods cannot be used (Longstaff and Schwartz, 2001; Broadie and Glasserman, 2004). For example, simulation (i.e. Monte Carlo sampling) allows the consideration of multiple stochastic factors – also referred to as uncertain or random variables –, stochastic processes of the underlying assets with complex characteristics, as well as possibly interdependent real options with complex exercise features (Pringles et al., 2015). Reviews of existing simulation methods were presented by Boyle et al. (1997); Glasserman (2003); Broadie and Detemple (2004); Kind (2005).

Even though initial attempts demonstrated that simulation is a powerful tool to value higher-dimensional American-type options, which were long believed to be computationally intractable (Broadie and Glasserman, 2004), accurately and efficiently pricing options with multiple exercise features – American in continuous and Bermudan² in discrete time – remained a challenging problem. This is because simulation generally generates sample paths forward in time, whereas pricing an American option, i.e. solving the optimal stopping problem by determining the optimal exercise policy, requires a backward-style dynamic programming approach (Broadie and Detemple, 2004). Yet applying backward

¹Other important works include (Rust, 1997; Keane and Wolpin, 1994; Broadie and Glasserman, 2004).

²To value a Bermudan option, the optimal stopping problem in continuous time, which is solved when valuing an American option, is replaced by its discrete time equivalent.

dynamic programming requires the determination of the conditional expected payoff to the optionholder from continuation – that is the value of the option if not exercised immediately –, which is practically impossible to calculate in multi-dimensional, multi-period settings (Powell, 2011). In mathematical terms, assuming the option can still be exercised at time t , its current value given stock price $X_t(\omega)$ along path $\omega \in \Omega$, $V_t(X_t(\omega))$, is determined by:

$$V_t(X_t(\omega)) = \begin{cases} \Pi_t(X_t(\omega)), & \text{if } \Pi_t(X_t(\omega)) \geq \Phi_t(X_t(\omega)), \\ \Phi_t(X_t(\omega)), & \text{if } \Pi_t(X_t(\omega)) < \Phi_t(X_t(\omega)), \end{cases} \quad (2.1)$$

where $\Pi_t(X_t(\omega))$ is the immediate payoff from exercising the option at t , and

$$\Phi_t(X_t) = \mathbb{E} \left[e^{-k\Delta} V_{t+\Delta}(X_{t+\Delta}) | X_t \right], \quad (2.2)$$

is the (usually hard or impossible to compute) continuation function, with k and Δ being the risk-free rate and time step, respectively.

To overcome this limitation, a variety of regression-based methods have been developed that approximate the continuation function with a statistical model of the future. In fact, already Bellman and Dreyfus (1959) advocated for the use of functional approximations in dynamic programming to decrease computation time requirements, with the early works of Bellman et al. (1963) and Daniel (1976) using polynomial (i.e. a parametric model) and spline approximations (i.e. a non-parametric model), respectively. For an overview see Rust (1996); Judd (1996). The three most-widely cited works in this area are (Carriere, 1996; Tsitsiklis and Van Roy, 1999, 2001; Longstaff and Schwartz, 2001), which can be classified as approximate dynamic programming (ADP) strategies (Powell, 2011). Of these works, Carriere (1996) applies a non-parametric approach by using regression with both splines and a local polynomial smoother, whereas Tsitsiklis and Van Roy (1999, 2001); Longstaff and Schwartz (2001) apply parametric models and a least-squares method to estimate the continuation function. Stentoft (2014) compared the regression-based algorithms of Carriere (1996); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001) and recommends the least squares Monte Carlo (LSM) method of Longstaff and Schwartz (2001) noting that it has a “a smaller absolute bias and less error accumulation”, thereby justifying its popularity among both academics and practitioners.

The main idea behind the LSM approach is to estimate the options’ expected payoff from continuation using a parametric regression model. This model is obtained by regressing the discounted payoff from optimally exercising the option on functions of the state variables – i.e. the stochastic factor(s) –, more specifically, on linear combinations of so-called basis functions. Using the optimal coefficients of these functions, which are obtained through a least squares method, the fitted value of the parametric model can then be used to determine the optimal exercise strategy of the option. In mathematical terms, this means

$\Phi_t(X_t)$ at time t is approximated by the following parametric model:

$$\hat{\Phi}_t^L(X_t) = \sum_{l=0}^L \hat{\alpha}_{tl} \phi_l(X_t), \quad (2.3)$$

where L is the model's (finite) dimension, the functions $\{\phi_l(X_t)\}_{l=0}^L$ are called basis functions (or features), and the optimal values of the coefficients (or weights) at time t , $(\alpha_{tl})_{l=0}^L$, are estimated using least-squares regression as follows:

$$(\hat{\alpha}_{tl})_{l=0}^L = \arg \min_{(\alpha_{tl})_{l=0}^L} \left\{ \sum_{\omega \in \Omega} \left[e^{-k\Delta} \bar{V}_{t+\Delta}(X_{t+\Delta}(\omega)) - \sum_{l=0}^L \alpha_{tl} \phi_l(X_t(\omega)) \right]^2 \right\} \quad (2.4)$$

Adapting the decision criteria in (2.1) by replacing $\Phi_t(X_t)$ with $\hat{\Phi}_t^L(X_t)$, the approximated option value at time t given stock price $X_t(\omega)$ along path $\omega \in \Omega$, $\bar{V}_t(X_t(\omega))$, is determined by:

$$\bar{V}_t(X_t(\omega)) = \begin{cases} \Pi_t(X_t(\omega)), & \text{if } \Pi_t(X_t(\omega)) \geq \hat{\Phi}_t^L(X_t(\omega)), \\ \Phi_t(X_t(\omega)), & \text{if } \Pi_t(X_t(\omega)) < \hat{\Phi}_t^L(X_t(\omega)). \end{cases} \quad (2.5)$$

Proceeding backwards to $t = 0$, the approximated option value is determined by taking averages of the path-wise continuation values over all $|\Omega|$ paths, giving $\bar{V}_0(X_0)$. It is important to note that the algorithm proposed by Longstaff and Schwartz (2001) uses only “in-the-money paths”, i.e. paths $\omega \in \Omega$ where $\Pi_t(X_t(\omega)) > 0$, both when estimating the coefficients by the least squares regression (2.4) and when determining the optimal exercise decision at each exercise time and path in (2.5). As noted by the authors, this is sufficient because the exercise decision is only required for in-the-money-paths.

Given its popularity, various researchers have investigated the quality of the approximation of the LSM method in a range of different settings. Considering a one-dimensional setting (i.e. a single stochastic factor), Longstaff and Schwartz (2001) presented convergence results limited to two exercise dates, whereas Clément et al. (2002) showed that, for an arbitrary number of option exercise times (implied by Δ)³ and for fixed $\{\phi_l(\cdot)\}_{l=0}^L$ and L , the approximation obtained by the LSM algorithm almost surely converges to the true option value as $|\Omega|$ goes to infinity. Although still focusing on one-dimensional problems, Glasserman and Yu (2004a) investigated the convergence of the LSM method as both L and $|\Omega|$ increase, and for a Brownian motion found that $|\Omega|$ needs to grow exponentially in L to ensure worst case convergence. Under general assumptions and applying the family of shifted Legendre polynomials, Stentoft (2004b) proved that the LSM algorithm also converges in a multi-dimensional setting (i.e. with multiple stochastic factors) if both $L \rightarrow \infty$ and $|\Omega| \rightarrow \infty$ provided that $L^3/|\Omega| \rightarrow 0$. For a recent discussion of related works

³Obviously, the value of the Bermudan option converges to the one of the American option as the time step, Δ , goes to 0. However, although true in theory, Tsitsiklis and Van Roy (2001) found that, due to accumulation errors, the overall approximation error of their algorithm increased when Δ was reduced.

see Zanger (2013).

Other works have numerically analysed computational issues related to the LSM method's valuation performance. To the best of our knowledge, the first authors to do so were Moreno and Navas (2003), who analysed the method's robustness to the choice of parametric model – i.e. to the polynomial family of $\{\phi_l(\cdot)\}_{l=0}^L$ and L – and, for an American put option using different polynomials with fixed L ($3 \leq L \leq 20$), found that the method is very robust, but noted that using more terms (i.e. $L > 20$) may lead to numerical problems related to the least-squares regression in (2.4). For more complex options – call option on the maximum of five uncorrelated assets and call option on the arithmetic average of the stock price – the authors concluded that the choice of parametric model is not clear and robustness of the LSM method may not be guaranteed. Also considering a complex, multi-dimensional setting, Stentoft (2004a) examined the LSM method's comparative performance and found that it is not only much easier to extend than Binomial techniques, but also superior when it comes to the trade-off between precision and computational cost. More recently, Areal et al. (2008) studied a range of different approaches to improve the LSM method: two different regression techniques⁴ – LFIT (general linear least-squares fit) and SVD (singular value decomposition); 11 different polynomial families⁵; and three variance reduction techniques⁶ – antithetic variates, control variates and moment matching, as well as low discrepancy sequences, which are also known as quasi-Monte Carlo or quasi-random numbers.

Besides the error caused by the discretisation of the state space through Monte Carlo sampling, there are two valuation biases directly related to the LSM method. According to Fabozzi et al. (2017), the downward bias of this method is caused by the finite-, and usually low-dimensional polynomial approximation of the continuation function, whereas the upward bias stems from the circumstance that the same paths are used in (2.5) both to make the exercise decision and to update the option value (in-sample overfitting). To reduce the low (downward) and high (upward) bias, Létourneau and Stentoft (2014) suggest to increase $L/|\Omega|$ and $|\Omega|$, respectively. Doing the latter has the additional advantage of reducing the standard error of the simulated paths, thereby reducing the overall valuation error (Areal et al., 2008). Interestingly, Moreno and Navas (2003) performed an out-of-sample analysis of the LSM method but did not find any substantial high bias when comparing in- and out-of-sample option values noting “prices are very similar and standard errors are low in both cases”. Referring to the earlier work of Glasserman and Yu (2004a), Areal et al. (2008) found that with a certain value for L , further improvement in

⁴In comparison, Longstaff and Schwartz (2001) used the double precision DLSBRR algorithm and mentioned the QR-algorithm and Cholesky-decomposition least-squares techniques.

⁵These are: weighted Laguerre, Powers, Legendre, Laguerre, Hermite-A, Hermite-B, Chebyshev 1st kind A, Chebyshev 1st kind B, Chebyshev 1st kind C, Chebyshev 2nd kind A and Chebyshev 2nd kind B.

⁶For a more thorough discussion of and comparison between these see Boyle et al. (1997), who additionally considered stratified and latin hypercube sampling.

accuracy can only be achieved by increasing $|\Omega|$. Nevertheless, Létourneau and Stentoft (2014) conclude that reducing the two biases to a reasonable level will not always be possible as a large L may result in both numerical errors (and hence a reduced option value approximation) due to the least-squares regression algorithm and in an increased high bias due to increased in-sample overfitting. As a consequence, several authors including Wang and Caffisch (2010); Létourneau and Stentoft (2014); Fabozzi et al. (2017) have presented extensions of the original LSM algorithm to partly address these biases.

Although positively influenced by a number of advantageous parameter choices, the LSM method still relies on an approximation of the conditional expectation. This leads almost inevitably to non-optimal exercise decisions and consequently to a non-optimal option value. In other words, the approximated option value, \bar{V}_0 , is a lower bound on the true option value, V_0 , as it follows a sub-optimal exercise policy. To provide performance bounds, several researchers presented valuation approaches that provide an upper bound on the option price, which, together with the lower bound, can then be used to characterise the quality of solutions. Applying duality-theory, Haugh and Kogan (2004); Rogers (2002) developed upper-bound algorithms that generate high-biased approximations whilst formulating the American option pricing problem as a minimisation problem. Unlike the optimal stopping problem, which represents the American option pricing problem as a problem of maximising over stopping times, Haugh and Kogan (2004); Rogers (2002) formulated the dual problem as one of minimising over a class of martingales or supermartingales (Glasserman, 2003).

Subsequently, Andersen and Broadie (2004); Meinshausen and Hambly (2004) proposed computationally efficient methods to generate bounds on the true option value. More recently, Broadie and Cao (2008) noted that many existing duality-based, upper bound algorithms require time-consuming nested simulations to approximate the option's continuation value at every possible exercise date. Nadarajah et al. (2017) claimed that this is due to their origin in continuation function approximation as applied by the LSM method and known as “regression now”. By contrast, the non standard LSM method proposed by Glasserman and Yu (2004b), also known as “regression later”, which applies value function approximation instead, avoids these computationally expensive calculations when estimating duality-based upper bounds as shown by Broadie and Cao (2008). Nadarajah et al. (2017) also compared the regression now and later variants, as well as provided numerical evidence that confirms the computational advantages of the latter when it comes to estimating dual (upper) bounds. See Kohler (2010) for a recent review of regression-based Monte Carlo methods for American option pricing.

2.3 Portfolios of Real Options Approaches

Early studies on portfolios of real options⁷ explored the concept rather qualitatively and in the context of corporate strategy and planning (Myers, 1984; Trigeorgis and Kananen, 1991; Bowman and Hurry, 1993; Luehrman, 1998). For example, Smit and Trigeorgis (2006b); Anand et al. (2007) presented strategic planning frameworks that aim at supporting corporations with the management of portfolios of real options. Using a generic example, Trigeorgis (1993a) investigated the nature of interactions between a firm's real options and found that options' individual values are generally non-additive; in other words, their combined value within a portfolio differs from the sum of the real options valued independently. More recently, Trigeorgis (2005) argued that investment decision problems represented by a portfolio of interdependent real options can be decomposed into a few basic-building blocks (i.e. individual real options) and then combined by one of four commonly encountered basic decision operators. These are "or" (max), "and", "average" and "multi-stage" (or compound), and represent choosing the maximum of mutually exclusive alternatives, the sum of independent options, the probabilistically weighted sum of options and the value of a sequential option, respectively.

Various scholars have presented approaches to value portfolios of real options in the context of specific practical applications. One of the first publications to do so was Rose (1998), who valued two interacting options embedded in a toll-road project and found that ignoring the embedded options' interactions results in significantly underestimated project values. Other relevant articles presented portfolio approaches in the context of R&D projects (Vassolo et al., 2004; McGrath and Nerkar, 2004; van Bekkum et al., 2009; Zapata and Reklaitis, 2010), maritime investments (Bendall and Stent, 2007), IT investments (Pendharkar, 2010), and transmission network expansion (Loureiro et al., 2012). While these approaches have applied ROA in a portfolio context, they have developed rather inflexible and restricted quantitative approaches since they are tailored to specific applications and limited to problems instances with specific features in terms of both options portfolio and uncertainties; by contrast, this work takes a fundamentally different approach by proposing a holistic and general valuation approach for portfolios of interdependent real options applicable to a wide range of complex practical investment problems.

A number of publications aimed at presenting more general quantitative frameworks for the problem of valuing portfolios of real options. In an important early theoretical contribution, Childs et al. (1998) provided closed-form (analytic) solutions for the value of two investment project that can be developed either in parallel or in sequence. Also Smith and Thompson (2008) provided analytic solutions whilst considering the problem of valuing a portfolio of sequencing options that represent projects. Considering one underlying source of uncertainty, Meier et al. (2001) proposed two models: the first combines contingent

⁷For a synthesis of the concept of portfolio decision analysis see Salo et al. (2011).

claims analysis (replicating portfolios) with the well-known Knapsack problem to select projects, whereas the second is essentially an “Asset Liability Model” solved by means of stochastic programming whilst combining Monte Carlo simulation with a binomial tree for scenario generation. Despite having presented more general portfolio-based real options approaches, neither of these publications addresses this work’s problem in such a holistic and general way we do. Furthermore, these approaches are simply impractical in most real-life situations.

Several recent articles that to some extent propose holistic approaches similar to ours include Gamba (2003); Wang and de Neufville (2004); Brosch (2008). Most recently, the work of Brosch (2008) addressed portfolios of real options by proposing a forward-backward looking algorithm based on stochastic mixed-integer programming and lattice/tree modelling, where the forward looking element captures the budget constraint by making sure only feasible paths can be chosen from. The framework presented by Wang and de Neufville (2004) consists of an options identification stage which contains a screening and simulation model, as well as of an options analysis stage which applies a stochastic mixed-integer programming model and a binomial technique for scenario generation. Interestingly, both authors discussed computational issues with respect to the optimality of solutions, but only Brosch (2008) also discussed simulation, yet dismissed it as an alternative technique in his model. While these works have presented important contributions, for example the global, dynamic budget constraints in (Brosch, 2008) as well as the identification and definition of real options in physical systems by Wang and de Neufville (2004), the combined complexity of applying both binomial techniques for scenario generation and stochastic mixed-integer programming for optimal real options timing as well as the resulting adverse computational issues make both of them impractical in most real-life situations.

In contrast, the framework presented by Gamba (2003) overcomes the computational limitations of the binomial techniques used in Wang and de Neufville (2004); Brosch (2008) by applying simulation and parametric regression. In fact, the author considered a portfolio of interdependent real options that can be decomposed into a set of simple real options, which can be independent, mutually exclusive, compound, or of the switching type, and presented decision rules for each of the four cases. These rules were then used within a valuation procedure that applies the LSM algorithm to analyse the four sub-problems individually. Although this author has proposed an interesting extension of the LSM algorithm, his paper neither addresses the modelling of portfolios of interdependent real options, nor presents a holistic and general approach. Our work differs in a number of ways when compared with (Gamba, 2003; Wang and de Neufville, 2004; Brosch, 2008). For example, we use influence diagrams (IDs) to model portfolios of interdependent real options and mathematically translate the interdependencies into linear constraints. Furthermore, we integrate both the linear constraints and the directly modelled dynamics of all underlying uncertainties into a portfolio optimisation problem which is formulated as

a multi-stage stochastic integer programme.

2.4 Uncertainty Classification and Modelling

The classification of uncertainties into exogenous and endogenous has received considerable attention in different branches of literature, and importantly in the operational research as well as in the finance and management literature. With regard to the former, to the best of our knowledge, the work of Jonsbråten et al. (1998) was the first to classify⁸ the formulation of stochastic programs into “standard” formulations with decision independent random variables and “manageable” formulations, in which the distribution of the random variables is dependent on decisions. Calling the former “exogenous uncertainty” and the latter “endogenous uncertainty” (Goel and Grossmann, 2004), Goel and Grossmann (2006) specified the way in which decisions can affect the stochastic process – which describes the evolution of an uncertain parameter (see Kirschenmann et al. (2014)) – by presenting two types of endogenous uncertainty. The first is when the decision alters the probability distribution (e.g. parameters of family), whereas the second relates to the decision affecting the timing of uncertainty resolution, a process often described as information revelation.

Considering the above specification of endogenous uncertainties, several relevant works have appeared in the operations research literature over the last few decades. As for the first type of endogenous uncertainty, Pflug (1990) was the first to take into account decision dependent probabilities in a stochastic optimization problem by considering a controlled Markov chain where the transition operator depends on the control, i.e. the decision. Other relevant articles related to this type are in the context of stochastic network problems (Held and Woodruff, 2005; Peeta et al., 2010), global climate policy (Webster et al., 2012) and natural gas markets (Devine et al., 2016). By contrast, the second type of endogenous uncertainty has received considerably more attention in the literature. The first work related to this type was (Goel and Grossmann, 2004), who presented a stochastic programming approach for the planning of an investment into a gas field with uncertain reserves represented through a decision-dependent scenario tree. Similar studies in terms of both uncertainty representation and application domain include (Tarhan et al., 2009; Terrazas-Moreno et al., 2012; Gupta and Grossmann, 2014). Other relevant works include the optimisation of R&D project portfolios (Solak et al., 2010) and clinical trial planning in the pharmaceutical R&D pipeline (Colvin and Maravelias, 2008, 2010, 2011).

Moreover, several works have incorporated both the second type of endogenous uncertainty and exogenous uncertainty in the formulation of stochastic programmes. For generic problem formulations and solution strategies see the rather theoretical works of Dupačová (2006); Goel and Grossmann (2006); Tarhan et al. (2013). Recent advances

⁸Another well-known uncertainty classification is to distinguish between aleatory and epistemic uncertainty (Kiureghian and Ditlevsen, 2009).

and summaries over existing computational strategies have been presented by Apap and Grossmann (2016); Grossmann et al. (2016). However, although almost all publications of this branch of literature refer to the classification and specification of Jonsbråten et al. (1998) and Goel and Grossmann (2006), respectively, Mercier and Van Hentenryck (2011) argued that problems in which merely the observation of an uncertainty depends on the decisions, but the actual underlying uncertainty is still exogenous (= second type of endogenous uncertainty) should be classified as “stochastic optimization problems with exogenous uncertainty and endogenous observations”. According to their redefined classification, problems with exogenous and the first type of endogenous uncertainty are referred to as *purely exogenous* and *purely endogenous*, respectively.

Unlike the operational research literature, the finance and management literature appears to be rather ambiguous, even somewhat inconsistent when it comes the classification of uncertainties. Indeed, although both the classification of uncertainties into exogenous or endogenous (Hirshleifer and Riley, 1979) and the importance of taking this distinction into account have been widely recognised in this branch of literature, especially in works related to the field of real options (Bowman and Hurry, 1993; Folta, 1998; Li et al., 2007; Li, 2007; Oriani and Sobrero, 2008), there is no clear and widely accepted definition. For example, Pindyck (1993); Dixit and Pindyck (1994) distinguish between technical and input cost uncertainty while noting their different effects on investment decisions as these incentivise investing and waiting, respectively. Building upon this distinction, McGrath (1997) called for a third form of uncertainty that lies in-between, and Folta (1998) stated (italics in their work) that “*exogenous uncertainty can be decreased by actions of the firm*”, while “*endogenous uncertainty is largely unaffected by firm actions*”. Furthermore, McGrath and Nerkar (2004) refers to the exogenous and endogenous resolution of uncertainty through the passing of time and learning, respectively. By contrast, Van der Hoek and Elliott (2006) took note of uncertainties that are state-dependent rather than dependent on the option holder’s actions (i.e. decisions).

A few studies aimed at presenting a more continuous classification of how various sources of uncertainty are affected by the option holder’s actions. Based on the overview of Micalizzi and Trigeorgis (1999), Scialdone (2007) presented an “uncertainty-mapping” (similar to Bräutigam et al. (2003)) that indicates the extent to which uncertainty categories (e.g. operational, market demand, price, financial, and industry) are exogenous or endogenous. In addition, the author qualitatively showed the categories’ relevance to single well-defined real options (e.g. options to wait, stage, switch and abandon). While these studies have linked different sources of uncertainty to individual option’s relative performance, they have presented rather unsatisfactory and ambiguous qualitative approaches; in contrast, this work takes a fundamentally different approach by presenting a holistic and general portfolio of real options approach that accounts for multiple, possibly interacting, exogenous and endogenous sources of uncertainty, as well as their influence on the performance

of the portfolio’s interdependent real options.

Various researchers have applied real option approaches to valuation problems with both exogenous and endogenous uncertainty⁹. Generalising the work of Roberts and Weitzman (1981), Pindyck (1993) evaluated a staged-investment with technical (endogenous) and input cost (exogenous) uncertainty using a finite difference method. Other relevant articles considered both types of uncertainty in the context of information technology investment projects (Schwartz and Zozaya-Gorostiza, 2003), patents and R&D projects (Schwartz, 2004), pharmaceutical R&D projects (Hsu and Schwartz, 2008; Pennings and Sereno, 2011), product platform flexibility planning (Jiao, 2012), and nuclear power plant investments (Zhu, 2012). With regard to state-dependent uncertainty, Sbuelz and Caliori (2012) studied the influence of state-dependent cashflow volatility on the investment decisions related to corporate growth options, whereas Palczewski et al. (2015) examined optimal portfolio strategies under stock price dynamics with state-dependent drift. Despite having applied real option approaches to valuation problems with both types of uncertainty, these quantitative approaches are rather inflexible and restricted in terms of the size of the real options portfolio, the number and types of uncertainties as well as the valuation technique(s) applied. Furthermore, they do not provide the holistic and unified approach presented in this thesis.

2.5 Summary

This literature review provided a selection of relevant works in areas directly related and important to this thesis. The chapter began with presenting a brief overview of ROA applied to infrastructure investments in order to provide both background and context for this research. It found that despite the growing number of publications over the last one-and-a-half decades addressing a range of applications in this area, the great potential of ROA remains largely unexploited. Since infrastructure investments are “ripe with flexibility” and have to be made in the context of enormous uncertainty, we argued that a practical and powerful appraisal technique will have to be capable of valuing the investments’ underlying portfolio of interdependent real options whilst accounting for the multiple sources of uncertainty. Based on these requirements, this chapter subsequently presented an overview of simulation-and-regression-based option pricing methods. Combining simulation (i.e. Monte Carlo sampling) and parametric regression, these methods are widely regarded flexible and powerful, and more importantly capable of dealing, amongst other things, with multiple sources of uncertainty and possibly complex interdependent real options. So they have the potential to be profitably used in real options applications relevant for this research.

⁹Interestingly, Adner and Levinthal (2004); Cuypers and Martin (2007, 2010) argued that real options theory cannot be applied to problems with endogenous uncertainty since, amongst other things, the real options’ discrete nature would be eroded.

From a methodological perspective, a comprehensive review of portfolio of real options approaches and endogenous uncertainties was presented in Sections 2.3 and 2.4, respectively. With regard to the former, starting with a presentation of early studies on option portfolios, which explored the concept rather qualitatively, and moving towards more quantitative approaches, it identified three works that presented holistic portfolio approaches similar to the one proposed here and it critically discussed their limitations. As mentioned, the present study differs in a number of important ways when compared with these three works. Most importantly, in the approach presented here IDs are used to graphically model portfolios of interdependent real options and the options' interdependencies are mathematically translated into linear constraints and then integrated in a portfolio optimisation problem. This problem is formulated as a multi-stage stochastic integer programme and is approximately solved by a simulation-and-regression-based valuation algorithm. Lastly, the classification of uncertainties into exogenous and endogenous in the related literature was reviewed and the limitations of relevant real option approaches were examined. In contrast to these approaches, the one proposed here, which extends our portfolio of real options approach to include endogenous uncertainty, is powerful and flexible in terms of the size of the real options portfolio as well as the number and types of uncertainties that can be considered.

Chapter 3

Modelling and Valuing Portfolios of Interdependent Real Options

In this chapter we present a new approach to model and approximate the value of portfolios of interdependent real options using influence diagrams (IDs) and simulation-and-regression. This chapter is organised as follows: In Section 3.1 we present the conceptual framework to model and value portfolios of interdependent real options. We proceed in Section 3.2 with the problem formulation, which contains the modelling of flexibilities using IDs, the formulation of the portfolio optimisation problem as a multi-stage stochastic integer programme, and a discussion of the underlying curses of dimensionality. Section 3.3 then presents the valuation algorithm by both describing how the optimisation problem's continuation function is approximated through parametric regression and developing a simulation-and-regression-based algorithm, as well as discusses the algorithm's computational efficiency and numerical accuracy in light of Section 2.2. Finally, Section 3.4 summarises this chapter.

3.1 Conceptual Framework

Although it is rarely used by real options analysts, the ID is a promising alternative to the traditionally-applied decision tree¹/lattice and has many advantages as a framework for identifying, defining, and modelling interdependent flexibilities inherent in investment projects. For example, IDs are intuitive and can be readily applied by decision makers to identify flexibilities (Lander and Shenoy, 1999), thereby focusing on the decisions the manager can make, rather than the risk modelling (Sick and Gamba, 2010). From a modelling perspective, IDs allow a more compact representation than lattice/tree techniques (Charnes and Shenoy, 2004), particularly in situations where there are multiple sources of uncertainty and a sequence of decisions, or path dependency (Demirer et al., 2003). This is because IDs do not scale with the number of uncertainties and grow linearly rather than combinatorially in the number of decision variables considered (Lander and Pinches, 1998). Finally, ID representations are simple, intuitive, transparent and flexible. In our approach,

¹See (Howard and Matheson, 2005) for a recent discussion from a decision analysis perspective.

we use IDs to graphically model the flexibilities contained in a portfolio of interdependent real options. The interdependencies between flexibilities are then mathematically translated into linear constraints and integrated into a portfolio optimisation problem, which is formulated as a multi-stage stochastic integer programme.

In order to approximate the value of portfolios of interdependent real options we apply simulation combined with parametric regression. It is widely acknowledged that Monte Carlo simulation² techniques, despite their computational complexity, have significant advantages over traditional option pricing techniques such as analytical and lattice-based methods. Amongst other things, simulation allows the consideration of many sources of uncertainty and the direct modelling of the uncertainties’ risk-neutral dynamics as well as real options with complex features (Pringles et al., 2015). Several authors have proposed numerical methods for the valuation of American options using simulation and regression; Stentoft (2014) recently compared the approaches of Carriere (1996); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001) and recommends the least squares Monte Carlo (LSM) approach of Longstaff and Schwartz (2001) for computational reasons. Like the LSM approach, which was developed for single American-style options, we apply “continuation function approximation” by using both a parametric model and a least-squares method³. In this way, we are able to approximate the complex continuation functions that describe the expected future contributions associated with transitions in the ID. These approximations are then used in a simulation-based valuation algorithm to determine optimal pathwise decisions for all available transitions at each possible resource state, subject to the linear constraints that describe interdependencies.

3.2 Problem Formulation

In this section, we present our approach to both the modelling of portfolios of interdependent real options and the formulation of the related portfolio optimisation problem as a multi-stage stochastic integer programme. A summary of the notation used is presented in Appendix A.

3.2.1 Modelling Flexibilities with Influence Diagrams

We consider the valuation of an investment project that is represented by a portfolio of interdependent real options. The flexibilities contained in this portfolio of interdependent real options are then modelled through an ID, which is composed of both a graphical and a numerical part. The former is composed of two elements: a set of (decision and terminal) nodes $\mathcal{N} = \{1, 2, \dots, N\}$, which may represent stages of development or operating modes,

²Another related article is Charnes and Shenoy (2004), which applies simulation to solve IDs using local computation.

³In fact, functional approximations have been used at least since Bellman and Dreyfus (1959) to increase the computational efficiency of dynamic programming, which is widely applied in decision analysis.

as well as a set of directed edges $\mathcal{H} = \{1, 2, \dots, H\}$, which represents the transitions linking the nodes in the ID, or in other words the flexibilities available to the decision maker. Unlike most modelling approaches for IDs in the context of real options (e.g. see Lander and Pinches (1998); Lander and Shenoy (1999); Charnes and Shenoy (2004), we allow for cycles in the ID, which is then represented through a directed cyclic graph $(\mathcal{N}, \mathcal{H})$ instead of an acyclic one. Although we only consider decision nodes, so apply the deterministic use case of IDs as noted by Howard and Matheson (2005), our specification can be easily extended to the probabilistic use by including chance nodes in the ID, as shown by Charnes and Shenoy (2004), thus allowing for the consideration of influencing factors such as technical uncertainty. Figure 6.1 contains an example of a district heating network expansion investment to illustrate the graphical part of the modelling approach.

The numerical part of the ID is specified by information associated with both nodes and transitions. Let the state of the system at time t , S_t , be composed of a resource and an information component denoted by $R_t \in \mathcal{R}_t$ and $I_t \in \mathcal{I}_t$, respectively, thus having $S_t = (R_t, I_t)$, where \mathcal{R}_t and \mathcal{I}_t are the corresponding state spaces. In general, R_t is an endogenous component (evolves deterministically), whereas I_t is an exogenous component (evolves stochastically). The former is modelled to contain at least information about the current decision node $N_t \in \mathcal{N}$, but, generally, will contain further problem-dependent resource state variables. On the other hand, I_t contains one or several stochastic factors (or random variables) that describe the value of the problem's uncertain parameters at t whose evolution can be modelled directly using (Markovian) stochastic processes. Most existing real option valuation approaches consider I_t to represent the “state” and do not explicitly model R_t . The few exceptions usually only consider either a discrete (Nadarajah et al., 2017) or continuous (Denault et al., 2013) scalar for the endogenous component. Here, in order to deal with complexities of portfolios of real options, including path-dependencies and interdependencies between options, we explicitly model R_t to be a vector made up of multiple resource state variables that characterise the valuation problem.

With regard to the information associated with nodes, in order to simplify the valuation algorithm presented in the next section, we assume that \mathcal{N} contains exactly one beginning node (no incoming transition(s)), but may have several terminal nodes, which are characterised through not having outgoing transitions. The value of a terminal node at t is given by its terminal value $G_t^T(S_t)$, for all $S_t \in \{S'_t \in \mathcal{S}_t : b^D(N'_t) = \emptyset\}$, where the action space $b^D(N_t)$ gives the set of outgoing transitions of node N_t . The set of decision times, which is often referred to as decision epochs in the Markov Decision Process (MDP) literature, is denoted by \mathcal{T} .

With regard to the information associated with transitions, there are three elements to any transition $h \in \mathcal{H}$:

1. The feasible region \mathcal{A}_{S_t} , which is composed of one or more linear constraints that describe the interdependencies between flexibilities, defines the transition(s) one can

make given state S_t . Let the decision to make any transition $h \in b^D(N_t)$ at node N_t be represented by a binary decision variable $a_{th} \in \{0, 1\}$, where $a_{th} = 1$ means transition h is made at time t and vice versa, as well as let the duration of transition h be Δ_h . Then, the vector $a_t = (a_{th})_{h \in b^D(N_t)}$ has to satisfy all constraints defined in \mathcal{A}_{S_t} , in other words $a_t \in \mathcal{A}_{S_t}$.

2. The transition function, which is generically written as $S^M(S_t, a_t, W_{t+\Delta_h})$, describes the evolution of the state S_t from time t to $t + \Delta_h$ when making transition h and given new exogenous information $W_{t+\Delta_h}$ that is learned between t and $t + \Delta_h$. In terms of the state's two components, $S^M(\cdot)$ can be interpreted as a composition of both a resource transition function $S^R(\cdot)$ and an information transition function $S^I(\cdot)$ which describe individually the evolution of R_t and I_t to $R_{t+\Delta_h}$ and $I_{t+\Delta_h}$, respectively, when making transition h at time t .
3. The immediate payoff $\Pi_t(S_t, a_t)$ is obtained at time t when making decision $a_t = (a_{th})_{h \in b^D(N_t)}$ given state S_t . Note that $\Pi_t(\cdot)$ depends only on variables whose value is known at time t , so is deterministic, and is being received at the beginning of the period t to $t + \Delta_h$.

3.2.2 Portfolio Optimisation Problem

Building upon the numerical part of the ID, the problem of determining the optimal value of the portfolio of interdependent real options is formulated as a multi-stage stochastic integer programme. Unlike the approach taken by Gamba (2003), who decomposed a portfolio of interacting real options into a set of independent, compound, mutually exclusive and switching options and then valued these sub-problems individually, this work proposes a single framework to value a portfolio of interdependent real options whilst using both linear constraints and binary decision variables to model strategic interdependencies and exercise decisions of real options. This modelling approach opens up the realm of integer programming with its powerful and flexible modelling techniques, see (Wolsey, 1998; Williams, 2006). Let the optimal value of the portfolio of real options at time t given state S_t be denoted by $G_t(S_t)$. The optimal value of the portfolio of interdependent real options, $G_0(S_0)$, is determined by the following multi-stage stochastic integer programme:

$$G_0(S_0) = \max_{(a_t)_{t \in \mathcal{T}}} \mathbb{E} \left[\sum_{t \in \mathcal{T}} e^{-kt} \Pi_t(S_t, a_t) | S_0 \right], \quad (3.1)$$

where S_0 is the state at time 0, $a_t = (a_{th})_{h \in b^D(N_t)}$, $a_t \in \mathcal{A}_{S_t}$, $a_{th} \in \{0, 1\}$, k is the risk-free rate, and $S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h})$.

The above optimisation problem with objective (3.1) can be solved recursively by using

the following value function for each state $S_t \in \{S'_t \in \mathcal{S}_t : b^D(N'_t) \neq \emptyset\}$ at time t :

$$G_t(S_t) = \max_{a_t} \Pi_t(S_t, a_t) + \mathbb{E} \left[e^{-k\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) | S_t, a_t \right] \quad (3.2)$$

$$\text{s.t. } a_t \in \mathcal{A}_{S_t}, \quad (3.3)$$

$$a_{th} \in \{0, 1\}, \quad \forall h \in b^D(N_t), \quad (3.4)$$

$$S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h}), \quad \forall h \in b^D(N_t), \quad (3.5)$$

with the boundary (terminal) condition $G_t(S_t) = G_t^T(S_t)$, for all $S_t \in \{S'_t \in \mathcal{S}_t : b^D(N'_t) = \emptyset\}$, $t \in \mathcal{T}$. Ultimately, the aim of the valuation problem is to determine the optimal value of the portfolio of interdependent real options given state S_0 at time 0, $G_0(S_0)$. It is important to note that, unlike traditional solution approaches for MDPs, the boundary condition used in (3.2)-(3.5) is not directly dependent on time t , but on whether the current node N_t in the ID is a terminal node, which may or may not have to be reached at a certain $t \in \mathcal{T}$. In the context of portfolios of real options discussed here, one may decide, for instance, to irreversibly abandon a project at any point in time $t \in \mathcal{T}$, thereby reaching the corresponding terminal node potentially well before $\max \mathcal{T}$.

3.2.3 Curses of Dimensionality

In general, solving the recursion in the (3.2)-(3.5) can be computationally expensive, even intractable due to at least three curses of dimensionality: (i) the high dimensionality of the resource state space \mathcal{R}_t and information state space \mathcal{I}_t ; (ii) the inability to (exactly) compute the conditional expectation in (3.2); and (iii) the high-dimensionality of the decision vector a_t and the feasible region \mathcal{A}_{S_t} . Of these, the curses related to both \mathcal{I}_t and (ii) are being addressed through the simulation and parametric regression approach⁴ described in Section 3.3. Also, although a vector, in most practical real option portfolio problems a_t will be a rather low-dimensional vector of binary variables and \mathcal{A}_{S_t} will be small in size and only depend on the resource state R_t , so (iii) can be neglected. In the context of real option portfolios, however, R_t is generally a vector of discrete variables (e.g., instead of a discrete scalar as in (Nadarajah et al., 2017)) with a possibly large state space \mathcal{R}_t . However, as demonstrated by the complex problem considered in Chapter 5, in which R_t has 4 dimensions, in general by appropriately modelling the problem at hand and carefully choosing relevant parameters a large state space \mathcal{R}_t is prevented and a computationally manageable valuation process is ensured.

⁴As noted by Tsekrekos et al. (2012), however, such a parametric regression approach is not entirely free of the curse of dimensionality related to \mathcal{I}_t because the number of basis functions needed (e.g. multivariate polynomials) and the computational cost of estimating the parametric model's coefficients are not linear in the dimension of \mathcal{I}_t . The authors offered the approach of Rust (1997) to overcome this issue.

3.3 The Valuation Algorithm

This section contains the approach to approximate the value of portfolios of interdependent real options as well as the corresponding simulation-based valuation algorithm.

3.3.1 Approximating the Continuation Function by Parametric Regression

The strategy chosen in this work is to approximate the value of the conditional expectation in (3.2), which represents the continuation value, using a parametric regression model. Using such an approximation of the continuation value is a commonly used strategy in the “Approximate Dynamic Programming” literature (Tsitsiklis and Van Roy, 2001; Longstaff and Schwartz, 2001; Glasserman, 2003; Powell, 2011) and directly tackles the curses of dimensionality related to \mathcal{I}_t ⁵ and the outcome space, as highlighted by Powell (2011). In particular, conditional upon being in state $S_t = (R_t, I_t)$ at time t and making decision $a_t \in \mathcal{A}_{S_t}$, we approximate

$$\Phi_t(S_t, a_t) = \mathbb{E} \left[e^{-k\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) | S_t, a_t \right], \quad (3.6)$$

by the following finite-dimensional, continuous function:

$$\hat{\Phi}_t^L(S_t, a_t) = \sum_{l=0}^L \hat{\alpha}_{tl}(S^R(R_t, a_t)) \phi_l(I_t), \quad (3.7)$$

where L is the parametric model’s dimension, the functions $\{\phi_l(I_t)\}_{l=0}^L$ are called basis functions or features, which are assumed to be independent of t and R_t , and the optimal values of the coefficients (or weights) at time t , $(\alpha_{tl}(S^R(R_t, a_t)))_{l=0}^L$, are estimated for all $a_t \in \mathcal{A}_{S_t}$ using least-squares regression as described in the following subsection. See Figure 6.2 in Powell (2011) for a simple illustration of a function approximation using basis functions.

3.3.2 The Simulation-and-Regression-based Valuation Algorithm

To approximate the value of the multi-stage stochastic integer programme (3.2)-(3.5), we apply a simulation-based algorithm that consists of both a forward and a backward induction procedure. Having specified all required inputs including the resource and information state at time 0, R_0 and I_0 , respectively, the number of sample paths $|\Omega|$, as well as the number (implied by L) and set of basis functions $\{\phi_l(I_t)\}_{l=0}^L$, the forward induction procedure initialises (and discretises) the state space \mathcal{S}_t for all $t \in \mathcal{T}$. More specifically, using the numerical part of the ID and starting at $t = 0$, the state space of the resource

⁵See comment in previous footnote.

state variable, \mathcal{R}_t , is being initialised for all $t \in \mathcal{T}$ through simple “exploration” to find all feasible resource states subject to \mathcal{A}_{S_t} , and by using the resource transition function $S^R(\cdot)$ to step forward in time. On the other hand, the state space of the information state variable I_t , which contains the realisations of the problem’s random variable(s), is generated by simulation (i.e. Monte Carlo sampling) while applying the information transition function $S^I(\cdot)$, resulting in a set of $|\Omega|$ independent sample realisations $\{I_t(\omega) : \omega \in \Omega\}$ for all $t \in \mathcal{T}$. Subsequently, the value of all terminal nodes $S_t \in \{S'_t \in \mathcal{S}_t : b^D(N'_t) = \emptyset\}$ in the ID is then initialised with $G_t^T(S_t)$, for all $t \in \mathcal{T}$. While the latter part of the above forward induction procedure is standard in the literature (e.g. see Glasserman (2003)), the former is a direct necessity of our portfolio approach yet uncommon in the real options literature, in which resource states are either not modelled explicitly or scalar.

The backward induction procedure determines an approximate value $\bar{G}_0(S_0)$ of the portfolio optimisation problem given by (3.2)-(3.5). Starting at time $\max \mathcal{T}$ and moving backwards to time $\min\{\mathcal{T} \setminus 0\}$, at each time t do the following three steps for each resource state $R_t \in \{R'_t \in \mathcal{R}_t : b^D(N'_t) \neq \emptyset\}$:

- (i) Applying least-squares regression, determine the optimal values of the coefficients $(\alpha_{tl}(S^R(R_t, a_t)))_{l=0}^L$ for all $a_t \in \mathcal{A}_{S_t}$ ⁶:

$$(\hat{\alpha}_{tl}(R_{t+\Delta_h}))_{l=0}^L = \arg \min_{(\alpha_{tl}(\cdot))_{l=0}^L} \left\{ \sum_{\omega \in \Omega} \left[e^{-k\Delta_h} \bar{G}_{t+\Delta_h}(S_{t+\Delta_h}(\omega)) - \sum_{l=0}^L \alpha_{tl}(R_{t+\Delta_h}) \phi_l(I_t(\omega)) \right]^2 \right\}, \quad (3.8)$$

where $R_{t+\Delta_h} = S^R(R_t, a_t)$ and $S_{t+\Delta_h}(\omega) = (R_{t+\Delta_h}, I_{t+\Delta_h}(\omega))$.

- (ii) Using the result of (i) with $\hat{\Phi}_t^L(S_t, a_t)$ as in (3.7), compute the pathwise optimisers $\hat{a}_t(\omega)$ of the pathwise approximation of the problem (3.2)-(3.5) for all $\omega \in \Omega$:

$$\hat{a}_t(\omega) = \arg \max_{a_t(\omega)} \Pi_t(S_t(\omega), a_t(\omega)) + \max \left\{ \check{\Phi}_t(R_t, a_t(\omega)), \hat{\Phi}_t^L(S_t(\omega), a_t(\omega)) \right\} \quad (3.9)$$

$$\text{s.t.} \quad a_t(\omega) \in \mathcal{A}_{S_t(\omega)}, \quad (3.10)$$

$$a_{th}(\omega) \in \{0, 1\}, \quad \forall h \in b^D(N_t). \quad (3.11)$$

where $\check{\Phi}_t(R_t, a_t(\omega))$ is a lower bound on the continuation value, given R_t and $a_t(\omega)$.

- (iii) Using the result of (ii), approximate the optimal portfolio value $G_t(S_t)$ given S_t at time t along each path $\omega \in \Omega$ by:

$$\bar{G}_t(S_t(\omega)) = \Pi_t(S_t(\omega), \hat{a}_t(\omega)) + e^{-k\Delta_h} \bar{G}_{t+\Delta_h}(S^R(R_t, \hat{a}_t(\omega)), I_{t+\Delta_h}(\omega)) \quad (3.12)$$

⁶For simplicity, the special case of transitions leading to terminal nodes with deterministic values is not treated separately here. See Algorithm 2 in Chapter 6 for a simple and straightforward way to algorithmically deal with such a case.

At time 0, however, the above three steps cannot be applied as $S_0 = S_0(\omega)$, for all $\omega \in \Omega$, but now the conditional expectation in (3.6) can be computed directly by taking the average over all $|\Omega|$ pathwise continuation values $\bar{G}_{\Delta_h}(S^R(R_0, a_0), I_{\Delta_h}(\omega))$, so the approximate value of the portfolio of interdependent real options is then:

$$\bar{G}_0(S_0) = \max_{a_0 \in \mathcal{A}_{S_0}} \left(\Pi_0(S_0, a_0) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{-k\Delta_h} \bar{G}_{\Delta_h}(S^R(R_0, a_0), I_{\Delta_h}(\omega)) \right), \quad (3.13)$$

where $a_{0,h} \in \{0, 1\}$, for all $h \in b^D(N_0)$.

In the first and third step above as well as in (3.13), the pathwise (approximated) continuation values $\bar{G}_{t+\Delta_h}(\cdot)$ and $\bar{G}_{\Delta_h}(\cdot)$ are already known at times t and 0, respectively, since these are defined recursively. Importantly, unlike Tsitsiklis and Van Roy (2001), we use $\hat{\Phi}_t^L(\cdot)$ only for the sake of computing the pathwise optimal decisions in step (ii), but the actually realised, pathwise continuation values, $\bar{G}_{t+\Delta_h}(\cdot)$, to approximate $G_t(\cdot)$ by $\bar{G}_t(\cdot)$ as in step (iii), which is in accordance with the approach of Longstaff and Schwartz (2001) and results in a comparatively smaller absolute bias as well as less and much slower accumulation of approximation errors, as recently demonstrated by Stentoft (2014). In addition, assuming a deterministic lower bound exists, we correct obviously erroneous approximations of the continuation value by replacing $\hat{\Phi}_t^L(\cdot)$ in (3.9) with $\max\{\check{\Phi}_t(\cdot), \hat{\Phi}_t^L(\cdot)\}$; e.g., $\check{\Phi}_t = 0$ for an American option, as in Glasserman (2003). A summary of the backward induction procedure is shown by Algorithm 1.

Although backward induction procedures are widely and standardly applied, the one described above contains several important features that allow us to approximate the value of a portfolio of interdependent real options. Firstly, the backward induction has to be applied for each resource state $R_t \in \mathcal{R}_t$ that does not correspond with a terminal node (no decision needed there), which is a direct consequence of our portfolio approach. By contrast, this is generally not needed for the regression-based pricing of single (real) options, where R_t is commonly not modelled explicitly. Secondly, two of the procedure's three nested loops perform particular portfolio-related tasks.

In step (i), the optimal coefficients $(\hat{\alpha}_{tl}(S^R(R_t, a_t)))_{l=0}^L$ are determined for every feasible decision a_t , given R_t , which consequently satisfies the linear constraints describing the interdependencies between real options in the portfolio. But this is generally not necessary for the pricing of single, well-defined options, which often feature trivial decision spaces; for instance, consider the simple “hold vs. exercise” decision underlying an American-type real option. Also, unlike the approach of Longstaff and Schwartz (2001), we include in the regression (3.8) all $|\Omega|$ simulation paths thereby improving approximation accuracy (Gamba, 2003; Areal et al., 2008; Tsekrekos et al., 2012; Stentoft, 2014). It is important to note, however, that doing so is in fact necessary (Areal et al., 2008) here as the structure and pathwise value of $\Pi_t(\cdot)$ generally differ for compound (i.e. path-dependent) real options within the portfolio.

Algorithm 1: Approximation of optimal value of problem (3.2)-(3.5)

Data: From forward induction procedure and problem specific inputs

Result: $\bar{G}_0(S_0)$

```

1 for  $t = \max\{\mathcal{T} \setminus 0\}$  do
2   forall  $R_t \in \{R'_t \in \mathcal{R}_t : b^D(N'_t) \neq \emptyset\}$  do
3     forall  $a_t \in \mathcal{A}_{S_t}$  do
4       Use both  $\check{\Phi}_t(R_t, a_t)$  and (3.7)-(3.8) to determine:
          $F_t(S_t(\omega), a_t) \leftarrow \max \left\{ \check{\Phi}_t(R_t, a_t), \hat{\Phi}_t^L(S_t(\omega), a_t) \right\}, \forall \omega \in \Omega$ 
5     end
6     forall  $\omega \in \Omega$  do
7       Compute pathwise optimisers:
          $\hat{a}_t(\omega) \leftarrow \arg \max_{a_t(\omega) \in \mathcal{A}_{S_t(\omega)}} \left\{ \Pi_t(S_t(\omega), a_t(\omega)) + F_t(S_t(\omega), a_t(\omega)) \right\}$ 
8       Approximate optimal portfolio value along each path  $\omega$ :
          $\bar{G}_t(S_t(\omega)) \leftarrow \Pi_t(S_t(\omega), \hat{a}_t(\omega)) + e^{-k\Delta_h} \bar{G}_{t+\Delta_h} \left( S^R(S_t(\omega), \hat{a}_t(\omega)), I_{t+\Delta_h}(\omega) \right)$ 
9     end
10  end
11   $\mathcal{T} \leftarrow \mathcal{T} \setminus t$ 
12 end
13 At  $t = 0$ , determine:
    
$$\bar{G}_0(S_0) \leftarrow \max_{a_0 \in \mathcal{A}_{S_0}} \left\{ \Pi_0(S_0, a_0) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{-k\Delta_h} \bar{G}_{\Delta_h} \left( S^R(S_0, a_0), I_{\Delta_h}(\omega) \right) \right\}$$


```

In step (ii), the optimal decision $\hat{a}_t(\omega)$ along path ω is computed by optimally solving the integer programme (3.9)-(3.11) for each path $\omega \in \Omega$, giving $|\Omega|$ pathwise optimisers $\hat{a}_t(\omega)$. These represent the binary decisions that maximise the pathwise approximated portfolio value given S_t at time t whilst satisfying the linear constraints in $\mathcal{A}_{S_t(\omega)}$. In contrast, the decision-making process underlying most existing regression-based pricing algorithms boils down to simply comparing $\Pi_t(\cdot)$ with $\hat{\Phi}_t^L(\cdot)$, e.g. by $\max\{\Pi_t(\cdot), \hat{\Phi}_t^L(\cdot)\}$, under total enumeration of all mutually exclusive alternatives; e.g., two in the case of an American/Bermudan option (Glasserman, 2003), several in the case of switching (Tsekrekos et al., 2012) and swing options (Nadarajah et al., 2017). Hence, in these existing algorithms the actual pathwise optimal decision generally plays a secondary role, rather than a fundamental one as in our algorithm.

3.3.3 Computational Efficiency and Numerical Accuracy

The efficiency of the simulation-based algorithm presented here and the accuracy of the approximation depend on a number of influencing factors as discussed in Section 2.2. With regard to the latter, a number of advantageous results obtained using the LSM approach are also relevant to our approach, which shares a comparable continuation function approx-

imation. For example, convergence results have been provided by Longstaff and Schwartz (2001); Clément et al. (2002); Stentoft (2004b); Glasserman and Yu (2004a). The robustness of the approach to different choices of basis functions has been shown by Moreno and Navas (2003) and more recently by Tsekrekos et al. (2012), as well as with an emphasis on the trade-off between computational time and precision by Stentoft (2004a). More recently, Areal et al. (2008) have demonstrated various ways to improve the accuracy of the valuation approach by investigating the influence of different regression algorithms, several variance reduction techniques, various polynomial families, as well as varying numbers of both model dimension (L) and paths ($|\Omega|$). Furthermore, Longstaff and Schwartz (2001) recommend appropriate scaling before performing the least-squares regression in order to avoid numerical errors and computational underflows.

By its nature, the proposed algorithm applies the same strategy regarding mutually exclusive options as the one of Areal et al. (2008), who has shown that this strategy provides faster and more accurate results than the algorithm presented by Gamba (2003). It is important to note that despite providing a lower bound on the value of the portfolio of interdependent real options, like the approaches of Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001), our approach can be extended by applying duality theory (e.g. see (Haugh and Kogan, 2007)) to allow for the estimation of precise and accurate upper (dual) bounds as shown by Andersen and Broadie (2004); Nadarajah et al. (2017). Furthermore, the integer programming problem (3.9)-(3.11) can be solved efficiently by standardly available solvers applying such algorithms as branch and bound (Ahmed et al., 2003). Lastly, while more advanced regression methods such as nonparametric models (e.g. kernel regression, local averaging, smoothing splines, and neural networks (Carriere, 1996; Judd, 1998; Powell, 2011)) can lead to more accurate results with lower computational efforts (see the review of Kohler (2010)), they are generally not readily applicable in high-dimensional settings and even low-dimensional problems can be comparatively complex (Pizzi and Pellizzari, 2002; Kohler, 2010; Powell, 2011).

3.4 Summary

This chapter presented a new approach for modelling and approximating the value of portfolios of interdependent real options using both IDs and simulation-and-regression. With regard to the proposed conceptual framework, IDs and simulation are powerful frameworks for representing flexibilities and valuing risky investments, respectively. They can be used in tandem in an optimisation framework to efficiently and correctly value portfolios of interdependent real options. Using IDs to model the flexibilities provided by portfolios of interdependent real options, their graphical and numerical part were specified. Our modelling technique is intuitive and compact; strategic interdependencies between real options are translated into linear constraints and the risk-neutral dynamics of all

underlying uncertainties are directly modelled using (Markovian) stochastic processes as proposed. These are then easily implemented in a portfolio optimisation problem which is formulated as a multi-stage stochastic integer programme.

To approximate the value of this optimisation problem, we apply simulation and parametric regression and have presented a transparent valuation algorithm. In contrast to existing regression-based valuation algorithms, ours explicitly takes into account the state variable’s multidimensional resource component that generally occurs in real option portfolios. In addition, the valuation algorithm presented here contains several important features specific to option portfolios; these features are adequately described and highlighted as well as compared with the state-of-the-art literature, thus clearly illustrating our contribution. In general, the corresponding valuation technique, which applies continuation function approximation, is accurate, flexible, robust and computationally efficient. To operationalise the presented approach, its application to three relatively simple numerical examples is presented in the next chapter, and Chapter 5 demonstrates its ability to evaluate complex and risky investment projects by evaluating a complex natural resource investment that features both a large portfolio of interdependent real options and four stochastic factors.

Given its generality, the presented approach can be applied to a wide range of complex and risky investment projects which have both inherent interdependent flexibilities and many sources of underlying uncertainties (such as in infrastructure), and it has the potential to lay the basis for further theoretical developments. For example, Chapter 6 demonstrates how to extend the approach presented in this chapter by integrating other types of uncertainties. This chapter also presented a discussion of limitations with respect to the three curses of dimensionality potentially affecting the computational tractability of the sequential decision problem as well as the computational efficiency and numerical accuracy of the simulation-and-regression-based valuation algorithm presented here. These discussions are not only relevant for informing (modelling-related) decisions surrounding the present work – e.g. possibilities to improve the algorithm’s efficiency in the presented applications –, but also indicate potentially rewarding directions for future research by giving rise to many interesting research questions.

Chapter 4

Numerical Examples

This chapter presents the operationalisation of the portfolio of real options approach presented in Chapter 3 in the context of three relevant examples. It is organised as follows: In Section 4.1 the approach is applied to the simple example of valuing an American put option and Section 4.2 demonstrates its ability to accurately model and correctly value a slightly more complex real option problem by evaluating a natural resource investment with a switching option – a portfolio of interdependent call and put options. This copper mine example is then extended from a one-factor to a three-factor setting in Section 4.3, which also presents an analysis of the effect of correlation on the investment’s value. Section 4.4 discusses the relationship of the proposed approach to existing option pricing and decision analysis approaches. Finally, Section 4.5 provides some summarising remarks.

4.1 Valuing an American Put Option

4.1.1 Problem Setting

To illustrate the use and implementation of the developed approach, we apply it to the simple numerical example of an American put option in a one-factor setting. Originally proposed by Longstaff and Schwartz (2001), this example has more recently been used by Stentoft (2014) to compare the methods of Carriere (1996); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001). A slightly modified yet almost identical example was used in (Powell, 2011) to illustrate the use of regression models to American option pricing. While this is clearly not a portfolio of options nor does it contain multifactor features, this simple example of an American put allows us to demonstrate the mechanics of our approach in a clear and easily comprehensible manner. In addition, since options with such an exercise feature are often integrated in more complex portfolios (e.g. see Chapters 5 and 6), this example will contribute to a better understanding of the bigger picture in more complex situations. Furthermore, options with (discrete) American-style exercise features, whether put or call, are arguably the most important problems in theory, so there is academic value in demonstrating the wide range of applications of our approach.

4.1.2 Modelling

The managerial flexibility provided through an American put option – the discrete-time version of this real option is sometimes referred to as Bermudan option – is represented by the ID in Figure 4.1. The ID consists of three nodes termed *Holding* (1), *Exercised* (2), and

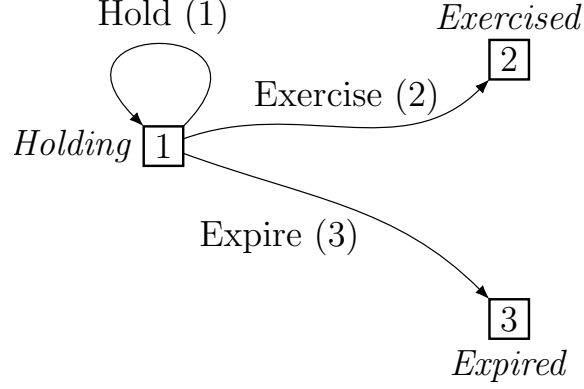


Figure 4.1: Influence diagram for American option.

Expired (3), of which only the first is a decision node, as well as three transitions termed Hold (1), Exercise (2), and Expire (3). Consequently, the sets of nodes and transitions are given by $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{H} = \{1, 2, 3\}$, respectively, and the duration of transition $h \in \mathcal{H}$ is Δ_h time period(s). With regard to the underlying sequential decision problem, when *Holding* at time $t > 0$, the decision maker has to decide whether to exercise the put option immediately by making transition 2 to become *Exercised*, or to hold the option until $t + \Delta_1$ and thus remain *Holding*. However, the latter decision can only be made at times $t < T^{max}$, whereas at the expiration date, T^{max} (i.e. the option's maturity date), this put option automatically expires if not exercised, which corresponds with making transition 3 to the *Expired* node. No (exercise) decision is made at $t = 0$.

Let the decision node at time t be denoted by N_t as well as let the stock price (of the underlying asset) at time t along path ω be denoted by $X_t(\omega)$. The endogenous part (resource component) of the state is then represented by $R_t = (t, N_t)$, whereas the exogenous part (information component) is given by $I_t = X_t$, which equals the set of stock price realisations at time t , $\{X_t(\omega) : \omega \in \Omega\}$, where Ω is the set of sample realisations. In the case of the American put option, the state variable is then given by $S_t = (t, N_t, X_t)$.

The decision variables available at node N_t , $a_t = (a_{th})_{h \in b^D(N_t)}$, where the action space $b^D(N_t)$ is given by

$$b^D(N_t) = \begin{cases} \{1, 2, 3\}, & \text{if } N_t = 1, \\ \{\}, & \text{otherwise,} \end{cases} \quad (4.1)$$

are defined in such a way that $a_{th} = 1$ means transition h is made at time t and 0 otherwise. The decision variables have to satisfy the following linear constraints, which

define the feasible region \mathcal{A}_{S_t} :

$$\begin{cases} a_{t1} + a_{t2} + a_{t3} = 1 & \text{if } N_t = 1, \end{cases} \quad (4.2)$$

$$\begin{cases} a_{t1}t < T^{max}, \end{cases} \quad (4.3)$$

$$\begin{cases} a_{t2} \leq t, \end{cases} \quad (4.4)$$

$$\begin{cases} a_{t3}T^{max} \leq t, \end{cases} \quad (4.5)$$

where $a_{th} \in \{0, 1\}, \forall h \in \mathcal{H}$. The meaning of these four constraints¹ is as follows: the first constraint, which could equally have been written as $\sum_{h \in \mathcal{H}} a_{th} = 1$, enforces that exactly one transition is made when $N_t = 1$; (4.3) and (4.4) ensure the option cannot be held until $t = 4$ and cannot be exercised at $t = 0$, respectively; and (4.5) makes sure the option can only expire at $t = T^{max}$ but not before.

The transition function $S^M(\cdot)$ is composed of a resource transition function $S^R(\cdot)$ and an information transition function $S^I(\cdot)$. With regard to the former, the evolution of t is straightforward as it simply evolves from t to $t + \Delta_h$ after having made transition h , and the transition of N_t is implicitly given by the adjacency matrix of the directed graph $(\mathcal{N}, \mathcal{H})$ underlying the ID:

$$\begin{array}{ccccc} & & 1 & 2 & 3 \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix} \end{array}$$

, where row and column indices represent starting and ending nodes, respectively, and the matrix's elements describe the transitions linking these nodes (“–” means there is no such transition). The information transition function, on the other hand, is represented by the information given in Table 4.1 of the next subsection.

The deterministic payoff obtained at time t when making decision a_t given S_t is:

$$\Pi_t(S_t, a_t) = \begin{cases} 0, & \text{if } a_{t1} = 1, \\ K - X_t, & \text{if } a_{t2} = 1, \\ 0, & \text{if } a_{t3} = 1, \end{cases} \quad (4.6)$$

where K is the option's strike price. Note that this payoff function models the “intrinsic value” of the American put option more generally than the commonly used notation in literature of $\max(0, K - X_t)$, which inherently assumes a decision maker would not exercise the option if its payoff was negative. While this feature could be easily implemented into the payoff function, e.g. to ensure non-negative option payoff values, we believe our formulation is more general and transparent with regard to the valuation problem's underlying sequential decision process.

¹It should be noted that all constraints in this thesis that are modelled as strict inequalities – such as (4.3) – can be easily transformed into (weak) inequality constraints if needed.

4.1.3 Valuation

For valuation, we use the same parameter values as Longstaff and Schwartz (2001): the put option's strike price is 1.10 (K); the risk-free rate is 6% (r); the option can be held up to its final expiration date at time 3 (T^{max}); and the following durations of transitions (in periods): $\Delta_1=1$ and $\Delta_2=\Delta_3=0$, giving $\mathcal{T} = \{0, 1, 2, 3\}$. In addition, Table 4.1 contains the 8 ($|\Omega|$) sample realisations of stock price paths, $\{X_t(\omega) : \omega \in \{1, 2, \dots, 8\}\}$, over four time periods ($|\mathcal{T}|$), which were pre-generated by Longstaff and Schwartz (2001) under the risk-neutral measure using $X_0 = 1.00$. As a consequence, since the information state space

Table 4.1: Sample realisations of stock prices $X_t(\omega)$ over four time periods.

ω	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

is given in this simple example, the forward induction procedure described in Section 3.3 is only used to generate the resource state space \mathcal{R}_t for all $t \in \mathcal{T}$ with $R_0 = (0, 1)$. In particular, we have:

$$\mathcal{R}_t = \begin{cases} \{(0, 1)\}, & \text{if } t = 0, \\ \{(1, 1), (1, 2)\}, & \text{if } t = 1, \\ \{(2, 1), (2, 2)\}, & \text{if } t = 2, \\ \{(3, 1), (3, 2), (3, 3)\}, & \text{if } t = 3. \end{cases}$$

Hence the full resource state space is given by $\mathcal{R} = \cup_{t=0}^3 \mathcal{R}_t$. Note that for the sake of both clarity and comprehensibility we included in \mathcal{R} also the resource states that do not correspond with decision nodes (only $N_t = 1$ here), but this is in general not needed when initialising \mathcal{R} as no decisions need to be made at these resource states.

As described in Section 3.3, we proceed with determining the pathwise portfolio values of states that correspond with terminal nodes. Since there is no value in being at the *Exercised* and *Expired* node, the terminal value $G_t^T(S_t)$ associated with these nodes is zero. That is, for all $S_t = (R_t, I_t)$, $R_t \in \{(1, 2), (2, 2), (3, 2), (3, 3)\}$, we have $G_t(S_t) = 0$. To illustrate the dynamics of the system, Figure 4.2 shows the relation between states and transitions (i.e. actions/decisions). Finally, we use as basis functions, as in (Longstaff and Schwartz, 2001), a constant term, the stock price and the square of this value, hence

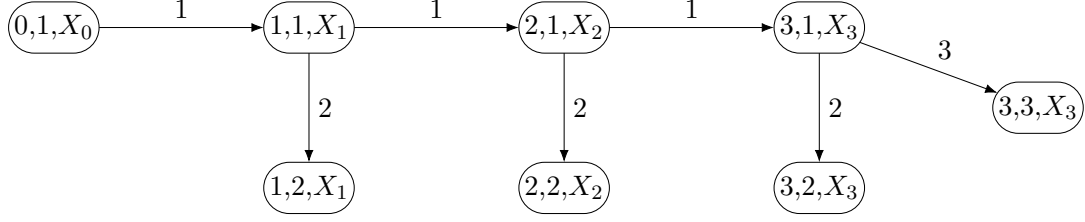


Figure 4.2: State transition diagram with nodes and arcs representing states $S_t = (t, N_t, X_t)$ and transitions $h \in \mathcal{H}$, respectively.

$L = 2$ and

$$\phi_l(I_t) = \begin{cases} 1, & \text{if } l = 0, \\ X_t, & \text{if } l = 1, \\ X_t^2, & \text{if } l = 2. \end{cases} \quad (4.7)$$

Having initialised all necessary inputs, the backward induction procedure starts with computing the pathwise optimal portfolio values at resource state $R_3 = (3, 1)$ at time $t = 3$. In doing so, the decision whether to exercise the option ($a_{3,2} = 1$) or to allow it to expire ($a_{3,3} = 1$) is made along each path $\omega \in \Omega$ by optimally comparing the pathwise values associated with each of these transitions subject to the constraints in $\mathcal{A}_{S_3(\omega)}$. Hence, for all $\omega \in \Omega$:

$$\hat{a}_3(\omega) = \arg \max_{a_{3,2}(\omega), a_{3,3}(\omega)} \left\{ \Pi_3(S_3(\omega), a_3(\omega)) + F_3(S_3(\omega), a_3(\omega)) : a_{3,2}(\omega) + a_{3,3}(\omega) = 1 \right\} \quad (4.8)$$

The pathwise optimisers $\hat{a}_3(\omega) = (\hat{a}_{3,2}(\omega), \hat{a}_{3,3}(\omega))$ are then used to determine the optimal portfolio value given resource state R_3 at time 3 along every path $\omega \in \Omega$, $\bar{G}_3(S_3(\omega))$. The steps undertaken are shown in Table 4.2.

Table 4.2: Steps performed at resource state $R_3 = (3, 1)$ at time $t = 3$.

ω	$\Pi_3(S_3(\omega), a_3)$		$F_3(S_3(\omega), a_3)^\dagger$		$\hat{a}_3(\omega)$		$\bar{G}_3(S_3(\omega))$
	$a_{3,2} = 1$	$a_{3,3} = 1$	$a_{3,2} = 1$	$a_{3,3} = 1$	$\hat{a}_{3,2}(\omega)$	$\hat{a}_{3,3}(\omega)$	
1	-0.24	0.00	0.00	0.00	0	1	0.00
2	-0.44	0.00	0.00	0.00	0	1	0.00
3	0.07	0.00	0.00	0.00	1	0	0.07
4	0.18	0.00	0.00	0.00	1	0	0.18
5	-0.42	0.00	0.00	0.00	0	1	0.00
6	0.20	0.00	0.00	0.00	1	0	0.20
7	0.09	0.00	0.00	0.00	1	0	0.09
8	-0.24	0.00	0.00	0.00	0	1	0.00

[†] Considering $\bar{\Phi}_3(R_3, a_3) = 0.00$ for both $a_{3,2} = 1$ and $a_{3,3} = 1$.

In resource state $R_2 = (2, 1)$ at time $t = 2$, the optionholder has to decide whether

to exercise the option immediately by making transition 2 (i.e. $a_{2,2} = 1$) or to hold the option by making transition 1 (i.e. $a_{2,1} = 1$), thereby keeping it alive until $t = 3$. In order to make these pathwise decisions, the continuation function $\Phi_2(S_2, a_2)$, which represents the value of being in state S_3 at time 3 conditional upon $S_2 = (2, 1, X_2)$ and $a_{2,1} = 1$, is approximated by $\hat{\Phi}_2^{(2)}(S_2, a_2)$. The fitted parametric model ($R^2 = 0.695$) is:

$$\hat{\Phi}_2^{(2)}(S_2, a_2) = 0.8215 - 1.1383 \cdot X_2 + 0.3896 \cdot X_2^2, \quad (4.9)$$

where the coefficients of the basis functions were determined by applying the least-squares regression method of (3.8) and using the $\bar{G}_3(S_3)$ -values of Table 4.2. On the other hand, for $a_{2,2} = 1$, no approximation is needed as the pathwise continuation values in this case are simply zero, resulting in $F_2(S_2(\omega), a_2) = 0, \forall \omega \in \Omega$. Then, for all $\omega \in \Omega$:

$$\hat{a}_2(\omega) = \arg \max_{a_{2,1}(\omega), a_{2,2}(\omega)} \left\{ \Pi_2(S_2(\omega), a_2(\omega)) + F_2(S_2(\omega), a_2(\omega)) : a_{2,1}(\omega) + a_{2,2}(\omega) = 1 \right\} \quad (4.10)$$

Applying both the pathwise optimal decisions, $\hat{a}_2(\omega)$, and the actually realised cash flow along path ω , $\Phi_t(S_2(\omega), \hat{a}_2(\omega))$, instead of the approximation $F_2(S_2(\omega), \hat{a}_2(\omega))$, the pathwise optimal values of the portfolio given $S_2(\omega) = (2, 1, X_2(\omega))$, $\bar{G}_2(S_2(\omega))$, are presented in the rightmost column of Table 4.3.

Table 4.3: Steps performed at resource state $R_2 = (2, 1)$ at time $t = 2$.

ω	$\Pi_2(S_2(\omega), a_2)$		$F_2(S_2(\omega), a_2)^\dagger$		$\hat{a}_2(\omega)$		$\bar{G}_2(S_2(\omega))$
	$a_{2,1} = 1$	$a_{2,2} = 1$	$a_{2,1} = 1$	$a_{2,2} = 1$	$\hat{a}_{2,1}(\omega)$	$\hat{a}_{2,2}(\omega)$	
1	0.00	0.02	0.0466	0.00	1	0	0.00
2	0.00	-0.16	0.0058	0.00	1	0	0.00
3	0.00	0.03	0.0496	0.00	1	0	0.0659
4	0.00	0.13	0.0839	0.00	0	1	0.13
5	0.00	-0.46	0.00 [‡]	0.00	1	0	0.00
6	0.00	0.33	0.1760	0.00	0	1	0.33
7	0.00	0.26	0.1402	0.00	0	1	0.26
8	0.00	-0.12	0.0127	0.00	1	0	0.00

[†] Considering $\check{\Phi}_2(R_2, a_2) = 0.00$ for both $a_{2,1} = 1$ and $a_{2,2} = 1$.

[‡] $\max\{0.00, -0.0061\}$.

In resource state $R_1 = (1, 1)$ at time $t = 1$, again, a decision needs to be made whether to immediately exercise the option or to continue its life (hold) until time 2. Like in the previous resource state, the decision to do the latter is based on an approximation of the discounted continuation value along path $\omega \in \Omega$, which is given by the following fit of the parametric regression model ($R^2 = 0.602$) conditional upon $S_1 = (1, 1, X_1)$ and $a_{1,1} = 1$:

$$\hat{\Phi}_1^{(2)}(S_1, a_1) = 2.6881 - 4.7491 \cdot X_1 + 2.1113 \cdot X_1^2 \quad (4.11)$$

Subsequently, determining the optimal decisions $\hat{a}_1(\omega) = (\hat{a}_{1,1}(\omega), \hat{a}_{1,2}(\omega))$ for all $\omega \in \Omega$ by solving the integer programme

$$\hat{a}_1(\omega) = \arg \max_{a_{1,1}(\omega), a_{1,2}(\omega)} \left\{ \Pi_1(S_1(\omega), a_1(\omega)) + F_1(S_1(\omega), a_1(\omega)) : a_{1,1}(\omega) + a_{1,2}(\omega) = 1 \right\}, \quad (4.12)$$

and using the actually realised cash flow along path ω , $\Phi_1(S_1(\omega), \hat{a}_1(\omega))$, instead of the approximation $F_1(S_1(\omega), \hat{a}_1(\omega))$, the pathwise optimal portfolio values given state $S_1(\omega) = (1, 1, X_1(\omega))$, $\bar{G}_1(S_1(\omega))$, are shown in the rightmost column of Table 4.4.

Table 4.4: Steps performed at resource state $R_1 = (1, 1)$ at time $t = 1$.

ω	$\Pi_1(S_1(\omega), a_1)$		$F_1(S_1(\omega), a_1)^\dagger$		$\hat{a}_1(\omega)$		$\bar{G}_1(S_1(\omega))$
	$a_{1,1} = 1$	$a_{1,2} = 1$	$a_{1,1} = 1$	$a_{1,2} = 1$	$\hat{a}_{1,1}(\omega)$	$\hat{a}_{1,2}(\omega)$	
1	0.00	0.01	0.0200	0.00	1	0	0.00
2	0.00	-0.06	0.0201	0.00	1	0	0.00
3	0.00	-0.12	0.0366	0.00	1	0	0.0621
4	0.00	0.17	0.0975	0.00	0	1	0.17
5	0.00	-0.01	0.0179	0.00	1	0	0.00
6	0.00	0.34	0.2982	0.00	0	1	0.34
7	0.00	0.18	0.1059	0.00	0	1	0.18
8	0.00	0.22	0.1439	0.00	0	1	0.22

[†] Considering $\check{\Phi}_1(R_1, a_1) = 0.00$ for both $a_{1,1} = 1$ and $a_{1,2} = 1$.

4.1.4 Results and Discussion

Finally, the approximated value of the American put option is determined by discounting the pathwise optimal portfolio values $\bar{G}_1(S_1(\omega))$ given state $S_1(\omega) = (1, 1, X_1(\omega))$ at time $t = 1$ back to time $t = 0$, and then taking the average over all 8 paths, that is $1/8 \sum_{\omega=1}^8 e^{-0.06} \bar{G}_1(S_1(\omega))$, totalling 0.1143. This is, not surprisingly, in line with the results previously obtained by Longstaff and Schwartz (2001); Stentoft (2014). However, unlike Longstaff and Schwartz (2001), who only used in the money paths, the algorithm proposed in this work and applied here uses all paths in the regression, as in Stentoft (2014), so the coefficients of the basis functions obtained here are slightly different. Also, as mentioned earlier, our payoff function defined in (4.6) allows negative intrinsic values, meaning we compare the pathwise values of holding (e.g., given by the sum of the second and fourth column of Table 4.3) with potentially negative exercise values (sum of third and fifth column of Table 4.3), which is in contrast to Longstaff and Schwartz (2001); Stentoft (2014), who only considered non-negative intrinsic values. Therefore it is important that $F_2(S_2(\omega), a_2)$ is bound below by $\check{\Phi}_2(R_2, a_2) = 0$. If this was not the case, that is if we had not corrected for erroneous continuation function approximations, then this could have potentially resulted in incorrect and hence non-optimal pathwise decisions. For example,

consider the row $\omega = 5$ of Table 4.3 but let us assume the option payoff from exercising (i.e. $a_{2,2} = 1$) was $-0.0061 < \Pi_2(S_2(5), a_2) < 0$ rather than -0.46 , then without lower bound the optimal decision for path $\omega = 5$ would have been to (incorrectly) exercise this put option (i.e. $\hat{a}_{2,2}(5) = 1$), though obviously out-of-the-money, resulting in a negative option value for this path, which clearly would not make sense as it contradicts using such an option in the first place.

4.2 Re-evaluating Natural Resource Investments

4.2.1 Problem Setting

In this section we demonstrate how the proposed approach can be used to evaluate a natural resource investment by considering the classical example of valuing a copper mine, which was originally proposed by Brennan and Schwartz (1985). Modelled and solved by the authors using partial differential equations and a finite difference method, respectively, this is one of the most highly cited works in the field of ROA. Furthermore, this example has been used by Gamba (2003); Abdel Sabour and Poulin (2006); Cortazar et al. (2008); Tsekrekos et al. (2012) as a benchmark to assess the LSM approach. While the operational flexibility available in this copper mine example represents a (small) portfolio of interdependent real options – a portfolio of interdependent call and put options –, it only contains one stochastic factor (copper price) and can be valued using finite differences. However, given the availability of widely-confirmed numerical results and the readily comprehensible setting, this well-known example is perfectly suited to allow the demonstration of the ability of the proposed approach to correctly value a risky investment slightly more complex than the previously considered American put option.

In this copper mine example, the decision maker has several possibilities to affect the mine's operation. Representing these flexibilities as a portfolio of interdependent real options, which we refer to as the option to switch, this and the portfolio's constituent real options are:

- (a) Option to switch: In addition to extracting the copper immediately until the mine inventory, Q_0 , is exhausted, the decision maker may decide to temporarily close the (operating) mine, to maintain or reopen the mine when it is currently closed, and/or to irreversibly abandon the copper mine before its inventory is fully exhausted, i.e. before Q_0/q years of operation at an annual output rate of q .
- (i) Option to temporarily mothball the mine: If the copper spot price at time t , X_t , is too low in relation to the mine's production costs, A_t , the decision maker can close down the opened (i.e. operating) mine at a cost of K_t^c , maintain the closed mine at an annual maintenance cost of M_t , and, if the copper price becomes favourable again, reopen the closed mine at a cost of K_t^o at time t .
- (ii) Option to abandon the mine: Whether opened or closed, the decision maker

retains the right to irreversibly abandon the copper mine at any time t without incurring any cost.

The value of this portfolio of interdependent real options is affected by the uncertainty surrounding future commodity prices, in other words by copper price uncertainty. The copper spot price at time t , X_t , is assumed to evolve according to the following discretised version of the continuous stochastic process used by Brennan and Schwartz (1985):

$$X_{t+\Delta} = X_t \exp \left\{ \left(r - \delta - \frac{\sigma_x^2}{2} \right) \Delta + \sigma_x \sqrt{\Delta} \epsilon_{t+\Delta}^x \right\}, \quad (4.13)$$

where Δ is the time step, r is the price trend, δ is the instantaneous convenience yield, σ_x is the standard deviation of price changes, and $\epsilon_{t+\Delta}^x$ is the driving zero-mean process – a standard normal random variable (mean 0, variance 1) whose increments are iid.

4.2.2 Modelling

The flexibilities inherent in the copper mine are illustrated by the ID in Figure 4.3. It con-

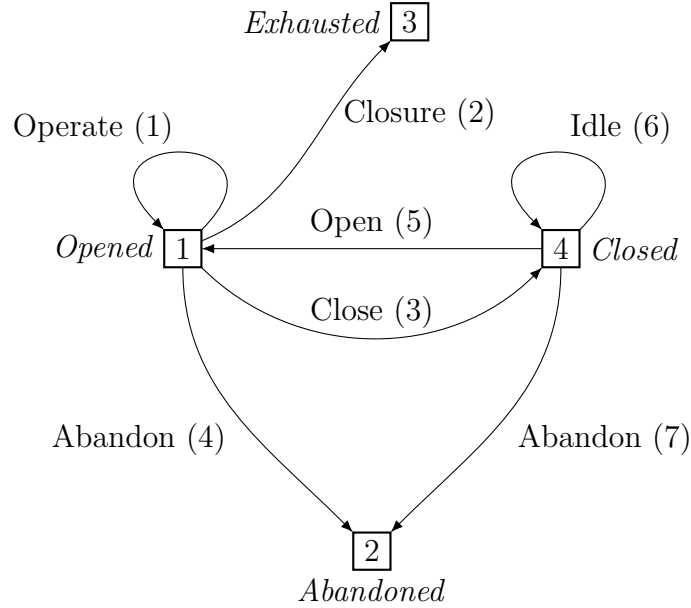


Figure 4.3: Influence diagram for the mine development project.

tains two decision nodes (*Opened* (1) and *Closed* (4)) and two terminal nodes (*Abandoned* (2) and *Exhausted* (3)), as well as seven transitions that link these nodes, resulting in $\mathcal{N} = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{1, 2, \dots, 7\}$. The duration (in years) of transition $h \in \mathcal{H}$ is Δ_h , with $\Delta_1 = \Delta_3 = \Delta_5 = \Delta_6$ and $\Delta_2 = \Delta_4 = \Delta_7 = 0$; in other words the durations of transitions $h \in \{1, 3, 5, 6\}$ are equal and positive, whereas the ones of transitions $h \in \{2, 4, 7\}$ are zero. When the mine is *Opened*, the decision maker has to decide whether to Operate (1) for Δ_1 year(s) whilst extracting $q\Delta_1$ of copper, temporarily Close (3), or irreversibly

Abandon (4) the copper mine project. On the other hand, if the mine is *Closed*, the available transitions are to keep the mine Idle (6), Open (5) it, or irreversibly Abandon (7) the project. However, the mine closures (2) if the commodity inventory is fully depleted and, as such, becomes *Exhausted*. Also, for the sake of definiteness, the mine has to be *Abandoned* when reaching its maximum lifetime of T^{max} years, thereby preventing the mine from having an infinite lifetime, e.g. by keeping it always closed.

Let the the decision node and the inventory of the mine at time t be denoted by N_t and Q_t , respectively, as well as let the copper spot price at time t along path ω be denoted by $X_t(\omega)$. Then, the resource and information component of the state variable S_t are given by $R_t = (t, N_t, Q_t)$ and $I_t = X_t$, respectively, with the latter representing the set of copper price realisations at time t , $\{X_t(\omega) : \omega \in \Omega\}$, where Ω is the set of sample realisations. The state variable is then written as $S_t = (t, N_t, Q_t, X_t)$. Since the mine's commodity inventory can be depleted at an annual output rate q , which is assumed constant, it takes at least Q_0/q years to empty the finite inventory. While the assumption of a finite horizon (through finite T^{max}) may result in an approximated numerical solution compared with the solution of Brennan and Schwartz (1985), which assumes not only continuous decisions (i.e. $\Delta \rightarrow 0$) but also an infinite time horizon (i.e. $T^{max} \rightarrow \infty$), this assumption is necessary due to the nature of our computational algorithm. However, adverse effects can be minimised, even avoided fully, by choosing $T^{max} \gg Q_0/q$.

The binary decision variables associated with the transitions available at decision node N_t at time t , which are given by the action space:

$$b^D(N_t) = \begin{cases} \{1, 2, 3, 4\}, & \text{if } N_t = 1, \\ \{5, 6, 7\}, & \text{if } N_t = 4, \\ \{\}, & \text{otherwise,} \end{cases} \quad (4.14)$$

have to satisfy the feasible region \mathcal{A}_{S_t} , which is defined by the following linear constraints:

$$\begin{cases} a_{t1} + a_{t2} + a_{t3} + a_{t4} = 1 & \text{if } N_t = 1, \end{cases} \quad (4.15)$$

$$\begin{cases} a_{t5} + a_{t6} + a_{t7} = 1 & \text{if } N_t = 4, \end{cases} \quad (4.16)$$

$$\begin{cases} a_{th}q\Delta_h \leq Q_t, & \forall h \in \{1, 5\}, \end{cases} \quad (4.17)$$

$$\begin{cases} 0 < a_{t2} + Q_t \leq 1 + Q_t(1 - Q_0^{-1}), \end{cases} \quad (4.18)$$

$$\begin{cases} t - T^{max} < a_{t2} + a_{t4}, \end{cases} \quad (4.19)$$

$$\begin{cases} t - T^{max} < a_{t7}, \end{cases} \quad (4.20)$$

where $a_{th} \in \{0, 1\}, \forall h \in \mathcal{H}$. These constraints accomplish the following: (4.15) and (4.16) ensure that exactly one transition is made at decision node 1 and 4, respectively; (4.17) makes sure the inventory does not become negative²; (4.18) requires the mine to closure

²For simplicity, we assume $Q_t \bmod (q\Delta_h) = 0$.

if and only if $Q_t = 0$; (4.19) ensures that if $t = T^{max}$ and $Q_{T^{max}} > 0$ then $a_{T^{max}4} = 1$; and, lastly, (4.20) makes sure the mine is abandoned if closed at $t = T^{max}$.

After having made a decision subject to these constraints, the resource state R_t evolves deterministically to $R_{t+\Delta_h}$ according to $S^R(\cdot)$, whereas the information state I_t evolves stochastically to $I_{t+\Delta_h}$ under the risk-neutral measure represented by $S^I(\cdot)$. With regard to $S^R(\cdot)$, the evolution of t is straightforward as it simply evolves from t to $t + \Delta_h$, the evolution of N_t is implicitly described by the adjacency matrix of the directed graph $(\mathcal{N}, \mathcal{H})$:

$$\begin{array}{c} \begin{array}{cccc} & 1 & 2 & 3 & 4 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{cccc} 1 & 4 & 2 & 3 \\ - & - & - & - \\ - & - & - & - \\ 5 & 7 & - & 6 \end{array} \right] \end{array}\end{array}$$

and the evolution of Q_t is specified by the following transition equation for all $h \in \mathcal{H}$:

$$Q_{t+\Delta_h} = Q_t - q\Delta_1(a_{t1} + a_{t5}), \quad (4.21)$$

Since decisions to exploit the mine may be made several times per year, the number of possible values Q_t can take is $1 + Q_0/(q\Delta_1)$. On the other hand, the copper price at time t , X_t , evolves stochastically to $X_{t+\Delta_h}$ according to the following discrete diffusion process:

$$X_{t+\Delta_h} = X_t \exp \left\{ \left(r - \delta - \frac{\sigma_x^2}{2} \right) \Delta_h + \sigma_x \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^x \right\} \quad (4.22)$$

The deterministic payoff obtained at time t when making decision a_t given state S_t is:

$$\Pi_t(S_t, a_t) = \begin{cases} q\Delta_1(X_t - A_t) - f(X_t)\Delta_1, & \text{if } a_{t1} = 1, \\ 0, & \text{if } a_{t2} = 1, \\ -M_t\Delta_3 - K_t^c, & \text{if } a_{t3} = 1, \\ 0, & \text{if } a_{t4} = 1, \\ q\Delta_5(X_t - A_t) - f(X_t)\Delta_5 - K_t^o, & \text{if } a_{t5} = 1, \\ -M_t\Delta_6, & \text{if } a_{t6} = 1, \\ 0, & \text{if } a_{t7} = 1, \end{cases} \quad (4.23)$$

where $A_t = A_0 e^{\pi t}$ is the average (per unit) production cost at time t with inflation rate π ; $f(X_t) = \tau_1 q X_t + \max\{\tau_2 q (X_t(1 - \tau_1) - A_t), 0\}$ is the sum of royalties and income tax paid at time t with τ_1 the royalty rate and τ_2 the income tax rate; $M_t = M_0 e^{\pi t}$ is the maintenance cost at time t ; and $K_t^c = K_0^c e^{\pi t}$ and $K_t^o = K_0^o e^{\pi t}$ are the costs to switch to the *Closed* and *Opened* node at time t , respectively.

4.2.3 Valuation

With regard to the valuation of this natural resource investment, we used the same parameter values as Brennan and Schwartz (1985), which are shown in Table 4.5. Furthermore,

Table 4.5: Input data for hypothetical copper mine of Brennan and Schwartz (1985).

Description	Parameter	Value	Unit
Mine			
Output rate	q	$10 \cdot 10^6$	lbs/year
Initial inventory	Q_0	$150 \cdot 10^6$	lbs
Initial average production cost	A_0	0.50	US\$/lbs
Initial cost of opening	K_0^o	$200 \cdot 10^3$	US\$
Initial cost of closing	K_0^c	$200 \cdot 10^3$	US\$
Initial maintenance cost	M_0	$500 \cdot 10^3$	US\$/year
Cost inflation rate	π	8%	year ⁻¹
Copper			
Convenience yield	δ	1%	year ⁻¹
Price variance	σ_x^2	8%	year ⁻¹
Nominal interest rate	r	10%	year ⁻¹
Taxes			
Royalty	τ_1	0%	—
Income	τ_2	50%	—
Property, <i>Opened/Closed</i>	λ_1	2%	year ⁻¹
Property, <i>Abandoned</i>	λ_3	0%	year ⁻¹

we considered five decisions to be made per year (i.e. $\Delta_h = 1/5, h \in \{1, 3, 5, 6\}$) and the first six (i.e. $L = 5$) generalized Chebyshev polynomials as basis functions. Also, we considered 100,000 ($=|\Omega|$) sample paths (half of which antithetic for variance reduction), as in Tsekrekos et al. (2012). While the inventory of the mine can be depleted as early as 15 years ($=Q_0/q$) after starting operation, a finite time horizon of $T^{max} = 60$ years was chosen for the time by which the right to extract copper from the mine expires. As mentioned earlier, while this could potentially result in slightly different values when compared with the infinite horizon solution of Brennan and Schwartz (1985), having chosen a sufficiently large time horizon is expected to eliminate the effects of this approximation. In fact, choosing $T^{max}=45$, i.e. three times the minimum time to depletion, already resulted in identical results, making it extremely unlikely that choosing $T^{max} > 60$ would further increase the value of the mine. For the sake of simplicity and since there is no payoff associated with transitions 4 and 7 nor a terminal value with the *Abandoned* node, we use a constant risk free rate of 12% ($= r + \lambda_1$) in our computations instead of a different rate of 10% ($= r + \lambda_3$) for values associated with transitions leading to the *Abandoned* node.

The forward induction procedure³ consists of the following steps:

1. Determine the set of decision times, \mathcal{T}_{N_t} , for all decisions nodes $N_t \in \{1, 4\}$, forming subsets of \mathcal{T} :

$$\mathcal{T}_{N_t} = \begin{cases} \{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T^{max}\}, & \text{if } N_t = 1, \\ \{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, \Delta_1 \leq i\Delta_1 \leq T^{max}\}, & \text{if } N_t = 4, \end{cases} \quad (4.24)$$

2. Use (4.22) to sample $|\Omega|$ paths of X_t giving $(X_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$
3. Generate the possible resource state space \mathcal{R}_t for each decision node and decision time:

$$\mathcal{R}_t = \begin{cases} \{(t, 1, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) \leq i\Delta_1 q \leq Q_0 - q \min(\Delta_1, t)\}, & \text{if } t \in \mathcal{T}_1, \\ \{(t, 4, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) + \Delta_1 q \leq i\Delta_1 q \leq Q_0\}, & \text{if } t \in \mathcal{T}_4. \end{cases} \quad (4.25)$$

Figure 4.4 shows the evolution of X_t for five generated paths considering $X_0 = 0.70$.

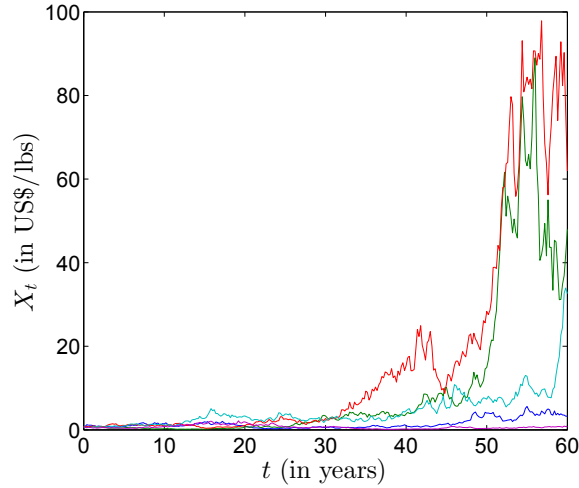


Figure 4.4: Selection of 5 equally likely paths for the evolution of the copper price, X_t .

Applying the backward induction procedure, results are presented and discussed in the following subsection.

4.2.4 Results and Discussion

This section begins with an analysis of the way in which the value of the (initially) opened and closed mine, $\bar{G}_0(S_0)$, characterised by $S_0 = (0, 1, Q_0, X_0)$ and $S_0 = (0, 4, Q_0, X_0)$, respectively, are affected by the initial price of copper, X_0 . The results are shown in Table 4.6 and compared with the ones obtained by Brennan and Schwartz (1985), who applied finite differences. In terms of the mine values (the switching decisions), our numerical results converge very closely (are identical) to the ones obtained by Brennan and Schwartz (1985) and are in line with the conclusions reached by Abdel Sabour and Poulin (2006);

³Assuming the mine is opened at time $t = 0$, i.e. $N_0 = 1$.

Table 4.6: Value (in US\$ millions) of mine for different copper prices according to alternative numerical methods.

Price (US\$/lbs) X_0	Brennan-Schwartz Finite Difference		Simulation-and- regression-based		Relative error (in %)	
	Opened	Closed	Opened	Closed	Opened	Closed
0.30	1.25 [†]	1.45	1.250 [†]	1.450	-0.020	-0.020
0.40	4.15 [†]	4.35	4.174 [†]	4.374	0.568	0.542
0.50	7.95	8.11	7.975	8.164	0.314	0.667
0.60	12.52	12.49	12.548	12.524	0.224	0.270
0.70	17.56	17.38	17.585	17.399	0.143	0.109
0.80	22.88	22.68 [‡]	22.914	22.714 [‡]	0.150	0.151
0.90	28.38	28.18 [‡]	28.439	28.239 [‡]	0.208	0.210
1.00	34.01	33.81 [‡]	34.101	33.901 [‡]	0.268	0.269

[†] Optimal to close mine.

[‡] Optimal to open mine.

Tsekrekos et al. (2012), thus confirming the adequacy of the proposed approach to correctly value complex and risky investment projects. In contrast, the switching decisions obtained by Gamba (2003); Cortazar et al. (2008) are not in line with (Brennan and Schwartz, 1985) and, as noted by Abdel Sabour and Poulin (2006), even confusing. This is because the switching policy of Gamba (2003) is cyclic in the initial copper price, X_0 , rather than being linear, and the policy implied by Cortazar et al. (2008) indicates that it is optimal to open the mine at $X_0 = 0.70$ since the difference between the value of the opened and closed mine equals K_0^o . Section B.1 contains a brief analysis of the effects of the copper price and its uncertainty on the value of this natural resource investment.

In order to demonstrate how the approach proposed here can be used to evaluate the individual real options available in this natural resource investment project, this section now analyses the extent to which the mine value with different configurations of option portfolios depends on the initial copper price, X_0 . Table 4.7 shows the sensitivity of the value of different portfolio configurations when X_0 is in the range from US\$ 0.30 to 1.00 per pound. Column (–) gives the expected value of the fixed-output-rate mine, which assumes it is opened at time $t = 0$ (i.e. $N_0 = 1$) and then operated at the rate of 10 million pounds per year until the 15-year inventory is fully exhausted. As can be seen, the value of the fixed-output-rate mine is negative for copper prices of US\$ 0.50 per pound and below, making operation unprofitable⁴. As described in Subsection 4.2.1, columns (a-i) and (a-ii) display the value of the mine if it can be temporarily mothballed and irreversibly abandoned, respectively. Having the flexibility provided by the former (latter)

⁴It should be noted that these are not the critical copper prices, i.e. the point at which it becomes optimal to invest, which largely depend on the chosen input data as mentioned by Brennan and Schwartz (1985); however, our approach can easily be used to accurately estimate these critical prices.

Table 4.7: Value (in US\$ millions) of copper mine for different copper price levels.

Copper Price (US\$/lbs) X_0	Value of Fixed-Output- Rate Mine (-)	In Isolation		Value of Option to Switch (5)	In Portfolio		Value with Option to Mothball (a-i)	Value with Option to Abandon (a-ii)	Value with Portfolio of Options (a)
		Mothball (3)	Value of Option to Abandon (4)		Mothball (6)	Abandon (7)			
0.30	-22.831*	18.793 (-)**	22.831 (-)	24.080 (-)	1.250 (5.19)	5.288 (21.96)	-4.038*	0*	1.250
0.40	-13.338*	13.322 (-)	15.515 (-)	17.512 (-)	1.997 (11.40)	4.190 (23.93)	-0.016*	2.177	4.174
0.50	-4.757*	9.205 (-)	11.259 (-)	12.732 (-)	1.473 (11.57)	3.527 (27.70)	4.448	6.502	7.975
0.60	2.950	6.582 (223.12)	8.527 (289.07)	9.598 (325.37)	1.071 (11.16)	3.016 (31.43)	9.532	11.477	12.548
0.70	10.142	4.836 (47.68)	6.716 (66.22)	7.443 (73.39)	0.727 (9.77)	2.607 (35.03)	14.978	16.858	17.585
0.80	17.017	3.618 (21.26)	5.441 (31.97)	5.898 (34.66)	0.457 (7.75)	2.279 (38.65)	20.635	22.457	22.914
0.90	23.685	2.784 (11.76)	4.481 (18.92)	4.754 (20.07)	0.273 (5.74)	1.970 (41.43)	26.470	28.166	28.439
1.00	30.213	2.210 (7.31)	3.736 (12.36)	3.888 (12.87)	0.152 (3.91)	1.678 (43.16)	32.423	33.949	34.101

* No investment.

** Added value of option(s) in %.

Note: the sets of transitions available in the different settings are as follows: $\mathcal{H}^- = \{1, 2\}$ in (-); $\mathcal{H}^- \cup \{3, 5, 6\}$ in (a-i); $\mathcal{H}^- \cup \{4\}$ in (a-ii); and \mathcal{H} in (a).

option enables the copper mine to become economically viable for copper prices of US\$ 0.50 (0.40) per pound and above, thus allowing the mine with such options to become viable in situations where the fixed-output-rate mine is not. By contrast, with the option to switch, whose value is shown in column (a) and which can be interpreted as the portfolio of options to mothball and abandon, the mine is economically viable for all copper prices under consideration. The IDs corresponding with the setting of columns (–), (a-i), (a-ii) and (a) are shown by Figures 4.5a, 4.5b, 4.5c and 4.3, respectively⁵.

In addition to reporting the value of the mine with different portfolio configurations, Table 4.7 also displays the value added by the portfolio's individual options – both in isolation and within the portfolio of options. Columns (3), (4) and (5) report the value of the option to temporarily mothball the mine, to abandon the mine and to switch, respectively. These values were determined by the difference between the mine values with these individual options – shown in columns (a-i), (a-ii) and (a) – and column (–), which gives the value of the fixed-output-rate mine. As can be seen, the real options considered add substantial value to this natural resource investment. For all copper prices under consideration, abandoning the mine was found to be more valuable than mothballing, and switching more valuable than abandoning. Importantly, the option to switch, which in itself is a portfolio of options containing the other two options, will always be at least as valuable than its constituent options. As expected, the values of these options decrease as the operating margin increases since operational flexibility becomes less attractive. Nevertheless, the added value of switching is still almost 13% for the highest copper price considered, which is twice the cost of production of US\$ 0.50 per pound.

Columns (6) and (7) of the table give the value of the option to mothball and to abandon the mine, respectively, within the portfolio of options. In other words, these columns report how much an individual real option adds to the portfolio assuming that the other individual option is already contained in the portfolio. To determine the value of one option, the value of the mine with the options portfolio was measured against the value of the mine with the other option. For example, the difference between column (a) and (a-ii) results in the values shown in column (6). Comparing the values shown in column (6) with the ones of column (7) shows that, while the value of either option in the portfolio generally decreases as X_0 increases⁶, abandoning adds substantially more value to the portfolio than mothballing, especially for high copper prices. This result is very intuitive given the above presented valuation of the portfolio's individual options – i.e. the options to mothball and abandon in isolation. It is interesting to note that, the relative portfolio value of mothballing – i.e. opening, closing and maintaining the closed mine – generally decreases as the initial price of copper increases, whereas the relative value of adding the option to

⁵Although not shown here for simplicity, the ID in Figure 4.5a (4.5b) also contains one (two) transition(s) to the *Abandoned* node, respectively, for the case $t = T^{max}$.

⁶The initial increase displayed in column (6) is due to the non-negative value of the mine with the option to abandon – see column (a-ii) –, which is bounded below by zero.

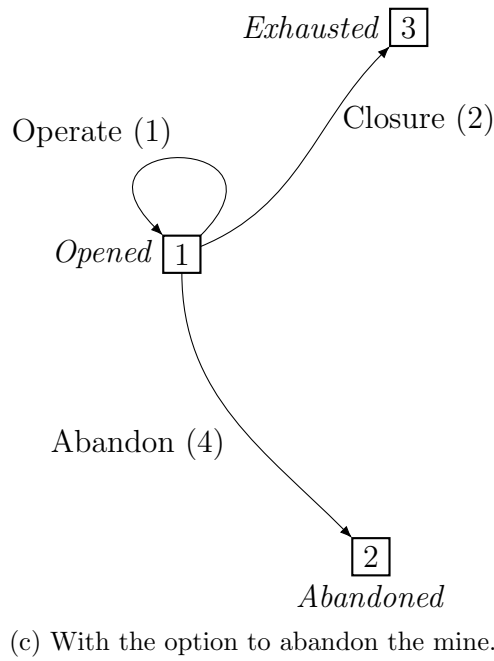
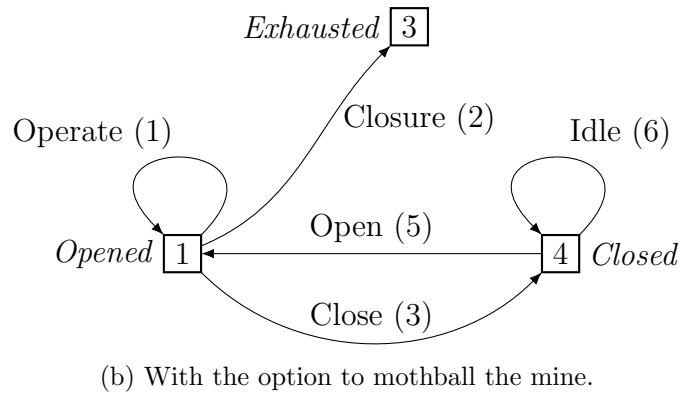
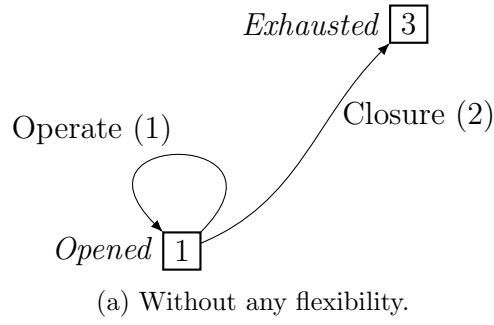


Figure 4.5: Influence diagrams for different copper mine settings.

abandon to the option to mothball the mine always increases in X_0 . This indicates that adding strategic flexibility (to limit downside risk by abandoning the mine early) to the mine that already has operational flexibility (to exploit upside risk by deviating from the immediate extraction of copper) is more valuable in this portfolio context than the other way around.

Although Brennan and Schwartz (1985) attempted to provide some insights into the valuation of this small options portfolio, there are some technical errors in their analysis and the respective results the authors presented are unfortunately incorrect. Table 4.8 reports the results of their analysis. According to Brennan and Schwartz (1985), the

Table 4.8: Value (in US\$ millions) of copper mine for different copper price levels according to Brennan and Schwartz (1985).

Copper Price (US\$/lbs) X_0	Mine Value		Value of Fixed-Output- Rate Mine (4)	Value of Closure Option (5)
	Open (2)	Closed (3)		
0.30	1.25 [†]	1.45	0.38	1.07
0.40	4.15 [†]	4.35	3.12	1.23
0.50	7.95	8.11	7.22	0.89
0.60	12.52	12.49	12.01	0.51
0.70	17.56	17.38	17.19	0.37
0.80	22.88	22.68 [‡]	22.61	0.27
0.90	28.38	28.18 [‡]	28.18	0.20
1.00	34.01	33.81 [‡]	33.85	0.16

[†] Optimal to close mine.

[‡] Optimal to open mine.

relevant columns of their table are defined as follows:

“Column 4 gives the value of the mine assuming that it cannot be closed down but must be operated at the rate of 10 million pounds per year until the inventory is exhausted in 15 years. The difference between column 4 and the greater of the values shown in columns 2 and 3 represents the value of the option to close down or abandon the mine if the price of copper falls far enough. The value of this closure option is shown in column 5”.

However, while the values reported in columns (2) and (3) have been widely confirmed in the literature including this thesis (see Table 4.6), there are technical errors related to both the definitions of and values shown in columns (4) and (5).

Firstly, the value of the fixed-output-rate mine shown in column (4) is incorrect. As can be seen, the authors found, somewhat counter-intuitively, this value to be positive and

convex in X_0 for all copper prices under consideration. By contrast, as we would expect, we have found that the expected value of the mine assuming copper must be extracted immediately until the 15-year inventory is fully exhausted is highly negative if operating margins are low and is an increasing yet concave function of X_0 . As seen in column (–) of Table 4.7, the value of the fixed-output-rate mine is negative for copper prices of US\$ 0.50 per pound and below. The nature of our results is in line with (Cortazar et al., 2008), who performed a comparative static analysis and showed, considering a three-factor commodity model (Cortazar and Schwartz, 2003), that the expected NPV of the mine without any flexibility (i.e. of the fixed-output-rate mine) is negative for low spot prices of copper and only breaks even at a price between US\$ 0.50 and 0.52 per pound. In addition to noting that the value of the opened and closed mine with the option to switch is convex in X_0 , the figure shown by Cortazar et al. (2008) also seems to illustrate that the value of the fixed-output-rate mine is concave in X_0 , whilst converging to the value of the opened mine for high commodity prices.

Fortunately, the comprehensive numerical analysis presented here gives insights into what might have gone wrong. Comparing the values shown in column (4) of Table 4.8 with our results of Table 4.7 shows that there is some similarity between these values and the ones we have obtained for the value of the mine with the option to abandon – see column (a-ii). Even though our results tend to be slightly low-biased, particularly for low copper prices, in relation to theirs, e.g. we obtained a mine value of zero for $X_0 = 0.30$ compared with their US\$ 0.38 millions, the patterns are arguably very similar. Surprisingly, our hypothesis even seems to be confirmed by Brennan and Schwartz (1985) themselves, who stated in their paper, two paragraphs below the one we cited above, that:

“Ownership of a mine that is not currently operating involves three distinct types of decision possibilities or options: first, the decision to begin operations; second, the decision to close the mine when it is currently operating (and possibly to reopen it later), which we have referred to as the closure option; and third, the decision to abandon the mine early, before the inventory is exhausted.”

According to this statement, which is clearly inconsistent with the authors’ earlier definition presented above, the case of their fixed-output-rate mine represents actually a mine with the early abandonment option, which we have referred to as the option to abandon the mine, and their closure option does not correspond with “the option to close down or abandon the mine” (Brennan and Schwartz, 1985), which we have referred to as the option to switch, but rather with an option that looks like our option to temporarily mothball the mine. Using IDs to graphically illustrate this inconsistency, for the fixed-output-rate mine, Brennan and Schwartz (1985) have incorrectly considered the case that corresponds with the ID of Figure 4.5c instead of the correct ID shown by Figure 4.5a, and the flexibility of their closure option, which is problematic in itself as discussed in the following paragraph, corresponds somewhat with the ID of Figure 4.5b rather than with the one of Figure 4.3.

Secondly, not only is its value incorrect, but the closure option is also ill-defined. Given by the “difference between column 4 and the greater of the values shown in columns 2 and 3” (Brennan and Schwartz, 1985), the value of this closure option is determined by subtracting a benchmark mine value – i.e. the value of the “fixed-output-rate mine” – from the maximum value of two different mines – and opened and a closed one. However, performing this subtraction is misleading, even incorrect, as the minuend of this subtraction is inconsistent with its subtrahend. This is because the latter assumes the mine is opened at time $t = 0$ ($N_0 = 1$), whereas the former represents a mine that is either opened ($N_0 = 1$) or closed ($N_0 = 4$) at the beginning. Comparing apples and oranges, this is, as a consequence, an ill-definition regardless of the benchmark applied, that is whether the (real) value of the fixed-output-rate mine (see Figure 4.5a) or the value of the mine with the option to abandon (see Figure 4.5c) is being used since both have the same initial state. Moreover, since their benchmark in column (4) seemingly corresponds, albeit inaccurately, with the value of the mine with the option to abandon, the values shown in column (5) do not represent, as implied by the authors’ definition, the value of the closure option in isolation but instead within the portfolio that contains the early abandonment option. Our analysis – the relevant values are given by column (6) of Table 4.7 – seems to confirm this when taking into account the above mentioned bias and ill-definition.

4.3 Re-evaluating Natural Resource Investments under Different Model Dynamics

4.3.1 Problem Setting

In this section we extend the example of valuing a copper mine presented in the previous section by considering three stochastic factors (i.e. random variables). In doing so we apply the well-known and highly cited three-factor model of Schwartz (1997), which builds upon the, also highly cited, two-factor model of Gibson and Schwartz (1990). The effects of both models have been investigated in the context of the original copper mine example by Tsekrekos et al. (2012), who applied the LSM approach. While the original copper mine example of Brennan and Schwartz (1985) considered a small portfolio of interdependent real options (option to temporarily mothball the mine and option to irreversibly abandon the mine), it only treated the commodity price, i.e. the price of copper, to be stochastic. Here we extend their example by additionally considering both the instantaneous convenience yield and the instantaneous interest rate to be stochastic. As such, in terms of options portfolio considered this setting is the same as the one described in Subsection 4.2.1.

In terms of uncertainties considered, however, we replace the one-factor setting of Brennan and Schwartz (1985) by the three-factor model of Schwartz (1997). Let the copper

spot price, the instantaneous convenience yield, and the instantaneous interest rate at time t be denoted by X_t , δ_t , and r_t , respectively. As described in (Tsekrekos et al., 2012), the discretised versions of the stochastic processes for the three stochastic factors of Schwartz (1997) are:

$$X_{t+\Delta} = X_t \exp \left\{ \left(r_t - \delta_t - \frac{\sigma_x^2}{2} \right) \Delta + \sigma_x \sqrt{\Delta} \epsilon_{t+\Delta}^x \right\}, \quad (4.26)$$

$$\delta_{t+\Delta} = \left(1 - e^{-\kappa_\delta \Delta} \right) \theta_\delta + e^{-\kappa_\delta \Delta} \delta_t + \sigma_\delta \sqrt{\frac{1 - e^{-2\kappa_\delta \Delta}}{2\kappa_\delta}} \epsilon_{t+\Delta}^\delta, \quad (4.27)$$

$$r_{t+\Delta} = \left(1 - e^{-\kappa_r \Delta} \right) \theta_r + e^{-\kappa_r \Delta} r_t + \sigma_r \sqrt{\frac{1 - e^{-2\kappa_r \Delta}}{2\kappa_r}} \epsilon_{t+\Delta}^r, \quad (4.28)$$

where Δ is the time step; σ_x , σ_δ and σ_r are the standard deviations of changes in X_t , δ_t and r_t , respectively; κ_δ and κ_r are positive mean reversion (speed of adjustment) coefficients; θ_δ and θ_r are the long run mean of convenience yield and interest rate, respectively; and $\epsilon_{t+\Delta}^x$, $\epsilon_{t+\Delta}^\delta$ and $\epsilon_{t+\Delta}^r$ are correlated standard normal random variables (mean 0, variance 1). Note that the two-factor model (copper price and convenience yield are stochastic) of Gibson and Schwartz (1990) is nested in the above presented three-factor model, and is obtained by making the interest rate constant, i.e. by setting $r_t = r, \forall t \in \mathcal{T}$.

4.3.2 Modelling

The modelling of this investment problem is to a large extent identical to the modelling of the previous example presented in Subsection 4.2.2. However, adaptations are necessary in the following two areas: the information state and its transition function. Let the copper spot price, the instantaneous convenience yield, and the instantaneous interest rate at time t along path ω be denoted by $X_t(\omega)$, $\delta_t(\omega)$, and $r_t(\omega)$, respectively. The information state component is then given by $I_t = (X_t, \delta_t, r_t)$ and represents the set of discrete random numbers $\{X_t(\omega), \delta_t(\omega), r_t(\omega) : \omega \in \Omega\}$, where Ω is the set of realisations. Hence, $S_t = (t, N_t, Q_t, X_t, \delta_t, r_t)$. The information state I_t evolves to $I_{t+\Delta_h}$ according to the following joint stochastic process:

$$X_{t+\Delta_h} = X_t \exp \left\{ \left(r_t - \delta_t - \frac{\sigma_x^2}{2} \right) \Delta_h + \sigma_x \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^x \right\}, \quad (4.29)$$

$$\delta_{t+\Delta_h} = \left(1 - e^{-\kappa_\delta \Delta_h} \right) \theta_\delta + e^{-\kappa_\delta \Delta_h} \delta_t + \sigma_\delta \sqrt{\frac{1 - e^{-2\kappa_\delta \Delta_h}}{2\kappa_\delta}} \epsilon_{t+\Delta_h}^\delta, \quad (4.30)$$

$$r_{t+\Delta_h} = \left(1 - e^{-\kappa_r \Delta_h} \right) \theta_r + e^{-\kappa_r \Delta_h} r_t + \sigma_r \sqrt{\frac{1 - e^{-2\kappa_r \Delta_h}}{2\kappa_r}} \epsilon_{t+\Delta_h}^r, \quad (4.31)$$

with correlation matrix (= covariance matrix Σ here, see Glasserman (2003)):

$$\begin{pmatrix} 1 & \rho_{x,\delta} & \rho_{x,r} \\ \rho_{x,\delta} & 1 & \rho_{r,\delta} \\ \rho_{x,r} & \rho_{r,\delta} & 1 \end{pmatrix}$$

4.3.3 Valuation

For the valuation of this extended copper mine example, we used the parameter values presented in Subsection 4.2.3 for the copper mine and, to ensure comparability, of Tsekrekos et al. (2012) for the three-factor model. For the sake of the numerical analysis presented here, yet without loss of generality, we focus on the three combinations of parameters of the convenience yield process shown in Table 4.9. These three specifications correspond

Table 4.9: Parameters of convenience yield process for different specifications as of Tsekrekos et al. (2012).

Spec.#	$\theta_\delta = \delta_0$	σ_δ	κ_δ
1	0.01	0.05	0.30
11	0.01	0.10	0.50
21	0.01	0.15	0.80

with the 1st, 11th, and 21st specification of Tables 3-6 of Tsekrekos et al. (2012) and are the most relevant specifications used by the authors, who analysed 81 different specifications in total. Our choice is sufficient for this analysis, more specifically, for studying the effects of different $\rho_{x,\delta}$ -values on the investment value under both the two- and three-factor model. Additional parameters used for the three-factor model are: $\kappa_r = 0.50$, $\theta_r = r_0 = 0.10$, $\sigma_r = 0.015$, $\rho_{r,\delta} = 0.10$ and $\rho_{x,r} = 0.15$. Also, as Tsekrekos et al. (2012), we considered 100,000 paths (half of which antithetic) and the complete set of polynomials in the parametric model, but unlike the authors used generalised Chebyshev polynomials with $L = 5$.

The adapted forward induction procedure consists of the following steps:

1. Determine the set of decision times, \mathcal{T}_{N_t} , for all decisions nodes $N_t \in \{1, 4\}$, forming subsets of \mathcal{T} :

$$\mathcal{T}_{N_t} = \begin{cases} \{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T^{max}\}, & \text{if } N_t = 1, \\ \{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, \Delta_1 \leq i\Delta_1 \leq T^{max}\}, & \text{if } N_t = 4, \end{cases} \quad (4.32)$$

2. Use (4.30) and (4.31) to sample $|\Omega|$ paths of δ_t and r_t , respectively, giving

$$(\delta_t(\omega), r_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$$

3. Use (4.29) to sample $|\Omega|$ paths of X_t giving $(X_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$

4. Generate the possible resource state space \mathcal{R}_t for each decision node and decision time:

$$\mathcal{R}_t = \begin{cases} \{(t, 1, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) \leq i\Delta_1 q \leq Q_0 - q \min(\Delta_1, t)\}, & \text{if } t \in \mathcal{T}_1, \\ \{(t, 4, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) + \Delta_1 q \leq i\Delta_1 q \leq Q_0\}, & \text{if } t \in \mathcal{T}_4. \end{cases} \quad (4.33)$$

Figures 4.6a, 4.6b and 4.6c show the evolution of X_t , δ_t and r_t , respectively, for five generated paths using $X_0 = 0.70$, which was used throughout this section. To simplify the

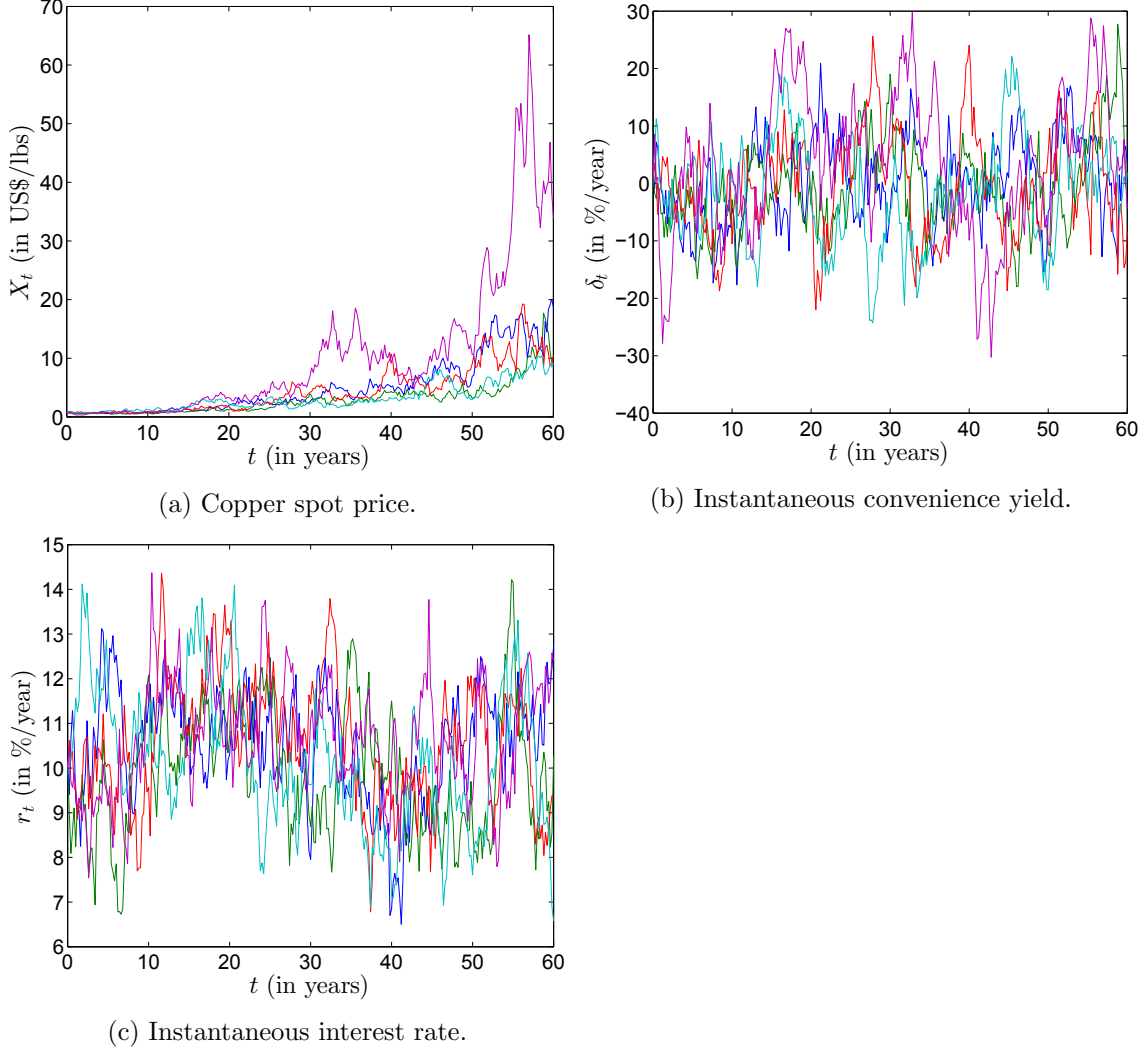


Figure 4.6: Selection of 5 equally likely paths for the evolution of the three stochastic factors with specification # 11 and $\rho_{x,\delta} = 0.60$.

presentation of the following Results and Discussion section, without loss of generality, we only present results for the opened mine, i.e. considering $N_0 = 1$ at $t = 0$.

4.3.4 Results and Discussion

The main purpose of this extended copper mine example is to operationalise our approach in the context of an investment project that represents a portfolio of interdependent real

options in a multi-dimensional setting. This subsection begins with an investigation of the way in which the value of the (opened) mine is affected by the different dynamics and stochastic behaviours of the two presented models. Tables 4.10 and 4.11 summarise the results under the two-factor model of Gibson and Schwartz (1990) – described by (4.29)-(4.30) with $r_t = r_0, \forall t \in \mathcal{T}$ – and the three-factor model of Schwartz (1997) given by (4.29)-(4.31), respectively, whilst considering the three specifications of Table 4.9 and three different values of the correlation between the copper price and convenience yield process, $\rho_{x,\delta}$. In addition, these tables report the corresponding results from Tsekrekos

Table 4.10: Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), under the two-factor model for the three specifications of Table 4.9 according to different numerical methods.

Spec.#	Our results			Tsekrekos et al. (2012) [†]		
	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$
1	16.210	13.960	11.958	22.625	23.221	25.060
11	16.703	13.518	10.768	23.497	25.095	27.271
21	16.328	13.239	10.500	23.532	24.079	25.789

[†] Obtained from Table 3 on page 552.

Table 4.11: Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), under the three-factor model for the three specifications of Table 4.9 according to different numerical methods.

Spec.#	Our results			Tsekrekos et al. (2012) [†]		
	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$
1	16.301	14.043	12.042	24.072	24.865	25.225
11	16.759	13.587	10.844	23.747	25.245	28.174
21	16.403	13.300	10.580	24.545	24.576	26.177

[†] Obtained from Table 5 on page 557.

et al. (2012). Comparing their results with ours shows that results are noticeably different. Not only are our mine values consistently lower than theirs, they also exhibit the opposite behaviour with respect to changes in $\rho_{x,\delta}$. Indeed, our mine values decrease in $\rho_{x,\delta}$, whereas Tsekrekos et al. (2012) found values to be increasing in $\rho_{x,\delta}$. This behaviour is confirmed by the authors⁷, who stated that:

“For the given set of parameters for the short-rate process, project values are found to be increasing [...] in the correlation between spot price and convenience yield changes, [...] much like in Section 3 where interest rates were assumed constant.”

⁷By contrast, results reported in an earlier yet similar conference paper (Tsekrekos et al., 2003) of the same authors, which only considered the two-factor model though, are in agreement with our results as far as the effect of $\rho_{x,\delta}$ on the mine value is concerned; however, their mine values are problematic as discussed by Abdel Sabour and Poulin (2006).

In general, the impact of correlation on investment value depends “critically on the assumed stochastic process of the underlying” factors (Schwartz, 1997) and on the specific investment problem at hand. While the value of a portfolio of interdependent real options can be affected positively or negatively by correlation between the underlying stochastic factors (Brosch, 2008), there is a non-negative relationship between real option value and underlying volatility, so higher volatility in the underlying asset generally results in a higher option value as flexibility becomes more valuable (Dixit and Pindyck, 2012). Considering the volatility of commodity futures returns, Schwartz (1997) derived expressions that describe the volatilities implied by the two- and three-factor model, which, in the limiting case⁸, converge to:

$$\sigma_{M_2}^2 = \sigma_x^2 + \frac{\sigma_\delta^2}{\kappa_\delta^2} - \frac{2\rho_{x,\delta}\sigma_x\sigma_\delta}{\kappa_\delta}, \quad (4.34)$$

and

$$\sigma_{M_3}^2 = \sigma_x^2 + \frac{\sigma_\delta^2}{\kappa_\delta^2} + \frac{\sigma_r^2}{\kappa_r^2} - \frac{2\rho_{x,\delta}\sigma_x\sigma_\delta}{\kappa_\delta} + \frac{2\rho_{x,r}\sigma_x\sigma_r}{\kappa_r} - \frac{2\rho_{r,\delta}\sigma_\delta\sigma_r}{\kappa_\delta\kappa_r}. \quad (4.35)$$

Even though equations (4.34) and (4.35) describe the volatility of futures returns in the two- and three-factor model, respectively, so consider commodity (i.e. copper) futures contracts rather than a natural resource investment project with managerial flexibility, it is reasonable to assume that the total volatility of the copper mine with the portfolio of options could be represented by a comparable functional relationship in terms of the two models’ parameters involved. In fact, since these were obtained by Schwartz (1997) from the solution to the partial differential equation that must be satisfied by futures prices in the respective model, the concept of contingent claims analysis (Schwartz, 1998) suggests that if the contingent claim is an investment project – in our case the copper mine with the options portfolio – instead of a futures contract, then term structure of the volatility may be obtained by expanding the valuation model’s partial differential equation(s) accordingly⁹. In this sense, it can be expected that, in the two-factor setting, (4.34) reflects the mine project’s actual volatility more accurately than (4.35) does in the three-factor setting, because in the former δ_t affects the valuation only indirectly through X_t , whereas when using the more complex three-factor model r_t has both indirect (through X_t) and direct (as a discount factor) effects on the valuation.

From (4.34) and (4.35) we can observe that the correlation coefficient $\rho_{x,\delta}$ negatively affects the volatility in both the two-factor ($\sigma_{M_2}^2$) and three-factor model ($\sigma_{M_3}^2$). As such, an increase in $\rho_{x,\delta}$ generally results in a decrease of the value of the mine as total volatility in the underlying decreases, which is, however, in contrast to what has been found by Tsekrekos et al. (2012). Intuitively, we would expect such a negative relationship con-

⁸For simplicity, we consider the case when time to maturity (of the futures contract) is infinity; the approximation error is negligible though.

⁹With regard to their valuation model, Brennan and Schwartz (1985) noted “in general there exists no analytic solution to the valuation model, though it is straightforward to solve it numerically”

sidering the way in which δ_t of (4.30) is nested in the dynamics of X_t in (4.29). For non-negative¹⁰ $\rho_{x,\delta}$, Figures 4.7 and 4.8 plot both the value of an opened mine ($\bar{G}_0(S_0)$) and the volatilities implied by the two- and three-factor model, respectively, as a function of $\rho_{x,\delta}$. It can be seen from these figures that the implied volatilities $\sigma_{M_2}^2$ and $\sigma_{M_3}^2$ and hence $\bar{G}_0(S_0)$ decrease as $\rho_{x,\delta}$ increases, for the three specifications under consideration. While the implied volatilities decrease linearly in $\rho_{x,\delta}$, as evident from (4.34)-(4.35), $\bar{G}_0(S_0)$ is a nonlinear function of $\rho_{x,\delta}$ and it is apparent that the decline in mine value, $\frac{\partial \bar{G}_0(S_0)}{\partial \rho_{x,\delta}}$, is larger – i.e. more negative – for lower (higher) values of $\rho_{x,\delta}$ ($\sigma_{M_2}^2$ and $\sigma_{M_3}^2$). This is consistent with the results reported in Figure B.1a and the nonlinear relationship¹¹ is in line with the discussion of Schwartz (1997), who predicted that

“When the option element of the investment is considered, the values obtained under the different models will be nonlinear functions of the spot price (and also of the other factors in the particular model)”

To further the numerical analysis of the impact of the correlation coefficient $\rho_{x,\delta}$ on the mine value we perform an equilibrium analysis of the three stochastic models. In doing so we investigate the influence of parameters of the convenience yield (σ_δ , κ_δ , $\rho_{x,\delta}$ and $\rho_{r,\delta}$) and interest rate process (σ_r , κ_r , and $\rho_{x,r}$) described by (4.30) and (4.31), respectively, on the implied volatilities of (4.34)-(4.35). In fact, we can eliminate the contribution of the convenience yield process to $\sigma_{M_2}^2$ of (4.34) and the contributions of both the convenience yield and interest rate process to $\sigma_{M_3}^2$ of (4.35) by determining the values of $\rho_{x,\delta}$ at which both $\sigma_{M_2}^2$ and $\sigma_{M_3}^2$ equal σ_x^2 . In other words, we can find the respective $\rho_{x,\delta}$ -values such that the sum of the 2nd and 3rd term of the right hand side of (4.34) becomes zero, and such that the sum of the 2nd to 6th term of the right hand side of (4.35) becomes zero, thus having $\sigma_{M_2}^2 = \sigma_x^2$ and $\sigma_{M_3}^2 = \sigma_x^2$. Analytical expressions for these values of the correlation coefficient $\rho_{x,\delta}$, which we refer to as equilibrium correlations, are given by:

$$\rho_{x,\delta}^* = \frac{\sigma_\delta}{2\kappa_\delta\sigma_x}, \quad (4.36)$$

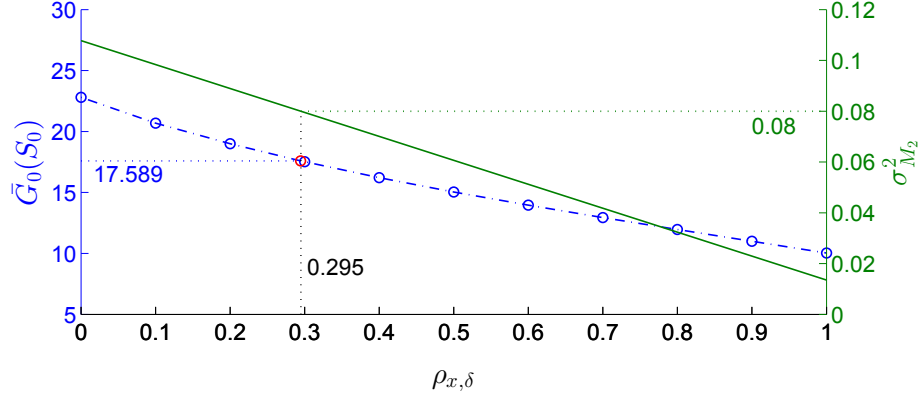
and

$$\rho_{x,\delta}^* = \frac{\sigma_\delta}{2\kappa_\delta\sigma_x} + \frac{\sigma_r}{\kappa_r} \left[\frac{\sigma_r\kappa_\delta}{2\kappa_r\sigma_\delta\sigma_x} + \frac{\rho_{x,r}\kappa_\delta}{\sigma_\delta} - \frac{\rho_{r,\delta}}{\sigma_x} \right]. \quad (4.37)$$

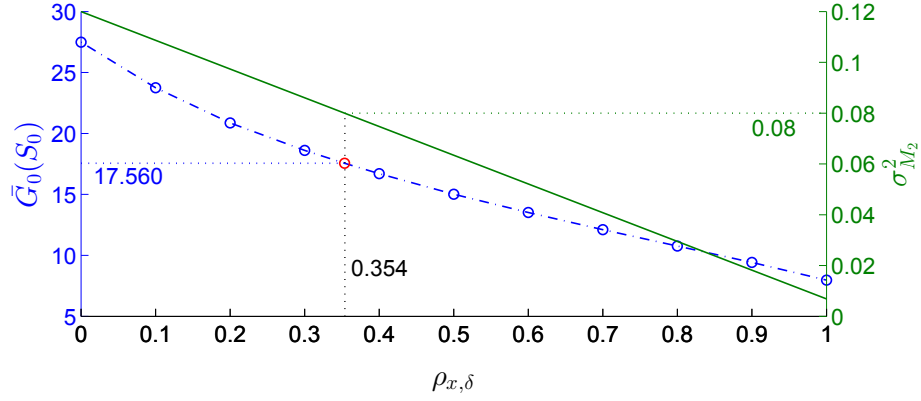
Hence, when $\rho_{x,\delta}$ equals the respective $\rho_{x,\delta}^*$ then the volatilities in the two- and three-factor model equal the copper price variance, σ_x^2 , of the one-factor model of Brennan and Schwartz (1985), in which only the price of copper (X_t) is stochastic. The mine value from (Brennan and Schwartz, 1985) is therefore used as benchmark in this equilibrium analysis.

¹⁰As argued for by Schwartz (1997).

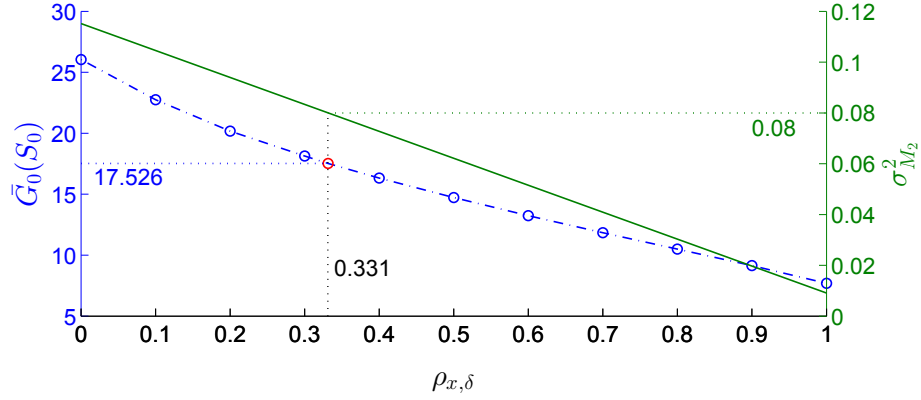
¹¹It should be noted that it is not entirely clear why Tsekrekos et al. (2012) obtained different results. Intuitively, one might expect results to converge at $\rho_{x,\delta} = 0$. However, comparing their results in Tables 4.10 and 4.11 with ours of Figures 4.7 and 4.8, respectively, indicates that our mine values are substantially larger than their values at $\rho_{x,\delta} = 0$, which, although not reported by the authors, can be estimated through extrapolation.



(a) Specification # 1.



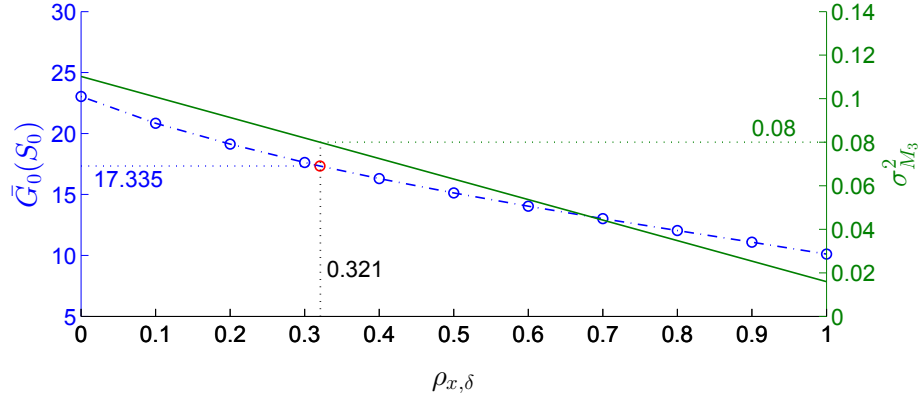
(b) Specification # 11.



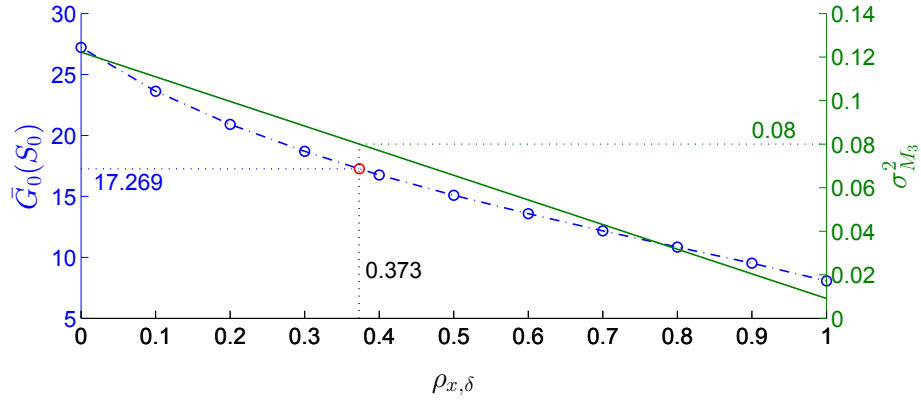
(c) Specification # 21.

Figure 4.7: Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), and volatility in two-factor model ($\sigma_{M_2}^2$) as a function of correlation between convenience yield and copper price process ($\rho_{x,\delta}$).

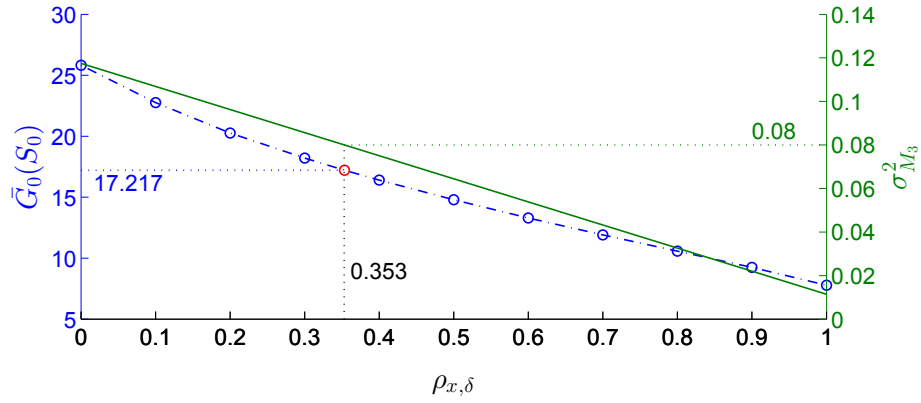
The results of our equilibrium analysis are shown by Table 4.12 and are also included in Figures 4.7 and 4.8. It can be observed in Table 4.12 that the values of opened mine, $\bar{G}_0(S_0)$, under the two-factor model converge very closely to the benchmark mine value, $\hat{G}_0(S_0)$, for all three specifications under consideration. Even though mine values under



(a) Specification # 1.



(b) Specification # 11.



(c) Specification # 21.

Figure 4.8: Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), and volatility in three-factor model ($\sigma_{M_3}^2$) as a function of correlation between convenience yield and copper price process ($\rho_{x,\delta}$).

the three-factor model are marginally below the benchmark value from the one-factor model, these results are in line with the previously mentioned (to be expected) differences in quality of the implied volatilities as predictors of the mine project's actual volatility due to the higher complexity of the three- over the two-factor model as well as other

Table 4.12: Results of equilibrium analysis for both the two- and three-factor model with the three specifications of Table 4.9.

Spec.#	One-factor model [†]		Two-factor model			Three-factor model [‡]		
	σ_x^2	$\hat{G}_0(S_0)$	$\rho_{x,\delta}^*$	$\sigma_{M_2}^2$	$\bar{G}_0(S_0)$	$\rho_{x,\delta}^*$	$\sigma_{M_3}^2$	$\bar{G}_0(S_0)$
1	0.08	17.56	0.295	0.08	17.589	0.321	0.08	17.335
11	0.08	17.56	0.354	0.08	17.560	0.373	0.08	17.269
21	0.08	17.56	0.331	0.08	17.526	0.353	0.08	17.217

[†] Obtained from Table 2, row $X_0 = 0.70$, of (Brennan and Schwartz, 1985).

[‡] With $\kappa_r = 0.50$, $\theta_r = r_0 = 0.10$, $\sigma_r = 0.015$, $\rho_{r,\delta} = 0.10$ and $\rho_{x,r} = 0.15$.

influencing factors. These factors are related to both the numerical procedure applied here and non-linearities in parameters such as X_0 , δ_0 and r_0 (e.g., see Figure B.1a).

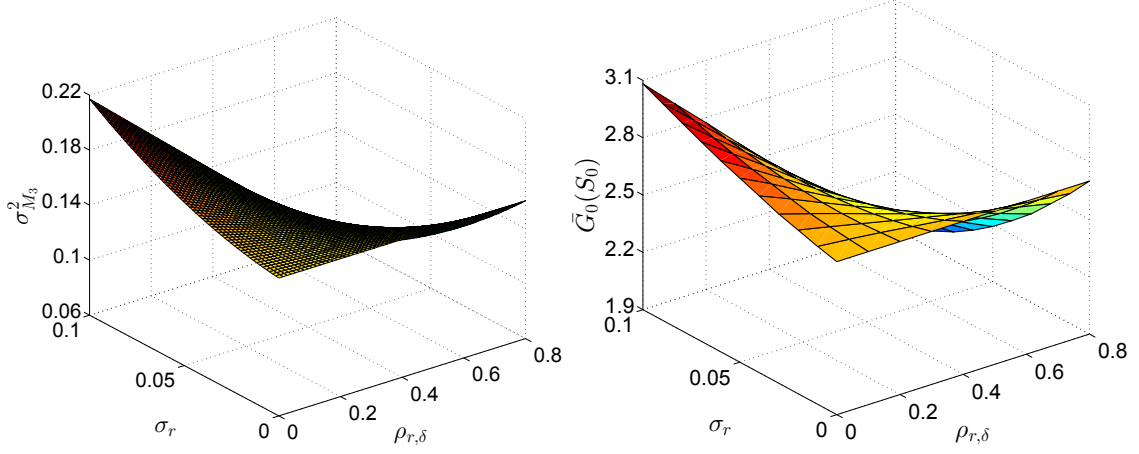
According to the above analysis, consistent with option pricing theory, the value of the opened copper mine decreases in the correlation coefficient $\rho_{x,\delta}$ as a consequence of the decrease in volatility in the models that describe the evolution of the underlying stochastic factors. Tsekrekos et al. (2012) also claimed that “values under a stochastic mean reverting convenience yield will be higher than those under a constant convenience yield assumption”. Unlike claimed by the authors, our analysis demonstrates that this is not always the case. As shown by our equilibrium analysis and indicated in Figures 4.7 and 4.8, for $\rho_{x,\delta}$ -values below the equilibrium correlation ($0 \leq \rho_{x,\delta} < \rho_{x,\delta}^*$), $\bar{G}_0(S_0)$ -values under both models are indeed higher than the benchmark mine value under the one-factor model, $\hat{G}_0(S_0)$, which assumes a constant convenience yield. At $\rho_{x,\delta} = \rho_{x,\delta}^*$, we approximately have $\bar{G}_0(S_0) = \hat{G}_0(S_0)$. However, for $\rho_{x,\delta}^* < \rho_{x,\delta} \leq 1$, mine values $\bar{G}_0(S_0)$ are lower than $\hat{G}_0(S_0)$ and, as $\rho_{x,\delta}$ approaches 1, these are even considerably lower than the constant benchmark, which was obtained under a deterministic convenience yield setting and is therefore independent of $\rho_{x,\delta}$.

With regard to the three-factor model, Tsekrekos et al. (2012) have also analysed how variations in both the standard deviation of changes in the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$) affect the value of the opened mine. The authors stated¹²:

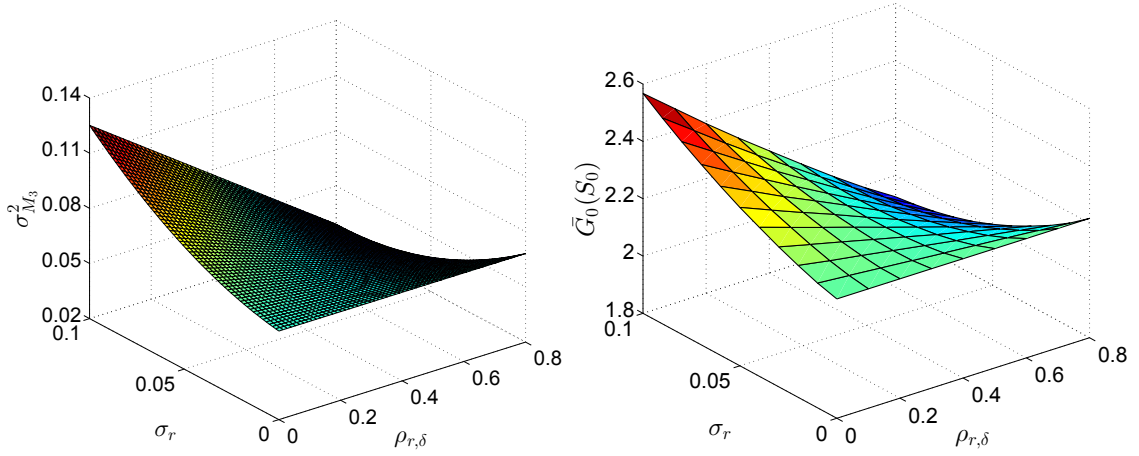
“Moreover, Panel (a) of Figure 3 demonstrates that the value of the investment is increasing in the volatility of the short rate and its correlation with convenience yield changes, since higher variability in expected project cash flows makes the flexibility to alter the operating mode of the project more valuable.”

We also performed this analysis and report results for θ_δ equalling 0.15 and 0.12 in Figures 4.9 and B.2, respectively. The figures on the left-hand (right-hand) side show $\sigma_{M_3}^2$ -values

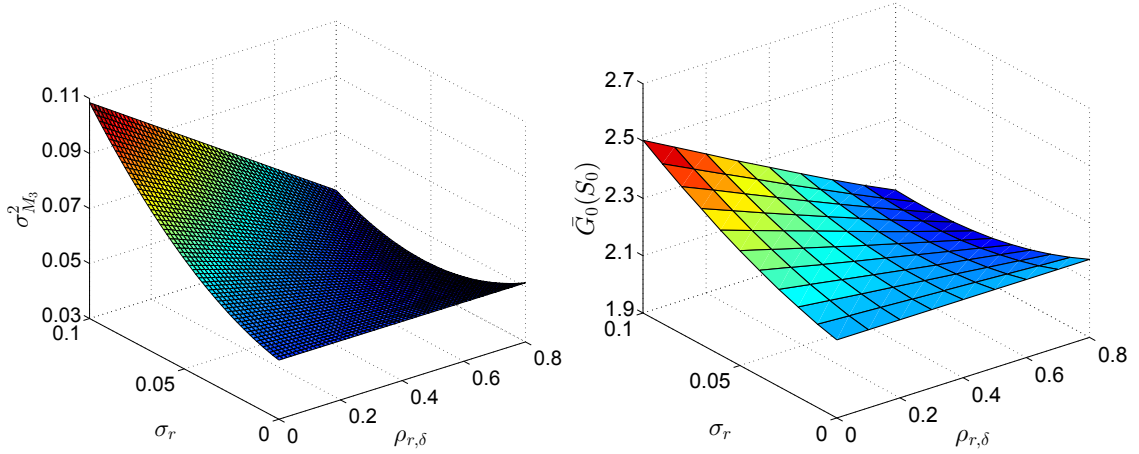
¹²It should, however, be noted that their statement is not consistent with their illustration because from their Figure it can be seen that the investment value is actually decreasing in the correlation coefficient.



(a) With $\kappa_\delta=0.30$.



(b) With $\kappa_\delta=0.50$.



(c) With $\kappa_\delta=0.80$.

Figure 4.9: Volatility in three-factor model ($\sigma^2_{M_3}$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.15$.

($\bar{G}_0(S_0)$ -values). Since the authors' choice of κ_δ is unclear, we ran our valuation algorithm for all three possible κ_δ -values of 0.30, 0.50 and 0.80, with results shown in Figures 4.9a, 4.9b and 4.9c, respectively. Of these, it appears that the results for $\kappa_\delta = 0.80$ are qualitatively most similar to the ones of Tsekrekos et al. (2012). It is evident that the mine value surfaces obtained here are in exceptionally good agreement with the volatility surfaces implied by the three-factor model. For $\sigma_r = 0$, $\bar{G}_0(S_0)$ -values are constant because $\sigma_{M_3}^2$ is, as evident from (4.35), independent of $\rho_{r,\delta}$. Also, as is apparent from Panel (a) of their figure, we also find that $\bar{G}_0(S_0)$ -values are decreasing in $\rho_{r,\delta}$.

In contrast to Tsekrekos et al. (2012), however, our results demonstrate that the investment value is not always increasing in the volatility of the interest rate process. As we can see from Figures (4.9a) and (4.9b), which consider $\kappa_\delta=0.30$ and $\kappa_\delta=0.50$, respectively, $\bar{G}_0(S_0)$ -values are increasing in σ_r for low values of $\rho_{r,\delta}$, yet decreasing for relatively high $\rho_{r,\delta}$ -values. Interestingly, we observe from Figure 4.9c ($\kappa_\delta=0.80$) that while $\bar{G}_0(S_0)$ -values increase in σ_r for the four lowest $\rho_{r,\delta}$ -values under consideration, there is a twofold effect of the degree of σ_r on the investment value for $0.4 \leq \rho_{r,\delta} \leq 0.8$: $\bar{G}_0(S_0)$ -values actually decrease in σ_r for low σ_r -values, but increase in σ_r for high σ_r -values; this change from decrease to increase seems to occur at higher σ_r -values the higher the value of the correlation coefficient $\rho_{r,\delta}$. The evidence provided by our analysis, particularly the illustration on the left-hand side of Figure 4.9c, seems to confirm that the nonlinear dependency of $\sigma_{M_3}^2$ on σ_r is the cause of this twofold effect. It can be inferred therefore that the volatility surfaces constructed using (4.35) can accurately describe how the value of the opened mine will be affected by changes in parameters of the stochastic processes.

4.4 Discussion

In this section, we discuss the significance and validity of the proposed portfolio of real options approach as well as its relationship to existing approaches used for modelling and valuing real options and, more generally, investments under uncertainty. We believe that the insights gained by this discussion are valuable not only in the fields of ROA and investment under uncertainty, but also when decision-makers need to select the best approach to model and value risky investment propositions.

In this work, we have developed a holistic approach for the modelling and valuation of portfolios of interdependent real options using both IDs and simulation-and-regression. Despite having many advantages as a framework to represent decision problems, IDs have rarely been used in the context of ROA. A reason for this might be, as Wallace (2010) suggests, that real option analysts, like their financial counterparts, are generally interested in determining the value of a single well-defined option, rather than identifying and defining the portfolio of options. This focus on valuing single options is perhaps derived from financial option theory, which addresses decision making problems in which the representation

of the investment proposition requires less sophistication than when considering complex physical assets. For example, as we have shown in Section 4.1, the ID (see Figure 4.1) that represents the sequential decision problem underlying the valuation of an American put option – i.e. optimal stopping problem – is quite simple and straightforward. However, realistic and practical real option problems are generally more complex, so their analysis benefits from the more sophisticated representation of their underlying decision problems that can be addressed via IDs. Indeed, as we have demonstrated in Section 4.2 for the valuation of a natural resource investment, the flexibilities available to decision makers can be simply and intuitively represented graphically by an ID (see Figure 4.3).

In order to approximate the value of portfolios of interdependent real options, we applied simulation in combination with parametric regression. The simulation procedure consists of directly modelling the risk-neutral dynamics of all the underlying uncertainties of the investment project, and then using simulation to generate sample paths that describe the evolution of all uncertainties over time. While more advanced parametric models may be considered, the valuation algorithm presented in Section 3.3 used a parametric regression model and a least-squares method to approximate the complex continuation functions that describe the expected future contributions associated with transitions in the ID. These approximations are then used in our portfolio optimisation framework to determine optimal decisions for all available transitions at each possible state, subject to the constraints that describe interdependencies. In contrast, the valuation approach of Gamba (2003), which also applied simulation and parametric regression whilst considering four types of strategic interdependencies between real options, only presented decision rules for each of the four cases to be applied within a valuation procedure, thereby decomposed the valuation problem into sub-problems that can be analysed individually. By contrast, our portfolio approach, which can be easily implemented¹³ and efficiently applied (e.g. using parallel computing), represents a single, coherent and flexible valuation framework to approximate the value of portfolios of interdependent real options.

A controversial issue in the real options community is whether to apply option pricing or decision analysis approaches. Adequately evaluating complex investments under uncertainty involving real options requires the proper modelling of the corresponding sequential decision problem, including a representation of both the flexibilities available to the decision maker and the investment project’s underlying uncertainties. Only then is it possible to devise and apply adequate and powerful strategies and algorithms for the computational valuation of complex and risky investments like portfolios of real options. We agree with Wallace (2010) in that option pricing theory has traditionally tended to focus on valuation, whilst neglecting the modelling of the underlying sequential decision problem, whereas decision analysis and the related tools generally have the decision context as a starting point. An example of this can be found in Christiansen and Wallace (1998), who

¹³Requirements are a standardly available solver (integer programming) and simple least squares.

compared a decision analysis approach (decision tree solved via dynamic programming) and an option pricing approach (valuation by arbitrage via a replication argument) using a simple example, and showed that although both approaches deliver the same result, they are methodologically different, with the latter focusing on determining the optimal value whilst delivering the optimal decisions as a consequence, and vice versa.

So are option pricing and decision analysis approaches just two sides of the same coin? This question not only highlights one of the more contentious debates in the field of ROA, but also implies that the approach taken here of directly modelling the risk-neutral dynamics of all the underlying uncertainties may well be the more appropriate approach. Smith and Nau (1995) were one of the first to integrate an option pricing (contingent claims analysis) and a decision analysis method (decision tree) whilst distinguishing between the nature of risks (private, market, or a mix). Addressing the work of Smith and Nau (1995), Brandão et al. (2005b) claim that there exist investment projects where “uncertainties fall somewhere in between the notions of private and market risks”. To address this limitation, Brandão et al. (2005b) presented an alternative approach based on traditional decision analysis tools, and after Smith (2005) criticised their proposed approach from a fundamental perspective, Brandão et al. (2005a) responded by agreeing that a fully risk-neutral approach, which does not separate between public (hedged) and private (unhedged) risks, like the one proposed here, should be the first choice when valuing a risky project.

To conclude this discussion, we believe our proposed portfolio of real options approach directly addresses and responds to a number of open and important research questions in the field of ROA, both in terms of modelling and valuation. While there is certainly no “magic bullet” (Smith, 2005) for simplifying complex real option problems, this study presented a simple yet powerful approach to both model and value a portfolio of interdependent real options. Furthermore, given its generality and holistic nature, we believe the proposed approach presents an important contribution to a more credible portfolio theory for interdependent real options, thus addressing one of the long-standing research challenges identified by Trigeorgis (2005). Lastly, Triantis and Borison (2001) predicted convergence among “the ‘decision analytic’ and ‘option pricing’ approaches”. Based on the above discussions, we believe the proposed approach, which focuses equally on modelling and valuation, may be regarded as a decision analysis approach as well as an option pricing approach, and hence represents convergence among these approaches.

4.5 Summary

In this chapter we operationalise the framework presented in Chapter 3 in the context of three well-known relevant examples of increasing complexity. First, we consider the widely studied example of valuing an American put option in a simple single-factor setting. While this example does clearly not contain any portfolio characteristics nor does the put

option's value depend on multiple stochastic factors, it is perfectly suited to demonstrate the individual steps – i.e. modelling and valuation – of our proposed approach in a clear and easily comprehensible manner. Moreover, since real options with an American-style exercise feature are important in both the theory and practice of option pricing, the consideration of this example is also valuable from an academic perspective as well as contributes to a better understanding in more complex situations.

Secondly, we demonstrate the ability of the proposed approach to accurately and correctly value a slightly more complex real option portfolio by re-evaluating a natural resource investment. Proposed by Brennan and Schwartz (1985) and modelled/solved by the authors using PDEs/finite differences, and although still a single-factor setting (stochastic copper price), the flexibilities available in this copper mine indeed represent a portfolio of options: to temporarily mothball the mine and irreversibly abandon the mine. Using the approach presented here we obtain results for both the mine values and the switching decisions that are in line with the ones of their work and of other works. Quite unexpectedly, however, we detect and highlight some technical errors when re-assessing the results in (Brennan and Schwartz, 1985). Using the comprehensive portfolio analysis – enabled by our holistic approach –, we provide detailed insights into their flawed analysis and the reasons for this as well as evidence of the benefits of our portfolio of real options approach.

In the third example we replace the one-factor setting of the previous example of valuing the portfolio of options available in the copper mine by the three-factor model of Schwartz (1997). Investigated by Tsekrekos et al. (2012) using the LSM approach, this problem setting features both a portfolio of interdependent real options and three stochastic factors: the copper spot price, the convenience yield and the interest rate. Combining the seminal work of Brennan and Schwartz (1985) with the highly cited paper of Schwartz (1997), Tsekrekos et al. (2012) bridge an important gap in literature. Unfortunately, however, we detect several errors when re-examining some of the analyses of Tsekrekos et al. (2012). Applying our proposed numerical approach together with analytical expressions from literature that describe the volatilities in the stochastic models applied, we perform a creative and simple yet powerful equilibrium analysis that enables use to disprove the related findings of Tsekrekos et al. (2012) whilst strongly confirming our own results.

Finally, we also discuss our approach in the context of existing option pricing and decision analysis approaches. In doing so, we show that the theoretical grounding of existing approaches may determine and possibly restrict the way in which the investment project's underlying sequential decision problem and uncertainties are modelled and, subsequently, how the relevant investments are valued. In addition, this theoretical basis may limit the ability of existing approaches to tackle complex real option problems. This comparison with existing approaches demonstrates that our approach is not only more transparent and intuitive, but also resolves many important research challenges by creating a convergence between a decision analysis and an option pricing approach.

Chapter 5

Evaluating Complex Natural Resource Investments

In this chapter we present the application of the proposed portfolio of real options approach to the example of evaluating a complex natural resource investment. This chapter is organised as follows: Section 5.1 describes the investment problem whilst specifying both the portfolio of interdependent real options and the underlying uncertainties as well as states five hypotheses to be tested. The modelling of this investment problem as a sequential decision problem is described in Section 5.2, and the parameter values used and the forward induction procedure applied for valuation are presented in Section 5.3. Section 5.4 presents and discusses the results in the light of the five hypotheses. Finally, Section 5.5 summarises this chapter.

5.1 The Investment Problem

The problem considered in this chapter is a complex yet important and realistic extension (Savolainen, 2016) of the classic example of valuing a copper mine, which was originally proposed by Brennan and Schwartz (1985) and solved by the authors using a finite difference method, and which has been used by Abdel Sabour and Poulin (2006); Cortazar et al. (2008); Tsekrekos et al. (2012) as a benchmark to assess the LSM approach. The original copper mine example of Brennan and Schwartz has only considered a limited set of options (option to temporarily close the mine and possibly reopen as well as early abandonment option) and only treated the price of copper to be uncertain. Here we substantially extend their example by considering both a large portfolio of interdependent real options and four stochastic factors (or uncertain/random variables). It is important to note that traditional valuation methods (e.g. binomial/lattice and finite difference) are impractical for the problem considered here given its large size in terms of both portfolio of options and sources of uncertainty.

5.1.1 Portfolio of Interdependent Real Options

In terms of portfolio of real options considered, we extend the setting of Brennan and Schwartz by integrating the option to defer (or delay) the development of the copper mine as proposed by Gamba (2003) and valued by the author using the LSM approach; however, Abdel Sabour and Poulin (2006) discussed the findings of Gamba and showed that there are some inconsistencies related to both numerical results (mine value cyclic in copper price) and switching decisions obtained. In addition to the option to defer, our example takes into account the option to irreversibly expand production capacity of the copper mine, which was proposed by Cortazar and Casassus (1998) and solved by the authors using partial differential equations; however, the authors did not consider the option to abandon the mine project.

This investment's portfolio of interdependent real options is composed of the following single, well-defined options:

- (a) Option to defer development: Instead of developing the copper mine immediately (i.e. at time 0), the decision maker may choose to defer the development of the mine until the expiration of the right to develop the mine in T_1^{max} years, without incurring any direct costs associated with deferring; with I_t^d being the development cost at time t .
- (b) Option to switch (during operation – mode I): Once developed, the decision maker may decide to temporarily close the (operating) mine, to maintain or reopen the mine when it is currently closed, and/or to irreversibly abandon the copper mine before its inventory, with initial level Q_0 , is fully exhausted.
 - (i) Option to temporarily mothball the developed mine: If operation of the mine becomes unprofitable – e.g. because the copper spot price at time t , X_t , is too low in relation to the mine's production costs, A_t – the decision maker can close down the opened (i.e. operating) mine at a cost of $K_t^{c,I}$, maintain the closed mine at an annual maintenance cost of M_t^I , and, if the copper price becomes favourable again, reopen the closed mine at a cost of $K_t^{o,I}$ at time t .
 - (ii) Option to abandon the developed mine: Whether opened or closed, the decision maker retains the right to irreversibly abandon the copper mine at any time t without incurring any cost.
- (c) Option to expand: Once developed, and if operating, the decision maker can increase (scale up) the mine's annual output rate from q^I to q^{II} at a cost of I_t^e at time t , thereby expanding production capacity.
- (d) Option to switch (during operation – mode II): After the expansion, the (operating) copper mine can also be temporarily closed, maintained or reopened when closed, and/or abandoned altogether, before the inventory is exhausted.
 - (i) Option to temporarily mothball the expanded mine: As in (b-i), if immediate extraction becomes unprofitable, the opened mine can be closed down at a cost

of $K_t^{c,II}$, the closed mine can be maintained at an annual cost of M_t^{II} , and, if profitable again, the mine can be reopened at a cost of $K_t^{o,II}$ at time t .

- (ii) Option to abandon the expanded mine: Whether opened or closed, the decision maker retains the right to irreversibly abandon the expanded copper mine at no cost.

To improve the presentation of this chapter – i.e. the demonstration of the ability of the proposed approach to evaluate the above presented portfolio of interdependent real options under four sources of uncertainty –, several testable hypotheses are considered. The five hypotheses, tested using the subsequent numerical example, are: (H1) the value of the option to defer development (in isolation) decreases as the operating margin increases; (H2) switching is at least as valuable than both mothballing and abandoning; (H3) the option to expand is the only individual option whose value (if positive) increases as the operating margin increases; (H4) the value of the mine with the options portfolio is positively affected by the underlying volatility; (H5) the option to defer development is the most valuable individual option within the portfolio.

5.1.2 Characterisation of Uncertainties

In terms of uncertainties considered, we replace the one-factor setting considered in Brennan and Schwartz (stochastic copper price) by applying the three-factor model (copper price, instantaneous convenience yield and instantaneous interest rate) of Schwartz (1997), which extends the two-factor model of Gibson and Schwartz (1990). Both models have been analysed in the context of the original copper mine example by Tsekrekos et al. (2012) using the LSM approach. In addition to these three stochastic factors, we introduce a fourth factor by treating the extraction (production) costs of copper to be uncertain, as argued for by Slade (2001). Let the copper spot price, the instantaneous convenience yield, the instantaneous interest rate, and the (per-unit) production cost at time t be denoted by X_t , δ_t , r_t , and A_t , respectively. Table 5.1 summarises the stochastic factors considered in this example of a complex natural resource investment.

Table 5.1: Summary of stochastic factors considered in this example.

Description	Factor	Defining eq.	Dynamics	Driving process ^a
Copper spot price	X_t	(5.1)	Exogenous	GWN, correlated with (5.2),(5.3)
Convenience yield	δ_t	(5.2)	Exogenous	GWN, correlated with (5.1),(5.3)
Interest rate	r_t	(5.3)	Exogenous	GWN, correlated with (5.1),(5.2)
Production cost	A_t	(5.4)	Exogenous	GWN, independent of (5.1)-(5.3)

^a Gaussian white noise (GWN).

As reported by Tsekrekos et al. (2012), the discretised version of the joint stochastic

process for the three factors of Schwartz (1997) is given by:

$$X_{t+\Delta} = X_t \exp \left\{ \left(r_t - \delta_t - \frac{\sigma_x^2}{2} \right) \Delta + \sigma_x \sqrt{\Delta} \epsilon_{t+\Delta}^x \right\}, \quad (5.1)$$

$$\delta_{t+\Delta} = \left(1 - e^{-\kappa_\delta \Delta} \right) \theta_\delta + e^{-\kappa_\delta \Delta} \delta_t + \sigma_\delta \sqrt{\frac{1 - e^{-2\kappa_\delta \Delta}}{2\kappa_\delta}} \epsilon_{t+\Delta}^\delta, \quad (5.2)$$

$$r_{t+\Delta} = \left(1 - e^{-\kappa_r \Delta} \right) \theta_r + e^{-\kappa_r \Delta} r_t + \sigma_r \sqrt{\frac{1 - e^{-2\kappa_r \Delta}}{2\kappa_r}} \epsilon_{t+\Delta}^r, \quad (5.3)$$

where Δ is the time step; σ_x , σ_δ and σ_r are the standard deviations of changes in X_t , δ_t and r_t , respectively; κ_δ and κ_r are positive mean reversion (speed of adjustment) coefficients; θ_δ and θ_r are the long run mean of convenience yield and interest rate, respectively; and $\epsilon_{t+\Delta}^x$, $\epsilon_{t+\Delta}^\delta$ and $\epsilon_{t+\Delta}^r$ are correlated standard normal random variables (mean 0, variance 1). For the evolution of A_t , as suggested by Slade (2001), we consider a mean-reverting process, in particular we use an Euler approximation (e.g. see Glasserman (2003)) of the geometric mean reversion described by Metcalf and Hassett (1995) giving:

$$A_{t+\Delta} = \left(A_0 e^{\pi t} - A_t \right) \kappa_a A_t \Delta + e^{\pi \Delta} A_t + \sigma_a A_t \sqrt{\Delta} \epsilon_{t+\Delta}^a, \quad (5.4)$$

where π is the cost inflation rate, κ_a is a positive mean reversion coefficient, σ_a is the standard deviation of the production cost, and $\epsilon_{t+\Delta}^a$ is a standard normal random variable (mean 0, variance 1), which is assumed to be uncorrelated with the ones above and whose increments are independently and identically distributed.

5.2 Modelling

The flexibilities inherent in the copper mine project are represented by the ID in Figure 5.1. It contains 5 decision nodes (1, 3, 5, 7, and 8) and 3 terminal nodes (2, 4, and 6), as well as 18 transitions that link these nodes, resulting in $\mathcal{N} = \{1, 2, \dots, 8\}$ and $\mathcal{H} = \{1, 2, \dots, 18\}$. The duration of transition $h \in \mathcal{H}$ is Δ_h year(s). When the mine is *Undeveloped*, the decision maker may decide either to Defer (1) development or Develop (2) the mine, both of which can be done for up to T_1^{max} years, after which the right to develop the mine expires. Once developed and in mode *Opened-I*, the decision maker has to decide whether to Operate (4) for the duration of Δ_4 while extracting an amount $q^I \Delta_4$ of copper, irreversibly Expand (5) the mine operation by increasing extraction rate from q^I to q^{II} , temporarily Close (6), or irreversibly Abandon (7) the project. On the other hand, if the mine is *Closed-I* (or *Closed-II*), the available transitions are to keep the mine Idle (9 or 15), Open (8 or 14) it again, or irreversibly Abandon (10 or 16) the project. In either operating mode, however, the mine closes if the commodity inventory with initial

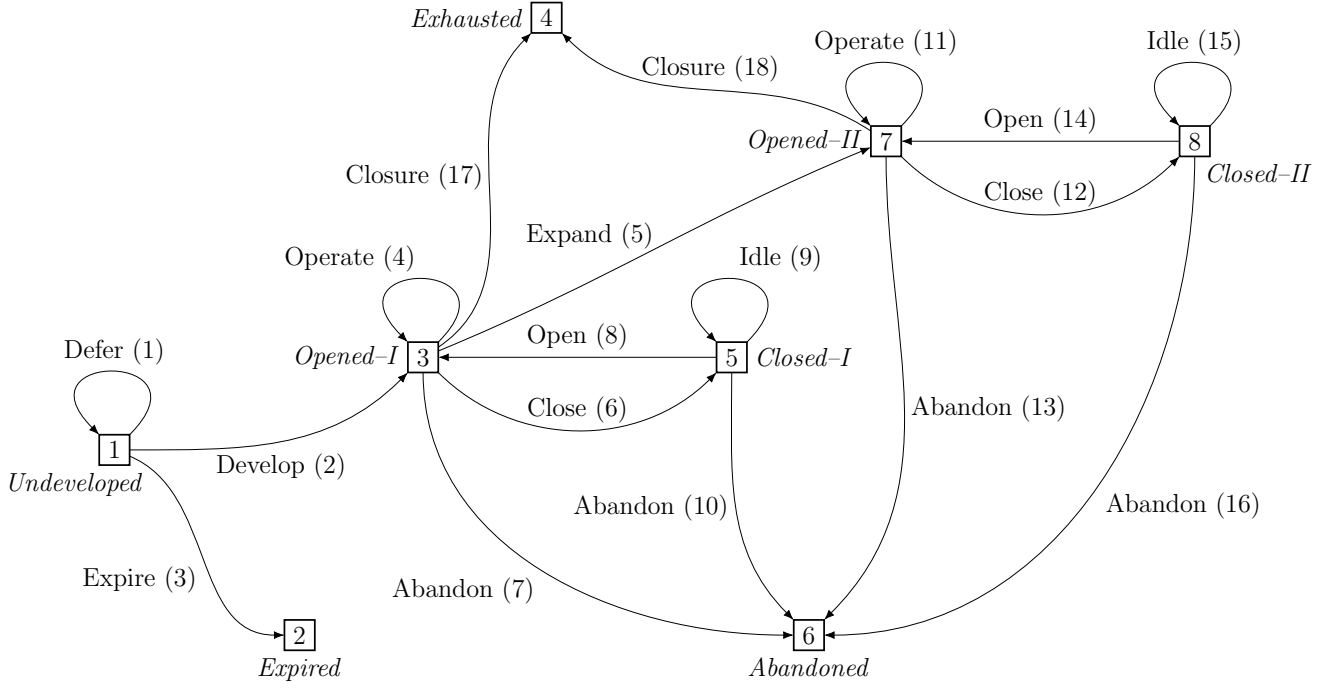


Figure 5.1: Influence diagram for the complex mine development project.

inventory Q_0 is fully depleted. Also, the project has to be *Abandoned* when reaching its lifetime of T_2^{max} years.

Let the decision node, the inventory of the mine, and the remaining time to develop the mine/lifetime of the mine at time t be denoted by N_t , Q_t , and T_t , respectively, as well as let the copper spot price, the instantaneous convenience yield, the instantaneous (risk-free) interest rate, and the (per-unit) production cost at time t along path ω be denoted by $X_t(\omega)$, $\delta_t(\omega)$, $r_t(\omega)$, and $A_t(\omega)$, respectively. Thus, the resource and information state component are given by $R_t = (t, N_t, Q_t, T_t)$ and $I_t = (X_t, \delta_t, r_t, A_t)$, respectively, with the latter representing the set of discrete random variables at time t , $\{X_t(\omega), \delta_t(\omega), r_t(\omega), A_t(\omega) : \omega \in \Omega\}$, where Ω is the set of sample realisations. Hence, the state at time t is then written as $S_t = (t, N_t, Q_t, T_t, X_t, \delta_t, r_t, A_t)$.

The binary decision variables related to the transitions available at decision node N_t at time t , $a_t = (a_{th})_{h \in b^D(N_t)}$, have to satisfy the feasible region \mathcal{A}_{S_t} , which is defined by the

following set of constraints:

$$\sum_{h \in b^D(N_t)} a_{th} = 1, \quad \forall N_t \in \mathcal{N} \setminus \{2, 4, 6\}, \quad (5.5)$$

$$a_{t1} T_1^{max} < T_1^{max} + T_t, \quad (5.6)$$

$$a_{t3} T_t = 0, \quad (5.7)$$

$$a_{th} q^I \Delta_h \leq Q_t, \quad \forall h \in \{4, 8\}, \quad (5.8)$$

$$a_{th} q^{II} \Delta_h \leq Q_t, \quad \forall h \in \{11, 14\}, \quad (5.9)$$

$$a_{th_1} + a_{th_2} + T_t > 0, \quad \forall (h_1, h_2) \in \{(7, 17), (13, 18)\}, \quad (5.10)$$

$$a_{th} + T_t > 0, \quad \forall h \in \{10, 16\}, \quad (5.11)$$

$$a_{th} + Q_t > 0, \quad \forall h \in \{17, 18\}, \quad (5.12)$$

$$a_{th} Q_t = 0, \quad \forall h \in \{17, 18\}, \quad (5.13)$$

where $a_{th} \in \{0, 1\}, \forall h \in \mathcal{H}$, and the action space is:

$$b^D(N_t) = \begin{cases} \{1, 2, 3\}, & \text{if } N_t = 1, \\ \{4, 5, 6, 7, 17\}, & \text{if } N_t = 3, \\ \{8, 9, 10\}, & \text{if } N_t = 5, \\ \{11, 12, 13, 18\}, & \text{if } N_t = 7, \\ \{14, 15, 16\}, & \text{if } N_t = 8, \\ \{\}, & \text{otherwise.} \end{cases} \quad (5.14)$$

Subsequently, the resource state R_t evolves deterministically to $R_{t+\Delta_h}$, with the transition of t being rather trivial as it simply evolves from t to $t + \Delta_h$ after having made transition h . The evolution of N_t is implicitly described by the adjacency matrix of the directed graph $(\mathcal{N}, \mathcal{H})$, which is not shown here for the sake of brevity. The evolution of Q_t and T_t is specified by the following transition equations for all $h \in \mathcal{H}$:

$$Q_{t+\Delta_h} = \max \left\{ Q_t - q^I \Delta_h (a_{t4} + a_{t8}) - q^{II} \Delta_h (a_{t11} + a_{t14}), 0 \right\}, \quad (5.15)$$

$$T_{t+\Delta_h} = \begin{cases} T_2^{max} - (T_1^{max} - T_t) - \Delta_2, & \text{if } a_{t2} = 1, \\ \max(T_t - \Delta_h, 0), & \text{otherwise,} \end{cases} \quad (5.16)$$

where $T_0 = T_1^{max}$. On the other hand, the information state I_t evolves stochastically to $I_{t+\Delta_h}$ under the risk-neutral measure according to the following discrete diffusion processes:

$$X_{t+\Delta_h} = X_t \exp \left\{ \left(r_t - \delta_t - \frac{\sigma_x^2}{2} \right) \Delta_h + \sigma_x \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^x \right\}, \quad (5.17)$$

$$\delta_{t+\Delta_h} = \left(1 - e^{-\kappa_\delta \Delta_h}\right) \theta_\delta + e^{-\kappa_\delta \Delta_h} \delta_t + \sigma_\delta \sqrt{\frac{1 - e^{-2\kappa_\delta \Delta_h}}{2\kappa_\delta}} \epsilon_{t+\Delta_h}^\delta, \quad (5.18)$$

$$r_{t+\Delta_h} = \left(1 - e^{-\kappa_r \Delta_h}\right) \theta_r + e^{-\kappa_r \Delta_h} r_t + \sigma_r \sqrt{\frac{1 - e^{-2\kappa_r \Delta_h}}{2\kappa_r}} \epsilon_{t+\Delta_h}^r, \quad (5.19)$$

$$A_{t+\Delta_h} = \left(A_0 e^{\pi t} - A_t\right) \kappa_a A_t \Delta_h + e^{\pi \Delta_h} A_t + \sigma_a A_t \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^a, \quad (5.20)$$

with correlation matrix (= covariance matrix Σ here, see Glasserman (2003)):

$$\begin{pmatrix} 1 & \rho_{x,\delta} & \rho_{x,r} \\ \rho_{x,\delta} & 1 & \rho_{r,\delta} \\ \rho_{x,r} & \rho_{r,\delta} & 1 \end{pmatrix}$$

The deterministic payoff obtained at time t when making decision a_t given I_t is:

$$\begin{aligned} \Pi_t(I_t, a_t) = & -I_t^d a_{t2} + [q^I(X_t - A_t) - f^I(X_t, A_t)] \Delta_h(a_{t4} + a_{t8}) - K_t^{c,I} a_{t6} - K_t^{o,I} a_{t8} \\ & - M_t^I \Delta_h(a_{t6} + a_{t9}) - I_t^e a_{t5} + [q^{II}(X_t - A_t) - f^{II}(X_t, A_t)] \Delta_h(a_{t11} + a_{t14}) \\ & - K_t^{c,II} a_{t12} - K_t^{o,II} a_{t14} - M_t^{II} \Delta_h(a_{t12} + a_{t15}), \end{aligned} \quad (5.21)$$

where $I_t^d = I_0^d e^{\pi t}$ is the development cost at time t , $f^I(X_t, A_t) = \tau_1 q^I X_t + \max\{\tau_2 q^I (X_t(1 - \tau_1) - A_t), 0\}$ is the sum of royalties and income tax paid at time t with τ_1 the royalty rate and τ_2 the income tax rate; $M_t^I = M_0^I e^{\pi t}$ is the maintenance cost at time t ; $K_t^{c,I} = K_0^{c,I} e^{\pi t}$ and $K_t^{o,I} = K_0^{o,I} e^{\pi t}$ are the costs to switch to the *Closed-I* and *Opened-I* node at time t , respectively; and $I_t^e = I_0^e e^{\pi t}$ is the expansion cost at time t . For costs/revenues related to the *Closed-II* and *Opened-II* nodes simply replace “I” with “II” in the above definitions. For the sake of simplicity, if $Q_t < q^{II} \Delta_h$ then the payoff associated with transitions 11 and 14 equals the one of transitions 4 and 8, respectively.

5.3 Valuation

For the valuation of this natural resource investment, we used the parameter values of Brennan and Schwartz (1985) for the copper mine and of Tsekrekos et al. (2012) for the three-factor model. The initial development cost and the initial expansion cost are estimated at US\$ 8 millions (I_0^d) and US\$ 4 millions (I_0^e), respectively. In addition, we consider the following: the possibility to defer development for up to two years (i.e. $T_1^{max}=2$); a lifetime of 45 years (T_2^{max}); and 5 decisions to be made per year (i.e. $\Delta_h = 1/5, h \in \mathcal{H} \setminus \{3, 7, 10, 13, 16, 17, 18\}$, and 0 otherwise). Also, we considered 100,000 ($|\Omega|$) sample paths (half of which antithetic for variance reduction) and complete sets of the first five (i.e. $L = 4$) Legendre/Hermite polynomials, as well as applied a singular value decomposition (SVD) algorithm with properly scaled basis functions to avoid numerical

problems when solving the least-squares regression in (3.8). For an analysis of the effects of different parametric models and their validation see Appendix C.1. The chosen input data for this example are summarised in Table 5.2.

Table 5.2: Input data for complex mine development project adapted from Brennan and Schwartz (1985); Tsekrekos et al. (2012) and own estimates.

Description	Parameter	Value	Unit
Mine			
Output rate	$q^I (q^{II})$	10 (20)	Mlbs/year
Initial inventory	Q_0	150	Mlbs
Initial cost of opening	$K_0^{o,I} (K_0^{o,II})$	0.20 (0.40)	US\$m
Initial cost of closing	$K_0^{c,I} (K_0^{c,II})$	0.20 (0.40)	US\$m
Initial maintenance cost	$M_0^I (M_0^{II})$	0.50 (1.00)	US\$m/year
Cost inflation rate	π	8%	year ⁻¹
Initial development cost	I_0^d	8	US\$m
Initial expansion cost	I_0^e	4	US\$m
Expiration of development right	T_1^{max}	2	year
Lifetime of copper mine project	T_2^{max}	45	year
Production cost			
Initial average cost of production	A_0	0.50	US\$/lbs
Speed of mean reversion in production cost	κ_a	0.20	—
Standard deviation of production cost	σ_a	15%	year ⁻¹
Copper^a			
Price variance	σ_x^2	8%	year ⁻¹
Initial convenience yield	δ_0	1%	year ⁻¹
Speed of mean reversion in convenience yield	κ_δ	0.30	—
Long-run mean convenience yield level	θ_δ	1%	year ⁻¹
Standard deviation of convenience yield	σ_δ	5%	year ⁻¹
Initial short-term interest rate	r_0	10%	year ⁻¹
Speed of mean reversion in interest rate	κ_r	0.50	—
Long-run mean interest rate level	θ_r	10%	year ⁻¹
Standard deviation of interest rate	σ_r	1.5%	year ⁻¹
Taxes			
Royalty	τ_1	0%	—
Income	τ_2	50%	—
Property, ^b <i>Opened/Closed</i>	λ_1	2%	year ⁻¹
Property, <i>Abandoned</i>	λ_3	0%	year ⁻¹

^a The values of the correlation coefficients are: $\rho_{x,r} = 0.15$, $\rho_{x,\delta} = 0.40$, and $\rho_{\delta,r} = 0.10$.

^b The value of the discount rate at time t , k_t , is $r_t + \lambda_1$.

The forward induction procedure consists of the following steps:

1. Determine the set of decision times, \mathcal{T}_{N_t} , for all decisions nodes $N_t \in \{1, 3, 5, 7, 8\}$,

forming subsets of \mathcal{T} :

$$\mathcal{T}_{N_t} = \begin{cases} \{i\Delta_1: i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T_1^{max}\}, & \text{if } N_t = 1, \\ \{i\Delta_1: i \in \mathbb{Z}_{\geq 0}, \tau_1 \in \mathcal{T}_1, \tau_1 + \Delta_1 \leq i\Delta_1 \leq T_2^{max}\}, & \text{if } N_t = 3, \\ \{i\Delta_1: i \in \mathbb{Z}_{\geq 0}, \tau_1 \in \mathcal{T}_1, \tau_1 + 2\Delta_1 \leq i\Delta_1 \leq T_2^{max}\}, & \text{if } N_t \in \{5, 7\} \\ \{i\Delta_1: i \in \mathbb{Z}_{\geq 0}, \tau_1 \in \mathcal{T}_1, \tau_1 + 3\Delta_1 \leq i\Delta_1 \leq T_2^{max}\}, & \text{if } N_t = 8, \end{cases} \quad (5.22)$$

2. Use (5.18), (5.19) and (5.17) to sample $|\Omega|$ paths of δ_t , r_t and X_t , respectively, giving $(X_t(\omega), \delta_t(\omega), r_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$
3. Use (5.20) to sample $|\Omega|$ paths of A_t , giving $(A_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$
4. Generate the possible resource state space \mathcal{R}_t for each decision node and decision time:

$$\mathcal{R}_t = \begin{cases} (t, 1, Q_0, T_1^{max} - t), & \text{if } t \in \mathcal{T}_1, \\ \{(t, 3, i\Delta_1 q^I, T_2^{max} - t): \tau_1 \in \mathcal{T}_1, i \in \mathbb{Z}_{\geq 0}, \tau_1 \leq t - \Delta_1, \\ \quad Q_0 - \min((t - \tau_1 - \Delta_1)q^I, Q_0) \leq i\Delta_1 q^I, \\ \quad i\Delta_1 q^I \leq Q_0 - q^I \min(\Delta_1, \max(t - T_1^{max} - \Delta_1, 0))\}, & \text{if } t \in \mathcal{T}_3, \\ \{(t, 5, i\Delta_1 q^I, T_2^{max} - t): \tau_1 \in \mathcal{T}_1, i \in \mathbb{Z}_{\geq 0}, \tau_1 \leq t - 2\Delta_1, Q_0 - \\ \quad \min((t - \tau_1 - 2\Delta_1)q^I, Q_0 - \Delta_1 q^I) \min(1, t/\Delta_1 - 2) \\ \quad \leq i\Delta_1 q^I \leq Q_0\}, & \text{if } t \in \mathcal{T}_5, \\ \{(t, 7, i\Delta_1 q^I, T_2^{max} - t): \tau_1 \in \mathcal{T}_1, i \in \mathbb{Z}_{\geq 0}, \tau_1 \leq t - 2\Delta_1, \\ \quad Q_0 - \min((t - \tau_1 - 2\Delta_1)q^{II}, Q_0) \leq i\Delta_1 q^I, \\ \quad i\Delta_1 q^I \leq Q_0 - q^I \min(\Delta_1, \max(t - T_1^{max} - 2\Delta_1, 0))\}, & \text{if } t \in \mathcal{T}_7, \\ \{(t, 8, i\Delta_1 q^I, T_2^{max} - t): \tau_1 \in \mathcal{T}_1, i \in \mathbb{Z}_{\geq 0}, \tau_1 \leq t - 3\Delta_1, Q_0 - \\ \quad \min((t - \tau_1 - 3\Delta_1)q^{II}, Q_0 - \Delta_1 q^I) \min(1, t/\Delta_1 - 3) \\ \quad \leq i\Delta_1 q^I \leq Q_0\}, & \text{if } t \in \mathcal{T}_8, \end{cases} \quad (5.23)$$

Figures 5.2a, 5.2b, 5.2c and 5.2d illustrate for $X_0 = 0.70$ the evolution of X_t , δ_t , r_t and A_t , respectively, for five generated paths.

5.4 Results and Discussion

This section begins with an analysis of the way in which the mine value with different configurations of option portfolios is affected by the initial copper price, X_0 . Table 5.3 summarises the results when X_0 is in the range from US\$ 0.30 to 1.00 per pound. Columns (†) and (‡) give the expected values of the mine under “now-or-never strategies”, which assume it must be either (developed and in case (‡) expanded immediately and then) operated at the rate of 10 and 20 Mlbs/year, respectively, until the inventory is fully exhausted, or left undeveloped. As can be seen, the value of the mine with fixed-output-

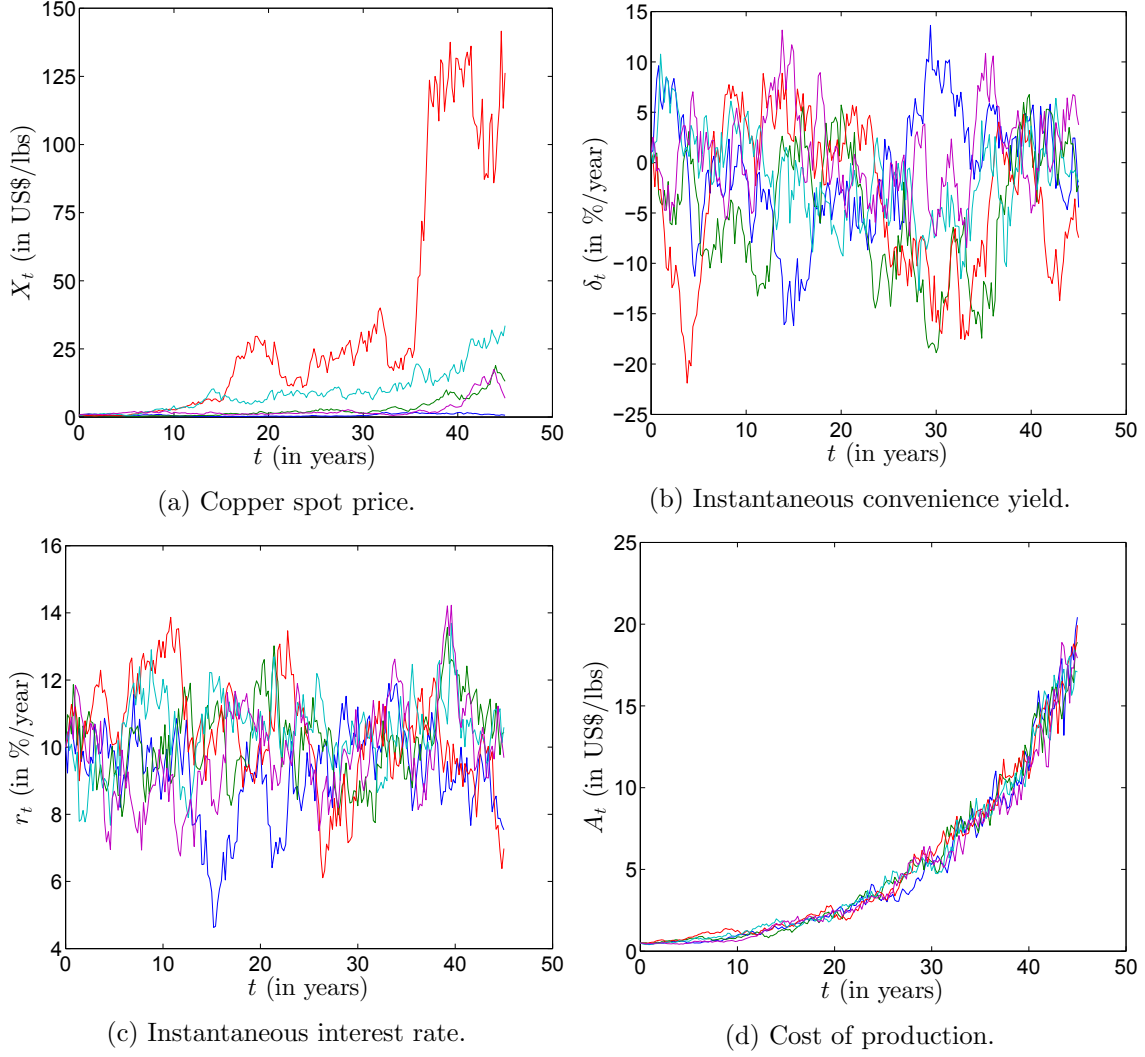


Figure 5.2: Selection of 5 equally likely paths for the evolution of X_t , δ_t , r_t and A_t .

rate q^I (q^{II}) is positive for copper prices of US\$ 0.70 (0.80) per pound and above, making development (and expansion) of the mine viable, but only for $X_0 = 1.00$ is it optimal to have an expanded mine with fixed-output-rate q^{II} . While these price levels are not the critical prices (i.e. the point at which it becomes optimal to invest, which largely depends on the input data), these can be estimated simply and accurately through simulation.

As can be seen from Table 5.3, the flexibility provided by individual real options can add considerable value. Column (a) displays the value added to the mine with fixed-output-rate q^I when development can be deferred for up to 2 years. Determined by the difference between the value of the fixed-output-rate mine with the option to defer (not shown here) and column (\dagger), the value of the option to defer is positive for all copper prices listed. This means it adds value in every situation. It adds sufficient value even when $X_0 \leq 0.60$, enabling the mine to become economically viable by allowing the mine's development to

be deferred. For $X_0 \leq 0.60$, the value of this option would be much higher if we had used the actual NPV (which is highly negative in these situations) as a benchmark instead of the non-negative value of the fixed-output-rate mine. Considering this circumstance, as expected (see H1), the value of this option decreases as X_0 increases because the ability to defer development is economically less attractive when prices are high.

Considering a developed but not expanded mine, columns (b-i), (b-ii) and (b) report the value of the option to temporarily mothball the operation, to abandon the project during operation and to switch, respectively. These values were determined such that the mine values with these individual options were measured against the value of the mine with fixed-output-rate q^I as the benchmark. Having either of these individual options is valuable for copper prices of US\$ 0.60/lbs and above, showing the mine to be viable in situations where the fixed-output-rate mine does not. At the same time, abandoning the project was found to be more valuable than mothballing, and switching more valuable than abandoning. In fact, representing the portfolio of the option to mothball and to abandon, the option to switch will always be at least as valuable as its constituent options, thereby confirming H2. Although the values of these three options decline as the price increases, their levels remain comparatively high and they decline less strongly than the value of the option to defer. This indicates that operational flexibility is more beneficial than the flexibility associated with investment timing when price levels are high.

Column (c) reports the value of the option to expand, which enables the mine to double its fixed-output-rate to 20 Mlbs/year at any point during operation. Its value is given by the difference between the value of the mine with the option to expand and the maximum of the values shown in columns (†) and (‡). Considering a developed and immediately expanded mine, columns (d-i), (d-ii) and (d) give the value of the option to temporarily mothball the operation, the value of the option to abandon the project during operation and the value of the option to switch, respectively. These values were determined in a similar way to those shown in columns (b-i), (b-ii) and (b), but now measured against the value of the mine with fixed-output-rate q^{II} . Comparing the values of columns (d-i), (d-ii) and (d) with the ones of columns (b-i), (b-ii) and (b) shows that mothballing, abandoning and switching during operation of the developed but not expanded mine is more valuable. This implies that operational flexibility to deviate from the extraction of copper is less beneficial if the mine is expanded.

The value of the mine with the portfolio of interdependent real options is shown in column (a,b,c,d). As seen in Table 5.3 in most cases its value is considerably larger than the value of the mine without options or with only an individual real option. This highlights the substantial added value achieved by considering such a complex portfolio. While the value of the mine with the portfolio increases in X_0 , the absolute difference between this value and the value of the best-performing fixed-output-rate mine decreases as X_0 increases. This is because flexibility to deviate from the static now-or-never strategy

becomes less valuable. However, the relative difference is still over 28% for the highest copper price considered. Comparing the values of the individual options of columns (a) to (d) shows that the option to expand is the only option whose value increases in the copper price, all other options diminish in value, which confirms H3. These results, which are in line with the real options literature, demonstrate the ability of the option to expand to exploit upside potential, and the ability of the other options (i.e. to defer, to mothball, to abandon, and to switch) to limit downside risk when operating margins are lower.

To illustrate the effects of the degrees of different uncertainties on the value of the copper mine, Figure 5.3 shows for $X_0 = 0.70$ the way in which the standard deviations of the production cost, σ_a , and the copper price, σ_x , affect the investment value. As we

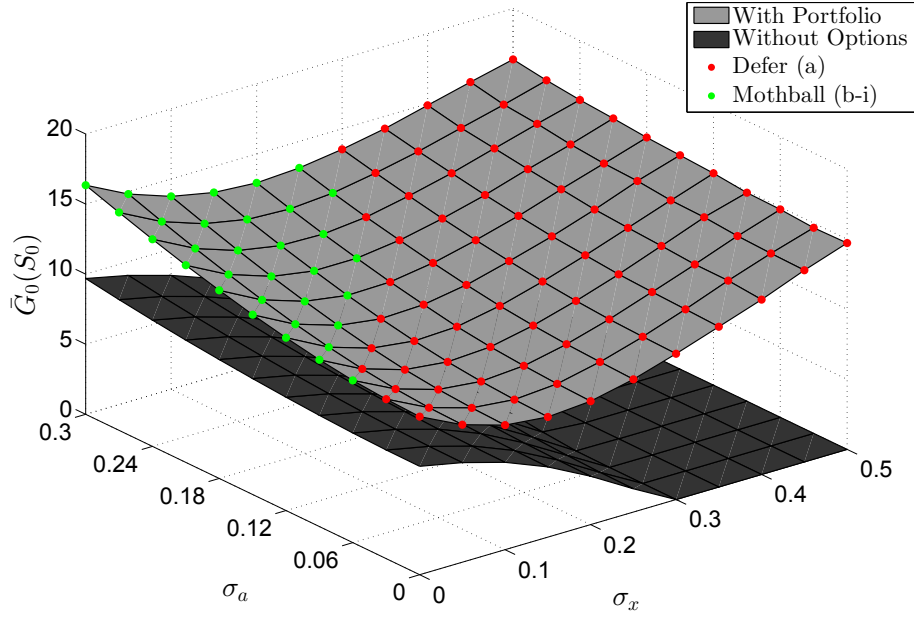


Figure 5.3: Value of investment project, $\bar{G}_0(S_0)$ (in US\$ millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of production cost (σ_a) and copper price (σ_x) uncertainty.

would expect, the value of the mine without options, which applies the best-performing static now-or-never strategy (always with fixed-output-rate q^I here) and hence does not consider any flexibility, decreases as price uncertainty increases and eventually reaches zero at $\sigma_x = 0.30$ ($\sigma_x = 0.35$ for $0.09 \leq \sigma_a \leq 0.24$, $\sigma_x = 0.40$ for $\sigma_a \geq 0.27$). On the other hand, the (expected) value of the mine without options, if positive, slightly increases in σ_a because, although production cost uncertainty increases, average production cost slightly decreases due to the characteristics/parameters of the stochastic process in (5.20).

Taking into account the portfolio of interdependent real options and thus allowing the decision-maker to exploit flexibilities adds substantial value to the investment project for all degrees of uncertainties considered, especially for high degrees of uncertainties¹.

¹Note that at $\sigma_x = 0$, X_t still evolves stochastically due to its dependence on the stochastic factors δ_t

Increasing σ_a from 0 to 0.30 generally results in appreciably higher values of the investment project with the portfolio and this effect tends to be stronger for lower σ_x . In contrast, there is a twofold effect of the degree of price uncertainty on the value of the mine with portfolio of options: values actually decrease in σ_x for low σ_x -values, but, beginning at a σ_x of 0.15 (0.20 for $\sigma_a \geq 0.24$), increase in σ_x ; this decrease (increase) tends to be steeper for higher (lower) σ_a -values. The reasons for this counterintuitive result (disproving H4) and somewhat intriguing twofold effect are believed to be the overall volatility in the three-factor model, which depends non-linearly on σ_x (see (4.35)) and features a similar pattern to the one shown in Figure 5.3, as well as portfolio effects which, as mentioned earlier and indicated by the coloured circles in Figures 5.3 and C.1, means that the portfolio's individual options are affected differently (beneficially or adversely) by changes of the underlying conditions resulting in positively or negatively affected portfolio values. As can be seen, the option to defer development is not always the portfolio's most valuable individual option as there are situations in which mothballing during operation of the developed mine is most valuable, which disproves the prediction of H5.

5.5 Summary

The main purpose of this chapter is to demonstrate the ability of the proposed portfolio of real options approach to evaluate a complex natural resource investment that features both a large portfolio of interdependent real options and multiple underlying stochastic factors. The options are to defer development, to temporarily mothball the developed mine, to irreversibly abandon the developed mine, to expand the mine's production capacity, to temporarily mothball the expanded mine, and to irreversibly abandon the expanded mine. The four underlying stochastic factors are the copper spot price, the instantaneous convenience yield, the instantaneous interest rate, and the cost of production. It is important to reiterate that this represents a setting where traditional methods (e.g. binomial/lattice and finite difference) are impractical.

Despite its complexity, the modelling and valuation of this complex yet realistic mine example are very cost-effective and straightforward. This example in some way combines the characteristics of the three examples studied in Chapter 4. In terms of portfolio of real options considered, it can be considered to contain twice the flexibility provided by the American-type option of Section 4.1 – for deferring and expanding – and, in addition, twice the flexibility provided by the option to switch of Section 4.2 – for mothballing and abandoning before and after expansion –, resulting in 6 individual options. In terms of uncertainties considered, this example can be considered as an extension of the three-factor model described in Section 4.3 to include an uncorrelated fourth stochastic factor.

and r_t in the three-factor model, with the value added by the portfolio at $\sigma_a = \sigma_x = 0$ amounting to almost 46% of the value without options.

Table 5.4 presents a comparison of the modelling complexity of the American option, the copper mine under the one- and three-factor model, and the complex mine development project of this chapter. As can be seen, with regard to the range of real option prob-

Table 5.4: Modelling complexity of different (real) option problems.

Example	Option(s)	Stochastic factor(s)	ID		Constraints	State variables ^c	Basis functions
			$ \mathcal{N} ^a$	$ \mathcal{H} ^b$			
American option	1	1	3(1)	3	4	2/1	3
Copper mine (I)	2	1	4(2)	7	7	3/1	6
Copper mine (II)	2	3	4(2)	7	7	3/3	56
Complex mine	6	4	8(5)	18	19	4/4	70

^a Number of decision nodes in brackets.

^b Equals the number of decision variables.

^c Resource/Information component.

lems considered in this work, our approach scales favourably as the problem’s complexity increases. While the number of basis functions used in the parametric model is directly linked to the number of information states (i.e. stochastic factors), which is an inherent characteristic of the parametric regression approach applied (see footnote in Subsection 3.2.3), the number of constraints and resource states, which are key elements when modelling sequential decision problems, as well as the size of the ID remain comparatively low. All valuation problems were implemented in Matlab, with the computational effort (time) required to solve the “Complex mine” and “Copper mine (II)” problem on average approximately 175% and 25% higher, respectively, than the requirements for “Copper mine (I)”. This complex mine example thereby demonstrates that our modelling technique is intuitive and compact, and capable of efficiently valuing complex and risky investments.

Using this example and considering five hypotheses, this chapter also demonstrates how the approach presented here can be used to investigate the way in which the value of the portfolio and its individual real options are effected by the underlying copper price level and the degrees of different uncertainties. This enables both the illustration and the interpretation of portfolio effects as well as the analysis of the comparative performance of our new approach and the expected NPV approach, which applies a static now-or-never strategy and hence does not take into account any flexibility. For example, the results demonstrate the ability of the complex portfolio considered here to pro-actively manage the risks involved – i.e. to exploit upside potential and to limit downside risk when operating margins are relatively high and low, respectively. Furthermore, using this example, we analyse the effect of different parametric models on the value of the portfolio of options by comparing different commonly used univariate orthogonal polynomial families and different numbers of basis functions (implied by L), discuss their choice in the light of approximation accuracy and computational time, as well as list potential limitations.

Chapter 6

Extending the Framework to Endogenous Uncertainties

In this chapter we extend our existing framework for modelling and approximating the value of interdependent real options to include endogenous, decision- and state-dependent uncertainties. This chapter is organised as follows: Section 6.1 presents the extension of the conceptual framework presented in Section 3.1 to include endogenous uncertainty. Section 6.2 describes the investment problem by specifying both the portfolio of interdependent real options and the set of uncertainties considered. In Section 6.3 we present the modelling and Section 6.4 contains the valuation approach together with the simulation-and-regression-based valuation algorithm. The approach and the algorithm are then applied to the real-case of a district heating network expansion investment in the London borough of Islington (Section 6.5). Results are presented and discussed in Section 6.6. Finally, some concluding remarks are provided in Section 6.7.

6.1 Extended Conceptual Framework

This chapter introduces a valuation approach for portfolios of interdependent real options under exogenous and endogenous sources of uncertainty. Considering the problem of a sequential and partially reversible investment project, we study a portfolio of options: to defer investment; stage investment; temporarily halt expansion; temporarily mothball the operation; and abandon the project during either construction or operation. In the problem studied here, the portfolio's value is affected by four underlying uncertainties. Of these, the project's actual cost to completion and its salvage value, are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the operating revenues and their growth rate evolve exogenously. The portfolio of real options approach presented in Chapter 3 proposed a multi-stage stochastic integer programming approach using influence diagrams and simulation-and-regression. To value such a complex portfolio under both types of uncertainty, we extend this approach to include endogenous sources of uncertainty. The dynamics of all four underlying uncertainties, which are modelled as stochastic (Markovian) processes, and the linear constraints modelling the interdependen-

cies between options are also integrated in this optimisation problem.

Our decision model is a stochastic dynamic, discrete-time (Markovian) model: the transition of the state S_t of the underlying system at time t to state $S_{t+\Delta}$ after a time increment Δ is driven by our decisions as well as by the random processes describing the uncertainties. Importantly, we distinguish between *exogenous* and *endogenous* sources of uncertainty. Modelled as stochastic Markovian processes, the evolution of endogenous uncertainties depends on the decision maker’s strategy or the system’s state, or both, while those of exogenous uncertainties are unaffected by decisions and states. Compared to standard models, models with decision/state dependent random variables are much more difficult to solve by simulation-optimisation methods since it is generally impossible to use random deviates which have been sampled once at the beginning. Thus, for a successful implementation, the optimization step(s) should be interleaved with random sampling steps. It should be mentioned, however, that for single-stage problems, if the objective is an expectation, one may use the likelihood ratios as correction terms and thus rely on just one sample, but the simulation error may get big and the likelihood ratio may be difficult to calculate. This change of measure technique is also known as Rubinstein’s “push-out method” (see (Rubinstein, 1992)). In our work, this method cannot be used because we deal with a multi-stage problem, so we have to resort to an interleaved simulation-optimisation method.

To approximate the value of this optimisation problem, we extend the simulation-and-regression-based valuation algorithm developed in Section 3.3 to include endogenous sources of uncertainty. Unlike the algorithms of Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), which are plain extensions of the algorithm proposed by Longstaff and Schwartz (2001) for American-style options, our algorithm takes into account the numerical implications of the state variables’ path-dependencies on the accuracy of the approximation. In order to avoid the negative numerical implications, we exploit the structure of the problem to be solved through dynamically and appropriately adapting the set of basis functions used in the parametric regression. Using an illustrative example of an urban infrastructure investment in London, we investigate the sensitivity of the optimal value of the portfolio and its individual options to the level of the initial annual revenues, as well as to the degrees of exogenous and endogenous uncertainty. In contrast to Miltersen and Schwartz (2007), who noted that the numerical solution techniques used by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008) “cannot easily handle temporary suspensions of the” investment project nor isolate the options’ values, this example demonstrates that our approach is flexible and powerful, and can be applied to value complex portfolios and their individual real options under both types of uncertainty.

6.2 The Investment Problem

In this section, we present the investment problem studied here by specifying both the portfolio of interdependent real options and the underlying set of uncertainties.

6.2.1 Portfolio of Interdependent Real Options

We study the problem of a decision maker wanting to determine the value of a sequential and partially reversible investment in a project whose stage-wise expansion (development) can be deferred, temporarily halted and/or abandoned altogether, and, once operating, whose cash flow generating asset can be used until the end of the asset's project life in T_3^{max} time periods, temporarily mothballed and/or abandoned early.

Representing the set of flexibilities as a portfolio of interdependent real options, the portfolio's single, well-defined options are:

- (a) Option to defer investment: Instead of starting immediately at time 0, the decision maker may choose to defer the start of the expansion until the expiration of the right to undertake this investment in T_1^{max} time periods, without incurring any direct costs associated with deferring.
- (b) Option to stage investment: As the development takes time to complete, the decision maker can invest at a rate of $0 < C_t \leq I^{max}$ in period t as long as the remaining investment cost at the beginning of period t , K_t , is greater than 0, i.e. while the project is under construction, where I^{max} and K_0 are the maximum rate of investment and the initial (expected) cost of completion, respectively.
 - (i) Option to temporarily halt expansion: If conditions turn out to be unfavourable, the decision maker can halt the expansion (i.e. set $C_t = 0$) at a cost of $C^{d,h}$, maintain the halted expansion for a total of T_2^{max} time periods at a periodic cost of C^h , and, if desirable, resume development at a cost of $C^{h,d}$.
 - (ii) Option to abandon the project during construction (i.e. when $K_t > 0$): Whether developing or halted, the project can be permanently abandoned at any given point in time t for the salvage value X_t , which is assumed to contain any costs that abandonment during construction involves.
- (c) Option to temporarily mothball the operation: If operation of the asset becomes uneconomic, the decision maker can mothball the operating asset at a cost of $C^{o,m}$, maintain the mothballed asset at a periodic cost of C^m , and, if conditions become favourable again, reactivate the asset at a cost of $C^{m,o}$.
- (d) Option to abandon the project during operation (i.e. when $K_t = 0$): Whether operating or mothballed, the decision maker retains the right to permanently abandon the project at any time t for its salvage value X_t , which is assumed to contain all costs related to abandoning during operation.

The above described individual real options are well-known and have been widely ex-

amined in the real options literature, for overviews see Trigeorgis (1993b, 1996). The first one, (a), is arguably the most-widely studied type of real option in the literature¹, e.g. see Trigeorgis (1993a); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001). Sequential investments, as per (b), have been studied in (Roberts and Weitzman, 1981; Majd and Pindyck, 1987; Pindyck, 1993; Trigeorgis, 1993a). Of these, the works of Majd and Pindyck (1987) and Pindyck (1993) explicitly and implicitly, respectively, considered the possibility to temporarily halt and later resume expansion – (b-i) – yet these authors ignored any direct costs associated with these decisions. With regard to (b-ii), these four works also allowed for abandonment during construction, but they neglected the project’s salvage value, which is over-simplistic; Trigeorgis (1993b, 1996) referred to (b-ii) as the “option to default during construction”. Categorised as an option “to alter operating scale” (Trigeorgis, 1993b), Brennan and Schwartz (1985) valued the option to temporarily shut down operations of a copper mine, which is practically the same as (c). Lastly, several works have analysed the flexibility related to (d). For example, building upon (Robichek and Van Horne, 1967; Dyl and Long, 1969) and considering an existing project with uncertain salvage value, Myers and Majd (1990) valued such option as an American put; Trigeorgis (1993a,b) referred to (d) as the “option to switch use”, where the salvage value represents the project’s value in its best alternative use.

6.2.2 Characterisation of Uncertainties

This study considers four sources of uncertainty – also referred to as stochastic factors or random variables – denoted by K_t , V_t , μ_t and X_t , representing the project’s actual cost to completion at time t , the revenues (net cash flow) generated by operation in period t , the growth rate of revenues in t and the salvage value at time t , respectively. The first and the fourth uncertainty are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the dynamics of the second and third factor are exogenous. The four stochastic factors are described by discrete-time random walks with drift, in a general form by:

$$M_{t+\Delta} = \varphi_t M_t + f_t(M_t, \theta_1) \Delta + \sigma_t(M_t, \theta_2) \sqrt{\Delta} \varepsilon_{t+\Delta}^m, \quad (6.1)$$

where Δ is the time step, φ_t is a multiplier, $f_t(\cdot)$ is the drift function, $\sigma_t(\cdot)$ is the diffusion function, and $\varepsilon_{t+\Delta}$ is the driving zero-mean process. Note that for endogenous stochastic factors, the parameters θ_1 or θ_2 , or both depend on the decisions or states, or both. The driving process $\varepsilon_{t+\Delta}^m$ is always Gaussian white noise (GWN), i.e. a standard normal random variable whose increments are iid, but drivers for different stochastic factors may be correlated. Table 6.1 summarises the stochastic factors considered here.

¹The continuous- and discrete-time version of this option are generally referred to as American and Bermudan call option, respectively

Table 6.1: Summary of stochastic factors considered in this investment problem.

Description	Factor	Defining eq.	Dynamics	Driving process ^a
Cost to completion	K_t	(6.2)	Decision-dep.	GWN, independent of (6.3)-(6.5)
Operating revenues	V_t	(6.3)	Exogenous	GWN, correlated with (6.4),(6.5)
Growth rate	μ_t	(6.4)	Exogenous	GWN, correlated with (6.3),(6.5)
Salvage value	X_t	(6.5)	State-dep.	GWN, correlated with (6.3),(6.4)

^a Gaussian white noise.

The dynamic of the project's actual cost to completion, K_t , depends on the rate of investment, $0 \leq C_t \leq I^{max}$, chosen by the decision maker, and is given by:

$$K_{t+\Delta} = K_t - C_t\Delta + \sigma_k \sqrt{C_t K_t \Delta} \varepsilon_{t+\Delta}^k, \quad (6.2)$$

where σ_k is the degree of technical uncertainty. The above equation is a discrete approximation of the controlled diffusion process proposed by Pindyck (1993). As analytically shown by Pindyck (1993); Schwartz and Zozaya-Gorostiza (2003) and referred to as “bang-bang policy” by Schwartz (2004), since the process (6.2) and the processes (6.3)-(6.5) are uncorrelated, the optimal rate of investment, C_t , is either I^{max} or 0.

The revenues received at time t for operation between t and $t + \Delta$, V_t , and their rate of growth, μ_t , evolve exogenously according to:

$$V_{t+\Delta} = e^{-\kappa_v \Delta} V_t + (1 - e^{-\kappa_v \Delta}) V_0(1 + \mu_t t) + \sigma_v \sqrt{\frac{1 - e^{-2\kappa_v \Delta}}{2\kappa_v}} \varepsilon_{t+\Delta}^v, \quad (6.3)$$

$$\mu_{t+\Delta} = e^{-\kappa_\mu \Delta} \mu_t + (1 - e^{-\kappa_\mu \Delta}) \bar{\mu} + \sigma_\mu \sqrt{\frac{1 - e^{-2\kappa_\mu \Delta}}{2\kappa_\mu}} \varepsilon_{t+\Delta}^\mu, \quad (6.4)$$

where σ_v and σ_μ are the standard deviations of changes in V_t and μ_t , respectively, as well as κ_v and κ_μ are positive mean reversion coefficients that describe the rate at which the corresponding factors converge to their linear trend, $V_0(1 + \mu_t t)$, and long-term average, $\bar{\mu}$, respectively. The nested model (6.3)-(6.4) is similar to the one of Schwartz and Moon (2001), who also used an *Ornstein-Uhlenbeck* process² to describe the evolution of μ_t . For the evolution of V_t , however, we apply an (arithmetic) Ornstein-Uhlenbeck model³ with linear – time-varying and stochastic – trend, which is adapted from the geometric mean reversion with exponential – constant and deterministic – trend of Metcalf and Hassett (1995).

The state-dependent salvage value obtained for abandoning the project at time t , X_t , is a function of both the expected asset value at time t , Z_t , which is a deterministic function

²This is the “simplest mean-reverting process” according to Dixit and Pindyck (1994).

³This model is more realistic, e.g., than a geometric Brownian motion, in the context of district heating networks as the level of V_t reflects both natural gas and electricity prices as well as heat demand.

of the state S_t (see (6.6)), and a *homoscedastic* noise term⁴ (i.e. error independent of the state), which is random and describes the percentage deviation as follows:

$$X_{t+\Delta} = Z_{t+\Delta} + \sigma_x Z_{t+\Delta} \varepsilon_{t+\Delta}^x, \quad (6.5)$$

where σ_x is the standard deviation of X_t . Unlike the existing approaches that allow for stochastic salvage (or abandonment) values, e.g see the works of Myers and Majd (1990); Adkins and Paxson (2017) and literature cited therein, which assume these values evolve exogenously, we introduce a state-dependent salvage value, as suggested in (Van der Hoek and Elliott, 2006). This example therefore represents one of the many practical situations in which the salvage value depends on endogenous factors (see (Trigeorgis, 1993a,b)), more specifically, on the state-dependent expected asset value. It is important to note that by “state” we actually mean its “resource” component (see Section 6.3), rather than its “information” component, specifically the latter’s three stochastic factors of (6.2)-(6.4), which are, of course, state-dependent too because Markovian.

6.3 Problem Formulation

This section contains the modelling of the investment problem as a sequential decision problem and the formulation of the valuation problem as a multi-stage stochastic integer programme.

6.3.1 Modelling Flexibilities with Influence Diagrams

The flexibilities available to the decision maker when having the portfolio of interdependent real options of Subsection 6.2.1 are shown by the ID in Figure 6.1. It contains nine nodes of which five are decision nodes and four are terminal nodes, as well as 18 transitions that link these nodes. The set of nodes and transitions is given by $\mathcal{N} = \{1, 2, \dots, 9\}$ and $\mathcal{H} = \{1, 2, \dots, 18\}$, respectively, and the duration of transition $h \in \mathcal{H}$ is Δ_h time period(s).

The state of the investment project at time t is written as:

$$S_t = (\underbrace{t, N_t, T_t, Q_t}_{R_t}, \underbrace{K_t, V_t, \mu_t, X_t}_{I_t}), \quad (6.6)$$

where $N_t \in \mathcal{N}$ is the node at time t ; T_t is the time left at t to defer investment/halt expansion/use the developed asset; Q_t is the amount invested up to time t ; and K_t , V_t , μ_t and X_t are as defined in Subsection 6.2.2. The first four variables of S_t are part of the resource state R_t , which is deterministic, whereas the information state I_t is made up of

⁴For a brief description of the modelling of both *homoscedastic* and *heteroscedastic* (i.e. state-dependent) noise see (Powell, 2011).

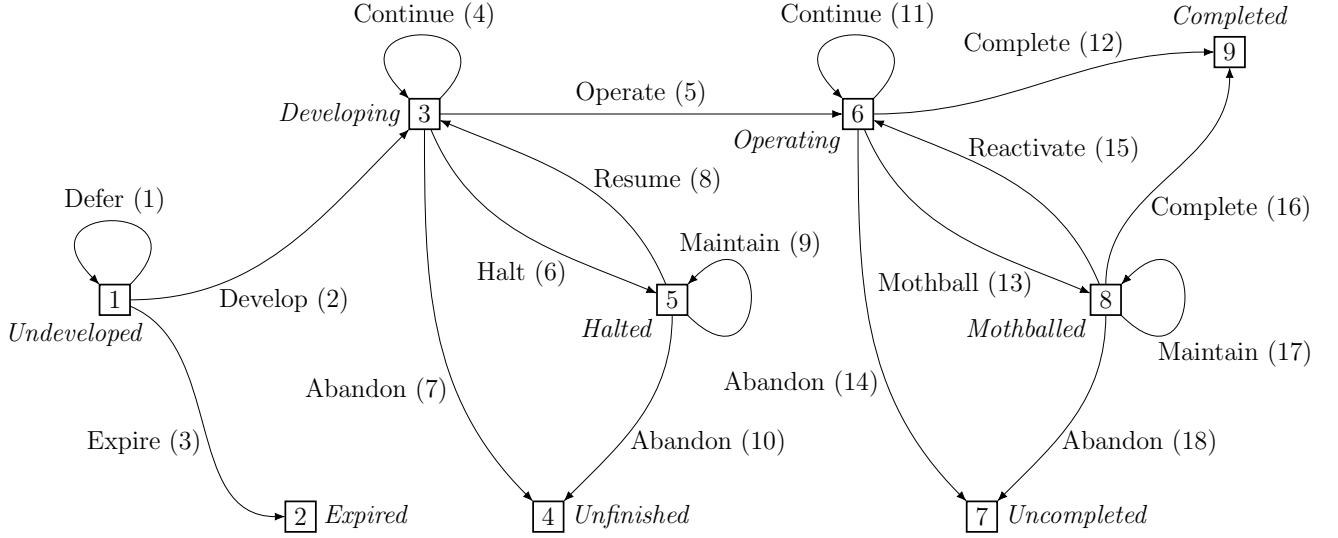


Figure 6.1: Flexibilities provided by portfolio of interdependent real options.

the problem's four random variables, two of which are exogenous and two are endogenous, decision- and state-dependent.

To each decision node we associate binary (0-1) variables a_{th} in such a way that $a_{th} = 1$ indicates that transition h is made at time t and 0 otherwise. It is clear that the action space $b^D(N_t)$ at node N_t is given by

$$b^D(N_t) = \begin{cases} \{1, 2, 3\}, & \text{if } N_t = 1, \\ \{4, 5, 6, 7\}, & \text{if } N_t = 3, \\ \{8, 9, 10\}, & \text{if } N_t = 5, \\ \{11, 12, 13, 14\}, & \text{if } N_t = 6, \\ \{15, 16, 17, 18\}, & \text{if } N_t = 8, \\ \{\}, & \text{otherwise.} \end{cases} \quad (6.7)$$

The decision variables $a_t = (a_{th})_{h \in b^D(N_t)}$ must then satisfy the feasible region \mathcal{A}_{S_t} , given

S_t , which is defined by the following set of constraints:

$$\begin{cases} \sum_{h \in b^D(N_t)} a_{th} = 1, & \forall N_t \in \{1, 3, 5, 6, 8\}, \end{cases} \quad (6.8)$$

$$a_{t1} T_1^{max} < T_t + T_1^{max}, \quad (6.9)$$

$$\begin{cases} a_{th} T_t = 0, & \forall h \in \{3, 12, 16\}, \end{cases} \quad (6.10)$$

$$\begin{cases} a_{t5} K_t = 0, \end{cases} \quad (6.11)$$

$$(1 - a_{t5} - a_{t7}) K_0 < K_t + K_0, \quad (6.12)$$

$$\begin{cases} a_{th} T_2^{max} < T_t + T_2^{max}, & \forall h \in \{6, 9\}, \end{cases} \quad (6.13)$$

$$\begin{cases} (1 - a_{th}) T_3^{max} < T_t + T_3^{max}, & \forall h \in \{12, 16\}, \end{cases} \quad (6.14)$$

where $a_{th} \in \{0, 1\}, \forall h \in \mathcal{H}$.

The transition function, which is generically written as $S^M(S_t, a_t, W_{t+\Delta_h})$ and describes the evolution of S_t from t to $t + \Delta_h$ after having made decision a_t with respect to \mathcal{A}_{S_t} and learned new information $W_{t+\Delta_h}$, is composed of the resource transition function $S^R(\cdot): R_t \rightarrow R_{t+\Delta_h}$ as well as the information transition function $S^I(\cdot): I_t \rightarrow I_{t+\Delta_h}$. With regard to the former, the transition of t is trivial as it simply evolves to $t + \Delta_h$; the transition of N_t is implicitly given by the adjacency matrix (not shown here) of the directed graph $(\mathcal{N}, \mathcal{H})$ underlying the influence diagram; the transition of T_t is given by:

$$T_{t+\Delta_h} = \begin{cases} \max\{T_t - \Delta_h, 0\}, & \text{if } a_{th} = 1, h \in \mathcal{H}_1, \\ T_2^{max}, & \text{if } a_{t2} = 1, \\ T_3^{max} - \Delta_5, & \text{if } a_{t5} = 1, \\ T_t, & \text{otherwise,} \end{cases} \quad (6.15)$$

where $T_0 = T_1^{max}$ and $\mathcal{H}_1 = \{1, 6, 9, 11, 13, 15, 17\}$; and the transition of Q_t is given by:

$$Q_{t+\Delta_h} = \begin{cases} Q_t + I^{max} \Delta_h, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ Q_t, & \text{otherwise,} \end{cases} \quad (6.16)$$

where $Q_0 = 0$. In contrast to the deterministic transitions of the variables of R_t , the information state variables evolve generally stochastically according to:

$$K_{t+\Delta_h} = \begin{cases} \max\{K_t - I^{max} \Delta_h + \sigma_k \sqrt{I^{max} K_t \Delta_h} \varepsilon_{t+\Delta_h}^k, 0\}, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ K_t, & \text{otherwise,} \end{cases} \quad (6.17)$$

$$V_{t+\Delta_h} = e^{-\kappa_v \Delta_h} V_t + (1 - e^{-\kappa_v \Delta_h}) V_0 (1 + \mu_t t) + \sigma_v \sqrt{\frac{1 - e^{-2\kappa_v \Delta_h}}{2\kappa_v}} \varepsilon_{t+\Delta_h}^v, \quad (6.18)$$

$$\mu_{t+\Delta_h} = e^{-\kappa_\mu \Delta_h} \mu_t + (1 - e^{-\kappa_\mu \Delta_h}) \bar{\mu} + \sigma_\mu \sqrt{\frac{1 - e^{-2\kappa_\mu \Delta_h}}{2\kappa_\mu}} \varepsilon_{t+\Delta_h}^\mu, \quad (6.19)$$

$$X_{t+\Delta_h} = Z_{t+\Delta_h}(S_{t+\Delta_h}) + \sigma_x Z_{t+\Delta_h}(S_{t+\Delta_h}) \varepsilon_{t+\Delta_h}^x, \quad (6.20)$$

where $Z_t(S_t)$, the expected asset value at time t , is given by:

$$Z_t(S_t) = \begin{cases} -\alpha I^{max}, & \text{if } N_t = 3 \wedge K_t > 0, \\ \gamma Q_t, & \text{if } N_t = 3 \wedge K_t = 0, \\ -\beta I^{max}, & \text{if } N_t = 5, \\ \gamma Q_t e^{-\xi(T_3^{max} - T_t)}, & \text{if } N_t = 6, \\ \delta Q_t e^{-\xi(T_3^{max} - T_t)}, & \text{if } N_t = 8, \\ 0, & \text{otherwise,} \end{cases} \quad (6.21)$$

where $\alpha \geq 0$ and $\beta \geq 0$ define the expected abandonment cost when *Developing* or *Halted*, respectively; $\gamma \geq 0$ and $\delta \geq 0$ are pay-out ratios determining the expected asset value when *Operating* or *Mothballed*, respectively; and ξ is the periodic depreciation rate describing the asset value's decline over time.

Lastly, the pay-off function is represented by:

$$\begin{aligned} \Pi_t(S_t, a_t) = & -I^{max}(\Delta_2 a_{t2} + \Delta_4 a_{t4}) + V_t(a_{t5} + a_{t11}) + X_t(a_{t7} + a_{t10} + a_{t14} + a_{t18}) \\ & - C^{d,h} a_{t6} - (C^{h,d} + I^{max} \Delta_8) a_{t8} - C^h \Delta_9 a_{t9} + X_t(a_{t12} + a_{t16}) \\ & - C^{o,m} a_{t13} + (V_t - C^{m,o}) a_{t15} - C^m \Delta_{17} a_{t17}. \end{aligned} \quad (6.22)$$

Note that, for the sake of simplicity, it is assumed that completing the project – by making either transition 12 (when *Operating*) or transition 16 (when *Mothballed*) – results in a pay-off of X_t , which thus represents the project's residual value.

6.3.2 Portfolio Optimisation Problem

Having fully modelled the sequential decision problem, similar to Section 3.2, the value of the portfolio of interdependent real options at time 0 given state S_0 , $G_0(S_0)$, is obtained by solving the following multi-stage stochastic integer programme:

$$G_0(S_0) = \max_{(a_t)_{t \in \mathcal{T}}} \mathbb{E} \left[\sum_{t \in \mathcal{T}} e^{-rt} \Pi_t(S_t, a_t) | S_0 \right], \quad (6.23)$$

where $S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$, $a_t = (a_{th})_{h \in b^D(N_t)}$, $a_t \in \mathcal{A}_{S_t}$, \mathcal{T} is the set of decision times (or decision epochs), $S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h})$, and r is the risk-free rate.

Applying Bellman's well-known “principle of optimality”, the optimisation problem in

(6.23) can be solved recursively with the optimal value of being in state S_t given by:

$$G_t(S_t) = \max_{a_t} \Pi_t(S_t, a_t) + \mathbb{E} \left[e^{-r\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) | S_t, a_t \right] \quad (6.24)$$

$$\text{s.t. } a_{th} \in \{0, 1\}, \quad \forall h \in b^D(N_t), \quad (6.25)$$

$$a_t \in \mathcal{A}_{S_t}, \quad (6.26)$$

$$S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h}), \quad \forall h \in b^D(N_t), \quad (6.27)$$

where $W_{t+\Delta_h} = (\varepsilon_{t+\Delta_h}^k, \varepsilon_{t+\Delta_h}^v, \varepsilon_{t+\Delta_h}^\mu, \varepsilon_{t+\Delta_h}^x)$ describes the information that arrives between time t and $t + \Delta_h$. The aim is then to determine $G_0(S_0)$, given the boundary (or terminal) condition $G_t(S_t) = 0, \forall t \in \mathcal{T}, N_t \in \{2, 4, 7, 9\}$.

6.4 The Valuation Algorithm

In this section we describe the approach to approximate the value of portfolios of interdependent real options under both exogenous and endogenous uncertainties as well as the simulation-based valuation algorithm and the solution procedure.

6.4.1 The Simulation-and-Regression-based Valuation Algorithm

In order to approximate the value of the portfolio of interdependent real options given by the optimisation problem (6.24)-(6.27), we extend the simulation-and-regression-based valuation algorithm presented in Section 3.3 to include endogenous sources of uncertainty. Furthermore, our proposed algorithm is both a generalisation and formalisation of the solution procedures offered by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), which are plain extensions of the algorithm proposed by Longstaff and Schwartz (2001) for single American-style options. While our algorithm also consists of an induction procedure with a forward and a backward pass as in Subsection 3.3.2, the procedure's individual steps were adapted to include endogenous uncertainty and to explicitly account for the negative numerical implications of the state variables' path dependencies on the accuracy of the approximation. See Subsection 6.4.2 for a description of the solution procedure's steps in which we assumed, for the sake of simplicity, that $\Delta_1 = \Delta_2 = \Delta_4 = \Delta_6 = \Delta_8 = \Delta_9$ and $\Delta_5 = \Delta_{11} = \Delta_{13} = \Delta_{15} = \Delta_{17}$.

The forward induction procedure generates the discrete state space \mathcal{S}_t through “exploration” of the resource state space \mathcal{R}_t and simulation (Monte Carlo sampling) of the information state space \mathcal{I}_t for all $t \in \mathcal{T}$. However, in addition to the path dependency of R_t because of the sequential decision process underlying the portfolio of real options (see Chapter 3), now both R_t and I_t are path-dependent because of the decision-dependent cost to completion, K_t . In fact, whether a resource state and its corresponding information state can be reached at time t (and are therefore part of \mathcal{R}_t and \mathcal{I}_t , respectively) does

not solely depend on the sequence of decisions made up to this point, but also on how K_t evolves stochastically; for instance, it might be that a particular R_t can be reached in only $\Omega_{R_t} \subseteq \Omega$, where $|\Omega_{R_t}| < |\Omega|$. Moreover, since the stochastic cost to completion can be directly translated into a stochastic time to completion, the decision times in \mathcal{T} are also path-dependent.

As a strategy in our procedure to overcome the curse of dimensionality related to both \mathcal{I}_t and the outcome space (for a discussion see Subsection 3.2.3 and Powell (2011); Nadarajah et al. (2017)), whenever needed we approximate the conditional expectation in (6.24), which represents the continuation function

$$\Phi_t(S_t, a_t) = \mathbb{E} \left[e^{-r\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) | S_t, a_t \right], \quad (6.28)$$

by the following continuous, finite-dimensional function (“the parametric model”):

$$\hat{\Phi}_t^{L_{S_t}}(S_t, a_t) = \sum_{l=0}^{L_{S_t}} \hat{\alpha}_{tl}(S^R(R_t, a_t)) \phi_{S_t l}(I_t), \quad (6.29)$$

where L_{S_t} is the model’s dimension; $\{\phi_{S_t l}\}_{l=0}^{L_{S_t}}$ are called basis functions (or features), which depend only on I_t and not the full S_t ; and the coefficients $(\hat{\alpha}_{tl}(S^R(R_t, a_t)))_{l=0}^{L_{S_t}}$ are obtained by the least-squares regression in (6.33). Unlike the parametric model of Subsection 3.3.1, here L_{S_t} and $\phi_{S_t l}$ depend on S_t , which enables us to reduce the model’s dimension if $N_t = 1$ ($N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\}$) by omitting functions of K_t and X_t (K_t) in the regression, thus reducing computational cost. Importantly, the function (6.29) is determined separately for each relevant and feasible decision a_t , given state S_t , whilst taking into account the set of paths Ω_{R_t} in which R_t can actually be reached. By contrast, in the setting of Chapter 3, every R_t can be reached along each path $\omega \in \Omega$ as it only considered exogenous uncertainty.

The valuation procedure shown in Algorithm 2 applies a standard backward induction to approximate the value of the multi-stage stochastic integer programme (6.24)-(6.27). Starting at $t = \max \mathcal{T}$ and moving backwards to $t = \min \mathcal{T} \setminus 0$, for each state $S_t \in \mathcal{S}_t$ perform the following three steps: (i) approximate (6.28) by both (6.29)-(6.33) and $\check{\Phi}_t(R_t, a_t)$ separately for all feasible a_t that do not lead to a terminal node, otherwise set them to 0, where $\check{\Phi}_t(R_t, a_t)$ is a deterministic lower bound on $\Phi_t(S_t, a_t)$, given R_t and a_t (*lines 3-9*); (ii) compute the pathwise optimisers $\hat{a}_t(\omega)$ for all $\omega \in \Omega_{R_t}$ in which R_t can be reached (*line 11*); (iii) using these pathwise optimisers, determine the approximation $\bar{G}_t(S_t(\omega))$ for each path $\omega \in \Omega_{R_t}$ (*line 12*). At $t = 0$, we have $(K_0, V_0, \mu_0, X_0) = (K_0(\omega), V_0(\omega), \mu_0(\omega), X_0(\omega))$, so we can simply calculate the value of (6.28) by taking averages of the path-wise continuation values over all $|\Omega|$ paths, and make optimal decisions based on these average values, giving $\bar{G}_0(S_0)$ (*line 17*).

6.4.2 Solution Procedure

The forward induction procedure consists of the following steps:

1. Starting at time 0 and using (6.17), sample $|\Omega|$ paths of K_t conditional on $a_{t2} = 1$ or $a_{t4} = 1$ until $K_t(\omega) = 0, \forall \omega \in \Omega$, where $\Delta^{con}(\omega) = \{\min t : K_t(\omega) = 0\}$ and $\mathcal{T}^{con} = \{\Delta^{con}(\omega) : \omega \in \Omega\}$ denote the construction time in path ω and the set of construction times, respectively.
2. Determine the set of decision times, \mathcal{T}_{N_t} , for all decisions nodes $N_t \in \{1, 3, 5, 6, 8\}$, forming subsets of \mathcal{T} :

$$\mathcal{T}_{N_t} = \begin{cases} \{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T_1^{max}\}, & \text{if } N_t = 1, \\ \{\tau_1 + \Delta_1(1 + i + 2j + m) : \tau_1 \in \mathcal{T}_1, i, j, m \in \mathbb{Z}_{\geq 0}, \Delta_1(1 + i + j) \leq \max \mathcal{T}^{con}, \\ \Delta_1(j + m) \leq T_2^{max} \max(0, \min(1, j))\}, & \text{if } N_t = 3, \\ \{\tau_1 + \Delta_1(2 + i + 2j + m) : \tau_1 \in \mathcal{T}_1, i, j, m \in \mathbb{Z}_{\geq 0}, \Delta_1(1 + i + j) < \max \mathcal{T}^{con}, \\ \Delta_1(1 + j + m) \leq T_2^{max}\}, & \text{if } N_t = 5, \\ \{\tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(1 + j) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i, j \in \mathbb{Z}_{\geq 0}, \\ \Delta_1 i \leq T_2^{max}, \Delta_5(1 + j) \leq T_3^{max}\}, & \text{if } N_t = 6, \\ \{\tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(2 + j) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i, j \in \mathbb{Z}_{\geq 0}, \\ \Delta_1 i \leq T_2^{max}, \Delta_5(2 + j) \leq T_3^{max}\}, & \text{if } N_t = 8, \end{cases} \quad (6.30)$$

3. Generate the possible resource state space \mathcal{R}_t for each decision node and decision time:

$$\mathcal{R}_t = \begin{cases} (t, 1, T_1^{max} - t/\Delta_1, 0), & \text{if } t \in \mathcal{T}_1, \\ \{(t, 3, T, Q) : \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, \\ \tau_1 < t, \Delta_1 \leq Q/I^{max} \leq \Delta^{con}(\omega), \exists \omega \in \Omega, \\ 0 \leq T_2^{max} - T \leq \max(t - \tau_1 - 2\Delta_1, 0)\}, & \text{if } t \in \mathcal{T}_3, \\ \{(t, 5, T, Q) : \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, \\ \tau_1 < t, \Delta_1 \leq Q/I^{max} < \Delta^{con}(\omega), \exists \omega \in \Omega, \\ \Delta_1 \leq T_2^{max} - T \leq \max(t - \tau_1 - \Delta_1, \Delta_1)\}, & \text{if } t \in \mathcal{T}_5, \\ \{(t, 6, T, Q) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, \\ T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - \Delta_5, \\ T \bmod \Delta_5 = 0, i \leq T_2^{max}\}, & \text{if } t \in \mathcal{T}_6, \\ \{(t, 8, T, Q) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, \\ T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - 2\Delta_5, \\ T \bmod \Delta_5 = 0, i \leq T_2^{max}\}, & \text{if } t \in \mathcal{T}_8, \end{cases} \quad (6.31)$$

5. For all $R_t \in \mathcal{R}_t, t \in \mathcal{T}$, compute the set of paths Ω_{R_t} in which resource state $R_t =$

(t, N_t, T_t, Q_t) is reachable:

$$\Omega_{R_t} = \begin{cases} \Omega, & \text{if } N_t = 1, \\ \{\omega \in \Omega : t - \tau_1 - T_2^{max} + T \leq \Delta^{con}(\omega), Q/I^{max} \leq \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\}, & \text{if } N_t = 3, \\ \{\omega \in \Omega : t - \tau_1 - T_2^{max} + T < \Delta^{con}(\omega), Q/I^{max} < \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\}, & \text{if } N_t = 5, \\ \{\omega \in \Omega : \Delta^{con}(\omega) = Q/I^{max}\}, & \text{if } N_t \in \{6, 8\}. \end{cases} \quad (6.32)$$

6. Use (6.19) and (6.18) to sample $|\Omega|$ paths of μ_t and V_t , respectively, giving

$$(V_t(\omega), \mu_t(\omega))_{\omega \in \Omega}, \forall t \in \mathcal{T}$$

7. Use (6.20)-(6.21) to sample $|\Omega_{R_t}|$ realisations of X_t

$$\text{giving } (X_t(\omega))_{\omega \in \Omega_{R_t}}, \forall R_t \in \mathcal{R}_t, t \in \mathcal{T}.$$

It is important to note that, unlike the forward induction procedure applied when there is only exogenous uncertainty, now the generation of the resource state space has to be interleaved with the random sampling steps (see Section 6.1 and Subsection 6.4.1).

The backward induction procedure is shown by Algorithm 2, with the optimal values of the coefficients $(\alpha_{tl}(S^R(R_t, a_t)))_{l=0}^{L_{S_t}}$, given R_t and a_t , in *line* 7 determined by (6.33).

$$(\hat{\alpha}_{tl}(R_{t+\Delta_h}))_{l=0}^{L_{S_t}} = \arg \min_{(\alpha_{tl}(\cdot))_{l=0}^{L_{S_t}}} \left\{ \sum_{\omega \in \Omega_{R_t}} \left[e^{-r\Delta_h} \bar{G}_{t+\Delta_h}(S_{t+\Delta_h}(\omega)) - \sum_{l=0}^{L_{S_t}} \alpha_{tl}(R_{t+\Delta_h}) \phi_{S_{tl}}(I_t(\omega)) \right]^2 \right\}, \quad (6.33)$$

where $R_{t+\Delta_h} = S^R(R_t, a_t)$ and $S_{t+\Delta_h}(\omega) = (R_{t+\Delta_h}, I_{t+\Delta_h}(\omega))$. Figure 6.2 illustrates the main steps of the solution approach given state S_t at time t .

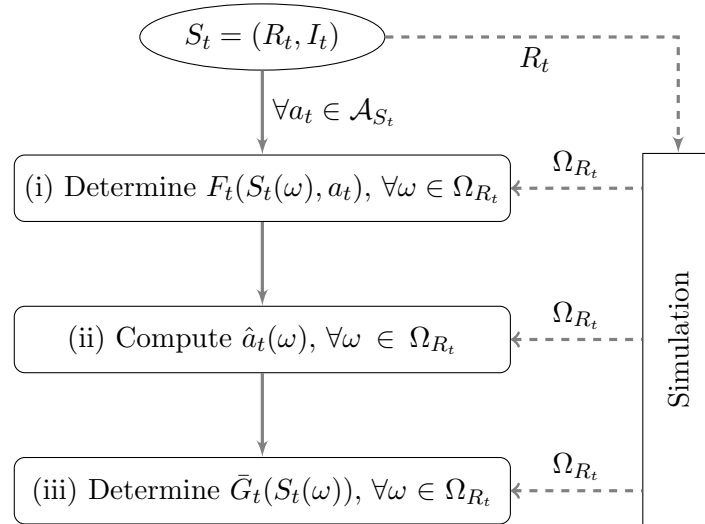


Figure 6.2: Main steps of interleaved solution approach given we are in state S_t at time t .

Algorithm 2: Approximation of optimal value of problem (6.24)-(6.27)

Data: From forward induction procedure and problem specific inputs

Result: $\bar{G}_0(S_0)$

```

1 for  $t = \max\{\mathcal{T} \setminus 0\}$  do
2   forall  $S_t \in \mathcal{S}_t$  do
3     forall  $a_t \in \mathcal{A}_{S_t}$  do
4       if  $a_{th} = 1, h \in \{3, 7, 10, 12, 14, 16, 18\}$  then
5          $F_t(S_t(\omega), a_t) \leftarrow 0, \forall \omega \in \Omega_{R_t}$ 
6       else
7         Use both (6.29)-(6.33) and  $\check{\Phi}_t(R_t, a_t)$  to determine:
8          $F_t(S_t(\omega), a_t) \leftarrow \max \left\{ \check{\Phi}_t(R_t, a_t), \hat{\Phi}_t^{L_{S_t}}(S_t(\omega), a_t) \right\}, \forall \omega \in \Omega_{R_t}$ 
9       end
10    end
11    forall  $\omega \in \Omega_{R_t}$  do
12      Compute pathwise optimisers:
13       $\hat{a}_t(\omega) \leftarrow \arg \max_{a_t(\omega) \in \mathcal{A}_{S_t(\omega)}} \left\{ \Pi_t(S_t(\omega), a_t(\omega)) + F_t(S_t(\omega), a_t(\omega)) \right\}$ 
14      Approximate optimal portfolio value along each path  $\omega$ :
15       $\bar{G}_t(S_t(\omega)) \leftarrow \Pi_t(S_t(\omega), \hat{a}_t(\omega)) + e^{-r\Delta_h} \bar{G}_{t+\Delta_h} \left( S^M(S_t(\omega), \hat{a}_t(\omega), W_{t+\Delta_h}(\omega)) \right)$ 
16    end
17  end
18   $\mathcal{T} \leftarrow \mathcal{T} \setminus t$ 
19 end
20 At  $t = 0, S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$ , determine:
21  $\bar{G}_0(S_0) \leftarrow \max_{a_0 \in \mathcal{A}_{S_0}} \left\{ \Pi_0(S_0, a_0) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{-r\Delta_h} \bar{G}_{\Delta_h} \left( S^M(S_0, a_0, W_{\Delta_h}(\omega)) \right) \right\}$ 

```

6.4.3 Computational Efficiency and Numerical Accuracy

While the efficiency and the accuracy of simulation and (parametric) regression approaches generally depend on a range of factors (e.g., see Subsection 3.3.3 for a discussion), here the actual number of paths ($|\Omega_{R_t}|$) available in the regression for state $S_t = (R_t, I_t)$ is particularly critical. Indeed, although disregarded by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), the additional path-dependency of both R_t and I_t caused by the decision-dependent uncertainty K_t may result in $|\Omega_{R_t}| \ll |\Omega|$, which, in turn, generally reduces the accuracy of the parametric regression model⁵. Considering polynomials as basis functions in the parametric model, Glasserman and Yu (2004a) examined the relationship between the number of simulated paths and the number of basis functions (implied by L_{S_t}), and showed that the required $|\Omega_{R_t}|$ for ensuring

⁵ A fundamentally different approach is to use the simulated evolution of K_t to determine the probability distribution that describes the probability that construction will be completed after a certain amount of cumulative investment, e.g. see (Cortazar et al., 2001; Pennings and Sereno, 2011).

convergence increases exponentially in L_{S_t} . However, Cortazar et al. (2008) have shown that taking advantage of the problem structure and carefully choosing an appropriate set of basis functions (e.g. call and put options on the expected spot price (Andersen and Broadie, 2004; Nadarajah et al., 2017)), rather than simply using high-order polynomials of information state variables as in (Glasserman and Yu, 2004a), allows one to substantially reduce the required L_{S_t} for a given level of accuracy, and is computationally more efficient. Hence, in general by exploiting the structure of the problem to be solved and choosing the set of basis functions appropriately, both the efficiency of the algorithm and the accuracy of the approximation are improved.

6.5 Case Study

This section provides specific details about the numerical example and computational implementation of the presented algorithm as well as presents the stochastic input data.

6.5.1 Expansion of District Heating Network

We consider the real case of an investment into the expansion of the district heating network in the London borough of Islington. We focus here on the development of the network’s “north extension”, as identified in a recent report (Grainger and Etherington, 2014) which investigated the development of a borough-wide network on behalf of the local council. According to this report, the capital expenditure of this expansion and the initial, annual operating revenues are estimated at £9.94 millions (K_0) and £564,600 (V_0), respectively. The report also noted that the asset can be used for up to 25 years (i.e. $T_3^{max}=300$). The risk-free rate, used to discount monetary values, is 3.5% per year (i.e. $r = 3.5\%/12$), as recommended by HM Treasury (2011). In addition, we assume the following: a maximum rate of investment of £1.0 million per month (I^{max}); the possibility of deferring development for up to one year (i.e. $T_1^{max}=11$); the possibility of halting expansion for up to one year (i.e. $T_2^{max}=11$); and the following durations of transitions (in months): $\Delta_h = 1, \forall h \in \{1, 2, 4, 6, 8, 9\}$; $\Delta_h = 12, \forall h \in \{5, 11, 13, 15, 17\}$; and 0 for the remainder of the transitions. Table 6.2 summarises the chosen input values for this example.

6.5.2 Generated State Space and Utilised Basis Functions

The discrete state space was generated by applying the forward induction procedure described in Subsection 6.4.1 (and 6.4.2) and using the data of Subsection 6.5.1. More specifically, 100,000 paths ($|\Omega|$) were generated to describe the stochastic evolution of the four factors K_t , V_t , μ and X_t for all $t \in \mathcal{T}$. Figure 6.3 shows the evolution of these for 5 equally likely paths. As can be seen in Figure 6.3a, while the expected duration of the

Table 6.2: Input data for district heating network expansion adapted from Grainger and Etherington (2014); HM Treasury (2011) and own estimates.

Description	Parameter	Value	Unit
Network expansion			
Cost of halting	$C^{d,h}$	$10 \cdot 10^3$	£
Cost of resuming	$C^{h,d}$	$10 \cdot 10^3$	£
Maintenance cost (halted)	C^h	$10 \cdot 10^3$	£/month
Cost of mothballing	$C^{o,m}$	$20 \cdot 10^3$	£
Cost of reactivating	$C^{m,o}$	$20 \cdot 10^3$	£
Maintenance cost (mothballed)	C^m	$20 \cdot 10^3$	£/month
Risk-free rate	r	0.035/12	month ⁻¹
Expiration of development right	T_1^{max}	11	month
Maximum period to halt expansion	T_2^{max}	11	month
Project life of developed asset	T_3^{max}	300	month
Investment cost			
Initial (expected) cost to completion	K_0	$9.94 \cdot 10^6$	£
Maximum rate of investment	I^{max}	$1.0 \cdot 10^6$	£/month
Degree of technical uncertainty	σ_k	35%	–
Revenue^a			
Initial annual operating revenue	V_0	564,600	£
Speed of mean reversion in revenue	κ_v	0.90	–
Standard deviation of revenue	σ_v	10%	month ⁻¹
Initial revenue growth rate	μ_0	0.10%	month ⁻¹
Speed of mean reversion in growth rate	κ_μ	0.90	–
Long-run mean growth rate level	$\bar{\mu}$	0.10%	month ⁻¹
Standard deviation of growth rate	σ_μ	0.01%	month ⁻¹
Salvage^a			
Depreciation rate	ξ	0.50%	month ⁻¹
Cost ratios	α, β	0.30	–
Pay-out ratios	δ, γ	0.70	–
Standard deviation of salvage value	σ_x	25%	–

^a The correlations between processes are: $\rho_{v,\mu} = -0.8$, $\rho_{v,x} = 0$, and $\rho_{\mu,x} = 0$.

expansion is 10 months, the actual time to build can vary substantially. Figures 6.4a and 6.4b show the total number of resource states $|\mathcal{R}_t|$ for all $t \in \mathcal{T}$ and the number of paths $|\Omega_{R_t}|$ in which every $R_t \in \mathcal{R}_t$ can be reached at $t \in \mathcal{T}$, respectively, supporting the claim made in Subsection 6.4.1 that some resource states may not be reachable in every simulation path. Furthermore, the total number of resource states, i.e. $\sum_{t \in \mathcal{T}} |\mathcal{R}_t|$, increased more than tenfold⁶ (from 3,635 to 41,815) as σ_k increased from 0.00 to 0.35, highlighting the computational complexity (cost) introduced by the decision-dependent uncertainty.

⁶ At the same time, the computational effort (time) required to solve the valuation problem increased by approximately 165%.

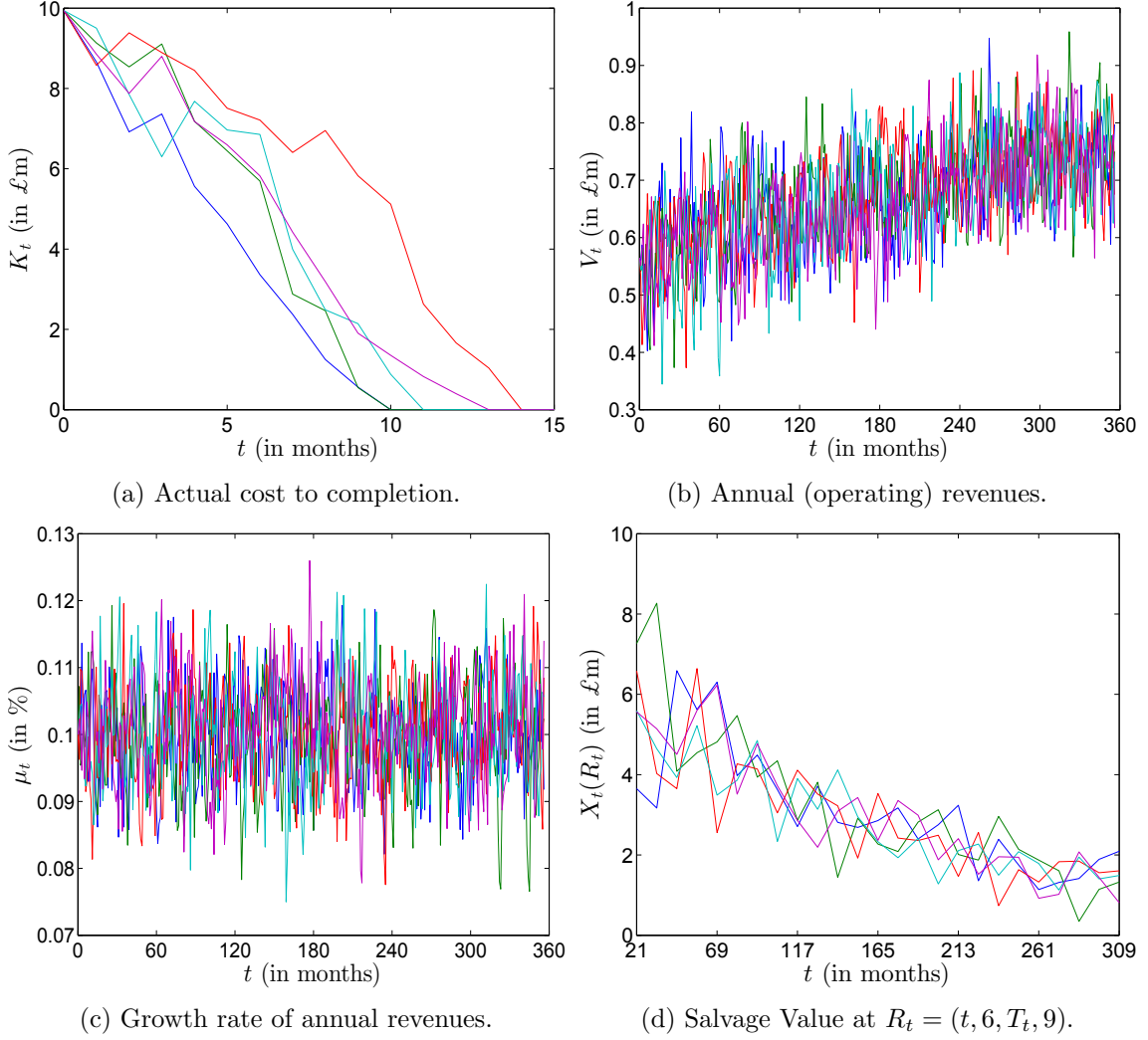


Figure 6.3: Selection of 5 equally likely paths for the evolution of K_t , V_t , μ_t and X_t .

With regard to the parametric model in (6.29), we apply as basis functions polynomials of the information state variables as well as both call and put options on the expected value of these variables partially based on (Longstaff and Schwartz, 2001; Andersen and Broadie, 2004; Cortazar et al., 2008; Nadarajah et al., 2017). In case $(N_t = 3 \wedge K_t > 0) \vee N_t = 5$, we use a set of 51 basis functions composed of a constant term, the four information state variables, polynomials of degree two (i.e. the squares of each variable and their cross products), polynomials of degree three, as well as the value of call and put options on the expected value of each variable and the square of this value. Otherwise, if $N_t = 1$ ($N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\}$), as mentioned in Subsection 6.4.3, we can reduce the number of basis functions used to 18 (32) by eliminating all the functions of K_t and X_t (K_t) because $K_t = K_0$ and X_t is non-existent ($K_t = 0$), so these variables do not add any information value to the least-squares regression. In order to avoid numerical problems the basis functions were properly scaled before performing the least-squares regression, which

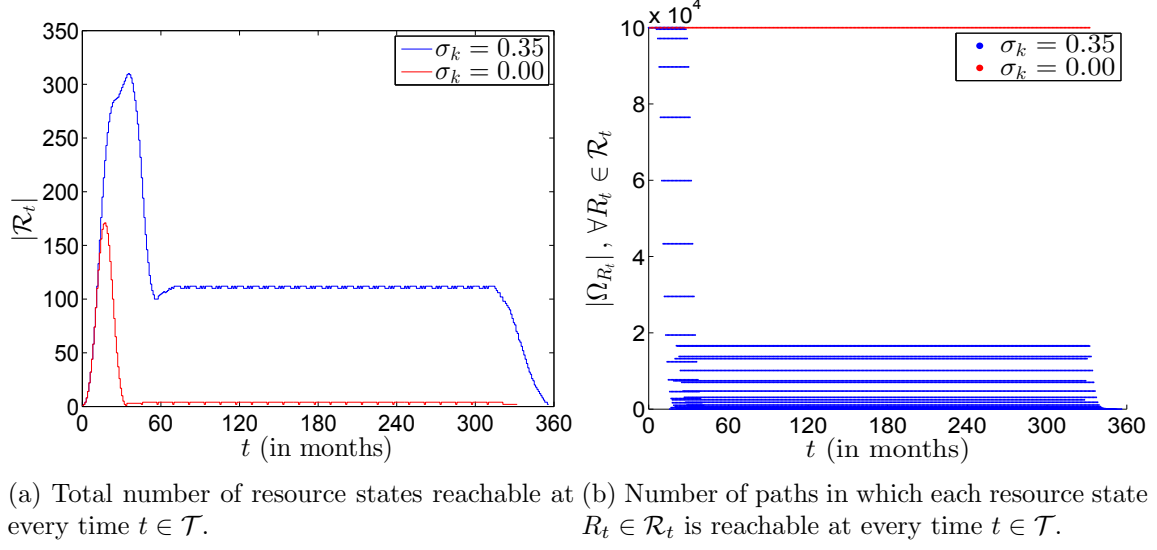


Figure 6.4: Results of reachability analysis.

is based on a singular value decomposition (SVD) algorithm. The valuation algorithm was implemented in Matlab.

6.6 Results and Discussion

In order to illustrate the extent to which the profitability of the district heating investment project depends on the initial value of the annual revenues, V_0 , Table 6.3 shows the sensitivity of the value of different portfolio configurations to varying levels of V_0 . As can be seen, for values of V_0 of £0.50 millions and below, the value of the investment project without options, configuration (-), is 0. This is because the expected NPV of the project is -£2.2060 millions, -£1.2751 millions, and -£0.3441 millions for values of V_0 of £0.40 millions, 0.45 millions, and 0.50 millions, respectively, so the optimal “now-or-never strategy”, which does not take any flexibility into account, is to leave the project undeveloped. The same strategy is optimal for the project with portfolio of options (a,b,c,d) for the lowest value of V_0 under consideration. However, for levels of V_0 of £0.45 millions and 0.50 millions, the value of the project with (a,b,c,d) is positive, reflecting the substantial value of having the flexibility provided by the portfolio of interdependent real options. Interestingly, in the first case, although the portfolio with all options achieves a positive value there is no individual option that provides sufficient added value on its own (i.e. in isolation), whereas in the case $V_0 = £0.50$ millions, having the option to defer alone – configuration (a) – also results in an economically viable project.

As can be seen from Table 6.3, beginning at a V_0 of £0.55 millions, the values of both the project without any flexibilities and almost all portfolio configurations are positive. In most cases the value of the project with (a,b,c,d) is considerable larger than without

Table 6.3: Value of investment project (in £millions) for different levels of initial annual revenues.

Annual Revenue (£m)	Value Without Options	Value of Option to Defer	During expansion			During operation			Value with Portfolio of Options
			Halt (b-i)	Abandon (b-ii)	Stage (b)	Mothball (c)	Value of Option to Abandon Switch (d)		
V_0	(-)	(a)	(b-i)	(b-ii)	(b)	(c)	(d)	(c,d)	(a,b,c,d)
0.40	0*	0	0	0	0	0	0	0	0*
		(-)**	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.45	0*	0	0	0	0	0	0	0	£42
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.50	0*	0.0006	0	0	0	0	0	0	0.1448
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.55	0.5868	0.0760	0.0045	0.1321	0.1618	0	0.1643	0.1643	0.9110
		(12.95)	(0.77)	(22.51)	(27.57)	(0)	(28.00)	(28.00)	(55.24)
0.5646	0.8586	0.0702	0.0043	0.1112	0.1419	0	0.1412	0.1412	1.1438
		(8.18)	(0.50)	(12.95)	(16.53)	(0)	(16.44)	(16.44)	(33.22)
0.60	1.5178	0.0556	0.0040	0.0754	0.1029	0	0.0978	0.0978	1.7296
		(3.66)	(0.26)	(4.97)	(6.78)	(0)	(6.45)	(6.45)	(13.96)
0.65	2.4487	0.0353	0.0035	0.0442	0.0658	0	0.0590	0.0590	2.5838
		(1.44)	(0.14)	(1.80)	(2.69)	(0)	(2.41)	(2.41)	(5.52)
0.70	3.3797	0.0161	0.0029	0.0260	0.0433	0	0.0361	0.0361	3.4621
		(0.48)	(0.08)	(0.77)	(1.28)	(0)	(1.07)	(1.07)	(2.44)
0.75	4.3106	0.0008	0.0027	0.0153	0.0306	0	0.0222	0.0222	4.3588
		(0.02)	(0.06)	(0.36)	(0.71)	(0)	(0.51)	(0.51)	(1.12)

* No investment.

** Added value of option(s) in %.

Note: the sets of transitions available in the different settings are as follows: $\mathcal{H}^- = \{2, 3, 4, 5, 11, 12\}$ in (-); $\mathcal{H}^- \cup \{1\}$ in (a); $\mathcal{H}^- \cup \{6, 8, 9\}$ in (b-i); $\mathcal{H}^- \cup \{7, 10\}$ in (b-ii); $\mathcal{H}^- \cup \{6, \dots, 10\}$ in (b); $\mathcal{H}^- \cup \{13, 15, 16, 17\}$ in (c); $\mathcal{H}^- \cup \{14, 18\}$ in (d); $\mathcal{H}^- \cup \{13, \dots, 18\}$ in (c,d); and \mathcal{H} in (a,b,c,d).

options (-), revealing the significant added value that is obtained by considering such a complex portfolio. While the values of the project without any options and the portfolio with all options both increase in V_0 , the values of almost all of the individual options in isolation show a different trend. Indeed, the values of the options to defer (a), to halt (b-i), and to abandon the project during construction (b-ii) and operation (d) are decreasing in V_0 , meaning there is less value in deferring, halting, and abandoning as the value of initial annual revenues increases. This is because the annual revenues, although still uncertain (i.e. stochastic), revert now to a linear trend that is shifted upwards, so their level is generally higher, which makes deviating from the static now-or-never strategy, and consequently the flexibility provided by individual real options less valuable. For all values of V_0 under consideration, the option to temporarily mothball the operation – configuration (c) – is of no value as the simulated values of V_t are always positive.

The effects of the degrees of exogenous and endogenous uncertainty on both the value of the portfolio of options and the comparative performance of the portfolio's individual options are particularly important for understanding the influence of different underlying uncertainties. In order to illustrate these effects for the exogenous annual revenues, V_t , and the endogenous, decision-dependent cost to completion, K_t , Figure 6.5 shows for $C^{o,m} = C^{m,o} = C^m = 0$ the way in which the standard deviation of changes in revenues, σ_v , and the degree of technical uncertainty, σ_k , effect the value of the investment project. While the effects of changes of σ_v on the value of the project without options is negligible,

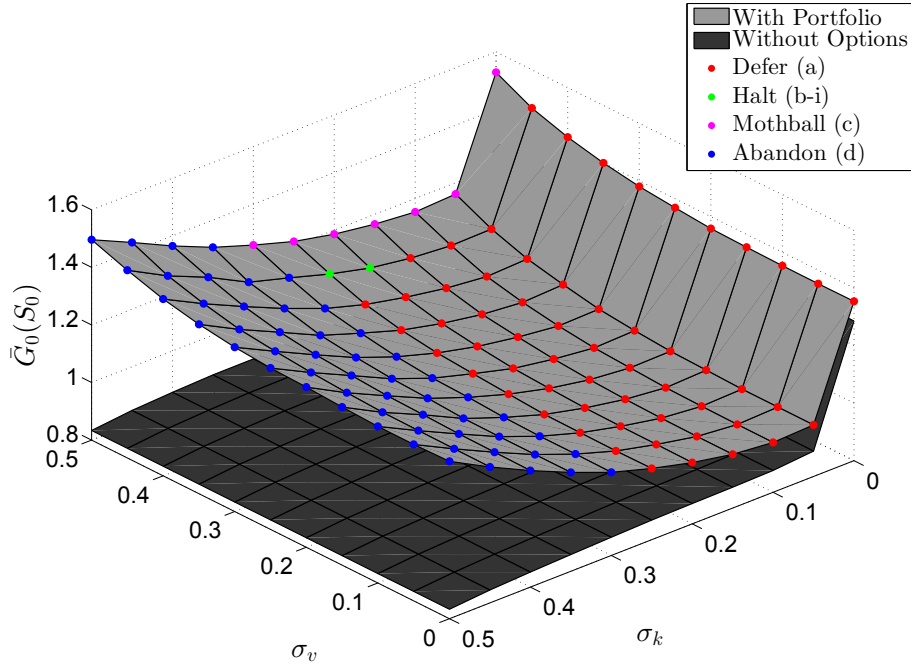


Figure 6.5: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty.

the value of the portfolio is generally increasing in σ_v , particularly steep for higher levels of σ_v and it seems the increase is more pronounced for lower values of σ_k . This increase in project value results from the flexibilities provided by the portfolio of real options, which allow a decision maker to exploit the upside potential of increased annual revenues, as compared to the negligibly affected value of the investment project without options, which applies a static now-or-never strategy.

On the other hand, increasing σ_k from 0 to 0.05 (i.e. introducing some construction cost uncertainty) results in a sharp decline in values of the investment project, but the decline is smaller for the project with the portfolio of real options. The reason for this sharp decline is mainly due to the increase in actual cost of completion caused by the introduction of technical uncertainty, but also because of the discretised investment expenditures. Unlike the investment project without options, whose value is always decreasing in σ_k , beginning at a σ_k of 0.1, the value of the portfolio is increasing in σ_k . This is because the flexibility provided by the portfolio, particularly by its option to abandon during operation (d), allows one to partially reverse the investment by recovering increased investment expenditures in situations with high values of σ_k , thereby taking advantage of relatively high state-dependent salvage values. This seems to explain why option (d) is the portfolio's most valuable individual option when the degree of technical uncertainty is high, whereas in most other situations, the option to defer (a) is the portfolio's most valuable option. Interestingly, for high values of σ_v , there are even situations in which options (b-i) and (c) are most-valuable, reflecting the ability of such a complex portfolio of real options to manage exogenous and endogenous uncertainties simultaneously in a wide range of uncertain environments. Figure D.1 displays the portfolio's least valuable individual option in these situations.

To show the effect of the endogenous, state-dependent salvage value, X_t , on investment decisions, Figure 6.6 shows the extent to which the value of the investment project is affected by the pay-out ratios γ and δ as well as by the standard deviation σ_x . The value of the project without options – where X_t is received as residual value when completing the project after 25 years of operation – is positive for all parameters under consideration. Furthermore, its value increases virtually linearly in (γ, δ) because of the linear dependence of the expected asset value, Z_t , on (γ, δ) , but is practically unaffected by changes in σ_x simply because the expected value of X_t does not change. Although the value of the project with the portfolio of options is always greater than the value of the project without options, the difference remains relatively constant for low values of (γ, δ) and for both low σ_x and moderate (γ, δ) , with the option to defer (a) being the portfolio's most valuable individual option in these situations (see Figure D.2 for its least valuable individual option). As can be seen, however, for high expected asset values and fairly high yet risky salvage values, the portfolio considered here is capable of extracting considerable value from flexibilities, especially from abandoning the project during either construction (b-ii) or operation (d).

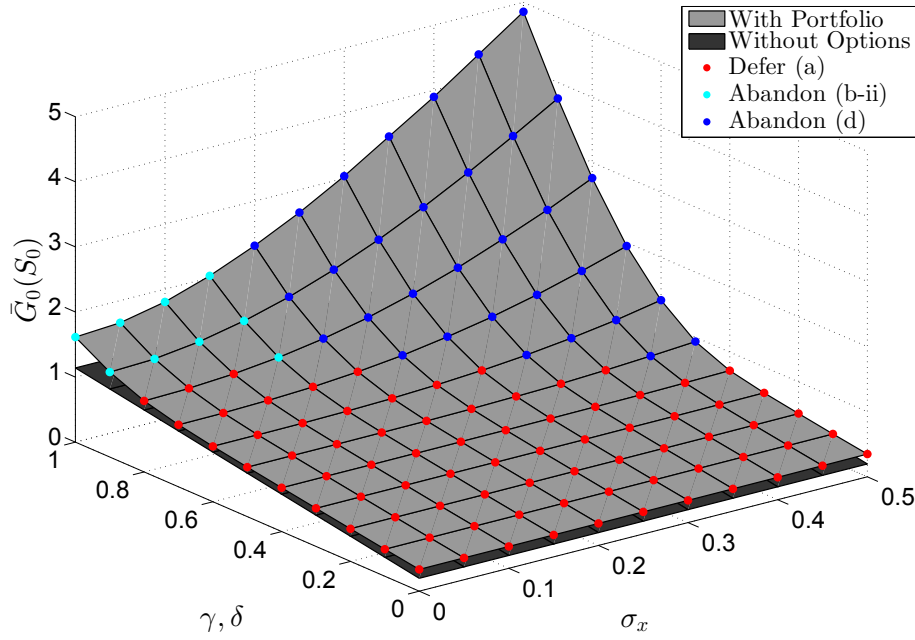


Figure 6.6: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x .

The above results therefore highlight the importance of applying such a portfolio of real options approach when there is both exogenous and endogenous uncertainty.

To reveal the effects of the depreciation rate (ξ) and the pay-out ratios (γ and δ ; only in the case of Figure 6.5) on the valuation, Figures D.3 and D.4 illustrate the same analyses as Figures 6.5 and 6.6, respectively, but with a higher depreciation rate – now $\xi = 0.80\%$ (before 0.50%) – and lower pay-out ratios – now $\gamma = \delta = 0.50$ (before 0.70). Comparing Figures 6.5 and D.3a as well as Figures 6.6 and D.4a, it is obvious that the values of the investment project – both with options portfolio and without options – are, as expected, substantially lower due to lower and faster-depreciating (and hence more rapidly devaluing) expected asset values. At the same time, the option to abandon the project during operation (d) is comparatively less valuable within the portfolio of options. The plotted trends are qualitatively identical to those obtained using $\xi = 0.50\%$ and $\gamma = \delta = 0.70$, and the results are in line with the previously discussed effects of uncertainties and pay-out ratios on the value of the investment project.

6.7 Summary

This chapter presents an approach for approximating the value of portfolios of interdependent real options under both exogenous and endogenous uncertainties. The approach is illustrated by valuing a complex urban infrastructure investment in London. Unlike existing

valuation approaches, which have considered only exogenous uncertainty or rather inflexible and restricted portfolios, this chapter has studied a complex yet practical portfolio of real options under conditions of four underlying uncertainties. The options were: to defer investment; stage investment; temporarily halt expansion; temporarily mothball the operation; and abandon the project. Two of the underlying uncertainties, decision-dependent cost to completion and state-dependent salvage value, were endogenous, whereas the other two, operating revenues and their growth rate, were exogenous. We have extended our previously presented approach for valuing portfolios of interdependent real options to include endogenous uncertainties. In the extended approach, the directly-modelled dynamics of all four uncertainties and the linear constraints modelling the real options' interdependencies are also integrated in a multi-stage stochastic integer programme. This chapter has presented an efficient valuation algorithm to approximate the value of this portfolio using simulation and parametric regression. In contrast to existing valuation algorithms, ours explicitly accounts for the negative numerical implications of the state variables' path dependencies on the accuracy of the approximation. We do so by exploiting the structure of the investment problem to be solved by dynamically and appropriately adapting the basis functions used in the parametric model. The illustrative example shows that our approach is flexible and powerful, and can be used to value both complex portfolios and their individual real options under both types of uncertainty.

Chapter 7

Conclusions

Addressing historically unprecedented global challenges including mitigation of and adaptation to climate change, ensuring urban resilience and liveability, as well as ageing populations, infrastructures and supply networks, will require massive capital investments globally over the next few decades in efficient and resilient, low-carbon infrastructures (Hoornweg, 2010; Bhattacharya et al., 2015). However, infrastructure investments are very capital intensive (sunk costs), involve a range of timescales – e.g. long lead times and even longer lifespans alongside ultra-rapid operational decisions regarding services and innovations (Flyvbjerg et al., 2009) –, and must be made in the context of enormous uncertainty engendered by increasingly complex economic, technological and policy environments. The valuation of infrastructure investment needs to account correctly for the multiple sources of uncertainty inherent in these investments. The real options analysis (ROA) approach aims to pro-actively manage risks by valuing the flexibilities (or options) inherent in uncertain and irreversible investments (e.g. with respect to design, financing, construction and operation). However, most existing ROA approaches consider only single, well-defined options and a few (exogenous) sources of uncertainty, so are inadequate and of little value when it comes to valuing infrastructure investments.

The aim of this study has been to develop a real options-based framework for the valuation of infrastructure investments as portfolios of interdependent real options under both exogenous and endogenous sources of uncertainty, and to illustrate its application. The four research objectives defined in the Introduction of this thesis, restated here to ease their mapping to this work’s contributions, are:

1. To develop an approach for the modelling and valuation of a portfolio of interdependent real options under exogenous uncertainty that is capable of accounting for multiple, possibly interdependent real options and various, possibly interlinked, sources of uncertainty.
2. To operationalise the approach using practical, relevant examples of increasing complexity in terms of both the portfolio of real options and the uncertainties considered, and to comprehensively evaluate the comparative performance of the conventional and new approach.

3. To demonstrate the ability of the approach to evaluate a complex natural resource investment project that features both a large portfolio of interdependent real options and multiple underlying uncertainties.
4. To extend the portfolio-based real options approach to include endogenous, decision and state dependent uncertainties, and to illustrate the applicability of the extended approach by valuing a district heating network expansion investment.

These four objectives were addressed in Chapters 3 to 6. Chapter 3 presented a simple yet powerful new approach for modelling and approximating the value of portfolios of interdependent real options under multiple, possibly interlinked exogenous uncertainties, using both influence diagrams and simulation-and-regression. Chapter 4 presented the operationalisation of our portfolio of real options approach using three well-known relevant examples: an American put option in a single-factor setting; a natural resource investment with a switching option (i.e a portfolio of options to mothball and abandon); and the same natural resource investment considering three stochastic factors instead of just one. Chapter 5 demonstrated the ability of this approach to evaluate complex and risky investment projects by considering a complex natural resource investment that features both a large portfolio of interdependent real options containing 6 individual options and four stochastic factors, of which three are correlated. Lastly, Chapter 6 presented an extension of the approach in order to approximate the value of portfolios of interdependent real options that include endogenous, decision and state dependent uncertainties. The extended approach was illustrated by valuing a complex urban infrastructure investment in the London borough of Islington. In this investment problem the options portfolio contained five individual options; two of the underlying uncertainties were exogenous and the other two were endogenous.

In conclusion, this research project has successfully achieved its four objectives. Section 7.1 summarises the main contributions of this research, while Section 7.2 discusses its limitations and suggests directions for further research.

7.1 Summary of Contributions

The main contributions of this research project are in the following areas:

- This work introduces a framework for modelling and valuing portfolios of interdependent real options using influence diagrams and simulation-and-regression. To approximate the value of this portfolio optimisation problem, which is formulated as a multi-stage stochastic integer programme, we present a transparent valuation algorithm that consists of a forward and backward induction procedure whilst applying simulation and parametric regression. In contrast to existing regression-based pricing methods, the valuation algorithm presented here contains several important

features specific to option portfolios; for example, ours explicitly takes into account the state variable’s multidimensional resource component that generally occurs in real option portfolios.

- We illustrate the application of the portfolio of real options approach presented here using the example of an American put option, and demonstrate its ability to accurately model and correctly value more complex real option problems by re-evaluating a natural resource investment in a one-, two- and three-factor setting. We show in detail how to model – e.g. how the portfolio’s real options and their interdependencies can be mathematically translated into linear constraints – and value – e.g. the specific steps of the forward induction procedure – these relevant examples. While re-evaluating a well-known copper mine example, we detect multiple errors in a highly-cited seminal work and in another important study. Using our approach and analytical solutions from literature, we perform a combined portfolio and equilibrium analysis which enables us to disprove their results whilst validating and confirming our own. In addition, this analysis provides interesting insights into the relationship between portfolio value and underlying volatility. We also discuss our approach in the context of existing option pricing and decision analysis approaches and highlight some key restrictions and limitations in existing approaches.
- We demonstrate the ability of the proposed approach to evaluate a complex yet realistic and important natural resource investment that features both a large portfolio of interdependent real options and four underlying uncertainties; importantly, this represents a setting where traditional methods (e.g. binomial/lattice and finite difference) are impractical. Using this example, we show how the approach presented here can be used to investigate the way in which the value of the portfolio and its individual real options are effected by the underlying operating margin and the degrees of different uncertainties, enabling both the illustration and the interpretation of portfolio effects. In addition, we analyse the effect of different parametric models on the value of the portfolio of options by comparing different commonly used univariate orthogonal polynomial families and different numbers of basis functions, and discuss their choice in the light of approximation accuracy and computational time.
- This work also presents an extension of the framework for modelling and valuing portfolios of interdependent real options to include endogenous uncertainty. Unlike existing valuation approaches, which have considered only exogenous uncertainty or rather inflexible and restricted option portfolios, this extension studies a complex yet practical portfolio of real options under conditions of four underlying uncertainties – two exogenous and two endogenous (one decision- and one state-dependent) uncertainties. To our knowledge, it is the first work to present a holistic and unified approach to model and value portfolios of interdependent real options under

exogenous and endogenous uncertainties. We present an efficient valuation algorithm to approximate the value of this portfolio using simulation and parametric regression. In contrast to existing valuation algorithms, ours explicitly accounts for the negative numerical implications of the state variables' path dependencies on the accuracy of the approximation by exploiting the structure of the investment problem to be solved by dynamically and appropriately adapting the basis functions used in the parametric model. We demonstrate the applicability of the proposed approach by valuing an urban infrastructure investment in London and show, for example, how the optimal value of the portfolio and its individual options are affected by the degrees of exogenous and endogenous uncertainty.

It is important to note that, although motivated by the valuation of infrastructure investments and the challenges surrounding them, our portfolio of real options approach is generic, so can be applied to a wide range of complex and risky investment projects which have both inherent interdependent flexibilities and many sources of underlying uncertainties. In conclusion, we believe our approach has the potential both to contribute to the methodology of decision making under uncertainty and to enhance the applicability of real options analysis to a wide range of important investment problems (such as in infrastructure), as well as to lay the basis for further theoretical developments.

Elements of this thesis have been submitted for consideration for publication in peer-reviewed journals and disseminated at various national and international conferences; see Appendix E for a list of publications relating to this thesis.

7.2 Limitations and Future Work

This research has mainly focused on the development of a new portfolio of real options approach and the demonstration of its applicability in multiple contexts, so naturally it has some limitations – both from a theoretical and practical perspective – and these provide suggestions and directions for future research. Relevant limitations are related to the modelling, valuation and numerical analyses, whereas potential future work concerns both methodological extensions and illustrating the wide applicability of the approach in terms of scope and scale.

The main limitation of the developed modelling approach is that it only considers binary (decision) variables and discrete resources states. While these considerations are sufficient and adequate to address many important real option problems – as demonstrated in this thesis –, there are problems that require a more nuanced modelling of the underlying sequential decision problems. In addition (or in contrast) to binary actions and discrete resource states, decisions may be discrete (i.e. general integer) or continuous, scalar/vector-valued and possibly high-dimensional, and resource state variables may take on continuous values. For example, real option problems that involve optimal capacity

choice generally involve decisions that are either discrete (e.g., see Décamps et al. (2006) or continuous (e.g., see Dangl (1999)) and resource state where one of the dimensions is continuous. In general, it should be relatively straightforward to model option problems with such features using our portfolio of real options approach; for example, by adding and/or adapting the relevant decision and resource state variables and then modifying the constraints accordingly. Further investigations are needed to analyse, amongst other things, the implications of such considerations on the computational tractability in the light of the discussed curses of dimensionality.

There are several limitations relating to the valuation approach presented. Many of these concern the approximation technique applied in the valuation algorithm. The computational efficiency and numerical accuracy of our simulation-and-regression-based algorithm is influenced by a variety of factors. These include the number of sample paths used as well as the number and type of basis functions used in the parametric model to approximate the continuation function. As discussed, the effects of these factors have been widely studied in literature using the LSM approach, confirming the method’s robustness and convergence of results. Since our approach also uses a continuation function approximation, it shares a number of desirable properties with the LSM approach. However, this work considers portfolios of interdependent real options which generally require complex sequential decision problems to be solved where sub-optimal decisions at a given state, e.g. due to non-optimal approximations at earlier states, may propagate through this process given the path-dependency of decisions in the portfolio, possibly leading to larger errors in the approximated portfolio values. In addition to the preliminary analysis of the effect of parametric model choice on portfolio value presented in this thesis, future work might therefore further analyse – both computationally and theoretically – our portfolio of real options approach, particularly with respect to robustness, convergence and efficiency as well as the free (or exercise) boundary and validation of the continuation function approximation.

Furthermore, the approximated value determined by our simulation-and-regression-based algorithm is a lower bound on the true value of the portfolio of interdependent real options. Although one could argue that such a low-biased value is sufficient when valuing infrastructure investments given the prevalence of the standard NPV method, which does not take into account any flexibility and as such provides generally much lower values, this circumstance can be unsatisfactory, even problematic for different reasons. For example, there could be a situation where our low-biased portfolio value is just below zero but the actual, true value is positive, thereby potentially contributing to the decision not to invest in an actually worthwhile infrastructure. From a computational perspective, in the (inevitable) absence of an exact benchmark value for comparison, having only a lower bound on the true value somewhat impedes the characterisation of the quality of our approximation. To provide a performance bound, future work will explore the extension of

our approach by applying duality theory to determine upper (dual) bounds. Also, in the context of endogenous uncertainty, the implications of our reachability analysis and, in particular, the benefits of dynamically adapting the set of basis functions – to account for the negative numerical implications of path dependencies in our algorithm – remain to be fully investigated and analysed.

The third set of limitations is related to the numerical examples and their analyses. Given that our quantitative valuation approach is numerical in nature (or rather by construction), there are obviously some unavoidable limitations inherent in the numerical analyses. For example, although we performed myriad sensitivity analyses to illustrate how the options portfolio’s value is affected by changes in parameters, this cannot replace the rigour and insights provided by analytical solutions with respect to real options analysis (e.g. whether values are convex/concave in input parameters). It should, however, be reiterated that such analytical solutions will not be attainable in most complex yet practical real-life situations, so we have to rely on numerical approaches like ours. Furthermore, even though being highly promising and encouraging, our numerical analysis of the relationship between volatility and value of the options portfolio was based on equations describing the volatility of futures returns implied in the two- and three-factor model, and only focused on the influence of a few parameters. Also, the equilibrium analysis was performed with respect to one specific correlation coefficient, but could have been easily extended to other parameters to further demonstrate its powerfulness. Future work should therefore develop the theoretical foundation and a general theory of the relationship between total volatility and portfolio value, thereby contributing to a more general portfolio theory, perhaps partially grounded in optimisation theory.

In addition to the future studies suggested to address limitations, other possible directions are methodological extensions and wide-ranging applications. With regard to extending the methodology, there are several fruitful directions. For example, the performance of infrastructure investments (and many other investments for that matter) is frequently affected by strategic interactions of multiple decision makers with often competing interests. Also, their performance is often affected by multiple interdependencies among real options and/or investment projects. In addition to the considered strategic interdependencies between options in the portfolio, interdependencies can be physical, digital, geographical and financial. Future work will investigate ways to integrate other types of interdependencies into the modelling framework presented here as well as explore possibilities to extend the current approach from a single-decision maker to a multi-decision maker context and to account for a decision maker’s attitude to risk (e.g., risk aversion (Chronopoulos et al., 2011)). Given its genericity, the proposed portfolio of real options approach can be applied to a wide range of important applications. Future work will evaluate the operational flexibility in interdependent district energy systems and the flexibility in complex natural resource investments under endogenous uncertainty.

Bibliography

- Abdel Sabour, S. A. and Poulin, R. (2006). Valuing real capital investments using the least-squares monte carlo method. *The Engineering Economist*, 51(2):141–160.
- Abramowitz, M. and Stegun, I. A., editors (1972). *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, volume 55 of *Applied Mathematics Series*. National Bureau of Standards, Washington, D.C., USA. 10th Printing.
- Adkins, R. and Paxson, D. (2017). Replacement decisions with multiple stochastic values and depreciation. *European Journal of Operational Research*, 257(1):174 – 184.
- Adner, R. and Levinthal, D. A. (2004). What is not a real option: Considering boundaries for the application of real options to business strategy. *Academy of Management Review*, 29(1):74–85.
- Ahmed, S., King, A. J., and Parija, G. (2003). A multi-stage stochastic integer programming approach for capacity expansion under uncertainty. *Journal of Global Optimization*, 26(1):3–24.
- Ahsan, M. and Musteen, M. (2011). Multinational enterprises’ entry mode strategies and uncertainty: A review and extension. *International Journal of Management Reviews*, 13(4):376–392.
- Alonso-Conde, A. B., Brown, C., and Rojo-Suarez, J. (2007). Public private partnerships: Incentives, risk transfer and real options. *Review of Financial Economics*, 16(4):335 – 349.
- Anand, J., Oriani, R., and Vassolo, R. S. (2007). Managing a portfolio of real options. In Reuer, J. J. and Tong, T. W., editors, *Real Options Theory: Advances in Strategic Management*, volume 24, chapter 10, pages 275–303. Elsevier Ltd.
- Andersen, L. and Broadie, M. (2004). Primal-dual simulation algorithm for pricing multidimensional american options. *Management Science*, 50(9):1222–1234.
- Apap, R. M. and Grossmann, I. E. (2016). Models and computational strategies for multistage stochastic programming under endogenous and exogenous uncertainties. *Computers & Chemical Engineering*, 103:233–274.
- Areal, N., Rodrigues, A., and Armada, M. R. (2008). On improving the least squares Monte Carlo option valuation method. *Review of Derivatives Research*, 11(1-2):119–151.
- Arrow, K. J. and Lind, R. C. (1970). Uncertainty and the evaluation of public investment decisions. *The American Economic Review*, 60(3):364–378.

- Azevedo, A. and Paxson, D. (2014). Developing real option game models. *European Journal of Operational Research*, 237(3):909 – 920.
- Barraquand, J. and Martineau, D. (1995). Numerical valuation of high dimensional multivariate american securities. *Journal of Financial and Quantitative Analysis*, 30(3):383–405.
- BBC (2014). Edinburgh’s trams roll into action. Available at: <http://www.bbc.co.uk/news/uk-scotland-27602618>. accessed on 31/05/2014.
- Bellman, R. and Dreyfus, S. (1959). Functional approximations and dynamic programming. *Mathematical Tables and Other Aids to Computation*, 13(68):247–251.
- Bellman, R., Kalaba, R., and Kotkin, B. (1963). Polynomial approximation—a new computational technique in dynamic programming: Allocation processes. *Mathematics of Computation*, 17(82):155–161.
- Ben-Horim, M. and Levy, H. (1980). Total risk, diversifiable risk and nondiversifiable risk: A pedagogic note. *Journal of Financial and Quantitative Analysis*, 15(2):289–297.
- Benaroch, M. (2002). Managing information technology investment risk: A real options perspective. *Journal of management information systems*, 19(2):43–84.
- Benaroch, M. and Kauffman, R. J. (1999). A case for using real options pricing analysis to evaluate information technology project investments. *Information Systems Research*, 10(1):70–86.
- Bendall, H. B. and Stent, A. F. (2007). Maritime investment strategies with a portfolio of real options. *Maritime Policy & Management*, 34(5):441–452.
- Bennouna, K., Meredith, G. G., and Marchant, T. (2010). Improved capital budgeting decision making: evidence from Canada. *Management Decision*, 48(2):225–247.
- Bhattacharya, A., Oppenheim, J., and Stern, N. (2015). Driving sustainable development through better infrastructure: Key elements of a transformation program. Working Paper 91, The Brookings Institution, Washington, DC, USA.
- Block, S. (2007). Are “real options” actually used in the real world? *The Engineering Economist*, 52(3):255–267.
- Borgonovo, E. and Gatti, S. (2013). Risk analysis with contractual default. does covenant breach matter? *European Journal of Operational Research*, 230(2):431 – 443.
- Borgonovo, E., Gatti, S., and Peccati, L. (2010). What drives value creation in investment projects? an application of sensitivity analysis to project finance transactions. *European Journal of Operational Research*, 205(1):227 – 236.
- Borgonovo, E. and Plischke, E. (2016). Sensitivity analysis: A review of recent advances. *European Journal of Operational Research*, 248(3):869 – 887.
- Bowman, E. H. and Hurry, D. (1993). Strategy through the option lens: An integrated view of resource investments and the incremental-choice process. *The Academy of Management Review*, 18(4):760–782.
- Boyle, P., Broadie, M., and Glasserman, P. (1997). Monte carlo methods for security

- pricing. *Journal of Economic Dynamics and Control*, 21(8–9):1267 – 1321.
- Boyle, P. P. (1977). Options: A monte carlo approach. *Journal of Financial Economics*, 4(3):323 – 338.
- Brandão, L. E., Dyer, J. S., and Hahn, W. J. (2005a). Response to comments on Brandão et al. (2005). *Decision Analysis*, 2(2):103–109.
- Brandão, L. E., Dyer, J. S., and Hahn, W. J. (2005b). Using binomial decision trees to solve real-option valuation problems. *Decision Analysis*, 2(2):69–88.
- Bräutigam, J., Esche, C., and Mehler-Bicher, A. (2003). Uncertainty as a key value driver of real options. In *Proceedings of the 5th Annual International Real Options Conference*, Washington, DC, USA.
- Brennan, M. J. and Schwartz, E. S. (1985). Evaluating natural resource investments. *The Journal of Business*, 58(2):135–157.
- Broadie, M. and Cao, M. (2008). Improved lower and upper bound algorithms for pricing american options by simulation. *Quantitative Finance*, 8(8):845–861.
- Broadie, M. and Detemple, J. B. (2004). Option pricing: Valuation models and applications. *Management Science*, 50(9):1145–1177.
- Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management Science*, 42(2):269–285.
- Broadie, M. and Glasserman, P. (1997). Pricing american-style securities using simulation. *Journal of Economic Dynamics and Control*, 21(8–9):1323 – 1352.
- Broadie, M. and Glasserman, P. (2004). A stochastic mesh method for pricing high-dimensional american options. *Journal of Computational Finance*, 7:35–72.
- Brosch, R. (2008). *Portfolios of Real Options*, volume 611 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag Berlin Heidelberg.
- Carriere, J. F. (1996). Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance: Mathematics and Economics*, 19(1):19 – 30.
- Cesena, E. A. M. and Davalos, F. R. (2011). Evaluation of investments in electricity infrastructure using real options and multiobjective formulation. *Latin America Transactions, IEEE (Revista IEEE America Latina)*, 9(5):767–773.
- Charnes, J. M. and Shenoy, P. P. (2004). Multistage Monte Carlo method for solving influence diagrams using local computation. *Management Science*, 50(3):405–418.
- Cheah, C. Y. J. and Garvin, M. J. (2009). Application of real options in ppp infrastructure projects: Opportunities and challenges. In Akintoye, A. and Beck, M., editors, *Policy, Finance & Management for Public-Private Partnerships*, chapter 13, pages 229–249. Wiley-Blackwell.
- Cheah, C. Y. J. and Liu, J. (2006). Valuing governmental support in infrastructure projects as real options using monte carlo simulation. *Construction Management and Economics*, 24(5):545–554.
- Chevalier-Roignant, B., Flath, C. M., Huchzermeier, A., and Trigeorgis, L. (2011). Strate-

- gic investment under uncertainty: A synthesis. *European Journal of Operational Research*, 215(3):639 – 650.
- Chiara, N., Garvin, M. J., and Vecer, J. (2007). Valuing simple multiple-exercise real options in infrastructure projects. *Journal of Infrastructure Systems*, 13(2):97–104.
- Childs, P. D., Ott, S. H., and Triantis, A. J. (1998). Capital budgeting for interrelated projects: A real options approach. *Journal of Financial and Quantitative Analysis*, 33(3):305–334.
- Christiansen, D. S. and Wallace, S. W. (1998). Option theory and modeling under uncertainty. *Annals of Operations Research*, 82(0):59–82.
- Chronopoulos, M., De Reyck, B., and Siddiqui, A. (2011). Optimal investment under operational flexibility, risk aversion, and uncertainty. *European Journal of Operational Research*, 213(1):221 – 237.
- Clément, E., Lamberton, D., and Protter, P. (2002). An analysis of a least squares regression method for American option pricing. *Finance and Stochastics*, 6(4):449–471.
- Colvin, M. and Maravelias, C. T. (2008). A stochastic programming approach for clinical trial planning in new drug development. *Computers & Chemical Engineering*, 32(11):2626 – 2642.
- Colvin, M. and Maravelias, C. T. (2010). Modeling methods and a branch and cut algorithm for pharmaceutical clinical trial planning using stochastic programming. *European Journal of Operational Research*, 203(1):205 – 215.
- Colvin, M. and Maravelias, C. T. (2011). R&D pipeline management: Task interdependencies and risk management. *European Journal of Operational Research*, 215(3):616 – 628.
- Cortazar, G. and Casassus, J. (1998). Optimal timing of a mine expansion: Implementing a real options model. *The Quarterly Review of Economics and Finance*, 38(3):755 – 769.
- Cortazar, G., Gravet, M., and Urzua, J. (2008). The valuation of multidimensional american real options using the (LSM) simulation method. *Computers & Operations Research*, 35(1):113 – 129.
- Cortazar, G. and Schwartz, E. S. (2003). Implementing a stochastic model for oil futures prices. *Energy Economics*, 25(3):215 – 238.
- Cortazar, G., Schwartz, E. S., and Casassus, J. (2001). Optimal exploration investments under price and geological-technical uncertainty: a real options model. *R&D Management*, 31(2):181–189.
- Cox Jr, L. A. (2009). *Risk analysis of complex and uncertain systems*, volume 129 of *International Series in Operations Research & Management Science*. Springer.
- Cuyppers, I. R. P. and Martin, X. (2007). Joint ventures and real options: An integrated perspective. In Reuer, J. J. and Tong, T. W., editors, *Real Options Theory*, volume 24 of *Advances in Strategic Management*, pages 103–144. Emerald Group Publishing Limited.

- Cuyppers, I. R. P. and Martin, X. (2010). What makes and what does not make a real option? a study of equity shares in international joint ventures. *Journal of International Business Studies*, 41(1):47–69.
- Dangl, T. (1999). Investment and capacity choice under uncertain demand. *European Journal of Operational Research*, 117(3):415 – 428.
- Daniel, J. W. (1976). Splines and efficiency in dynamic programming. *Journal of Mathematical Analysis and Applications*, 54(2):402 – 407.
- de Neufville, R. (2003). Real options: Dealing with uncertainty in systems planning and design. *Integrated Assessment*, 4(1):26–34.
- de Neufville, R. and Scholtes, S. (2011). *Flexibility in engineering design*. Engineering Systems. MIT Press.
- de Neufville, R., Scholtes, S., and Wang, T. (2006). Real options by spreadsheet: Parking garage case example. *Journal of Infrastructure Systems*, 12(2):107–111.
- De Reyck, B., Degraeve, Z., and Vandenborre, R. (2008). Project options valuation with net present value and decision tree analysis. *European Journal of Operational Research*, 184(1):341 – 355.
- Décamps, J.-P., Mariotti, T., and Villeneuve, S. (2006). Irreversible investment in alternative projects. *Economic Theory*, 28(2):425–448.
- Della Croce, R. and Gatti, S. (2014). Financing infrastructure – international trends. *OECD Journal: Financial Market Trends*, 2014(1):123–138.
- Demirer, R., Charnes, J. M., and Kellogg, D. (2003). Influence diagrams for real options valuation. Available at: <http://www.siue.edu/business/economicsandfinance/pdf/demirer-agouron.pdf>.
- Denault, M., Simonato, J.-G., and Stentoft, L. (2013). A simulation-and-regression approach for stochastic dynamic programs with endogenous state variables. *Computers & Operations Research*, 40(11):2760 – 2769.
- Devine, M. T., Gabriel, S. A., and Moryadee, S. (2016). A rolling horizon approach for stochastic mixed complementarity problems with endogenous learning: Application to natural gas markets. *Computers & Operations Research*, 68:1 – 15.
- Dixit, A. K. and Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton university press.
- Dixit, A. K. and Pindyck, R. S. (2012). *Investment under uncertainty*. Princeton University Press, Princeton, New Jersey, USA.
- Dupačová, J. (2006). Optimization under exogenous and endogenous uncertainty. In Lukás, L., editor, *Proceedings of MME06*, University of West Bohemia in Pilsen.
- Dyl, E. A. and Long, H. W. (1969). Abandonment value and capital budgeting: Comment. *The Journal of Finance*, 24(1):88–95.
- European Investment Bank (2013). The Economic Appraisal of Investment Projects at the EIB. Projects Directorate.

- Fabozzi, F. J., Paletta, T., and Tunaru, R. (2017). An improved least squares monte carlo valuation method based on heteroscedasticity. *European Journal of Operational Research*, 263(2):698 – 706.
- Flyvbjerg, B., Bruzelius, N., and Rothengatter, W. (2003). *Megaprojects and Risk: An Anatomy of Ambition*. Cambridge University Press, Cambridge, UK.
- Flyvbjerg, B., Garbuio, M., and Lovallo, D. (2009). Delusion and deception in large infrastructure projects: Two models for explaining and preventing executive disaster. *California Management Review*, 51(2):170–193.
- Folta, T. B. (1998). Governance and uncertainty: The trade-off between administrative control and commitment. *Strategic Management Journal*, 19(11):1007–1028.
- Gamba, A. (2003). Real options valuation: A Monte Carlo approach. Working Paper 2002/3, Faculty of Management, University of Calgary.
- Garvin, M. J. and Cheah, C. Y. J. (2004). Valuation techniques for infrastructure investment decisions. *Construction Management and Economics*, 22(4):373–383.
- Garvin, M. J. and Ford, D. N. (2012). Real options in infrastructure projects: theory, practice and prospects. *Engineering Project Organization Journal*, 2(1-2):97–108.
- Gibson, R. and Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *The Journal of Finance*, 45(3):959–976.
- Gil, N. (2007). On the value of project safeguards: Embedding real options in complex products and systems. *Research Policy*, 36(7):980 – 999.
- Gil, N. and Beckman, S. (2009). Infrastructure meets business: Building new bridges, mending old ones. *California Management Review*, 51(2):6–29.
- Glasserman, P. (2003). *Monte Carlo Methods in Financial Engineering*, volume 53 of *Applications of mathematics*. Springer-Verlag New York, 1st edition.
- Glasserman, P. and Yu, B. (2004a). Number of paths versus number of basis functions in american option pricing. *Ann. Appl. Probab.*, 14(4):2090–2119.
- Glasserman, P. and Yu, B. (2004b). Simulation for american options: Regression now or regression later? In Niederreiter, H., editor, *Monte Carlo and Quasi-Monte Carlo Methods 2002: Proceedings of a Conference held at the National University of Singapore, Republic of Singapore, November 25–28, 2002*, pages 213–226. Springer Berlin Heidelberg.
- Goel, V. and Grossmann, I. E. (2004). A stochastic programming approach to planning of offshore gas field developments under uncertainty in reserves. *Computers & Chemical Engineering*, 28(8):1409 – 1429.
- Goel, V. and Grossmann, I. E. (2006). A class of stochastic programs with decision dependent uncertainty. *Mathematical Programming*, 108(2):355–394.
- Grainger, C. and Etherington, J. (2014). *Islington Borough Energy Mapping: Phase 3 Development of a Strategic Masterplan Vision*. Buro Happold Limited. Available at <https://www.islington.gov.uk/~media/sharepoint-lists/>

- public-records/environmentalprotection/information/guidance/20152016/20150717energymasterplanburohappold2014 ; accessed on 04/11/2014.
- Grant, D., Vora, G., and Weeks, D. (1997). Path-dependent options: Extending the monte carlo simulation approach. *Management Science*, 43(11):1589–1602.
- Grayburn, J. (2012). *Real Options and Investment Decision Making*. Ofgem/Ofgem E-Serve, London, UK. Consultation 32/12.
- Grossmann, I. E., Apap, R. M., Calfa, B. A., Garca-Herreros, P., and Zhang, Q. (2016). Recent advances in mathematical programming techniques for the optimization of process systems under uncertainty. *Computers & Chemical Engineering*, 91:3 – 14.
- Gupta, V. and Grossmann, I. E. (2014). Multistage stochastic programming approach for offshore oilfield infrastructure planning under production sharing agreements and endogenous uncertainties. *Journal of Petroleum Science and Engineering*, 124:180 – 197.
- Haugh, M. B. and Kogan, L. (2004). Pricing american options: A duality approach. *Operations Research*, 52(2):258–270.
- Haugh, M. B. and Kogan, L. (2007). Duality theory and approximate dynamic programming for pricing american options and portfolio optimization. In Birge, J. R. and Linetsky, V., editors, *Financial Engineering*, volume 15 of *Handbooks in Operations Research and Management Science*, chapter 22, pages 925 – 948. Elsevier.
- Held, H. and Woodruff, D. L. (2005). Heuristics for multi-stage interdiction of stochastic networks. *Journal of Heuristics*, 11(5):483–500.
- Hermes, N., Smid, P., and Yao, L. (2007). Capital budgeting practices: A comparative study of the Netherlands and China. *International Business Review*, 16(5):630 – 654.
- Hirshleifer, J. (1961). Risk, the discount rate, and investment decisions. *The American Economic Review*, 51(2):112–120.
- Hirshleifer, J. and Riley, J. G. (1979). The analytics of uncertainty and information-an expository survey. *Journal of Economic Literature*, 17(4):1375–1421.
- HM Treasury (2009). Accounting for the effects of climate change: Supplementary green book guidance. Available at: https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/191501/Accounting_for_the_effects_of_climate_change.pdf ; accessed on 04/10/2017.
- HM Treasury (2011). The green book: appraisal and evaluation in central government. Available at: https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/220541/green_book_complete.pdf ; accessed on 04/10/2017.
- HM Treasury (2015). Valuing infrastructure spend: Supplementary guidance to the green book. Available at: https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/417822/PU1798_Valuing_Infrastructure_Spend_-_lastest_draft.pdf ; accessed on 04/10/2017.
- Ho, S. P. and Liu, L. Y. (2002). An option pricing-based model for evaluating the fi-

- nancial viability of privatized infrastructure projects. *Construction Management and Economics*, 20(2):143–156.
- Hoornweg, D. (2010). *Cities and Climate Change: An Urgent Agenda*, volume 10. The International Bank for Reconstruction and Development/The World Bank, Washington DC, USA.
- Howard, R. A. and Matheson, J. E. (2005). Influence diagrams. *Decision Analysis*, 2(3):127–143.
- Hsu, J. C. and Schwartz, E. S. (2008). A model of R&D valuation and the design of research incentives. *Insurance: Mathematics and Economics*, 43(3):350 – 367.
- Jiao, J. R. (2012). Product platform flexibility planning by hybrid real options analysis. *IIE Transactions*, 44(6):431–445.
- Jonsbråten, T. W., Wets, R. J.-B., and Woodruff, D. L. (1998). A class of stochastic programs with decision dependent random elements. *Annals of Operations Research*, 82(0):83–106.
- Judd, K. L. (1996). Approximation, perturbation, and projection methods in economic analysis. In Amman, H., Kendrick, D., and Rust, J., editors, *Handbook of Computational Economics*, volume 1, chapter 12, pages 509–585. Elsevier B.V.
- Judd, K. L. (1998). *Numerical Methods in Economics*. The MIT Press.
- Kambil, A., Henderson, J. C., and Mohsenzadeh, H. (1991). *Strategic management of information technology investments: an options perspective*. Center for Information Systems Research, Sloan School of Management, Massachusetts Institute of Technology. CISR WP No. 222, Sloan WP No. 3319.
- Keane, M. P. and Wolpin, K. I. (1994). The solution and estimation of discrete choice dynamic programming models by simulation and interpolation: Monte carlo evidence. *The Review of Economics and Statistics*, 76(4):648–672.
- Kind, A. (2005). Pricing american-style options by simulation. *Financial Markets and Portfolio Management*, 19(1):109–116.
- Kirschenmann, T., Popova, E., Damien, P., and Hanson, T. (2014). Decision dependent stochastic processes. *European Journal of Operational Research*, 234(3):731 – 742.
- Kiureghian, A. D. and Ditlevsen, O. (2009). Aleatory or epistemic? does it matter? *Structural Safety*, 31(2):105 – 112.
- Knight, F. H. (1964). *Risk, Uncertainty, and Profit*. Reprints of Economic Classics. Augustus M. Kelley, Bookseller, New York.
- Kohler, M. (2010). A review on regression-based monte carlo methods for pricing american options. In Devroye, L., Karasözen, B., Kohler, M., and Korn, R., editors, *Recent Developments in Applied Probability and Statistics*, pages 37–58. Physica-Verlag HD.
- Krüger, N. A. (2012). To kill a real option – incomplete contracts, real options and PPP. *Transportation Research Part A: Policy and Practice*, 46(8):1359 – 1371.
- Kumar, R. L. (2004). A framework for assessing the business value of information tech-

- nology infrastructures. *Journal of Management Information Systems*, 21(2):11–32.
- Lander, D. M. and Pinches, G. E. (1998). Challenges to the practical implementation of modeling and valuing real options. *The Quarterly Review of Economics and Finance*, 38(3):537 – 567.
- Lander, D. M. and Shenoy, P. P. (1999). Modeling and valuing real options using influence diagrams. Working Paper 283, School of Business, University of Kansas. Available at: <http://pshenoy.faculty.ku.edu/Papers/WP283.pdf>.
- Létourneau, P. and Stentoft, L. (2014). Refining the least squares monte carlo method by imposing structure. *Quantitative Finance*, 14(3):495–507.
- Li, J. (2007). Real options theory and international strategy: A critical review. In Reuer, J. J. and Tong, T. W., editors, *Real Options Theory*, volume 24 of *Advances in Strategic Management*, pages 67–101. Emerald Group Publishing Limited.
- Li, Y., James, B. E., Madhavan, R., and Mahoney, J. T. (2007). Real options: Taking stock and looking ahead. In Reuer, J. J. and Tong, T. W., editors, *Real Options Theory*, volume 24 of *Advances in Strategic Management*, pages 31–66. Emerald Group Publishing Limited.
- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, 14(1):113–147.
- Loureiro, M. V., Pereira, P. J., and Claro, J. (2012). Portfolios of real options and capacity expansion in transmission network expansion planning. In *Proceedings of the 16th Annual International Real Options Conference*, London, England.
- Luehrman, T. A. (1998). Strategy as a portfolio of real options. *Harvard business review*, 76:89–101.
- Maier, S. (2016). Risk-managing a portfolio of systemic low-carbon urban infrastructure investments using approximate dynamic programming with decision dependent uncertainties. YSSP Report, International Institute for Applied Systems Analysis (IIASA).
- Maier, S. (2017). An approach for valuing portfolios of interdependent real options under both exogenous and endogenous uncertainties. In *Proceedings of the 21st Annual International Real Options Conference*, Boston MA, USA.
- Maier, S., Pflug, G. C., Gann, D. M., and Polak, J. W. (2017a). An approach for valuing portfolios of interdependent real options under both exogenous and endogenous uncertainties. Submitted.
- Maier, S., Polak, J. W., and Gann, D. M. (2015). Appraising a portfolio of interdependent physical and digital urban infrastructure investments: A real options approach. In *Proceedings of the 19th Annual International Real Options Conference*, Athens and Monemvasia, Greece.
- Maier, S., Polak, J. W., and Gann, D. M. (2016). Valuing strategic interactions in systemic urban infrastructure investments using option games: A portfolio-based approximate dynamic programming approach. In *Proceedings of the 20th Annual International Real*

- Options Conference*, Oslo and Trondheim, Norway.
- Maier, S., Polak, J. W., and Gann, D. M. (2017b). Valuing portfolios of interdependent real options using influence diagrams and simulation-and-regression: A multi-stage stochastic integer programming approach. Submitted.
- Majd, S. and Pindyck, R. S. (1987). Time to build, option value, and investment decisions. *Journal of Financial Economics*, 18(1):7 – 27.
- Marques, J., Cunha, M., and Savić, D. A. (2015). Multi-objective optimization of water distribution systems based on a real options approach. *Environmental Modelling & Software*, 63(0):1 – 13.
- Martins, J., Marques, R., and Cruz, C. (2015). Real options in infrastructure: Revisiting the literature. *Journal of Infrastructure Systems*, 21(1):1–10.
- McGrath, R. G. (1997). A real options logic for initiating technology positioning investments. *The Academy of Management Review*, 22(4):974–996.
- McGrath, R. G. and Nerkar, A. (2004). Real options reasoning and a new look at the R&D investment strategies of pharmaceutical firms. *Strategic Management Journal*, 25(1):1–21.
- Meier, H., Christofides, N., and Salkin, G. (2001). Capital budgeting under uncertainty – an integrated approach using contingent claims analysis and integer programming. *Operations Research*, 49(2):196–206.
- Meinshausen, N. and Hambly, B. M. (2004). Monte carlo methods for the valuation of multiple-exercise options. *Mathematical Finance*, 14(4):557–583.
- Mercier, L. and Van Hentenryck, P. (2011). An anytime multistep anticipatory algorithm for online stochastic combinatorial optimization. *Annals of Operations Research*, 184(1):233–271.
- Metcalf, G. E. and Hassett, K. A. (1995). Investment under alternative return assumptions comparing random walks and mean reversion. *Journal of Economic Dynamics and Control*, 19(8):1471 – 1488.
- Micalizzi, A. and Trigeorgis, L. (1999). Project evaluation, strategy, and real options. In Trigeorgis, L., editor, *Real Options and Business Strategy: Applications to Decision-making*, pages 1–19. Risk Books.
- Miles, J. A. and Ezzell, J. R. (1980). The weighted average cost of capital, perfect capital markets, and project life: A clarification. *The Journal of Financial and Quantitative Analysis*, 15(3):719–730.
- Miller, L., Choi, S. H., and Park, C. S. (2004). Using an options approach to evaluate korean information technology infrastructure. *The Engineering Economist*, 49(3):199–219.
- Miltersen, K. R. and Schwartz, E. S. (2004). R&D investments with competitive interactions. *Review of Finance*, 8(3):355.
- Miltersen, K. R. and Schwartz, E. S. (2007). Real options with uncertain maturity and

- competition. NBER Working Paper 12990, National Bureau of Economic Research.
- Moreno, M. and Navas, J. F. (2003). On the robustness of least-squares Monte Carlo (LSM) for pricing american derivatives. *Review of Derivatives Research*, 6(2):107–128.
- Munoz, F. D., Hobbs, B. F., Ho, J. L., and Kasina, S. (2014). An engineering-economic approach to transmission planning under market and regulatory uncertainties: Wecc case study. *IEEE Transactions on Power Systems*, 29(1):307–317.
- Myers, S. C. (1974). Interactions of corporate financing and investment decisions-implications for capital budgeting. *The Journal of Finance*, 29(1):1–25.
- Myers, S. C. (1984). Finance theory and financial strategy. *Interfaces*, 14(1):126–137.
- Myers, S. C. and Majd, S. (1990). Abandonment value and project life. *Advances in Futures and Options Research*, 4:1–21.
- Nadarajah, S., Margot, F., and Secomandi, N. (2017). Comparison of least squares monte carlo methods with applications to energy real options. *European Journal of Operational Research*, 256(1):196 – 204.
- Oriani, R. and Sobrero, M. (2008). Uncertainty and the market valuation of R&D within a real options logic. *Strategic Management Journal*, 29(4):343–361.
- Otim, S. and Grover, V. (2012). Resolving uncertainty and creating value from the exercise of e-commerce investment options. *Information Systems Journal*, 22(4):261–287.
- Palczewski, J., Poulsen, R., Schenk-Hopp, K. R., and Wang, H. (2015). Dynamic portfolio optimization with transaction costs and state-dependent drift. *European Journal of Operational Research*, 243(3):921 – 931.
- Panayi, S. and Trigeorgis, L. (1998). Multi-stage real options: The cases of information technology infrastructure and international bank expansion. *The Quarterly Review of Economics and Finance*, 38(3):675 – 692.
- Pearce, D., Atkinson, G., and Mourato, S. (2006). *Cost-Benefit Analysis and the Environment*. OECD Publishing, Paris, France.
- Peeta, S., Salman, F. S., Gunneer, D., and Viswanath, K. (2010). Pre-disaster investment decisions for strengthening a highway network. *Computers & Operations Research*, 37(10):1708 – 1719.
- Pendharkar, P. C. (2010). Valuing interdependent multi-stage IT investments: A real options approach. *European Journal of Operational Research*, 201(3):847 – 859.
- Pennings, E. and Sereno, L. (2011). Evaluating pharmaceutical R&D under technical and economic uncertainty. *European Journal of Operational Research*, 212(2):374 – 385.
- Pflug, G. C. (1990). On-line optimization of simulated markovian processes. *Mathematics of Operations Research*, 15(3):381–395.
- Pindyck, R. S. (1993). Investments of uncertain cost. *Journal of Financial Economics*, 34(1):53 – 76.
- Pizzi, C. and Pellizzari, P. (2002). Monte carlo pricing of american options using nonparametric regression. *Rendiconti per gli Studi Economici Quantitativi*, 1:75–91.

- Powell, W. B. (2011). *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., 2nd edition.
- Pringles, R., Olsina, F., and Garcs, F. (2015). Real option valuation of power transmission investments by stochastic simulation. *Energy Economics*, 47(0):215 – 226.
- Roberts, K. and Weitzman, M. L. (1981). Funding criteria for research, development, and exploration projects. *Econometrica*, 49(5):1261–1288.
- Robichek, A. A. and Van Horne, J. C. (1967). Abandonment value and capital budgeting. *The Journal of Finance*, 22(4):577–589.
- Rogers, L. C. G. (2002). Monte carlo valuation of american options. *Mathematical Finance*, 12(3):271–286.
- Rose, S. (1998). Valuation of interacting real options in a tollroad infrastructure project. *The Quarterly Review of Economics and Finance*, 38(3):711 – 723.
- Rubinstein, R. Y. (1992). Sensitivity analysis of discrete event systems by the “push out” method. *Annals of Operations Research*, 39(1):229–250.
- Rust, J. (1996). Numerical dynamic programming in economics. In Amman, H., Kendrick, D., and Rust, J., editors, *Handbook of Computational Economics*, volume 1, chapter 14, pages 619 – 729. Elsevier B.V.
- Rust, J. (1997). Using randomization to break the curse of dimensionality. *Econometrica*, 65(3):487–516.
- Ryan, P. A. and Ryan, G. P. (2002). Capital budgeting practices of the Fortune 1000: How have things changed? *Journal of Business and Management*, 8(4):355–364.
- Salo, A., Keisler, J., and Morton, A., editors (2011). *Portfolio Decision Analysis: Improved Methods for Resource Allocation*, volume 162 of *International Series in Operations Research & Management Science*. Springer New York, NY, USA.
- Saphores, J.-D. and Boarnet, M. G. (2004). Investing in urban transportation infrastructure under uncertainty. In *Proceedings of 8th Annual International Real Options Conference*, Montréal, Canada.
- Savolainen, J. (2016). Real options in metal mining project valuation: Review of literature. *Resources Policy*, 50:49 – 65.
- Sbuelz, A. and Caliori, M. (2012). Revisiting corporate growth options in the presence of state-dependent cashflow risk. *European Journal of Operational Research*, 220(1):286 – 294.
- Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *The Journal of Finance*, 52(3):923–973.
- Schwartz, E. S. (1998). Valuing long-term commodity assets. *Journal of Energy Finance & Development*, 3(2):85 – 99.
- Schwartz, E. S. (2004). Patents and R&D as real options. *Economic Notes*, 33(1):23–54.
- Schwartz, E. S. and Moon, M. (2001). Rational pricing of internet companies revisited.

- Financial Review*, 36(4):7–26.
- Schwartz, E. S. and Zozaya-Gorostiza, C. (2003). Investment under uncertainty in information technology: Acquisition and development projects. *Management Science*, 49(1):57–70.
- Scialdone, P. (2007). *Valuing managerial flexibility: Challenges and opportunities of the real option approach in practice*. Cuvillier Verlag.
- Sick, G. and Gamba, A. (2010). Some important issues involving real options: an overview. *Multinational Finance Journal*, 14(1/2):73–123.
- Slade, M. E. (2001). Valuing managerial flexibility: An application of real-option theory to mining investments. *Journal of Environmental Economics and Management*, 41(2):193 – 233.
- Smit, H. T. J. (2003). Infrastructure investment as a real options game: The case of european airport expansion. *Financial Management*, 32(4):27–57.
- Smit, H. T. J. and Trigeorgis, L. (2006a). Real options and games: Competition, alliances and other applications of valuation and strategy. *Review of Financial Economics*, 15(2):95 – 112.
- Smit, H. T. J. and Trigeorgis, L. (2006b). Strategic planning: valuing and managing portfolios of real options. *R&D Management*, 36(4):403–419.
- Smit, H. T. J. and Trigeorgis, L. (2007). Strategic options and games in analysing dynamic technology investments. *Long Range Planning*, 40(1):84 – 114.
- Smit, H. T. J. and Trigeorgis, L. (2009). Valuing infrastructure investment: An option games approach. *California Management Review*, 51(2):79–100.
- Smith, J. E. (2005). Alternative approaches for solving real-options problems (comment on Brandão et al. 2005). *Decision Analysis*, 2(2):89–102.
- Smith, J. E. and Nau, R. F. (1995). Valuing risky projects: Option pricing theory and decision analysis. *Management Science*, 41(5):795–816.
- Smith, J. L. and Thompson, R. (2008). Managing a portfolio of real options: Sequential exploration of dependent prospects. *The Energy Journal*, 29:43–61.
- Solak, S., Clarke, J.-P. B., Johnson, E. L., and Barnes, E. R. (2010). Optimization of R&D project portfolios under endogenous uncertainty. *European Journal of Operational Research*, 207(1):420 – 433.
- Stentoft, L. (2004a). Assessing the least squares Monte-Carlo approach to American option valuation. *Review of Derivatives Research*, 7(2):129–168.
- Stentoft, L. (2004b). Convergence of the least squares Monte Carlo approach to American option valuation. *Management Science*, 50(9):1193–1203.
- Stentoft, L. (2014). Value function approximation or stopping time approximation: A comparison of two recent numerical methods for American option pricing using simulation and regression. *Journal of Computational Finance*, 18(1):65–120.
- Suttinon, P., Bhatti, A., and Nasu, S. (2012). Option games in water infrastructure

- investment. *Journal of Water Resources Planning and Management*, 138(3):268–276.
- Tarhan, B., Grossmann, I. E., and Goel, V. (2009). Stochastic programming approach for the planning of offshore oil or gas field infrastructure under decision-dependent uncertainty. *Industrial & Engineering Chemistry Research*, 48(6):3078–3097.
- Tarhan, B., Grossmann, I. E., and Goel, V. (2013). Computational strategies for non-convex multistage minlp models with decision-dependent uncertainty and gradual uncertainty resolution. *Annals of Operations Research*, 203(1):141–166.
- Terrazas-Moreno, S., Grossmann, I. E., Wassick, J. M., Bury, S. J., and Akiya, N. (2012). An efficient method for optimal design of large-scale integrated chemical production sites with endogenous uncertainty. *Computers & Chemical Engineering*, 37:89 – 103.
- Tilley, J. A. (1993). Valuing american options in a path simulation model. *Transactions of Society of Actuaries*, 45:499–520.
- Tiwana, A., Keil, M., and Fichman, R. G. (2006). Information systems project continuation in escalation situations: A real options model. *Decision Sciences*, 37(3):357–391.
- Triantis, A. and Borison, A. (2001). Real options: State of the practice. *Journal of Applied Corporate Finance*, 14(2):8–24.
- Trigeorgis, L. (1993a). The nature of option interactions and the valuation of investments with multiple real options. *The Journal of Financial and Quantitative Analysis*, 28(1):1–20.
- Trigeorgis, L. (1993b). Real options and interactions with financial flexibility. *Financial Management*, 22(3):202–224.
- Trigeorgis, L. (1995). Real options: An overview. In Trigeorgis, L., editor, *Real Options in Capital Investment: Models, Strategies, and Applications*, chapter 1, pages 1–28. Greenwood Publishing Group.
- Trigeorgis, L. (1996). *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. The MIT Press, Cambridge, MA, USA.
- Trigeorgis, L. (2005). Making use of real options simple: An overview and applications in flexible/modular decision making. *The Engineering Economist*, 50(1):25–53.
- Trigeorgis, L. and Kananen, E. (1991). An integrated options-based strategic planning and control model. *Managerial Finance*, 17(2/3):16–28.
- Trigeorgis, L. and Reuer, J. J. (2017). Real options theory in strategic management. *Strategic Management Journal*, 38(1):42–63.
- Tsekrekos, A. E., Shackleton, M. B., and Wojakowski, R. (2003). Evaluating natural resource investments using the least-squares monte carlo simulation approach. In *Proceedings of 7th Annual International Real Options Conference*, Washington D.C., USA.
- Tsekrekos, A. E., Shackleton, M. B., and Wojakowski, R. (2012). Evaluating natural resource investments under different model dynamics: Managerial insights. *European Financial Management*, 18(4):543–575.
- Tsitsiklis, J. N. and Van Roy, B. (1999). Optimal stopping of markov processes: Hilbert

- space theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives. *IEEE Transactions on Automatic Control*, 44(10):1840–1851.
- Tsitsiklis, J. N. and Van Roy, B. (2001). Regression methods for pricing complex American-style options. *IEEE Transactions on Neural Networks*, 12(4):694–703.
- van Bekkum, S., Pennings, E., and Smit, H. T. J. (2009). A real options perspective on R&D portfolio diversification. *Research Policy*, 38(7):1150 – 1158.
- Van der Hoek, J. and Elliott, R. J. (2006). *Binomial models in finance*. Number 1 in Springer Finance Textbooks. Springer Science & Business Media.
- Vassolo, R. S., Anand, J., and Folta, T. B. (2004). Non-additivity in portfolios of exploration activities: a real options-based analysis of equity alliances in biotechnology. *Strategic Management Journal*, 25(11):1045–1061.
- Verbeeten, F. H. (2006). Do organizations adopt sophisticated capital budgeting practices to deal with uncertainty in the investment decision?: A research note. *Management Accounting Research*, 17(1):106 – 120.
- Vidal, C. J. and Goetschalckx, M. (1997). Strategic production-distribution models: A critical review with emphasis on global supply chain models. *European Journal of Operational Research*, 98(1):1 – 18.
- Wallace, S. W. (2010). Stochastic programming and the option of doing it differently. *Annals of Operations Research*, 177(1):3–8.
- Wang, T. and de Neufville, R. (2004). Building real options into physical systems with stochastic mixed-integer programming. In *Proceedings of the 8th Annual International Real Options Conference*, Montréal, Canada.
- Wang, Y. and Caffisch, R. (2010). Pricing and hedging american-style options: a simple simulation-based approach. *The Journal of Computational Finance*, 13(4):95–125.
- Webster, M., Santen, N., and Parpas, P. (2012). An approximate dynamic programming framework for modeling global climate policy under decision-dependent uncertainty. *Computational Management Science*, 9(3):339–362.
- Williams, H. P. (2006). The formulation and solution of discrete optimisation models. In Appa, G., Pitsoulis, L., and Williams, H. P., editors, *Handbook on Modelling for Discrete Optimization*, volume 88 of *International Series in Operations Research & Management Science*, chapter 1, pages 3–38. Springer US, Boston, MA.
- Wolsey, L. A. (1998). *Integer programming*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons.
- World Bank Group (2015). Press Release: World bank president outlines strategy to end poverty, welcomes new development partners. Available at: <http://www.worldbank.org/en/news/press-release/2015/04/07/world-bank-president-strategy-end-poverty-new-development-partners>. accessed on 02/10/2015.
- World Economic Forum (2012). *Strategic Infrastructure: Steps to Prioritize and Deliver*

- Infrastructure Effectively and Efficiently*. Geneva, SUI.
- World Economic Forum (2013). *Strategic Infrastructure: Steps to Prepare and Accelerate Public-Private Partnerships*. Geneva, SUI.
- World Economic Forum (2014). *Strategic Infrastructure: Steps to Operate and Maintain Infrastructure Efficiently and Effectively*. Geneva, SUI.
- Zanger, D. Z. (2013). Quantitative error estimates for a least-squares monte carlo algorithm for american option pricing. *Finance and Stochastics*, 17(3):503–534.
- Zapata, J. C. and Reklaitis, G. V. (2010). Valuation of project portfolios: An endogenously discounted method. *European Journal of Operational Research*, 206(3):653 – 666.
- Zhao, T. and Tseng, C.-L. (2003). Valuing flexibility in infrastructure expansion. *Journal of infrastructure systems*, 9(3):89–97.
- Zhu, L. (2012). A simulation based real options approach for the investment evaluation of nuclear power. *Computers & Industrial Engineering*, 63(3):585 – 593.

Appendix A

Nomenclature

This appendix contains a summary of the notation used in Chapter 3.

Sets and indices

\mathcal{N}	Set of nodes, $\{1, \dots, N\}$
\mathcal{H}	Set of edges (or transitions), $\{1, \dots, H\}$
t	Time index
\mathcal{T}	Set of decision times (or epochs)
\mathcal{S}_t	State space at time t
\mathcal{R}_t	Resource state space at time t
\mathcal{I}_t	Information state space at time t
ω	Sample path
Ω	Set of sample paths
l	Index of summation, $l = 0, \dots, L$, used to specify the l -th dimension of the parametric model, where $l = 0$ generally refers to a constant term

Parameters

N	Number of nodes
H	Number of edges (or transitions)
Δ_h	Duration of transition $h \in \mathcal{H}$
k	Risk-free rate (discount factor)
$G_t^T(S_t)$	Terminal value given S_t at time t
$\phi_l(I_t)$	A basis function (or feature) that extracts information from I_t
L	Dimension of parametric model
$\check{\Phi}_t(R_t, a_t)$	A lower bound on the continuation value at time t given that we are in resource state R_t and take action a_t

Variables

S_t	State at time t
S_0	Initial state (i.e. state at time 0)
R_t	Resource state variable
I_t	Information state variable
N_t	Node at time t

a_{th}	(Binary) decision at time t for transition h
a_t	Action (or decision) at time t
\mathcal{A}_{S_t}	Feasible region when in S_t at time t
$\alpha_{tl}(S^R(R_t, a_t))$	Regression coefficient (or weight) when we are in resource state R_t at time t and take action a_t
W_t	Exogenous information that first becomes known at time t

Functions and mappings

$b^D(N_t)$	Set of outgoing transitions of node N_t
$S^M(S_t, a_t, W_{t+\Delta_h})$	Transition function, giving state $S_{t+\Delta_h}$ given that we are in state S_t , take action a_t (i.e. make transition h), and then learn $W_{t+\Delta_h}$, which is revealed between t and $t + \Delta_h$
$S^R(R_t, a_t)$	Resource transition function, giving resource state $R_{t+\Delta_h}$ given that we are in resource state R_t and take action a_t (i.e. make transition h)
$\Pi_t(S_t, a_t)$	Payoff at time t given we are in state S_t and take action a_t
$G_t(S_t)$	Value of portfolio of real options when in state S_t at time t
$\bar{G}_t(S_t)$	Approximation of $G_t(S_t)$
$\Phi_t(S_t, a_t)$	Continuation value at time t when in state S_t and taking action a_t
$\hat{\Phi}_t^L(S_t, a_t)$	Approximation of $\Phi_t(S_t, a_t)$

Appendix B

Additional Analyses of Chapter 4

B.1 Effect of Copper Price and Uncertainty on Mine Value

To illustrate the combined effects of the degrees of the operating margin and copper price uncertainty on the value of the opened mine, Figure B.1 shows the way in which the initial copper price, X_0 , and the standard deviation of the copper price, σ_x , effect the investment value. As we would expect, once positive, the value of both the mine with the portfolio of options (i.e. the option to switch, see Figure B.1a) and the mine without options (see Figure B.1b), which applies a static “now-or-never strategy”, increases in the initial price of copper, X_0 , and the value of the former also increases as copper price uncertainty increases, whereas the latter decreases in σ_x . This is simply because the consideration of the option to switch enables the decision maker to pro-actively manage price uncertainty by exploiting managerial flexibilities, thereby limiting downside risk whilst benefiting from upside potential; but such risks cannot be managed when flexibilities are neglected, which explains why the mine without options is adversely affected by price uncertainty. It is interesting to note that, as can be seen in Figure B.1c, the flexibility provided by the option to switch is most valuable for average copper prices – more precisely, both slightly below average X_0 and low σ_x , both average X_0 and average σ_x , as well as both slightly above average X_0 and high σ_x –, that is in areas where the mine with the options portfolio is already of value but where the mine without options is still worthless.

B.2 Effect of Interest Rate Uncertainty and Correlation on Mine Value

Figure B.2 illustrates the same analysis as in Figure 4.9 but using $\theta_\delta = 0.12$ instead of $\theta_\delta = 0.15$, with results for κ_δ equalling 0.30, 0.50 and 0.80 shown by Figures B.2a, B.2b and B.2c, respectively. While the volatility surfaces on the left-hand side of Figure B.2 are independent of θ_δ and therefore equal to the ones of Figure 4.9, the mine values and consequently the mine value surfaces for $\theta_\delta = 0.12$, which are shown on the right-hand side of Figure B.2, are as expected consistently above those obtained for $\theta_\delta = 0.15$.

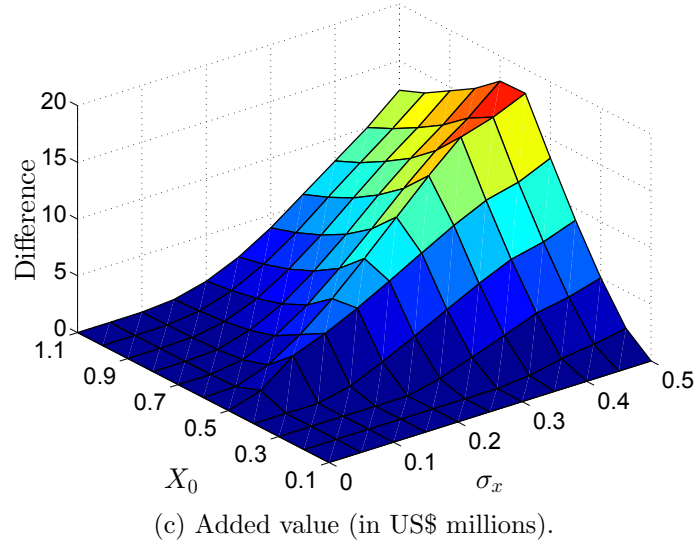
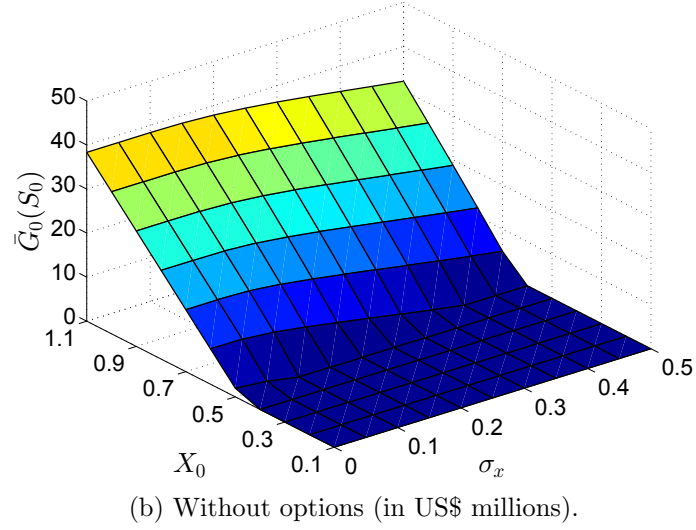
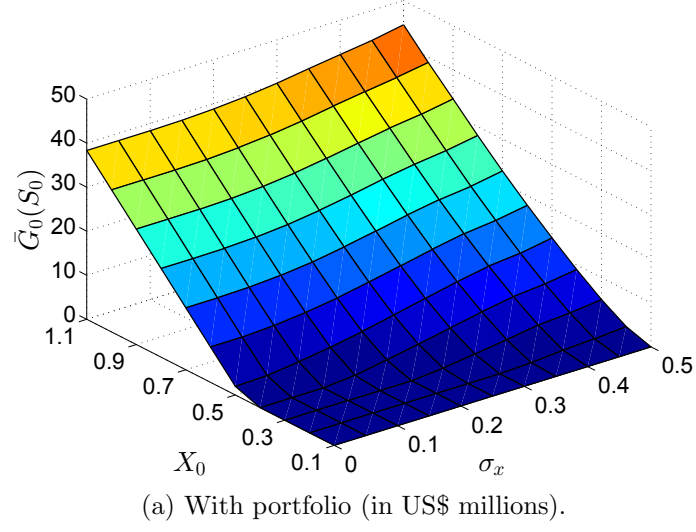
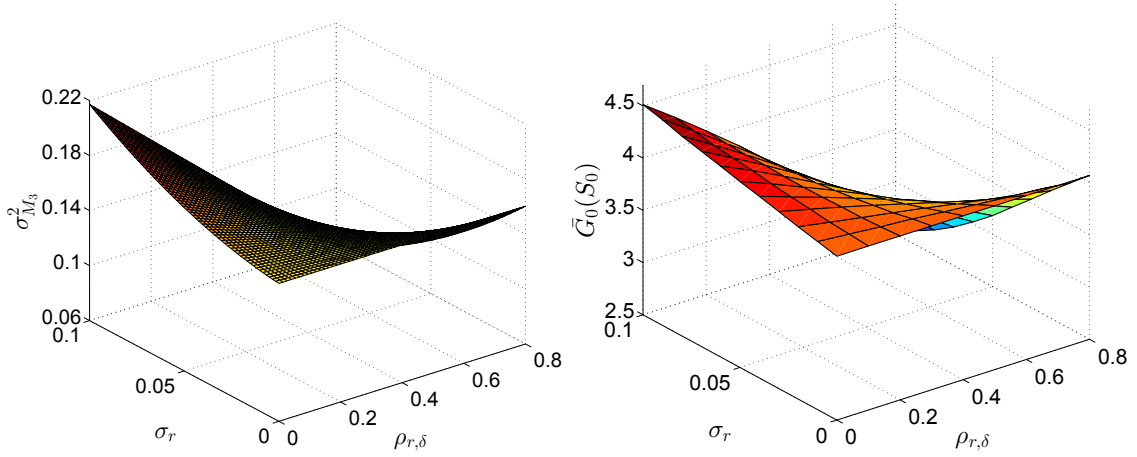
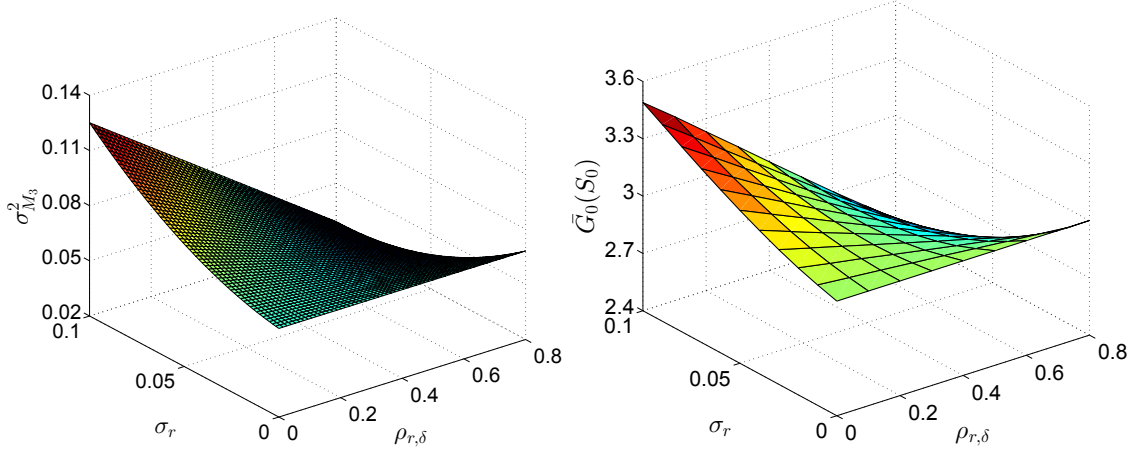


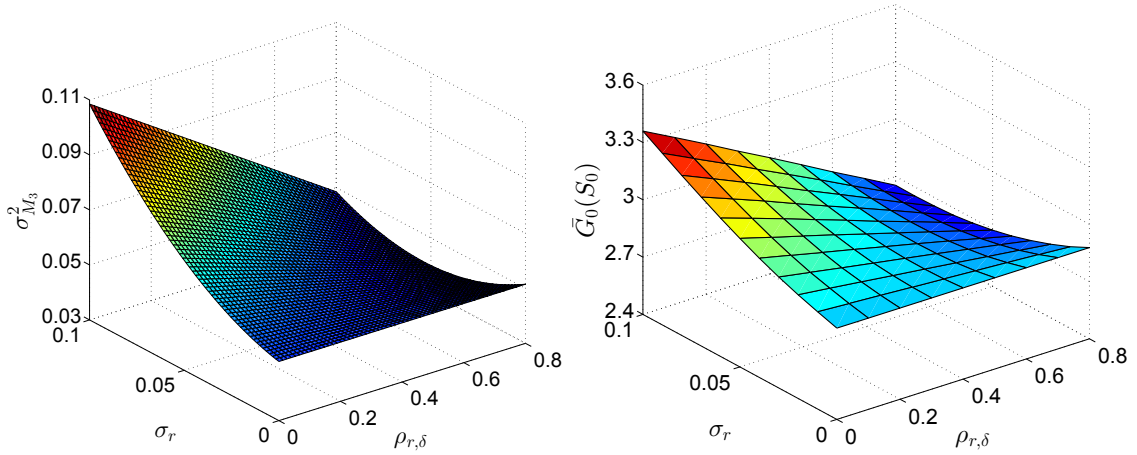
Figure B.1: Value of (opened) copper mine with portfolio of options and without options as well as added value of portfolio as a function of initial copper price (X_0) and copper price uncertainty (σ_x) – all values are in US\$ millions.



(a) With $\kappa_\delta=0.30$.



(b) With $\kappa_\delta=0.50$.



(c) With $\kappa_\delta=0.80$.

Figure B.2: Volatility in three-factor model ($\sigma^2_{M_3}$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.12$.

Appendix C

Additional Analyses of Chapter 5

C.1 Effect of Parametric Model Choices on Portfolio Value

In this appendix we investigate the effect of different parametric models on the value of the copper mine investment project. In particular, we study the Power series, Laguerre, Hermite, Legendre and generalized Chebyshev polynomials of degree L , with $L \in \{1, 2, 3, 4, 5, 6\}$. Considering the complete set of polynomials as described by Judd (1998) and the families of orthogonal polynomials as defined in (Abramowitz and Stegun, 1972), as well as using the parameters of Table C.1, Table C.2 shows the portfolio value for different univariate orthogonal polynomials and different dimensions, L , of the parametric model. Independent of the chosen family of polynomials, since we have four stochastic

Table C.1: Parameters of convenience yield process for different specifications of Tsekrekos et al. (2012).

Spec.	$\rho_{x,\delta}$	$\theta_\delta = \delta_0$	σ_δ	κ_δ
1	0.40	0.01	0.05	0.30
41	0.60	0.10	0.10	0.50
81	0.80	0.15	0.15	0.80

factors, setting L to 1, 2, 3, 4, 5, and 6, results in 5, 15, 35, 70, 126, and 210, respectively, basis functions (or regressors) in the respective parametric model.

It is evident from the results that in general the approximated value of the mine investment project (which is a lower bound on its true value) improved as the parametric model's dimension, L , increased, regardless of the family of polynomials used in the parametric model. This is in line with the multidimensional convergence results of Moreno and Navas (2003); Stentoft (2004b); Areal et al. (2008). As can be seen, the results for specifications 41 and 81 are essentially the same across polynomial families, suggesting that any polynomial family with sufficient degree L can be used in these two cases. In the case of specification 1, however, our analysis revealed considerable value differences amongst different polynomials for high model dimensions, with differences between the highest and lowest project values at around US\$ 32k and US\$ 83k for $L = 5$ and $L = 6$, respectively.

Table C.2: Value of investment project with portfolio of options (in US\$ millions) for the specifications of Table C.1 and different parametric models.

Complete set of polynomials						
Panel A: Power series						
Spec.	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
1	10.385 (0.083) *	10.742 (0.096)	10.858 (0.107)	10.993 (0.118)	11.091 (0.126)	11.201 (0.131)
41	0.546 (0.016)	0.602 (0.016)	0.637 (0.017)	0.650 (0.017)	0.655 (0.017)	0.661 (0.017)
81	0.002 (0.001)	0.020 (0.003)	0.020 (0.003)	0.023 (0.003)	0.024 (0.003)	0.026 (0.003)
Panel B: Laguerre						
Spec.	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
1	10.385 (0.083)	10.742 (0.096)	10.870 (0.107)	10.988 (0.117)	11.098 (0.124)	11.241 (0.133)
41	0.546 (0.016)	0.602 (0.016)	0.637 (0.017)	0.650 (0.017)	0.654 (0.017)	0.661 (0.017)
81	0.002 (0.001)	0.020 (0.003)	0.020 (0.003)	0.023 (0.003)	0.024 (0.003)	0.026 (0.003)
Panel C: Hermite						
Spec.	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
1	10.385 (0.083)	10.742 (0.096)	10.858 (0.107)	10.996 (0.117)	11.097 (0.126)	11.174 (0.129)
41	0.546 (0.016)	0.602 (0.016)	0.637 (0.017)	0.650 (0.017)	0.655 (0.017)	0.661 (0.017)
81	0.002 (0.001)	0.020 (0.003)	0.020 (0.003)	0.023 (0.003)	0.024 (0.003)	0.026 (0.003)
Panel D: Legendre						
Spec.	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
1	10.385 (0.083)	10.742 (0.096)	10.858 (0.107)	10.997 (0.118)	11.106 (0.123)	11.181 (0.129)
41	0.546 (0.016)	0.602 (0.016)	0.637 (0.017)	0.649 (0.017)	0.654 (0.017)	0.662 (0.017)
81	0.002 (0.001)	0.020 (0.003)	0.020 (0.003)	0.023 (0.003)	0.024 (0.003)	0.026 (0.003)
Panel E: Gen. Cheyshev						
Spec.	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
1	10.385 (0.083)	10.742 (0.096)	10.870 (0.107)	10.992 (0.116)	11.074 (0.123)	11.158 (0.128)
41	0.546 (0.016)	0.602 (0.016)	0.637 (0.017)	0.650 (0.017)	0.655 (0.017)	0.661 (0.017)
81	0.002 (0.001)	0.020 (0.003)	0.020 (0.003)	0.023 (0.003)	0.024 (0.003)	0.026 (0.003)

* Standard error in parentheses.

This suggests that the approximation can be improved considerably by choosing an appropriate polynomial family. Note that while standard errors in specification 1 are higher than in the other two specifications, project values are proportionally even higher.

Compared with values obtained for specifications 41 and 81, which eventually levelled off at high L , the trends corresponding with specification 1 seem to indicate that values have not converged yet. This suggests that an increase in L would further improve the lower bound. Although the Power series was found to perform surprisingly well and reduced computational time by about 40-45% (which is similar to the reduction found by Areal et al. (2008)), the number of basis functions used in the regression grows exponentially

in L . This substantially increased both the complexity of the parametric model and computational cost; e.g., plus 84 basis functions and 50% more time when L is increased from 5 to 6. In Section 5.4, complete sets of Legendre and Hermite polynomials with $L = 4$ are used for Table 5.3 and Figure 5.3, respectively. Although different parametric model choices may result in better approximations, our choice, which presents the best trade-off between accuracy and computational time amongst all models tested, is sufficient for the purposes of demonstration. Future work might therefore investigate the convergence properties and the computational efficiency of different parametric models in situations with both complex portfolios of interdependent real options and multiple stochastic factors.

C.2 Effect of Production Cost and Copper Price Uncertainty on Mine Value

Figure C.1 demonstrates the way in which the value of the copper mine (with options portfolio and without options) is affected by the standard deviations of the production cost and the copper price whilst highlighting the least valuable individual option in the portfolio – through colour of filled circles; by contrast, Figure 5.3 highlights the portfolio’s most valuable individual option.

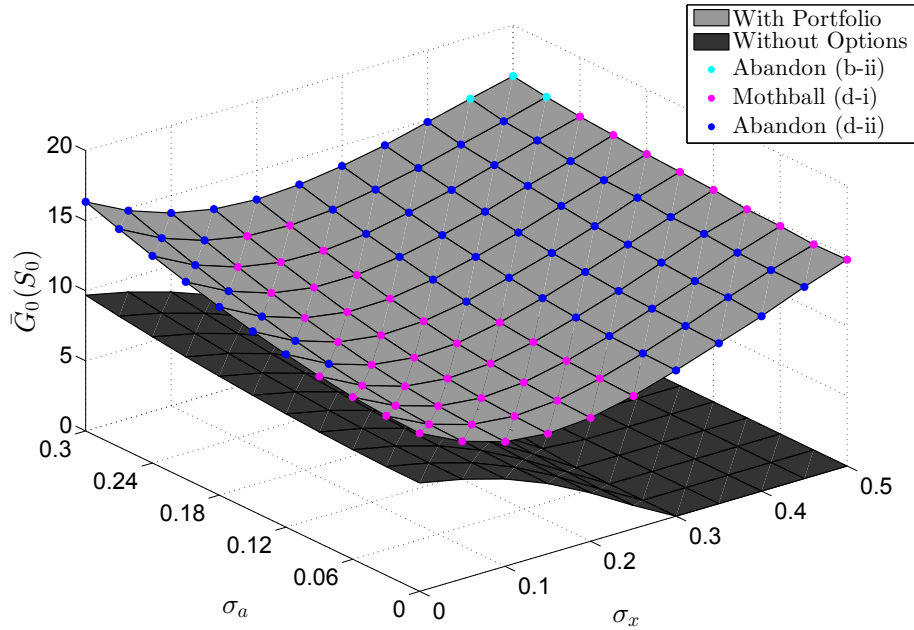


Figure C.1: Value of investment project, $\bar{G}_0(S_0)$ (in US\$ millions), with portfolio of real options and without options as well as portfolio’s least valuable individual options (filled circles), as a function of degrees of production cost (σ_a) and copper price (σ_x) uncertainty.

Appendix D

Additional Analyses of Chapter 6

D.1 Effect of Revenue and Technical Uncertainty on Investment Value

Figure D.1 demonstrates the way in which the standard deviation of changes in revenues, σ_v , and the degree of technical uncertainty, σ_k , effect the value of the investment project whilst displaying the portfolio's least valuable individual option, compared to Figure 6.5 showing its most valuable individual option.

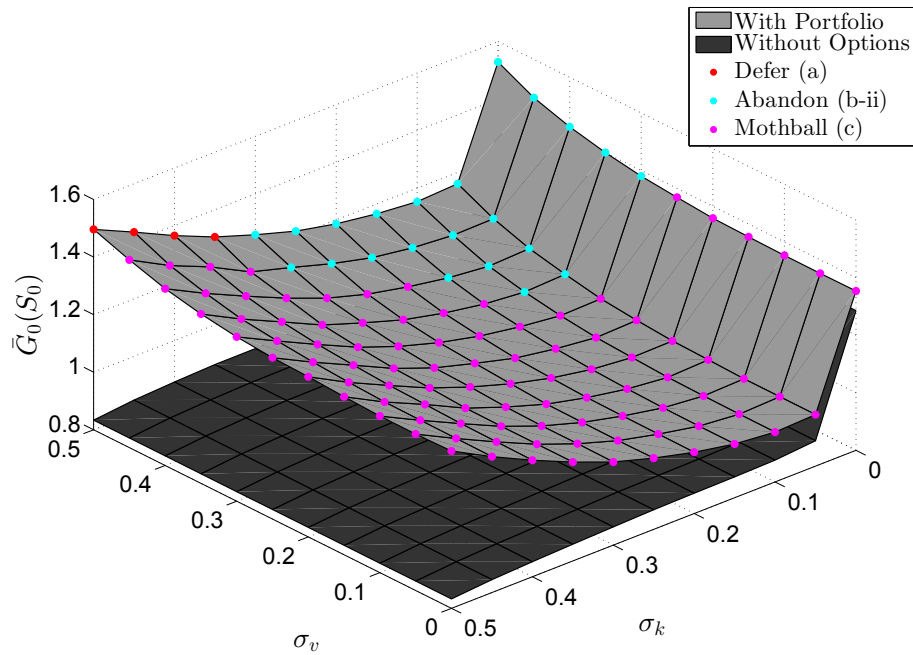


Figure D.1: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's least valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty.

D.2 Effect of Pay-out Ratios and Salvage Value Uncertainty on Investment Value

As in Figure 6.6, Figure D.2 shows the extent to which the value of the investment project is affected by the pay-out ratios γ and δ as well as by the standard deviation σ_x , but displays the portfolio's least valuable individual option instead.

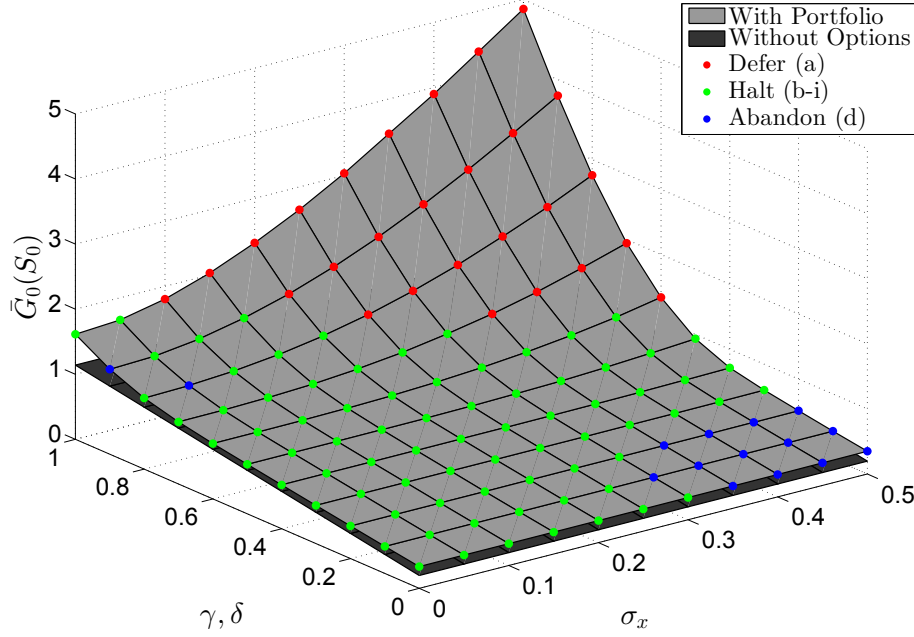
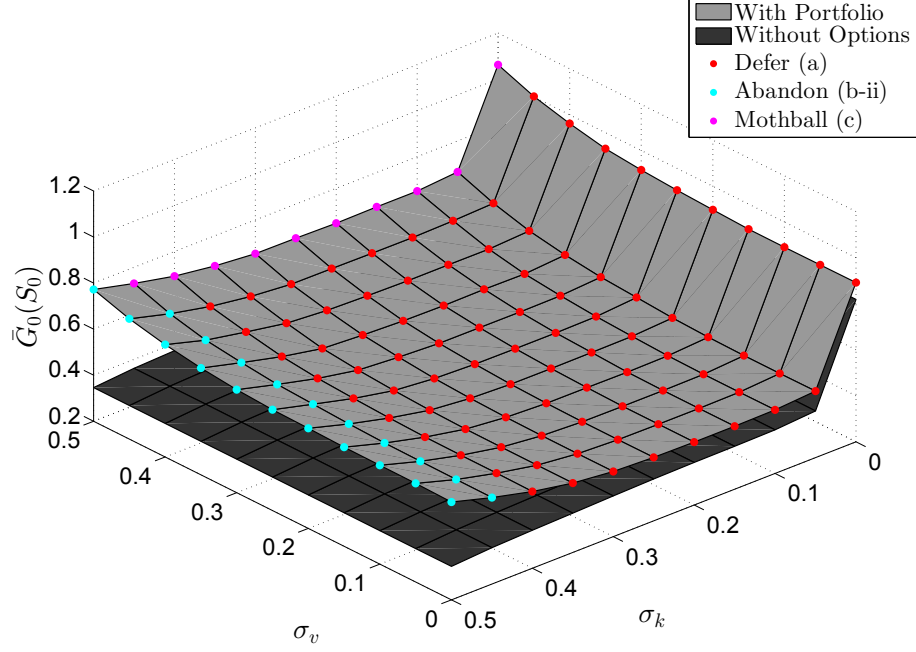


Figure D.2: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as well as portfolio's least valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x .

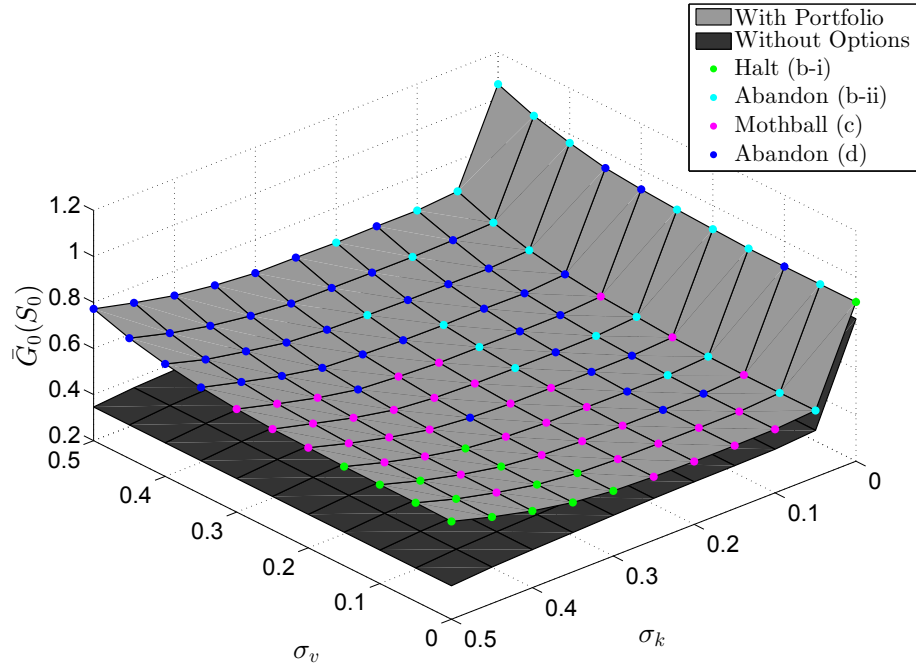
D.3 Effect of Depreciation Rate on Portfolio Value

Figure D.3 illustrates for $\xi = 0.80\%$ and $\gamma = \delta = 0.50$ the way in which the standard deviation of changes in revenues, σ_v , and the degree of technical uncertainty, σ_k , effect the value of the investment project. It is important to note that many of the portfolio's least valuable options displayed in Figure D.3b are not unique; in other words, in these situations several individual options are of equally little value in the portfolio (or even worthless).

Figure D.4 shows for $\xi = 0.80\%$ the extent to which the value of the investment project is affected by the pay-out ratios γ and δ as well as by the standard deviation σ_x .

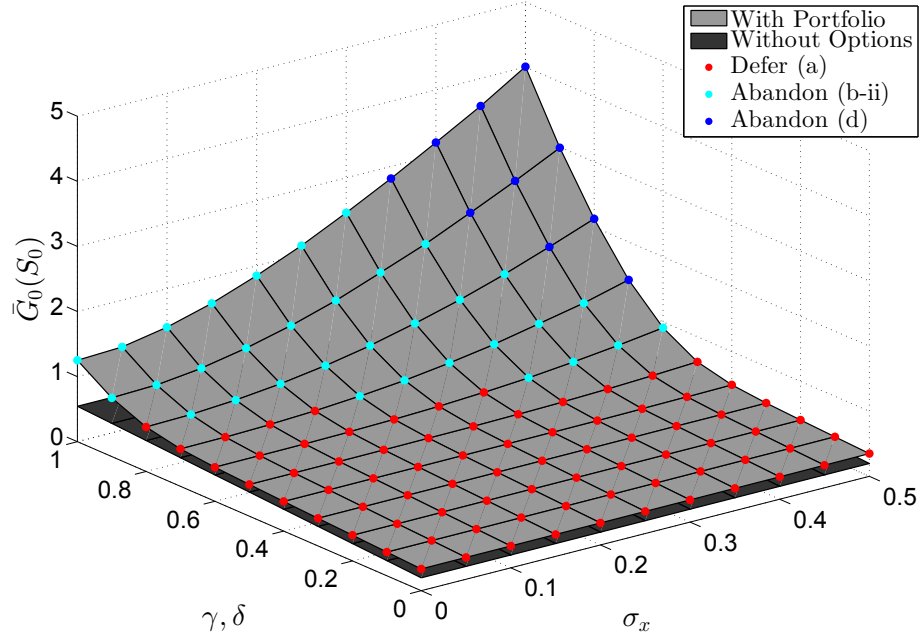


(a) Portfolio's most valuable individual option (filled circles).

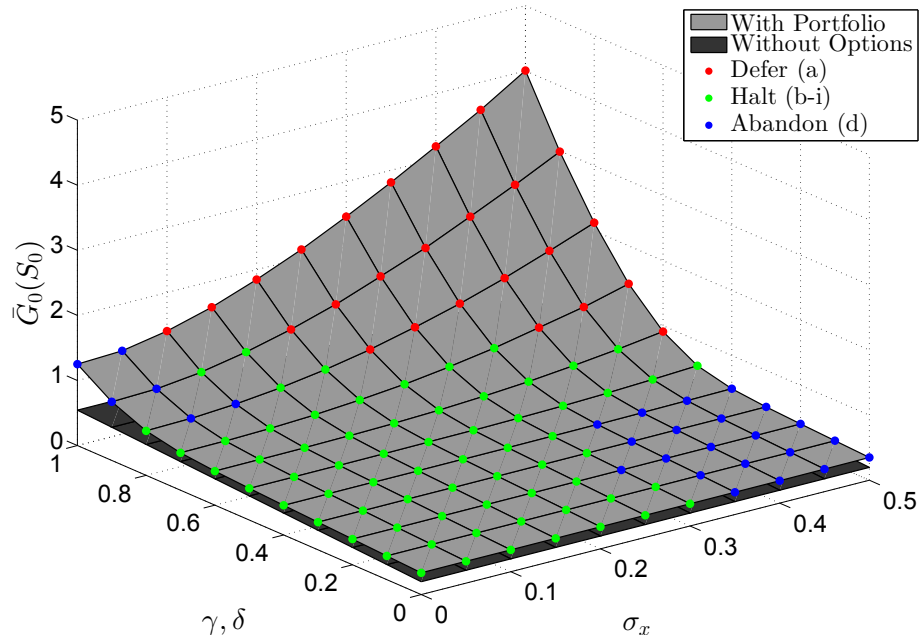


(b) Portfolio's least valuable individual option (filled circles).

Figure D.3: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty, for $\xi = 0.80\%$ and $\gamma = \delta = 0.50$.



(a) Portfolio's most valuable individual option (filled circles).



(b) Portfolio's least valuable individual option (filled circles).

Figure D.4: Value of investment project, $\bar{G}_0(S_0)$ (in £millions), with portfolio of real options and without options as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x , for $\xi = 0.80\%$.

Appendix E

List of Publications

E.1 Papers Currently under Review

Maier, S., Pflug, G. C., Gann, D. M., and Polak, J. W. (2017a). An approach for valuing portfolios of interdependent real options under both exogenous and endogenous uncertainties. Submitted.

Maier, S., Polak, J. W., and Gann, D. M. (2017b). Valuing portfolios of interdependent real options using influence diagrams and simulation-and-regression: A multi-stage stochastic integer programming approach. Submitted.

E.2 Accepted Papers in Conference Proceedings

Maier, S., Polak, J. W., and Gann, D. M. (2015). Appraising a portfolio of interdependent physical and digital urban infrastructure investments: A real options approach. In *Proceedings of the 19th Annual International Real Options Conference*, Athens and Monemvasia, Greece.

Maier, S., Polak, J. W., and Gann, D. M. (2016). Valuing strategic interactions in systemic urban infrastructure investments using option games: A portfolio-based approximate dynamic programming approach. In *Proceedings of the 20th Annual International Real Options Conference*, Oslo and Trondheim, Norway.

Maier, S. (2017). An approach for valuing portfolios of interdependent real options under both exogenous and endogenous uncertainties. In *Proceedings of the 21st Annual International Real Options Conference*, Boston MA, USA.

E.3 Research Reports

Maier, S. (2016). Risk-managing a portfolio of systemic low-carbon urban infrastructure investments using approximate dynamic programming with decision dependent uncertainties. YSSP Report, International Institute for Applied Systems Analysis (IIASA).