

Abstract of thesis entitled

# TOPICS IN PORTFOLIO MANAGEMENT

submitted by

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In this thesis, two topics in portfolio management have been studied: utility-risk portfolio selection and a paradox in time consistency in mean-variance problem.

The first topic is a comprehensive study on utility maximization subject to deviation risk constraints. Under the complete Black-Scholes framework, by using the martingale approach and mean-field heuristic, a static problem including a variational inequality and some constraints on nonlinear moments, called Nonlinear Moment Problem, has been obtained to completely characterize the optimal terminal payoff. By solving the Nonlinear Moment Problem, the various well-posed mean-risk problems already known in the literature have been revisited, and also the existence of the optimal solutions for both utility-downside-risk and utility-strictly-convex-risk problems has been established under the assumption that the underlying utility satisfies the Inada Condition. To the best of our knowledge, the positive answers to the latter two problems have long been absent in the literature. In particular, the existence of an optimal solution for utility-semivariance problem, an example of the utility-downside-risk problem, is in substantial contrast to the nonexistence of an optimal solution for the mean-

semivariance problem. This existence result allows us to utilize semivariance as a risk measure in portfolio management. Furthermore, it has been shown that the continuity of the optimal terminal wealth in pricing kernel, thus the solutions in the binomial tree models converge to the solution in the continuous-time Black-Scholes model. The convergence can be applied to provide a numerical method to compute the optimal solution for utility-deviation-risk problem by using the optimal portfolios in the binomial tree models, which are easily computed; such numerical algorithm for optimal solution to utility-risk problem has been absent in the literature.

In the second part of this thesis, a paradox in time consistency in mean-variance has been established. People often change their preference over time, so the maximizer for current preference may not be optimal in the future. We call this phenomenon time inconsistency or dynamic inconsistency. To manage the issues of time inconsistency, a game-theoretic approach is widely utilized to provide a time-consistent equilibrium solution for dynamic optimization problem. It has been established that, if investors with mean-variance preference adopt the equilibrium solutions, an investor facing short-selling prohibition can acquire a greater objective value than his counterpart without the prohibition in a buoyant market. It has been further shown that the pure strategy of solely investing in bond can sometimes simultaneously dominate both constrained and unconstrained equilibrium strategies. With numerical experiments, the constrained investor can dominate the unconstrained one for more than 90% of the time horizon. The source of paradox is rooted from the nature of game-theoretic approach on time consistency, which purposely seeks for an equilibrium solution but not the ultimate maximizer. Our obtained results actually advocate that, to properly implement the concept of time consistency in various financial problems, all economic aspects should be critically taken into account at a time.

# TOPICS IN PORTFOLIO MANAGEMENT

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A thesis submitted in partial fulfilment of the requirements for  
the Degree of Joint Doctor of Philosophy  
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## DECLARATION

I declare that this thesis represents my own work under the supervision of Prof. Hailiang Yang and Dr. Harry Zheng during the period 2013-2016 for the degree of Joint Doctor of Philosophy at Imperial College London and the University of Hong Kong. The work submitted has not been previously included in any thesis, dissertation or report submitted to these Universities or to any other institution for a degree, diploma or other qualification.

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WONG, KWOK CHUEN

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# Chapter 1

## Introduction

Since the first introduction in Markowitz (1952), portfolio management has become one of the key research topics in finance. In financial markets, investors always have to make decisions on asset allocations and plan for the future. Investing more proportion of wealth in a more risky asset can enhance the expected portfolio payoff, meanwhile, it also increases the uncertainty on the portfolio payoff, called “portfolio risk”. Portfolio management is a study on striving for an optimal portfolio which maintains an ideal balance between portfolio return and portfolio risk. Hence, the research on portfolio management can provide more effective portfolio management and monitor prevalent risks in financial markets.

Samuelson (1969) and Merton (1971) extended Markowitz’s portfolio selection problem from single-period framework to multi-period and continuous time settings respectively. Stochastic control theory is widely applied to dynamic decision making by formulating a stochastic control problem. A portfolio adopted by the investor is represented by a control function. An objective function, which is a functional of the control function, is used for representing the investor’s preference in the portfolio. Under this setting, the investor aims to choose the ideal control function which maximizes the objective function. There are two common approaches to solve the stochastic control problem and portfolio selection, namely,

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dynamic programming approach and martingale approach. The former approach makes use of dynamic programming principle (DPP) to obtain the Hamilton-Jacobi-Bellman (HJB) equation, which characterizes the optimal portfolio, by invoking the inherent tower property. The optimal portfolio can be obtained by solving the HJB equation using partial differential equation (PDE) methods; see Merton (1971), Touzi (2002), and Fleming and Soner (2006) for details. Alternatively, the martingale method can be applied to solve for this utility maximization problem in a complete market, where the existence of the optimal solution can be shown by using duality method, and then utilize the Clark-Ocone formula to seek for the optimal weight; see Pliska (1982, 1986), Karatzas et al. (1987, 1991), Cox and Huang (1989), and Deelstra et al. (2001) for details.

In this thesis, two topics in portfolio management have been investigated. The first topic is a comprehensive study of the utility-deviation-risk portfolio selection problem. In this framework, the investor aims to maximize the expected utility of portfolio terminal payoff and to minimize the portfolio risk induced by the deviation of the portfolio terminal payoff from the expected payoff at the same time. We establish the existence and characterization results of optimal solutions to the utility-risk problems. Furthermore, we provide a numerical algorithm to compute the optimal solution. As the result, an optimal asset allocation which maximizes investor's satisfaction gained from portfolio payoff and manages the risk of the underlying portfolio simultaneously can be obtained. The second topic concerns the time consistency in portfolio management. People often change their preference over time, thus, in dynamic decision making, a decision which is optimal for the current preference however may not be optimal for the future preference. We call this intertemporal conflict time inconsistency or dynamic inconsistency. The continuous-time mean-variance problem is an example of time-inconsistent problems. A game-theoretic approach is commonly used to provide a time-consistent solution, called equilibrium strategy. Our main contribution to the second topic

is a paradoxical result in the equilibrium solution to mean-variance problem: an investor facing more investment constraints can acquire a greater objective value than his counterpart with less. An overview of these two topics and a summary of our work in this thesis will be described in this chapter.

## 1.1 Utility-risk Portfolio Selection

Expected utility and mean-variance are two common criteria for evaluating portfolio performance. For example, Samuelson (1969) and Merton (1971) investigated utility maximization problems in multi-period and continuous time settings respectively, while Markowitz (1956) and Merton (1972) aimed to minimize the variance of the portfolio return subject to a constraint on the expected return of the terminal wealth, and they also established the efficient frontier.

The advantage of the utility maximization formulation allows a direct application of dynamic programming or via HJB by invoking the inherent tower property. The advantage of using mean-variance criteria is due to its relative computational simplicity and convenience in selling in bulk to accommodate market demand; indeed, different consumers possess different utilities towards return, but due to the limitation of resources available, it is more convenient to sell a uniform package which can cater for the needs of most people. Levy and Markowitz (1979) showed that the optimal portfolio in utility maximization can be approximated by the mean-variance efficient frontier over ranges of commonly used utilities, return rates and volatilities. Hence, the mean-variance portfolio can basically entertain the almost optimal satisfaction of common consumers. Further studies support this approximation; for instances, see Markowitz (1959, 2010), Pulley (1981), and Kroll et al. (1984).

Due to the nonlinear nature of the square function of the expectation of the terminal wealth involved in the variance, an immediate application of dynamic

programming principle is not viable, which results that the analytic research in mean-variance portfolio optimization is used to mainly focused on single-period models at the first stage. The embedding technique developed by Li and Zhou (2000) broke the ice by converting the mean-variance problems under both continuous time and multi-period settings into the canonical linear-quadratic stochastic control problems. From that point on, more complicated mean-variance problems have also been investigated, in work such as Lim and Zhou (2002), Bielecki et al. (2005), Chen et al. (2008) and Chen and Yang (2011).

Variance is not the only risk measure commonly adopted in the portfolio selection problem. Jin et al. (2005) consider a general convex risk function of the deviation of the terminal payoff from its own mean, by following the Lagrangian approach as proposed in Bielecki et al. (2005), to characterize the optimal terminal payoff, and then they applied the Clark-Ocone formula to determine the optimal portfolio weights. Besides, they also studied the mean-downside-risk problem and established the non-existence of an optimal solution by showing that the optimal value function is unattainable by any admissible control. The downside-risk measure can remedy the common criticism on incurring penalty on the upside return which happens in the use of variance. Markowitz (1990) also claims that “semivariance (an example of downside risk measure) seems more plausible than variance as a measure of risk since it is concerned only with adverse deviations”. In contrast to continuous time models, Jin et al. (2006) solved for the single-period mean-semivariance portfolio selection problem. After that, the study on the optimization problem subject to downside risk measure has been absent until the recent study by Cao et al. (2014), in which they showed that mean-lower-partial-moments problem possesses a positive solution if we impose a uniform upper bound on the terminal payoff. For the relevant literature in connection with downside risk measure and semivariance, see also Hogan and Warren (1972), Nantell and Price (1979), Nawrocki (1999), and Steinbach (2001). Apart from

using deviation risk measure, He et al. (2015) studied continuous-time mean-risk portfolio choice problems with general risk measures including VaR, CVaR, and law-invariant coherent risk measures.

Turning back to reality, a number of financial crises have been observed frequently over recent decades, so tighter government regulations have been enforced in the financial market. On the other hand, the intensive competition in the market pushes any old-fashioned profitable strategies to the edge; all of these urge most companies to provide more tailor-made investment products in order to maintain their profit margins. A uniform package such as the mean-variance portfolio mentioned above can barely satisfy the demand of sophisticated investors nowadays, and a definitive answer to utility maximization with minimal risk is eagerly sought. Nevertheless, before our present work, the solution to this most relevant optimization problem has still been long absent in the literature.

In Chapter 2, we first provide a comprehensive study of utility-risk portfolio selection problems: we suggest that the objective function of portfolio selection is not simply the expected value of a certain functional of the terminal payoff, but it also deals with the deviation risk caused by the underlying portfolio. Note that Jin et al. (2005) call their problem formulation mean-risk problem, though they only consider deviation risk measure in their work, so to avoid ambiguity, we use a single word “risk” to stand for the deviation risk measure throughout this thesis. Our proposed problem follows the recent trend of embedding various risk management criteria into the utility maximization framework. Such risk-monitoring mechanisms reduce the drawback caused by the ambitious investment strategy in pure utility maximization problems, which could lead to higher risk of potential pecuniary loss (see Zheng, 2009). To name a few along this direction, Basak and Shapiro (2001) first suggested implementing a Value-at-Risk (VaR) constraint into the portfolio optimization due to the prevailing regulation on VaR limitation. The research in Yiu (2004), Leippold et al. (2006), and Cuoco et al.



(2008) further turn the VaR limitation from a static constraint to a dynamic one in various utility-optimization problems. Besides, Zheng (2009) studied the efficient frontier problem of both maximizing the expected utility of the terminal wealth and minimizing the conditional VaR of any potential loss. To the best of our knowledge, our present work is the first attempt to apply risk management to utility maximization subject to the deviation risk measure.

More precisely, we model the objective function as the difference of deviation risk (function of the deviation of the terminal payoff from its own mean) from the utility (concave increasing function of the terminal payoff); see 2.1.2. We first follow the same idea as in Bielecki et al. (2005) and Jin et al. (2005) to convert our dynamic optimization problem into an equivalent static problem. By considering the first-order condition for the objective function, we can obtain a primitive static problem, called the *Nonlinear Moment Problem*, which characterizes the optimal terminal wealth with respect to the respective necessity and sufficiency results (Sections 2.2.1 and 2.3.1), which are fundamentally different, and not equivalent to each other. For necessity, the optimal terminal wealth satisfying two mild regularity conditions (Conditions 2.2.1 (i) and (ii)) solves for the Nonlinear Moment Problem; while for sufficiency, the solution of the Nonlinear Moment Problem that satisfies Condition 2.3.1 serves as the optimal terminal wealth. Note that this Nonlinear Moment Problem includes a variational inequality (2.2.4) with a set of constraints (2.2.5)-(2.2.7) involving the expectation of some nonlinear functions of the optimal terminal wealth and its own mean, or the “mean-field term” in the context of mean-field type control theory. The formulation of the Nonlinear Moment Problem is motivated by the mean-field approach developed in Bensoussan et al. (2014), in which the authors studied the classical mean-variance problem with the aid of a novel mean-field type HJB equation. Note that the same static problem may be obtained via the formal Lagrangian multiplier approach as in Bielecki et al. (2005) and Jin et al. (2005).

With the aid of the Nonlinear Moment Problem, our necessity conditions warrant an alternative deduction of the non-existence result of the mean-semivariance problem, first considered in Jin et al. (2005). On the other hand, for the application of the sufficiency conditions together with the Nonlinear Moment Problem, we replicate the explicit construction of the optimal solutions of various well-posed mean-risk problems in the existing literature. Furthermore, the novelty of our new approach allows us to establish new existence result for the optimal solutions for a variety of utility-risk problems, especially the utility-downside-risk (in Section 2.3.2) and the utility-strictly-convex-risk problems (in Section 2.3.3), in which the underlying utility satisfies the common Inada Condition. To the best of our knowledge, these problems have not been considered so far before our work. Note that by the sufficiency result in Theorem 2.3.2, we can conclude that there exists an optimal solution for the utility-downside-risk problem including utility-semivariance problem, and this result is in contrast to Jin et al. (2005), in which they find that the continuous-time mean-downside-risk problem possesses no optimal solution at all. As a consequence, the possibility of using semivariance as a natural risk measure in the portfolio selection can now be legitimately implemented.

## 1.2 Numerical Valuation of Optimal Utility-risk Portfolio Payoff

In Chapter 2, we characterize the optimal terminal wealth to utility-risk problem using Nonlinear Moment Problem. In order to obtain explicit solutions for utility-risk problems, we need to first obtain the explicit solution for the corresponding system of nonlinear equations simplified from the Nonlinear Moment Problem. Since the system consists of improper integrations of the implicit function of a variable over the positive real line, an explicit numerical solution of the equation

system is usually difficult to be obtained even in the simple cases such as power-utility-variance problems (the details can be refer to Remark 3.3.15). To compute integration numerically, a usual approach is to discretize the domain so that the integral can be approximated by a finite sum. In Chapter 3, we make use of this idea in a more intuitive way in the sense that the optimal terminal payoffs for continuous-time utility-risk problem in Chapter 2 is approximated by the optimal terminal payoffs in the binomial tree model.

A main objective of portfolio selection is to implement the optimal portfolio in practice, while the theoretical existence result is an intermediate achievement. The explicit form of optimal portfolio is eagerly sought as we can compute the optimal portfolio weight and implement the optimal portfolio directly. However, most problems in portfolio selection do not have an explicit form of optimal portfolio, so we may seek for some numerical method to trace the optimal solution. Through DPP approach, the optimal portfolio can be characterized by a HJB equation. The discrete approximation of HJB equation such as finite difference method can be applied to obtain the optimal portfolio and its corresponding value function. For instance, perpetual utility maximization with the respective convergence results was studied in Fitzpatrick and Fleming (1991). For the other literature which uses discrete approximation of HJB equation in portfolio selection paradigm, one can consult Brennan et al. (1997) and Forsyth and Wang (2010). Alternative, Campbell and Viceira (1999) obtain approximate closed-form solutions through log-linearizing the first-order condition characterization of optimal solution. Under the martingale approach, the optimal terminal wealth for portfolio selection can be obtained, but there is no information about the corresponding optimal portfolio. To resolve this problem, Cvitanić et al. (2003) and Detemple et al. (2003) utilize a Monte Carlo method to simulate the optimal solutions.

Apart from discretizing the analytic characterization of optimal solution directly, we can approximate the optimal solution under continuous-time setting

by using the solution in discrete time model which is easier to compute. He (1991) establish that the sequence of optimal solutions to utility maximization under discrete-time lattice models to continuous-time diffusion model with multi assets. The corresponding discrete-time solution is obtained by solving a partial differential equation derived through DPP approach. The book by Prigent (2003) provides a comprehensive study of the weak convergence of the optimal solution obtained through the martingale method from discrete-time models to continuous time model.

In Chapter 3, we first extend the utility-risk framework in Chapter 2 to a generalized model setting such that the dynamics of asset prices can be unspecified. We make assumptions on the existence of pricing kernel and complete market (see Section 3.1). In this case, the discrete binomial tree model, the continuous-time Black-Scholes model (which is used in Chapter 2), and more complicated stochastic interest rate model are covered (see examples in Section 3.1). In contrast with Chapter 2, we focus on the combinations between utility function which satisfies the Inada conditions and strictly concavity and two types of deviation risk measure, namely, downside risk and strictly convex risk. Under this problem formulation, two problems will be studied in Chapter 3:

- (i) The existence and uniqueness of the optimal solution.
- (ii) The conditions for the convergence of the optimal solutions under different model settings.

The numerical approximation of the optimal solution under the continuous-time Black-Scholes model by the solutions in the binomial tree models can be justified by verifying these models satisfy the conditions in Problem (ii).

Since the solution for Nonlinear Moment Problem in Chapter 2 depends on market settings in terms of the pricing kernel only, the similar Nonlinear Moment Problem with the corresponding necessity and sufficiency results can be obtained

(See Section 3.2) by using the martingale approach, even though the dynamics of asset prices are not specified. Following the similar arguments in Chapter 2, we establish the existence of optimal solutions for utility-downside-risk and utility-strictly-convex-risk problems under our generalized framework. Because of the strictly concavity assumption, any solution to our utility-risk problem is the unique solution. Hence, Problem (i) is resolved.

As for Problem (ii), we first establish that the optimal terminal payoff to utility risk problem is a continuous function of pricing kernel in the sense that, if a sequence of the pricing kernels converges weakly, the corresponding optimal terminal payoffs under the market with such pricing kernels converge weakly to optimal terminal payoff under the market with the corresponding pricing kernel being the limit of the sequence of pricing kernels (see Section 3.3). Larsen and Žitković (2007) conduct similar convergence studies on how small perturbations of the market coefficient processes changes the optimal solution for utility maximization. The proof of this stability result made use of convex-duality techniques. In contrast, since the optimal terminal wealth to our utility-risk problem is in terms of a solution of a nonlinear equation system, we prove our convergence result in Section 3.3 by establishing the limit of the sequence of the solutions of the equation system is finite, followed by applications of the Dominated Convergence Theorem.

Afterward, we shall verify that the conditions in Problem (ii) are satisfied by the pricing kernels under the discrete binomial tree market models which are constructed from the continuous-time model in Chapter 2 (see Section 3.3.3). The proof of weak convergence of pricing kernel is motivated by Föllmer and Schied (2004) which establish the weak convergence of stock pricing from the binomial-tree model to the continuous-time Black-Scholes model through an application of the Central Limit Theorem. With the results in Section 3.3, we have a numerical algorithm to approximate the optimal solution for the continuous-time utility-risk

problem by the optimal terminal payoffs in the binomial tree model.

## 1.3 A Paradox in Time Consistency in Mean-Variance Problem?

In the financial investment context, agents barely keep the same preference over time; in contrast, they usually make an investment decision that may only be optimal with respect to their contemporary utility while disregarding its distant future prospect. This myopic decision may lead to some potential substantial burden in the longer run as investors currently underestimate their future responsibility. This phenomenon is described as time inconsistency in decision making.

There is a long history of the study on time consistency with intense popularity. Strotz (1955) first raised the issues of time inconsistency, and considered a continuous-time deterministic consumption problem with non-exponential discount factor, which is not necessarily a constant. The Euler differential equation obtained in his setting varies with the initial time, and so the corresponding optimal path depends on the initial time (or evaluation time); however, this optimal solution would no longer be optimal for agent's future preference. Recall the Bellman optimality principle (Bellman, 1957):

*“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision.”*

The optimization problem under Strotz's framework is time inconsistent in the sense that it cannot satisfy the Bellman optimality principle. Loosely speaking, a dynamic optimization is time-consistent if its maximizer is dependent solely on the current states. With regard to this inconsistent matter, if the agent could have a choice of reconsidering his strategy in the future and there is no commitment

on his strategy, he should give up the original strategy which was only optimal at an earlier time, and adopt another one which is contemporarily optimal at that future time. Strotz described this agent as **spendthrift**<sup>1</sup> since his behavior is inconsistent with his original plans.

Furthermore, Strotz (1955) suggested two possible resolutions to deal with time-inconsistency matter, namely: precommitment and consistent strategies. Under the **precommitment** policy, an individual precommits his future activities so that he will implement the optimal path even though it will then no longer be optimal in the future. On the other hand, the consistent planning is one through which the agent is supposed to look for “*the best plan among those that he will actually follow*”. The main feature of Strotz’s consistent strategy is that the current action is the one that makes the agent to achieve the best possible value of his contemporary objective function under the assumption that all his future actions shall be optimal. The Strotz’s idea was later interpreted in terms of game theory by Peleg and Yaari (1973): we consider a time-inconsistent problem as a non-cooperative game, in which the players of the game represent different time points, and choose their strategy in order to maximize their own objective functions; then the notion of Nash subgame perfect equilibrium of this game problem is then used to define the “time-consistent” strategy for the original problem. Therefore, the Strotz’s consistent planning and the corresponding time-consistent strategy are named as **game-theoretic** approach and equilibrium strategy respectively. Pollak (1968) supplemented Strotz’s notion by suggesting backward recursive evaluation method, which can serve as an effective algorithm for finding the time-consistent equilibrium solution. Besides, under logarithmic utility, Pollak also showed that the equilibrium solution coincides with the spendthrift

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<sup>1</sup>In economic literature such as Pollak (1968) and Marín-Solano and Navas (2010), the notion of spendthrift is sometimes referred as “naive”, since the agent may be too naive to even not recognize the time inconsistency issue.

solution for Strotz's problem. Goldman (1980) provided some general conditions under which the subgame perfect equilibrium, as described in Strotz-Pollak's work, exists.

In summary, there are three approaches of tackling time-inconsistent problems as described above: 1) spendthrift; 2) precommitment; and 3) game theoretic one. In Section 4.1.2, we shall provide the mathematical descriptions of time inconsistency and the notion of these three approaches including the definition of equilibrium solution. The comparison among these three approaches will be discussed through a concrete example.

It is well known that the classical mean-variance portfolio selection problem has time-inconsistency aspects. In particular, due to the non-linearity in the expectation of the second moment in the objective function (see Björk and Murgoci, 2010), the usual Tower Property fails to hold, so the corresponding optimization problem can never admit the Bellman optimality. For instance, the maximizer for mean-variance problem depends on both the current and the initial states (see Li and Zhou (2000) or Example 4.1.2 in Section 4.1.1 of the present thesis). Mean-variance problem under multi-period and continuous-time frameworks were solved using precommitment approach in Li and Ng (2000) and Li and Zhou (2000) respectively. The literature provided in Sections 1.1 and 1.2 regarding the optimization problem which does not admit the Bellman optimality principle, such as Bielecki et al. (2005), Jin et al. (2005), and Chen et al. (2008), and also our work in Chapters 2 and 3 concern precommitment solution.

Apart from mean-variance problem, the consumption and saving problem subject to non-exponential discounting is a substantial topic in time consistency. The game-theoretic approach can be utilized to solve the time inconsistency issues in this time-inconsistent consumption and saving problem. Phelps and Pollak (1968) first used quasi-hyperbolic discount to model intergenerational time preference in discrete-time stationary Ramsey's saving problem and obtained an explicit time-



consistent equilibrium solution for isoelastic utility. This quasi-hyperbolic discount is geometrically decaying across all dates except at the current date and it has a sharp short-run drop at this time. This phenomenon reflects that people are more impatient in the shorter term so that their discount behaves like a hyperbolic function; to name a few, the empirical supports on this observation can be found in Thaler (1981), Ainslie (1992), and Loewenstein and Prelec (1992). The study on time consistency in Ramsey's problems under non-exponential discount was then followed by Laibson (1997), Harris and Laibson (2001), and Krusell and Smith (2003), over discrete-time settings; and Barro (1999), Karp (2007), and Ekeland and Lazrak (2010) over continuous-time settings. Laibson (1997) and Harris and Laibson (2001) characterized the equilibrium solutions by using Euler-type equations. Karp (2007) derived the dynamic programming equation as the necessary condition for equilibrium solutions. Ekeland and Lazrak (2010) carried the computations in addition to that in Karp (2007), and obtained the Hamilton-Jacobi-Bellman (HJB) equation with the corresponding verification theorem to characterize the equilibrium solution. The study on time consistency over stochastic frameworks was initiated by Harris and Laibson (2001), and they studied the Ramsey's problem with stochastic income subject to the quasi-hyperbolic discount. Harris and Laibson (2013) later extended their previous work in Ramsey's problem by introducing a randomness in the time duration of short-run drop. Using HJB equation, Ekeland and Pirvu (2008) and Marín-Solano and Navas (2010) solved the stochastic Merton investment and consumption problem under non-exponential discount, which includes quasi-hyperbolic discount as a special case. In addition, Marín-Solano and Navas (2010) compared the equilibrium strategy with precommitment and spendthrift strategies for specific utility functions, namely, logarithmic, power, and exponential utilities.

The notion of equilibrium solution for continuous-time optimization problem is more intricate than that under the discrete-time setting in which time units

together can be interpreted as a finitely many player non-cooperative game. Ekeland and Pirvu (2008) provided a precise mathematical definition of equilibrium solution for the continuous-time problems among all Markovian controls by considering the limiting case when the decision-maker can commit only over each infinitesimal unit of time. Björk and Murgoci (2010) supplemented the former work by extending to a general class of continuous-time time-inconsistent Markovian control problems. With this definition, an extended HJB equation system was derived to characterize the equilibrium solution. Those two works allow us to investigate more subtle problems in connection with time consistency, for instance, see further developments in Ekeland et al. (2012), Bensoussan et al. (2014), Björk et al. (2014), and Kronborg and Steffensen (2015).

Besides, Hu et al. (2012) extended the notion of the equilibrium control among all open-loop controls. By using the stochastic maximum principle, they derived a linear forward-backward stochastic differential equation to characterize the equilibrium control for linear-quadratic time-inconsistent control problems. They showed that the corresponding equilibrium strategy can sometimes, in a broader sense, be different from that under the Björk and Murgoci's framework.

The time-consistent solution for the classical mean-variance paradigm was first obtained in Basak and Chabakauri (2010) through the derivation of dynamic programming principle. On the other hand, Björk and Murgoci (2010) obtained the same time-consistent solution through the studying of the extended HJB system. Czichowsky (2013) later studied the mean-variance problem under a general semimartingale setting and obtained the time-consistent solution through taking limit of the solutions for the corresponding discrete-time models, which can be determined by backward recursive argument. Furthermore, Basak and Chabakauri (2012) solved the mean-variance problem on incomplete markets.

In contrast to the game theoretic approach, Pedersen and Peskir (2015) solved the classical mean-variance problem using an alternative notion of time-consistent

solution, which, in fact, aligns with the spendthrift approach, also see Example 4.1.4 in Section 4.1.2 for detail.

Since the common equilibrium solution of mean-variance problem with constant risk aversion in Basak and Chabakauri (2010) and Björk and Murgoci (2010) is completely independent of the current state, Björk et al. (2014) criticized that this state-independent solution is not economically sounding as the investor puts the same dollar amount in stock regardless of his current wealth. Therefore, Björk et al. (2014) and Björk and Murgoci (2014) revisited the mean-variance problem but with wealth-dependent risk aversion under continuous and discrete time settings respectively. Under these new settings, they established that the equilibrium controls are linear in the current wealth if the risk aversion varies inversely with the current wealth. Apart from considering the state-dependent risk aversion, Wei et al. (2013) obtained a state-dependent solution for mean-variance asset-liability problem, in which the liability process is somehow exogenous so that the market is incomplete.

It should be noted that the mean-variance problem with such a dependence of risk aversion on the current wealth in Björk et al. (2014) and Björk and Murgoci (2014) critically relies on the positivity of the current wealth all the time. In the absence of the confinement of short-selling prohibition, one can often find an admissible control resulting in a negative wealth process, especially over the discrete-time setting. The negative wealth will cause the investor's risk aversion to be negative which lead the investors to be risk seeking. In this case, the mean-variance utility becomes unbounded, and the problem becomes ill-posed. Since the obtained equilibrium portfolio in Björk et al. (2014) is only linear in wealth, its corresponding equilibrium wealth process is a geometric Brownian motion, and thus the process keeps positive over the time horizon almost surely. However, in the discrete time framework with the allowance of shortselling in Björk and Murgoci (2014), the positivity of the optimal wealth process is no longer guaran-

teed even when the equilibrium strategy is linear (see Remark 2.6 in Bensoussan et al., 2014). This ill-posed issue seems to expose the economic limitation of the framework in Björk and Murgoci (2014); to remedy this shortcoming yet with more economic relevance under the trend of setting tightening regulation in favor of shortselling prohibition after the recent financial crisis, forbidding shorting of both bonds and stocks can ensure the positivity of the corresponding equilibrium wealth process. Hence, with the enforcement of shortselling prohibition, Bensoussan et al. (2014) revisited the problems of Björk et al. (2014) and Björk and Murgoci (2014), and obtained the constrained equilibrium solutions. For other recently interesting works on the time consistent portfolio selection under mean-variance setting and more general time-inconsistent framework, one can consult with Yong (2012), Cui et al. (2015), and Gu et al. (2016).

Besides, Bensoussan et al. (2014) illuminated numerically an observation that the investor facing a shortselling prohibition (said to be constrained investor) can even acquire a better value function than another agent without the confinement (said to be unconstrained investor) for time-consistent mean-variance optimization with state-dependent risk aversion; also see Table 4.2 in Section 4.1.3 for details. This is a unusual numerical observation in optimization problem: whatever the constrained investor can do, the unconstrained investor must be able to do the same, so we should expect that the unconstrained investor always has a better value function than that of the constrained one. The similar observation was discovered in Forsyth and Wang (2011), in which they first showed with an interesting numerical evidence that the strategy of constrained investor could be sometimes more efficient than that of the unconstrained one. Nevertheless, the theoretical framework for the paradigm under which similar phenomena will reappear is still absent in the literature.

In Chapter 4, as motivated by the counter-intuitive numerical examples in Forsyth and Wang (2011) and Bensoussan et al. (2014), we aim at providing an

analytic study on how the constrained investor can outperform the unconstrained investor. We illuminate analytically that, under certain economically meaningful conditions which usually hold in a buoyant market, an investor facing more investment restrictions can outperform than an investor with less:

- (i) An agent can start off his constrained time-consistent strategy at a certain time in the past or in the future which can outperform the time-consistent one adopted by his unconstrained rival.
- (ii) If the commencement is allowed to be earlier, investing all the wealth in riskless bond can beat both equilibrium strategies adopted by constrained and unconstrained investors respectively.
- (iii) To a certain extreme, with properly chosen parameters such that the “goodness index” of the stock,  $\frac{\alpha_t}{\sigma_t}$ , studied in Shiryaev et al. (2008) and Du Toit and Peskir (2009), is close to the index value computed from the market data, there could even be more than 90% of positive commencement time over the whole time horizon so that the constrained time-consistent strategy outperforms the unconstrained one.

The source of these paradoxical results is rooted in the nature of a direct application of game-theoretic approach on time consistency, which seeks for the Nash subgame perfect equilibrium of the intertemporal games between different time-players but not the ultimate maximizer. Therefore, it is not necessary that the equilibrium solution among a larger admissible set to have a greater value of objective function. However, the inconsistency and paradox raised in Chapter 4 do not mean that we should give up time-consistent solution. The equilibrium strategy does resolve the intertemporal conflict which appears in precommitment solution, thus the study on time consistency should focus on how to obtain a time-consistent strategy in a proper manner. It is more important to have more extensive study on the economic meaning behind time-consistent solution, so that

we can construct a more sophisticated time-consistent strategy which takes account of immediate economic consideration.

## 1.4 Outline of this Thesis

Chapter 2 studies utility-deviation-risk portfolio selection problem under continuous-time Black-Scholes framework. In our framework, the investor aims to maximize the expected value of utility, which is a function of the portfolio terminal payoff representing his satisfaction, and to minimize the convex-deviation-risk measure, used in Jin et al. (2005), simultaneously. By an application of the Martingale Representation Theorem, we first convert our dynamic optimization problem into an equivalent static problem. By considering the first-order optimality conditions and mean-field heuristic, we characterize the optimal terminal payoff by Nonlinear Moment Problem, which includes a variational inequality and some equality constraints on nonlinear moments, with corresponding necessity and sufficiency results. The Nonlinear Moment Problem can be further simplified into a system of nonlinear equations. By assuming that utility function satisfies the Inada conditions, we establish the existence of optimal solutions for both utility-downside-risk and utility-strictly-convex-risk problems through solving the system of equations. Moreover, the necessity and sufficiency results regarding Nonlinear Moment Problem can be applied to revisit several mean-risk problems studied in Jin et al. (2005), including the non-existence of optimal solution to mean-semivariance problem. Chapter 2 is based on Wong et al. (2015).

Chapter 3 provides a numerical method to compute the optimal solution to utility-risk problem. We first extend the utility-risk framework in Chapter 2 from continuous-time Black-Scholes framework to generalized market framework where the dynamics of asset prices can be unspecified. Under our generalized market framework, the continuous-time Black-Scholes model, the discrete binomial tree

model and the stochastic interest rate model for asset prices can be covered. Under the generalized framework, the similar existence results as in Chapter 2 for utility-downside-risk and utility-strictly-convex-risk problem are obtained. In addition, we establish the continuity of optimal terminal wealth in terminal pricing kernel. The continuity result can be applied to show the convergence of optimal solution from the binomial tree model to the continuous-time Black-Scholes model. This convergence result provides a numerical method to approximate the optimal solution to the continuous-time utility-risk problem in Chapter 2. Chapter 3 is based on Wong et al. (2016).

Chapter 4 presents a paradox in time consistency in mean-variance problem. We first introduce the time consistency and revisit the literature about mean-variance problems through game theoretic approach, including Björk et al. (2014) and Bensoussan et al. (2014). Motivated by the counter-intuitive numerical observations in Forsyth and Wang (2011) and Bensoussan et al. (2014) that an investor with more investment constraints can acquire a greater objective value than an investor with less, we establish analytically that the similar phenomenon will reappear in a buoyant market. Moreover, we show that even the pure investment strategy of solely investing in riskless asset can beat both equilibrium strategies adopted by the constrained and unconstrained investors for a large enough time horizon. With numerical experiments, we also illustrate that the constrained investor can outperform the unconstrained one for more than 90% of the time horizon. Finally, we shall discuss the meaning beneath our paradoxical results in time consistency. Chapter 4 is based on Bensoussan et al. (2016).

# Chapter 2

## Utility-Risk Portfolio Selection

In this chapter, we first introduce the problem formulation in Section 2.1 and convert our continuous-time utility-risk problem into an equivalent static formulation as stated in Theorem 2.1.5. In Section 2.2, we derive the necessary condition that the optimal terminal wealth satisfying two mild regularity conditions, Conditions 2.2.1 (i) and (ii), solves for the Nonlinear Moment Problem in Theorem 2.2.2. We then apply this necessity result to revisit the non-existence result for the mean-semivariance problem (Section 2.2.2). In Section 2.3, we establish the verification theorem (Theorem 2.3.2), serving as the sufficient condition that the solution of the Nonlinear Moment Problem satisfying Condition 2.3.1 serves as the optimal terminal wealth. We then apply the sufficiency result to establish the existence of the corresponding optimal solutions for utility-downside-risk and utility-strictly-convex-risk problems in Sections 2.3.2 and 2.3.3 respectively; the technical proofs are deferred to the Appendix. Finally, we apply the Nonlinear Moment Problem to establish the sufficient condition for the existence of an optimal solution of mean-risk problem in Section 2.3.4. Such sufficient condition can be used to revisit the various well-known mean risk problems such as mean-variance (Example 2.3.17), mean-weighted-power-risk (Example 2.3.18) (which includes mean-weighted-variance and mean-variance as special cases, also



see Remark 2.3.20) and mean-exponential-risk problems (Example 2.3.21).

## 2.1 Problem Setting

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed complete probability space, over which  $W(t) = (W_1(t), \dots, W_m(t))^t$  denotes  $m$ -dimensional standard Brownian motion;  $M^t$  denotes the transpose of a matrix  $M$ . We adopt the same market modeling setting as in Jin et al. (2005). Define  $\mathcal{F}_t := \sigma(W(s) : s \leq t)$ . Suppose that the market has one riskless money account with price process  $B(t)$  and  $m$  risky assets with the joint price process,  $S(t) := (S_1(t), \dots, S_m(t))^t$ , such that the pair  $(B(t), S(t))$  satisfies the following equations:

$$\begin{cases} dB(t) &= r(t)B(t)dt, \quad B(0) = b_0 > 0, \\ dS_k(t) &= \mu_k(t)S_k(t)dt + S_k(t) \sum_{j=1}^m \sigma_{kj}(t)dW_j(t), \quad S_k(0) = s_k > 0, \\ &k = 1, \dots, m, \end{cases}$$

where  $r(t)$  is the riskless interest-rate,  $\mu_k(t)$  and  $\sigma_k(t) := (\sigma_{k1}(t), \dots, \sigma_{km}(t))$  are respectively the appreciation rate and volatility of the  $k$ -th risky asset, all assumed to be uniformly bounded. We also assume that the volatility matrix of assets  $\sigma(t) := (\sigma_{kj}(t))_{m \times m}$  is uniformly elliptic, so that  $\sigma(t)\sigma(t)^t \geq \delta I$  for some  $\delta > 0$ , so **the market is complete** and  $(\sigma(t))^{-1}$  exists for all  $t$ .

Let  $\pi(t) := (\pi_1(t), \dots, \pi_m(t))^t$ , where  $\pi_k(t)$  is the money amount invested in the  $k$ -th risky asset of the portfolio at time  $t$ . The dynamics of controlled wealth process is:

$$dX^\pi(t) = (r(t)X^\pi(t) + \pi(t)^t \alpha(t))dt + \pi(t)^t \sigma(t)dW(t), \quad X^\pi(0) = x_0 > 0, \quad (2.1.1)$$

where  $\alpha(t) := (\alpha_1(t), \dots, \alpha_m(t))^t$  and  $\alpha_k(t) := \mu_k(t) - r(t)$  for any  $k \in \{1, \dots, m\}$ .

The objective functional is:

$$J(\pi) := \mathbb{E}[U(X^\pi(T))] - \gamma \mathbb{E}[D(\mathbb{E}[X^\pi(T)] - X^\pi(T))], \quad (2.1.2)$$

where the terminal time  $T$  is finite and  $\gamma > 0$  denotes the risk-aversion coefficient. We denote by  $U$  a utility function such that  $U : \text{Dom}(U) \rightarrow \mathbb{R}$  is strictly increasing, concave and continuously differentiable in the interior; here the domain of  $U$ ,  $\mathcal{D} := \text{Dom}(U)$ , is a convex set in  $\mathbb{R}$ . Define the lower end point of the domain  $\mathcal{D}$ ,  $K := \inf(\mathcal{D}) \in [-\infty, \infty)$ . For completeness, we extend the definition of  $U$  over  $\mathbb{R}$  so that  $U(x) = -\infty$  for  $x \in \mathbb{R}/\mathcal{D}$  and  $U'(K) := \lim_{x \downarrow K} U'(x)$ . Here the function  $D : \mathbb{R} \rightarrow \mathbb{R}_+$  stands for a risk function which measures the deviation of the random return from its own expectation. We assume that  $D$  is non-negative, convex and continuously differentiable.

For any given  $p \geq 1$ , denote  $\mathcal{L}^p := \left\{ Z \mid \|Z\|_p := \mathbb{E}[|Z|^p]^{\frac{1}{p}} < \infty \right\}$  and  $\mathcal{L}^\infty := \{Z \mid \|Z\|_\infty := \sup_{\omega \in \Omega} |Z(\omega)| < \infty\}$ . Define  $\mathcal{H}^2$  to be the class of all  $\mathcal{F}_t$ -adapted processes  $\pi$ , equipped with a norm  $\|\pi\|_{\mathcal{H}^2}^2 := \mathbb{E} \left[ \int_0^T \pi(t)^t \pi(t) dt \right] < \infty$ .

**Definition 2.1.1.** We define the class of all admissible controls  $\pi \in \mathcal{A}$  as follows:

$$\mathcal{A} := \left\{ \pi \in \mathcal{H}^2 \mid X^\pi(T) \in \mathcal{X} \right\},$$

where  $\mathcal{X}$  is the class of all admissible terminal wealths, such that

$$\mathcal{X} := \left\{ X \in \mathcal{L}^2 \mid X \in \mathcal{F}_T, X \in \mathcal{D} \text{ a.s.}, U(X) \in \mathcal{L}^1, D(\mathbb{E}[X] - X) \in \mathcal{L}^1 \right\}.$$

Note that, for every admissible terminal wealth, both its expected utility and expected deviation risk are well-defined. Since  $U$  is increasing and  $D$  is convex, we have

$$\begin{aligned} |U(\theta x + (1 - \theta)y)| &\leq |U(x)| + |U(y)| && \text{for all } x, y \in \mathcal{D} \text{ and } \theta \in [0, 1]; \\ 0 \leq D(\theta x + (1 - \theta)y) &\leq \theta D(x) + (1 - \theta)D(y) && \text{for all } x, y \in \mathbb{R} \text{ and } \theta \in [0, 1]. \end{aligned}$$

Hence,  $\mathcal{X}$  is a convex subspace of  $\mathcal{L}^2$ . For any admissible control  $\pi$ , we have  $X^\pi \in \mathcal{H}^2$  and  $X^\pi(t) \in \mathcal{L}^2$  for any  $t \in [0, T]$  by Theorems 1.2 and 2.1 in Touzi (2013).

Under the above settings, our utility risk problem can be stated as follows:

**Problem 2.1.2.**

Maximize  $J(\pi)$ ,  
 subject to  $\pi \in \mathcal{A}$  and  $(X^\pi(\cdot), \pi(\cdot))$  satisfies (2.1.1) with initial wealth  $x_0$ .

We define  $\xi(t)$  as

$$\xi(t) := \exp \left( - \int_0^t \left( r(s) ds + \frac{1}{2} \alpha(s)^t (\sigma(s) \sigma(s)^t)^{-1} \alpha(s) ds + \alpha(s)^t (\sigma(s)^t)^{-1} dW(s) \right) \right).$$

By applying Itô's formula to  $\xi(t)X^\pi(t)$ , it is clear that  $\xi(t)$  is the pricing kernel. Denote  $\xi := \xi(T) \in \mathcal{L}^p$  for any  $p \geq 1$ . Hence, for a given initial condition  $X^\pi(0) = x_0$ ,  $\mathbb{E}[\xi X^\pi(T)] = x_0$  for any  $\pi \in \mathcal{A}$ . If  $x_0 < \mathbb{E}[\xi] K$ ,  $\mathcal{A}$  is empty<sup>1</sup>. If  $x_0 = \mathbb{E}[\xi] K$ , even when  $\mathcal{A}$  is non-empty, all such  $\pi \in \mathcal{A}$  will give the same terminal wealth,  $X^\pi(T) = K$  a.s.<sup>2</sup>, so no actual optimization is required, thus the corresponding problem becomes trivial. In the rest of this chapter, based on this observation, we only consider our problem under this natural assumption:

**Assumption 2.1.3.** The initial wealth  $x_0$ , the lower end point of  $\mathcal{D}$ ,  $K \in [-\infty, \infty)$ , and pricing kernel  $\xi := \xi(T)$  altogether satisfy:

$$x_0 > \mathbb{E}[\xi] K.$$

Note that if we choose  $U$  to be linear and  $D$  to be quadratic, i.e.  $U(x) = x$  and  $D(x) = x^2$ , then Problem 2.1.2 reduces to the classical mean-variance problem. If we only choose  $U$  to be linear, then Problem 2.1.2 reduces to the mean-risk problem as in Jin et al. (2005); in particular, if we alternatively choose  $D(x) = ax_+ + bx_-$ , then Problem 2.1.2 reduces to the mean-weighted-variance problem. If we just set  $D$  to be a convex function with  $D(x) = 0$  for  $x \leq 0$ , Problem 2.1.2 is to maximize utility and minimize the downside risk of terminal wealth; its resolution will be established in Subsection 2.3.2.

<sup>1</sup>If  $\mathcal{A}$  is non-empty, we have  $x_0 = \mathbb{E}[\xi X^\pi(T)] \geq \mathbb{E}[\xi] K$  since  $X^\pi(T) \in \mathcal{X}$  for any  $\pi \in \mathcal{A}$ , which implies  $X^\pi(T) \geq K$  a.s.

<sup>2</sup>If there exists  $\pi \in \mathcal{A}$  such that  $\mathbb{P}[X^\pi(T) > K] > 0$ , then  $x_0 = \mathbb{E}[\xi X^\pi(T)] > \mathbb{E}[\xi] K$ .

Since our market is complete, all  $\mathcal{L}^2$ -integrable and  $\mathcal{F}_T$ -measurable terminal wealth can be attained by an admissible control, in the light of Martingale Representation Theorem. Our dynamic utility-risk optimization problem 2.1.2 can be converted into the following static optimization problem:

Define  $\Psi : \mathcal{X} \rightarrow \mathbb{R}$  such that  $\Psi(X) := \mathbb{E}[U(X)] - \mathbb{E}[D(\mathbb{E}[X] - X)]$ .

**Problem 2.1.4.**

$$\begin{aligned} & \text{Maximize} && \Psi(X), && (2.1.3) \\ & \text{subject to} && X \in \mathcal{X} \text{ and } \mathbb{E}[\xi X] = x_0. \end{aligned}$$

Then, the optimal solution of Problem 2.1.4 is the optimal terminal wealth of Problem 2.1.2:

**Theorem 2.1.5** (Theorem 2.1 in Bielecki et al. (2005) and Theorem 2.1 in Jin et al. (2005)). *If  $\pi(t)$  is optimal for Problem 2.1.2, then  $X^\pi(T)$  is optimal for Problem 2.1.4. Conversely, if  $X \in \mathcal{X}$  is optimal for Problem 2.1.4, there exists  $\pi \in \mathcal{A}$  such that  $X^\pi(T) = X$  and  $\pi$  is optimal for Problem 2.1.2.*

Note that the maximization in Problem 2.1.4 is confined to the set  $\mathcal{X}$ , so that the solution obtained in Problem 2.1.4 is an admissible terminal wealth in Problem 2.1.2. Our present chapter aims to establish an admissible terminal wealth  $X \in \mathcal{X}$  that maximizes  $\Psi(X)$  under rather general scenarios, including those not yet covered in the existing literature.

## 2.2 Necessary Condition

### 2.2.1 Maximum Principle

To show the necessity for optimality, we assume that the optimal solution of Problem 2.1.4,  $\hat{X} \in \mathcal{X}$ , satisfies the following two very mild technical conditions:

**Condition 2.2.1.** (i) Both  $U'(Z) \in \mathcal{L}^1$  and  $D'(\mathbb{E}[Z] - Z) \in \mathcal{L}^1$ .

(ii) There exists  $\delta > 0$  such that  $D(\mathbb{E}[Z] - Z - \delta) \in \mathcal{L}^1$  and  $D(\mathbb{E}[Z] - Z + \delta) \in \mathcal{L}^1$ .

Now, it is necessary for  $\hat{X}$  to solve for the following auxiliary static problem, we call it the *Nonlinear Moment Problem*:

**Theorem 2.2.2** (Nonlinear Moment Problem). *If  $\hat{X}$  is the optimal solution of Problem 2.1.4 satisfying Conditions 2.2.1 (i) and (ii), then it is necessary that there exist constants  $Y, M, R \in \mathbb{R}$  such that the quadruple  $(\hat{X}, Y, M, R)$  solves for the following variational inequality:*

$$\begin{cases} Y\xi = U'(\hat{X}) - \gamma R + \gamma D'(M - \hat{X}), & \text{a.s. on } \{\hat{X} > K\}, \\ Y\xi \geq U'(\hat{X}) - \gamma R + \gamma D'(M - \hat{X}), & \text{a.s. on } \{\hat{X} = K\} \text{ (if } \mathbb{P}[\hat{X} = K] > 0), \end{cases} \quad (2.2.4)$$

subject to the nonlinear moment constraints

$$\mathbb{E}[\xi \hat{X}] = x_0, \quad (2.2.5)$$

$$\mathbb{E}[\hat{X}] = M, \quad (2.2.6)$$

$$\mathbb{E}[D'(M - \hat{X})] = R. \quad (2.2.7)$$

### 2.2.1.1 Proof of Theorem 2.2.2

Let  $\hat{X} \in \mathcal{X}$  be an optimal solution of Problem 2.1.4. We define  $\Gamma : \mathcal{L}^2 \times \mathcal{D} \rightarrow \mathbb{R}$  by

$$\Gamma(X, x) := U'(x) - \gamma \mathbb{E}[D'(\mathbb{E}[X] - X)] + \gamma D'(\mathbb{E}[X] - x). \quad (2.2.8)$$

For simplicity of notation, in the rest of this chapter, we shall denote the random variable  $\Gamma(\hat{X}, \hat{X})$  by  $\hat{\Gamma}$ .

To prove the necessity, we first apply the first-order conditions as stated in Proposition 2.2.3. Next, we make use of Proposition 2.2.3 to give a preliminary

result for characterizing the optimal solution of Problem 2.1.4,  $\hat{X}$ , in Lemma 2.2.5: if we can find a random variable,  $Z$ , as described in Lemma 2.2.5, then it is necessary that  $\hat{X}$  has to satisfy the variational inequality (2.2.4). Finally, in Proposition 2.2.10, we construct such a  $Z$ .

**Proposition 2.2.3.** *If  $\hat{X}$  is optimal for Problem 2.1.4 satisfying Condition 2.2.1 (i), then*

$$\mathbb{E} \left[ \tilde{X} \hat{\Gamma} \right] \leq 0,$$

for all  $\tilde{X} \in \Theta := \left\{ Z \in \mathcal{L}^\infty \mid \mathbb{E}[Z\xi] = 0 \text{ and } \hat{X} + Z \in \mathcal{X} \right\}$ .

For any  $\tilde{X} \in \Theta$ , by the convexity of  $\mathcal{X}$ ,  $\hat{X} + \theta\tilde{X} \in \mathcal{X}$  for all  $0 < \theta < 1$ . The directional derivative of  $\Psi(X)$  is

$$\begin{aligned} & \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} \\ &= \left. \frac{d}{d\theta} \left( \mathbb{E} \left[ U(\hat{X} + \theta\tilde{X}) \right] - \gamma \mathbb{E} \left[ D \left( \mathbb{E}[\hat{X} + \theta\tilde{X}] - (\hat{X} + \theta\tilde{X}) \right) \right] \right) \right|_{\theta=0}. \end{aligned}$$

Before we proceed on the proof of Proposition 2.2.3, we first justify the interchange of the order of differentiation and taking expectation of the above expression. To this end, we need the following lemma:

**Lemma 2.2.4.** *Given two random variables  $\hat{X} \in \mathcal{X}$  and  $\tilde{X} \in \mathcal{L}^2$  such that  $\hat{X} + \tilde{X} \in \mathcal{X}$ ,*

$$\begin{aligned} & \lim_{\theta \downarrow 0} \mathbb{E} \left[ \frac{U(\hat{X} + \theta\tilde{X}) - \gamma D \left( \mathbb{E}[\hat{X} + \theta\tilde{X}] - (\hat{X} + \theta\tilde{X}) \right) - \left( U(\hat{X}) - \gamma D \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \right)}{\theta} \right] \\ &= \mathbb{E} \left[ \lim_{\theta \downarrow 0} \frac{U(\hat{X} + \theta\tilde{X}) - \gamma D \left( \mathbb{E}[\hat{X} + \theta\tilde{X}] - (\hat{X} + \theta\tilde{X}) \right) - \left( U(\hat{X}) - \gamma D \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \right)}{\theta} \right]. \end{aligned}$$

*Proof.* Since  $U$  is concave and  $D$  is convex function, so  $f(\theta) := U(\hat{X} + \theta\tilde{X}) - \gamma D(\mathbb{E}[\hat{X} + \theta\tilde{X}] - (\hat{X} + \theta\tilde{X}))$  is concave in  $\theta > 0$ . Thus, for any  $\delta > 0$ , by concavity of  $f$ ,  $f(\theta) \geq \frac{\delta}{\theta+\delta}f(0) + \frac{\theta}{\theta+\delta}f(\theta + \delta)$ , so

$$\frac{f(\theta) - f(0)}{\theta} \geq \frac{f(\theta + \delta) - f(0)}{\theta + \delta}.$$

Hence,  $\frac{1}{\theta} \left( U(\hat{X} + \theta\tilde{X}) - \gamma D \left( \mathbb{E} [\hat{X} + \theta\tilde{X}] - (\hat{X} + \theta\tilde{X}) \right) - \left( U(\hat{X}) - \gamma D \left( \mathbb{E} [\hat{X}] - \hat{X} \right) \right) \right)$  is increasing as  $\theta$  decreases to 0. Since  $\hat{X}$  and  $\hat{X} + \tilde{X}$  are admissible terminal wealth, thus  $U(\hat{X} + \tilde{X}) - \gamma D \left( \mathbb{E} [\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X}) \right)$  and  $U(\hat{X}) - \gamma D \left( \mathbb{E} [\hat{X}] - \hat{X} \right)$  are both  $\mathcal{L}^1$ -integrable because  $\hat{X} + \tilde{X}, \hat{X} \in \mathcal{X}$ . Hence, this lemma follows from the Monotone Convergence Theorem.  $\square$

*Proof of Proposition 2.2.3.* By Lemma 2.2.4, the chain rule and Condition 2.2.1 (i), we have

$$\begin{aligned} \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} &= \mathbb{E} \left[ U'(\hat{X})\tilde{X} \right] - \gamma \mathbb{E} \left[ D' \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \left( \mathbb{E}[\tilde{X}] - \tilde{X} \right) \right] \\ &= \mathbb{E} \left[ \tilde{X}\hat{\Gamma} \right]. \end{aligned} \quad (2.2.9)$$

Our claim follows by the first-order necessary condition for optimality.  $\square$

To characterize the optimal solution  $\hat{X}$ , we first have the following lemma:

**Lemma 2.2.5.** *Given that  $\hat{X}$  is optimal for Problem 2.1.4 satisfying Condition 2.2.1 (i), if there exists a random variable,  $Z \in [0, 1]$ , such that the following three items hold:*

$$\begin{cases} Z > 0 & \text{a.s. on } \{\hat{X} > K\}, \\ Z = 0 & \text{a.s. on } \{\hat{X} = K\} \text{ (if } \mathbb{P}[\hat{X} = K] > 0), \end{cases} \quad (2.2.10)$$

$$Z\xi\hat{\Gamma} \in \mathcal{L}^1, \quad Z \left( \hat{\Gamma} - Y\xi \right) \in \mathcal{L}^\infty, \text{ and} \quad (2.2.11)$$

$$\hat{X} + Z \left( \hat{\Gamma} - Y\xi \right) \in \mathcal{X}, \quad (2.2.12)$$

where  $Y := \frac{\mathbb{E}[Z\xi\hat{\Gamma}]}{\mathbb{E}[Z\xi^2]}$ , then it is necessary that  $\hat{\Gamma}$  defined in (2.2.8) satisfies the following algebraic structure:

$$\begin{cases} \hat{\Gamma} = Y\xi & \text{a.s. on } \{\hat{X} > K\}, \\ \hat{\Gamma} \leq Y\xi & \text{a.s. on } \{\hat{X} = K\} \text{ (if } \mathbb{P}[\hat{X} = K] > 0). \end{cases} \quad (2.2.13)$$

*Proof.* We split our proof into two parts: (i)  $\hat{\Gamma} = Y\xi$  a.s. on  $\{\hat{X} > K\}$ , and (ii)  $\hat{\Gamma} \leq Y\xi$  a.s. on  $\{\hat{X} = K\}$  (if  $\mathbb{P}[\hat{X} = K] > 0$ ).

(i) Take

$$\tilde{X} = Z \left( \hat{\Gamma} - Y\xi \right), \quad (2.2.14)$$

(2.2.11) and (2.2.12) warrants that  $\tilde{X} \in \mathcal{L}^\infty$  with  $\hat{X} + \tilde{X} \in \mathcal{X}$  and

$$\mathbb{E}[\tilde{X}\xi] = \mathbb{E} \left[ Z\xi\hat{\Gamma} \right] - Y\mathbb{E}[Z\xi^2] = 0.$$

By Proposition 2.2.3 and the fact that  $Y = \mathbb{E} \left[ Z\xi\hat{\Gamma} \right] / \mathbb{E}[Z\xi^2]$ ,

$$\begin{aligned} 0 \geq \mathbb{E} \left[ \tilde{X}\hat{\Gamma} \right] &= \mathbb{E} \left[ Z \left( \hat{\Gamma} - Y\xi \right) \hat{\Gamma} \right] - Y\mathbb{E} \left[ Z\xi\hat{\Gamma} \right] + Y^2 \mathbb{E}[Z\xi^2] \\ &= \mathbb{E} \left[ Z \left( \hat{\Gamma} - Y\xi \right)^2 \right]. \end{aligned}$$

(2.2.10) ensures that  $Z \left( \hat{\Gamma} - Y\xi \right)^2 \geq 0$ , and therefore  $\mathbb{E} \left[ Z \left( \hat{\Gamma} - Y\xi \right)^2 \right] = 0$ , which implies that  $Z \left( \hat{\Gamma} - Y\xi \right)^2 = 0$  a.s. By (2.2.10), on  $\left\{ \hat{X} > K \right\}$ ,  $Z > 0$ , hence  $\hat{\Gamma} = Y\xi$  a.s. on  $\left\{ \hat{X} > K \right\}$ .

(ii) If  $\mathbb{P}[\hat{X} = K] > 0$ , assume the contrary that  $\mathbb{P} \left[ \mathbb{I} \left\{ \hat{X} = K \right\} \left( \hat{\Gamma} - Y\xi \right) > 0 \right] > 0$ . Consider

$$\tilde{X} = k\mathbb{I} \left\{ \hat{X} = K \right\} - \frac{\min\{\hat{X} - K, 1\}}{2}$$

where  $k := \mathbb{E} \left[ \left( \min\{\hat{X} - K, 1\} \right) \xi \right] / \left( 2\mathbb{E} \left[ \mathbb{I} \left\{ \hat{X} = K \right\} \xi \right] \right) > 0$  in light of the required feasibility of  $\mathcal{X}$  and our interest being only on non-trivial setting. We have

$$\hat{X} + \tilde{X} = \begin{cases} \max \left\{ \frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2} \right\}, & \text{if } \hat{X} > K, \\ K + k, & \text{if } \hat{X} = K. \end{cases}$$

Obviously,  $\tilde{X} \in \mathcal{L}^\infty$ ,  $K \leq \hat{X} + \tilde{X} \in \mathcal{D}$  a.s. and  $\mathbb{E} \left[ \tilde{X}\xi \right] = 0$ .

Since  $U$  is monotonic,

$$\begin{aligned} |U(\hat{X} + \tilde{X})| &\leq |U(K + k)| + \left| U \left( \max \left\{ \frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2} \right\} \right) \right| \\ &\leq |U(K + k)| + |U(K)| + |U(\hat{X})|. \end{aligned}$$



As  $\mathbb{P}[\hat{X} = K] > 0$  and  $\mathbb{P}[\hat{X} = K] \cdot |U(K)| \leq \mathbb{E}[|U(\hat{X})|]$  is finite, they prevent  $U(K)$  from taking  $-\infty$ . Clearly,  $K + k \in \mathcal{D}$ , and so  $U(K + k)$  is finite. Since  $\hat{X} \in \mathcal{X}$ , we also have  $U(\hat{X}) \in \mathcal{L}^1$ . These three claims altogether imply that  $U(\hat{X} + \tilde{X}) \in \mathcal{L}^1$ .

Note that

$$\begin{aligned} & \mathbb{E}[\hat{X} + \tilde{X}] \\ &= \mathbb{E}[(K + k)\mathbb{I}[\hat{X} = K]] + \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\} (1 - \mathbb{I}[\hat{X} = K])\right] \\ &= k\mathbb{P}[\hat{X} = K] + \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right], \end{aligned}$$

we then establish the upper bound of  $D\left(\mathbb{E}[\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X})\right)$  into three different cases: (1)  $\hat{X} = K$ , (2)  $K < \hat{X} < K + 1$ , and (3)  $\hat{X} \geq K + 1$ .

(1)  $D\left(\mathbb{E}[\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X})\right) = D\left(\mathbb{E}[\hat{X} + \tilde{X}] - K - k\right)$  is a finite constant.

(2) Note that

$$\begin{aligned} \mathbb{E}\left[\frac{\hat{X} + K}{2}\right] - \frac{\hat{X} + K}{2} &\leq \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right] - \frac{\hat{X} + K}{2} \\ &\leq \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right] - K. \end{aligned}$$

By convexity of  $D$ , we have

$$\begin{aligned}
& D\left(\mathbb{E}\left[\hat{X} + \tilde{X}\right] - \left(\hat{X} + \tilde{X}\right)\right) \\
&= D\left(k\mathbb{P}\left[\hat{X} = K\right] + \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right] - \frac{\hat{X} + K}{2}\right) \\
&\leq D\left(k\mathbb{P}\left[\hat{X} = K\right] + \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right] - K\right) \\
&\quad + D\left(k\mathbb{P}\left[\hat{X} = K\right] + \mathbb{E}\left[\frac{\hat{X} + K}{2}\right] - \frac{\hat{X} + K}{2}\right) \\
&\leq C + D\left(k\mathbb{P}\left[\hat{X} = K\right] + \frac{\mathbb{E}\left[\hat{X}\right] - \hat{X}}{2}\right) \\
&\leq C + D\left(k\mathbb{P}\left[\hat{X} = K\right] + \frac{\mathbb{E}\left[\hat{X}\right] - K}{2}\right) + D\left(\frac{\mathbb{E}\left[\hat{X}\right] - \hat{X}}{2}\right) \\
&\leq C + D(0) + D\left(\mathbb{E}\left[\hat{X}\right] - \hat{X}\right), \text{ where } C \text{ is a constant.}
\end{aligned}$$

(3) By the same arguments as that for case (ii), we have

$$\begin{aligned}
& D\left(\mathbb{E}\left[\hat{X} + \tilde{X}\right] - \left(\hat{X} + \tilde{X}\right)\right) \\
&\leq D\left(k\mathbb{P}\left[\hat{X} = K\right] + \mathbb{E}\left[\max\left\{\frac{\hat{X} + K}{2}, \hat{X} - \frac{1}{2}\right\}\right] - K\right) \\
&\quad + D\left(k\mathbb{P}\left[\hat{X} = K\right] + \mathbb{E}\left[\hat{X}\right] - K\right) + D\left(\mathbb{E}\left[\hat{X}\right] - \hat{X}\right).
\end{aligned}$$

Since  $\hat{X} \in \mathcal{X}$ , we have  $D(\mathbb{E}[\hat{X}] - \hat{X}) \in \mathcal{L}^1$ , so  $D(\mathbb{E}[\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X})) \in \mathcal{L}^1$ .

Finally,

$$\begin{aligned}
& \mathbb{E}\left[\hat{\Gamma}\tilde{X}\right] \\
&= k\mathbb{E}\left[\mathbb{I}\left\{\hat{X} = K\right\}\hat{\Gamma}\right] - \frac{\mathbb{E}\left[\left(\min\left\{\hat{X} - K, 1\right\}\right)\hat{\Gamma}\right]}{2} \\
&= k\mathbb{E}\left[\mathbb{I}\left\{\hat{X} = K\right\}\hat{\Gamma}\right] - \frac{\mathbb{E}\left[\left(\min\left\{\hat{X} - K, 1\right\}\right)Y\xi\right]}{2} \\
&= k\left(\mathbb{E}\left[\mathbb{I}\left\{\hat{X} = K\right\}\hat{\Gamma}\right] - \mathbb{E}\left[\mathbb{I}\left\{\hat{X} = K\right\}Y\xi\right]\right) > 0, \quad (2.2.15)
\end{aligned}$$

where the second equality follows because we have shown that  $\hat{\Gamma} = Y\xi$  when  $\hat{X} > K$  and the third equality follows because  $k = \frac{\mathbb{E}[(\min\{\hat{X}-K, 1\})\xi]}{(2\mathbb{E}[\mathbb{I}\{\hat{X}=K\}\xi])}$ . (2.2.15) violates Proposition 2.2.3, this implies that if  $\mathbb{P}[\hat{X} = K] > 0$ ,  $\mathbb{P}[\mathbb{I}\{\hat{X} = K\}(\hat{\Gamma} - Y\xi) > 0] > 0$  leads to a contradiction. We have

$$\mathbb{P}[\mathbb{I}\{\hat{X} = K\}(\hat{\Gamma} - Y\xi) \leq 0] = 1.$$

Therefore, the complete characterization as specified in (2.2.13) now follows.  $\square$

The overall necessity claim will be accomplished if the explicit construction of  $Z$  as described in the hypothesis in Lemma 2.2.5 can be obtained. Even the nature of such  $Z$  appears to be complicated and uncommon in the literature, we shall devote the remaining part of this subsection to the establishment of its existence.

In order to satisfy (2.2.12),  $\tilde{X}$  expressed in terms of  $Z$  as in (2.2.14) needs to be bounded so that the deviation of  $U(\hat{X} + \tilde{X})$  from  $U(\hat{X})$  and that of  $D(\mathbb{E}[\hat{X} + \tilde{X}] - \hat{X} - \tilde{X})$  from  $D(\mathbb{E}[\hat{X}] - \hat{X})$  are less than some constant, say 1 for simplicity, almost surely. To warrant this, we need the following lemma immediate from the continuity of  $U$ :

**Lemma 2.2.6.** *There exists  $\delta^U : \text{int}(\mathcal{D}) \rightarrow (0, 1]$  such that for any  $x_0 \in \text{int}(\mathcal{D})^3$ ,*

$$|U(x) - U(x_0)| \leq 1, \quad \forall x \in \mathcal{D} \text{ such that } |x - x_0| < \delta^U(x_0).$$

We shall make use of  $\delta^U$  defined in Lemma 2.2.6 to construct  $Z$  so that  $U(\hat{X} + \tilde{X}) \in \mathcal{L}^1$ , where  $\tilde{X}$  in terms of  $Z$  is given in (2.2.14). Beforehand, for any  $y \in$

<sup>3</sup>A possible choice of  $\delta^U$  can be constructed as the following:

$$\delta^U(x) := \min \left\{ \frac{1}{U^*(x)}, \frac{x-K}{2}, 1 \right\},$$

where

$$U^*(x) := \begin{cases} U'(x-1), & \text{if } x-2 \in \mathcal{D}, \\ U'(\frac{x+K}{2}), & \text{if } x-2 \notin \mathcal{D}. \end{cases}$$

$(0, \infty)$ , define a random variable  $Z_y \in [0, 1]$  by:

$$Z_y := \begin{cases} 0, & \text{if } \hat{X} = K, \\ 1, & \text{if } \hat{X} > K \text{ and } \hat{\Gamma} = y\xi, \\ \min \left\{ \frac{\min\{\delta^U(\hat{X}), \frac{1}{2}(\hat{X}-K), \frac{\delta}{2}\}}{|\hat{\Gamma}-y\xi|}, 1 \right\}, & \text{otherwise.} \end{cases} \quad (2.2.16)$$

First, we show that (2.2.11) is satisfied for any  $y \in (0, \infty)$ :

**Lemma 2.2.7.** *For any  $y \in (0, \infty)$ , we have  $Z_y \xi \hat{\Gamma} \in \mathcal{L}^1$  and  $Z_y (\hat{\Gamma} - y\xi) \in \mathcal{L}^\infty$ .*

*Proof.* By the definition of  $Z_y$  in (2.2.16),  $|Z_y (\hat{\Gamma} - y\xi)| \leq \delta$ , so  $Z_y (\hat{\Gamma} - y\xi) \in \mathcal{L}^\infty$ . Since

$$|Z_y \xi \hat{\Gamma}| \leq |Z_y \xi (\hat{\Gamma} - y\xi)| + |y\xi^2 Z_y| \leq \delta\xi + y\xi^2,$$

we have  $Z_y \xi \hat{\Gamma} \in \mathcal{L}^1$ . □

Next, we want to ensure one can find a  $y$  so that  $y = \frac{\mathbb{E}[\xi \hat{\Gamma} Z_y]}{\mathbb{E}[\xi^2 Z_y]}$  is satisfied.

**Lemma 2.2.8.** *Define  $f : (0, \infty) \rightarrow \mathbb{R}$  by*

$$f(y) := y\mathbb{E}[\xi^2 Z_y] - \mathbb{E}[\xi \hat{\Gamma} Z_y].$$

*There is a root  $y^* \in (0, \infty)$  such that  $f(y^*) = 0$ .*

Before we prove Lemma 2.2.8, we require to show the claim that  $\hat{\Gamma} > 0$ :

**Lemma 2.2.9.** *Given  $\hat{X}$  is optimal for Problem 2.1.4 satisfying Conditions 2.2.1 (i) and (ii), it is necessary that  $\hat{\Gamma} > 0$  almost surely.*

*Proof.* We consider two cases: (i)  $K > -\infty$  and (ii)  $K = -\infty$ , respectively.

We consider case (i)  $K > -\infty$ . With the optimal solution  $\hat{X}$  of Problem 2.1.4, we define  $h(x) := \Gamma(\hat{X}, x)$ , which is a decreasing continuous function. Since

$U' > 0$  on  $\{\hat{X} > K\}$  and  $U'$  is decreasing because of concavity of  $U$ , we have  $\lim_{x \downarrow K} U'(x) > 0$ . Hence,

$$\lim_{x \downarrow K} h(x) > -\gamma \mathbb{E} \left[ D' \left( \mathbb{E} [\hat{X}] - \hat{X} \right) \right] + \gamma D' \left( \mathbb{E} [\hat{X}] - K \right).$$

By definition,  $\hat{X} \geq K$  uniformly, and  $D'$  is increasing, thus  $D' \left( \mathbb{E} [\hat{X}] - K \right) \geq \mathbb{E} \left[ D' \left( \mathbb{E} [\hat{X}] - \hat{X} \right) \right]$ . Consequently, one have  $\lim_{x \downarrow K} h(x) > 0$ . Then, there exists  $k_0 := \inf \{x > K \mid h(x) \leq 0\} \in (K, \infty]$ , i.e.,  $\hat{\Gamma} \leq 0$  implies  $\hat{X} \geq k_0$ . If  $k_0 = \infty$ , it is immediate that  $\hat{\Gamma} > 0$  almost surely, so we consider the case that  $k_0 < \infty$ . Assume the contrary, that  $\mathbb{P} \left[ \hat{\Gamma} \leq 0 \right] > 0$ . Consider

$$\tilde{X} := \begin{cases} -\frac{k_0 - K}{2}, & \text{if } \hat{\Gamma} \leq 0, \\ \frac{k_0 - K}{2} \frac{\mathbb{E} [\mathbb{I} \{ \hat{\Gamma} \leq 0 \} \xi]}{\mathbb{E} [\mathbb{I} \{ \hat{\Gamma} > 0 \} \xi]}, & \text{if } \hat{\Gamma} > 0. \end{cases} \quad (2.2.17)$$

We have  $\hat{X} + \tilde{X} > K$  and  $\tilde{X} \in \mathcal{L}^\infty$ . Since  $U$  is concave,

$$U(\hat{X}) + \tilde{X} U'(\hat{X} + \tilde{X}) \leq U(\hat{X} + \tilde{X}) \leq U(\hat{X}) + \tilde{X} U'(\hat{X}).$$

Furthermore, when  $\hat{\Gamma} \leq 0$ , we have  $\tilde{X} < 0$  and  $\hat{X} \geq k_0$ , then

$$\tilde{X} U'(\hat{X} + \tilde{X}) \geq \tilde{X} U' \left( k_0 - \frac{k_0 - K}{2} \right);$$

while  $\hat{\Gamma} > 0$ , we have  $\tilde{X} > 0$  and  $\hat{X}(T) \leq k_0$ , then

$$\tilde{X} U'(\hat{X} + \tilde{X}) \geq \tilde{X} U' \left( k_0 + \frac{k_0 - K}{2} \frac{\mathbb{E} [\mathbb{I} \{ \hat{\Gamma} \leq 0 \} \xi]}{\mathbb{E} [\mathbb{I} \{ \hat{\Gamma} > 0 \} \xi]} \right).$$

Thus,  $U(\hat{X} + \tilde{X}) \in \mathcal{L}^1$ . On the other hand, since  $D$  is convex,

$$\begin{aligned} 0 &\leq D \left( \mathbb{E} [\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X}) \right) \\ &\leq D \left( \mathbb{E} [\hat{X}] - \hat{X} \right) + \left( \mathbb{E} [\tilde{X}] - \tilde{X} \right) D' \left( \mathbb{E} [\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X}) \right). \end{aligned}$$

Similar to showing  $\tilde{X} U'(\hat{X} + \tilde{X})$  being bounded from below, we can show that  $\left( \mathbb{E} [\tilde{X}] - \tilde{X} \right) D' \left( \mathbb{E} [\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X}) \right)$  is bounded from above, thus

$$D \left( \mathbb{E} [\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X}) \right) \in \mathcal{L}^1.$$

Hence,  $\hat{X} + \tilde{X} \in \mathcal{X}$ . Also,  $\mathbb{E} [\tilde{X}\xi] = 0$ , but  $\mathbb{E} [\tilde{X}\hat{\Gamma}] > 0$ , which altogether violates Proposition 2.2.3. As a result,  $\mathbb{P} [\hat{\Gamma} \leq 0] = 0$ .

Now, consider the case (ii)  $\mathcal{D} = \mathbb{R}$ ; the approach is similar as in the case (i). Firstly, there exists  $k_0 := \inf \{x > -\infty \mid h(x) \leq 0\} \in (-\infty, \infty]$ . If  $k_0 < \infty$ , we assume the contrary, that  $\mathbb{P} [\hat{\Gamma} \leq 0] > 0$ , and then as in case (i), we can show that Proposition 2.2.3 is violated by setting  $\tilde{X}$  as follows:

$$\tilde{X} := \begin{cases} -1, & \text{if } \hat{\Gamma} \leq 0, \\ \frac{\mathbb{E}[\mathbb{I}\{\hat{\Gamma} \leq 0\}\xi]}{\mathbb{E}[\mathbb{I}\{\hat{\Gamma} > 0\}\xi]}, & \text{if } \hat{\Gamma} > 0. \end{cases}$$

□

*Proof of Lemma 2.2.8.* For any  $y \in (0, \infty)$ , by (2.2.16), we have

$$|y\xi^2 Z_y - \xi\hat{\Gamma} Z_y| = |y\xi - \hat{\Gamma}| \cdot |Z_y| \xi \leq \delta\xi. \quad (2.2.18)$$

By the Dominated Convergence Theorem,  $f$  is continuous on  $(0, \infty)$ . Since  $\hat{\Gamma} > 0$  almost surely by Lemma 2.2.9,

$$\lim_{y \downarrow 0} Z_y = \min \left\{ \frac{\min \left\{ \delta^U(\hat{X}), \frac{1}{2}(\hat{X} - K), \frac{\delta}{2} \right\}}{\hat{\Gamma}}, 1 \right\} > 0$$

almost surely on  $\{\hat{X} > K\}$ . Since  $\mathbb{P} [\hat{X} > K] > 0$ , we have, by the Dominated Convergence Theorem,

$$\lim_{y \rightarrow 0} f(y) = \mathbb{E} \left[ \lim_{y \rightarrow 0} (y\xi^2 Z_y - \xi\hat{\Gamma} Z_y) \right] = -\mathbb{E} \left[ \xi\hat{\Gamma} \lim_{y \downarrow 0} Z_y \right] < 0.$$

Note that

$$\begin{aligned} & \lim_{y \rightarrow \infty} y\xi^2 Z_y \\ &= \xi \min \left\{ \delta^U(\hat{X}), \frac{1}{2}(\hat{X} - K), \frac{\delta}{2} \right\} > 0 \text{ a.s. on } \{\hat{X} > K\} \end{aligned}$$

and

$$\begin{aligned} & \lim_{y \rightarrow \infty} \xi \hat{\Gamma} Z_y \\ &= \xi \hat{\Gamma} \lim_{y \rightarrow \infty} \min \left\{ \frac{\min \left\{ \delta^U(\hat{X}), \frac{1}{2}(\hat{X} - K), \frac{\delta}{2} \right\}}{|\hat{\Gamma} - y\xi|}, 1 \right\} = 0. \end{aligned}$$

By applying the Dominated Convergence Theorem and the fact that  $\mathbb{P}[\hat{X} > K] > 0$  under Assumption 2.1.3, we have:

$$\lim_{y \rightarrow \infty} f(y) = \mathbb{E} \left[ \lim_{y \rightarrow \infty} (y\xi^2 Z_y - \xi \hat{\Gamma} Z_y) \right] = \mathbb{E} \left[ \lim_{y \rightarrow \infty} y\xi^2 Z_y \right] > 0,$$

Our claim follows by intermediate value theorem.  $\square$

**Proposition 2.2.10.** *Suppose the optimal solution of Problem 2.1.4,  $\hat{X}$ , satisfies Conditions 2.2.1 (i) and (ii). There exists a random variable  $Z \in [0, 1]$  satisfying (2.2.10)-(2.2.12) in Lemma 2.2.5.*

*Proof.* We shall verify that  $Z := Z_{y^*}$  with  $Z_y$  as defined in (2.2.16) and  $y^*$  obtained in Lemma 2.2.8 satisfies (2.2.10)-(2.2.12). Note that  $\delta^U$  in Lemma 2.2.6 only take positive values no matter what the corresponding arguments are; in particular, according to (2.2.16), when  $\hat{X} > K$ ,  $Z > 0$ . Therefore,  $Z$  satisfies (2.2.10).

Note that  $y^* = \mathbb{E}[Z\xi\hat{\Gamma}] / \mathbb{E}[Z\xi^2]$  by Lemma 2.2.8. By Lemma 2.2.7, we have  $Z\xi\hat{\Gamma} \in \mathcal{L}^1$  and  $Z(\hat{\Gamma} - y^*\xi) \in \mathcal{L}^\infty$ , thus (2.2.11) is satisfied.

By a simple calculation under the third case in (2.2.16),  $\left| Z(\hat{\Gamma} - y^*\xi) \right| \leq \frac{1}{2}(\hat{X} - K)$ , thus we have

$$\hat{X} + Z \left( \hat{\Gamma} - \frac{\mathbb{E}[Z\xi\hat{\Gamma}]}{\mathbb{E}[Z\xi^2]} \xi \right) \geq \hat{X} - \frac{1}{2}(\hat{X} - K) = \frac{1}{2}(\hat{X} + K) \in \mathcal{D}, \text{ a.s..}$$

Since  $\left| Z(\hat{\Gamma} - y^*\xi) \right| \leq \delta^U(\hat{X})$ , by a direct application of Lemma 2.2.6 (a), we have  $\left| U \left( \hat{X} + Z(\hat{\Gamma} - y^*\xi) \right) - U(\hat{X}) \right| \leq 1$ , and thus

$$\left| U \left( \hat{X} + Z \left( \hat{\Gamma} - \frac{\mathbb{E}[Z\xi\hat{\Gamma}]}{\mathbb{E}[Z\xi^2]} \xi \right) \right) \right| \leq |U(\hat{X})| + 1.$$

Hence,  $U \left( \hat{X} + Z \left( \hat{\Gamma} - \frac{\mathbb{E}[Z\xi\hat{\Gamma}]}{\mathbb{E}[Z\xi^2]}\xi \right) \right) \in \mathcal{L}^1$ .

Similarly, since we also have  $\left| \mathbb{E} \left[ Z \left( \hat{\Gamma} - y^*\xi \right) \right] - Z \left( \hat{\Gamma} - y^*\xi \right) \right| \leq \delta$ , we have

$$\begin{aligned} & D \left( \mathbb{E} \left[ \hat{X} + Z \left( \hat{\Gamma} - y^*\xi \right) \right] - \left( \hat{X} + Z \left( \hat{\Gamma} - y^*\xi \right) \right) \right) \\ &= D \left( \mathbb{E} \left[ \hat{X} \right] - \hat{X} - Z \left( \hat{\Gamma} - y^*\xi \right) + \mathbb{E} \left[ Z \left( \hat{\Gamma} - y^*\xi \right) \right] \right) \\ &\leq D \left( \mathbb{E} \left[ \hat{X} \right] - \hat{X} - \delta \right) + D \left( \mathbb{E} \left[ \hat{X} \right] - \hat{X} + \delta \right). \end{aligned}$$

Hence, by Condition 2.2.1 (ii),  $D \left( \mathbb{E} \left[ \hat{X} + Z \left( \hat{\Gamma} - y^*\xi \right) \right] - \left( \hat{X} + Z \left( \hat{\Gamma} - y^*\xi \right) \right) \right) \in \mathcal{L}^1$ .

We can now conclude  $\hat{X} + Z \left( \hat{\Gamma} - y^*\xi \right)$  satisfies all the admissibility conditions of  $\mathcal{X}$ , and hence  $Z$  satisfies (2.2.12).  $\square$

In summary, by Proposition 2.2.10, we have  $Z_{y^*}$  satisfying (2.2.10)-(2.2.12) in Lemma 2.2.5. By Lemma 2.2.5, it is necessary that  $\hat{\Gamma}$  in terms of  $\hat{X}$  as in (2.2.8) satisfies the following algebraic structure:

$$\begin{cases} \hat{\Gamma} = Y\xi & \text{a.s. on } \{ \hat{X} > K \}, \\ \hat{\Gamma} \leq Y\xi & \text{a.s. on } \{ \hat{X} = K \} \text{ (if } \mathbb{P}[\hat{X} = K] > 0), \end{cases}$$

where  $Y = \mathbb{E} \left[ Z\xi\hat{\Gamma} \right] / \mathbb{E} [Z\xi^2]$  and  $Z$  is obtained in Proposition 2.2.10. Now, by setting  $M := \mathbb{E}[\hat{X}]$  and  $R := \mathbb{E}[D'(M - \hat{X})]$  together with the constraint  $\mathbb{E}[\hat{X}\xi] = x_0$  given in Problem 2.1.4, the claim described in Theorem 2.2.2 follows.

**Remark 2.2.11** (Comments on the Proof of Theorem 2.2.2). The Condition 2.2.1(i) has been utilized in order to make  $\mathbb{E}[\tilde{X}\hat{\Gamma}]$  in Proposition 2.2.3 well-defined. After establishing Proposition 2.2.3, to show the Nonlinear Moment Problem, a natural method is to construct another admissible terminal wealth  $\hat{X} + \tilde{X}$ , where  $\tilde{X}$  is a perturbation, so that, if the NMP fails to hold, Proposition 2.2.3 will be violated. However, in order to make  $\hat{X} + \tilde{X}$  admissible, we require the validity of  $D'(\mathbb{E}[\hat{X} + \tilde{X}] - (\hat{X} + \tilde{X})) \in \mathcal{L}^1$  which is not immediate in general even  $\tilde{X}$  is small



because  $D$  is convex. Hence, the establishment of Condition 2.2.1(ii) is crucial for this purpose.

Condition 2.2.1(ii) is mild and is satisfied under the case of power utility function.

### 2.2.2 Application to the Mean-Semivariance Problem

In this subsection, we take  $U(x) = x$ ,  $D(x) = \frac{1}{2}x_+^2$ . Then  $D'(x) = x_+$ . We revisit the non-existence result first obtained in Jin et al. (2005) via our Theorem 2.2.2.

**Theorem 2.2.12.** *There is no optimal solution for the continuous-time mean-semivariance problem.*

*Proof.* Assume the contrary, that there exists an admissible optimal control  $\hat{\pi}$ ; then its corresponding optimal terminal wealth  $\hat{X} \in \mathcal{L}^2$  solves Problem 2.1.4 by Theorem 2.1.5. Since  $D$  and  $D'$  are bounded by quadratic and linear functions respectively, it is clear that  $\hat{X}$  satisfies Conditions 2.2.1 (i) and (ii). Hence, by Theorem 2.2.2, it is necessary that there exist constants  $Y, M, R \in \mathbb{R}$  such that the quadruple  $(\hat{X}, Y, M, R)$  solves for the following Nonlinear Moment Problem:

$$Y\xi + \gamma R - 1 = \gamma \left( M - \hat{X} \right)_+ \text{ a.s.}, \quad (2.2.19)$$

subject to the constraints:  $\mathbb{E} \left[ \xi \hat{X} \right] = x_0$ ,  $\mathbb{E} \left[ \hat{X} \right] = M$  and  $\mathbb{E} \left[ \left( M - \hat{X} \right)_+ \right] = R$ .

Firstly, by taking expectation on the both sides of (2.2.19), we immediately have  $Y = 1/\mathbb{E}[\xi] > 0$ . If  $\gamma R - 1 \geq 0$ , then by (2.2.19),  $\gamma \left( M - \hat{X} \right)_+ > 0$  a.s., and hence  $\mathbb{E}[\hat{X}] < M$  which is in conflict with the constraint  $\mathbb{E}[\hat{X}] = M$ . If  $\gamma R - 1 < 0$ , there exists some  $\xi_0 > 0$  such that  $\gamma R - 1 + Y\xi < 0$  for all  $\xi \in (0, \xi_0)$ , which contradicts the positivity of the right hand side in (2.2.19). Thus, the nonlinear moment problem has no solution. We conclude that mean-semivariance problem does not admit an optimal solution.  $\square$

**Remark 2.2.13.** The mean-semivariance problem has been investigated in Jin et al. (2005). The authors considered the semivariance minimization problem with a fixed mean, and showed that this problem does not have an optimal solution except for the trivial case in which the mean is equal to the terminal wealth, which is the initial wealth accumulated at riskless interest rate. The nonexistence was proven in their work by showing that the optimal value function is non-attainable. The constrained optimization problem in Jin et al. (2005) and in Problem 2.1.2 are equivalent for suitable values of mean and risk aversion parameter. The trivial riskless solution becomes optimal in Problem 2.1.2 only when  $\gamma = \infty$ . For  $\gamma < \infty$ , the riskless strategy is dominated by another strategy attaining  $\frac{x_0}{\mathbb{E}[\xi]} + \theta \left(1 - \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi^2]}\xi\right)$  as the corresponding terminal wealth for sufficiently small values of  $\theta$ : Because the mean of  $\frac{x_0}{\mathbb{E}[\xi]} + \theta \left(1 - \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi^2]}\xi\right)$  is greater than  $\frac{x_0}{\mathbb{E}[\xi]}$  in the order of  $O(\theta)$  while the semivariance of  $\frac{x_0}{\mathbb{E}[\xi]} + \theta \left(1 - \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi^2]}\xi\right)$  is of the order  $O(\theta^2)$ , therefore  $\frac{x_0}{\mathbb{E}[\xi]} + \theta \left(1 - \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi^2]}\xi\right)$  has a greater objective value for sufficiently small  $\theta$ .

## 2.3 Sufficient Condition

### 2.3.1 Verification Theorem

We first introduce the following technical condition:

In this subsection, we aim to show that any admissible terminal wealth  $\hat{X} \in \mathcal{X}$  solving the Nonlinear Moment Problem satisfying the following condition is optimal terminal wealth of Problem 2.1.2:

**Condition 2.3.1.** Both  $U'(Z) \in \mathcal{L}^2$  and  $D'(\mathbb{E}[Z] - Z) \in \mathcal{L}^2$ .

There is a fundamental difference between the necessary condition in Theorem 2.2.2 and the sufficient condition in the next theorem. Conditions 2.2.1 (i) and (ii) are needed for the optimal terminal wealth satisfying the Nonlinear Moment Problem in the necessity result, while Condition 2.3.1 is required for the sufficiency.

**Theorem 2.3.2.** *Suppose that there exists  $\hat{X} \in \mathcal{X}$  satisfying Condition 2.3.1 and there exist constants  $Y, M, R \in \mathbb{R}$  so that the quadruple  $(\hat{X}, Y, M, R)$  solves for the Nonlinear Moment Problem (2.2.4)-(2.2.7). Then,  $\hat{X}$  is optimal for Problem 2.1.4, and it is also the optimal terminal wealth of Problem 2.1.2.*

**Remark 2.3.3.** Theorem 2.3.2 boils the optimal control problem 2.1.2 down to a static problem. Suppose that there exists an implicit function  $I(m, y) \in \mathbb{R}$  satisfying:

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) = y, \quad \text{for any } (m, y). \quad (2.3.20)$$

Then the Nonlinear Moment Problem (2.2.4)-(2.2.7) will be solved by  $(\max\{I(M, \gamma R + Y\xi), K\}, Y, M, R)$ , where the constants  $Y, M$  and  $R$  satisfy the following system of nonlinear equations:

$$\mathbb{E}[\xi \max\{I(M, \gamma R + Y\xi), K\}] = x_0, \quad (2.3.21)$$

$$\mathbb{E}[\max\{I(M, \gamma R + Y\xi), K\}] = M, \quad (2.3.22)$$

$$\mathbb{E}[D'(M - \max\{I(M, \gamma R + Y\xi), K\})] = R. \quad (2.3.23)$$

After we verify that  $\max\{I(M, \gamma R + Y\xi), K\}$  belongs to  $\mathcal{X}$  and also satisfies Condition 2.3.1,  $\max\{I(M, \gamma R + Y\xi), K\}$  is the optimal solution for Problem 2.1.4.

*Proof of Theorem 2.3.2.* Let  $(\hat{X}, Y, M, R)$  be the solution of Nonlinear Moment Problem (2.2.4)-(2.2.7) and  $\tilde{X} \in \mathcal{L}^2$  be an arbitrary random variable such that  $\hat{X} + \tilde{X}$  is admissible for Problem 2.1.4, i.e.  $\hat{X} + \tilde{X} \in \mathcal{X}$  and  $\mathbb{E}[\xi(\hat{X} + \tilde{X})] = x_0$ . By (2.2.5), we have  $\mathbb{E}[\xi\tilde{X}] = 0$ . By Lemma 2.2.4, the chain rule and under our hypothesis that  $\hat{X}$  satisfies (2.2.4) and Condition 2.3.1, we have

$$\begin{aligned}
& \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} \\
&= \mathbb{E} \left[ U'(\hat{X})\tilde{X} \right] - \gamma \mathbb{E} \left[ D' \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \left( \mathbb{E}[\tilde{X}] - \tilde{X} \right) \right] \\
&= \mathbb{E} \left[ \tilde{X} \left( U'(\hat{X}) - \gamma \mathbb{E} \left[ D' \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \right] + \gamma D' \left( \mathbb{E}[\hat{X}] - \hat{X} \right) \right) \right] \\
&\leq \mathbb{E} \left[ \tilde{X}(Y\xi) \right] = 0.
\end{aligned} \tag{2.3.24}$$

By the concavity of  $U$  and convexity of  $D$ , it is clear that  $\Psi(\hat{X} + \theta\tilde{X}) \geq (1 - \theta)\Psi(\hat{X}) + \theta\Psi(\hat{X} + \tilde{X})$  for any  $\theta \in (0, 1]$ . Then

$$\Psi(\hat{X}) \geq \Psi(\hat{X} + \tilde{X}) - \frac{\Psi(\hat{X} + \theta\tilde{X}) - \Psi(\hat{X})}{\theta}. \tag{2.3.25}$$

By (2.3.24),  $\lim_{\theta \downarrow 0} \frac{\Psi(\hat{X} + \theta\tilde{X}) - \Psi(\hat{X})}{\theta} = \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} \leq 0$ . After taking limits on both sides of (2.3.25),  $\Psi(\hat{X}) \geq \Psi(\hat{X} + \tilde{X})$ , hence  $\hat{X}$  is optimal for Problem 2.1.4. By Theorem 2.1.5, We can now conclude that  $\hat{X}$  is the optimal terminal wealth of Problem 2.1.2.  $\square$

In the next three subsections, we apply Theorem 2.3.2 to establish the existence of optimal solutions for different utility-risk frameworks: (i) Utility-Downside-Risk, (ii) Utility-Strictly-Convex-Risk, and (iii) Mean-Risk. In particular, we remark that the positive answers to the first two problems have long been absent in the literature.

### 2.3.2 Application to the Utility-Downside-Risk Problem

In this subsection, we take  $\mathcal{D} = [0, \infty)$ . We assume that  $U : [0, \infty) \rightarrow [0, \infty)$  is strictly concave, and  $U$  and  $D : \mathbb{R} \rightarrow [0, \infty)$  are continuously differentiable. We consider  $D$  to be a downside risk function, so  $D$  is positive and strictly convex on  $(0, \infty)$  and  $D(x) = 0$  for  $x \leq 0$ . Thus, we have  $D'(x) > 0$  when  $x > 0$  and  $D'(x) = 0$  when  $x \leq 0$ . In this proposed model, the payoff greater than its mean will not be penalized, and only the downside risk would be taken into account.

Moreover, we assume that  $U$  and  $D$  satisfy the following conditions:

$$U'(0) = \infty, U'(\infty) = 0 \text{ and } D'(\infty) = \infty. \quad (2.3.26)$$

Thus any utility functions satisfying the Inada conditions can be covered. Note that this formulation can cover the utility-semivariance problem, its positive answer has a substantial contrast to the nonexistence of an optimal solution to the mean-semivariance problem. We further make the following assumption on the utility function:

**Assumption 2.3.4.** There exists  $k_0 > 0$  so that the inverse of the first-order derivative of  $U$ ,  $(U')^{-1}$ , satisfies  $(U')^{-1}(k_0\xi) \in \mathcal{L}^2$ .

Note that Assumption 2.3.4 can be satisfied if there exist  $\beta \in (0, 1)$  and  $\gamma > 1$  such that  $U'(\beta y) \leq \gamma U'(y)$  for all  $y > 0$ , and this condition has been adopted in Zheng (2009).

According to Remark 2.3.3, we first find an implicit function satisfying (2.3.20), then the Nonlinear Moment Problem (2.2.4)-(2.2.7) can be reduced into a nonlinear programming problem (2.3.21)-(2.3.23).

**Proposition 2.3.5.** *There exists an implicit function  $I : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$  satisfying:*

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R} \times (0, \infty). \quad (2.3.27)$$

Moreover, this function  $I$  possesses the following regularities:

- (a) (i) For each  $m \in \mathbb{R}$ ,  $I(m, y)$  is strictly decreasing in  $y$  on  $(0, \infty)$ .
- (ii) For each  $y \in (0, \infty)$ ,  $I(m, y)$  is strictly increasing in  $m$  on  $\{m \in \mathbb{R} \mid y \geq U'(m)\}$ ;  $I(m, y) = (U')^{-1}(y) \in (0, \infty)$  for all  $m \in \{m \in \mathbb{R} \mid y \leq U'(m)\}$ .
- (b)  $I(m, y)$  is jointly continuous in  $(m, y) \in \mathbb{R} \times (0, \infty)$ .

*Proof.* Fix  $(m, y) \in \mathbb{R} \times (0, \infty)$ . Since  $U$  is strictly concave and  $D$  is convex,  $U'(z) + \gamma D'(m - z) - y$  is strictly decreasing in  $z$ . Since  $U'$  and  $D'$  is continuous,  $U'(z) + \gamma D'(m - z) - y$  is continuous in  $z$ . Under Assumptions (2.3.26), we can also easily show that  $U'(z) + \gamma D'(m - z) - y$  is coercive in the sense that

$$\lim_{z \rightarrow 0} U'(z) + \gamma D'(m - z) - y = \infty, \quad \lim_{z \rightarrow \infty} U'(z) + \gamma D'(m - z) - y = -y < 0.$$

Thus, by the intermediate value theorem and strict monotonicity, for any  $(m, y) \in \mathbb{R} \times (0, \infty)$ , there exists a unique  $I(m, y) \in (0, \infty)$  such that

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0.$$

**(a) (i)** For fixed  $(m, y)$ ,  $U'(z) + \gamma D'(m - z) - y$  is strictly decreasing in  $z$ . When  $(z, m)$  is fixed,  $U'(z) + \gamma D'(m - z) - y$  is strictly decreasing in  $y$ , so  $I(m, y)$  is strictly decreasing in  $y$ .

**(ii)** We first claim that  $m \geq I(m, y)$  when  $y \geq U'(m)$ . Assume the contrary, that  $m < I(m, y)$ . We have  $D'(m - I(m, y)) = 0$ , and then  $y = U'(I(m, y)) \geq U'(m)$ , which contradicts to  $m < I(m, y)$  as  $U'$  is decreasing. Next, we assume another contrary, that there exists  $m_0, y_0$  with  $y_0 \leq U'(m_0)$  and  $\delta > 0$  such that  $I(m_0 + \delta, y_0) \leq I(m_0, y_0)$ . Then we have  $m_0 + \delta - I(m_0 + \delta, y_0) > m_0 - I(m_0, y_0) \geq 0$ , thus

$$\begin{aligned} & U'(I(m_0 + \delta, y_0)) + \gamma D'(m_0 + \delta - I(m_0 + \delta, y_0)) \\ & > U'(I(m_0, y_0)) + \gamma D'(m_0 - I(m_0, y_0)), \end{aligned}$$

which contradicts (2.3.27).

For the second assertion,  $y \leq U'(m)$  implies that  $m \leq (U')^{-1}(y)$ , thus  $I(m, y) = (U')^{-1}(y)$  satisfies (2.3.27), and it is the unique solution by the main result in this proposition.

**(b)** Fix  $(M_0, Y_0) \in \mathbb{R} \times (0, \infty)$ . By part (a), for any small enough  $\epsilon > 0$ ,

$$I(M_0 - \epsilon, Y_0 + \epsilon) \leq I(m, y) \leq I(M_0 + \epsilon, Y_0 - \epsilon) \quad (2.3.28)$$

for any  $|(m, y) - (M_0, Y_0)| < \epsilon$ .

It is straightforward to show that  $\lim_{\epsilon \downarrow 0} I(M_0 + \epsilon, Y_0 - \epsilon)$  and  $I(M_0, Y_0)$  satisfy the same equation in (2.3.27), so we have  $\lim_{\epsilon \downarrow 0} I(M_0 + \epsilon, Y_0 - \epsilon) = I(M_0, Y_0)$ . Similarly, we have  $\lim_{\epsilon \downarrow 0} I(M_0 - \epsilon, Y_0 + \epsilon) = I(M_0, Y_0)$ . Applying the sandwich theorem to (2.3.28), we can conclude that

$$\lim_{(m,y) \rightarrow (M_0, Y_0)} I(m, y) = I(M_0, Y_0).$$

□

Since the implicit function  $I$  never takes value in the boundary of  $\mathcal{D}$ , so we now look for numbers  $Y, M$  and  $R$  that solve the following system of equations as described in Remark 2.3.3:

$$\mathbb{E}[\xi I(M, \gamma R + Y\xi)] = x_0; \quad (2.3.29)$$

$$\mathbb{E}[I(M, \gamma R + Y\xi)] = M; \quad (2.3.30)$$

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] = R. \quad (2.3.31)$$

**Proposition 2.3.6.** *There exist numbers  $Y, M, R \in (0, \infty)$  such that the system of nonlinear equations of (2.3.29)-(2.3.31) is satisfied. Thus,  $(I(M, \gamma R + Y\xi), Y, M, R)$  is a solution of the system of Equations (2.2.4)-(2.2.7), where  $I$  is given in Proposition 2.3.5.*

We shall solve for roots  $Y, M$  and  $R$  one by one via applying the intermediate value theorem successively:

**Lemma 2.3.7.** *Given  $Y, M \in (0, \infty)$ , there exists a unique  $R = R_{Y,M} \in (0, D'(M))$  satisfying*

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] = R; \quad (2.3.32)$$

or equivalently by (2.3.27):

$$\mathbb{E}[U'(I(M, \gamma R + Y\xi))] = Y\mathbb{E}[\xi]. \quad (2.3.33)$$

Furthermore,  $R_{Y,M}$  is strictly increasing in  $M$  for a fixed  $Y$  and is also strictly increasing in  $Y$  for a fixed  $M$ .

*Proof.* We prove the followings in order:

- (a)  $\mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R]$  is strictly decreasing in  $R$ ,
- (b)  $\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))]$  is continuous in  $R$ ,
- (c1)  $\lim_{R \rightarrow 0} \mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R] > 0$ ,
- (c2)  $\lim_{R \rightarrow D'(M)} \mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R] < 0$ .

In light of (b), (c1) and (c2), then by the intermediate value theorem, there exists  $R = R_{Y,M}$  satisfying (2.3.32) while the uniqueness of  $R_{Y,M}$  is guaranteed by (a). Finally, we show that

- (d1)  $R_{Y,M}$  is strictly increasing in  $M$  for fixed  $Y$ ,
- (d2)  $R_{Y,M}$  is strictly increasing in  $Y$  for fixed  $M$ .

For each of the above items:

- (a) By Proposition 2.3.5 (a)(i),  $U'(I(M, \gamma R + Y\xi))$  is strictly increasing in  $R$  almost surely. By (2.3.27),  $D'(M - I(M, \gamma R + Y\xi)) - R = \frac{1}{\gamma}(Y\xi - U'(I(M, \gamma R + Y\xi)))$  is therefore strictly decreasing in  $R$  almost surely. Thus,  $\mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R]$  is strictly decreasing in  $R$ .
- (b) Since  $D'$  and  $I$  are both continuous, so  $D'(M - I(M, \gamma R + Y\xi))$  is continuous in  $R$ . Hence, the claim follows by an application of the Dominated Convergence Theorem.
- (c1) When  $Y\xi > U'(M)$ , we have  $M > I(M, Y\xi)$ , thus

$$D'(M - I(M, Y\xi)) > 0, \text{ a.s. on } \{Y\xi > U'(M)\}.$$



Since  $\frac{U'(M)}{Y} \in (0, \infty)$ , by the definition of  $\xi$ , we have

$$\mathbb{P}[D'(M - I(M, Y\xi)) > 0] \geq \mathbb{P}\left[\xi > \frac{U'(M)}{Y}\right] > 0.$$

By the Dominated Convergence Theorem,

$$\begin{aligned} & \lim_{R \rightarrow 0} \mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R] \\ &= \mathbb{E}\left[\lim_{R \rightarrow 0} D'(M - I(M, \gamma R + Y\xi))\right] \\ &= \mathbb{E}[D'(M - I(M, Y\xi))] > 0. \end{aligned}$$

(c2) By (2.3.27), we have  $\lim_{R \rightarrow D'(M)} I(M, \gamma R + Y\xi) > 0$  almost surely, so

$$\lim_{R \rightarrow D'(M)} D'(M - I(M, \gamma R + Y\xi)) < D'(M), \text{ a.s. .}$$

By the Monotone Convergence Theorem,

$$\lim_{R \rightarrow D'(M)} \mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] < D'(M),$$

thus this part follows.

(d1) Assume the contrary, that there exists a  $M_0 \in (0, \infty)$  and a  $\delta > 0$  such that

$$R_{Y, M_0} \geq R_{Y, M_0 + \delta}.$$

By Proposition 2.3.5 (a)(ii), when  $\gamma R + Y\xi \geq U'(M)$ ,  $I(M, \gamma R + Y\xi)$  is strictly increasing in  $M$ , thus  $D'(M - I(M, \gamma R + Y\xi)) - R$  is strictly increasing in  $M$  on  $\{\gamma R + Y\xi \geq U'(M)\}$  by (2.3.27). When  $\gamma R + Y\xi \leq U'(M)$ ,  $D'(M - I(M, \gamma R + Y\xi)) - R = -R$ . So given  $Y$  and  $R$ ,  $D'(M - I(M, \gamma R + Y\xi))$  is increasing in  $M$  and is strictly increasing on  $\{\gamma R + Y\xi \geq U'(M)\}$ .

Thus we have  $\mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R]$  is strictly increasing in  $M$ .

On the other hand, in (a),  $D'(M - I(M, \gamma R + Y\xi)) - R$  is strictly decreasing in  $R$  almost surely. Now, we get

$$\begin{aligned} 0 &= \mathbb{E}[D'(M_0 + \delta - I(M_0 + \delta, \gamma R_{Y, M_0 + \delta} + Y\xi)) - R_{Y, M_0 + \delta}] \\ &\geq \mathbb{E}[D'(M_0 + \delta - I(M_0 + \delta, \gamma R_{Y, M_0} + Y\xi)) - R_{Y, M_0}] \\ &> \mathbb{E}[D'(M_0 - I(M_0, \gamma R_{Y, M_0} + Y\xi)) - R_{Y, M_0}] = 0, \end{aligned}$$

which is a contradiction.

**(d2)** In Proposition 2.3.5 (a)(i), we have  $I(M, \gamma R + Y\xi)$  strictly decreasing in  $Y$  given a fixed point of  $M$  and  $R$ , thus  $\mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R]$  is also strictly increasing in  $Y$  almost surely given  $(M, R)$ . Thus, this part can be verified as in (d1).

□

**Lemma 2.3.8.** *Given  $Y \in (0, \infty)$  and  $R_{Y,M}$  as specified for each  $M \in (0, \infty)$  in Lemma 2.3.7, there exists a unique  $M = M_Y \in (0, \infty)$  such that*

$$\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] = M. \quad (2.3.34)$$

Furthermore,  $M_Y$  is strictly decreasing in  $Y$ .

*Proof.* In this lemma, we prove the following in order:

- (a)**  $\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M$  is strictly decreasing in  $M$ ,
- (b)**  $\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)]$  is continuous in  $M$ ,
- (c1)**  $\lim_{M \rightarrow 0} \mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi) - M] > 0$ ,
- (c2)**  $\lim_{M \rightarrow \infty} \mathbb{E}[M - I(M, \gamma R_{Y,M} + Y\xi)] = \infty$ .

By using (b), (c1) and (c2), in accordance with the intermediate value theorem and part (a), there exists a unique  $M = M_Y$  satisfying (2.3.34). Finally, we show that

- (d)**  $M_Y$  is strictly decreasing in  $Y$ .

For each of the above items:

(a) By Proposition 2.3.5 (a)(ii),  $D'(M - I(M, \gamma R + Y\xi)) - R = \frac{1}{\gamma}(Y\xi - U'(I(M, \gamma R + Y\xi)))$  is strictly increasing in  $M$  when  $\gamma R + Y\xi \geq U'(M)$ . Thus, because  $D'$  is strictly increasing for positive  $M - I(M, \gamma R + Y\xi)$ ,  $I(M, \gamma R + Y\xi) - M$  is strictly decreasing in  $M$  on  $\{\gamma R + Y\xi \geq U'(M)\}$ . On the other hand, on  $\{\gamma R + Y\xi \leq U'(M)\}$ ,  $I(M, \gamma R + Y\xi) - M = (U')^{-1}(\gamma R + Y\xi) - M$  is strictly decreasing in  $M$ .

By Proposition 2.3.5 (a)(i),  $I(M, \gamma R + Y\xi) - M$  is strictly decreasing in  $R$ . By Lemma 2.3.7,  $R_{Y,M}$  is strictly increasing in  $M$ . Thus, for any  $\delta > 0$ , it is almost surely that:

$$\begin{aligned} I(M, \gamma R_{Y,M} + Y\xi) - M &> I(M, \gamma R_{Y,M+\delta} + Y\xi) - M \\ &> I(M + \delta, \gamma R_{Y,M+\delta} + Y\xi) - (M + \delta) \end{aligned}$$

Thus,  $\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M$  is strictly decreasing in  $M$ .

(b) Fix  $M_0 \in (0, \infty)$ . By the continuity of  $D'$  and  $I$ , it is almost surely that:

$$\begin{aligned} &\lim_{M \downarrow M_0} (D'(M - I(M, \gamma R_{Y,M} + Y\xi)) - R_{Y,M}) \\ &= D' \left( M_0 - I \left( M_0, \gamma \lim_{M \downarrow M_0} R_{Y,M} + Y\xi \right) \right) - \lim_{M \downarrow M_0} R_{Y,M} \quad (2.3.35) \end{aligned}$$

By (2.3.32), (2.3.35) and the Dominated Convergence Theorem, we have

$$\begin{aligned} &\mathbb{E} \left[ D' \left( M_0 - I \left( M_0, \gamma \lim_{M \downarrow M_0} R_{Y,M} + Y\xi \right) \right) \right] - \lim_{M \downarrow M_0} R_{Y,M} \\ &= \lim_{M \downarrow M_0} (\mathbb{E}[D'(M - I(M, \gamma R_{Y,M} + Y\xi))] - R_{Y,M}) = 0. \end{aligned}$$

By the uniqueness result in Lemma 2.3.7, we conclude that  $\lim_{M \downarrow M_0} R_{Y,M} = R_{Y,M_0}$ . Similarly, we have  $\lim_{M \uparrow M_0} R_{Y,M} = R_{Y,M_0}$ . By continuity of  $I$ ,

$$\lim_{M \downarrow M_0} I(M, \gamma R_{Y,M} + Y\xi) = I(M_0, \gamma R_{Y,M_0} + Y\xi).$$

Similarly, the equality of limits from the opposite side can also be deduced, so

$$\lim_{M \rightarrow M_0} I(M, \gamma R_{Y,M} + Y\xi) = I(M_0, \gamma R_{Y,M_0} + Y\xi).$$

Finally, our claim follows by the Dominated Convergence Theorem.

(c1) Since  $I(M, \gamma R_{Y,M} + Y\xi) - M$  is decreasing in  $M$  by (a), thus

$$\lim_{M \rightarrow 0} I(M, \gamma R_{Y,M} + Y\xi) = \lim_{M \rightarrow 0} (I(M, \gamma R_{Y,M} + Y\xi) - M).$$

We claim that  $\lim_{M \rightarrow 0} I(M, \gamma R_{Y,M} + Y\xi) > 0$  almost surely. Assume the contrary, that there exists a sample value of  $\xi_0$  such that  $\lim_{M \rightarrow 0} I(M, \gamma R_{Y,M} + Y\xi_0) = 0$ . Then, we have

$$\begin{aligned} \lim_{M \rightarrow 0} D'(M - I(M, \gamma R_{Y,M} + Y\xi_0)) &= D'(0) \quad \text{and} \\ \lim_{M \rightarrow 0} U'(I(M, \gamma R_{Y,M} + Y\xi_0)) &= \infty. \end{aligned}$$

But by (2.3.27), we again have:

$$\begin{aligned} &\lim_{M \rightarrow 0} U'(I(M, \gamma R_{Y,M} + Y\xi_0)) \\ &= \xi_0 Y + \gamma \lim_{M \rightarrow 0} (R_{Y,M} - D'(M - I(M, \gamma R_{Y,M} + Y\xi_0))) \\ &\leq \xi_0 Y + \gamma \lim_{M \rightarrow 0} (D'(M) - D'(M - I(M, \gamma R_{Y,M} + Y\xi_0))) \\ &\leq \xi_0 Y + \gamma (D'(0) - D'(0)) < \infty, \end{aligned}$$

which leads to a contradiction. Hence,  $\lim_{M \rightarrow 0} I(M, \gamma R_{Y,M} + Y\xi) > 0$  almost surely. Since  $I(M, \gamma R_{Y,M} + Y\xi) - M$  is decreasing in  $M$ , by the Monotone Convergence Theorem,

$$\lim_{M \rightarrow 0} \mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi) - M] > 0.$$

(c2) By (2.3.27), we either have  $\lim_{M \rightarrow \infty} I(M, \gamma R_{Y,M} + Y\xi) = \infty$  almost surely or  $\lim_{M \rightarrow \infty} R_{Y,M} = \infty$ .

Assume that  $\lim_{M \rightarrow \infty} R_{Y,M} < \infty$ , thus  $\lim_{M \rightarrow \infty} I(M, \gamma R_{Y,M} + Y\xi) = \infty$  almost surely, then by continuity of  $U'$ ,

$$\lim_{M \rightarrow \infty} U'(I(M, \gamma R_{Y,M} + Y\xi)) = 0 \text{ a.s.} \quad (2.3.36)$$

Then, by the Dominated Convergence Theorem and (2.3.36),

$$\lim_{M \rightarrow \infty} \mathbb{E} [U'(I(M, \gamma R_{Y,M} + Y\xi))] = \mathbb{E} \left[ \lim_{M \rightarrow \infty} U'(I(M, \gamma R_{Y,M} + Y\xi)) \right] = 0.$$

Hence, we have  $\lim_{M \rightarrow \infty} \mathbb{E}[U'(I(M, \gamma R_{Y,M} + Y\xi)) - \xi Y] = -\mathbb{E}[\xi]Y < 0$ .

This contradicts (2.3.33) in Lemma 2.3.7, so we must have  $\lim_{M \rightarrow \infty} R_{Y,M} = \infty$ .

Given a sample  $\xi_0$ , assume the contrary that  $\lim_{M \rightarrow \infty} M - I(M, \gamma R_{Y,M} + Y\xi_0) < \infty$ , then  $\lim_{M \rightarrow \infty} D'(M - I(M, \gamma R_{Y,M} + Y\xi_0)) < \infty$ . By (2.3.27) and  $\lim_{M \rightarrow \infty} R_{Y,M} = \infty$ , we have  $\lim_{M \rightarrow \infty} U'(I(M, \gamma R_{Y,M} + Y\xi_0)) = \infty$ , thus  $\lim_{M \rightarrow \infty} I(M, \gamma R_{Y,M} + Y\xi_0) = 0$  and then it results in:

$$\lim_{M \rightarrow \infty} (M - I(M, \gamma R_{Y,M} + Y\xi_0)) = \infty,$$

a contradiction.

Now, we have  $\lim_{M \rightarrow \infty} M - I(M, \gamma R_{Y,M} + Y\xi) = \infty$  almost surely. By the Monotone Convergence Theorem,

$$\lim_{M \rightarrow \infty} \mathbb{E} [M - I(M, \gamma R_{Y,M} + Y\xi)] = \infty > 0.$$

- (d) In part (a), we have shown that  $\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M$  is strictly decreasing in  $M$  for a fixed  $Y$ . On the other hand, in Proposition 2.3.5 (a)(i) and Lemma 2.3.7 (d1), we can show that  $\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M$  is strictly decreasing in  $Y$  for a fixed  $M$ . By (2.3.34),  $\mathbb{E}[I(M_Y, \gamma R_{Y,M_Y} + Y\xi)] - M_Y = 0$  for all values of  $Y$ , thus,  $M_Y$  is strictly decreasing in  $Y$ .

□

**Lemma 2.3.9.** *Given  $R_{Y,M}$  and  $M_Y$  as specified in Lemmas 2.3.7 and 2.3.8 respectively for each  $Y, M \in (0, \infty)$ , there exists a (not necessarily unique)  $Y^* \in (0, \infty)$  such that*

$$\mathbb{E}[\xi I(M_Y, \gamma R_{Y,M_Y} + Y\xi)] = x_0. \quad (2.3.37)$$

*Proof.* In this lemma, we prove the following in order:

- (a)  $\mathbb{E} [\xi I (M_Y, \gamma R_{Y, M_Y} + Y \xi)]$  is continuous in  $Y$ ,
- (b1)  $\lim_{Y \rightarrow 0} \mathbb{E} [\xi I (M_Y, \gamma R_{Y, M_Y} + Y \xi)] = \infty$ ,
- (b2)  $\lim_{Y \rightarrow \infty} \mathbb{E} [\xi I (M_Y, \gamma R_{Y, M_Y} + Y \xi)] = 0$ .

In turn, by the intermediate value theorem with (a), (b1) and (b2), there exists  $Y$  satisfying (2.3.37).

For each of the above items:

- (a) Fix  $Y_0 \in (0, \infty)$ . For any  $\epsilon > 0$ , by Lemmas 2.3.7 and 2.3.8, we have

$$R_{Y_0 - \epsilon, M_{Y_0 + \epsilon}} < R_{Y, M_Y} < R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \text{ for any } Y_0 - \epsilon < Y < Y_0 + \epsilon, (2.3.38)$$

and  $R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}$  is increasing in  $\epsilon$ , we therefore have both the finite existence of  $\lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}$  and  $\lim_{\epsilon \downarrow 0} R_{Y_0 - \epsilon, M_{Y_0 + \epsilon}}$ . By Proposition 2.3.5 (b),  $I$  is jointly continuous,

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} I (M_{Y_0 - \epsilon}, (Y_0 + \epsilon) \xi + \gamma R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}) \\ &= I \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}, Y_0 \xi + \gamma \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} (D' (M_{Y_0 - \epsilon} - I (M_{Y_0 - \epsilon}, (Y_0 + \epsilon) \xi + \gamma R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}})) - R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}) \\ &= D' \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon} - I \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}, Y_0 \xi + \gamma \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right) \right) \\ & \quad - \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}. \end{aligned}$$

By standard application of the Dominated Convergence Theorem,

$$\begin{aligned} & \mathbb{E} \left[ D' \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon} - I \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}, Y_0 \xi + \gamma \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right) \right) - \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right] \\ &= \lim_{\epsilon \downarrow 0} \mathbb{E} \left[ D' (M_{Y_0 - \epsilon} - I (M_{Y_0 - \epsilon}, (Y_0 + \epsilon) \xi + \gamma R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}})) - R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right] = 0. \end{aligned}$$

Since  $R$  in Lemma 2.3.7 is uniquely defined in (2.3.32), thus  $R_{Y_0, \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}} = \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}}$ . Similarly, we have  $R_{Y_0, \lim_{\epsilon \downarrow 0} M_{Y_0 + \epsilon}} = \lim_{\epsilon \downarrow 0} R_{Y_0 - \epsilon, M_{Y_0 + \epsilon}}$ . By the Dominated Convergence Theorem and (2.3.34),

$$\begin{aligned} & \mathbb{E} \left[ I \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}, Y_0 \xi + \gamma R_{Y_0, \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}} \right) - \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon} \right] \\ &= \mathbb{E} \left[ I \left( \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}, Y_0 \xi + \gamma \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right) - \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon} \right] \\ &= \mathbb{E} \left[ \lim_{\epsilon \downarrow 0} \left( I \left( M_{Y_0 - \epsilon}, (Y_0 + \epsilon) \xi + \gamma R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} \right) - M_{Y_0 - \epsilon} \right) \right] = 0. \end{aligned}$$

Since  $M$  in Lemma 2.3.8 is uniquely defined in (2.3.34), thus  $\lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon} = M_{Y_0}$ . Similarly, we have  $\lim_{\epsilon \uparrow 0} M_{Y_0 - \epsilon} = M_{Y_0}$  and  $\lim_{Y \rightarrow Y_0} M_Y = M_{Y_0}$ . Hence,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} R_{Y_0 + \epsilon, M_{Y_0 - \epsilon}} &= R_{Y_0, \lim_{\epsilon \downarrow 0} M_{Y_0 - \epsilon}} = R_{Y_0, M_{Y_0}} \\ &= R_{Y_0, \lim_{\epsilon \downarrow 0} M_{Y_0 + \epsilon}} = \lim_{\epsilon \downarrow 0} R_{Y_0 - \epsilon, M_{Y_0 + \epsilon}}. \end{aligned}$$

By (2.3.38), we have  $\lim_{Y \rightarrow Y_0} R_{Y, M_Y} = R_{Y_0, M_{Y_0}}$ . Then we have

$$\begin{aligned} \lim_{Y \rightarrow Y_0} I \left( M_Y, Y \xi + \gamma R_{Y, M_Y} \right) &= I \left( \lim_{Y \rightarrow Y_0} M_Y, Y_0 \xi + \gamma \lim_{Y \rightarrow Y_0} R_{Y, M_Y} \right) \\ &= I \left( M_{Y_0}, Y_0 \xi + \gamma R_{Y_0, M_{Y_0}} \right). \end{aligned}$$

Finally, our claim follows from another application of the Dominated Convergence Theorem.

**(b1)** For an arbitrary a sample value  $\xi_0 \in (0, \infty)$ . Assume the contrary that

$$\liminf_{Y \rightarrow 0} I \left( M_Y, Y \xi_0 + \gamma R_{Y, M_Y} \right) < \infty,$$

then there exists a sequence  $\{y_n\}$  with  $y_n \rightarrow 0$  such that

$$\liminf_{Y \rightarrow 0} I \left( M_Y, Y \xi_0 + \gamma R_{Y, M_Y} \right) = \lim_{n \rightarrow \infty} I \left( M_{y_n}, y_n \xi_0 + \gamma R_{y_n, M_{y_n}} \right) < \infty.$$

Clearly,  $\lim_{n \rightarrow \infty} U' \left( I \left( M_{y_n}, y_n \xi_0 + \gamma R_{y_n, M_{y_n}} \right) \right) > 0$ . Furthermore, since  $U'(I(m, \cdot))$  is increasing, thus for any  $\xi > \xi_0$ ,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} U' \left( I \left( M_{y_n}, y_n \xi + \gamma R_{y_n, M_{y_n}} \right) \right) \\ & \geq \lim_{n \rightarrow \infty} U' \left( I \left( M_{y_n}, y_n \xi_0 + \gamma R_{y_n, M_{y_n}} \right) \right) > 0. \end{aligned}$$

By Fatou's Lemma and (2.3.33),

$$\begin{aligned} \liminf_{n \rightarrow \infty} y_n \mathbb{E}[\xi] &= \liminf_{n \rightarrow \infty} \mathbb{E} \left[ U' \left( I \left( M_{y_n}, y_n \xi + \gamma R_{y_n, M_{y_n}} \right) \right) \right] \\ &\geq \mathbb{E} \left[ \liminf_{n \rightarrow \infty} U' \left( I \left( M_{y_n}, y_n \xi + \gamma R_{y_n, M_{y_n}} \right) \right) \right] \\ &\geq \lim_{n \rightarrow \infty} U' \left( I \left( M_{y_n}, y_n \xi_0 + \gamma R_{y_n, M_{y_n}} \right) \right) \mathbb{P}[\xi > \xi_0] > 0, \end{aligned}$$

which contradict  $\liminf_{n \rightarrow \infty} y_n \mathbb{E}[\xi] = 0$ .

Therefore,  $\liminf_{Y \rightarrow 0} I(M_Y, Y\xi_0 + \gamma R_{Y, M_Y}) = \infty$ , for any  $\xi_0 \in (0, \infty)$ . Hence,  $\liminf_{Y \rightarrow 0} \xi I(M_Y, Y\xi + \gamma R_{Y, M_Y}) = \infty$  almost surely. By Fatou's Lemma,

$$\liminf_{Y \rightarrow 0} \mathbb{E}[\xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi)] \geq \mathbb{E} \left[ \liminf_{Y \rightarrow 0} \xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi) \right] = \infty.$$

**(b2)** Since  $M_Y$  is decreasing in  $Y$  as shown in Lemma 2.3.8, thus  $\lim_{Y \rightarrow \infty} M_Y$  exists and is finite.

For any  $N \in (0, \infty)$ , it is clear from its definition that  $\lim_{Y \rightarrow \infty} R_{N, M_Y} \geq 0$ .

Therefore,  $\lim_{Y \rightarrow \infty} I(M_Y, N\xi + \gamma R_{N, M_Y}) = I(\lim_{Y \rightarrow \infty} M_Y, N\xi + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y})$  is finite almost surely for any  $N \in (0, \infty)$ . Since  $I(\lim_{Y \rightarrow \infty} M_Y, N\xi + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y})$  is decreasing in  $N$ , then  $\lim_{N \rightarrow \infty} I(\lim_{Y \rightarrow \infty} M_Y, N\xi + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y})$  exists and is finite since  $I$  is always non-negative.

For if there exists a sample  $\xi_0 \in (0, \infty)$  such that

$$\lim_{N \rightarrow \infty} I \left( \lim_{Y \rightarrow \infty} M_Y, N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \right) > 0, \quad (2.3.39)$$

then  $\lim_{N \rightarrow \infty} D'(\lim_{Y \rightarrow \infty} M_Y - I(\lim_{Y \rightarrow \infty} M_Y, N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y})) < \infty$  and  $\lim_{N \rightarrow \infty} U'(I(\lim_{Y \rightarrow \infty} M_Y, N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y})) < \infty$  if (2.3.39) holds. By (2.3.27),

$$\begin{aligned} &N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \\ &= U' \left( I \left( \lim_{Y \rightarrow \infty} M_Y, N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \right) \right) \\ &\quad + D' \left( \lim_{Y \rightarrow \infty} M_Y - I \left( \lim_{Y \rightarrow \infty} M_Y, N\xi_0 + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \right) \right). \end{aligned}$$



Then taking  $N \rightarrow \infty$  into the both sides, the limit in the left hand side tends to infinity as  $\lim_{Y \rightarrow \infty} R_{N, M_Y} \geq 0 > -\infty$  for any finite fixed  $N$ , while the limit in the right hand side is finite, a contradiction. So

$$\lim_{N \rightarrow \infty} I \left( \lim_{Y \rightarrow \infty} M_Y, N\xi + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \right) = 0, \quad a.s. \quad (2.3.40)$$

On the other hand, since  $I(M_Y, Y\xi + \gamma R_{Y, M_Y}) \leq I(M_Y, N\xi + \gamma R_{N, M_Y})$  for all  $N \leq Y < \infty$  and  $\lim_{Y \rightarrow \infty} I(M_Y, N\xi + \gamma R_{N, M_Y})$  exists, we have almost surely, for any  $N$ ,

$$\begin{aligned} \limsup_{Y \rightarrow \infty} I(M_Y, Y\xi + \gamma R_{Y, M_Y}) &\leq \lim_{Y \rightarrow \infty} I(M_Y, N\xi + \gamma R_{N, M_Y}) \\ &= I \left( \lim_{Y \rightarrow \infty} M_Y, N\xi + \gamma \lim_{Y \rightarrow \infty} R_{N, M_Y} \right). \end{aligned}$$

By (2.3.40),  $\limsup_{Y \rightarrow \infty} I(M_Y, Y\xi + \gamma R_{Y, M_Y}) = 0$  almost surely. Finally, by reverse Fatou's lemma, since  $I(M_Y, Y\xi + \gamma R_{Y, M_Y}) \leq I(M_{k_0}, k_0\xi) \leq I(M_{k_0}, U'(M_{k_0})) + (U')^{-1}(k_0\xi) \in \mathcal{L}^2$  for  $Y \geq k_0$ , where  $k_0$  is given in Assumption 2.3.4, we have

$$\begin{aligned} &\limsup_{Y \rightarrow \infty} \mathbb{E} [\xi I(M_Y, Y\xi + \gamma R_{Y, M_Y})] \\ &\leq \mathbb{E} \left[ \limsup_{Y \rightarrow \infty} \xi I(M_Y, Y\xi + \gamma R_{Y, M_Y}) \right] = 0. \end{aligned} \quad (2.3.41)$$

□

*Proof of Proposition 2.3.6.* According to Lemmas 2.3.7, 2.3.8 and 2.3.9, the triple  $(Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$  solves the system of nonlinear equations in (2.3.29)-(2.3.31).

□

Next, we shall verify that  $\hat{X} = I(M, \gamma R + Y\xi)$ , where  $I$  is given in Proposition 2.3.5 and the numbers  $Y, M$  and  $R$  are warranted in Proposition 2.3.6, belongs to  $\mathcal{X}$  and satisfies Condition 2.3.1. Then, the optimal terminal wealth can be found by Theorem 2.3.2:

**Theorem 2.3.10.**  $\hat{X} = I(M, \gamma R + Y\xi)$  is an optimal terminal wealth to the utility-downside-risk problem, where  $I$  is given in Proposition 2.3.5 and the numbers  $Y, M$  and  $R$  are warranted in Proposition 2.3.6.

*Proof.* According to Proposition 2.3.5,  $I(m, y)$  is finite on  $\mathbb{R} \times (0, \infty)$  and is strictly decreasing in  $y$  for a fixed  $m$ . We have  $0 \leq I(M, Y\xi + \gamma R) \leq I(M, \gamma R) < \infty$ , thus  $U(I(M, Y\xi + \gamma R))$  and  $D(M - I(M, Y\xi + \gamma R))$  are both uniformly bounded. Hence,  $X = I(M, Y\xi + \gamma R) \in \mathcal{X}$ . Since  $D'$  is increasing,  $D'(M - I(M, \gamma R)) \leq D'(M - I(M, Y\xi + \gamma R)) \leq D'(M)$ , and hence  $D'(M - I(M, Y\xi + \gamma R))$  is uniformly bounded and in  $\mathcal{L}^2$ . Furthermore, by (2.3.27),  $U'(I(M, Y\xi + \gamma R)) = Y\xi + \gamma R - \gamma D'(M - I(M, Y\xi + \gamma R))$ , which is in  $\mathcal{L}^2$ , hence  $\hat{X} = I(M, Y\xi + \gamma R)$  satisfies Condition 2.3.1.

With  $(Y, M, R)$  as warranted in Proposition 2.3.6,  $(I(M, Y\xi + \gamma R), Y, M, R)$  solves the Nonlinear Moment Problem (2.2.4)-(2.2.7). Then, by Theorem 2.3.2,  $\hat{X} = I(M, Y\xi + \gamma R)$  is an optimal solution to Problem 2.1.4 with downside risk function  $D$ . Finally, by Theorem 2.1.5,  $\hat{X} = I(M, Y\xi + \gamma R)$  is an optimal terminal wealth of utility-downside-risk problem.  $\square$

In the proof of Theorem 2.3.10, it seems not immediate that the optimal terminal wealth is uniformly bounded even if the risk measure is a downside one. By Theorems 2.1.5 and 2.2.2, there exist numbers  $Y, M$  and  $R$  so that any optimal terminal wealth  $\hat{X}$  (satisfying Conditions 2.2.1 (i) and (ii)) satisfies:

$$\begin{cases} Y\xi = f_{M,R}(\hat{X}), & \text{a.s. on } \{\hat{X} > 0\}, \\ Y\xi \leq f_{M,R}(\hat{X}), & \text{a.s. on } \{\hat{X} = 0\}, \end{cases} \quad (2.3.42)$$

where  $f_{M,R}(x) := U'(x) - \gamma R + \gamma D'(M - x)$ . By taking expectation on both sides of (2.3.42) with some terms being eliminated in accordance with (2.2.7), we have  $Y \geq \mathbb{E}[U'(\hat{X})]/\mathbb{E}[\xi] > 0$ . By the definition of  $I$  in Proposition 2.3.5 and the fact that  $f_{M,R}$  is decreasing,  $f_{M,R}(x) \leq 0$  whenever  $x \geq I(M, \gamma R) > 0$ , together with the facts that  $Y\xi > 0$  and  $\hat{X}$  has to satisfy (2.3.42) a.s., there is

no possibility that  $\hat{X}$  takes value greater than  $I(M, \gamma R)$ . In other words,  $\hat{X}$  has to be bounded above by the finite deterministic number  $I(M, \gamma R)$ . Note that the optimal terminal wealth is bounded when the risk function is strictly convex. In particular, the optimal terminal payoff in our utility-risk problem in Theorem 2.3.10 is counter-monotonic with the pricing kernel, which is a commonly found property in the portfolio selection literature.

To motivate the claim of the boundedness of the optimal payoff from a financial perspective, we consider a simple single period example. Based on the previous observation, it is justifiable to simply take the optimal terminal payoff under this example to be also counter-monotonic with the pricing kernel.

We suppose that the payoff is a random variable  $Z$  with two possible outcomes, 0 and a number  $z > 1$ , and their respective probabilities are  $p_0 := 1 - p_z$  and  $p_z$ . For simplicity, we assume the riskless rate is zero. Our objective function is

$$J(Z) := \mathbb{E}[U(Z)] - \mathbb{E}[D(\mathbb{E}[Z] - Z)] := \mathbb{E}[Z^\theta] - \mathbb{E}[(\mathbb{E}[Z] - Z)_+^\rho], \quad (2.3.43)$$

where  $\theta < 1$  and  $\rho > 1$ . There is a budget constraint on the payoff, namely

$$\mathbb{E}[\xi Z] = q_z z = 1, \quad (2.3.44)$$

where  $q_z$  is the risk neutral probability of  $Z = z$ . We look for the optimal  $z$  so that the corresponding payoff  $Z$  maximizes (2.3.43). Since we assume that the payoff  $Z$  and the pricing kernel  $\xi$  are counter-monotonic, there exist a  $\xi_0(z) \in (0, \infty)$  such that  $\{Z = z\} = \{\xi < \xi_0(z)\}$ , thus  $q_z = \int_0^{\xi_0(z)} \xi \mathbb{P}[d\xi]$ . Then, in order to maintain the budget constraint at the same level, increasing  $z$  has to be balanced off with a smaller risk neutral probability  $q_z$ , thus  $\xi_0(z)$  decreases in  $z$ . Therefore,  $p_z = \int_0^{\xi_0(z)} \mathbb{P}[d\xi]$  decreases in  $z$ . Define  $h(z) := \frac{q_z}{p_z} = \frac{\int_0^{\xi_0(z)} \xi \mathbb{P}[d\xi]}{\int_0^{\xi_0(z)} \mathbb{P}[d\xi]}$ , simple calculus concludes that  $h'(z) < 0$ , and therefore  $h(z)$  decreases in  $z$ . By (2.3.44),  $p_z = \frac{h(z)^{-1}}{z}$  and  $\mathbb{E}[Z] = h(z)^{-1}$ . Then, we have

- (i)  $\mathbb{E}[U(Z)] = \mathbb{E}[Z^\theta] = \frac{h(z)^{-1}}{z^{1-\theta}}$  and
- (ii)  $\mathbb{E}[D(\mathbb{E}[Z] - Z)] = \mathbb{E}[(\mathbb{E}[Z] - Z)_+^\rho] = (\mathbb{E}[Z])^\rho (1 - p_z) \geq \delta_0 (h(z)^{-1})^\rho,$

for some  $\delta_0 > 0$ . Next, we consider two cases: (i)  $\lim_{z \rightarrow \infty} h(z)^{-1} < \infty$  or (ii)  $\lim_{z \rightarrow \infty} h(z)^{-1} = \infty$ .

(i) We have  $\lim_{z \rightarrow \infty} \mathbb{E}[U(Z)] = \lim_{z \rightarrow \infty} \frac{h(z)^{-1}}{z^{1-\theta}} = 0$ , i.e. a bounded  $Z$  is optimal even in the ordinary utility maximization.

(ii) We have

$$\lim_{z \rightarrow \infty} \frac{\mathbb{E}[D(\mathbb{E}[Z] - Z)]}{\mathbb{E}[U(Z)]} \geq \lim_{z \rightarrow \infty} \frac{\delta_0 (h(z)^{-1})^\rho}{\frac{h(z)^{-1}}{z^{1-\theta}}} \geq \lim_{z \rightarrow \infty} \delta_0 (h(z)^{-1})^{\rho-1} = \infty,$$

which means that  $\mathbb{E}[D(\mathbb{E}[Z] - Z)]$  grows with  $z$  faster than  $\mathbb{E}[U(Z)]$  due to the diminishing marginal value of utility  $U$  and the increasing marginal value of deviation risk  $D(\mathbb{E}[Z] - Z)$ . As a result, taking an arbitrary large value in  $z$  actually causes a negative effect on the objective value. It is noted that even the downside risk does not penalize on the upside payoff, the deviation risk of  $Z$  still increases in  $z$  because  $\mathbb{E}[Z]$  increases in  $z$ .

### 2.3.3 Application to the Utility-Strictly-Convex-Risk Problem

In this subsection, we take  $\mathcal{D} = [0, \infty)$ . We assume that  $U : [0, \infty) \rightarrow [0, \infty)$  is strictly concave and continuously differentiable, while  $D : \mathbb{R} \rightarrow [0, \infty)$  is strictly convex and continuously differentiable. Moreover, we assume that  $U$  and  $D$  satisfy (2.3.26) and  $D'(-\infty) = -\infty$ . Thus any utility functions satisfying the Inada conditions can be covered.

We can establish the existence of the solution of the nonlinear moment problem in (2.2.4) by using the same approach as in Subsection 2.3.2. Since most derivations are similar, we only indicate here the major differences from the last subsection.

**Proposition 2.3.11.**

There exists an implicit function  $I : \mathbb{R}^2 \rightarrow (0, \infty)$  satisfying:

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R}^2. \quad (2.3.45)$$

Moreover, this function  $I$  possesses the following regularities:

- (a) (i) For each  $y \in \mathbb{R}$ ,  $I(m, y)$  is strictly increasing in  $m$ .
- (ii) For each  $m \in \mathbb{R}$ ,  $I(m, y)$  is strictly decreasing in  $y$ .
- (b)  $I(m, y)$  is jointly continuous in  $(m, y) \in \mathbb{R}^2$ .

*Proof.* The proof is essentially the same as the proof of Proposition 2.3.5 except that  $\lim_{Z \rightarrow \infty} U'(Z) + \gamma D'(m - Z) - y = -\infty$  for any  $(m, y) \in \mathbb{R}^2$ .  $\square$

**Proposition 2.3.12.** *There exist constants  $Y, M \in (0, \infty)$  and  $R \in \mathbb{R}$  satisfying a system of nonlinear equations in (2.3.29)-(2.3.31). Thus,  $(I(M, \gamma R + Y\xi), Y, M, R)$  is the solution of system of Equations (2.2.4)-(2.2.7), where  $I$  is given in Proposition 2.3.11.*

*Proof.* The approach is again the same as that of Proposition 2.3.6 with the following major changes:

- (i) Since  $R$  can take values in  $(-\infty, D'(M))$  instead. We have to prove the following in place of that in (c1) in Lemma 2.3.7:

$$\lim_{R \rightarrow -\infty} \mathbb{E}[D'(M - I(M, \gamma R + Y\xi)) - R] > 0.$$

By (2.3.45), we have  $\lim_{R \rightarrow -\infty} I(M, \gamma R + Y\xi) = \infty$  almost surely. Since  $U'$  is continuous, so we have  $\lim_{R \rightarrow -\infty} Y\xi - U'(I(M, \gamma R + Y\xi)) = Y\xi$  almost surely by the Inada condition. Since  $Y\xi - U'(I(M, \gamma R + Y\xi))$  is strictly increasing as  $R \rightarrow -\infty$  and  $Y\xi - U'(I(M, \gamma R + Y\xi)) \geq D'(M - I(M, 0))$

for  $R < 0$  by (2.3.45), then by the Monotone Convergence Theorem,

$$\begin{aligned} & \lim_{R \rightarrow -\infty} \mathbb{E} [D'(M - I(M, \gamma R + Y\xi)) - R] \\ &= \lim_{R \rightarrow -\infty} \mathbb{E} \left[ \frac{1}{\gamma} (Y\xi - U'(I(M, \gamma R + Y\xi))) \right] \\ &= \frac{1}{\gamma} Y \mathbb{E} [\xi] > 0. \end{aligned}$$

- (ii) In part (d1) in Lemma 2.3.7, by Proposition 2.3.11 (a),  $I(M, \gamma R + Y\xi)$  is strictly increasing in  $M$ , so  $D'(M - I(M, \gamma R + Y\xi)) - R$  is strictly increasing in  $M$ .
- (iii) In part (a) in Lemma 2.3.8, by (2.3.27),  $D'(M - I(M, \gamma R + Y\xi)) - R$  is strictly increasing in  $M$  almost surely. Since  $D'$  is strictly increasing,  $I(M, \gamma R + Y\xi) - M$  is strictly decreasing in  $M$  for fixed  $Y, R$ .
- (iv) In part (b2) in Lemma 2.3.9, since  $R$  can be negative, we no longer have  $\lim_{Y \rightarrow \infty} R_{N, M_Y} \geq 0$ . Instead, we claim that for any  $N \in (0, \infty)$ ,  $\lim_{Y \rightarrow \infty} R_{N, M_Y} > -\infty$ . Assume the contrary, that there exists  $N$  such that  $\lim_{Y \rightarrow \infty} R_{N, M_Y} = -\infty$ . Then

$$\lim_{Y \rightarrow \infty} (\xi N + \gamma R_{N, M_Y}) = -\infty \text{ for all } \xi \in (0, \infty). \quad (2.3.46)$$

Fix a sample  $\xi_0 \in (0, \infty)$ . Let  $\{y_n\}$  be a sequence with  $y_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} I(M_{y_n}, N\xi_0 + \gamma R_{N, M_{y_n}}) = \liminf_{Y \rightarrow \infty} I(M_Y, N\xi_0 + \gamma R_{N, M_Y}).$$

If  $\lim_{n \rightarrow \infty} I(M_{y_n}, N\xi_0 + \gamma R_{N, M_{y_n}}) < \infty$ , then

$$\lim_{n \rightarrow \infty} U'(I(M_{y_n}, N\xi_0 + \gamma R_{N, M_{y_n}})) > 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} D'(M_{y_n} - I(M_{y_n}, N\xi_0 + \gamma R_{N, M_{y_n}})) > -\infty.$$

With (2.3.45), they contradict to (2.3.46). So

$$\liminf_{Y \rightarrow \infty} I(M_Y, N\xi_0 + \gamma R_{N, M_Y}) = \lim_{n \rightarrow \infty} I(M_{y_n}, N\xi_0 + \gamma R_{N, M_{y_n}}) = \infty.$$

Thus, we have  $\liminf_{Y \rightarrow \infty} I(M_Y, N\xi + \gamma R_{N, M_Y}) = \infty$  almost surely. Then,

$$\lim_{Y \rightarrow \infty} U'(I(M_Y, N\xi + \gamma R_{N, M_Y})) = U' \left( \lim_{Y \rightarrow \infty} I(M_Y, N\xi + \gamma R_{N, M_Y}) \right) = 0.$$

Since  $U'(I(M_Y, N\xi + \gamma R_{N, M_Y})) \leq N\xi + \gamma R_{N, M_N} - D'(-I(0, \gamma R_{N, M_N}))$  for  $Y > N$ , then by the reverse Fatou lemma, we have

$$\begin{aligned} 0 &= \mathbb{E} \left[ \limsup_{Y \rightarrow \infty} U'(I(M_Y, N\xi + \gamma R_{N, M_Y})) \right] \\ &\geq \limsup_{Y \rightarrow \infty} \mathbb{E} [U'(I(M_Y, N\xi + \gamma R_{N, M_Y}))] \\ &= N\mathbb{E}[\xi] > 0, \end{aligned}$$

which is a contradiction; thus for any  $N \in (0, \infty)$ ,  $\lim_{Y \rightarrow \infty} R_{N, M_Y} > -\infty$ . From here on the rest of the proof is the same as that for part (b2) in Proof for Lemma 2.3.9 until the last line,

$$I(M_Y, Y\xi + \gamma R_{Y, M_Y}) \leq I \left( M_1, \gamma \lim_{Y \rightarrow \infty} R_{1, M_Y} \right) < \infty,$$

which is independent of  $\xi$ . Hence, the reverse Fatou Lemma still works in (2.3.41).

□

Using the same argument as in Section 2.3.2, we can draw the same existence conclusion:

**Theorem 2.3.13.**  $\hat{X} = I(M, \gamma R + Y\xi)$  is an optimal terminal wealth of the utility-strictly-convex-risk problem, where  $I$  is specified in Proposition 2.3.11 and the numbers  $Y, M$  and  $R$  are warranted in Proposition 2.3.12.

Furthermore, if we specify risk function  $D$  to be the square function, i.e.  $D(x) = x^2$ , and hence variance of the terminal payoff is the risk measure concerned, then the solution of the Nonlinear Moment Problem in Theorem 2.3.2 is unique:

**Proposition 2.3.14.** *There exists a unique set of numbers  $Y, M \in (0, \infty)$  and  $R \in \mathbb{R}$  such that a system of nonlinear equations in (2.3.29)-(2.3.31) is satisfied. Thus,  $X = I(M, \gamma R + Y\xi)$ , where  $I$  is a function defined in Proposition 2.3.11, is the unique optimal terminal wealth of the utility-variance problem.*

*Proof.* Now,  $D'(x) = 2x$ , then  $R_{Y, M_Y} = 0$  for all  $Y$  by (2.3.30) and (2.3.31). Since  $M_Y$  is strictly decreasing in  $Y$ , by Proposition 2.3.11 (b),  $I(M_Y, \gamma R_{Y, M_Y} + Y\xi) = I(M_Y, Y\xi)$  is strictly decreasing in  $Y$ . Because  $\xi$  is absolute continuous with no point mass and its support is  $\mathbb{R}$ , hence  $\mathbb{E}[\xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi)]$  is strictly decreasing in  $Y$ . Therefore,  $Y^*$  obtained in Proposition 2.3.12 is unique. Thus, (2.3.29)-(2.3.31) is uniquely solved by  $(Y^*, M_{Y^*}, 0)$ .

By remark 2.3.3,  $(I(M_{Y^*}, Y^*\xi), Y^*, M_{Y^*}, 0)$  solve the Nonlinear Moment Problem. By Theorems 2.1.5 and 2.3.2, the second assertion follows.  $\square$

### 2.3.4 Application to the Mean-Risk Problem

In this subsection, we assume the utility function to be linear, i.e.  $U(x) = x$ , and we set  $\mathcal{D} = \mathbb{R}$ . Our Problem 2.1.2 reduces to a mean-risk optimization problem:

$$\max_{\pi \in \mathcal{A}} \mathbb{E}[X^\pi(T)] - \gamma \mathbb{E}[D(\mathbb{E}[X^\pi(T)] - X^\pi(T))]. \quad (2.3.47)$$

As the Inada conditions in (2.3.26) do not hold in this case, the method developed in the previous subsection cannot be directly translated here. Suppose that there is an inverse function for the first-order derivative of risk function,  $I_2 := (D')^{-1}$ . The Nonlinear Moment Problem (2.2.4) corresponding to (2.3.47) can be simplified as follows:

$$Y\xi = 1 - \gamma R + \gamma D'(M - \hat{X}), \quad (2.3.48)$$



where the numbers  $Y, M, R \in \mathbb{R}$  satisfy

$$\mathbb{E}[\xi \hat{X}] = x_0, \quad (2.3.49)$$

$$\mathbb{E}[\hat{X}] = M, \quad (2.3.50)$$

$$\mathbb{E} \left[ D' \left( M - \hat{X} \right) \right] = R. \quad (2.3.51)$$

In accordance with Theorem 2.3.2, we are going to show that the reduced Nonlinear Moment Problem (2.3.48) admits a solution, so that the corresponding  $\hat{X}$  will be an optimal terminal wealth for the mean-risk problem (2.3.47).

**Theorem 2.3.15.** *If there exists a unique  $R \in \mathbb{R}$  so that:*

$$I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \in \mathcal{L}^2 \quad \text{and} \quad (2.3.52)$$

$$\mathbb{E} \left[ I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right] = 0, \quad (2.3.53)$$

then by setting

$$\hat{X} := M - I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right), \quad (2.3.54)$$

$$M := \frac{x_0}{\mathbb{E}[\xi]} + \frac{\mathbb{E} \left[ \xi I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) \right]}{\mathbb{E}[\xi]}, \quad (2.3.55)$$

$$Y := \frac{1}{\mathbb{E}[\xi]}, \quad (2.3.56)$$

together with  $R$ , they will solve the reduced Nonlinear Moment Problem (2.3.48)–(2.3.51).

*Proof.* Condition (2.3.52) guarantees that  $X$  defined in (2.3.54) is in  $\mathcal{L}^2$  and  $\mathbb{E} \left[ \xi I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) \right]$  is finite, by the Cauchy-Schwarz inequality. It is clear that Condition 2.3.1 is satisfied. Since  $D$  is convex,

$$D \left( I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) \right) \leq D(0) + \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right),$$

hence  $I_2 \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) \in \mathcal{X}$ . (2.3.48)–(2.3.51) can be verified easily by direct substitution of (2.3.54)–(2.3.56).  $\square$

In Theorem 2.3.15, we solve the mean-risk optimization problem for any risk function satisfying (2.3.52) and (2.3.53). (2.3.53) can be obviously satisfied when  $I_2$  is both continuous and coercive. Note that the uniqueness of  $R$  can be warranted by the strict convexity of  $D$  and (2.3.52) can be satisfied when  $I_2$  is of polynomial growth.

**Remark 2.3.16.** In Jin et al. (2005), they studied the same mean-risk optimization problem by using the Lagrangian approach, and they also formulated the problem as follows:

$$\min D(\mathbb{E}[X(T)] - X(T)), \text{ subject to } \mathbb{E}[X(T)] = z.$$

This problem is equivalent to (2.3.47) for appropriate relationship between  $\gamma$  and  $z$ . The work Jin et al. (2005) shows that if the mean-risk problem has a solution, the optimal terminal wealth  $X = z - I_2(\mu\xi - \lambda)$ , where  $\lambda$  and  $\mu$  satisfy the equations

$$\begin{aligned} \mathbb{E}[I_2(\mu\xi - \lambda)] &= 0, \\ \mathbb{E}[\xi I_2(\mu\xi - \lambda)] &= z\mathbb{E}[\xi] - x_0. \end{aligned}$$

For any  $z$  such that there exists  $\gamma > 0$  satisfying

$$z = \frac{x_0}{\mathbb{E}[\xi]} + \frac{\mathbb{E} \left[ \xi I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right]}{\mathbb{E}[\xi]},$$

if we set

$$\mu = \frac{1}{\gamma \mathbb{E}[\xi]}, \lambda = \frac{1}{\gamma} - R,$$

where  $R$  is as obtained in (2.3.53), the solution in Jin et al. (2005) can then be recovered.

**Example 2.3.17** (Mean-Variance Case). We further set  $D(x) = \frac{1}{2}x^2$ , the general utility risk problem (2.1.2) boils down to the classical mean-variance problem. Clearly, we have  $D'(x) = I_2(x) = x$ . By setting  $\rho = 1$  in (2.3.62), we have  $R = 0$ ,

and also in light of (2.3.55),  $M = \frac{x_0}{\mathbb{E}[\xi]} + \frac{1}{\gamma} \left( \frac{\mathbb{E}[\xi^2]}{(\mathbb{E}[\xi])^2} - 1 \right)$ , one can derive the optimal terminal wealth from (2.3.61),

$$X = \frac{x_0}{\mathbb{E}[\xi]} + \frac{1}{\gamma} \left( \frac{\mathbb{E}[\xi^2]}{(\mathbb{E}[\xi])^2} - \frac{\xi}{\mathbb{E}[\xi]} \right). \quad (2.3.57)$$

Then the optimal trading strategy can be obtained with an application of Martingale Representation Theorem since the market is complete. Now, we aim to show that if all market coefficients are deterministic, we can obtain the explicit form of optimal control. With deterministic market parameters, we have:

$$\begin{aligned} \mathbb{E}[\xi] &= \exp \left[ - \int_0^T r(s) ds \right] \\ \mathbb{E}[\xi^2] &= \exp \left[ \int_0^T (-2r(s) + \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)) ds \right]. \end{aligned}$$

Using (2.3.57), we have

$$X = C - \frac{1}{\gamma} \exp \left[ \int_0^T r(s) ds \right] \xi,$$

where  $C := \exp \left[ \int_0^T r(s) ds \right] x_0 + \frac{1}{\gamma} \exp \left[ \int_0^T \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s) ds \right]$ . Since  $\{\hat{X}(t)\xi(t)\}$  is a martingale, and we have

$$\begin{aligned} \hat{X}(t)\xi(t) &= \mathbb{E} \left[ \hat{X}(T)\xi(T) | \mathcal{F}_t \right] \\ &= C \mathbb{E}[\xi | \mathcal{F}_t] - \frac{1}{\gamma} e^{\int_0^T r(s) ds} \mathbb{E}[\xi^2 | \mathcal{F}_t] \\ &= C \xi(t) e^{-\int_t^T r(s) ds} - \frac{1}{\gamma} e^{\int_0^T r(s) ds} (\xi(t))^2 e^{\int_t^T (-2r(s) + \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)) ds}. \end{aligned} \quad (2.3.58)$$

Thus, through an application of Itô's formula to (2.3.58), we have

$$\begin{aligned} &d \left( \hat{X}(t)\xi(t) \right) \\ &= -C \xi(t) e^{\int_t^T -r(s) ds} \alpha(t)^t (\sigma(t)^t)^{-1} dW(t) \\ &\quad + \frac{1}{\gamma} e^{\int_0^T r(s) ds} (\xi(t))^2 e^{\int_t^T (-2r(s) + \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)) ds} \alpha(t)^t (\sigma(t)^t)^{-1} dW(t) \\ &= C \xi(t) e^{\int_t^T -r(s) ds} \alpha(t)^t (\sigma(t)^t)^{-1} - 2 \hat{X}(t)\xi(t) \alpha(t)^t (\sigma(t)^t)^{-1} dW(t). \end{aligned} \quad (2.3.59)$$

On the other hand, by applying Itô's formula to  $\hat{X}(t)\xi(t)$  directly, we have

$$d\left(\hat{X}(t)\xi(t)\right) = \xi(t) \left(-\alpha(t)^t (\sigma(t)^t)^{-1} \hat{X}(t) + \pi(t)^t \sigma(t)\right) dW(t), \quad (2.3.60)$$

By comparing the coefficients of (2.3.59) and (2.3.60) and knowing  $\xi(t) > 0$ , we can obtain the optimal control which coincides the result as obtained in Li and Zhou (2000) and Bensoussan et al. (2014):

$$\hat{\pi}(t) = (\sigma(t)\sigma(t)^t)^{-1} \alpha(t) \left(-\hat{X}(t) + Ke^{\int_t^T -r(s)ds}\right).$$

**Example 2.3.18** (Mean-Weighted-Power-Risk Function case). Consider

$$D(x) = \frac{a x_+^{\rho+1}}{2\rho+1} - \frac{b x_-^{\rho+1}}{2\rho+1}$$

for  $\rho > 0$  and  $a \geq b > 0$ .  $a \geq b$  means that the risk incurred when the return is less than the expectation will be greater than that when the return is greater than the expectation. Now,  $D'(x) = ax_+^\rho - bx_-^\rho$ , and  $I_2(x) = \frac{1}{a}x_+^{\frac{1}{\rho}} - \frac{1}{b}x_-^{\frac{1}{\rho}}$ . To verify (2.3.52), we consider two cases: (i)  $\rho \leq 2$  and (ii)  $\rho > 2$  respectively.

(i) If  $\rho \leq 2$ , by Minkowski's inequality, for any  $R \in \mathbb{R}$ ,

$$\begin{aligned} \mathbb{E} \left[ \left( I_2 \left( \frac{1}{\gamma \mathbb{E}[\xi]} \xi + R - \frac{1}{\gamma} \right) \right)^2 \right] &\leq \mathbb{E} \left[ \frac{1}{b^2} \left| \frac{1}{\gamma \mathbb{E}[\xi]} \xi + R - \frac{1}{\gamma} \right|^{\frac{2}{\rho}} \right] \\ &\leq \frac{1}{b^2} \left( \mathbb{E} \left[ \left| \frac{\xi}{\gamma \mathbb{E}[\xi]} \right|^{\frac{2}{\rho}} \right]^{\frac{\rho}{2}} + \left| R - \frac{1}{\gamma} \right| \right)^{\frac{2}{\rho}}. \end{aligned}$$

We next show that  $\xi(t)^k$  is bounded for all  $k \in \mathbb{R}$  and  $t \in [0, T]$ :

**Lemma 2.3.19.** *For any  $k \in \mathbb{R}$ ,  $t \in [0, T]$ ,  $\xi(t)^k$  is integrable.*

*Proof.*

$$\begin{aligned} &\mathbb{E} [\xi(t)^k] \\ &= \mathbb{E} \left[ e^{-\int_0^t (kr(s)ds + \frac{k}{2}\alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds + k\alpha(s)^t (\sigma(s)^t)^{-1} dW(s))} \right] \\ &= \mathbb{E} \left[ e^{-\int_0^t (kr(s)ds + \frac{k-k^2}{2}\alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds} \right. \\ &\quad \left. \times e^{-\int_0^t (\frac{k^2}{2}\alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds + k\alpha(s)^t (\sigma(s)^t)^{-1} dW(s)} \right]. \end{aligned}$$

Due to the uniform boundedness of the underlying market parameters, there are some  $r$  and  $K$  such that  $|r(s)| \leq r$  and  $|\alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)| \leq K$  for any  $s \in [0, T]$ , we thus have

$$e^{-\int_0^t (kr(s)ds + \frac{k-k^2}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds)} \leq e^{T(|k|r + |k-k^2|K)}$$

Because  $\left\{ e^{-\int_0^t (\frac{k^2}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds + k\alpha(s)^t (\sigma(s)^t)^{-1} dW(s)} \right\}$  is a martingale, we can show that for any  $k \in \mathbb{R}$ ,  $t \in [0, T]$ ,  $\mathbb{E} [\xi(t)^k]$  is bounded; indeed,

$$\begin{aligned} & \mathbb{E} [\xi(t)^k] \\ = & \mathbb{E} \left[ e^{-\int_0^t (kr(s)ds + \frac{k-k^2}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds)} \right. \\ & \left. \times e^{-\int_0^t (\frac{k^2}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds + k\alpha(s)^t (\sigma(s)^t)^{-1} dW(s)} \right] \\ \leq & e^{T(|k|r + |k-k^2|K)} \mathbb{E} \left[ e^{-\int_0^t (\frac{k^2}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s)ds + k\alpha(s)^t (\sigma(s)^t)^{-1} dW(s)} \right] \\ = & e^{T(|k|r + |k-k^2|K)}. \end{aligned}$$

□

Since  $\mathbb{E} [\xi^k]$  is bounded for any  $k \in \mathbb{R}$ ,  $I_2 \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \in \mathcal{L}^2$ , i.e. (2.3.52) is satisfied.

(ii) If  $\rho > 2$ , for any  $R \in \mathbb{R}$ ,

$$\begin{aligned} & \mathbb{E} \left[ \left( I_2 \left( \frac{1}{\gamma \mathbb{E}[\xi]} \xi + R - \frac{1}{\gamma} \right) \right)^2 \right] \\ \leq & \frac{1}{b^2} \left( \frac{2^{\frac{1}{\rho}}}{\gamma^{\frac{2}{\rho}} (\mathbb{E}[\xi])^{\frac{2}{\rho}}} \mathbb{E} [\xi^{\frac{2}{\rho}}] + \frac{2}{p} \left( \frac{2}{\gamma^2 (\mathbb{E}[\xi])^2} \right)^{\frac{1}{\rho}-1} \left( R - \frac{1}{\gamma} \right)^2 \mathbb{E} [\xi^{2(\frac{1}{\rho}-1)}] \right), \end{aligned}$$

by concavity of  $x^{\frac{1}{\rho}}$ . By the fact that  $\mathbb{E} [\xi^k]$  is bounded for any  $k \in \mathbb{R}$ , (2.3.52) is satisfied.

Note that the expression

$$I_2 \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) = \frac{1}{a} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_+^{\frac{1}{\rho}} - \frac{1}{b} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_-^{\frac{1}{\rho}}$$

is increasing in  $R$  and  $\mathcal{L}^1$ -integrable by Jensen's inequality, for all  $R \in \mathbb{R}$ . Thus,  $\mathbb{E} \left[ I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right]$  is continuous in  $R$  by the Dominated Convergence Theorem. It is not difficult to use the Monotone Convergence Theorem to show that  $\mathbb{E} \left[ I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right]$  is coercive in the sense that

$$\lim_{R \rightarrow -\infty} \mathbb{E} \left[ I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right] = -\infty, \quad \lim_{R \rightarrow \infty} \mathbb{E} \left[ I_2 \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right) \right] = \infty.$$

By the intermediate value theorem, there exists a unique  $R \in \mathbb{R}$  so that (2.3.53) and (2.3.52) are satisfied. The solution of the mean-weighted-power-risk problem is:

$$\begin{aligned} \hat{X} &= \frac{1}{\mathbb{E}[\xi]} \left( x_0 + \mathbb{E} \left[ \frac{\xi}{a} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_+^{\frac{1}{\rho}} - \frac{\xi}{b} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_-^{\frac{1}{\rho}} \right] \right) \\ &\quad - \frac{1}{a} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_+^{\frac{1}{\rho}} + \frac{1}{b} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_-^{\frac{1}{\rho}}, \end{aligned} \quad (2.3.61)$$

where  $R$  is the unique root of the equation

$$\mathbb{E} \left[ \frac{1}{a} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_+^{\frac{1}{\rho}} - \frac{1}{b} \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right)_-^{\frac{1}{\rho}} \right] = 0. \quad (2.3.62)$$

**Remark 2.3.20.** If  $\rho = 1$ , this mean-weighted-power-risk model becomes the mean-weighted-variance one, studied in Jin et al. (2005). The results in Jin et al. (2005) can be recovered by choosing  $\mu = \frac{1}{\gamma \mathbb{E}[\xi]}$ ,  $\lambda = \frac{1}{\gamma} - R$  where  $\frac{1}{\gamma}$  is selected such that

$$z = \frac{1}{\mathbb{E}[\xi]} \left( x_0 + \mathbb{E} \left[ \frac{\xi}{a} \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right)_+ - \frac{\xi}{b} \left( R + \frac{\xi}{\gamma \mathbb{E}[\xi]} - \frac{1}{\gamma} \right)_- \right] \right).$$

If  $a = b = 1$ , this mean-weighted-variance model further becomes the classical mean-variance setting. We can easily get that  $R = 0$  from (2.3.62). Then we can recover the following solution:

$$\hat{X} = \frac{x_0}{\mathbb{E}[\xi]} + \frac{1}{\gamma} \left( \frac{\mathbb{E}[\xi^2]}{(\mathbb{E}[\xi])^2} - \frac{\xi}{\mathbb{E}[\xi]} \right).$$

This result can coincide with the solution on P.226–227 in Bielecki et al. (2005) by choosing

$$\mu = \frac{1}{\gamma \mathbb{E}[\xi]}, \lambda = \frac{x_0}{\mathbb{E}[\xi]} + \frac{\mathbb{E}[\xi^2]}{\gamma (\mathbb{E}[\xi])^2}, \text{ where } \frac{1}{\gamma} = \frac{\mathbb{E}[\xi] (z \mathbb{E}[\xi] - x_0)}{\text{Var}[\xi]}.$$

**Example 2.3.21** (Mean-Exponential-Risk Function Case). We further revisit another example found in Jin et al. (2005). Consider the exponential risk function  $D(x) = e^x$ . Then  $D'(x) = e^x$  and  $I_2(x) = \ln x$  for  $x > 0$ .

**Proposition 2.3.22.** *Mean-Exponential-Risk Problem possesses an optimal solution if and only if  $\gamma \geq \exp\left(\mathbb{E}\left[\ln\left(\frac{\xi}{\mathbb{E}[\xi]}\right)\right]\right)^4$ . Furthermore, if the problem possesses an optimal solution, the optimal terminal wealth is*

$$\hat{X} = \frac{1}{\mathbb{E}[\xi]} \left( x_0 + \mathbb{E} \left[ \xi \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right] \right) - \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right), \quad (2.3.63)$$

where  $R \in [\frac{1}{\gamma}, \infty)$  is the unique root of the equation:

$$\mathbb{E} \left[ \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right] = 0.$$

*Proof.* Firstly, we know that given  $x, y > 0$ , if  $0 < x + y \leq 1$ , then  $|\ln(x + y)| \leq |\ln x|$ ; on the other hand, if  $x + y > 1$ , then  $0 < \ln(x + y) < \ln x + \frac{y}{x}$ . Combining,  $|\ln(x + y)| \leq |\ln(x)| + |\frac{y}{x}|$ . For fixed  $R \in [\frac{1}{\gamma}, \infty)$ ,

$$\begin{aligned} & \mathbb{E} \left[ \left| \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right|^2 \right] \\ & \leq \mathbb{E} \left[ \left( \left| \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} \right) \right| + \left| \mathbb{E}[\xi] \frac{\gamma R - 1}{\xi} \right| \right)^2 \right] \\ & \leq \mathbb{E} \left[ (|\ln \xi| + |\ln(\gamma \mathbb{E}[\xi])| + (\gamma R - 1) \mathbb{E}[\xi] \xi^{-1})^2 \right] \\ & \leq 3 \left( \mathbb{E} [|\ln \xi|^2] + \ln(\gamma \mathbb{E}[\xi])^2 + (\gamma R - 1)^2 \mathbb{E}[\xi]^2 \mathbb{E} [\xi^{-2}] \right). \quad (2.3.64) \end{aligned}$$

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<sup>4</sup> $\mathbb{E} \left[ \ln \left( \frac{\mathbb{E}[\xi]}{\xi} \right) \right]$  is known to be the Kullback-Leibler Divergence (relative entropy) from  $\mathbb{P}$  to  $\mathbb{Q}$ , the risk neutral measure.

Clearly, by a simple calculation,

$$\begin{aligned}
& \mathbb{E} [|\ln \xi|^2] \\
& \leq \mathbb{E} \left[ 2T \int_0^T \left| r(s) + \frac{1}{2} \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s) \right|^2 ds \right. \\
& \quad \left. + 2 \int_0^T \alpha(s)^t (\sigma(s)\sigma(s)^t)^{-1} \alpha(s) ds \right] \\
& \leq 2T^2 \left( r + \frac{C}{2} \right)^2 + 2CT,
\end{aligned}$$

where  $C$  is a constant. With this last result and the boundedness of  $\mathbb{E}[\xi^{-2}]$ , we can show that (2.3.64) is bounded, and hence  $\ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \in \mathcal{L}^2$  for any  $R \in [\frac{1}{\gamma}, \infty)$ .

Now, we consider three different cases: (i)  $\gamma > \exp \left( \mathbb{E} \left[ \ln \left( \frac{\xi}{\mathbb{E}[\xi]} \right) \right] \right)$ , (ii)  $\gamma = \exp \left( \mathbb{E} \left[ \ln \left( \frac{\xi}{\mathbb{E}[\xi]} \right) \right] \right)$ , and (iii)  $\gamma < \exp \left( \mathbb{E} \left[ \ln \left( \frac{\xi}{\mathbb{E}[\xi]} \right) \right] \right)$ .

- (i) Obviously,  $\mathbb{E} \left[ \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right]$  is strictly increasing and continuous in  $R \in \left( \frac{1}{\gamma}, \infty \right)$ . By the Monotone Convergence Theorem,

$$\lim_{R \rightarrow \infty} \mathbb{E} \left[ \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right] = \infty.$$

On the other hand,  $\lim_{R \rightarrow \frac{1}{\gamma}} \mathbb{E} \left[ \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} + R - \frac{1}{\gamma} \right) \right] = \mathbb{E} \left[ \ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} \right) \right] < 0$ . Hence, by intermediate value theorem, there exist an unique  $R$  satisfying (2.3.52) and (2.3.53).

- (ii)  $R = \frac{1}{\gamma}$  is the unique solution satisfying (2.3.52) and (2.3.53).
- (iii) Assume the contrary that there exists an admissible solution  $\hat{X}$  being an optimal terminal wealth for this mean-exponential-risk problem. Since  $D = D'$ , so  $\hat{X}$  satisfies Condition 2.2.1 (i). It is clear that  $\hat{X}$  satisfies Condition 2.2.1 (ii). Hence, by Theorem 2.2.2, it is necessary that there exist numbers  $Y, M, R$  such that  $(\hat{X}, Y, M, R)$  solves the Nonlinear Moment Problem



(2.3.48)-(2.3.51). By (2.3.48) and (2.3.51), we first have

$$\hat{X} = M - \ln \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right), \quad Y = \frac{1}{\mathbb{E}[\xi]}. \quad (2.3.65)$$

Given that  $\gamma < \exp \left( \mathbb{E} \left[ \ln \left( \frac{\xi}{\mathbb{E}[\xi]} \right) \right] \right)$ , we have  $\ln \left( \frac{\xi}{\gamma \mathbb{E}[\xi]} \right) > \ln[\xi] - \mathbb{E}[\ln[\xi]]$ . Taking expectation on both sides of  $\hat{X}$  in (2.3.65), for any  $R \in \left[ \frac{1}{\gamma}, \infty \right)$ , we have

$$\begin{aligned} \mathbb{E} \left[ \hat{X} \right] &= M - \mathbb{E} \left[ \ln \left( R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma} \right) \right] \geq M - \mathbb{E} \left[ \ln \left( \frac{\xi Y}{\gamma} \right) \right] \\ &> M - \mathbb{E} \left[ \ln[\xi] - \mathbb{E}[\ln[\xi]] \right] = M, \end{aligned}$$

which contradicts with (2.3.50). Therefore, there is no solution for the Non-linear Moment Problem, and hence, this mean-exponential risk problem has no optimal solution. □

## 2.4 Conclusion

In this chapter, we studied the utility risk portfolio selection problem. We derived the Nonlinear Moment Problem in (2.2.4)-(2.2.7), whose solution can completely characterize the optimal terminal wealth by the necessity and sufficiency results in Theorems 2.2.2 and 2.3.2 respectively. The nonexistence of optimal solution for the mean-semivariance problem can be revisited by the application of Theorem 2.2.2. Furthermore, we applied Theorem 2.3.2 to establish the existence of optimal solutions for the utility-downside-risk and utility-strictly-convex-risk problems. Their resolutions have long been missing in the literature, and the positive answer in utility-downside-risk problem is in big contrast to the negative answer in mean-downside-risk problem; with our present result, we can now use semivariance as a proper risk measure in portfolio selection. Finally, we established the sufficient condition for the Nonlinear Moment Problem through which the existence of optimal solution of mean-risk problem can be ensured.

In this chapter, the derivation of the Nonlinear Moment Problem with necessity and sufficiency theorem and the existence results of optimal solution to various specific utility-risk problems rely on the market completeness and existence of pricing kernel, thus these results can be extended from the present continuous-time Black-Scholes model to a more general framework. In the next chapter, I shall investigate this extension.

Also, we have shown that some specific utility-risk problems possess an optimal solution, however, how to numerically compute the optimal solution for the present utility-risk problem has not been studied. In the next chapter, I shall develop an numerical method to compute the optimal terminal wealth.

## Chapter 3

# Numerical Valuation of Optimal Utility-risk Portfolio Payoff

In this chapter, we first introduce the problem formulation and some assumptions on market model in Section 3.1 and convert our dynamic utility-risk problem into an equivalent static formulation as stated in Theorem 3.1.9. We will verify that the aforementioned assumptions can be satisfied by various asset price models. In Section 3.2, we revisit the results in Chapter 2 including the theory about Non-linear Moment Problem and the unique existence of optimal solution for utility-downside-risk problems (Theorem 3.2.8) and utility-strictly-convex-risk problems (Theorem 3.2.14) under the generalized setting. In Section 3.3, we show that the sequence of the optimal terminal payoffs in different model settings converges under some conditions on the pricing kernel for the market model. Such conditions are satisfied by discrete binomial tree model in Section 3.3.3 so that we can conclude that the optimal solution for utility-risk problem under continuous-time Black-Scholes model can be approximated by the solutions in discrete binomial tree models. Since there is fundamental difference between downside risk and strictly convex risk, all the aforementioned results are essentially presented in a separate way. In Section 3.4, the convergence results in Section 3.3 are realized to

compute the numerical solution for utility-risk problem under discrete binomial tree model in Example 3.1.11.

### 3.1 Problem Setting

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed complete probability space. Suppose that the market has  $m+1$  assets with the joint price process  $S(t) := (S_0(t), \dots, S_m(t))^t$ ;  $M^t$  denotes the transpose of a matrix  $M$ . Define the information filtration  $\mathcal{F}_t := \sigma(S(s) : s \leq t)$ .

Let  $\pi(t) \triangleq (\pi_1(t), \dots, \pi_m(t))^t$ , where  $\pi_k(t)$  be the money amount invested in the  $k$ -th risky asset of the portfolio at time  $t$ . The dynamics of controlled wealth process is:

$$\begin{aligned} dX^\pi(t) &= (X^\pi(t) - \pi(t)^t \mathbf{1}_m) \frac{dS_0(t)}{S_0(t)} + \sum_{k=1}^m \pi_k(t) \frac{dS_k(t)}{S(t)}, \quad (3.1.1) \\ X^\pi(0) &= x_0 > 0, \end{aligned}$$

where  $\mathbf{1}_m$  is  $m$ -dimension vector.

The objective functional is:

$$J(\pi) := \mathbb{E}[U(X^\pi(T))] - \gamma \mathbb{E}[D(\mathbb{E}[X^\pi(T)] - X^\pi(T))], \quad (3.1.2)$$

where the terminal time  $T$  is finite and  $\gamma > 0$  denotes the risk aversion coefficient.

The utility function  $U$  is defined as follows:

**Definition 3.1.1.** We define a utility function  $U$  such that  $U : (0, \infty) \rightarrow \mathbb{R}$  is strictly increasing, strictly concave and continuously differentiable in the interior. Furthermore, we assume that  $U'(0) = \infty, U'(\infty) = 0$ .

Note that any utility functions satisfying Inada conditions can be covered. For the completeness, we extend the definition of  $U$  over  $\mathbb{R}$  so that  $U(0) := \lim_{x \downarrow 0} U(x)$  and  $U(x) := -\infty$  for  $x < 0$ .

The deviation risk function  $D$  is used to measure the investor's dissatisfaction on the deviation of the random return from its own expectation. We consider

two different types of deviation risk functions: downside risk function and strictly convex risk function. They are to be defined in Definitions 3.1.2 and 3.1.3 respectively:

**Definition 3.1.2.** We define a downside risk function  $D$  such that  $D : \mathbb{R} \rightarrow \mathbb{R}_+$  is positive, strictly convex and continuously differentiable on  $(0, \infty)$  and  $D(x) = 0$  for  $x \leq 0$ . Furthermore, we assume that  $D'(\infty) = \infty$  and  $D'(0) = 0$ .

**Definition 3.1.3.** We define a strictly convex risk function  $D$  such that  $D : \mathbb{R} \rightarrow \mathbb{R}_+$  is non-negative, strictly convex and continuously differentiable. Furthermore, we assume that  $D'(\infty) = \infty$  and  $D'(-\infty) = -\infty$ .

Under Definition 3.1.2, the payoff greater than its mean will not be penalized, and only the downside deviation risk would be taken into account. We also have  $D'(x) > 0$  when  $x > 0$  and  $D'(x) = 0$  when  $x \leq 0$  in this case.

For any given,  $p \geq 1$ ,  $\mathcal{L}^p := \left\{ Z \mid \|Z\|_p := \mathbb{E}[|Z|^p]^{\frac{1}{p}} < \infty \right\}$ . Define  $\mathcal{H}^2$  to be the class of all  $\mathcal{F}_t$ -adapted processes  $\pi$ , equipped with a norm  $\|\pi\|_{\mathcal{H}^2}^2 := \mathbb{E} \left[ \int_0^T \pi(t)^t \pi(t) dt \right] < \infty$ .

**Definition 3.1.4.** We define the class of all admissible controls  $\pi \in \mathcal{A}$  as follows:

$$\mathcal{A} := \left\{ \pi \in \mathcal{H}^2 \mid X^\pi(T) \in \mathcal{X} \right\},$$

here  $\mathcal{X}$  is the class of all admissible terminal wealths, such that

$$\mathcal{X} := \left\{ X \in \mathcal{L}^2 \mid X \in \mathcal{F}_T, X \geq 0 \text{ a.s.}, U(X) \in \mathcal{L}^1, D(\mathbb{E}[X] - X) \in \mathcal{L}^1 \right\}.$$

Note that, for every admissible terminal wealth, both its expected utility and expected deviation risk are well-defined. It is clear that  $\mathcal{X}$  is a convex subspace of  $\mathcal{L}^2$ .

Under the above settings, our utility risk problem can be stated as follows:

**Problem 3.1.5.**

Maximize  $J(\pi)$ ,  
 subject to  $\pi \in \mathcal{A}$  and  $(X^\pi(\cdot), \pi(\cdot))$  satisfies (3.1.1) with initial wealth  $x_0$ .

We have following assumptions on the market:

**Assumption 3.1.6** (Pricing Kernel). Assume there exists a pricing kernel  $\xi(\cdot) \in \mathcal{H}^2$  such that  $\xi := \xi(T) \in \mathcal{L}^2$  and  $\xi(t)X^\pi(t)$  is martingale for all  $\pi \in \mathcal{A}$ . We also assume that  $\xi(0) = 1$  and for each  $t \in [0, T]$ ,  $\xi(t) \in (0, \infty)$  a.s. In addition, we define  $\underline{\xi} := \text{essinf } \xi$  and  $\bar{\xi} := \text{esssup } \xi$ .

**Assumption 3.1.7** (Complete Market). For every  $X \in \mathcal{L}^2$  and measurable by  $\mathcal{F}_T$ , there exists a  $\pi \in \mathcal{H}^2$  such that  $X^\pi(T) = X$ .

Since our market is complete by Assumption 3.1.7, all  $\mathcal{L}^2$ -integrable and  $\mathcal{F}_T$ -measurable terminal wealth can be attained by an admissible control. Our dynamic utility-risk optimization problem 3.1.5 can be converted into the following static optimization problem:

Define  $\Psi : \mathcal{X} \rightarrow \mathbb{R}$  such that  $\Psi(X) := \mathbb{E}[U(X)] - \mathbb{E}[D(\mathbb{E}[X] - X)]$ .

**Problem 3.1.8.**

Maximize  $\Psi(X)$ , (3.1.3)  
 subject to  $X \in \mathcal{X}$  and  $\mathbb{E}[\xi X] = x_0$ ,

Under the complete market assumption in Assumption 3.1.7, same as Theorem 2.1.5, the optimal solution of Problem 3.1.8 is the optimal terminal wealth of Problem 3.1.5:

**Theorem 3.1.9.** *If  $\pi(t)$  is optimal for Problem 3.1.5, then  $X^\pi(T)$  is optimal for Problem 3.1.8. Conversely, if  $X \in \mathcal{X}$  is optimal for Problem 3.1.8, there exists  $\pi \in \mathcal{A}$  such that  $X^\pi(T) = X$  and  $\pi$  is optimal for Problem 3.1.5.*

Note that the maximization in Problem 3.1.8 is confined to the set  $\mathcal{X}$ , so that the solution obtained in Problem 3.1.8 is an admissible terminal wealth in Problem 3.1.5. Since Theorem 3.1.9 provide the equivalence between Problems 3.1.5 and 3.1.8, our present chapter now aims to establish an admissible terminal wealth  $X \in \mathcal{X}$  that maximizes  $\Psi(X)$ .

**Example 3.1.10** (Continuous time Black-Scholes Model). Assume that the market has one bond and one stock. We consider the bond price and stock price satisfy the following SDE:

$$\begin{cases} dB(t) = rB(t)dt, & B(0) = b_0 > 0, \\ dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t), & S(0) = s_0 > 0, \end{cases}$$

where  $W(t)$  is a standard Brownian motion. Let  $\pi(t)$  be the money amount of risky investment, thus the dynamic of wealth process becomes:

$$dX^\pi(t) = (rX^\pi(t) + \alpha\pi(t))dt + \sigma\pi(t)dW(t), \quad X^\pi(0) = x_0 > 0.$$

The continuous time pricing kernel  $\xi(\cdot)$  becomes the following:

$$\xi(t) := \exp \left[ -rt - \frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W_t \right], \quad (3.1.4)$$

Clearly,  $\xi(t)$  satisfies Assumption 3.1.6. Assumption 3.1.7 can be satisfied by the application of Martingale Representation Theorem. Note that all assumptions are satisfied even when the market parameters are  $\mathcal{F}_t$ -adapted processes.

**Example 3.1.11** (Binomial Tree Model). Assume that the market has one bond and one stock, we consider a  $N$ -period discrete time framework with time interval  $\Delta t := \frac{T}{N}$ . The price processes of bond  $B^{(N)}(t)$  and stock  $S_n^{(N)}$  are given by  $B^{(N)}(t) := B_{[t\Delta t]}^{(N)}$  and  $S^{(N)}(t) := S_{[t\Delta t]}^{(N)}$  respectively, where  $B_n^{(N)}$  and  $S_n^{(N)}$  have following dynamics:

$$\begin{aligned} B_{n+1}^{(N)} &= e^{r\Delta t} B_n^{(N)}, & B_0^{(N)} &= b_0 \\ S_{n+1}^{(N)} &= e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z_{n+1}^{(N)}} S_n^{(N)}, & S_0^{(N)} &= b_0 \end{aligned}$$

where  $Z_n^{(N)}$  is an iid Bernoulli random variable measurable by  $\mathcal{F}_{n\Delta t}$  with probability:  $\mathbb{P}\left[Z_n^{(N)} = 1\right] = \mathbb{P}\left[Z_n^{(N)} = -1\right] = \frac{1}{2}$ . Let  $\pi_n^{(N)}$  be the money amount of risky investment in  $n$ -th period, the corresponding wealth process  $X^{(N),\pi}(t) := X_{[t\Delta t]}^{(N),\pi}$  has the following the dynamics:

$$\begin{aligned} X_{n+1}^{(N),\pi} &= e^{r\Delta t} X_n^{(N),\pi} + \left( e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z_{n+1}^{(N)}} - e^{r\Delta t} \right) \pi_n^{(N)}, \\ X_0^{(N),\pi} &= x_0 \end{aligned} \quad (3.1.5)$$

In this binomial tree model, the pricing kernel is given by  $\xi^{(N)}(t) := \xi_{[t\Delta t]}^{(N)}$ , where:

$$\xi_n^{(N)} := \prod_{k=1}^n \frac{2}{e^{r\Delta t}} \left( \frac{e^{-(\alpha - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_k^{(N)}} - 1}{e^{2\sigma\sqrt{\Delta t}Z_k^{(N)}} - 1} \right), \quad (3.1.6)$$

where  $\alpha = \mu - r$ . Hence, assumption 3.1.6 is obviously satisfied.

For Assumption 3.1.7, given  $X \in \mathcal{L}^2$  and measurable by  $\mathcal{F}_T$ , the attaining wealth process can be obtained by

$$X_n^{(N),\pi} = \mathbb{E} \left[ \frac{\xi_N^{(N)}}{\xi_n^{(N)}} X \middle| \xi_n^{(N)} \right] = \mathbb{E} \left[ \prod_{k=n+1}^N \frac{2}{e^{r\Delta t}} \left( \frac{e^{-(\alpha - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_k^{(N)}} - 1}{e^{2\sigma\sqrt{\Delta t}Z_k^{(N)}} - 1} \right) X \right],$$

and the attaining trading strategy can be obtained from (3.1.5).

**Example 3.1.12** (Stochastic Interest Rate Model). We follow the market setting in Bajoux-Besnainou et al. (2003). Assume that the market has one riskless money account (cash), one stock, and one zero coupon bond fund and their prices at  $t$  are denote by  $S_0(t)$ ,  $S_1(t)$ , and  $S_2(t)$  respectively. The asset prices satisfy the following SDE:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, & S_0(t) = s_0 > 0, \\ dS_1(t) = (r(t) + \alpha_1)S_1(t)dt + \sigma_{11}S_1(t)dW_1(t) + \sigma_{12}S_1(t)dW_2(t), \\ S_1(0) = s_1 > 0, \\ dS_2(t) = (r(t) + \alpha_2)S_2(t)dt + \sigma_{21}S_2(t)dW_1(t), & S_2(0) = s_2 > 0, \end{cases}$$



where  $W_1(t)$  and  $W_2(t)$  are independent standard Brownian motions and the interest rate  $r(t)$  follows an Ornstein-Uhlenbeck process with constant parameters given by

$$dr(t) = a_r (b_r - r(t)) dt - \sigma_r dW_1(t), \quad r(0) = r_0$$

Let  $\pi := (\pi_1, \pi_2)$  and  $\pi_1(t)$  and  $\pi_2(t)$  be the money amount of the investment in stock and bond fund respectively, thus the dynamic of wealth process becomes:

$$\begin{aligned} dX^\pi(t) &= (X^\pi(t) - \pi_1(t) - \pi_2(t)) \frac{dS_0(t)}{S_0(t)} + \pi_1(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t) \frac{dS_2(t)}{S_2(t)} \\ &= (r(t)X^\pi(t) + \alpha_1\pi_1(t) + \alpha_2\pi_2(t)) dt \\ &\quad + (\sigma_{11}\pi_1(t) + \sigma_{21}\pi_2(t)) dW_1(t) + \sigma_{12}\pi_1(t)dW_2(t), \\ X^\pi(0) &= x_0 > 0 \end{aligned}$$

The pricing kernel  $\xi(\cdot)$  for the market setting becomes the following:

$$\begin{aligned} \xi(t) := \exp &\left[ - \int_0^t r(s) ds - \frac{1}{2} \left( \frac{(\alpha_1 - \alpha_2 \frac{\sigma_{12}}{\sigma_{21}})^2}{\sigma_{11}^2} + \frac{\alpha_2^2}{\sigma_{21}^2} \right) t \right. \\ &\left. - \frac{\alpha_2}{\sigma_{21}} W_1(t) - \frac{\alpha_1 - \alpha_2 \frac{\sigma_{11}}{\sigma_{21}}}{\sigma_{12}} W_2(t) \right], \end{aligned}$$

By applying Itô's formula to  $\xi_t X^\pi(t)$ , it is martingale, so Assumption 3.1.6 follows.

The Brownian motions  $W_1(t)$  and  $W_2(t)$  can be hedged perfectly through trading the stock and the zero-coupon bond continuously and freely, so the market is complete and Assumption 3.1.7 is satisfied.

## 3.2 Nonlinear Moment Problem (NMP)

In this section, we shall establish the Nonlinear Moment Problem for our generalized market framework where the dynamics asset price processes are unspecified. They can be proven by using the same argument as in Chapter 2, the essential changes and key steps needed will be provided.

### 3.2.1 Necessary and Sufficient Optimality Conditions

First, we shall derive a static problem, called *Nonlinear Moment Problem* to characterize the optimal solution for Problem 3.1.8.

To show the necessity for optimality, we assume that the optimal solution of Problem 3.1.8,  $\hat{X} \in \mathcal{X}$ , satisfies the following two very mild technical conditions:

**Condition 3.2.1.** Both  $U'(Z) \in \mathcal{L}^1$  and  $D'(\mathbb{E}[Z] - Z) \in \mathcal{L}^1$ .

**Condition 3.2.2.** There exists  $\delta > 0$  such that  $D(\mathbb{E}[Z] - Z - \delta) \in \mathcal{L}^1$  and  $D(\mathbb{E}[Z] - Z + \delta) \in \mathcal{L}^1$ .

Now, it is necessary for  $\hat{X}$  to solve the *Nonlinear Moment Problem*:

**Theorem 3.2.3** (Nonlinear Moment Problem). *If  $\hat{X}$  is the optimal solution of Problem 3.1.8 satisfying Conditions 3.2.1 and 3.2.2, then it is necessary that there exist constants  $Y, M, R \in \mathbb{R}$  such that the quadruple  $(\hat{X}, Y, M, R)$  solves the following equality:*

$$Y\xi = U'(\hat{X}) - \gamma R + \gamma D'(M - \hat{X}) \quad a.s., \quad (3.2.7)$$

subject to the nonlinear moment constraints

$$\mathbb{E}[\xi \hat{X}] = x_0, \quad (3.2.8)$$

$$\mathbb{E}[\hat{X}] = M, \quad (3.2.9)$$

$$\mathbb{E}\left[D'(M - \hat{X})\right] = R. \quad (3.2.10)$$

*Proof.* The proof is exactly the same as Theorem 2.2.2. Note that Condition 3.2.1 confines  $\hat{X} > 0$  a.s. □

We denote the static problem (3.2.7)-(3.2.10) as  $\text{NMP}(\xi)$  as its solution depends on the choice of the terminal random variable of the pricing kernel,  $\xi$ . Next, we establish the sufficiency result regarding to the Nonlinear Moment Problem.

In this subsection, we aim to show that any admissible terminal wealth  $\hat{X} \in \mathcal{X}$  solving the Nonlinear Moment Problem satisfying the following condition is optimal terminal wealth of Problem 3.1.5:

**Condition 3.2.4.** Both  $U'(Z) \in \mathcal{L}^2$  and  $D'(\mathbb{E}[Z] - Z) \in \mathcal{L}^2$ .

There is a fundamental difference between the necessary condition in Theorem 3.2.3 and the sufficient condition in the next theorem. Conditions 3.2.1 and 3.2.2 are needed for the optimal terminal wealth satisfying the Nonlinear Moment Problem in the necessity result, while Condition 3.2.4 is required for the sufficiency.

**Theorem 3.2.5.** Suppose that there exists  $\hat{X} \in \mathcal{X}$  satisfying Condition 3.2.4 and there exist constants  $Y, M, R \in \mathbb{R}$  so that the quadruple  $(\hat{X}, Y, M, R)$  solves for the Nonlinear Moment Problem (3.2.7)-(3.2.10). Then,  $\hat{X}$  is the unique optimal solution for Problem 3.1.8, and it is also the unique optimal terminal wealth of Problem 3.1.5.

*Proof.* Let  $(\hat{X}, Y, M, R)$  be the solution of Nonlinear Moment Problem (3.2.7)-(3.2.10) and  $\tilde{X} \in \mathcal{L}^2$  be an arbitrary nontrivial random variable such that  $\hat{X} + \tilde{X}$  is admissible for Problem 3.1.8, i.e.  $\mathbb{P}[\tilde{X} \neq 0] > 0$ ,  $\hat{X} + \tilde{X} \in \mathcal{X}$  and  $\mathbb{E}[\xi(\hat{X} + \tilde{X})] = x_0$ . By (3.2.8), we have  $\mathbb{E}[\xi\tilde{X}] = 0$ .

By the strict concavity of  $U$  and convexity of  $D$ , it is clear that  $\Psi$  is strictly concave, i.e.

$$\Psi(\hat{X} + \theta\tilde{X}) > (1 - \theta)\Psi(\hat{X}) + \theta\Psi(\hat{X} + \tilde{X}) \text{ for any } \theta \in (0, 1). \quad (3.2.11)$$

By the concavity of  $\Psi$ , the chain rule, and under our hypothesis that  $\hat{X}$  satisfies (3.2.7) and Condition 3.2.4, we have

$$\begin{aligned} & \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} \\ &= \mathbb{E} \left[ \lim_{\theta \downarrow 0} \frac{\Psi(\hat{X} + \theta\tilde{X}) - \Psi(\hat{X})}{\theta} \right] \\ &= \mathbb{E} \left[ U'(\hat{X})\tilde{X} \right] - \gamma \mathbb{E} \left[ D'(\mathbb{E}[\hat{X}] - \hat{X}) (\mathbb{E}[\tilde{X}] - \tilde{X}) \right] \\ &= \mathbb{E} \left[ \tilde{X} \left( U'(\hat{X}) - \gamma \mathbb{E} \left[ D'(\mathbb{E}[\hat{X}] - \hat{X}) \right] + \gamma D'(\mathbb{E}[\hat{X}] - \hat{X}) \right) \right] \\ &= \mathbb{E} \left[ \tilde{X}(Y\xi) \right] = 0. \end{aligned} \quad (3.2.12)$$

The first equality follows by Lemma 2.2.4. Since  $\Psi$  is strictly concave, we have

$$\frac{\Psi(\hat{X} + \theta\tilde{X}) - \Psi(\hat{X})}{\theta} \leq \left. \frac{d}{d\theta} \Psi(\hat{X} + \theta\tilde{X}) \right|_{\theta=0} = 0.$$

By (3.2.11), we have

$$\Psi(\hat{X}) > \Psi(\hat{X} + \tilde{X}) - \frac{\Psi(\hat{X} + \theta\tilde{X}) - \Psi(\hat{X})}{\theta} \geq \Psi(\hat{X} + \tilde{X}).$$

Hence  $\hat{X}$  is the unique solution for Problem 3.1.8. By Theorem 3.1.9, we can now conclude that  $\hat{X}$  is the unique optimal terminal wealth of Problem 3.1.5.  $\square$

**Remark 3.2.6.** Theorem 3.2.5 boils the optimal control problem 3.1.5 down to a static problem. Suppose that there exists an implicit function  $I(m, y) \in \mathbb{R}$  satisfying:

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) = y, \quad \text{for any } (m, y). \quad (3.2.13)$$

Then the Nonlinear Moment Problem (3.2.7)-(3.2.10) will be solved by  $(I(M, \gamma R + Y\xi), Y, M, R)$ , where the constants  $Y, M$  and  $R$  satisfy the following system of nonlinear equations:

$$\mathbb{E}[\xi I(M, \gamma R + Y\xi)] = x_0, \quad (3.2.14)$$

$$\mathbb{E}[I(M, \gamma R + Y\xi)] = M, \quad (3.2.15)$$

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] = R. \quad (3.2.16)$$

After we verify that  $I(M, \gamma R + Y\xi)$  belongs to  $\mathcal{X}$  and also satisfies Condition 3.2.4,  $I(M, \gamma R + Y\xi)$  is the unique optimal solution for Problem 3.1.8.

Note that the optimal solution for the utility risk problem in Problem 3.1.8 depends on the choice of the terminal random variable of pricing kernel,  $\xi \in \mathcal{L}^2$ .

### 3.2.2 Existence of the Solution of NMP: Case of Downside Risk

After we obtained the sufficiency result in Theorem 3.2.5, we shall show the existence of the optimal solution of the utility-risk portfolio selection problems.

According to Remark 3.2.6, we first find an implicit function satisfying (3.2.13), then the Nonlinear Moment Problem (3.2.7)-(3.2.10) can be reduced into a nonlinear programming problem (3.2.14)-(3.2.16). The desired implicit function can be warranted by the same argument in Proposition 2.3.5:

**Proposition 3.2.7.** *Let  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.2 respectively. There exists an implicit function  $I : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$  satisfying:*

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R} \times (0, \infty). \quad (3.2.17)$$

Moreover, this function  $I$  possesses the following regularities:

- (a) (i) For each  $m \in \mathbb{R}$ ,  $I(m, y)$  is strictly decreasing in  $y$  on  $(0, \infty)$ .
- (ii) For each  $y \in (0, \infty)$ ,  $I(m, y)$  is strictly increasing in  $m$  on  $\{m \in \mathbb{R} \mid y \geq U'(m)\}$ ;  $I(m, y) = (U')^{-1}(y) \in (0, \infty)$  for all  $m \in \{m \in \mathbb{R} \mid y \leq U'(m)\}$ .
- (b)  $I(m, y)$  is jointly continuous in  $(m, y) \in \mathbb{R} \times (0, \infty)$ .

Then, the overall existence claim will be accomplished if we can solve the nonlinear programming problem (3.2.14)-(3.2.16):

**Theorem 3.2.8.** *Given that Assumption 2.3.4 holds and  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.2 respectively, there exists the unique set of numbers  $Y, M, R \in (0, \infty)$  such that the system of nonlinear equations of (3.2.14)-(3.2.16) is satisfied. Thus,  $(I(M, \gamma R + Y\xi), Y, M, R)$  is the unique solution for the Nonlinear Moment Problem (3.2.7)-(3.2.10),  $NMP(\xi)$ , where  $I$  is given in Proposition 3.2.7. Hence,  $\hat{X} = I(M, \gamma R + Y\xi)$  is the unique optimal terminal wealth of Utility-Downside-Risk problem.*

*Proof.* We will solve for roots  $Y, M$  and  $R$  for (3.2.14)-(3.2.16) one by one via applying the intermediate value theorem successively as in Proposition 2.3.6. We shall indicate the key steps and necessary changes in our proof.

**Lemma 3.2.9.** *Given  $Y \in (0, \infty)$ , we consider two cases: (i)  $M \in ((U')^{-1}(Y\bar{\xi}), \infty)$  and (ii)  $M \in (0, (U')^{-1}(Y\bar{\xi}))$ .*

(i) *there exists a unique  $R = R_{Y,M} \in (0, D'(M))$  satisfying*

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] = R; \quad (3.2.18)$$

*or equivalently by (3.2.17):*

$$\mathbb{E}[U'(I(M, \gamma R + Y\xi))] = Y\mathbb{E}[\xi]. \quad (3.2.19)$$

*Furthermore,  $R_{Y,M}$  is strictly increasing in  $M$  for a fixed  $Y$  and is also strictly increasing in  $Y$  for a fixed  $M$ .*

(ii)  *$R = R_{Y,M} = 0$  uniquely solve (3.2.18).*

*Proof.* The proof of case (i) is the same as Lemma 2.3.7.

For case (ii),  $Y\bar{\xi} \leq U'(M)$ , thus for all  $\xi$ , we have  $I(M, Y\xi) = (U')^{-1}(Y\xi) \geq (U')^{-1}(Y\bar{\xi}) \geq M$ , so  $D'(M - I(M, Y\xi)) = 0$  almost surely. Thus  $R_{Y,M} = 0$  satisfy (3.2.18). Since  $R - D'(M - I(M, \gamma R + Y\xi))$  is strictly increasing, thus  $R = R_{Y,M} = 0$  uniquely solve (3.2.18).  $\square$

**Lemma 3.2.10.** *Given  $Y \in (0, \infty)$  and  $R_{Y,M}$  as specified for each  $M \in (0, \infty)$  in Lemma 3.2.9, there exists a unique  $M = M_Y \in ((U')^{-1}(Y\bar{\xi}), \infty)$  such that*

$$\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] = M. \quad (3.2.20)$$

*Furthermore,  $M_Y$  is strictly decreasing in  $Y$ .*

*Proof.* Denote  $\underline{M} := (U')^{-1}(Y\bar{\xi})$ . We only verify  $\lim_{M \downarrow \underline{M}} (\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M) > 0$ , the rest are the same as the proof of Lemma 2.3.8.

By the continuity of  $D'$  and  $I$ , it is almost surely that:

$$\begin{aligned} & \lim_{M \downarrow \underline{M}} (D'(M - I(M, \gamma R_{Y,M} + Y\xi)) - R_{Y,M}) \\ &= D' \left( \underline{M} - I \left( \underline{M}, \gamma \lim_{M \downarrow \underline{M}} R_{Y,M} + Y\xi \right) \right) - \lim_{M \downarrow \underline{M}} R_{Y,M} \end{aligned} \quad (3.2.21)$$

By (3.2.18), (3.2.21) and the Dominated Convergence Theorem, we have

$$\begin{aligned} & \mathbb{E} \left[ D' \left( \underline{M} - I \left( \underline{M}, \gamma \lim_{M \downarrow \underline{M}} R_{Y,M} + Y\xi \right) \right) \right] - \lim_{M \downarrow \underline{M}} R_{Y,M} \\ &= \lim_{M \downarrow \underline{M}} (\mathbb{E}[D'(M - I(M, \gamma R_{Y,M} + Y\xi))] - R_{Y,M}) = 0. \end{aligned}$$

By the uniqueness result in Lemma 3.2.9, we conclude that  $\lim_{M \downarrow \underline{M}} R_{Y,M} = 0$ . Because  $I$  is continuous, for  $\xi < \bar{\xi}$ , we have  $\lim_{M \downarrow \underline{M}} I(M, \gamma R_{Y,M} + Y\xi) = I(\underline{M}, Y\xi) = (U')^{-1}(Y\xi) > (U')^{-1}(Y\bar{\xi}) = \underline{M}$ . Using the same approach as in part (a) in the proof of Lemma 2.3.8, we have  $I(M, \gamma R_{Y,M} + Y\xi) - M$  is strictly decreasing in  $M$ . Finally, by Monotone Convergence Theorem,

$$\lim_{M \downarrow \underline{M}} (\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] - M) > 0.$$

□

**Lemma 3.2.11.** *Given  $R_{Y,M}$  and  $M_Y$  as specified in Lemmas 3.2.9 and 3.2.10 respectively for each  $Y, M \in (0, \infty)$ , there exists a  $Y^* \in (0, \infty)$  such that*

$$\mathbb{E}[\xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi)] = x_0. \quad (3.2.22)$$

*Proof.* The proof is the same as that of Lemma 2.3.7 except a small change is needed in part (b1):

The arbitrary sample value in the proof,  $\xi_0$ , should only be considered in the range  $(\underline{\xi}, \bar{\xi})$ . In this case,  $\mathbb{P}[\xi > \xi_0] > 0$ . Then we can follow the same arguments to verify that  $\liminf_{Y \rightarrow 0} I(M_Y, Y\xi_0 + \gamma R_{Y, M_Y}) = \infty$ , for any  $\xi_0 \in (\underline{\xi}, \bar{\xi})$ , and hence we conclude that  $\liminf_{Y \rightarrow 0} \xi I(M_Y, Y\xi + \gamma R_{Y, M_Y}) = \infty$  for any  $\xi \in (\underline{\xi}, \bar{\xi})$ . Finally, by Fatou's Lemma,

$$\liminf_{Y \rightarrow 0} \mathbb{E}[\xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi)] \geq \mathbb{E} \left[ \liminf_{Y \rightarrow 0} \xi I(M_Y, \gamma R_{Y, M_Y} + Y\xi) \right] = \infty.$$

□

*Proof of Theorem 3.2.8.* According to Lemmas 3.2.9, 3.2.10 and 3.2.11, the triple  $(Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$  solves the system of nonlinear equations in (3.2.14)-(3.2.16).

Next, we can verify that  $\hat{X} = I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^* \xi)$  belongs to  $\mathcal{X}$  and satisfies Condition 3.2.4 using the same argument as in Theorem 2.3.10. Then, by Theorem 3.2.5,  $\hat{X} = I(M_{Y^*}, Y^* \xi + \gamma R_{Y^*, M_{Y^*}})$  is the unique optimal solution for Problem 3.1.8 with downside risk function  $D$ .

Since  $\hat{X}$  is the unique solution and  $Y = \mathbb{E} \left[ U' \left( \hat{X} \right) \right] / \mathbb{E}[\xi]$  by (3.2.19),  $Y$  obtained in Lemma 3.2.11 is unique. Hence, only a unique set of triple  $(Y, M, R)$  solves (3.2.14)-(3.2.16). Finally, by Theorem 3.1.9,  $\hat{X} = I(M, Y \xi + \gamma R)$  is the unique optimal terminal wealth of utility-downside-risk problem. □

**Remark 3.2.12.** Proposition 3.2.7 and Theorem 3.2.8 can be reduced to Proposition 2.3.5 and Theorem 2.3.10 respectively by setting  $\underline{\xi} = 0$  and  $\bar{\xi} = \infty$ , which are warranted under the Black-Scholes setting.

Our formulation can cover the utility-semivariance problem. The existence result in this section has a substantial contrast to the nonexistence of an optimal solution to the mean-semivariance problem.

### 3.2.3 Existence of the Solution of NMP: Case of Strictly-Convex Risk

We can establish the similar existence result by using the same approach as in the case of downside risk.

**Proposition 3.2.13.** *Let  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.3 respectively. There exists an implicit function  $I : \mathbb{R}^2 \rightarrow (0, \infty)$  satisfying:*

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R}^2. \quad (3.2.23)$$

Moreover, this function  $I$  possesses the following regularities:



- (a) (i) For each  $y \in \mathbb{R}$ ,  $I(m, y)$  is strictly increasing in  $m$ .  
(ii) For each  $m \in \mathbb{R}$ ,  $I(m, y)$  is strictly decreasing in  $y$ .  
(b)  $I(m, y)$  is jointly continuous in  $(m, y) \in \mathbb{R}^2$ .

*Proof.* Same as Proposition 2.3.11. □

Using the same argument as in the case of downside risk, we can draw the same existence conclusion:

**Theorem 3.2.14.** *Given that  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.3 respectively, there exists the unique set of numbers  $Y, M \in (0, \infty)$  and  $R \in \mathbb{R}$  such that the system of nonlinear equations of (3.2.14)-(3.2.16) is satisfied. Thus,  $(I(M, \gamma R + Y\xi), Y, M, R)$  is the unique solution for the Nonlinear Moment Problem (3.2.7)-(3.2.10), where  $I$  is given in Proposition 3.2.13. Hence,  $\hat{X} = I(M, \gamma R + Y\xi)$  is the unique optimal terminal wealth of Utility-Strictly-Convex-Risk problem.*

*Proof.* The proof of existence of  $(Y, M, R)$  solving the system of equations (3.2.14)-(3.2.16) is similar in Proposition 2.3.12. Following the similar arguments, we can prove the following lemmas:

**Lemma 3.2.15.** *Given  $Y, M \in (0, \infty)$ , there exists a unique  $R = R_{Y,M} \in (-\infty, D'(M))$  satisfying*

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi))] = R; \quad (3.2.24)$$

*or equivalently by (3.2.23):*

$$\mathbb{E}[U'(I(M, \gamma R + Y\xi))] = Y\mathbb{E}[\xi]. \quad (3.2.25)$$

*Furthermore,  $R_{Y,M}$  is strictly increasing in  $M$  for a fixed  $Y$  and is also strictly increasing in  $Y$  for a fixed  $M$ .*

**Lemma 3.2.16.** *Given  $Y \in (0, \infty)$  and  $R_{Y,M}$  as specified for each  $M \in (0, \infty)$  in Lemma 3.2.15, there exists a unique  $M = M_Y \in (0, \infty)$  such that*

$$\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi)] = M. \quad (3.2.26)$$

*Furthermore,  $M_Y$  is strictly decreasing in  $Y$ .*

**Lemma 3.2.17.** *Given  $R_{Y,M}$  and  $M_Y$  as specified in Lemmas 3.2.15 and 3.2.16 respectively for each  $Y, M \in (0, \infty)$ , there exists a  $Y^* \in (0, \infty)$  such that*

$$\mathbb{E}[\xi I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi)] = x_0. \quad (3.2.27)$$

Now, we have a set of constants  $Y, M \in (0, \infty)$  and  $R \in \mathbb{R}$  satisfying a system of nonlinear equations in (3.2.14)-(3.2.16). The remaining assertions can be proven using the same arguments in Theorem 3.2.8, and we can conclude that  $\hat{X} = I(M, Y\xi + \gamma R)$  is an optimal terminal wealth of utility-strictly-convex-risk problem.  $\square$

### 3.3 Main Results

In this section, we first shall establish the continuity of optimal terminal wealth for utility risk problem: the sequence of optimal terminal payoffs converges weakly with their corresponding terminal random variables of pricing kernel (or simply call them terminal pricing kernels for convenience). Then, we show that the sequence of terminal pricing kernels for binomial tree models converges weakly to the terminal pricing kernels of the continuous-time Black-Scholes model. Hence, we can apply these results to compute the optimal solution of utility-risk problem under the Black-Scholes model using the solutions under the binomial tree models, which are easily computed.

### 3.3.1 Convergence of Optimal Solution: Case of Downside Risk

We consider a sequence of markets which satisfy Conditions 3.1.6 and 3.1.7, denoted by  $\{\Pi^{(N)}\}$ , thus each market  $\Pi_N$  has its unique pricing kernel  $\xi^{(N)}(\cdot)$ , and we have a sequence of the terminal random variables of pricing kernel  $\{\xi^{(N)}\}$ , where  $\xi^{(N)} := \xi^{(N)}(T)$ . By Theorem 3.2.8, we know that each market possesses the unique optimal terminal wealth for the utility risk problem  $\hat{X}^{(N)}$ , and we have a sequence of optimal terminal payoffs  $\{\hat{X}^{(N)}\}$ . We aim to show that if this sequence of terminal pricing kernels converges weakly to a terminal pricing kernel  $\{\hat{X}^*\}$  corresponding to a market  $\Pi^*$ , then the weak limit of the sequence of optimal terminal payoffs solves the utility-downside-risk problem under the market  $\Pi^*$ .

**Theorem 3.3.1.** *Given that  $U$  and  $D$  satisfy Definitions 3.1.1 and 3.1.2 respectively, Assumption 2.3.4 holds, and all markets  $\Pi^{(N)}$  and  $\Pi^*$  satisfying Conditions 3.1.6 and 3.1.7 with corresponding unique terminal pricing kernels  $\xi_N$  and  $\xi^*$  and optimal terminal payoffs  $\hat{X}^{(N)}$  and  $\hat{X}^*$ , as described in Theorem 3.2.8. Suppose that a sequence of their terminal pricing kernels  $\xi^{(N)}$  satisfy the following conditions:*

(i)  $\xi^{(N)} \in \mathcal{L}^2$  converges weakly to  $\xi^* \in \mathcal{L}^2$  as  $N \rightarrow \infty$ .

(ii) For any  $K > 0$ ,  $\{\xi^{(N)}((U')^{-1}(\xi^{(N)} + K))\}_{N \in \mathbb{N}}$  is uniformly integrable.

Then the sequence  $\{\hat{X}^{(N)}\}$  converges weakly to  $\hat{X}^*$ . Hence,  $\Psi(\hat{X}^{(N)}) \rightarrow \Psi(\hat{X}^*)$ , i.e., the sequence of the optimal value functions of utility-risk problem under  $\Pi^{(N)}$  converges to the optimal value functions under  $\Pi^*$ .

*Proof.* By Theorem 3.2.8, the unique optimal terminal wealth for utility-downside-risk problem under  $\Pi_N$  is given by

$$\hat{X}^{(N)} = I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)}),$$

where the function  $I$  is given in Proposition 3.2.7 and the triple  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfies the system of equations in (3.2.14)-(3.2.16) under  $\xi = \xi^{(N)}$ . Similarly, the unique optimal terminal wealth under  $\Pi^*$  is given by

$$\hat{X}^* = I(M^*, \gamma R^* + Y^* \xi^*).$$

Note that the choice of  $I$  in Proposition 3.2.7 is independent of the pricing kernel  $\xi$ .

By Skorokhod's representation theorem and condition (i), there exists a sequence of random variable  $\zeta^{(N)}$  which has the same distribution as  $\xi^{(N)}$  and  $\zeta^{(N)} \rightarrow \xi^*$  almost surely.

The following proposition shows that the sequence of  $(Y^{(N)}, M^{(N)}, R^{(N)})$  converges to  $(Y^*, M^*, R^*)$ ; its technical proof will be postponed to Section 3.3.1.1.

**Proposition 3.3.2.**  $\lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^*, M^*, R^*)$ .

By Proposition 3.3.2,  $I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}) \rightarrow I(M^*, \gamma R^* + Y^* \xi^*)$  almost surely. Together with the fact that  $I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)})$  and  $I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)})$  have the same distribution,

$$\hat{X}^{(N)} = I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)}) \rightarrow I(M^*, \gamma R^* + Y^* \xi^*) = \hat{X}^* \text{ weakly.}$$

Since  $I(m, y)$  is decreasing in  $y$  for fixed  $m$ , for each  $N$ ,

$$I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}) \leq I(M^{(N)}, \gamma R^{(N)})$$

By Proposition 3.3.2,  $I(M^{(N)}, \gamma R^{(N)}) \rightarrow I(M^*, \gamma R^*) < \infty$ , so we have

$$I(M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}) \leq I(M^*, \gamma R^*) + 1 \text{ a.s., for all large enough } N.$$

By the application of the Dominated Convergence Theorem,

$$\Psi(I(M^{(N)}, Y^{(N)} \zeta^{(N)} + \gamma R^{(N)})) \rightarrow \Psi(\hat{X}^*).$$

Together with the fact that  $I(M^{(N)}, Y^{(N)} \zeta^{(N)} + \gamma R^{(N)})$  and  $\hat{X}^{(N)}$  have the same distribution, the convergence of value functions follows.  $\square$

### 3.3.1.1 Proof of Proposition 3.3.2

Since, for each  $N$ ,  $\zeta^{(N)}$  has the same distribution as  $\xi^{(N)}$ , then the systems of nonlinear equations in (3.2.14)-(3.2.16) with  $\xi = \zeta^{(N)}$  and  $\xi = \xi^{(N)}$  share the same set of solution  $(Y^{(N)}, M^{(N)}, R^{(N)})$ , i.e.  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfy:

$$\mathbb{E} [D' (M^{(N)} - I (M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)}))] = R^{(N)}, \quad (3.3.28)$$

$$\mathbb{E} [I (M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)})] = M^{(N)}, \quad (3.3.29)$$

$$\mathbb{E} [\zeta^{(N)} I (M^{(N)}, \gamma R^{(N)} + Y^{(N)} \zeta^{(N)})] = x_0. \quad (3.3.30)$$

Given the terminal pricing kernel  $\xi = \xi^*$ , we can define  $R_{Y,M}^*$  and  $M_Y^*$  from Lemmas 3.2.9 and 3.2.10. We know that  $(Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$  solves the system of nonlinear equations of (3.2.14)-(3.2.16) with  $\xi = \xi^*$ . Since the solution of the system is unique, we have  $(Y^*, M^*, R^*) = (Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$

Similarly, for the case of  $\xi = \zeta^{(N)}$ , we can define  $R_{Y,M}^{(N)}$  and  $M_Y^{(N)}$  in the same way and we have  $(Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^{(N)}, M_{Y^{(N)}}^{(N)}, R_{Y^{(N)}, M_{Y^{(N)}}^{(N)}}^{(N)})$ .

We first consider the next three lemmas which show  $R^{(N)}, M^{(N)}, Y^{(N)}$  convergent.

**Lemma 3.3.3.** *Given a sequence  $\{(M_N, Y_N)\} \in (0, \infty)^2$ , assume there exists a subsequence  $\{N_k\}$  and  $M_0, Y_0 \in (0, \infty)$  such that  $M_0 = \lim_{k \rightarrow \infty} M_{N_k}$  and  $Y_0 = \lim_{k \rightarrow \infty} Y_{N_k}$ , then  $\lim_{k \rightarrow \infty} R_{Y_{N_k}, M_{N_k}}^{(N_k)} = R_{Y_0, M_0}^*$  where  $R_{Y,M}^*$  and  $R_{Y,M}^{(N)}$  are defined in Lemma 3.2.9 with  $\xi = \xi^*$  and  $\xi = \zeta^{(N)}$  respectively.*

*Proof.* For simplicity of notation, we denote  $R_N := R_{Y_N, M_N}^{(N)}$ . By Bolzano-Weierstrass theorem, there exists a subsequence  $\{N_i\} \subset \{N_k\}$  such that  $R_0 := \lim_{i \rightarrow \infty} R_{N_i} \in [0, D'(M_0)]$  exists.

Since  $I$  and  $D'$  are continuous, thus

$$\lim_{i \rightarrow \infty} D' (M_{N_i} - I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) = D' (M_0 - I (M_0, \gamma R_0 + Y_0 \xi^*)).$$

Then, by the Dominated Convergence Theorem and Lemma 3.2.9 with  $\xi = \zeta^{(N)}$ ,

$$\begin{aligned}
 & \mathbb{E}[D'(M_0 - I(M_0, \gamma R_0 + Y_0 \xi^*))] \\
 &= \lim_{i \rightarrow \infty} \mathbb{E}[D'(M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}))] \\
 &= \lim_{i \rightarrow \infty} R_{N_i} = R_0
 \end{aligned} \tag{3.3.31}$$

Since Lemma 3.2.9 with  $\xi = \xi^*$  is uniquely solved by  $R_{Y_0, M_0}^*$ , we have  $R_0 = R_{Y_0, M_0}^*$ . Since every subsequential limits of  $R_{N_k}$  agree with  $R_{Y_0, M_0}^*$ , we are done.  $\square$

**Lemma 3.3.4.** *Given a sequence  $\{Y_N\} \in (0, \infty)$ , assume there exists a subsequence  $\{N_k\}$  and  $Y_0 \in (0, \infty)$  such that  $Y_0 = \lim_{k \rightarrow \infty} Y_{N_k}$ , then  $\lim_{k \rightarrow \infty} M_{Y_{N_k}}^{(N_k)} = M_{Y_0}^*$  where  $M_Y^*$  and  $M_Y^{(N)}$  are defined in Lemma 3.2.10 with  $\xi = \xi^*$  and  $\xi = \zeta^{(N)}$  respectively.*

*Proof.* For simplicity of notation, we denote  $M_N := M_{Y_N}^{(N)}$  and  $R_N := R_{Y_N, M_N}^{(N)} = R_{Y_N, M_{Y_N}^{(N)}}^{(N)}$ . By Bolzano-Weierstrass theorem, there exists a subsequence  $\{N_i\} \subset \{N_k\}$  such that  $M_0 := \lim_{i \rightarrow \infty} M_{N_i} \in [0, \infty]$  exists.

We first show that  $M_0 \in (0, \infty)$  by using following 4 steps:

Step 1: Prove if  $M_0 = 0$ , then  $\lim_{i \rightarrow \infty} I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) > 0$  almost surely.

Step 2: Prove  $M_0 > 0$ .

Step 3: Prove if  $M_0 = \infty$ , then  $\lim_{i \rightarrow \infty} R_{N_i} = \infty$ .

Step 4: Prove  $M_0 < \infty$ .

Step 1:

Given a sample  $\omega \in \Omega$ , assume a contrary that there exists a subsequence  $\{N_j\} \subset \{N_i\}$  such that

$$\lim_{j \rightarrow \infty} I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega)) = 0.$$

Then, we have  $\lim_{j \rightarrow \infty} D' (M_{N_j} - I (M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) = D'(0)$

and

$\lim_{j \rightarrow \infty} U' (I (M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) = \infty$ . But by (3.2.17) and the fact that  $R_{N_j} = R_{Y_{N_j}, M_{N_j}}^{(N_j)} \leq D'(M_{N_j})$  from Lemma 3.2.9 with  $\xi = \zeta^{(N)}$ , we have:

$$\begin{aligned} \infty &= \lim_{j \rightarrow \infty} U' (I (M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) \\ &= \xi^*(\omega) Y_0 + \gamma \lim_{j \rightarrow \infty} (R_{N_j} - D' (M_{N_j} - I (M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega)))) \\ &\leq \xi^*(\omega) Y_0 \\ &\quad + \gamma \lim_{j \rightarrow \infty} (D'(M_{N_j}) - D' (M_{N_j} - I (M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega)))) \\ &\leq \xi^*(\omega) Y_0 + \gamma (D'(0) - D'(0)) < \infty, \end{aligned}$$

which leads to a contradiction. Hence,  $\lim_{i \rightarrow \infty} I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) > 0$  almost surely.

Step 2: By Step 1, we have  $\lim_{i \rightarrow \infty} I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) > 0$  almost surely.

By Fatou's Lemma and Lemma 3.2.10 with  $\xi = \zeta^{(N)}$ , we have

$$\begin{aligned} 0 = \liminf_{i \rightarrow \infty} M_{N_i} &= \liminf_{i \rightarrow \infty} \mathbb{E} [I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})] \\ &\geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) \right] > 0. \end{aligned}$$

We have a contradiction, hence  $M_0 > 0$ .

Step 3: By (3.2.17), we first have

$$\begin{aligned} &U' (I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) - Y_{N_i} \zeta^{(N_i)} \\ &= \gamma R_{N_i} - \gamma D' (M_{N_i} - I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) \end{aligned} \quad (3.3.32)$$

If  $M_0 = \lim_{i \rightarrow \infty} M_{N_i} = \infty$ , then either  $\lim_{i \rightarrow \infty} R_{N_i} = \infty$  or

$\lim_{i \rightarrow \infty} I (M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) = \infty$  almost surely. We further assume a contrary that there exist a subsequence  $\{N_j\} \subset \{N_i\}$  such that  $\lim_{j \rightarrow \infty} R_{N_j} <$

$\infty$ , then  $\lim_{j \rightarrow \infty} I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}) = \infty$ , thus

$$\lim_{j \rightarrow \infty} (U'(I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)})) - Y_{N_j} \zeta^{(N_j)}) = -Y_0 \xi^* \text{ a.s.} \quad (3.3.33)$$

Since, for large enough  $j$ , by (3.3.32),

$$\begin{aligned} & U'(I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)})) - Y_{N_j} \zeta^{(N_j)} \\ & \leq U'(I(1, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)})) - Y_{N_j} \zeta^{(N_j)} \\ & \leq \gamma \left( \lim_{j \rightarrow \infty} R_{N_j} + 1 \right) - \gamma D' \left( 1 - I \left( 1, \frac{\gamma}{2} \lim_{j \rightarrow \infty} R_{N_j} \right) \right), \end{aligned} \quad (3.3.34)$$

then by Reverse Fatou's lemma, (3.3.33), and Lemma 3.2.9 with  $\xi = \zeta^{(N)}$ , we have:

$$\begin{aligned} 0 > -Y_0 \mathbb{E}[\xi^*] &= \mathbb{E} \left[ \limsup_{j \rightarrow \infty} (U'(I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)})) - Y_{N_j} \zeta^{(N_j)}) \right] \\ &\geq \limsup_{j \rightarrow \infty} \mathbb{E} [U'(I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)})) - Y_{N_j} \zeta^{(N_j)}] \\ &= 0. \end{aligned} \quad (3.3.35)$$

We have a contradiction, therefore,  $\lim_{i \rightarrow \infty} R_{N_i} = \infty$ .

Step 4:

Assume  $M_0 = \infty$ , we claim that  $\lim_{i \rightarrow \infty} (M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) = \infty$  almost surely. Given a sample  $\omega \in \Omega$ , assume a contrary that there exists a subsequence  $\{N_j\} \subset \{N_i\}$  such that  $\lim_{j \rightarrow \infty} (M_{N_j} - I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) < \infty$ , then

$$\lim_{j \rightarrow \infty} D'(M_{N_j} - I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) < \infty.$$

Since  $\lim_{j \rightarrow \infty} R_{N_j} = \infty$  by Step 3, we have

$$\lim_{j \rightarrow \infty} U'(I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) = \infty,$$

thus  $\lim_{j \rightarrow \infty} I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega)) = 0$  and then it results in

$$\lim_{j \rightarrow \infty} (M_{N_j} - I(M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)}(\omega))) = \infty, \quad (3.3.36)$$



which contradicts the assumption on its finiteness at the first place. Now, we have  $\lim_{i \rightarrow \infty} (M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) = \infty$  almost surely.

On  $\{(Y, M) | Y \geq U'(M)\}$ , we have  $D'(M - I(M, Y)) = Y - U'(I(M, Y))$  is strictly increasing in  $M$ . Thus, because  $D'$  is strictly increasing for positive  $M - I(M, Y)$ ,  $M - I(M, Y)$  is strictly increasing in  $M$  on  $\{(Y, M) | Y \leq U'(M)\}$ .

On the other hand, on  $\{(Y, M) | Y \leq U'(M)\}$ ,  $M - I(M, Y) = M - (U')^{-1}(Y)$  is strictly increasing in  $M$ . Hence,  $M - I(M, Y)$  increases in  $M$  for any fixed  $Y$ , then we have

$$M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) \geq 1 - I(1, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) \geq -I(1, 1),$$

for large enough  $i$ , where the last inequality follows because  $\lim_{i \rightarrow \infty} R_{N_i} = \infty$  by Step 3, and  $I(M, Y)$  decreases in  $Y$  for any fixed  $M$ .

By Fatou's Lemma, (3.3.36), and Lemma 3.2.10 with  $\xi = \zeta^{(N)}$ , we have contradiction:

$$\begin{aligned} 0 &= \liminf_{i \rightarrow \infty} \mathbb{E} [M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})] \\ &\geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} (M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) \right] = \infty. \end{aligned}$$

By Steps 2 and 4, we have  $M_0 \in (0, \infty)$ . By Lemma 3.3.3,  $\lim_{i \rightarrow \infty} R_{N_i} = \lim_{i \rightarrow \infty} R_{Y_{N_i}, M_{N_i}}^{(N_i)} = R_{Y_0, M_0}^*$ , thus

$$\lim_{i \rightarrow \infty} I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) = I(M_0, \gamma R_{Y_0, M_0}^* + Y_0 \xi^*).$$

By the Dominated Convergence Theorem and Lemma 3.2.10 with  $\xi = \zeta^{(N)}$ ,

$$\mathbb{E} [I(M_0, \gamma R_{Y_0, M_0}^* + Y_0 \xi^*)] = \lim_{i \rightarrow \infty} \mathbb{E} [I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})] = \lim_{i \rightarrow \infty} M_{N_i} = M_0.$$

By the uniqueness of Lemma 3.2.10 with  $\xi = \xi^*$ , we have  $M_0 = M_{Y_0}^*$ .

Since every subsequential limits of  $\{M_{N_k}\}$  agree with  $M_0 = \lim_{i \rightarrow \infty} M_{N_i} = M_{Y_0}^*$ , we are done. □

**Lemma 3.3.5.**  $\lim_{N \rightarrow \infty} Y^{(N)} = Y^*$ .

*Proof.* By Bolzano-Weierstrass theorem, there exist a subsequence  $\{N_k\}$  such that  $Y_0 := \lim_{k \rightarrow \infty} Y^{(N_k)} \in [0, \infty]$  exists.

We will prove  $Y_0 \in (0, \infty)$  by using following 4 steps:

Step 1: Prove if  $Y_0 = 0$ ,  $\mathbb{P} [\lim_{k \rightarrow \infty} I (M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)}) = \infty] > 0$ .

Step 2: Prove  $Y_0 > 0$ .

Step 3: Prove if  $Y_0 = \infty$ ,  $\lim_{k \rightarrow \infty} I (M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)}) = 0$  almost surely.

Step 4: Prove  $Y_0 < \infty$ .

Step 1:

Assume the contrary that there exists an arbitrary a sample  $\omega_0 \in \{\omega \in \Omega \mid \xi^*(\omega) < \bar{\xi}^*\}$  and a sequence  $\{N_i\} \subset \{N_k\}$  such that

$$\lim_{i \rightarrow \infty} I (M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} (\omega_0)) < \infty.$$

For any sample  $\omega \in \{\omega \in \Omega \mid \xi^*(\omega) > \xi^*(\omega_0)\}$ , we have  $\zeta^{(N_i)}(\omega) > \zeta^{(N_i)}(\omega_0)$  for large enough  $i$ , hence

$$\begin{aligned} & U' (I (M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} (\omega))) \\ & > U' (I (M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} (\omega_0))). \end{aligned}$$

Therefore, for any  $\omega \in \{\omega \in \Omega \mid \xi^*(\omega) > \xi^*(\omega_0)\}$ ,

$$\begin{aligned} & \liminf_{i \rightarrow \infty} U' (I (M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} (\omega))) \\ & \geq \lim_{i \rightarrow \infty} U' (I (M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} (\omega_0))) > 0. \end{aligned}$$

The satisfaction of Condition (ii) in Theorem 3.2.8 implies that  $\mathbb{E}[\zeta^{(N_i)}]$  is uniformly bounded from above, then, by Fatou's Lemma, (3.2.17), and (3.3.28),

$$\begin{aligned}
 0 &= \liminf_{i \rightarrow \infty} Y^{(N_i)} \mathbb{E}[\zeta^{(N_i)}] \\
 &= \liminf_{i \rightarrow \infty} \mathbb{E} \left[ U' \left( I \left( M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) \right) \right] \\
 &\geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} U' \left( I \left( M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) \right) \right] \\
 &\geq \lim_{i \rightarrow \infty} U' \left( I \left( M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)}(\omega_0) \right) \right) \mathbb{P}[\xi^* > \xi^*(\omega_0)] > 0.
 \end{aligned}$$

We have contradiction, hence

$$\lim_{k \rightarrow \infty} I \left( M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)}(\omega_0) \right) = \infty$$

for all  $\omega_0 \in \{\omega \in \Omega \mid \xi^*(\omega) < \bar{\xi}^*\}$ .

Step 2: Assume a contrary that  $Y_0 = 0$ , by Step 1, we further have

$\mathbb{P} \left[ \lim_{k \rightarrow \infty} \left( \zeta^{(N_k)} I \left( M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)} \right) \right) = \infty \right] > 0$ . By Fatou's Lemma and (3.3.30), we have

$$\begin{aligned}
 x_0 &= \liminf_{k \rightarrow \infty} \mathbb{E} \left[ \zeta^{(N_k)} I \left( M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)} \right) \right] \\
 &\geq \mathbb{E} \left[ \liminf_{k \rightarrow \infty} \zeta^{(N_k)} I \left( M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)} \right) \right] = \infty.
 \end{aligned}$$

We have contradiction, thus  $Y_0 > 0$ .

Step 3: Since  $M_Y$  is decreasing in  $Y$  as shown in Lemma 3.2.10, and  $\lim_{k \rightarrow \infty} M_1^{(N_k)} = M_1^*$  by Lemma 3.3.4, we have

$$M^{(N_k)} = M_{Y^{(N_k)}}^{(N_k)} \leq M_1^{(N_k)} \leq 2M_1^* < \infty \text{ for large enough } k \quad (3.3.37)$$

. Then, since  $D'$  is increasing, we have

$$\limsup_{k \rightarrow \infty} D' \left( M^{(N_k)} \right) \leq D'(2M_1^*) < \infty. \quad (3.3.38)$$

Assume a contrary that there exists a sample  $\omega \in \Omega$  and a sequence  $\{N_j\} \subset \{N_k\}$  with  $Y^{(N_j)} \rightarrow \infty$  such that

$$\lim_{j \rightarrow \infty} I(M^{(N_j)}, Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma R^{(N_j)}) > 0. \quad (3.3.39)$$

By (3.2.17), we have

$$\begin{aligned} & Y^{(N_j)} \zeta^{(N_j)}(\omega) - U'(I(M^{(N_j)}, Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma R^{(N_j)})) \\ &= D'(M^{(N_j)} - I(M^{(N_j)}, Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma R^{(N_j)})) - \gamma R^{(N_j)} \\ &\leq D'(M^{(N_j)}) - \gamma R^{(N_j)}, \end{aligned} \quad (3.3.40)$$

where the last inequality follows because  $I(M^{(N_j)}, Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma R^{(N_j)})$  is positive. Taking the limit superior with  $j \rightarrow \infty$  in the both sides of (3.3.40), the limit in the left hand side tends to infinity because  $\lim_{j \rightarrow \infty} U'(I(M^{(N_j)}, Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma R^{(N_j)})) < \infty$  by (3.3.39), while the limit in the right hand side is finite due to (3.3.38) and the non-negativity of  $R^{(N_j)}$ , we have contradiction. Therefore, we have

$$\lim_{k \rightarrow \infty} I(M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)}) = 0, \quad a.s. \quad (3.3.41)$$

Step 4:

Assume  $Y_0 = \infty$ . By step 3,  $\lim_{k \rightarrow \infty} I(M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)}) = 0$  almost surely.

Since  $M^{(N_k)} \leq 2M_1^*$  for large enough  $k$  by (3.3.37) and  $R^{(N_k)} \geq 0$  for all  $k$ , we have

$$\begin{aligned} I(M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)}) &\leq I(2M_1^*, \zeta^{(N_k)}) \\ &\leq I(2M_1^*, U'(2M_1^*)) + (U')^{-1}(\zeta^{(N_k)}), \end{aligned}$$

for large enough  $k$ .

By the satisfaction of Condition (iii) in Theorem 3.2.8,

$\{\zeta^{(N_k)} I(M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)})\}_{k \in \mathbb{N}}$  is uniformly integrable. By the

Dominated Convergence Theorem under uniform integrability assumption, (3.3.30), and Step 3,

$$\begin{aligned} x_0 &= \lim_{k \rightarrow \infty} \mathbb{E} [\zeta^{(N_k)} I (M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)})] \\ &= \mathbb{E} \left[ \lim_{k \rightarrow \infty} \zeta^{(N_k)} I (M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)}) \right] = 0. \end{aligned} \quad (3.3.42)$$

We have a contradiction, so  $Y_0 < \infty$ .

By Steps 2 and 4, we have  $Y_0 \in (0, \infty)$ . By Lemmas 3.3.3 and 3.3.4,

$$\begin{aligned} \lim_{k \rightarrow \infty} (Y^{(N_k)}, M^{(N_k)}, R^{(N_k)}) &= \left( Y_0, \lim_{k \rightarrow \infty} M_{Y^{(N_k)}}^{(N_k)}, \lim_{k \rightarrow \infty} R_{Y^{(N_k)}, M_{Y^{(N_k)}}}^{(N_k)} \right) \\ &= \left( Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^* \right), \end{aligned}$$

therefore, we have

$$\lim_{k \rightarrow \infty} \zeta^{(N_k)} I (M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)}) = \xi^* I (M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \xi^*).$$

Since the satisfaction of Condition (ii) in Theorem 3.2.8 implies that  $\{\zeta^{(N_k)}\}_{k \in \mathbb{N}}$  is uniformly integrable, by Dominated Convergence Theorem and (3.3.30),

$$\begin{aligned} x_0 &= \lim_{k \rightarrow \infty} \mathbb{E} [\zeta^{(N_k)} I (M^{(N_k)}, \gamma R^{(N_k)} + Y^{(N_k)} \zeta^{(N_k)})] \\ &= \mathbb{E} \left[ \xi^* I (M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \xi^*) \right]. \end{aligned}$$

Together with the fact that  $R_{Y, M}^*$  and  $M_Y^*$  solve (3.2.18) and (3.2.20) with  $\xi = \xi^*$ ,  $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*)$  solves the system of equations (3.2.14)-(3.2.16) with  $\xi = \xi^*$ . Since the solution of the system is unique, so we have  $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*) = (Y^*, M^*, R^*)$ .

Since every subsequential limits of  $\{Y^{(N)}\}$  agree with  $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)} = Y^*$ , we have  $Y^* = \lim_{N \rightarrow \infty} Y^{(N)}$ .

□

By Lemmas 3.3.3 and 3.3.4, we have

$$\begin{aligned}
 \lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) &= \left( Y^*, \lim_{N \rightarrow \infty} M_{Y^{(N)}}^{(N)}, \lim_{N \rightarrow \infty} R_{Y^{(N)}, M_{Y^{(N)}}^{(N)}}^{(N)} \right) \\
 &= \left( Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^* \right) \\
 &= (Y^*, M^*, R^*). \tag{3.3.43}
 \end{aligned}$$

Hence, this proposition follows.

### 3.3.2 Convergence of Optimal Solution: Case of Strictly-Convex Risk

We aim to show the similar convergence result as the case of downside risk.  $\{\Pi^{(N)}\}$  denotes a sequence of markets which satisfy Conditions 3.1.6 and 3.1.7, thus we have a sequence of terminal pricing kernels  $\{\xi^{(N)}\}$ . Similar argument applies to  $\hat{X}^*$  and  $\Pi^*$ . By Theorem 3.2.14, there exist optimal terminal payoffs of utility-strictly-convex-risk problems  $\hat{X}^{(N)}$  and  $\hat{X}^*$  under the markets  $\Pi^{(N)}$  and  $\Pi^*$  respectively.

**Theorem 3.3.6.** *Given that  $U$  and  $D$  satisfy Definitions 3.1.1 and 3.1.3 respectively and all markets  $\Pi^{(N)}$  and  $\Pi^*$  satisfying Conditions 3.1.6 and 3.1.7 with corresponding unique terminal pricing kernels  $\xi_N$  and  $\xi^*$  and optimal terminal payoffs  $\hat{X}^{(N)}$  and  $\hat{X}^*$ , as described in Theorem 3.2.14. Suppose that a sequence of their terminal pricing kernels  $\{\xi^{(N)}\}$  satisfy the following conditions:*

- (i)  $\xi^{(N)} \in \mathcal{L}^2$  converges weakly to  $\xi^* \in \mathcal{L}^2$  as  $N \rightarrow \infty$ .
- (ii)  $\{\xi^{(N)}\}_{N \in \mathbb{N}}$  is uniformly integrable.

Then the sequence  $\{\hat{X}^{(N)}\}$  converges weakly to  $\hat{X}^*$ . Hence,  $\Psi(\hat{X}^{(N)}) \rightarrow \Psi(\hat{X}^*)$ , i.e., the sequence of the optimal value functions of utility-risk problem under  $\Pi^{(N)}$  converges to the optimal value functions under  $\Pi^*$ .

*Proof.* The proof is similar to Theorem 3.3.1, the essential changes are the follows:

- Lemma 3.3.3. To apply the Dominated Convergence Theorem in (3.3.31), we need to first verify that  $R_0 > -\infty$ .

We assume the contrary that  $R_0 = -\infty$  for strictly convex risk  $D$ , then  $\lim_{i \rightarrow \infty} I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}) = \infty$  almost surely by (3.2.23). Since  $U'$  is continuous, by Inada condition, we have

$$\lim_{i \rightarrow \infty} Y_{N_i} \zeta^{(N_i)} - U'(I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) = Y_0 \xi^* \text{ a.s.} \quad (3.3.44)$$

By (3.2.23), for all  $i$  such that  $R_{N_i} < 0$ ,

$$\begin{aligned} & Y_{N_i} \zeta^{(N_i)} - U'(I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) \\ & \geq Y_{N_i} \zeta^{(N_i)} - U'(I(M_{N_i}, Y_{N_i} \zeta^{(N_i)})) \\ & = \gamma D'(M_{N_i} - I(M_{N_i}, Y_{N_i} \zeta^{(N_i)})) \\ & \geq \gamma D'(-I(M_0 + 1, 0)) \end{aligned}$$

By Fatou's lemma, (3.3.44), and Lemma 3.2.15 with  $\xi = \zeta^{(N)}$ , we have

$$\begin{aligned} 0 & = \lim_{i \rightarrow \infty} \mathbb{E} [D'(M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)})) - R_{N_i}] \\ & \geq \liminf_{i \rightarrow \infty} \mathbb{E} \left[ \frac{1}{\gamma} (Y_{N_i} \zeta^{(N_i)} - U'(I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}))) \right] \\ & \geq \frac{1}{\gamma} Y_0 \mathbb{E} [\xi^*] > 0. \end{aligned}$$

We have contradiction, therefore,  $R_0 > -\infty$ .

Then, we can show that  $D'(M_{N_i} - I(M_{N_i}, \gamma R_{N_i} + Y_{N_i} \zeta^{(N_i)}))$  is uniformly bounded by  $D'(\frac{M_0}{2} - I(2M_0, \gamma(R_0 - 1)))$  and  $D'(2M_0)$  in large enough  $i$ , so we can apply the Dominated Convergence Theorem in (3.3.31).

- Step 3 in Lemma 3.3.4. Since  $\lim_{j \rightarrow \infty} R^{(N_j)}$  may not be non-negative in the case of strictly convex risk, we can prove the boundedness as (3.3.34) by follows:

for large enough  $j$ ,

$$\begin{aligned}
 & U' \left( I \left( M_{N_j}, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)} \right) \right) - Y_{N_j} \zeta^{(N_j)} \\
 & \leq U' \left( I \left( 1, \gamma R_{N_j} + Y_{N_j} \zeta^{(N_j)} \right) \right) - Y_{N_j} \zeta^{(N_j)} \\
 & \leq \gamma \left( \lim_{j \rightarrow \infty} R^{(N_j)} + 1 \right) - D' \left( 1 - I \left( 1, \gamma \lim_{j \rightarrow \infty} R^{(N_j)} - 1 \right) \right).
 \end{aligned}$$

Then, we can apply reverse Fatou's Lemma to have (3.3.35).

- Step 3 in Lemma 3.3.5. Since the non-negativity of  $R^{(N_k)}$  does not hold in the case of strictly convex risk, we have to verify that if  $Y_0 = \infty$ ,

$$\liminf_{k \rightarrow \infty} R^{(N_k)} > -\infty$$

in order to show the right hand size in (3.3.40) is finite as  $j \rightarrow \infty$ .

By Lemma 3.2.15 with  $\xi = \zeta^{(N_i)}$ ,  $R_{Y,M}^{(N_i)}$  is increasing in  $Y$ ,

$$\liminf_{k \rightarrow \infty} R^{(N_k)} = \liminf_{k \rightarrow \infty} R_{Y^{(N_k)}, M^{(N_k)}}^{(N_k)} \geq \liminf_{k \rightarrow \infty} R_{K, M^{(N_k)}}^{(N_k)} \text{ for all } K > 0,$$

so we claim that  $\liminf_{k \rightarrow \infty} R_{K, M^{(N_k)}}^{(N_k)} > -\infty$  for all  $K > 0$ .

Assume the contrary that there exists  $K > 0$  and a subsequence  $\{N_i\} \subset \{N_k\}$  such that  $\lim_{i \rightarrow \infty} R_{K, M^{(N_i)}}^{(N_i)} = -\infty$ . Then,

$$\lim_{i \rightarrow \infty} \left( \zeta^{(N_i)} K + \gamma R_{K, M^{(N_i)}}^{(N_i)} \right) = -\infty.$$

Therefore,  $\lim_{i \rightarrow \infty} I \left( M^{(N_i)}, \zeta^{(N_i)} K + \gamma R_{K, M^{(N_i)}}^{(N_i)} \right) = \infty$ . Then,

$$\lim_{i \rightarrow \infty} U' \left( I \left( M^{(N_i)}, \zeta^{(N_i)} K + \gamma R_{K, M^{(N_i)}}^{(N_i)} \right) \right) = 0. \quad (3.3.45)$$

Since

$$\begin{aligned}
 U' \left( I \left( M_{Y^{(N_i)}}, \zeta^{(N_i)} K + \gamma R_{K, M_{Y^{(N_i)}}}^{(N_i)} \right) \right) & \leq U' \left( I \left( 0, \zeta^{(N_i)} K \right) \right) \\
 & \leq \zeta^{(N_i)} K - D' \left( -I \left( 0, 0 \right) \right)
 \end{aligned}$$



for large enough  $i$ ,  $\mathcal{L}^1$  boundedness is preserved by the uniform integrability of  $\xi'_N$ . Then, applying reverse Fatou's lemma with (3.3.45) and using the fact that  $R_{Y,M}^{(N_i)}$  satisfies (3.2.25), we have:

$$\begin{aligned} 0 &= \mathbb{E} \left[ \limsup_{i \rightarrow \infty} U' \left( I \left( M^{(N_i)}, \zeta^{(N_i)} K + \gamma R_{K,M^{(N_i)}}^{(N_i)} \right) \right) \right] \\ &\geq \limsup_{i \rightarrow \infty} \mathbb{E} \left[ U' \left( I \left( M^{(N_i)}, \zeta^{(N_i)} K + \gamma R_{K,M^{(N_i)}}^{(N_i)} \right) \right) \right] \\ &= K \limsup_{i \rightarrow \infty} \mathbb{E} [\zeta^{(N_i)}] > 0. \end{aligned}$$

We have a contradiction, thus

$$\liminf_{k \rightarrow \infty} R_{K,M^{(N_k)}}^{(N_k)} > -\infty \text{ for any } K \in (0, \infty), \quad (3.3.46)$$

and hence, by Lemma 3.3.45, we have

$$\liminf_{k \rightarrow \infty} R^{(N_k)} > -\infty.$$

From here on, the rest of the proof of Step 3 in Lemma 3.3.5 is the same as the case of downside risk.

- Step 4 in Lemma 3.3.5. By Lemma 3.2.15, for any  $K \in (0, \infty)$ ,  $R^{(N_k)} = R_{Y^{(N_k)}, M^{(N_k)}}^{(N_k)} \geq R_{K, M^{(N_k)}}^{(N_k)}$  for large enough  $k$ . Thus,  $\liminf_{k \rightarrow \infty} R^{(N_k)} \geq \liminf_{k \rightarrow \infty} R_{K, M^{(N_k)}}^{(N_k)} > -\infty$ , where the last inequality has been verified in (3.3.46). Since  $M^{(N_k)} = M_{Y^{(N_k)}}^{(N_k)} \leq M_1^{(N_k)} \leq 2M_1^*$  by Lemmas 3.2.16 and 3.3.4, we have

$$I \left( M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)} \right) \leq I \left( 2M_1^*, \gamma \liminf_{k \rightarrow \infty} R_{K, M^{(N_k)}}^{(N_k)} \right) < \infty.$$

By the uniform integrability of  $\zeta^{(N_k)}$ , we can apply the Dominated Convergence Theorem under uniform integrability assumption to obtain the same contradiction in (3.3.42).

□

### 3.3.3 Application: Approximation of Optimal Solution to Continuous-Time Utility-Risk Problem

In this section, we will show that the sequence of the terminal pricing kernels  $\xi^{(N)} := \xi_N^{(N)}$ , where  $\xi_n^{(N)}$  defined in (3.1.6) satisfies the conditions stated in Theorems 3.3.1 and 3.3.6 with  $\xi^* := \xi_T$ , where  $\xi_t$  is defined in (3.1.4). Then, by Theorems 3.3.1 and 3.3.6, the optimal terminal wealth of utility-risk problems under continuous-time Black-Scholes model can be approximated by the optimal solution in discrete-time binomial-tree model.

To show the terminal pricing kernels  $\xi^{(N)} := \xi_N^{(N)}$  satisfying the conditions stated in Theorem 3.3.1 for utility-downside-risk problem, the following assumption stronger than Assumption 2.3.4, which is adopted in Remark 4.4 in Karatzas and Shreve (1998), is required:

**Assumption 3.3.7.** There exist  $\eta \in (0, 1)$  and  $\beta > 1$  such that  $U'(\eta y) \leq \beta U'(y)$  for all  $y > 0$ .

**Theorem 3.3.8.** *Given that Assumption 3.3.7 holds and  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.2, the sequence of the terminal pricing kernels  $\xi^{(N)} := \xi_N^{(N)}$  defined in (3.1.6) satisfies the conditions stated in Theorem 3.3.1 with  $\xi^* := \xi_T$  where  $\xi_t$  is defined in (3.1.4). Hence, the sequence of the optimal terminal wealth for utility-downside-risk problem under binomial tree models converge to the optimal terminal wealth under the Black-Scholes model described in Example 3.1.10.*

Before we proceed to the proof, we first need the following two technical lemmas for verifying the conditions in Theorem 3.3.8.

**Lemma 3.3.9.**  $\xi^{(N)}$  defined in (3.1.6) converges weakly to  $\xi^*$  defined in (3.1.4).

**Lemma 3.3.10.** *Given that  $\xi^{(N)}$  is defined in (3.1.6), we have*

(i)  $\mathbb{E} [(\xi^{(N)})^3] < e^{KT}$ , where  $K$  is a constant independent of  $N$ .

(ii) For any fixed  $n \in \mathbb{N}$ , there exist a constant  $M_n$  such that  $\mathbb{E} [(\xi^{(N)})^{-n}] \leq M_n$  for all  $N \in \mathbb{N}$ .

**Remark 3.3.11.** Since the pricing kernel can be expressed as a function of stock price, the weak convergence of stock prices from the binomial-tree models to the continuous-time Black-Scholes model is sufficient for Lemma 3.3.9. The convergence of stock price has been studied in plenty of literature; such as Prigent (2003) and Föllmer and Schied (2004). Alternatively, we can show the convergence of pricing kernels directly by the application of the Central Limit Theorem; the proof is provided for the sake of convenience of reader.

*Proof of Lemma 3.3.9.*

To show the weak convergence of pricing kernels, we first need the following alternative version of central limit theorem:

**Theorem 3.3.12** (Theorem A.36 in Föllmer and Schied (2004)). *Suppose that for each  $N \in \mathbb{N}$ , we are given  $N$  independent random variables  $Y_1^{(N)}, \dots, Y_N^{(N)}$  which satisfy the following conditions:*

- (i) *There are constants  $\gamma_N$  such that  $\gamma_N \rightarrow 0$  and  $|Y_k^{(N)}| \leq \gamma_N$  a.s. for all  $k$ .*
- (ii)  $\sum_{k=1}^N \mathbb{E} [Y_k^{(N)}] \rightarrow m$
- (iii)  $\sum_{k=1}^N \text{Var} [Y_k^{(N)}] \rightarrow \sigma^2$

*Then the distributions of  $\sum_{k=1}^N Y_k^{(N)}$  converge weakly to the normal distribution with mean  $m$  and variance  $\sigma^2$ .*

By considering  $\xi_n^{(N)}$  defined in (3.1.6), we have  $\xi^{(N)} = e^{-rT} \prod_{k=1}^N (1 + R_k^{(N)})$ , where

$$R_k^{(N)} := 2 \frac{e^{-\lambda \Delta t + \sigma \sqrt{\Delta t} Z_k^{(N)}} - 1}{e^{2\sigma \sqrt{\Delta t} Z_k^{(N)}} - 1} - 1.$$

Hence,

$$R_k^{(N)} = \begin{cases} \alpha_N := 2 \frac{e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}} - 1}{e^{2\sigma\sqrt{\Delta t}} - 1} - 1, & \text{if } Z_k^{(N)} = 1; \\ \beta_N := 2 \frac{e^{2\sigma\sqrt{\Delta t}} - e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}}}{e^{2\sigma\sqrt{\Delta t}} - 1} - 1, & \text{if } Z_k^{(N)} = -1; \end{cases} \quad \text{for all } k = 1, 2, \dots, N.$$

Since

$$\frac{e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}} - 1}{e^{2\sigma\sqrt{\Delta t}} - 1} = \frac{\sigma\sqrt{\Delta t} + O(\Delta t)}{2\sigma\sqrt{\Delta t} + O(\Delta t)} = \frac{1}{2} + O(\sqrt{\Delta t})$$

and similarly

$$\frac{e^{2\sigma\sqrt{\Delta t}} - e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}}}{e^{2\sigma\sqrt{\Delta t}} - 1} = \left( \sigma\sqrt{\Delta t} + O(\Delta t) \right) \left( \frac{1}{2\sigma\sqrt{\Delta t}} + O(1) \right) = \frac{1}{2} + O(\sqrt{\Delta t}),$$

we have  $\alpha_N = \beta_N = O(\sqrt{\Delta t}) \rightarrow 0$  as  $N \rightarrow \infty$ .

Then by Taylor expansion,

$$Y_k^{(N)} := \ln \left( 1 + R_k^{(N)} \right) = R_k^{(N)} - \frac{1}{2} \left( R_k^{(N)} \right)^2 + O(\Delta t^{\frac{3}{2}}) \text{ a.s.} \quad (3.3.47)$$

Since  $R_k^{(N)}$  is bounded by  $\gamma_N := \max\{\alpha_N, \beta_N\} = O(\sqrt{\Delta t})$ ,

$$|Y_k^{(N)}| = \gamma_N + \frac{1}{2} \gamma_N^2 + O(\Delta t^{\frac{3}{2}}) \rightarrow 0.$$

Hence, Condition (i) in Theorem 3.3.12 follows.

Next, to compute  $\mathbb{E} \left[ Y_k^{(N)} \right]$ , we consider  $\mathbb{E} [R_k^{(N)}]$  and  $\mathbb{E} \left[ \left( R_k^{(N)} \right)^2 \right]$  first. By a standard computation, we have  $\mathbb{E} [R_k^{(N)}] = 0$  for all  $k, N$ . Next,

$$\begin{aligned} \mathbb{E} \left[ \left( R_k^{(N)} \right)^2 \right] &= \frac{\alpha_N^2 + \beta_N^2}{2} = \left( \frac{e^{2\sigma\sqrt{\Delta t}} - 2e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}} + 1}{e^{2\sigma\sqrt{\Delta t}} - 1} \right)^2 \\ &= \left( \left( 2\alpha\Delta t + O(\Delta t^{\frac{3}{2}}) \right) \left( \frac{1}{2\sigma\sqrt{\Delta t}} + O(1) \right) \right)^2 \\ &= \left( \frac{\alpha}{\sigma} \sqrt{\Delta t} + O(\Delta t) \right)^2 \\ &= \frac{\alpha^2}{\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}). \end{aligned}$$

Then, by (3.3.47), we have

$$\begin{aligned} \mathbb{E} \left[ Y_k^{(N)} \right] &= \mathbb{E} \left[ R_k^{(N)} \right] - \frac{1}{2} \mathbb{E} \left[ \left( R_k^{(N)} \right)^2 \right] + O(\Delta t^{\frac{3}{2}}) \\ &= -\frac{\alpha^2}{2\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}) \end{aligned} \quad (3.3.48)$$

Hence,

$$\sum_{k=1}^N \mathbb{E} \left[ Y_k^{(N)} \right] = -\frac{\alpha^2}{2\sigma^2} T + O(\Delta t^{\frac{1}{2}}) \rightarrow -\frac{\alpha^2}{2\sigma^2} T \text{ as } N \rightarrow \infty.$$

By (3.3.47) and the fact that  $R_k^{(N)}$  has an order of  $O(\sqrt{\Delta t})$ , we have  $\left( Y_k^{(N)} \right)^2 = \left( R_k^{(N)} \right)^2 + O(\Delta t^{\frac{3}{2}})$  almost surely. Then, by (3.3.48),

$$\begin{aligned} \text{Var} \left[ Y_k^{(N)} \right] &= \mathbb{E} \left[ \left( Y_k^{(N)} \right)^2 \right] - \left( \mathbb{E} \left[ Y_k^{(N)} \right] \right)^2 = \mathbb{E} \left[ \left( R_k^{(N)} \right)^2 \right] + O(\Delta t^{\frac{3}{2}}) \\ &= \frac{\alpha^2}{\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}). \end{aligned}$$

Hence,

$$\sum_{k=1}^N \text{Var} \left[ Y_k^{(N)} \right] = \frac{\alpha^2}{\sigma^2} T + O(\Delta t^{\frac{1}{2}}) \rightarrow \frac{\alpha^2}{\sigma^2} T \text{ as } N \rightarrow \infty.$$

By Theorem 3.3.12, we have  $\sum_{k=1}^N Y_k^{(N)}$  converge weakly to the normal distribution with mean  $-\frac{\alpha^2}{2\sigma^2} T$  and variance  $\frac{\alpha^2}{\sigma^2} T$ . Hence,  $\xi^{(N)} = e^{-rT} e^{\sum_{k=1}^N Y_k^{(N)}}$  converge weakly to log-normally distributed random variable with mean  $-rT - \frac{\alpha^2}{2\sigma^2} T$  and variance  $\frac{\alpha^2}{\sigma^2} T$ , which has the same distribution as  $\xi_T$ .

□

*Proof of Lemma 3.3.10.*

By Taylor series expansion, we have

$$\begin{aligned}
 & \mathbb{E} \left[ \left( 1 + R_k^{(N)} \right)^3 \right] \\
 &= \frac{4 \left( e^{6\sigma\sqrt{\Delta t}} - 3e^{5\sigma\sqrt{\Delta t} - \lambda\Delta t} + 3e^{4\sigma\sqrt{\Delta t} - 2\lambda\Delta t} - 3e^{2\sigma\sqrt{\Delta t} - 2\lambda\Delta t} + 3e^{\sigma\sqrt{\Delta t} - \lambda\Delta t} - 1 \right)}{\left( e^{2\sigma\sqrt{\Delta t}} - 1 \right)^3} \\
 &\leq 1 + 3 \left( e^{\sigma\sqrt{\Delta t}} - 1 \right)^2 \\
 &\quad + \frac{3e^{\sigma\sqrt{\Delta t} - \lambda\Delta t} \left( e^{\lambda\Delta t} - 1 \right) \left( e^{2\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t} - \lambda\Delta t} \left( e^{\lambda\Delta t} + 1 \right) + 1 \right)}{\sigma^2 \Delta t} \\
 &= 1 + 3 \left( \sigma\sqrt{\Delta t} + O(\Delta t) \right)^2 \\
 &\quad + \frac{3e^{\sigma\sqrt{\Delta t} - \lambda\Delta t} \left( \lambda\Delta t + O(\Delta t^2) \right) \left( (2\sigma^2 + \lambda)\Delta t + O(\Delta t^{\frac{3}{2}}) \right)}{\sigma^2 \Delta t} \\
 &= 1 + K\Delta t
 \end{aligned}$$

for some constant  $K$  independent of  $N$ . Then,

$$\mathbb{E} \left[ \left( \xi^{(N)} \right)^3 \right] = \mathbb{E} \left[ \prod_{i=1}^N \left( 1 + R_k^{(N)} \right)^3 \right] \leq (1 + K\Delta t)^N \leq e^{KT} \text{ for all } N.$$

Therefore, (i) follows.

Given fixed  $n \in \mathbb{N}$ , by standard computations with the application of Taylor series expansions, we have:

$$\begin{aligned}
 & \mathbb{E} \left[ \left( 1 + R_k^{(N)} \right)^{-n} \right] \\
 &= \frac{\left( e^{2\sigma\sqrt{\Delta t}} - 1 \right)^n}{2^{n+1}} \left( \frac{1}{\left( e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}} - 1 \right)^n} + \frac{1}{\left( e^{2\sigma\sqrt{\Delta t}} - e^{-\lambda\Delta t + \sigma\sqrt{\Delta t}} \right)^n} \right) \\
 &= 1 + \frac{(n+1)n}{2\sigma^2} \left( \frac{1}{2}\sigma^2 + \lambda \right)^2 \Delta t + O(\Delta t^{\frac{3}{2}}).
 \end{aligned}$$

Then, there exist a constant  $K_n$  independent of  $N$  such that

$$\begin{aligned}
 \mathbb{E} \left[ \left( \xi^{(N)} \right)^{-n} \right] &= \mathbb{E} \left[ \prod_{i=1}^N \left( 1 + R_k^{(N)} \right)^{-n} \right] \\
 &= \left( 1 + \frac{(n+1)n}{2\sigma^2} \left( \frac{1}{2}\sigma^2 + \lambda \right)^2 \Delta t + O(\Delta t^{\frac{3}{2}}) \right)^N \\
 &\leq e^{K_n T}.
 \end{aligned}$$

□

*Proof of Theorem 3.3.8.*

By Lemma 3.3.9, Condition (i) in Theorem 3.3.1 follows.

By Assumption 3.3.7, its inverse function  $(U')^{-1}$  satisfy the property:  $(U')^{-1}(\eta x) \leq \beta(U')^{-1}(x)$  for all  $x > 0$ . Thus,  $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x)$  for all  $x, y > 0$  and further we have  $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x) + (U')^{-1}(x)$  for all  $y \in \mathbb{R}, x > 0$ . Hence,

$$(U')^{-1}(z) \leq \beta z^{\frac{\ln \beta}{\ln \eta}} (U')^{-1}(1) + (U')^{-1}(1) \text{ for any } z > 0.$$

Then, by Minkowski inequality, we have

$$\left( \mathbb{E} \left[ \left( (U')^{-1}(\xi^{(N)}) \right)^3 \right] \right)^{\frac{1}{3}} \leq (U')^{-1}(1) \left( \beta^3 \mathbb{E} \left[ \left( \xi^{(N)} \right)^{3 \frac{\ln \beta}{\ln \eta}} \right] + 1 \right)^{\frac{1}{3}}.$$

In turn, by Lemma 3.3.10(ii), we have  $\mathbb{E} \left[ \left( (U')^{-1}(\xi^{(N)}) \right)^3 \right] < \infty$ .

Consider that

$$\begin{aligned} & \mathbb{E} \left[ \left( \xi^{(N)} \left( (U')^{-1}(\xi^{(N)}) + M \right) \right)^{\frac{3}{2}} \right] \\ & \leq \mathbb{E} \left[ \left( \frac{1}{2} \left( \left( \xi^{(N)} \right)^2 + 2 \left( (U')^{-1}(\xi^{(N)}) \right)^2 + 2M^2 \right) \right)^{\frac{3}{2}} \right], \end{aligned}$$

then, by Minkowski inequality and Lemma 3.3.10(i), we can show that

$$\mathbb{E} \left[ \left( \xi^{(N)} \left( (U')^{-1}(\xi^{(N)}) + M \right) \right)^{\frac{3}{2}} \right] < \infty.$$

Hence, Condition (ii) in Theorem 3.3.1 follows by 13.3(a) in Williams (1991).

By Theorem 3.3.1, the desired convergence result of the optimal terminal wealth follows.

□

Similarly, we can show that the terminal pricing kernels  $\xi^{(N)} := \xi_N^{(N)}$  satisfies the conditions stated in Theorem 3.3.1 for utility-strictly-convex-risk problem.

**Theorem 3.3.13.** *Let  $U$  and  $D$  are given by Definitions 3.1.1 and 3.1.3. The sequence of the terminal pricing kernels  $\xi^{(N)} := \xi_N^{(N)}$  defined in (3.1.6) satisfies*

the conditions stated in Theorem 3.3.6 with  $\xi^* := \xi_T$  where  $\xi_t$  is defined in (3.1.4). Hence, the sequence of the optimal terminal wealth for utility-strictly-convex-risk problem under binomial tree models converge to the optimal terminal wealth under the Black-Scholes model described in Example 3.1.10.

*Proof.* Condition (i) in Theorem 3.3.6 follows in Lemma 3.3.9.

By Lemma 3.3.10(i),  $\mathbb{E} [(\xi^{(N)})^3] < e^{KT}$ , where  $K$  is a constant independent of  $N$ . Hence, Condition (ii) in Theorem 3.3.6 follows by 13.3(a) in Williams (1991).

By Theorem 3.3.6, we have the desired convergence result of the optimal terminal wealth. □

Furthermore, similar approximation of optimal value function can be done:

**Corollary 3.3.14.** (i) *Given that Assumption 3.3.7 holds, the optimal value function for utility-downside-risk problem under discrete binomial tree model as in Example 3.1.11 converges to the optimal value function under continuous-time Black-Scholes model as in Example 3.1.10 as time interval decreases (i.e.  $N \rightarrow \infty$ ).*

(ii) *The optimal value function for utility-strictly-convex-risk problem under discrete binomial tree model as in Example 3.1.11 converges to the optimal value function under continuous-time Black-Scholes model as in Example 3.1.10 as time interval decreases (i.e.  $N \rightarrow \infty$ ).*

**Remark 3.3.15.** Chapter 2 provided a comprehensive study of utility-risk portfolio selection under the continuous-time Black-Scholes framework. By Remark 3.2.6, the optimal terminal wealth obtained in Chapter 2 is in the analytical form of  $I(M, Y\xi + \gamma R)$  where  $I$  is an implicit function satisfying (3.2.13) and  $Y, M$  and  $R$  are constants satisfying the nonlinear moment constraints in (3.2.14)-(3.2.16). However, it is difficult to numerically compute the solution in Chapter 2 even we



have an explicit form of  $I$ . It is because  $\xi$  is a continuous random variable with a support of the positive half real line. Hence, the expectations in (3.2.14)-(3.2.16) become improper integrals in which the integrand is a nonlinear function of unknown parameters  $Y, M$  and  $R$ . Thus, it is difficult to compute the improper integrals numerically and so do the nonlinear programming problem for  $Y, M$  and  $R$  under the continuous-time Black-Scholes framework.

Under the binomial-tree model,  $\xi$  has finite possibilities, so the expectations in (3.2.14)-(3.2.16) become finite sums. Therefore, it is easier to solve the system of equations in (3.2.14)-(3.2.16) numerically. With Theorems 3.3.8 and 3.3.13, we can approximate the optimal terminal wealth in continuous-time Black-Scholes model by the optimal terminal wealth in discrete binomial tree model, which is more easier to compute.

## 3.4 Numerical Simulation

In this subsection, we compute the numerical solution for utility risk problem under binomial tree model as described in Example 3.1.11. Without any specific instruction, we set the parameters to be:  $r = 0.03, \alpha = 0.07, \sigma = 0.2, T = 1, x_0 = 1$ . We consider utility function to be a power function:  $U(x) = 2x^{\frac{1}{2}}$ , then  $U'(x) = x^{-\frac{1}{2}}$ , we consider risk function to be either variance ( $D(x) = \frac{x^2}{2}$ ) or semivariance ( $D(x) = \frac{x_+^2}{2}$ ).

### 3.4.1 Convergence by Decreasing Time Interval

In this subsection, we will illustrate numerically Corollary 3.3.14 that the optimal terminal wealth and value function will converges when the number of period  $N$  increases so that  $\Delta = \frac{T}{N}$  decreases, then by Theorems 3.3.8 and 3.3.13. We will verify this result numerically. We fix  $\gamma = 0.1$ . We shall compute the solution of the nonlinear system in (3.2.14)-(3.2.16) ( $Y, M, R$ ), optimal control at  $t = 0$

Table 3.1: The Convergence of Optimal Solutions of Utility-Variance Problems as Decreasing Time Interval

N	Y	M	R	$\pi^*(0)$	Utility	Variance	$J^*$	% Change
25	1.2041	2.9321	0	1.8851	3.1548	1.8665	2.9682	-0.0563%
50	1.2055	2.9243	0	1.8727	3.1517	1.8519	2.9665	-0.0279%
100	1.2061	2.9205	0	1.8667	3.1502	1.8448	2.9657	-0.0139%
200	1.2065	2.9186	0	1.8636	3.1494	1.8412	2.9653	-0.0069%
400	1.2066	2.9176	0	1.8621	3.1490	1.8394	2.9651	-0.0035%
800	1.2067	2.9171	0	1.8613	3.1488	1.8385	2.9650	

( $\pi^*(0)$ ), optimal expected utility, optimal variance, optimal objective value ( $J^*$ ), and the relative change of optimal objective value if the number of period is further doubled for different number of period (N).

From the numerical result in Tables 3.1 and 3.2, we see that whenever we double the number of period  $N$ , the change in value function is approximately halved. Under this trend, the results in  $N = 50$  is a good approximation to the optimal solution for utility risk problem under continuous time Black-Scholes model as in Example 3.1.10 with the error less than 0.1% in terms of the optimal objective value.

### 3.4.2 Utility-Risk Efficient Frontiers

In this subsection, we plot the utility-variance and utility-semivariance efficient frontiers for different volatility ( $\sigma = 0.1, 0.2$ , and  $0.4$ ). Smaller  $\gamma$  lead more risky investment, which increases both utility and risk, then we can obtain an efficient frontier by varying  $\gamma$ .

In Figure 3.1, we observe that given same value of risk, smaller  $\sigma$  will give

Table 3.2: The Convergence of Optimal Solutions to Utility-Semivariance Problems as Decreasing Time Interval

N	Y	M	R	$\pi^*(0)$	Utility	Semivariance	$J^*$	% Change
25	1.2551	3.8750	1.5450	2.2335	3.3928	2.2067	3.1721	-0.0752%
50	1.2563	3.8656	1.5397	2.2172	3.3888	2.1908	3.1698	-0.0384%
100	1.2571	3.8587	1.5399	2.2091	3.3867	2.1813	3.1685	-0.0193%
200	1.2573	3.8567	1.5379	2.2050	3.3857	2.1777	3.1679	-0.0095%
400	1.2575	3.8553	1.5376	2.2030	3.3852	2.1756	3.1676	-0.0050%
800	1.2575	3.8547	1.5372	2.2020	3.3849	2.1746	3.1675	

greater utility. It is because, when  $\sigma$  is doubled, the investor has to bear more risk in order to have additional expected return or expected utility through increasing risky investment.

In Figure 3.2, we observe that smaller volatility of the risky asset encourages investor to have more risky investment because, under the smaller volatility, risky investment induces comparatively smaller deviation risk, hence the risky investment becomes comparatively more profitable.

Now, we fix  $\sigma = 0.2$ , we examine extreme value of  $\gamma$ . If  $\gamma = 0$ , our utility risk problem will reduce to a canonical utility maximization, this topic will be discussed in the next subsection. If  $\gamma \rightarrow \infty$ , then we put all of the focus on the deviation risk, so the solution will tend to riskless investment. Hence, in Table 3.3, we observe that the utility will tend to  $U(e^{rT}x_0) = 2e^{\frac{rT}{2}}$ , which is the utility of terminal payoff under riskless strategy. Meanwhile, the deviation risk and the optimal control will tend to 0.

Table 3.3: The Convergence of Optimal Solution for Utility-Risk Problems to Riskless Solution

The Case of Utility-Variance			
$\gamma$	$\pi^*(0)$	Utility	Variance
100	$3.145 \times 10^{-2}$	2.2677	$6.341 \times 10^{-5}$
10000	$3.174 \times 10^{-4}$	2.2551	$6.458 \times 10^{-9}$
1000000	$3.174 \times 10^{-6}$	2.2550	$6.459 \times 10^{-13}$
$\infty$ (riskless)	0	2.2550	0
The Case of Utility-Semivariance			
$\gamma$	$\pi^*(0)$	Utility	Semivariance
100	$3.376 \times 10^{-2}$	2.2737	$9.186 \times 10^{-5}$
10000	$3.305 \times 10^{-4}$	2.2552	$1.007 \times 10^{-8}$
1000000	$3.243 \times 10^{-6}$	2.2550	$1.026 \times 10^{-12}$
$\infty$ (riskless)	0	2.2550	0

Table 3.4: The Convergence of Optimal Solution from Utility-Variance Problem to Solely Utility Maximization

$\gamma$	$\pi^*(0)$	Utility	Utility loss	Variance	Risk reduced
0	3.5119	3.6829		1681.1068	
0.0001	3.3407	3.6692	0.37%	219.8164	86.92%
0.001	3.0599	3.6181	1.76%	62.9261	96.26%
0.01	2.5704	3.4663	5.88%	12.9751	99.23%
0.1	1.8667	3.1502	14.47%	1.8448	99.89%

### 3.4.3 Comparison between Utility-Risk Optimization and Solely Utility Maximization

When  $\gamma = 0$ , our utility risk problem will reduce to a canonical solely utility maximization. The optimal terminal wealth of the utility maximization is  $X = \frac{1}{\mathbb{E}[\frac{1}{\xi}]e^{\xi^2}}$ . Furthermore, under the continuous-time Black-Scholes model,  $X = e^{rT + \frac{\alpha^2}{\sigma^2}T + 2\frac{\alpha}{\sigma}W_T}$ , then the corresponding expected utility and variance in diffusion model will become  $2e^{\frac{r}{2}T + \frac{\alpha^2}{2\sigma^2}T}$  and  $\frac{1}{2}e^{2rT + 4\frac{\alpha^2}{\sigma^2}T} \left( e^{4\frac{\alpha^2}{\sigma^2}T} - 1 \right)$  respectively.

As  $\gamma \rightarrow 0$ , the expected utility in utility-risk problem will converge to the expected utility in solely utility maximization (we set  $N = 100$ ) in Tables 3.4 and 3.5.

From Tables 4a and 4b, we observe that adding a risk term to utility maximization problem can help to reduce the variance risk and semivariance risk. For example, in Table 4a, adding variance with  $\gamma = 0.01$  can reduce 99% of variance in simple utility maximization with less than 6% utility loss as expense. In Table 4b, the additional semivariance term can reduce more 90% of semivariance in simple utility maximization with less than 10% utility loss as expense.

We also notice that there are limit points in the efficient frontiers when  $\sigma = 0.4$

Table 3.5: The Convergence of Optimal Solution from Utility-Semivariance Problem to Solely Utility Maximization

The Case of Utility-Semivariance					
$\gamma$	$\pi^*(0)$	Utility	Utility loss	Semivariance	Risk reduced
0	3.5119	3.6829		23.6906	
0.0001	3.5011	3.6829	0.0007%	23.1534	2.27%
0.001	3.4203	3.6811	0.05%	19.6849	16.91%
0.01	3.0437	3.6403	1.16%	9.8944	58.23%
0.1	2.2091	3.3867	8.04%	2.1813	90.79%

in Figure 3.1, it is because the efficient frontiers reach the points representing canonical utility maximizations. In contrast to mean-variance case that the efficient frontier will go to infinity when we set  $\gamma \rightarrow 0$ , the expected utility in utility-risk problem is bounded above by the expected utility in simple utility maximization, thus there is always a limit point in the utility-risk efficient frontier, where the limit point represents the case of  $\gamma = 0$ , i.e. solely utility maximization. In the case in Figure 3.1(a), it becomes (3.443, 2.548).

### 3.4.4 Comparison between Variance and Semivariance

In this subsection, we compare the ratio of downside semivariance to upside semivariance ( $\frac{\mathbb{E}[(\mathbb{E}[X]-X)_+]^2}{\mathbb{E}[(\mathbb{E}[X]-X)_-]^2}$ ) and the corresponding trading strategy given a fixed utility target between utility-variance and utility-semivariance investors.

In Figures 3.3 (a), utility-semivariance investor has smaller the ratio of downside semivariance to upside semivariance than utility-variance investor because the former investor specially focus on reducing downside semivariance. Two curves converge as the utility target increase, since risk aversion toward the deviation

risks decrease and will tend to zero when the utility target reaches the optimal utility value for solely utility maximization. In Figure 3.3 (b), utility-semivariance investor has more conservative investment than utility-variance investor under the same utility target.

## 3.5 Conclusion

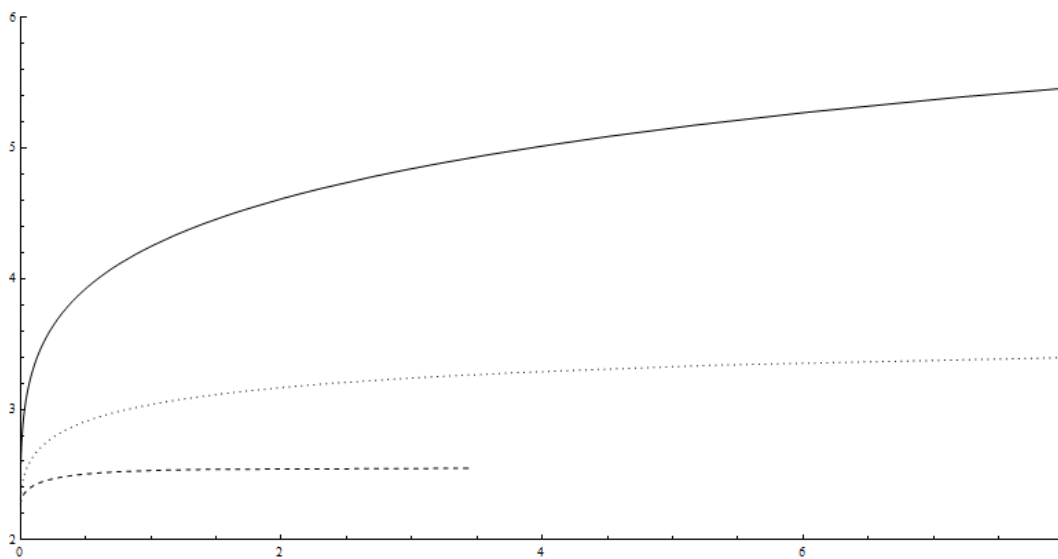
In this chapter, we considered dynamic utility-deviation-risk portfolio selection under a generalized model setting where the dynamics of asset prices can be unspecified. Our generalized model can cover the discrete binomial tree model, the continuous-time Black-Scholes model and stochastic interest rate model.

Under the complete market assumption in Assumption 3.1.7, we first converted our dynamic optimization problem into an equivalent static problem by Theorem 3.1.9. We further derived Nonlinear Moment Problem, which includes a equation involving terminal pricing kernel described in Assumption 3.1.6 and three equality constraints on nonlinear moments, to characterize the optimal terminal wealth for utility-risk problem. The corresponding necessary and sufficient optimality theorems related to the Nonlinear Moment Problem were given in Theorems 3.2.3 and 3.2.5 respectively. Under the satisfaction of the Inada Conditions, we established the existence of optimal solutions for utility-downside-risk problems, and utility-strictly-convex-risk problems in Theorems 3.2.8 and 3.2.14 respectively under our generalized framework. The existence and uniqueness of the optimal solution for utility-risk problem have been resolved.

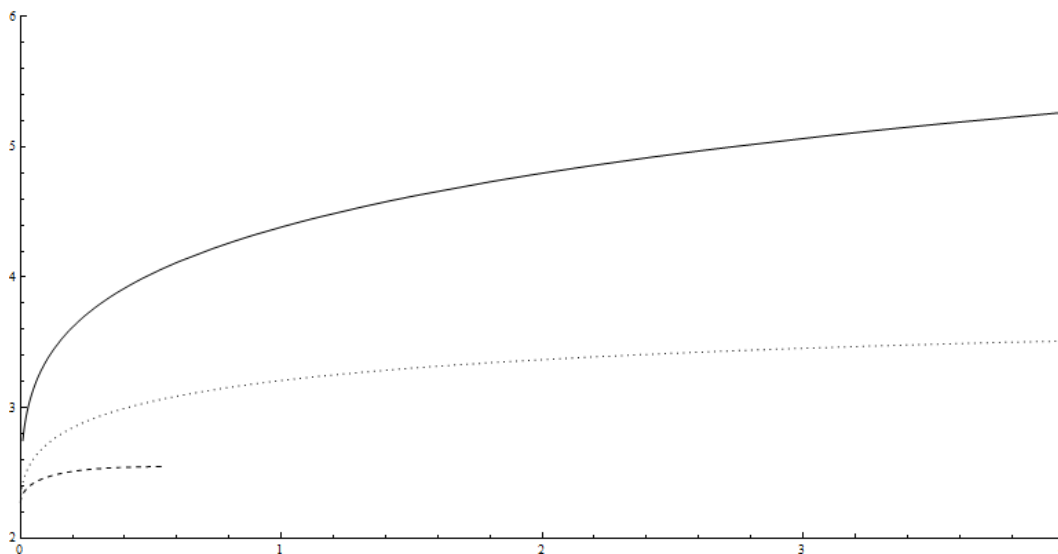
In Theorems 3.3.1 and 3.3.6, we established the continuity of optimal terminal payoff in terminal pricing kernel in the sense that the sequence of optimal terminal payoffs converges weakly as the terminal pricing kernels. The limit of such sequence of payoffs is the terminal payoff under a market with the limit of terminal pricing kernel. These convergence results were then applied to establish

the weak convergence of optimal terminal payoffs from the discrete binomial tree model to the continuous-time Black-Scholes model in Theorems 3.3.8 and 3.3.13. As the result, we have a numerical algorithm to compute the optimal solution for the continuous-time utility-risk problem numerically. In numerical examples in Section 3.4, we observe that, after adding a risk management term such variance or semivariance of the terminal payoff, the deviation risk incurred in the case of solely utility maximization can be reduced by more than 90% with less than 10% loss in utility as a trade-off.



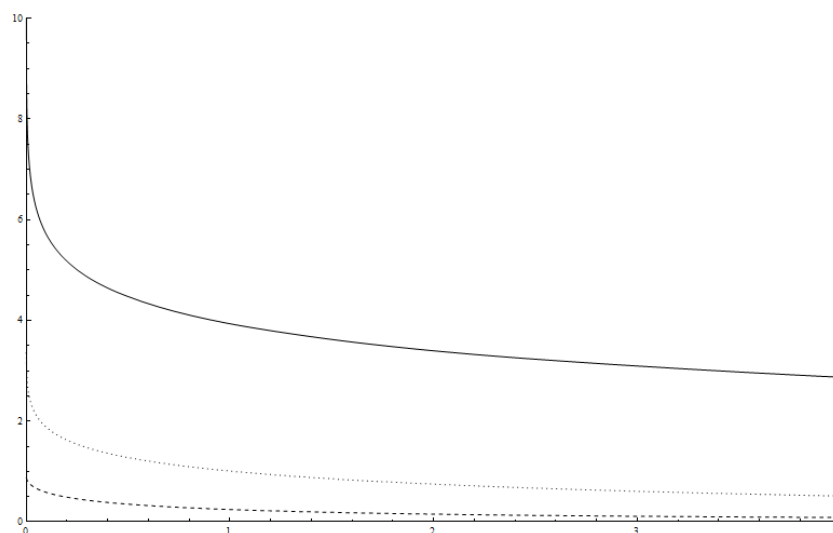


(a)

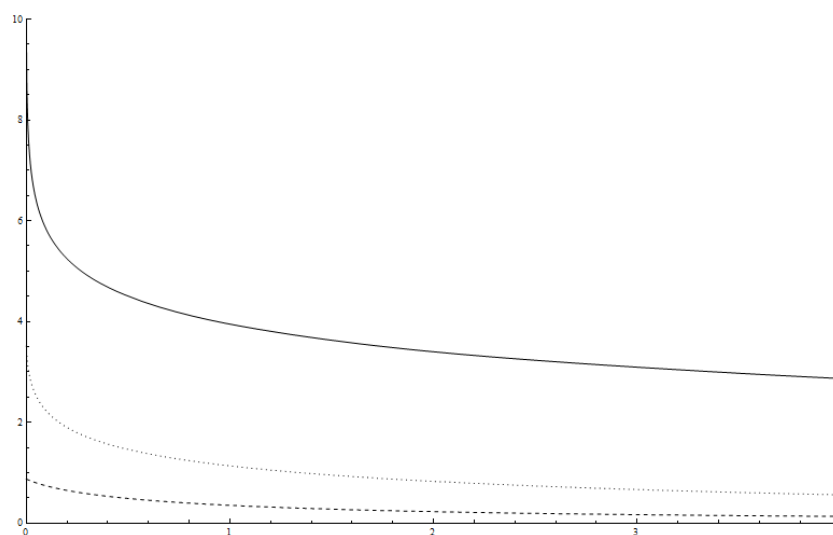


(b)

Figure 3.1: Utility-Risk Efficient Frontiers. (a) The comparison of utility-variance efficient frontiers between different  $\sigma$ , (b) The comparison of utility-semivariance efficient frontiers between different  $\sigma$ . Horizontal-axis represents variance in (a) and semivariance in (b). Vertical-axis represents utility. Solid lines represent  $\sigma = 0.1$ , dotted line represents  $\sigma = 0.2$ , and dashed lines represent  $\sigma = 0.4$ .

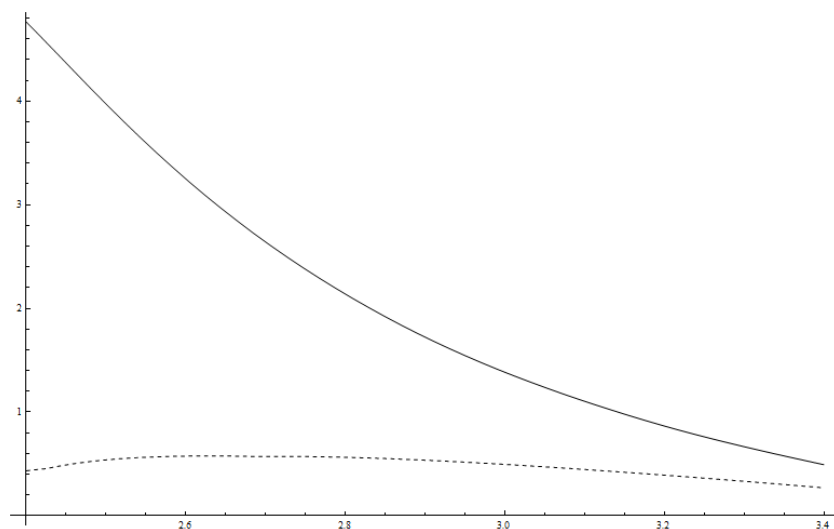


(a)

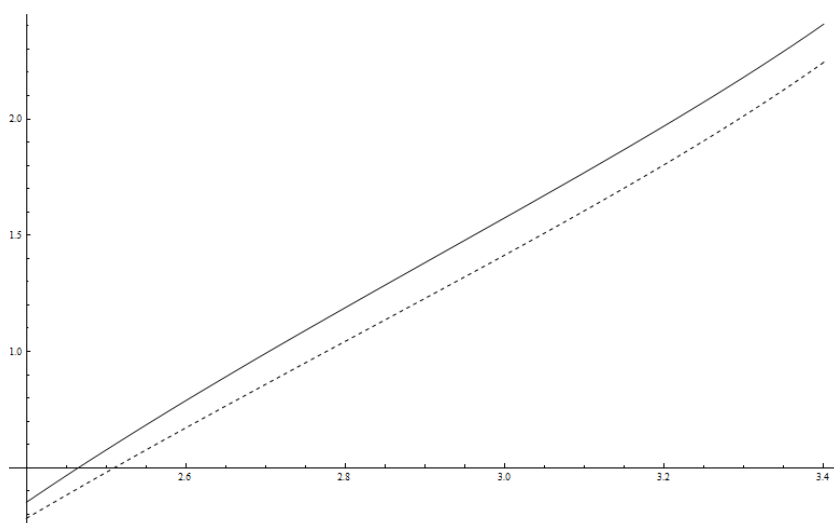


(b)

Figure 3.2: Optimal Utility-Risk Portfolio. (a) The comparison of optimal control at  $t = 0$ ,  $\pi^*(0)$ , for utility-variance problem against risk aversion,  $\gamma$ , between different  $\sigma$ . (b) The comparison of optimal control at  $t = 0$  for utility-semivariance problem against  $\gamma$  between different  $\sigma$ . Horizontal-axis represents  $\gamma$ . Vertical-axis represents  $\pi^*(0)$ . Solid lines represent  $\sigma = 0.1$ , dotted lines represent  $\sigma = 0.2$ , and dashed lines represent  $\sigma = 0.4$ .



(a)



(b)

Figure 3.3: (a) The comparison of the ratio of downside semivariance to upside semivariance,  $\frac{\mathbb{E}[(\mathbb{E}[X]-X)_+^2]}{\mathbb{E}[(\mathbb{E}[X]-X)_-^2]}$ , against a given utility target,  $2E[X^{1/2}]$ , between different deviation risk functions. (b) The comparison of optimal control at  $t = 0$ ,  $\pi^*(0)$ , against a given utility target,  $2E[X^{1/2}]$ , between different risk functions. Horizontal-axis represents  $2E[X^{1/2}]$ . Vertical-axis represents  $\frac{\mathbb{E}[(\mathbb{E}[X]-X)_+^2]}{\mathbb{E}[(\mathbb{E}[X]-X)_-^2]}$  in (a) and  $\pi^*(0)$  in (b). Solid lines represent the case of variance and dashed lines represent the case of semivariance.

# Chapter 4

## A Paradox in Time Consistency in Mean-Variance Problem?

In Section 4.1.1, we shall introduce the market framework and the investors' preference. The three solution approaches to tackle time inconsistent problems are described and compared using the classical mean-variance problem in Section 4.1.2. In Section 4.1.3, we shall provide an overview of the theoretical results of time-consistent mean-variance optimization under state-dependent risk-aversion in Bensoussan et al. (2014) and Björk et al. (2014) and the non-intuitive numerical results in Bensoussan et al. (2014). In Section 4.2, we shall first list out the sufficient conditions together with their economic implications, under which the mentioned obscure phenomena will appear as shown in the main theorems stated at the end of the section; namely, one can start off his/her constrained equilibrium strategy at a certain time to beat the unconstrained counterpart (Theorem 4.2.4), even more, the pure strategy of solely investing in bond can sometimes simultaneously dominate both constrained and unconstrained equilibrium strategies (Theorems 4.2.5 and 4.2.6). In Section 4.3, we shall establish the main results. Further numerical illustrations will be provided in Section 4.4; and we observe that the constrained time-consistent strategy dominates the unconstrained one

for over 90% of the whole time horizon. We finally conclude in Section 4.5.

## 4.1 Model Setting and Time Consistency

### 4.1.1 Market Model and Investors' Preference

In this chapter, we adopt the continuous time framework as in Section 5 of Bensoussan et al. (2014). We fix a finite terminal time  $T > 0$ . Given the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{E}$  denotes the expectation with respect to  $\mathbb{P}$ , and  $W_t$  denotes the standard one-dimensional  $\mathbb{P}$ -Brownian motion which then generates a natural filtration  $\mathcal{F}_t := \sigma(W_t : s \leq t) \vee \mathcal{N}_0$  on  $\mathcal{F}$ , where  $\mathcal{N}_0$  is  $\mathbb{P}$ -null set. We assume that there are one bond and a single stock in the market. The dynamics of bond and stock are given by:

$$\begin{aligned} dB_t &= r_t B_t dt, & B_0 &= b_0; \\ dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t, & S_0 &= s_0, \end{aligned}$$

where  $r_t$  is the riskless return rate,  $\mu_t$  and  $\sigma_t$  are the appreciation and the volatility rates of the stock respectively. Also assume that  $\alpha_t := \mu_t - r_t > 0$ . All market parameters  $r_t, \alpha_t$  and  $\sigma_t$  are time dependent, deterministic, differentiable, and uniformly bounded on  $\mathbb{R}$ , i.e.  $0 < \underline{r} \leq r_t \leq \bar{r} < \infty$ ,  $0 < \underline{\alpha} \leq \alpha_t \leq \bar{\alpha} < \infty$ , and  $0 < \underline{\sigma} \leq \sigma_t \leq \bar{\sigma} < \infty$  for all  $t \in \mathbb{R}$ . Let  $u_t$  (the admissible control) be the amount of money invested in the stock at time  $t$ . The dynamics of the controlled wealth process is:

$$dX_t^u = (r_t X_t^u + \alpha_t u_t) dt + \sigma_t u_t dW_t, \quad X_0^u = x_0. \quad (4.1.1)$$

We now consider two different investors: unconstrained and constrained investors. The former one can shortsell both bond and stock; while the latter one cannot. Their common objective function at the commencement  $t \in \mathbb{R}$  with the current wealth  $x > 0$  is:

$$J(t, x; u) := \mathbb{E}_{t,x}[X_T^u] - \frac{\gamma_t}{2x} \text{Var}_{t,x}[X_T^u], \quad (4.1.2)$$

where  $\mathbb{E}_{t,x}$  and  $Var_{t,x}$  denote the conditional expectation and conditional variance on the event  $\{X_t^u = x\}$ , and the risk aversion coefficient  $\gamma_t$  is assumed to be positive, increasing, and differentiable in  $t \in \mathbb{R}$ . Note that it is reasonable to assume an increasing  $\gamma_t$  since people usually look for a more stable income when they are getting elder, so they become more risk-averse, relative to the current wealth, on their own terminal payoffs.

Furthermore, in this chapter, the admissible controls are confined to be Markovian:  $u_t = u(t, X_t)$ , i.e. the admissible control is a function, which is a feedback one, in both the current time  $t$  and the current wealth  $X_t$  only. The sole difference in the setting between unconstrained and constrained investors is the set of admissible controls: the risky investment strategy of the unconstrained investor,  $u^{(U)}(t, x)$ , can be chosen on  $\mathbb{R}$ ; while that of the constrained investor,  $u^{(C)}$ , is confined by the shortselling prohibition, so that  $0 \leq u^{(C)}(t, x) \leq x$ . More precisely, the collection of the admissible controls of the unconstrained and constrained investors,  $\mathcal{A}^{(U)}$  and  $\mathcal{A}^{(C)}$ , are defined respectively as follows:

$$\begin{aligned} \mathcal{A}^{(U)} &:= \left\{ u^{(U)} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R} \mid u^{(U)}(t, x) \in \mathbb{R} \right\}; \\ \mathcal{A}^{(C)} &:= \left\{ u^{(C)} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R} \mid 0 \leq u^{(C)}(t, x) \leq x \right\}. \end{aligned}$$

In summary, an unconstrained investor looks for an admissible strategy  $u_t \in \mathcal{A}^{(U)}$  to maximize the objective function (4.1.2), where the portfolio wealth is generated by dynamics (4.1.1). A constrained investor concerned a similar portfolio management problem as the unconstrained investor except that the class of admissible strategies is instead restricted to  $\mathcal{A}^{(C)}$ .

The optimization problems of unconstrained and constrained investors are both time-inconsistent (which will be elaborated in Section 4.1.2), and we seek for time-consistent solutions for these optimization problems in Problems 4.1.7 and 4.1.9 in Section 4.1.3 respectively.

### 4.1.2 Time Consistency and Mean-Variance Problems

In this section, we shall describe three solution approaches for time inconsistent problems, namely: (i) precommitment; (ii) spendthrift; and (iii) game-theoretic. And as an illustration on their usefulness, we shall compare them through the celebrated classical mean-variance model.

#### (i) Precommitment Approach

For the precommitment approach, we fix an initial state and then find a maximizer for the objective function at that initial state; the maximizer is called precommitment solution. Under this approach, the maximizer may depend on initial states, then it may not be optimal for the objective function at any future state. Hence, the solution obtained through precommitment approach is still time-inconsistent.

**Definition 4.1.1.** Given a continuous-time Markovian control problem with an objective function  $J(t, x; u)$  and an admissible control set  $\mathcal{A}$ ,  $\bar{u}^{t_0, x_0} \in \mathcal{A}$  is said to be a precommitment strategy at  $(t_0, x_0)$  if

$$\bar{u}^{t_0, x_0} := \arg \max_{u \in \mathcal{A}} J(t_0, x_0; u).$$

**Example 4.1.2** (Time-Inconsistent Mean-Variance Problem, Li and Zhou (2000)).

Consider the following maximization problem:

$$\max_{u \in \mathcal{A}(U)} J_0(t_0, x_0; u), \text{ where } J_0(t, x; u) := \mathbb{E}_{t,x}[X_T^u] - \gamma \text{Var}_{t,x}[X_T^u], \quad (4.1.3)$$

subject to (4.1.1).

The precommitment solution, that maximize  $J_0(t_0, x_0; u)$ ,

$$\bar{u}^{t_0, x_0}(t, x) = -\frac{\alpha_t}{\sigma_t^2} \left( x - e^{\int_{t_0}^t r_s ds} \left[ x_0 + \frac{1}{2\gamma} e^{-\int_{t_0}^T \left( r_s - \frac{\alpha_s^2}{\sigma_s^2} \right) ds} \right] \right). \quad (4.1.4)$$

Since the maximizer  $\bar{u}^{t_0, x_0}$  for mean-variance problem in Example 4.1.2 **depends** on the initial state  $(t_0, x_0)$ , so this problem is **time-inconsistent**. To be

frank, The optimality of the precommitment solution only makes sense at only one moment - the initial time. For instance, we know that  $\bar{u}^{t_0, x_0}$  is optimal for  $J(t_0, x_0; u)$ , but  $\bar{u}^{t_0, x_0}$  may not be optimal for  $J(t_1, X_{t_1}^{\bar{u}^{t_0, x_0}}; u)$  whenever  $t_1 > t_0$ , which should be maximized by  $\bar{u}^{t_1, X_{t_1}^{\bar{u}^{t_0, x_0}}}$  instead.

(ii) *Spendthrift Approach*

At  $(t_1, x_1)$ , if an individual re-evaluate his plan and he has no commitment on the plan, it should be rational for him to give up  $\bar{u}^{t_0, x_0}$  and adopt  $\bar{u}^{t_1, x_1}$ . If the plan is re-evaluated continuously, any single plan  $\bar{u}^{t_0, x_0}$  chosen can only have validity at  $t_0$ , and his actual strategy becomes  $\tilde{u}(t, x) := \bar{u}^{t, x}(t, x)$  for  $(t, x) \in \mathbb{R}^2$ . In this case, the agent keeps changing his strategy to the currently optimal one for every time point, Strotz (1955) termed this behavior spendthrift and Pedersen and Peskir (2015) termed the corresponding strategy dynamically optimal strategy:

**Definition 4.1.3.** Given a continuous-time Markovian control problem with an objective function  $J(t, x; u)$  and an admissible control set  $\mathcal{A}$ ,  $\tilde{u} \in \mathcal{A}$  is said to be a dynamically optimal strategy if

$$\tilde{u}(t, x) := \bar{u}^{t, x}(t, x) \text{ for } (t, x) \in \mathbb{R}^2,$$

where

$$\bar{u}^{t, x} := \arg \max_{u \in \mathcal{A}} J(t, x; u), \text{ for each } (t, x) \in \mathbb{R}^2$$

Under the spendthrift approach, the individual can apparently maintain his objective value to be the maximum one,  $J(t, x; \bar{u}^{t, x})$ , over the whole time horizon; however, it can actually never be the case. Indeed, the maximized objective function value can only be achieved by the investor who commits to adopt the corresponding maximizing strategy **without revising** it at **all future time points**. Hence, the investor who keeps changing his strategy to the currently optimal one can never achieve the maximum objective value as specified at an earlier time. For instance,  $J(t_0, x_0; \bar{u}^{t_0, x_0})$  is attained by the investor who adopts  $\bar{u}^{t_0, x_0}(t, x)$  for



all  $t \in [t_0, T]$  and  $x \in \mathbb{R}$ , but can never be attained by the investor who adopts  $\tilde{u}(t, x)$  for  $t \in [t_0, T]$  and  $x \in \mathbb{R}$ .

Nevertheless, the solution from the spendthrift approach is meaningful in some aspects. First, as mentioned in Strotz (1955), this spendthrift strategy is adopted by any individual who cannot be aware of the time-inconsistency matter or has no intention on resolving this inconsistency issue. Thus, this myopic individual always revises his strategy to satisfy his contemporary desire. Indeed,  $\tilde{u}$  is **independent of the initial states**, so it is **time-consistent**. In particular, Pedersen and Peskir (2015) treated  $\tilde{u}$  as a time-consistent strategy:

**Example 4.1.4** (Time-Consistent Mean-Variance Problem (Spendthrift), Pedersen and Peskir (2015)). Consider the following portfolio selection problem:

$$\max_{u \in \mathcal{A}^{(U)}} J_0(t_0, x_0; u), \text{ where } J_0(t, x; u) \text{ is defined in (4.1.3), subject to (4.1.1),}$$

the dynamically optimal solution (under Definition 4.1.3 with objective function  $J_0$  and admissible set  $\mathcal{A}^{(U)}$ ) is:

$$\tilde{u}(t, x) = \frac{\alpha_t}{2\gamma\sigma_t^2} e^{-\int_t^T \left(r_s - \frac{\alpha_s^2}{\sigma_s^2}\right) ds}. \quad (4.1.5)$$

The solution under the spendthrift notion  $\tilde{u}$  can be obtained from the precommitment solution  $\bar{u}^{t_0, x_0}$  by replacing the initial states  $(t_0, x_0)$  by the current states  $(t, x)$ . Therefore, the solutions under precommitment and spendthrift can be obtained by the same mathematical arguments. The difference between the two notions is how to implement the mathematical solution: whether or not the investor will revise his strategy during the re-evaluation in some future time points.

As the mathematical context for spendthrift strategy is essentially the same as the precommitment strategy, the spendthrift strategy is mostly considered as the strategy adopted by the investor who cannot be aware the time inconsistency issues, the spendthrift strategy is perceived to be inferior and its effect on actual behavior in market trading is usually overlooked which is only partially covered in

the literature. The spendthrift strategy was mainly studied in order to compare with other strategies, such as precommitment and equilibrium strategies, under various time inconsistent problems: Pollak (1968) for Ramsey's consumption and saving problem, Marín-Solano and Navas (2010) for Merton's consumption and investment problem, and Pedersen and Peskir (2015) for Mean-variance problem.

(iii) *Game-theoretic Approach*

Alternatively, game-theoretic approach is more widely used to recommend a time-consistent solution. This approach purposely seeks for a solution that one will consistently follow but it is not the ultimate maximizer. Under this approach, the portfolio selection problem is converted into a non-cooperative intertemporal game, in which every time point in the time horizon is represented by the corresponding time player. The time player chooses his strategy, which becomes the portfolio allocation at the time point he representing, to maximize the objective function at this time point. The strategies by all time players form a solution path for the original dynamic optimization problem. After formulating the game, the Nash equilibrium of the intertemporal game is then obtained and utilized to be a time-consistent solution, called equilibrium solution, for the portfolio selection problem (see Table 4.1). At the equilibrium point, all the time players have no incentive to choose the strategy other than  $\hat{u}$  given that all later time players have chosen their own equilibrium ones. Hence, the equilibrium control is a solution that we will consistently follow over time and thus time-consistent.

The following definition of equilibrium control is provided by Ekeland and Pirvu (2008) as a time-consistent solution for continuous-time control problem:

**Definition 4.1.5** (Ekeland and Pirvu (2008)). Given a continuous-time Markovian control problem with an objective function  $J(t, x; u)$  and an admissible control set  $\mathcal{A}$ , a Markovian control  $\hat{u} \in \mathcal{A}$  is said to be an equilibrium control if for

Non-cooperative game	Dynamic optimization
Time player $t$	Time point $t$
Objective function of Player $t$	Objective function at $t$ , $J(t, x; u)$
Strategy by Player $t$	Portfolio at $t$ , $u_t \in \mathbb{R}$
Strategies by all players	Solution path over time horizon, $u : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$
Nash equilibrium	Time-consistent solution

Table 4.1: The Construction of Non-cooperative Game Problem under Game-theoretic Approach

every admissible  $u \in \mathcal{A}$ ,

$$\liminf_{h \rightarrow 0^+} \frac{J(t, x; \hat{u}) - J(t, x; u_h)}{h} \geq 0, \text{ for any } t < T \text{ and } x \in \mathbb{R}, \quad (4.1.6)$$

where  $u_h$  is given by

$$u_h(s, x) := \begin{cases} u(s, x), & \text{for } t \leq s < t + h, x > 0; \\ \hat{u}(s, x), & \text{for } t + h \leq s \leq T, x > 0. \end{cases}$$

With the equilibrium control, we can further define the equilibrium value function,  $V(t, x) := J(t, x; \hat{u})$ , attained at  $\hat{u}$ .

To solve a dynamic optimization problem using game-theoretic approach, we look for an admissible solution satisfying Definition 4.1.5. This precise definition allows us to find a time-consistent solution for dynamic decision problems through modeling the equivalent stochastic control problems. To elaborate Definition 4.1.5 using the language of game theory, we consider our dynamic optimization problem as a non-cooperative game problem with a continuum of players: For each  $t \in [0, T]$ , there is a time player, player  $t$ , who chooses a strategy  $u(t)$  (only at  $t$ ; the state variable is omitted) to maximize his objective function  $J(t; u)$  which depends not only on  $u(t)$ , the strategy chosen by player  $t$ , but also on all  $u(s)$  with  $s \geq t$ , the strategies chosen by player  $s$ . Following the notion of equilibrium

control in Definition 4.1.5, provided that all players  $s > t$  have already chosen  $\hat{u}$ , it is optimal for the player  $t$  to also choose  $\hat{u}$ . Based on this game with a continuum of players, the time-consistent strategy of the original dynamic optimization can be defined using the concept of a “subgame perfect Nash equilibrium point”. For further motivations of Definition 4.1.5, one can consult the works by Peleg and Yaari (1973), Ekeland and Pirvu (2008), and Björk and Murgoci (2010).

For the application of time-consistent solution approach to mean-variance setting, Basak and Chabakauri (2010) and Björk and Murgoci (2010) are the first to obtain the equilibrium solution:

**Example 4.1.6** (Time-Consistent Mean-Variance Problem (Equilibrium Solution), Basak and Chabakauri (2010) and Björk and Murgoci (2010)). Consider the following portfolio selection problem:

$$\max_{u \in \mathcal{A}^{(U)}} J_0(t_0, x_0; u), \text{ where } J_0(t, x; u) \text{ is defined in (4.1.3), subject to (4.1.1),}$$

the equilibrium solution (under Definition 4.1.5 with objective function  $J_0$  and admissible set  $\mathcal{A}^{(U)}$ ) is:

$$\hat{u}(t, x) = \frac{\alpha_t}{2\gamma\sigma_t^2} e^{-\int_t^T r_s ds}. \quad (4.1.7)$$

Although the solutions obtained from both spendthrift and game theoretic approaches are time-consistent, they are completely different. The value functions of dynamically optimal and equilibrium strategies are established respectively in Pedersen and Peskir (2015) and Björk and Murgoci (2010) as follows:

$$\begin{aligned} \tilde{V}(t, x) := J(t, x; \tilde{u}) &= e^{\int_t^T r_s ds} x + \frac{1}{2\gamma} \left( e^{\int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds} - \frac{1}{4} e^{2\int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds} - \frac{3}{4} \right), \\ \hat{V}(t, x) := J(t, x; \hat{u}) &= e^{\int_t^T r_s ds} x + \frac{1}{4\gamma} \int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds, \end{aligned}$$

where  $\tilde{u}$  and  $\hat{u}$  are given in (4.1.5) and (4.1.7) respectively.

By an immediate application of the mean-value theorem, the value function evaluated at the equilibrium strategy is actually greater than that at dynamically

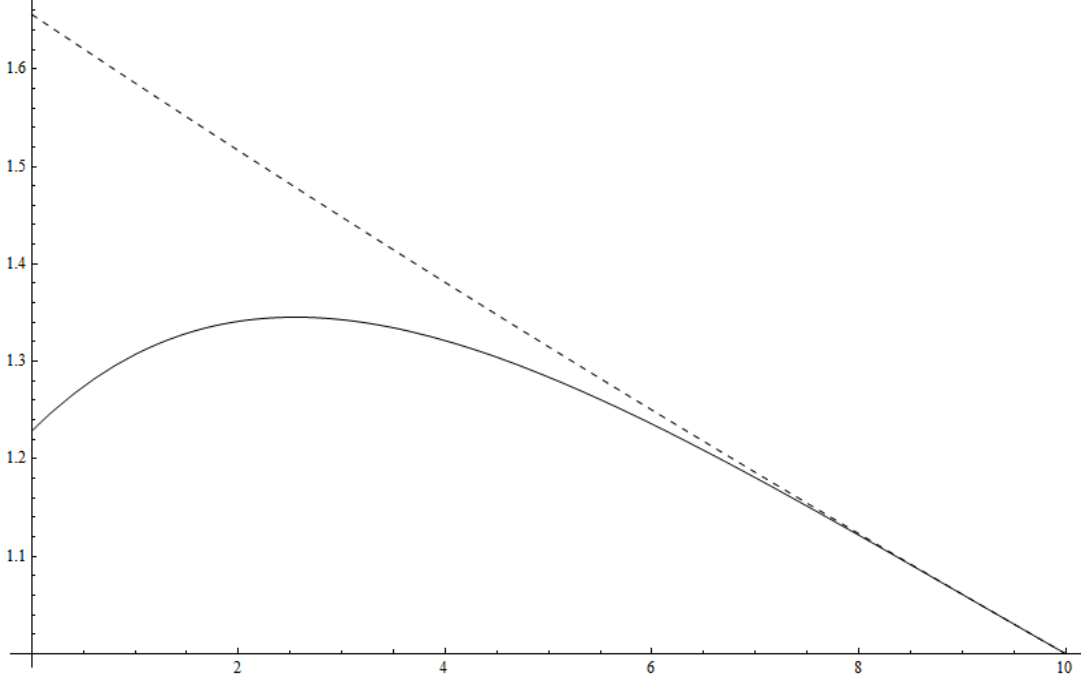


Figure 4.1: The value functions of dynamically optimal solution,  $\tilde{V}(t, 1)$  (solid line) and equilibrium solution,  $\hat{V}(t, 1)$  (dashed line), against the current time  $t$ .

optimal strategy:

$$\begin{aligned} \hat{V}(t, x) - \tilde{V}(t, x) &= \frac{1}{2\gamma} \left( \frac{1}{2} \int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds - e^{\int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds} + \frac{1}{4} e^{2 \int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds} + \frac{3}{4} \right) \\ &= \frac{1}{2\gamma} \left( \int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds \right) \left( \frac{1}{2} - e^\eta + \frac{1}{2} e^{2\eta} \right) > 0, \end{aligned}$$

for some  $\eta \in \left(0, \int_t^T \frac{\alpha_s^2}{\sigma_s^2} ds\right)$ . A numerical illustration is shown in Figure 4.1 with  $r_t = 0.03$ ,  $\mu_t = 0.1$ , and  $\sigma_t = 0.2$  for all  $t \in [0, 10]$  and  $\gamma = 1$ ,  $x = 1$ , and  $T = 10$ , where the value functions of dynamically optimal and equilibrium solutions,  $\tilde{V}(t, x)$  and  $\hat{V}(t, x)$ , are shown. The dynamically optimal solution has a greater expected value, as this spendthrift solution actually always suggests to invest more in risky asset than that using the equilibrium solution. However, in the meanwhile, this more aggressive strategy will also give a greater variance in magnitude than that under the equilibrium strategy. In sum, the value function of the dynamically optimal solution, which the investor concerns, will be smaller than that using

equilibrium solution.

To the best of our knowledge, the comparative study on the performance between equilibrium and dynamically optimal strategies has been up to specific time-inconsistent problems. We here illustrate that difference in performance through the classical mean-variance models; nevertheless, we still look forward to the further similar analysis over the general time-inconsistent settings, and up to this point, we cannot completely claim on which one could be uniformly better than another. From our viewpoint, the importance of spendthrift approach is not to provide a strategy which gives out a better objective value, but to mimic the actual behavior in the market. Most practitioners are not aware about the time inconsistency issues, so they usually behave as spendthrift strategy. The inferior performance of the spendthrift strategy may explain why the managers in the market perform weaker than widely expected, because they are incapable of refraining from their deeply rooted inconsistent and spendthrift habit. Therefore, the study on spendthrift strategy is subtle. It may explain the actual behavioral bias of practitioners and helps us to look for superior strategy that transcends against the “market”.

In this chapter, we shall concern on the equilibrium solution from the game-theoretic approach only and treat it as the time-consistent solution. In the next subsection, we shall introduce the motivation behind our observed paradoxical results that appear in (time-consistent) equilibrium approach.

### **4.1.3 Motivation: Time-Consistent Mean-Variance Optimization with Wealth-Dependent Risk Aversion**

Note that, when the risk aversion stays constant over the whole time horizon, the equilibrium solution for the classical mean-variance problem as shown in Example 4.1.6 is state-independent. Later, Björk et al. (2014) discovered that, under a more realistic mean-variance framework under which the risk aversion is inversely

proportional to the current wealth, the equilibrium solution varies linearly with the current wealth, and hence the solution is state-dependent which seems more economically sounding. More precisely, Björk et al. (2014) studied the following time-consistent optimization problem:

**Problem 4.1.7** (Unconstrained investor's problem, Björk et al. (2014)). Find an equilibrium control  $\hat{u}^{(U)} \in \mathcal{A}^{(U)}$  according to Definition 4.1.5, where the objective function  $J$  is given by (4.1.2), and the portfolio wealth is generated by dynamics (4.1.1), and the admissible class is given by  $\mathcal{A}^{(U)}$ .

Björk et al. (2014) characterized the equilibrium solution by the extended HJB equations systems, and then, by using a suitable *Ansatz*, a semi-explicit form of the equilibrium solution could be obtained:

**Theorem 4.1.8** (Theorem 4.6 in Björk et al. (2014)). *The equilibrium solution to Problem 4.1.7 is given by  $\hat{u}^{(U)}(t, x) = c_t^{(U)} x$ , where  $c_t^{(U)}$  satisfies the following integral equation:*

$$\begin{cases} c_t^{(U)} = \frac{\alpha_t}{\sigma_t^2} d_t^{(U)}; \\ d_t^{(U)} = \frac{1}{\gamma t} e^{-\int_t^T (r_s + \alpha_s c_s^{(U)} + \sigma_s^2 (c_s^{(U)})^2) ds} + e^{-\int_t^T \sigma_s^2 (c_s^{(U)})^2 ds} - 1, \end{cases} \quad (4.1.8)$$

The equilibrium value function is given by

$$V^{(U)}(t, x) = e^{\int_t^T (r_s + \alpha_s c_s^{(U)}) ds} x - \frac{\gamma t}{2} \left( e^{\int_t^T 2(r_s + \alpha_s c_s^{(U)} + \sigma_s^2 (c_s^{(U)})^2) ds} - e^{\int_t^T 2(r_s + \alpha_s c_s^{(U)}) ds} \right) x. \quad (4.1.9)$$

Since there is no shortselling restrictions in Problem 4.1.7, the equilibrium control  $\hat{u}^{(C)}$  in Theorem 4.1.8 can take value outside the range of  $[0, x]$ , i.e. the investor can shortsell when he implements the equilibrium strategy. However, the shortselling in discrete framework as in Björk and Murgoci (2014) can cause the wealth in the next period to take a non-positive value, making the mean-variance maximization problem with wealth-dependent risk version ill-posed, and

this observation was explained in detail in Bensoussan et al. (2014). Therefore, in Bensoussan et al. (2014), we looked for an equilibrium solution for mean-variance problem with wealth-dependent risk aversion subject to the shortselling prohibition on both stock and bond as recall as below again:

**Problem 4.1.9** (Constrained investor's problem, Bensoussan et al. (2014)). Find an equilibrium control  $\hat{u}^{(C)} \in \mathcal{A}^{(C)}$  according to Definition 4.1.5, where the objective function  $J$  is given by (4.1.2), where the portfolio wealth is generated by dynamics (4.1.1), and the admissible class is given by  $\mathcal{A}^{(C)}$ .

**Remark 4.1.10.** The objective functions in Problems 4.1.7 and 4.1.9 are the same, their only difference is the admissible set. With different admissible set, the equilibrium solution obtained are different. As we are not seeking for the ultimate maximizer, it is not necessary that the equilibrium solution with larger admissible set can achieve a greater objective value.

By solving the extended HJB system for the constrained Problem 4.1.9, Bensoussan et al. (2014) obtained an equilibrium solution:

**Theorem 4.1.11** (Theorem 6.1 in Bensoussan et al. (2014)). *The equilibrium solution to Problem 4.1.7 is given by  $\hat{u}^{(C)}(t, x) = c_t^{(C)}x$ , where  $c_t^{(C)}$  satisfies the following integral equation:*

$$\begin{cases} c_t^{(C)} = G\left(\frac{\alpha_t}{\sigma_t^2}d_t^{(C)}\right); \\ d_t^{(C)} = \frac{1}{\gamma t}e^{-\int_t^T (r_s + \alpha_s c_s^{(C)} + \sigma_s^2 (c_s^{(C)})^2) ds} + e^{-\int_t^T \sigma_s^2 (c_s^{(C)})^2 ds} - 1, \end{cases} \quad (4.1.10)$$

where  $G$  is a layer function defined as:

$$G(x) := \begin{cases} 1 & \text{if } x > 1, \\ x & \text{if } x \in [0, 1], \\ 0 & \text{if } x < 0. \end{cases}$$



Equilibrium objective value for different investors		
in different $T$ with $t = 0, x = 1$ .		
Investor	Unconstrained	Constrained
T=1	<b>1.2839</b>	1.2069
T=10	2.4730	<b>4.1623</b>
T=20	5.3994	<b>9.0910</b>
T=50	78.8514	<b>128.2294</b>

Table 4.2: The Counter-intuitive Numerical Observation in Bensoussan et al. (2014).

The equilibrium value function is given by

$$V^{(C)}(t, x) = e^{\int_t^T (r_s + \alpha_s c_s^{(C)}) ds} x - \frac{\gamma t}{2} \left( e^{\int_t^T 2(r_s + \alpha_s c_s^{(C)}) + \sigma_s^2 (c_s^{(C)})^2 ds} - e^{\int_t^T 2(r_s + \alpha_s c_s^{(C)}) ds} \right) x. \quad (4.1.11)$$

In the numerical studies in Bensoussan et al. (2014), we further compare the performances of equilibrium strategies adopted by unconstrained and constrained investors; their semi-explicit forms are those stated in Theorems 4.1.8 and 4.1.11. In Bensoussan et al. (2014), we fixed  $r = 0.05$ ,  $\mu = 0.2$ ,  $\sigma = 0.2$ , and

$$\gamma(t; T) := \frac{1}{1 + e^{-0.1(t-T)}},$$

we observed that the constrained investor can acquire a greater mean-variance objective value than that of the unconstrained investor as quoted in Table 4.2.

These numerical observation is surprising because whatever the constrained investor can do, it is supposed that the unconstrained investor can do the same. Therefore, we expect that the unconstrained investor should perform better than his constrained counterpart.

Motivated by our counter-intuitive numerical observation, in this chapter, we shall provide an analytical support to these non-intuitive numerical observations.

We investigate the economically meaningful conditions under which the same non-typical phenomenon reappear: the constrained investor dominates the unconstrained investor. An analytical comparison on the equilibrium mean-variance value functions for unconstrained and constrained investors will be demonstrated. Furthermore, we shall compare the performance of equilibrium strategies (for both constrained and unconstrained investors) with the pure strategy of solely investing in bond.

## 4.2 Main Results

We set the following conditions for our paradoxical main results :

**Condition 4.2.1.**

$$\frac{\alpha_T}{\sigma_T^2} > \gamma_T.$$

**Condition 4.2.2.** *There exists  $\delta_0 > 0$  such that*

$$\underline{r} - \frac{\gamma'_t}{\gamma_t} + \frac{\min \left\{ \left( \frac{\alpha_t}{\sigma_t^2} \right)', 0 \right\}}{\frac{\alpha_t}{\sigma_t^2}} > \delta_0 \quad \text{for all } t \in \mathbb{R}.$$

We further set a more relaxing condition replacing Condition 4.2.2:

**Condition 4.2.3.** *There exists  $\delta_1 > 0$  such that*

$$\underline{r} - \frac{\gamma'_t}{\gamma_t} > \delta_1 \quad \text{for all } t \in \mathbb{R}.$$

Obviously, for taking  $\delta_1 \geq \delta_0 > 0$ , Condition 4.2.2 implies Condition 4.2.3, and Condition 4.2.3 implies that  $\gamma_T e^{-(\underline{r}-\delta_1)(T-t)} \leq \gamma_t \leq \gamma_T$  for all  $t \leq T$ .

Assume that the riskfree rate  $r_t$ , the appreciation rate  $\mu_t$  and the volatility  $\sigma_t$  of the stock is constant over  $t$ . In Shiryaev et al. (2008) (also see Du Toit and Peskir, 2009), the ratio  $\frac{\alpha}{\sigma^2}$  is described as the “goodness index” of the stock, which justifies a particular stock on whether it should be sold-at-once or bought-and-hold. In particular, they showed that if the ratio  $\frac{\alpha}{\sigma^2} \geq \frac{1}{2}$ , the investor should

hold the stock until the expiry of the predetermined time horizon; otherwise, he should sell the stock at once (i.e. to invest solely in bond). The strategy based on such “goodness index” can maximize the expected ratio of the discounted stock selling price to the ultimate maximum over the planned time horizon. Hence, the ratio  $\frac{\alpha}{\sigma^2}$  can indicate the favorable performance or not of the stock over the bond. Our Condition 4.2.1 can be interpreted as comparing whether the terminal performance of the stock is beyond the investor’s own risk aversion  $\gamma_T$  at the expiry. Loosely speaking, it is expected that the validity of Condition 4.2.1 would normally encourage the investor to buy more stock as the outperformance of the stock may compensate his fear towards risk.

Condition 4.2.2 can be interpreted as comparing whether the sum of the relative decrease in “goodness index” and the relative increase in risk aversion coefficient is bounded above by the riskfree rate. When the market is going well, most investors have an optimistic anticipation on the continuing market appreciation, their risk aversion of the investor should expect to enjoy a more gentle progressive growth over time, while the “goodness index” of the stock against the bond should show a better performance in the future. In reality, in order to avoid any irrational exuberance in the security market during the rapid economic boom, it is common to set the riskfree rate high enough to slow down any over-investment. In this case, Condition 4.2.2 is likely to be satisfied.

In contrast, in a lull market, investors’ pessimistic view on the market growth guides them to have a substantial increase in risk aversion over time while the stock may probably possess a sharp decline in its own “goodness index”; to remedy the economic downturn, reducing borrowing interest rate can help to boost up the investment environment. Hence, Condition 4.2.2 is less likely to be satisfied in a weak market.

In the financial world, it is commonly observed that under a growing market, such that Conditions 4.2.1 and 4.2.2 could be naturally satisfied as discussed

above, would prefer investors to hold a relatively larger proportion in stock, while any restriction on shortselling will set extra hurdle for investors on leveraging to a more decent profit. However, our mathematical result demonstrates a paradoxical claim that any time-consistent investor should sometimes prefer the confined strategy even when the market goes very well: under Conditions 4.2.1 and 4.2.2, the constrained strategy dominates the unconstrained one.

**Theorem 4.2.4.** *Under Conditions 4.2.1 and 4.2.2, there exists some  $t < T$  such that  $V^{(C)}(t, x) > V^{(U)}(t, x)$  for all  $x > 0$ .*

Its proof will be given in Section 4.3.2. Note that the expression of the equilibrium mean-variance value functions for unconstrained and constrained investors, denoted by  $V^{(U)}$  and  $V^{(C)}$  respectively, are given in Theorems 4.1.8 and 4.1.11.

As a relatively less restrictive version of Condition 4.2.2, Condition 4.2.3 can be interpreted as comparing whether the relative increase in risk aversion coefficient is bounded above by the riskfree rate. Again, the latter condition is likely to appear when the market is performing well, which can be argued as above for Condition 4.2.2. Similarly, in the prevailing understanding of the market behavior, investors should allocate at least a noticeable portion in stock, especially in a very good economy. However, the following two main theorems provide another paradoxical, yet mathematically precise, assertion that the time-consistent strategy, no matter confined or unconfined one, is sometimes beaten by the pure strategy of solely investing in bond even in a well-performed market:

**Theorem 4.2.5.** *Under Condition 4.2.3, there exists a  $t^* < T$  such that  $V^{(Rf)}(t, x) > V^{(U)}(t, x)$  for all  $x > 0$  and  $t < t^*$ , where  $V^{(Rf)}$  is value function of the pure strategy of solely investing in bond given by*

$$V^{(Rf)}(t, x) := e^{\int_t^T r_s ds} x. \quad (4.2.12)$$

**Theorem 4.2.6.** *Under Condition 4.2.3, there exists a  $t^\dagger < T$  such that  $V^{(Rf)}(t, x) > V^{(C)}(t, x)$  for all  $x > 0$  and  $t < t^\dagger$ , where  $V^{(Rf)}$  is given in (4.2.12)*

Their proofs of above theorems will be given in Sections 4.3.3 and 4.3.4 respectively.

## 4.3 Proof of Main Results

Note that the ratios of investment to wealth of equilibrium strategy for unconstrained and constrained investors, denoted by  $c^{(U)}$  and  $c^{(C)}$  respectively, are given by (4.1.8) in Theorem 4.1.8 and (4.1.10) in Theorem 4.1.11; the corresponding equilibrium mean-variance value functions for unconstrained and constrained investors, denoted by  $V^{(U)}$  and  $V^{(C)}$  respectively, are given by (4.1.9) and (4.1.11).

### 4.3.1 Preliminary Lemmas for Main Theorems

Before we proceed to the proof of the main theorems, we first establish some preliminary lemmas.

#### 4.3.1.1 Unconstrained case

We first establish that under Conditions 4.2.1 and 4.2.2, the ratio of investment to wealth,  $c_t^{(U)}$ , is increasing in  $t$  whenever one is holding some of the stock at  $t$ ; similar results for  $d_t^{(U)}$  hold under Condition 4.2.3:

**Lemma 4.3.1. (i)** *Suppose that Conditions 4.2.1 and 4.2.2 hold. For any  $t \leq T$ , whenever  $c_t^{(U)} > 0$ , we have  $(c_t^{(U)})' > 0$ , i.e.  $c_t^{(U)}$  is increasing corresponding to those  $t$ 's. Hence,*

$$c_t^{(U)} \leq \overline{c^{(U)}} := \frac{\bar{\alpha}}{\gamma_T \sigma^2}, \text{ for all } \infty < t \leq T \quad (4.3.13)$$

**(ii)** *Suppose that Condition 4.2.3 holds. For any  $t \leq T$ , whenever  $d_t^{(U)} > -\frac{\sigma_t^2}{\alpha_t} \delta_1$ , we have  $(d_t^{(U)})' > 0$ .*

*Proof.*

$$\begin{aligned}
 & \left( c_t^{(U)} \right)' \\
 = & \left( \frac{\alpha_t}{\sigma_t^2} \right) \left\{ \frac{1}{\gamma_t} \left( r_t + \alpha_t c_t^{(U)} + \sigma_t^2 \left( c_t^{(U)} \right)^2 - \frac{\gamma_t'}{\gamma_t} \right) e^{-\int_t^T \left( r_s + \alpha_s c_s^{(U)} + \sigma_s^2 \left( c_s^{(U)} \right)^2 \right) ds} \right. \\
 & \left. + \sigma_t^2 \left( c_t^{(U)} \right)^2 e^{-\int_t^T \sigma_s^2 \left( c_s^{(U)} \right)^2 ds} \right\} + \left( \frac{\alpha_t}{\sigma_t^2} \right)' d_t^{(U)} \\
 > & \frac{\alpha_t}{\sigma_t^2} \left( \left( r_t + \alpha_t c_t^{(U)} - \frac{\gamma_t'}{\gamma_t} \right) + \frac{\min \left\{ \left( \frac{\alpha_t}{\sigma_t^2} \right)', 0 \right\}}{\frac{\alpha_t}{\sigma_t^2}} \right) \\
 & \times \left( \frac{1}{\gamma_t} e^{-\int_t^T \left( r_s + \alpha_s c_s^{(U)} + \sigma_s^2 \left( c_s^{(U)} \right)^2 \right) ds} \right) \\
 > & 0.
 \end{aligned} \tag{4.3.14}$$

The first inequality follows because  $d_t^{(U)}$  is positively proportional to  $c_t^{(U)}$  by (4.1.8) and so  $d_t^{(U)} > 0$ , and  $e^{-\int_t^T \sigma_s^2 \left( c_s^{(U)} \right)^2 ds} - 1 < 0$ . The last inequality follows after Condition 4.2.2. By (4.3.14), we have  $c_t^{(U)} \leq c_T^{(U)} = \frac{\alpha_T}{\gamma_T \sigma_T^2} \leq \frac{\bar{\alpha}}{\gamma_T \underline{\sigma}^2} = \bar{c}^{(U)}$ , so the second assertion in (i) follows. The case in (ii) can also be proven similarly.  $\square$

For any  $k \in (0, \frac{1}{\gamma_T})$ , define

$$\tau_k^{(U)} := \sup \left\{ t < T \mid d_t^{(U)} = k \right\} \in [-\infty, T). \tag{4.3.15}$$

Note that Lemma 4.3.1 (ii) implies that  $d_t^{(U)} > k$  if and only if  $t > \tau_k^{(U)}$ . We shall then establish some finite upper and lower bounds for  $\tau_k^{(U)}$  as follows.

**Lemma 4.3.2.** *Suppose that Condition 4.2.3 holds. For any  $k \in (0, \frac{1}{\gamma_T})$ ,  $\underline{\tau}_k^{(U)} \leq \tau_k^{(U)} \leq \bar{\tau}_k^{(U)}$ , where  $\tau_k^{(U)}$  is given by (4.3.15),*

$$\begin{aligned}
 T - \underline{\tau}_k^{(U)} & := \min \left\{ \frac{1}{\delta_1} \ln \left[ \frac{1}{\gamma_T k} \right], \frac{\bar{\sigma}^2}{k^2 \underline{\alpha}^2} \ln \left[ \frac{1 + \frac{1}{\gamma_T}}{1 + k} \right] \right\}, \\
 T - \bar{\tau}_k^{(U)} & := \frac{1}{\bar{r} + \bar{\alpha} \bar{c}^{(U)} + \bar{\sigma}^2 \left( \bar{c}^{(U)} \right)^2} \ln \left[ \frac{1 + \frac{1}{\gamma_T}}{1 + k} \right],
 \end{aligned}$$

and  $\overline{c^{(U)}}$  is defined in (4.3.13).

*Proof.* Since  $d^{(U)}$  is increasing by Lemma 4.3.1 (ii), we have  $c_t^{(U)} = \frac{\alpha_t}{\sigma_t^2} d_t^{(U)} \leq \frac{\overline{\alpha}}{\overline{\sigma^2}} d_T^{(U)} \leq \overline{c^{(U)}}$ . By considering (4.1.8), for any  $t$  such that  $d_t^{(U)} > 0$ , we have

$$\begin{aligned} & e^{-\left(\overline{\bar{r}} + \overline{\alpha c^{(U)}} + \overline{\sigma^2} \left(\overline{c^{(U)}}\right)^2\right)(T-t)} \left(1 + \frac{1}{\gamma_T}\right) - 1 \\ & \leq d_t^{(U)} \leq \min \left\{ e^{-\frac{\alpha^2}{\sigma^2} k^2 (T - \max\{t, \tau_k^{(U)}\})} \left(1 + \frac{1}{\gamma_T}\right) - 1, \frac{e^{-\delta_1(T-t)}}{\gamma_T} \right\}. \end{aligned} \quad (4.3.16)$$

By taking  $t = \overline{\tau_k^{(U)}}$  in the last-handed inequality in (4.3.16), we have

$$\frac{d^{(U)}}{\overline{\tau_k^{(U)}}} \geq e^{-\left(\overline{\bar{r}} + \overline{\alpha c^{(U)}} + \overline{\sigma^2} \left(\overline{c^{(U)}}\right)^2\right) \left(T - \overline{\tau_k^{(U)}}\right)} \left(1 + \frac{1}{\gamma_T}\right) - 1 = k,$$

then  $\tau_k^{(U)} \leq \overline{\tau_k^{(U)}}$  follows by recalling that  $d^{(U)}$  is increasing in accordance with Lemma 4.3.1 (ii).

To show that  $\tau_k^{(U)} \leq \underline{\tau_k^{(U)}}$ , we consider two cases: (i)  $T - \underline{\tau_k^{(U)}} = \frac{1}{\delta_1} \ln \left[ \frac{1}{k\gamma_T} \right]$ ;  
(ii)  $T - \underline{\tau_k^{(U)}} = \frac{\overline{\sigma^2}}{k^2 \overline{\alpha^2}} \ln \left[ \frac{1 + \frac{1}{\gamma_T}}{1+k} \right]$ .

(i) By taking  $t = \underline{\tau_k^{(U)}}$  in the right-handed inequality in (4.3.16), we have

$$\frac{d^{(U)}}{\underline{\tau_k^{(U)}}} \leq \frac{e^{-\delta_1 \left(T - \underline{\tau_k^{(U)}}\right)}}{\gamma_T} = k,$$

then  $\tau_k^{(U)} \geq \underline{\tau_k^{(U)}}$  follows by recalling that  $d^{(U)}$  is increasing in light of Lemma 4.3.1 (ii).

(ii) Assume the contrary that  $\tau_k^{(U)} < \underline{\tau_k^{(U)}}$ . Since  $d^{(U)}$  is strictly increasing according to Lemma 4.3.1 (ii),  $d_{\underline{\tau_k^{(U)}}}^{(U)} > d_{\tau_k^{(U)}}^{(U)} = k$ , where the last equality is due to the definition of  $\tau_k^{(U)}$  in (4.3.15). On the other hand, by taking  $t = \underline{\tau_k^{(U)}}$  in the right-handed inequality in (4.3.16) and using the assumption that  $\tau_k^{(U)} < \underline{\tau_k^{(U)}}$ , we have

$$\begin{aligned} \frac{d^{(U)}}{\underline{\tau_k^{(U)}}} & \leq e^{-\frac{\alpha^2}{\sigma^2} k^2 \left(T - \max\{\underline{\tau_k^{(U)}}, \tau_k^{(U)}\}\right)} \left(1 + \frac{1}{\gamma_T}\right) - 1 \\ & = e^{-\frac{\alpha^2}{\sigma^2} k^2 \left(T - \underline{\tau_k^{(U)}}\right)} \left(1 + \frac{1}{\gamma_T}\right) - 1 = k, \end{aligned}$$

which leads to a contradiction.

Hence, both cases can result in  $\tau_k^{(U)} \geq \underline{\tau}_k^{(U)}$ .

□

By Lemma 4.3.2, for  $t \leq \tau_k^{(U)}$ , we have

$$1 - e^{-\int_t^T \sigma_s^2 (c_s^{(U)})^2 ds} > 1 - e^{-\frac{\alpha^2}{\sigma^2} k^2 (T - \tau_k^{(U)})} \geq 1 - e^{-\frac{\alpha^2}{\sigma^2} k^2 (T - \overline{\tau}_k^{(U)})} =: p_k, \quad (4.3.17)$$

and so  $p_k = 1 - \left( \frac{1+k}{1+\frac{1}{\gamma T}} \right)^{\frac{\alpha^2 k^2}{\sigma^2 (\overline{\tau} + \overline{\alpha} c^{(U)} + \sigma^2 (c^{(U)})^2)}$ .

Define  $k^*$  such that  $p_{k^*} = \max_{k \in (0, \frac{1}{\gamma T})} p_k$ . Next, we shall show that  $d_t^{(U)}$  will have a negative upper bound for large enough  $T - t$ . Define

$$-L^* := \max \left\{ -\frac{\delta_1 \sigma^2}{2\alpha^2}, -\frac{p_{k^*}}{2} \right\} < 0 \text{ and } \tau^* := \sup \left\{ t < T \mid d_t^{(U)} = -L^* \right\} \in [-\infty, T), \quad (4.3.18)$$

**Lemma 4.3.3.** *Suppose that Condition 4.2.3 holds.  $T - \tau^* \leq T - \underline{\tau}^*$ , where  $T - \underline{\tau}^* := \max \left\{ \frac{2}{\delta_1} \ln \left[ \frac{2}{\gamma T p_{k^*}} \right], T - \tau_{k^*}^{(U)} \right\}$  and,  $\tau^*$  and  $\tau_{k^*}^{(U)}$  are defined in (4.3.18) and (4.3.15) respectively.*

*Proof.* Assume the contrary that  $\tau^* < \underline{\tau}^*$ , by the fact that  $d_t^{(U)} > L^*$  for all  $t \in [\underline{\tau}^*, T)$  in accordance with Lemma 4.3.1 (ii), we have  $c_t^{(U)} > -\frac{\alpha t}{\sigma_t^2} L^*$ . On the other hand, we have

$$\frac{1}{\gamma_{\underline{\tau}^*}} e^{-\int_{\underline{\tau}^*}^T (r_s + \alpha_s c_s^{(U)} + \sigma_s^2 (c_s^{(U)})^2) ds} < \frac{1}{\gamma_T} e^{-(\delta_1 - \frac{\alpha^2}{\sigma^2} L^*) (T - \underline{\tau}^*)} \leq \frac{1}{\gamma_T} e^{-\frac{1}{2} \delta_1 (T - \underline{\tau}^*)} \leq \frac{p_{k^*}}{2};$$

indeed, the first inequality follows from Condition 4.2.3 and the fact that  $c_t^{(U)} > -\frac{\alpha t}{\sigma_t^2} L^*$  for all  $t \in [\underline{\tau}^*, T)$ ; while the second and third inequalities follow from the definitions of  $L^*$  and  $\underline{\tau}^*$  respectively.

Since  $\underline{\tau}^* \leq \tau_{k^*}^{(U)}$ , then by (4.3.17), we have

$$d_{\underline{\tau}^*}^{(U)} = \frac{1}{\gamma_{\underline{\tau}^*}} e^{-\int_{\underline{\tau}^*}^T (r_s + \alpha_s c_s^{(U)} + \sigma_s^2 (c_s^{(U)})^2) ds} + e^{-\int_{\underline{\tau}^*}^T \sigma_s^2 (c_s^{(U)})^2 ds} - 1 < -\frac{p_{k^*}}{2} \leq -L^*,$$



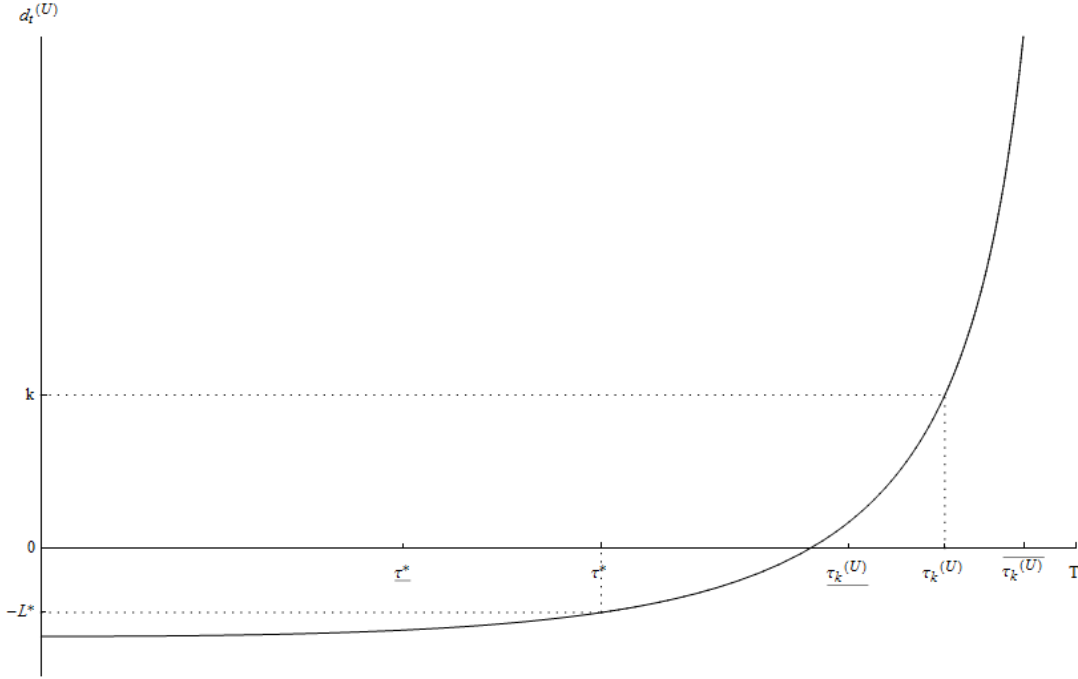


Figure 4.2: The graphical illustration for  $\underline{\tau}^*$ ,  $\tau^*$ ,  $\underline{\tau}_k^{(U)}$ ,  $\tau_k^{(U)}$  and  $\overline{\tau}_k^{(U)}$ . The black line represents  $d_t^{(U)}$ .  $\underline{\tau}_k^{(U)}$ ,  $\tau_k^{(U)}$  and  $\overline{\tau}_k^{(U)}$  are defined in Lemma 4.3.2.  $L^*$ ,  $\tau^*$  and  $\underline{\tau}^*$  are defined in 4.3.4.

which contradicts that  $\tau^* < \underline{\tau}^*$ .

□

**Lemma 4.3.4.** *Suppose that Condition 4.2.3 holds.  $d_t^{(U)} \leq -L^*$  for all  $t < \tau^*$ , where  $L^*$  and  $\tau^*$  are defined in (4.3.18).*

*Proof.* Since  $-L^* > -\frac{\delta_1 \sigma_t^2}{\alpha_t^2}$  and  $(d_t^{(U)})' > 0$  whenever  $d_t^{(U)} > -\frac{\delta_1 \sigma_t^2}{\alpha_t^2}$  by Lemma 4.3.1 (ii), so  $d_t^{(U)} \leq -L^*$  for all  $t \leq \tau^*$ , hence our claim follows.

□

Figure 4.2 illustrates the relative magnitude between  $\underline{\tau}^*$ ,  $\tau^*$ ,  $\underline{\tau}_k^{(U)}$ ,  $\tau_k^{(U)}$  and  $\overline{\tau}_k^{(U)}$ .

### 4.3.1.2 Constrained case

The results established in the previous subsection are also valid for the constrained setting. The following lemma can be proven using the same argument as Lemma 4.3.1.

**Lemma 4.3.5. (i)** *Suppose that Conditions 4.2.1 and 4.2.2 hold. For any  $t \leq T$ , whenever  $c_t^{(C)} \in (0, 1)$ , we have  $(c_t^{(C)})' > 0$ , i.e.  $c_t^{(C)}$  is increasing corresponding to those  $t$ 's.*

**(ii)** *Suppose that Condition 4.2.3 holds. For any  $t \leq T$ , whenever  $d_t^{(C)} > -\frac{\sigma_t^2}{\alpha_t} \delta_1$ , we have  $(d_t^{(C)})' > 0$ .*

For any  $k \in \left(0, \min \left\{ \frac{1}{\gamma_T}, \frac{\sigma^2}{\alpha} \right\}\right)$ , define

$$\tau_k^{(C)} := \sup \left\{ t < T \mid d_t^{(C)} = k \right\} \in [-\infty, T), \quad (4.3.19)$$

we shall then establish some finite upper and lower bounds for  $\tau_k^{(C)}$  as follows using the same argument as Lemma 4.3.2.

**Lemma 4.3.6.** *Suppose that Condition 4.2.3 holds. For any  $k \in \left(0, \min \left\{ \frac{1}{\gamma_T}, \frac{\sigma^2}{\alpha} \right\}\right)$ ,  $\underline{\tau}_k^{(C)} \leq \tau_k^{(C)} \leq \overline{\tau}_k^{(C)}$ , where  $\tau_k^{(C)}$  is defined in (4.3.19),*

$$\begin{aligned} T - \underline{\tau}_k^{(C)} &:= T - \tau_k^{(U)} = \min \left\{ \frac{1}{\delta_1} \ln \left[ \frac{1}{\gamma_T k} \right], \frac{\overline{\sigma}^2}{k^2 \underline{\alpha}^2} \ln \left[ \frac{1 + \frac{1}{\gamma_T}}{1 + k} \right] \right\}, \\ T - \overline{\tau}_k^{(C)} &:= \frac{1}{\overline{\gamma} + \overline{\alpha} + \overline{\sigma}^2} \ln \left[ \frac{1 + \frac{1}{\gamma_T}}{1 + k} \right]. \end{aligned}$$

By Lemma 4.3.6, for  $t < \tau_k^{(C)}$ , we have

$$1 - e^{-\int_t^T \sigma_s^2 (c_s^{(C)})^2 ds} > 1 - e^{-\frac{\alpha^2}{\sigma^2} k^2 (T - \overline{\tau}_k^{(C)})} =: q_k, \quad (4.3.20)$$

and so  $q_k = 1 - \left( \frac{1+k}{1+\frac{1}{\gamma_T}} \right)^{\frac{\alpha^2 k^2}{\sigma^2 (\overline{\gamma} + \overline{\alpha} + \overline{\sigma}^2)}}$ . Define  $k^\dagger$  such that  $q_{k^\dagger} = \max_{k \in (0, \min\{\frac{1}{\gamma_T}, \frac{\sigma^2}{\alpha}\})} q_k$ . Similar to the unconstrained case, we shall show that  $c_t^{(C)}$  will be zero for large enough  $T - t$ .

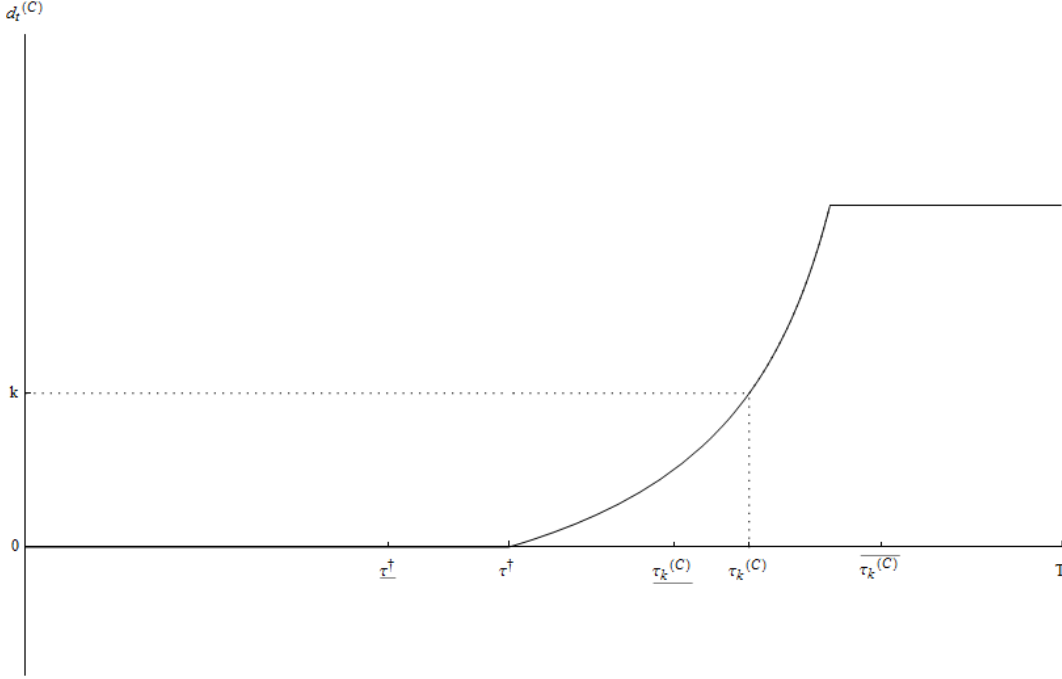


Figure 4.3: The graphical illustration for  $\underline{\tau}^\dagger$ ,  $\tau^\dagger$ ,  $\underline{\tau}_k^{(C)}$ ,  $\tau_k^{(C)}$  and  $\overline{\tau}_k^{(C)}$ . The black line represents  $d_t^{(C)}$ .  $\underline{\tau}_k^{(C)}$ ,  $\tau_k^{(C)}$  and  $\overline{\tau}_k^{(C)}$  are defined in Lemma 4.3.6.  $\tau^\dagger := \sup \{t < T \mid d_t^{(C)} = 0\}$  and  $\underline{\tau}^\dagger$  are defined in Lemma 4.3.7.

**Lemma 4.3.7.** *Suppose that Condition 4.2.3 holds.  $c_t^{(C)} = 0$  for all  $t < \underline{\tau}^\dagger$ , where  $T - \underline{\tau}^\dagger := \max \left\{ \frac{1}{\delta_1} \ln \left[ \frac{1}{\gamma_T q_{k^\dagger}} \right], T - \tau_{k^\dagger}^{(C)} \right\}$ , and  $\tau_k^{(C)}$  is defined in (4.3.19).*

*Proof.* By (4.3.20), for any  $t < \underline{\tau}^\dagger \leq \tau_{k^\dagger}^{(C)}$ ,

$$d_t^{(C)} = \frac{1}{\gamma_t} e^{-\int_t^T (r_s + \alpha_s c_s^{(C)} + \sigma_s^2 (c_s^{(C)})^2) ds} + e^{-\int_t^T \sigma_s^2 (c_s^{(C)})^2 ds} - 1 < \frac{1}{\gamma_T} e^{-\delta_1 (T - \underline{\tau}^\dagger)} - q_{k^\dagger} \leq 0,$$

where the second last term being less than zero is followed by the fact that  $T - \underline{\tau}^\dagger \geq \frac{1}{\delta_1} \ln \left[ \frac{1}{\gamma_T q_{k^\dagger}} \right]$ .  $\square$

Figure 4.3 illustrates the relative magnitude between  $\underline{\tau}^\dagger$ ,  $\tau^\dagger$ ,  $\underline{\tau}_k^{(C)}$ ,  $\tau_k^{(C)}$  and  $\overline{\tau}_k^{(C)}$ , where  $\tau^\dagger := \sup \{t < T \mid d_t^{(C)} = 0\} \in [-\infty, T)$ .

### 4.3.2 Proof of Theorem 4.2.4

Define

$$t_0 := \sup \left\{ t < T \mid \int_t^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds = 0 \right\} \in [-\infty, T), \quad (4.3.21)$$

in particular,  $t_0 = -\infty$  means  $\int_t^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds > 0$  for all  $t < T$ . Under Condition 4.2.1, it is obvious that  $1 = c_T^{(C)} < c_T^{(U)} = \frac{\alpha_T}{\gamma_T \sigma_T^2}$  (i.e. the unconstrained investor will shortsell bond to buy more stock at  $T$ ), thus by the continuity of  $c$ 's,

$$\int_t^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds > 0 \text{ when } t \text{ is close enough to } T. \quad (4.3.22)$$

We now establish that the constrained solution can outperform the unconstrained one when the investor commences the strategies at  $t_0$ ; for the justification of the existence of a finite  $t_0$ , it will be shown in Proposition 4.3.10.

**Proposition 4.3.8.** *Given that Conditions 4.2.1 and 4.2.2 hold. With such a finite  $t_0 < T$  defined in (4.3.21), we have that  $V^{(C)}(t_0, x) > V^{(U)}(t_0, x)$  for all  $x > 0$ .*

Define

$$t_1 := \sup \left\{ t < T \mid c_t^{(U)} = c_t^{(C)} \right\} \in [-\infty, T). \quad (4.3.23)$$

Before we proceed on its proof, we first establish the following useful lemma:

**Lemma 4.3.9.** *Under Conditions 4.2.1 and 4.2.2, the followings hold:*

(i)  $t_1 \in (t_0, T)$ ; (ii)  $c_t^{(C)} = 1$  for all  $t \in [t_1, T]$ ; (iii) There exists  $\epsilon > 0$  such that  $c_t^{(U)} < c_t^{(C)}$  for all  $t \in (t_1 - \epsilon, t_1)$ ; (iv)  $c_t^{(U)} < c_t^{(C)}$  for all  $t \in (t_0, t_1)$ .

*Proof.* (i) Assume the contrary that  $t_1 \leq t_0$ ,  $c_t^{(C)} < c_t^{(U)}$  for all  $t \in (t_0, T)$ , then

$$\int_{t_0}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds > 0, \text{ which contradicts to the definition of } t_0.$$

(ii) Since  $c_t^{(C)} < c_t^{(U)}$  for all  $t \in (t_1, T)$ , by (4.1.8) and (4.1.10), we have  $\frac{\alpha_t}{\sigma_t^2} d_t^{(C)} > \frac{\alpha_t}{\sigma_t^2} d_t^{(U)} = c_t^{(U)} \geq c_t^{(C)}$  for all  $t \in (t_1, T)$ , which implies that  $G$  takes its maximum value 1 and so  $c_t^{(C)} = 1$  for all  $t \in (t_1, T)$ .

(iii) By the proof of (ii),  $\frac{\alpha_{t_1}}{\sigma_{t_1}^2} d_{t_1}^{(C)} > 1$ . The continuity of  $d$  ensures that there exists  $\epsilon_1 > 0$  such that  $c_t^{(C)} = 1$  for all  $t \in (t_1 - \epsilon_1, t_1)$ . By Lemma 4.3.1(i),  $\left(c_{t_1}^{(U)}\right)' > 0$  implies that there exists  $\epsilon_2 > 0$  such that  $c_t^{(U)} < 1 = c_t^{(C)}$  for all  $t \in (t_1 - \epsilon_2, t_1)$ . The result follows by choosing  $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ .

(iv) By Lemma 4.3.5 (ii) and (4.1.10),  $\left(\frac{\sigma_t^2}{\alpha_t} c_t^{(C)}\right)' \geq 0$  for all  $t \in (-\infty, T]$  almost everywhere (except  $t = \tau^\dagger$  or  $t = \inf\{t | c_t^{(C)} = 1\}$ ). Assume the contrary, so as there exists  $t_2 := \sup\{t < t_1 | c_t^{(U)} = c_t^{(C)}\} \in (t_0, t_1)$ . By (iii), we have that  $c_t^{(U)} > c_t^{(C)}$  for all  $t \in (t_1, T)$  and  $c_t^{(U)} < c_t^{(C)}$  for all  $t \in (t_2, t_1)$ . Then

$$\begin{aligned} & \int_{t_2}^T \sigma_s^2 \left( (c_s^{(U)})^2 - (c_s^{(C)})^2 \right) ds \\ &= 2 \int_{t_2}^T \left( \frac{\sigma_s^2}{\alpha_s} c_s^{(C)} \right) \alpha_s (c_s^{(U)} - c_s^{(C)}) ds + \int_{t_2}^T \sigma_s^2 (c_s^{(U)} - c_s^{(C)})^2 ds \\ &> 2 \left( \frac{\sigma_{t_1}^2}{\alpha_{t_1}} c_{t_1}^{(C)} \right) \int_{t_2}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds, \end{aligned} \quad (4.3.24)$$

where the last inequality follows by the fact that  $\left(\frac{\sigma_t^2}{\alpha_t} c_t^{(C)}\right)' \geq 0$  for all  $t$ .

By considering the definition of  $t_0$  in (4.3.21),  $t_0 < t_2$ , and the fact of (4.3.22),  $\int_{t_2}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds > 0$ , and hence  $\int_{t_2}^T \sigma_s^2 \left( (c_s^{(U)})^2 - (c_s^{(C)})^2 \right) ds > 0$ . Then,  $d_{t_2}^{(U)} < d_{t_2}^{(C)}$  by comparing (4.1.8) and (4.1.10). Due to (iii) and  $c_t^{(U)}$  is increasing whenever  $c_t^{(U)} > 0$  in accordance with Lemma 4.3.1 (i), we have  $c_{t_2}^{(U)} < 1$ . However, because  $d_{t_2}^{(U)} < d_{t_2}^{(C)}$ , it is impossible that  $c_{t_2}^{(U)} = c_{t_2}^{(C)}$ , which contradicts to the definition of  $t_2$ . □

*Proof of Proposition 4.3.8.* Following the same computation as in (4.3.24), we have

$$\int_{t_0}^T \sigma_s^2 \left( (c_s^{(U)})^2 - (c_s^{(C)})^2 \right) ds > 2 \left( \frac{\sigma_{t_1}^2}{\alpha_{t_1}} c_{t_1}^{(C)} \right) \int_{t_0}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds = 0.$$

Our claim follows by comparing (4.1.9) and (4.1.11). □

We finally establish the existence of  $t_0$  (i.e.  $t_0 > -\infty$ ) by constructing a finite lower bound.

**Proposition 4.3.10.** *Given that Conditions 4.2.1 and 4.2.2 hold. There exists  $t_0 \in [\underline{t}_0, T)$  defined in (4.3.21), where  $\underline{t}_0 := \underline{\tau}^* - \frac{\overline{\alpha}\sigma^2(c_T^{(U)} - 1)(T - \underline{\tau}_\theta^{(U)})}{\underline{\alpha}^2 L^*} > -\infty$ ,  $\theta := \frac{\sigma^2}{\underline{\alpha}}$ ,  $\underline{\tau}^{(U)}$  and  $\underline{\tau}^*$  are defined in Lemmas 4.3.2 and 4.3.3, and  $L^*$  is given in (4.3.18). Hence,  $t_0 > -\infty$ .*

*Proof.* Assume the contrary that  $t_0 < \underline{t}_0$ . Since  $\underline{t}_0 < \underline{\tau}^* < \tau^*$  (by Lemma 4.3.3), then  $c_{\underline{t}_0}^{(U)} < 0$  (by Lemma 4.3.4). By continuity of  $c^{(U)}$  and  $c^{(C)}$ ,  $c_T^{(U)} > 1$  and  $c^{(C)} \in [0, 1]$ , the Intermediate Value Theorem ensures that  $t_1 > \underline{t}_0$ . By Lemma 4.3.9(ii), we have  $c_{t_1}^{(U)} = c_{t_1}^{(C)} = 1$ . By Lemma 4.3.1 and the definition of  $t_1$  in (4.3.23), we have  $c_T^{(U)} > c_t^{(U)} > c_t^{(C)} = 1$  for all  $t \in (t_1, T)$ . By Lemma 4.3.9(iv) and the assumption that  $t_0 < \underline{t}_0$ ,  $c_t^{(U)} < c_t^{(C)}$  for all  $t \in (t_0, t_1)$ . By definition of  $\tau_\theta^{(U)}$  in (4.3.15) and definition of  $\theta$ ,  $d_{\tau_\theta^{(U)}}^{(U)} = \frac{\sigma^2}{\underline{\alpha}}$ , then  $c_{\tau_\theta^{(U)}}^{(U)} \leq 1 = c_{t_1}^{(U)}$ . By Lemma 4.3.1(i) and Lemma 4.3.2, we have  $t_1 \geq \tau_\theta^{(U)} \geq \underline{\tau}_\theta^{(U)}$ .

With the above results, we have

$$\begin{aligned}
 & \int_{\underline{t}_0}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds \\
 & \leq \overline{\alpha} \int_{t_1}^T (c_s^{(U)} - c_s^{(C)}) ds + \underline{\alpha} \int_{\underline{t}_0}^{t_1} (c_s^{(U)} - c_s^{(C)}) ds \\
 & < \overline{\alpha} \int_{t_1}^T (c_T^{(U)} - 1) ds + \underline{\alpha} \int_{\underline{t}_0}^{\tau^*} \frac{\alpha_s}{\sigma_s^2} d_s^{(U)} ds \\
 & < \overline{\alpha} (c_T^{(U)} - 1) (T - \underline{\tau}_\theta^{(U)}) - \frac{\underline{\alpha}^2}{\sigma^2} L^* (\tau^* - \underline{t}_0) = 0, \tag{4.3.25}
 \end{aligned}$$

where the second inequality follows from  $c_t^{(C)} = 1$  for all  $t \in (t_1, T)$  and  $c_t^{(U)} < c_t^{(C)}$  for all  $t \in (\tau^*, t_1)$ ; the third inequality follows from  $t_1 \geq \underline{\tau}_\theta^{(U)}$  and Lemmas 4.3.3 and 4.3.4; and the last equality follows from the definition of  $\underline{t}_0$ . Note that (4.3.25) contradicts to the assumption that  $t_0 < \underline{t}_0$ .  $\square$

Since the finiteness of  $t_0$ , defined in (4.3.21), is warranted by Proposition 4.3.10, Theorem 4.2.4 follows from Proposition 4.3.8.

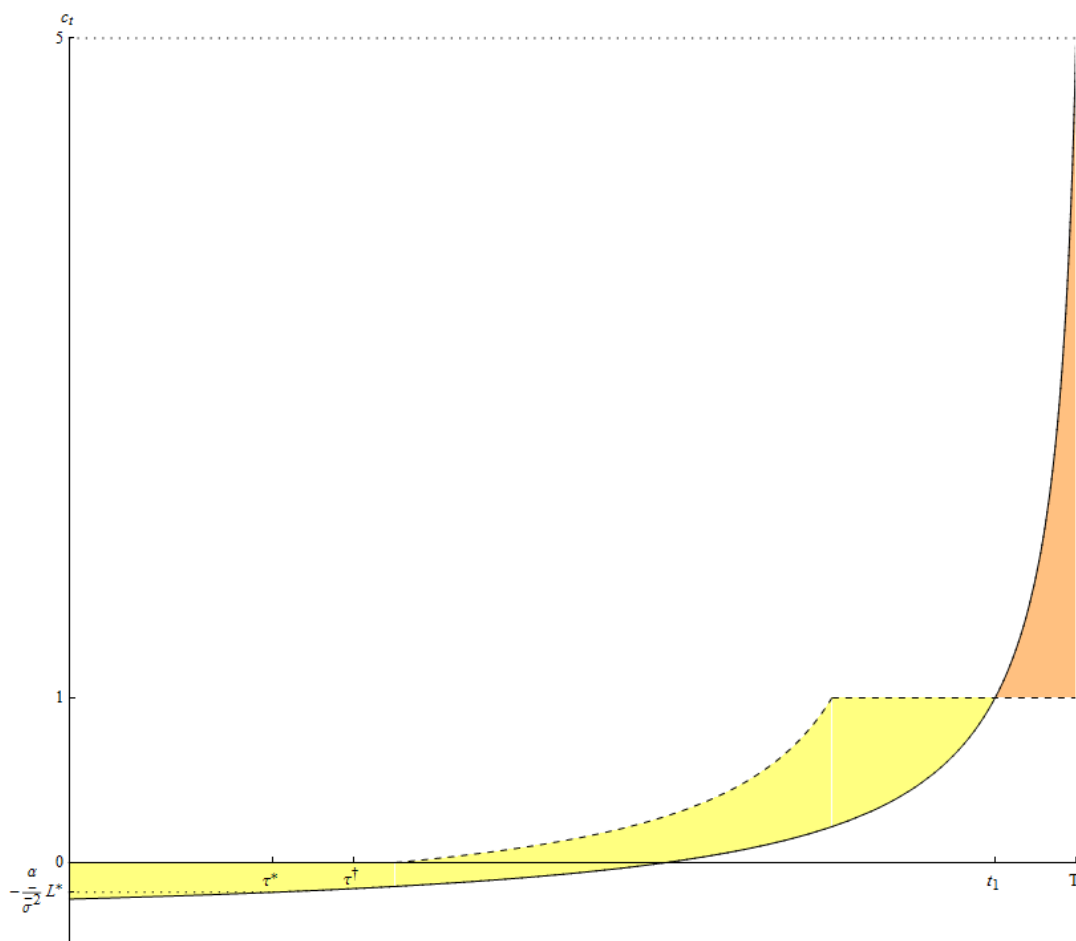


Figure 4.4: The graphical illustration to showing how equilibrium strategy changes with commencement time. The solid line and the dashed line represent the equilibrium investment to the wealth ratio of unconstrained and constrained investor respectively.  $L^*$ ,  $\tau^*$  and  $\tau^\dagger$  are defined in Lemmas 4.3.4 and 4.3.7.  $t_1$  is defined in Lemma 4.3.9.

**Remark 4.3.11.** The idea of the proof of Theorem 4.2.4 is to seek for a  $t_0$  such that  $\int_{t_0}^T \alpha_s (c_s^{(U)} - c_s^{(C)}) ds = 0$  subject to the sufficient condition assumed in Proposition 4.3.8. Assume that the market parameters  $r, \alpha$  and  $\sigma$  are constant. In Figure 4.4,  $\int_t^T (c_s^{(U)} - c_s^{(C)}) ds$  represents the difference of the area of the yellow region from the area of the orange one between  $t$  and  $T$ , so  $t_0$  will be taken to be the first  $t < T$  such that this difference vanishes. To obtain such  $t_0$ , we can argue as follows. By Lemma 4.3.9, we know that there is exactly one finite time  $t_1$  such that  $c^{(U)}$  and  $c^{(C)}$  intercept with each other after  $t_0$ . Hence, as  $c^{(U)}$  is bounded from above, there exists a  $M > 0$  so that, for any  $t \in (t_0, t_1)$ ,  $\int_t^T \max \{c_s^{(U)} - c_s^{(C)}, 0\} ds = \int_{t_1}^T (c_s^{(U)} - c_s^{(C)}) ds \leq M$ , i.e. the area of the orange region in Figure 4.4 is bounded. By Lemma 4.3.4, we know that before  $\tau^*$ ,  $c^{(U)} < -\frac{\alpha}{\sigma^2} L^*$ , thus  $c^{(U)} - c^{(C)} < -\frac{\alpha}{\sigma^2} L^*$ , so we can find a large enough  $T - t$  such that  $\int_t^T \max \left\{ -\left( c_s^{(U)} - c_s^{(C)} \right), 0 \right\} ds > M$  (In Figure 4.4, when  $t < \tau^*$  moves to the left, the area of the yellow region increases at a rate  $= |c^{(U)} - c^{(C)}| > \frac{\alpha}{\sigma^2} L^*$ , so we can choose an earlier enough  $t$  such that the area of the yellow region from  $t$  to  $T$  is greater than  $M$ ), hence  $t_0$  can be identified.

### 4.3.3 Proof of Theorem 4.2.5

By Lemmas 4.3.3 and 4.3.4, for  $t < \underline{\tau}^* \leq \tau^*$ ,  $c_t^{(U)} \leq -\frac{\alpha}{\sigma^2} L^*$ , where  $L^*$  is given in (4.3.18). Define  $t^* := \underline{\tau}^* - \frac{\overline{\alpha c^{(U)}}(T - \underline{\tau}^*)}{\alpha^2 L^*}$ , where  $\overline{c^{(U)}}$  and  $\underline{\tau}^*$  are defined in (4.3.13) and Lemma 4.3.3 respectively. For all  $t < t^*$ , we have

$$\begin{aligned} \int_t^T \alpha_s c_s^{(U)} ds &= \int_{\underline{\tau}^*}^T \alpha_s c_s^{(U)} ds + \int_t^{\underline{\tau}^*} \alpha_s c_s^{(U)} ds \\ &\leq \overline{\alpha c^{(U)}}(T - \underline{\tau}^*) - \frac{\alpha^2}{\sigma^2} L^*(\underline{\tau}^* - t) < 0. \end{aligned}$$



Recall the expression of  $V^{(U)}$  in (4.1.9), for any  $t < t^*$ ,

$$\begin{aligned} & V^{(U)}(t, x) \\ &= e^{\int_t^T (r_s + \alpha_s c_s^{(U)}) ds} x - \gamma_t \left( e^{\int_t^T 2(r_s + \alpha_s c_s^{(U)}) + \sigma_s^2 (c_s^{(U)})^2 ds} - e^{\int_t^T 2(r_s + \alpha_s c_s^{(U)}) ds} \right) x \\ &< e^{\int_t^T (r_s + \alpha_s c_s^{(U)}) ds} x < e^{\int_t^T r_s ds} x = V^{(Rf)}(t, x). \end{aligned}$$

**Remark 4.3.12.** By Theorem 4.2.5, the value function of riskless strategy is greater than that of unconstrained strategy whenever the commencing time  $t < t^*$ . Considering the expected terminal wealth with the ratio  $c$  of investment to wealth, i.e.  $\mathbb{E}_{t,x}[X_T] = e^{\int_t^T (r_s + \alpha_s c_s) ds} x$ , any holding of stock in an amount of  $c > 0$  can boost up this expectation through a factor of  $e^{\int_t^T \alpha_s c_s ds}$ . By Lemma 4.3.4, we know that the unconstrained investor will shortsell stock to invest more in bond for any time before  $\tau^*$  with  $c$  less than a negative constant, so that  $\int_t^T \alpha_s c_s ds$  is increasing in  $t$  before  $\tau^*$ . Therefore, there exists a earlier enough  $t$  so that  $\int_t^T \alpha_s c_s ds < 0$  making the factor  $e^{\int_t^T \alpha_s c_s ds} < 1$ . As a result, the unconstrained strategy at such commencement time  $t < t^*$  will be beaten by the riskless strategy of solely investing in bond.

#### 4.3.4 Proof of Theorem 4.2.6

By Lemma 4.3.7, we know for  $t < \underline{\tau}^\dagger$ ,  $c_t^{(C)} = 0$ . Define  $t^\dagger$  such that

$$T - t^\dagger := \max \left\{ \frac{1}{\delta_1} \ln \left[ \frac{2 \left( 1 - e^{-\bar{\alpha}(T - \underline{\tau}^\dagger)} \right)}{\gamma T q_{k^\dagger}} \right], T - \underline{\tau}^\dagger \right\},$$

where  $\underline{\tau}^\dagger$  is defined in Lemma 4.3.7.

For any  $t < t^\dagger$ , we have  $t < t^\dagger \leq \underline{\tau}^\dagger \leq \tau_{k^\dagger}^{(C)}$ , so

$$\begin{aligned} 1 - \frac{\gamma t}{2} e^{\int_t^T (r_s + \alpha_s c_s^{(C)}) ds} \left( e^{\int_t^T \sigma_s^2 (c_s^{(C)})^2 ds} - 1 \right) &< 1 - \frac{\gamma T q_{k^\dagger}}{2} e^{\delta_1(T-t)} < e^{-\bar{\alpha}(T-t^\dagger)} \\ &\leq e^{-\int_t^T \alpha_s c_s^{(C)} ds}, \end{aligned}$$

where the first inequality follows from (4.3.20), the second inequality follows from the definition of  $t^\dagger$ , and the last inequality is a consequence of the facts that  $c_s^{(C)} \leq 1$  for all  $s$  and  $c_s^{(C)} = 0$  for  $s < \underline{\tau}$ .

Recall the expression of  $V^{(C)}$  in (4.1.11), for any  $t < t^\dagger$ ,

$$\begin{aligned} & V^{(C)}(t, x) \\ &= e^{\int_t^T (r_s + \alpha_s c_s^{(C)}) ds} x - \frac{\gamma_t}{2} \left( e^{\int_t^T 2(r_s + \alpha_s c_s^{(C)}) + \sigma_s^2 (c_s^{(C)})^2 ds} - e^{\int_t^T 2(r_s + \alpha_s c_s^{(C)}) ds} \right) x \\ &= e^{\int_t^T (r_s + \alpha_s c_s^{(C)}) ds} x \left( 1 - \frac{\gamma_t}{2} e^{\int_t^T (r_s + \alpha_s c_s^{(C)}) ds} \left( e^{\int_t^T \sigma_s^2 (c_s^{(C)})^2 ds} - 1 \right) \right) \\ &< e^{\int_t^T r_s ds} x = V^{(Rf)}(t, x). \end{aligned}$$

**Remark 4.3.13.** Similar to Remark 4.3.12, by Lemma 4.3.7, we know that the constrained investor will adopt riskfree strategy for any time before  $\tau^\dagger$ , thus by (4.1.11), whenever  $t < \tau^\dagger$ ,

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{V^{(C)}(t, x)}{e^{\int_t^T r_s ds} x} \right) &= \frac{\gamma_t}{2} r_t \left( e^{\int_t^T (r_s + 2\alpha_s c_s^{(C)}) + 2\sigma_s^2 (c_s^{(C)})^2 ds} - e^{\int_t^T (r_s + 2\alpha_s c_s^{(C)}) ds} \right) \\ &\geq \underline{r} \frac{\gamma_T}{2} \left( e^{\int_{\tau^\dagger}^T 2\sigma_s^2 (c_s^{(C)})^2 ds} - 1 \right) > 0 \end{aligned}$$

Therefore, there exists a small enough  $t$  such that  $\frac{V^{(C)}(t, x)}{e^{\int_t^T r_s ds} x} < 1$ . Again, the constrained strategy at such commencement time  $t < \tau^\dagger$  will be beaten by the riskless strategy, as claimed in Theorem 4.2.6.

## 4.4 Numerical Illustration

In this section, for different value of risk aversion  $\gamma_t$ , we provide a graphical illustration between the ratios of equilibrium investment to wealth,  $c_t^{(U)}$  and  $c_t^{(C)}$ , and the equilibrium value functions,  $V^{(U)}(t, x)$  and  $V^{(C)}(t, x)$ , for unconstrained and constrained investors. Their expressions are shown in Theorems 4.1.8 and 4.1.11. In each plot, the respective ratios and the value functions of both unconstrained

Table 4.3: The Parameters Used in  $\gamma_t$  for Figures

	$\gamma_T = 0.2$	$\gamma_T = 1$	$\gamma_T = 3$
k=0.025	Figure 4.5	Figure 4.7	Figure 4.9
k=0	Figure 4.6	Figure 4.8	Figure 4.10

and constrained investor against the commencement time are shown. The performance of the investor who takes the pure strategy by simply putting all his wealth in bank is considered and compared with that of unconstrained and constrained investors, for illustrating the implication of Theorems 4.2.5 and 4.2.6.

We set risk aversion  $\gamma_t$  to be time-varying such that an investor behaves more risk averse over time. For illustration, we propose to model  $\gamma_t$  by the following logistic function with some known parameters  $\gamma_T$  and  $k$ :

$$\gamma_t := \frac{2\gamma_T}{1 + e^{-k(t-T)}}.$$

Larger value of  $k$  will result more considerable increase in risk aversion as time goes on; so the risk aversion of an investor with long time to the expiry will be smaller for larger value of  $k$ . We consider 6 different functions of  $\gamma(t)$  with  $\gamma_T \in \{0.2, 1, 3\}$  and  $k \in \{0.025, 0\}$  (see Table 4.3). Note that the risk aversion coefficient keeps constant if  $k = 0$ . For  $\gamma_T = 0.2, 1$ , Condition 4.2.1 is clearly satisfied; while for  $\gamma_T = 3$ , Condition 4.2.1 is not satisfied, through the example, we show that the claim in Theorem 4.2.4 also very likely fails to hold, while suggest that our proposed Condition 4.2.1 could be “optimal” in form. Besides, we fix  $r = 0.03$ ,  $\mu = 0.1$ ,  $\sigma = 0.2$ , and  $T = 40$ .

Furthermore, we illustrate Theorems 4.2.4, 4.2.5 and 4.2.6 by computing the following constants:

- $\bar{t}^* := \sup \{t < T | V^{(U)}(t, x) \leq V^{(Rf)}(t, x)\};$
- $\bar{t}^\dagger := \sup \{t < T | V^{(C)}(t, x) \leq V^{(Rf)}(t, x)\};$

- $\bar{t}_0 := \sup \{t < T | V^{(U)}(t, x) \leq V^{(C)}(t, x)\};$
- $t_0 := \sup \left\{ t < T \mid \int_t^T \alpha_s \left( c_s^{(U)} - c_s^{(C)} \right) ds = 0 \right\}$  as defined in (4.3.21);
- $\Delta V(t_0, 1) := V^{(U)}(t_0, 1) - V^{(C)}(t_0, 1);$

where  $V^{(Rf)}$  is given in (4.2.12).

All 6 figures show that the ratios of investment to wealth of both unconstrained and constrained investors increase in commencement time as in Lemma 4.3.1 (i). Furthermore,  $c_t^{(U)}$  becomes negative and  $c_t^{(C)}$  stays at zero for all earlier enough  $t$ , which is consistent with Lemmas 4.3.4 and 4.3.7.

Theorem 4.2.4 says that there exists some commencement time such that from that time on, the constrained equilibrium strategy performs better than the unconstrained one. In Table 4.4, the feasible set of such commencement times occupy more than 90% of the whole time horizon. This means that there is more than 90% of the time that the constrained equilibrium strategy performs better than the unconstrained one. Furthermore, we have shown that the outperformance of the constrained strategy against the unconstrained one if both commence at  $t_0$  as defined in (4.3.21) in Proposition 4.3.8, this explains that  $\Delta V(t_0, 1)$  is negative in Table 4.4.

$\bar{t}^*$  and  $\bar{t}^\dagger$  in Table 4.5 indicate the respective time points where the riskfree investor performs better than unconstrained and constrained investors before which, as established in Theorems 4.2.5 and 4.2.6 respectively. It is reasonable to describe a strategy, which underperforms than the riskless one, to be inferior, and so the unconstrained equilibrium strategy and constrained equilibrium strategy are both less favorable if one starts to invest before  $\bar{t}^*$  and  $\bar{t}^\dagger$  respectively. Note that when the terminal risk aversion coefficient  $\gamma_T$  decreases,  $\bar{t}^*$  would get closer to the expiry. Hence, the time portion over which the unconstrained strategy remains more favorable than the riskless one reduces, as  $\gamma_T$  decreases. In contrast, as  $\gamma_T$  decreases,  $\bar{t}^\dagger$  gets departed from  $\bar{t}^*$ . The shortselling constraints enlarges the time portion

Table 4.4: When the Constrained Investor Dominates the Unconstrained Investor for Different  $\gamma_t$

$\gamma_T$	$k$	$\bar{t}_0$	% of $t$ which $V^{(C)} > V^{(U)}$	$t_0$	$\Delta V(t_0, 1)$
0.2	0.025	36.9271	92.32%	29.3924	-0.8408
0.2	0	37.1294	92.82%	30.0075	-0.8328
1	0.025	38.5363	96.34%	31.7436	-0.0580
1	0	38.6208	96.55%	31.8949	-0.0605

Table 4.5: When the Riskless Strategy Dominate the Equilibrium Strategies for Different  $\gamma_t$

$\gamma_T$	$k$	$\bar{t}^*$	$\bar{t}^\dagger$	$k$	$\bar{t}^*$	$\bar{t}^\dagger$
0.2	0.025	15.6295	-26.9883	0	25.6088	4.3168
0.5	0.025	12.3127	-9.2159	0	23.5456	12.6319
1	0.025	7.6871	1.7609	0	21.0064	17.7313
1.75	0.025	2.1902	2.1902	0	18.3645	18.3645
3	0.025	-4.7595	-4.7595	0	15.5073	15.5073

of the equilibrium strategy being more favorable than the riskless one. So one can interpret that the shortselling constraints are more useful for time-consistent investor when he becomes less risk averse. This observation can be interpreted as the less risk averse attitude misleading to make over-investment, while shortselling constraints can save the investor from running into dangerous investment which may actually be unfavorable in term of the other objective functions.

## 4.5 Discussion

Our result suggests some reasonable economic conditions, that could often happen in the reality, under which, the constrained equilibrium strategy can dominate the unconstrained one at a certain commencement time (Theorem 4.2.4); furthermore both of these strategies can be beaten by the pure one of solely investing in bond for early enough commencement time (Theorems 4.2.5 and 4.2.6). The numerical study also illustrate that the unconstrained strategy is dominated by another time-consistent one in the most of the time over the time horizon.

Theorem 4.2.4 shows that the equilibrium solution for the unconstrained investor is not a maximizer because it cannot beat that of the constrained counterpart which is still time-consistent, and commonly accessible and admissible to the unconstrained investor. Even worse, both the unconstrained and constrained equilibrium strategies are beaten by the riskless strategy in Theorems 4.2.5 and 4.2.6. Hence, game theoretic approach may not provide an alternative maximizer even among all time-consistent strategies, even though our objective is apparent to looking for a “maximizer” (actually just an equilibrium solution).

There is no mathematical paradox for the present time-consistent solution. The equilibrium strategies of both unconstrained and constrained investors do satisfy the Nash equilibrium nature under game theoretic approach. From this idea of game theoretic approach, any such equilibrium strategy is a time-consistent strategy which has a portion being optimal at the commencement time close to the expiry time  $T$ ; this also explains why we observe that the unconstrained strategy has the best possible performance against that of the constrained and riskless strategies in Figures 4.5-4.10 when  $t$  close to  $T$ . In the meanwhile, the later time players set a routine to which the optimal control of the earlier time players refers, so the preference of the earlier time players will be overlooked. On the one hand, the appearance of the paradox suggests that the strategy of the later time players in the unconstrained optimization can be more unfavorable for

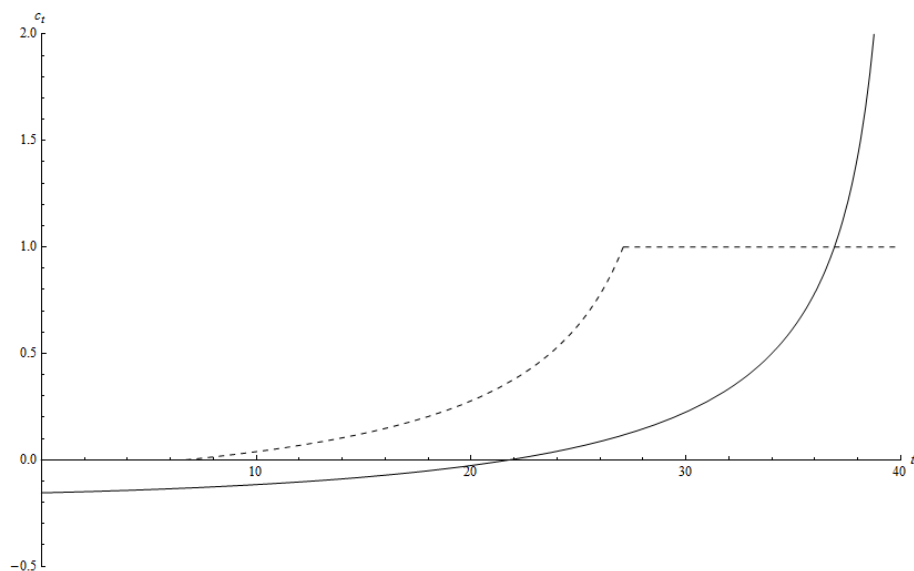
the earlier time players than those in the constrained case. While on the other hand, the game theoretic approach assigns a “precommitment” strategy to the later time players. In principle, if this time-consistent approach is appropriate, a sophisticated investor should not care that his unconstrained strategy being beaten by another (constrained) time-consistent strategy, and even by a riskless strategy, at the earlier time point. However, based on our results obtained, it is too tempting to make a change, isn’t it?

The rationale behind the present acclaimed paradox should be rooted in the economic interpretation of game theoretic approach. Strotz (1955) first suggested to use game theoretic approach as a “time-consistent plan”, and said that it should be “the best plan among those that he (would) actually follow”. However, we have established that the equilibrium strategy of unconstrained investor obtained by game theoretic approach, which is described as the “best time-consistent strategy” in Strotz (1955), is essentially beaten by another admissible time-consistent strategy (the strategy adopted by constrained investor). Even worse, in the numerical examples in Table 4.4, the constrained strategy outperforms against the unconstrained one for more than 90% of the whole time horizon, which is just like an unconstrained equilibrium investor giving up the victory over his constrained counterpart for more than 18 years in order to attain a triumph only at the final year. Of course, there is no correct answer whether it is worthwhile to achieve a perfect ending. Nevertheless, before becoming the premier, the unconstrained investor needs to bear the stress from the inferior performance of unconstrained equilibrium strategy, especially even an investor who puts all his wealth into the bank account can gain more satisfaction than adopting the unconstrained strategy by Table 4.5 and Theorem 4.2.5. Actually, sometimes the unconstrained investor can also acquire the performance under the constrained strategy by confining his admissible set, it seems too tempting for him to give up the unconstrained strategy and pursue his “best plan among consistent plans”.

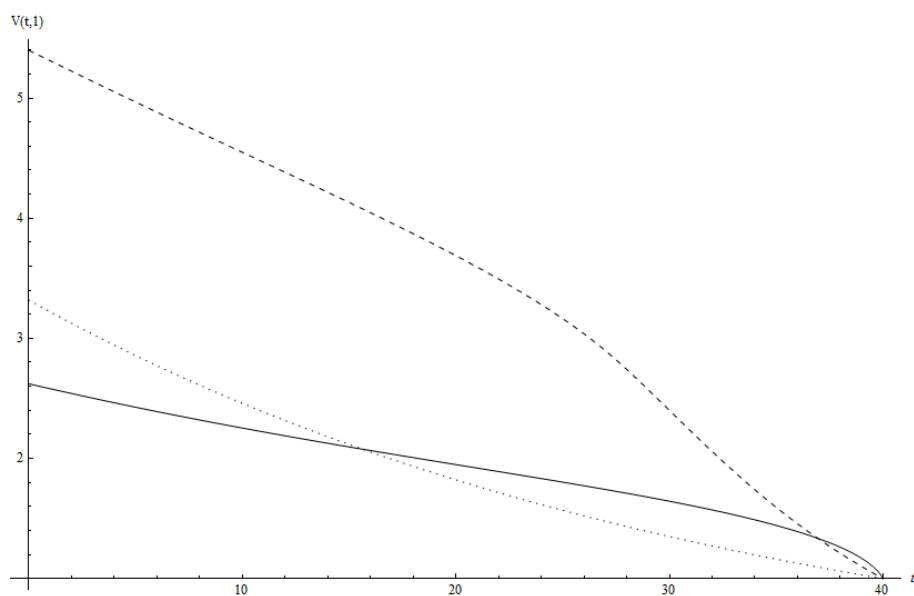
On the other hand, for open-loop controls in light of the definition of equilibrium control in Hu et al. (2012), whether the equilibrium control over a smaller admissible set can out-perform the one over a larger set is an open problem. Also, it is an interesting study to compare the performance of the respective equilibrium control among the class of open-loop ones and the class of Markovian ones.

Game theoretic approach can help us to find a time-consistent solution, but whether it is a good solution for an investor is a question that we have to think about carefully. The inconsistency and paradox raised in this Chapter show that the solution concept of equilibrium control is subtle, so an intensive study on underlying economic interpretation behind the time-consistent solution should be encouraged. It is interesting to have a more comprehensive empirical study on the connection between the mathematical solution and investors' behavior for equilibrium strategy, and we should reconsider carefully whether the solution via the simple non-cooperative game theoretic approach is justifiable as an ideal strategy used by a time-consistent investor. As pointed out by Schweizer (2010), to find a time-consistent formulation in general is an open problem, which means that there could be other ideal formats of time-consistent strategy. Since time inconsistent problems can be intricate, different problems should be tackled by their own tailor-made "time consistent" approach in accordance with their respective economic considerations. We hope that our present work can motivate more substantial research on time consistency that can cunningly link mathematics and economics in a proper manner.



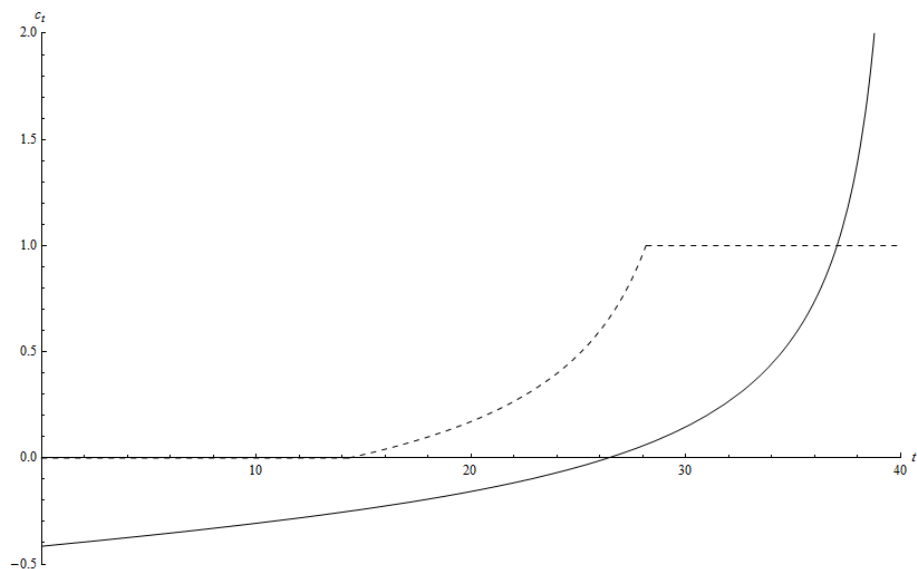


(a)

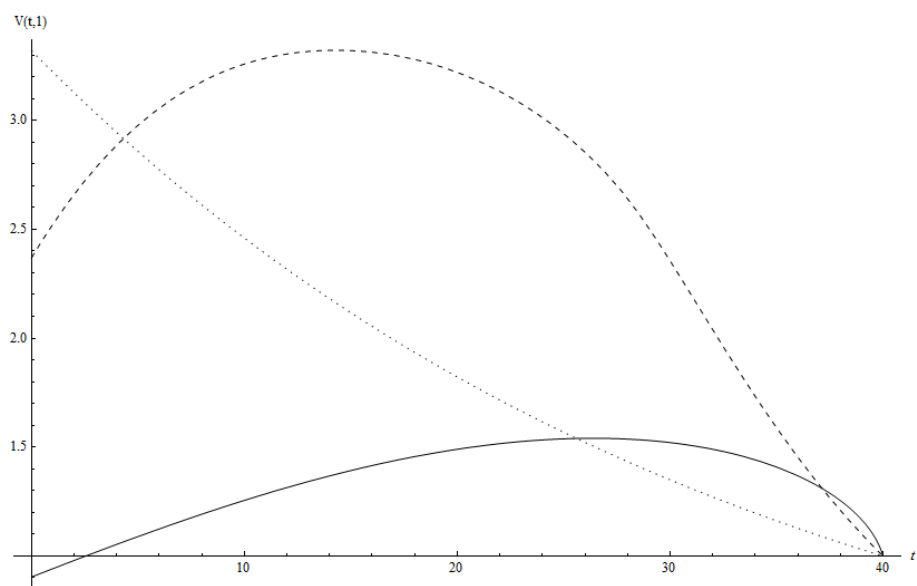


(b)

Figure 4.5: Fix  $\gamma_T = 0.2, k = 0.025$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).

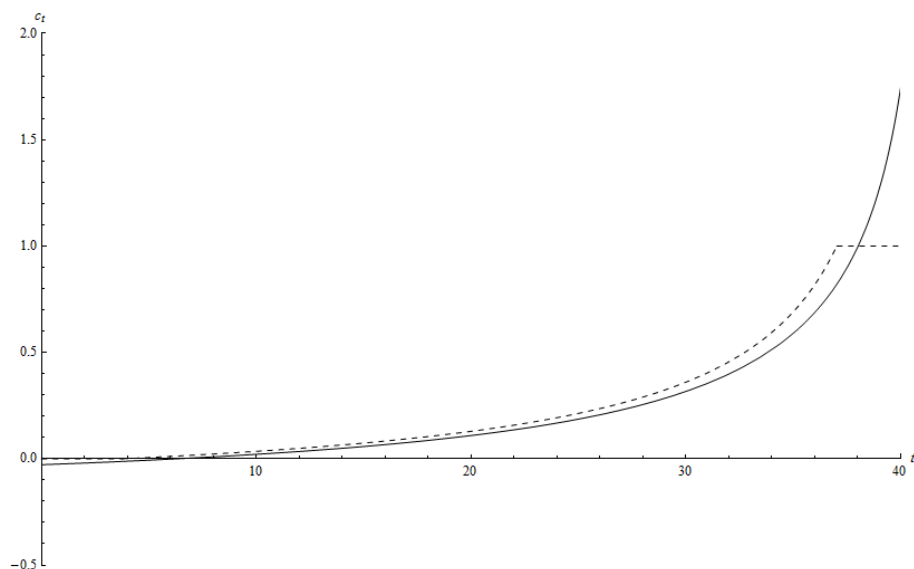


(a)

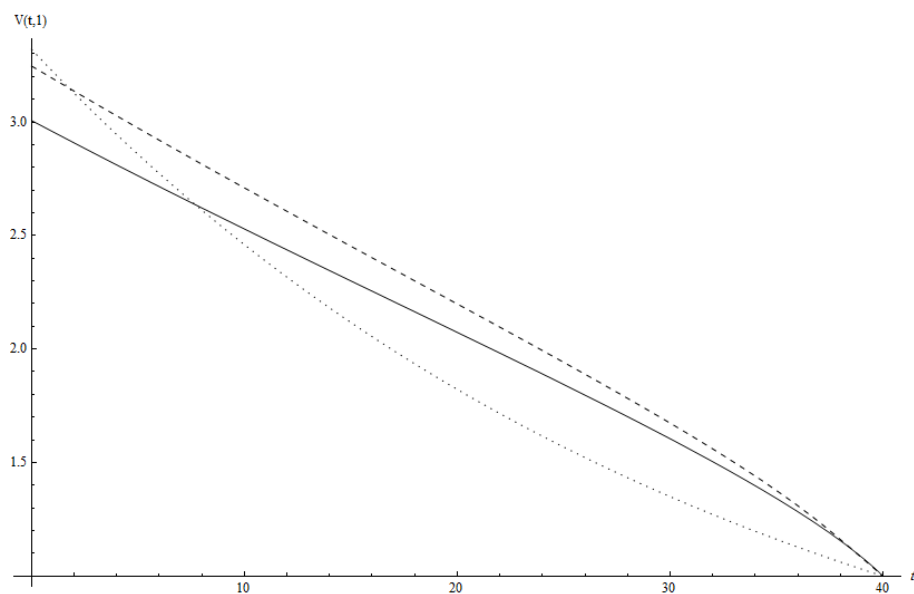


(b)

Figure 4.6: Fix  $\gamma_T = 0.2, k = 0$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).

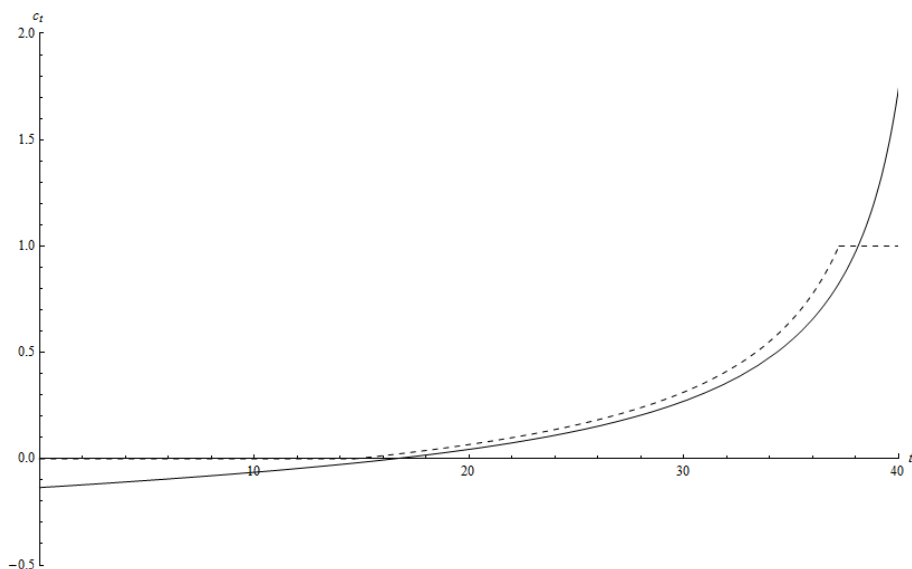


(a)

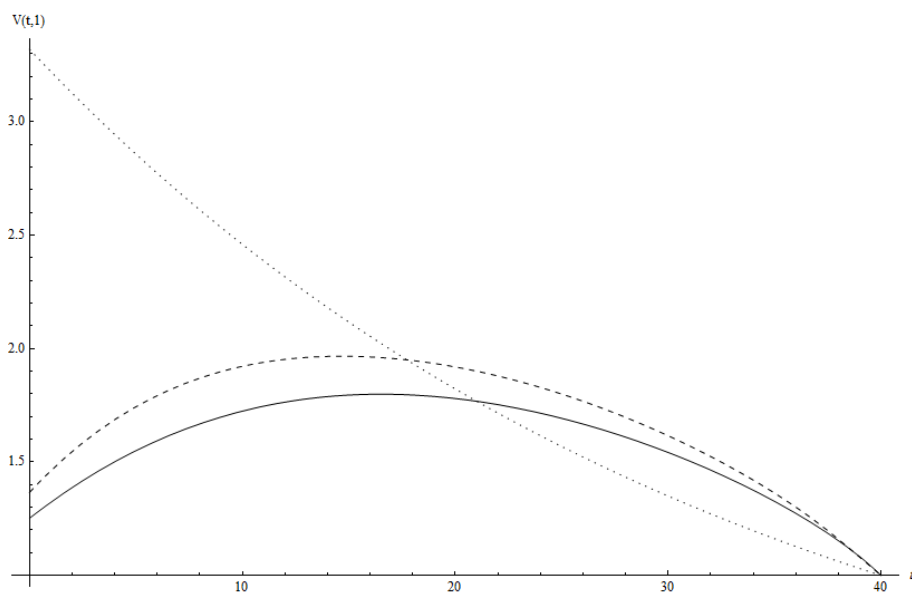


(b)

Figure 4.7: Fix  $\gamma_T = 1, k = 0.025$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).

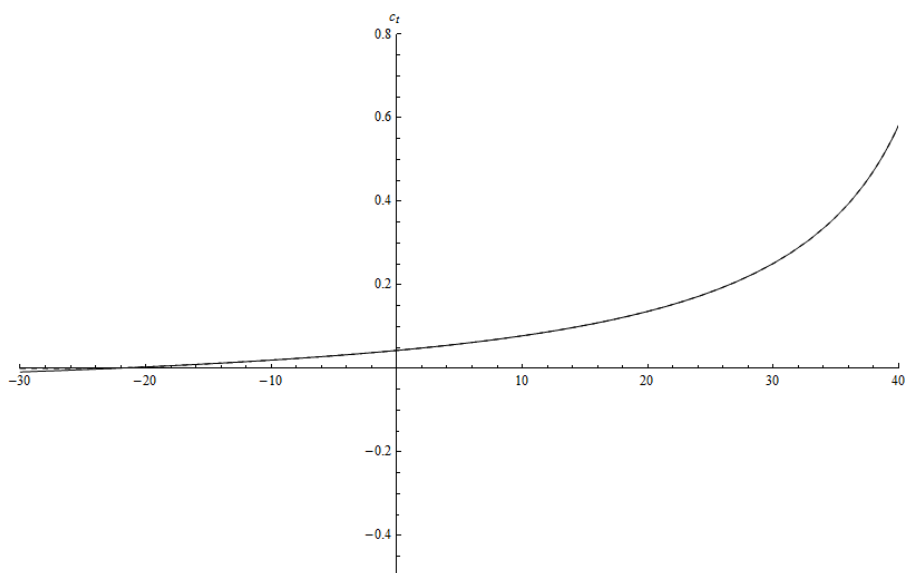


(a)

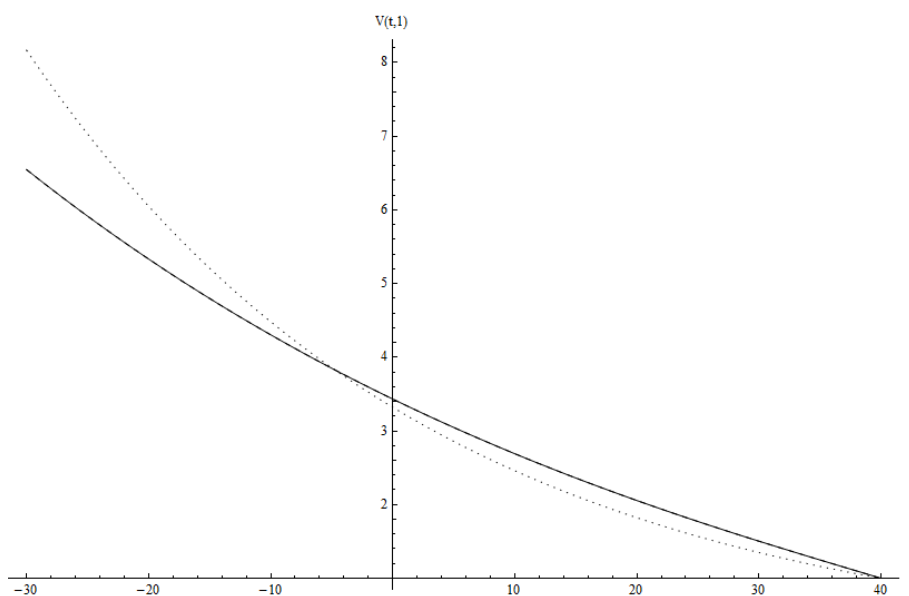


(b)

Figure 4.8: Fix  $\gamma_T = 1, k = 0$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).

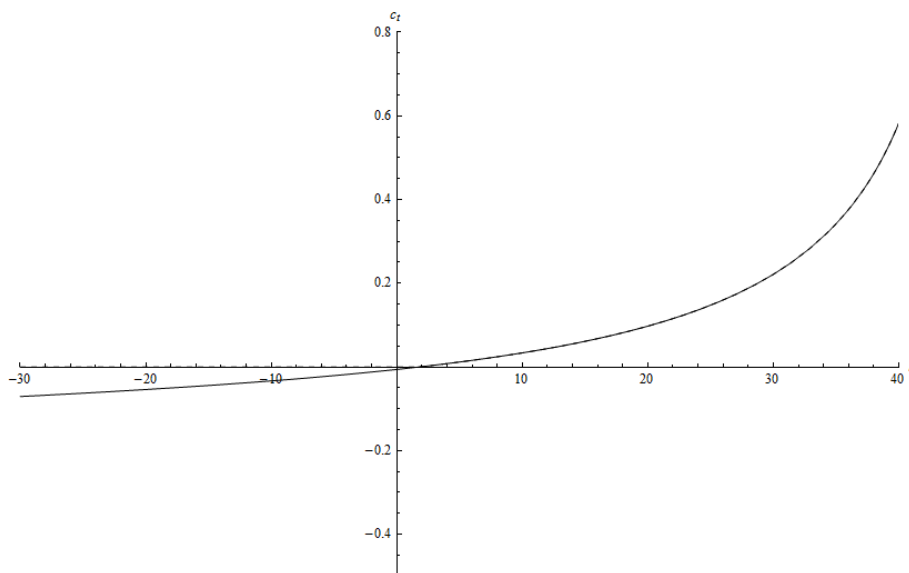


(a)

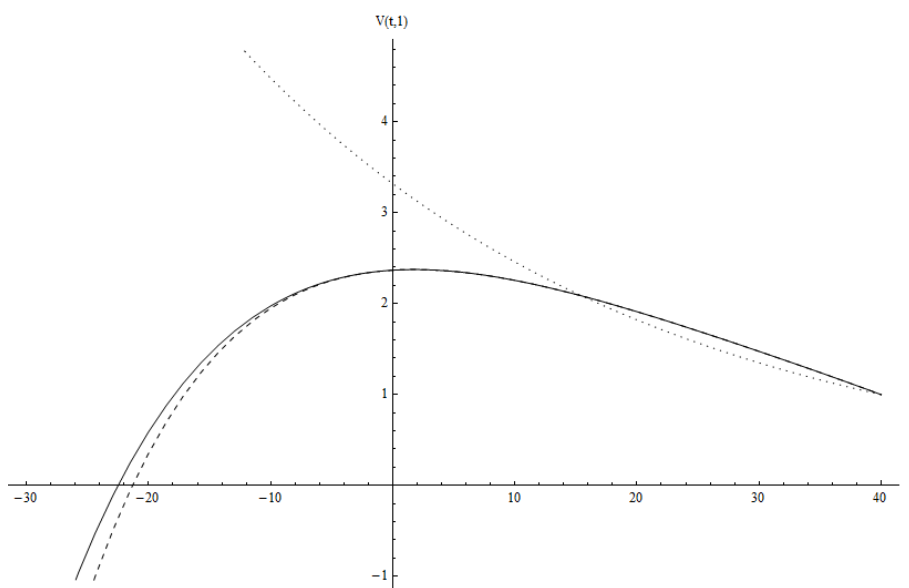


(b)

Figure 4.9: Fix  $\gamma_T = 3, k = 0.025$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).



(a)



(b)

Figure 4.10: Fix  $\gamma_T = 3, k = 0$ : (a) The ratio of investment to wealth,  $c_t$ , and (b) the equilibrium value function,  $V(t, 1)$ , against the commencement time  $t$ , for unconstrained (solid line), constrained (dashed line) and riskless investors (dotted line).

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