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Third-graders' problem solving performance and teachers' actions

Laine, Anu

UMERC, Umeå University

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Content

Learning Problem Solving
And
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In early September 2011 a group of about 20 mathematic education researchers gathered in Umeå for the 13th ProMath conference. The participants came from a large number of countries and represented a great variety of research traditions and educational systems. The common interest in problem solving in mathematics was visible through all 13 presentations.

The idea communicated in the conference theme, Learning Problem Solving and Learning Through Problem Solving, often came up in discussions, both in connection to the presentations and during coffee breaks and social activities.

The organizing committee would like to thank all participants for their contributions and for coming to Umeå to discuss what we all have a close relationship to: problem solving in mathematics education.

On behalf of Umeå Mathematics Education Research Centre:

Tomas Bergqvist.

The solving of problems and the problem of meaning
The case with grade eight adolescent students

Sharada Gade

Umeå Mathematics Education Research Centre, Umeå University

The problem of loss of meaning in schooling and teaching-learning of mathematics is explored in a study with adolescent students at two grade eight classes in Sweden with five frames of reference: deploying CHAT theoretical perspectives, incorporating student agency and identity, conduct of an action strategy, the design of meaningful mathematical tasks and the situatedness of these in local contexts of classroom and school. Exemplary of second-order action research, the conduct of five mathematical tasks enables reformulating the situated social practice in the classrooms, evidencing overt display of student identity in the fifth and final task. The addressing of problems posed by students in this open-ended task e.g. What is your favorite sport? Have you tested smoking? allows students to combine mathematical knowing and a sense of achievement, along with their selves as perceived in their local contexts. The inclusion of problems/mathematical tasks related to students' self is thus sought for in the curriculum of mathematics for adolescent students.

Key words: CHAT, situated learning, mathematical tasks, action research, agency and identity
ZDM: C70 - Teaching-learning-processes; D40 - Teaching methods and classroom techniques

Introduction

This paper explores the recognised problem of loss of meaning in schooling and teaching-learning of mathematics by drawing upon five frames of reference: the deploying of cultural-historical and activity or CHAT perspectives, the bringing forth of student agency and identity in their learning, the conduct of an action strategy to affect change, the conduct of mathematical tasks in succession and the situatedness of these in local contexts of classroom and school. Prior research in each of these areas serve as relevant points of departure. First, del Rio & Alvarez (2002) argue student interest as the most significant aspect that could bring about change, given

that students are found to be deeply dissatisfied with schooling. Drawing on CHAT perspectives, which I elaborate in the next section, they seek student participation in activities that have meaning, include action and emotion and provide for the development of students' identity. Second, Grootenboer & Jorgensen (2009) argue student agency and identity depend upon providing task opportunities, wherein a sense of achievement can be had by drawing upon prior mathematical knowledge by them. They refer to Boaler (2003) who seeks classroom practices that allow for interchange of agency of students with that of the discipline of mathematics. Third, Altrichter et al. (1993) characterise action strategies as co-ordinated actions taken in local contexts of classrooms, aimed at improving educational quality. The conduct of any strategy, they say, proceeds with no expectation of preconceived or immediate results. Fourth, the conception of mathematical task and activity conducive to perspectives that are adopted in this study follow Watson & Mason who argue:

> Task in the full sense includes the activity which results from learners embarking on a task, including how they alter the task in order to make sense of it, the ways in which the teacher directs and redirects learner attention to aspects arising, and how learners are encouraged to reflect or otherwise learn from the experience of engaging in the activity initiated by the task. (Watson & Mason; 2007, p 207)

Finally, the design of such mathematical tasks and ensuing activity in my study follows Lave (1990) who points to mutually constitutive nature of students learning and their social and cultural world asserting "what is to be learned is integrally implicated in the forms in which it is appropriated, so that, for example, *how* math is learned in school depends on its being learned *there"* (p. 310).

Taken together, the above arguments underpin conduct of an action strategy in collaboration with two teachers Greta and Marcus (All names are pseudonyms) in their Grade eight classrooms. This strategy was made up of five mathematical tasks conducted in succession, wherein each subsequent task was designed after conduct of the prior. It was in such conduct that Greta and Marcus' students evidenced an overt display of identity in the fifth and final task, which was open-ended and lent voice to the agency that they encountered as individuals in their respective classrooms. Shedding light on the search of meaning by students of schooling (Rio & Alvarez, 2002) the conduct of mathematical tasks as action strategy (Altrichter et al., 1993) allowed for interchange of agency between students and the mathematics they were learning (Boaler, 2003). It was by incorporating social and cultural aspects prevalent in their local contexts (Lave, 1990)

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that led the final task to allow students to pose problems, the pursuit of which enabled them to combine mathematical knowledge with a sense of achievement (Grootenboer & Jorgensen, 2009). What nature of agency and identity did students display when provided opportunity to pose meaningful problems in an open-ended mathematical task, within an action strategy, is the research question.

Theoretical underpinnings

Under ongoing exploration, CHAT perspectives perceive education as a process of simultaneous enculturation and transformation, alongside development of understanding and formation of minds and identities. Conducive to turbulent times such as ours, Wells & Claxton (2002) highlight three features that have bearing on my study. First, the role of cultural tools and artefacts which mediate understanding and afford means with which to know and share wisdom accumulated in any culture. It is learning to appropriate cultural and conceptual resources and the use of these with others, that provides for a learning that leads human development (Vygotsky, 1978). Second, they point out that values, goals and willingness of people who collaborate while using cultural tools and artefacts need not either be the same or coincide, thus providing opportunities for both enculturation as well as transformation. Finally, CHAT they stress is concerned not only with cognitive development but also of a person's mind and spirit as a whole. Any understanding of other's thought processes they stress needs to include one's interest, affect, emotion and volition. It is by drawing on these views that del Rio & Alvarez argue against fragmented approaches in education and favour the conduct of personally significant and socially meaningful activities:

> In meaningful practical activities, the object and purpose of the activity are apparent, the result of the action is contingent and feedback is immediate. When the activities are also productive, the results merge into a product that strengthens participants' identity and sense of self-efficacy. The produced artifact also becomes an external, stable symbol of the processes involved in producing it. (del Rio & Alvarez; 2002, p 64)

It was also the case that Greta and Marcus' classrooms and school were located in an industrial area, where at the time of conduct of the study there was considerable discussion in the press of possible closure of industry and possible loss of jobs for parents of students at the school. It followed that participation by Greta and Marcus' students in classroom activities depended on the manner in which mathematics was available for their appropriation in these local contexts of their

school. In agreement with Grootenboer & Jorgensen (2009) and with relevance to students learning in their local contexts, Lave (1990) also points out that routine instructional practices *of* classrooms could alienate learners, who would alternately gain from a curriculum designed *for* practice in which students are active agents. It was these arguments that formed backdrop to the design of the five-task action strategy which privileged active participation of students, moving attention away from a normative attention to their textbook. Lave (1992) has further highlighted the hypothetical nature of mathematical word problems in curricula which leave students, she says, to look upon everyday mathematics negatively by implication. Lave therefore argues for students' ownership of problems in a dilemma motivated manner in classroom activity, as is the case with problems encountered in everyday life. As outlined in the next section the design of five successive tasks enabled students to voice such concerns and address issues as faced by them in their respective classrooms.

CHAT perspectives significantly argue in addition that social practices produce not only knowledge but also participant identities, constituted through active relations with their social world. Students' identity Stetsenko (2010) argues is real work, in which their self is born and enacted in the activities that they participate. Human subjectivity and thinking she clarifies is a threefold process in which cultural tools and artefacts are provided through teaching, their use learnt by students, which in turn provides opportunity to transform their life's agendas. Such a view underpins the interchange of agency of students and mathematics (Boaler, 2003) its being situated in local contexts (Lave; 1990) and underlines providing for meaningful activities (del Rio & Alvarez 2002). With pedagogical implications of CHAT in mind, Stetsenko specifies teaching-learning to be:

> organized in ways where knowledge is revealed: (a) as stemming *out of social practice* - as its constituent tools; (b) *through social practice* - where tools are rediscovered through students' active explorations and inquiry; and (c) *for social practice* - where knowledge is rendered meaningful in light of its relevance in activities significant to students, that is, where knowledge is turned into a tool of identity development. (Stetsenko; 2010, p 13)

Methodology and methods

CHAT perspectives premise practical activities in which individuals participate, use cultural tools, gain agency, develop identity and transform their social world as comprehensive unit of

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analysis. These activities as Vygotsky (1978) argued are simultaneously object, tool and result of any study. The units of analysis in my study is thus participation of students in each of the mathematical tasks that constituted the action strategy deployed, where such conduct was a result of collaboration that Greta and Marcus and myself had come to agree upon. On my approaching their Rektor and seeking a grade seven for study at their school I was offered a grade eight instead, since this grade had demanding parents voicing concerns about the quality of their children's schooling. I visited Greta's class which was organised for regular students and later Marcus' class organised for more basic students. In Greta and Marcus' school offering specialised training in sports and music, it was also the case that Greta's class had the presence of a handful of boys who trained professionally for hockey. In a year ahead interview Greta mentioned that within instruction their presence demanded inordinate amount of her time and classroom space. While I deliberate my drawing upon cultural studies to theorise these concrete circumstances elsewhere (Gade, 2012) I now turn to perspectives that informed the design and conduct of the five mathematical tasks in succession.

Altrichter et al. (1993) outline action strategies as falling in an action research paradigm wherein questions about everyday work are asked so as to study and improve teaching-learning. Recognising the need to draw on situated theories that can inform action, they acknowledge too that social situations are complex and cannot be changed by any single action. They thus suggest criteria that could guide any sequence of actions that form an action strategy including (1) planning (2) acting and observing (3) reflecting and (4) replanning. Encouraging flexibility in one's approach with also not expecting predetermined results, Altrichter et al. importantly seek inclusion of voices of all stakeholders during design and conduct. It was to gather these voices in my study that I adopted narrative inquiry which led me to ascertain the experiences that Greta, Marcus and their students had in their local contexts. Alasuutari (1997) argues narrating in everyday life as a phenomenon to be studied in its own right, since the selves of individuals are not mere object in a physical world but importantly constructions lived by in existing social realities. Such manner of attention to these accompanied by my other observations of students' complaints about being tired, listening to music or being playful to avoid instruction lead me to surmise their lack of interest in mathematics or loss of meaning in school, or both, in agreement with del Rio & Alvarez (2002). In addition to drawing upon narrative inquiry I considered

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students' working in groups as pedagogical aim in my study. This followed Vygotsky's dictum that peer interaction is the leading activity amongst adolescents, instrumental in the development of their self-consciousness (Karpov, 2005). Designing my tasks for such conduct I was careful to have instructional content area also in mind, to avoid burden from conduct of the action strategy. Such manner of action, inclusive and not independent of stakeholder voice, is termed second-order action research (Elliott, 1991). I now offer background to the tasks, of which I dwell only upon the fifth one in detail within data and discussion.

I premised the design of my first task on the possibility that students may be resentful of using their textbooks, given that many of them seemed to display disinterest. I turned to non-routine tasks such as those from the Kängaru competition (http://ncm.gu.se/kanguru) and asked students to find area and perimeter of figures shown alongside Task 1 in the Table below:

Task 1   Task 2
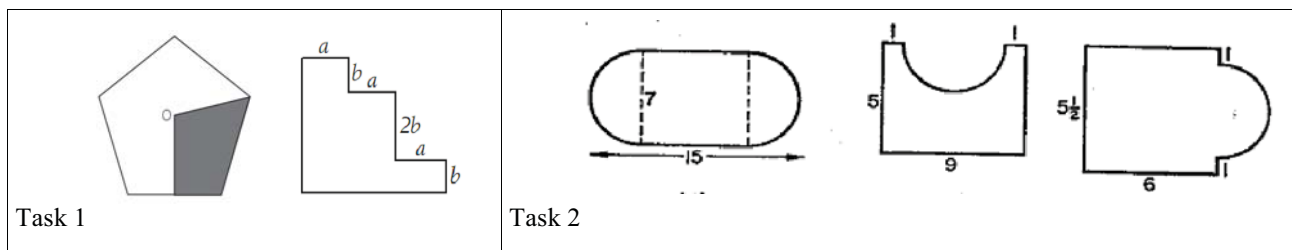
The conduct of Task 1 involved students first discussing their solutions in their respective groups, followed by their sharing these at the whiteboard with their classmates. This provided opportunity for student peers to observe and listen to alternate solutions and was indicative of initiating group work in Greta and Marcus' classroom culture. With intention of verifying my premise of students' possible aversion to the textbook I retained the goal of finding area and perimeter in group work in Task 2, yet offered figures that were from a text-book (Channon et al., 1970, p. 174). The conduct of this task strengthened my earlier premise, since I found the more basic students in Marcus' class to have difficulty in attempting this task. I was informed by Marcus that he found them struggling with their attempts, with one of them even coming up to me, expressing disappointment with facial expression and reporting "We need help." I surmised this feedback of students to come with a sense of their being let down by me, as their attempts at Task 1 may have given them a sense of hope in meeting the demands of mathematics expected of them. I thus reverted to everyday contexts while designing Task 3 and chose to work with maps taken from Internet search engine Google. Offering three maps that showed directions from the city centre (1) to their school (2) to a nearby town and (3) to the country's capital, I asked

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students to calculate the scales that were used in each map, in their respective groups. Being highly relevant to the experience of each student the conduct of this task was met with a lot of interest, with students asking if they could measure distances *as the crow flies* as well as taking pride in greater accuracy of scales that they calculated. Encouraged by such responses, I based Task 4 on various containers they encountered in their everyday and asked students to first estimate and then calculate their volume. This task was in fact better received by more basic students in Marcus' class, who felt no hesitation in guessing the volume in terms of number of dice or milk packets say, where those in Greta's class were cautious and wanted to be accurate in their estimation. My combined observation of such evidence of agency in students prepared ground for their acting with emotion in their final task, set in the topic of statistics.

Task 3	Task 4
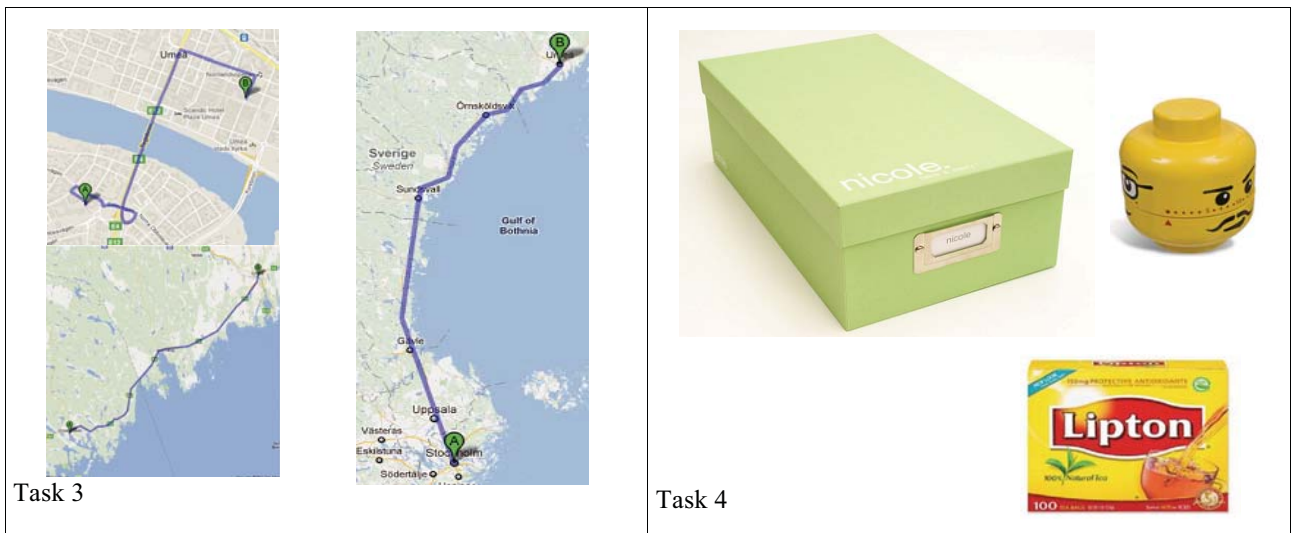
Data – The fifth task

With marked reformulation in students' agency in Greta and Marcus' instructional practice via the conduct of the first four tasks, I decided to give their students greater voice in the fifth task. It was with this in mind that I designed Task 5 to be open-ended and gave them opportunity to pose their own problems. In conducting this task myself, Greta and Marcus gave the following instructions (1) Work in groups of two or three (2) Decide on a question/pose a problem of your own choice (3) Collect data from other groups in the classroom and (4) Display your results in a column graph or pie chart. The sense of excitement displayed by students in either class while attempting this task was palpable. Greta, Marcus and me observed students groups to first formulate questions and then seek data from other groups towards addressing their problem,

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which understandably incorporated a sense of ownership. I present examples of students questions and graphs below.

  Vilken månad är du född i? Which month were you born?	 Vad är din favorit genre av film? What is you favorite genre of film
  50% What brand of cellphone do you own ?	 Favorit farger. What is your favorite colour?
  What is the country you have most travelled to?	 What brand of four wheelers does your family own?
  How much does you get as monthly pocket money?	 How many brothers and sisters do you have?

The eight graphs I present evidence the variety of problems that the majority of students in Greta and Marcus' class sought solutions to. However two particular solutions stood out against this

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norm and overtly expressed students' self or identity as experienced by them in the social practice of their classroom. The first of these which asked *What is your favourite sport?* was pursued in Greta's class in which boys playing hockey were present. As mentioned earlier on, it was the presence of these boys that demanded a lot of attention both symbolically and in reality within Greta's instruction. The second which asked *Have you tried smoking?* was pursued by a group in Marcus' class. This later group consisted of Alba who smoked cigarettes and was a regular student enrolled in Greta's class in the beginning of the year. At the time of conduct of this task Alba had moved, or may have even been asked to move to the more basic group in Marcus' class, leading to possible feelings of her resentment. I was aware that Alba's habit worried Greta, who as her teacher felt she was unable to do anything beyond speaking about it with Alba's parents. I argue that students responses to these two questions were real and meaningful to them in their local contexts, as was any interpretation of these as researcher also was. By overtly addressing self and identity, I argue that student groups in either class utilised Task 5 and demonstrated, or voiced as it were, that hockey was not the most favourite sport and that it was a large majority of students who had tried smoking. That this seemed to be the case can be seen from the first graph where hockey is represented by only four students with the football, curling, handball, badminton, basketball, riding and innebandy represented by the majority. Alba's graph showed too that more than three quarters, or 77% of students in her class had tried smoking, something that she had a history of being singled out for alone.

What is your favourite sport?	Have you tried smoking?
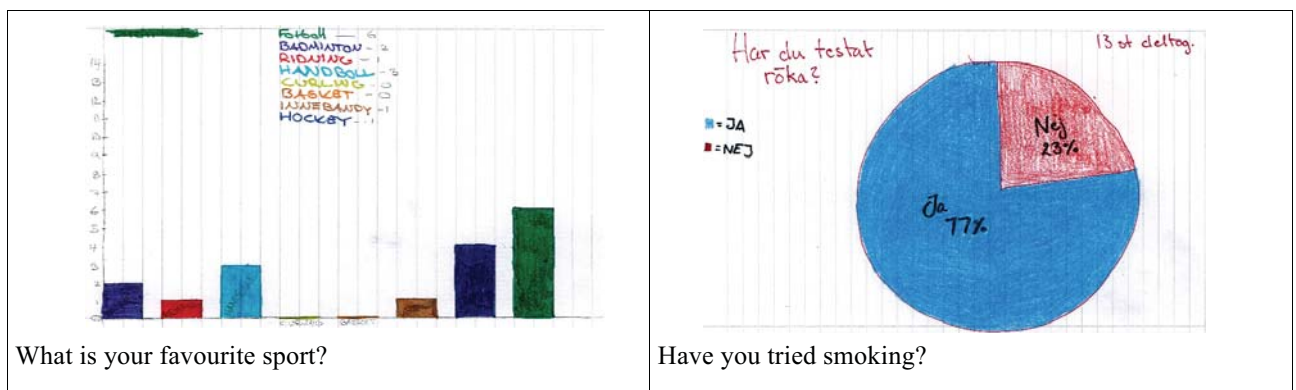
Discussion – The fifth and final task

I consider most student responses to the fifth and final task as quite normative, as can be expected in any Grade eight, except for the overt display of students' self and identity in the last two cases I report above. Central to the five frames of reference deployed in this paper I discuss implications

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of these graphs in their reverse order. It was drawing upon Lave (1992) that I first shifted focus away from students' textbook, which ultimately resulted in the last two solutions and problems posed as being meaningful to their selves in the social practice of *their* classroom, addressing dilemmas *they* faced within. Such problems designed specifically *for* their classroom practice, I argue, resulted in students not feeling alienated, voicing concerns and dilemmas *being* faced in their social reality (Lave, 1990; Alasuutari, 1997). Such overt display of self and identity was representative of how students learning and their social world were mutually constitutive. The participation of Greta and Marcus' students also exemplified Watson & Mason's (2007) notion of activity that surrounded a mathematical task, within which it was that students displayed visible shifts in their agency. Greta and Marcus' guidance in conduct of these was no less significant as in speaking native Swedish they were able to seek engagement of students in each and every task. In fact the overt display of self and identity in the fifth and final task was neither anticipated nor planned. Following Altrichter et al. (1993) our actions taken to change and improve educational quality was not a single one, but many successive actions that vitally took stakeholder voice into account. This study thus evidences how it is possible to bring about greater student engagement both in classroom teaching-learning and the discipline of mathematics. A visible representative of interchange of student agency and mathematics in particular, were exemplified by the two graphs about students' favorite sport and their attempts at smoking (Boaler, 2003). It was via these two graphs that student groups showcased their combining a sense of accomplishment with their mathematical knowledge (Grootenboer & Jorgensen, 2009). Following CHAT perspectives, the fifth and final task was not only a cultural tool and artefact whose use students were being enculturated into, but also one they were transforming as means of expressing self, identity and their very being (Wells & Claxton, 2002). Finally the design and conduct of tasks based upon the loss of meaning in mathematics and schooling that del Rio & Alvarez (2002) alluded to, was a viable strategy that led to greater agency and resulted in students voicing their selves and their identity. These actions were those that *became* personally significant and socially meaningful. My drawing on voices grounded in social practices within local contexts, lent finally to the immediacy and nature of change that any second-order action research, it is argued, has potential to bring about (Elliott, 1991).

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In conclusion

My attempt to address the problem of loss of meaning in schooling and the teaching-learning of mathematics in and through my study has led to an approach situated in the social realities of local contexts of classroom and school. Towards any resolution of this issue I have found it imperative to take all stakeholders voices into account. Besides Greta, Marcus, their students and their Rektor, at Greta's request I agreed to meet parents of students at their parent-teacher meeting. My rationale for agreeing to this was based on the ethical need for the practice of educational research to stand up to societal scrutiny. Towards this, my drawing upon situated stakeholder narratives was means with which to not only make personal sense of how these were situated, but also how my study itself was to be situated in wider society. Narratives, following Alasuutari (1997), are phenomena which enable research to attend to how individual selves became personalities in social realities. Towards this, attention in my study to activities that accompanied the mathematical tasks (Watson & Mason, 2007) provided opportunity for Greta, Marcus and me to direct as well as redirect various aspects of these very realities. Not achieved by a single action, as Altrichter et al. (1997) rightly point out, the incidence of this was possible only by a sequence of tasks in which the importance of allowing for group work is also noteworthy. Following Vygostky, I argue that it was such manner of conduct that gave students many an opportunity to not only develop self-consciousness, but also its display as self and identity (Karpov, 2005). Such an holistic approach to solving problems, inclusive of the social being and emotions of students, is I find often overlooked in cognitive studies of problem solving. In light of Stetsenko's (2010) arguments that student identity is real work, born and enacted in activities being participated, my study shows how students' identity was born out of their social practice, through social practice and for the social practice that locally prevailed. It was successive changes brought about in instructional practice via conduct of an action strategy, that the tasks and ensuing activities *became* meaningful for Greta and Marcus' students (Rio & Alvarez, 2002). Based on my study, I thus seek inclusion of problems and/or tasks related to students' self in mathematics curriculum for adolescent students. Not allowing for such opportunities, would risk leaving learner as well as that which is learnt unchanged and unaltered in education.

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Investigating the Problem Field of Triangular Pyramids

Günter Graumann
University of Bielefeld

Working with geometry in space is a very important task for school especially to develop spatial perception. The triangular pyramids (general tetrahedrons) are the simplest geometrical solids (analogue to the triangles in the plane geometry) and can be produced easily as solid body or/and as net or surface body. In contradiction to the triangle the triangular pyramids deliver more different shapes. So it is an interesting problem field to discuss different types of triangular pyramids and make an ordering system of these. Here we will find out all symmetric triangular pyramids.

Introduction

Besides the development of knowledge and the training of special skills a fundamental aim of school is the development of general competences which can help to master life. In mathematics education you can gear towards several such general competences.

In the German educational standards (Bildungsstandards) from 2003 e.g. the following general competences for mathematics education are stated: Arguing, communicating, problem solving, modelling, picturing and dealing with symbols and formal or technical elements.

In the problem field of triangular pyramids we will focus on the general competence of the development of perception based on handlings, especially spatial perception, as well as the willingness and ability to work positive with problems, applying systematisation and discussion in respect to all possible cases concerning a complex problem.

The fundamental figures in plane geometry besides points, segments and straight lines are the triangles. They are defined by three points which do not lie on a straight line. The analogues of triangles in space are triangular pyramids (general tetrahedron). They are defined by four points which do not lie on a plane.

Though the triangular pyramids are simple and fundamental figures mostly they are not discussed in school. I will advocate here for investigating triangular pyramids in school. They represent a problem field which is not to difficult to picture on paper but can deepen spatial perception as

well as the training of abilities like problem solving and working systematically. Especially with looking out for symmetric triangular pyramids we can tie in with usual discussions on symmetry and symmetric quadrilaterals. In respect hereof I am tying to my presentation on the ProMath conference last year.

Looking out for all possible symmetries of a triangular pyramid

A triangular pyramid ABCD is constituted by a set of the four vertices {A, B, C, D} and its line-connections. Thus *all permutations of the four vertices* can represent a symmetry of ABCD.

For the regular triangular pyramid (the regular tetrahedron) all permutations of the four vertices really build a symmetry mapping. If the triangular pyramid is not the regular one then we have to choose the symmetry mappings out of this set of all permutations.

To be sure not having missed a permutation we easily can find out by combinatorial considerations that there exist exactly *24 permutations* of four different points A, B, C, D. Noting only the image of ABCD these twenty-four permutations can be pictured by

ABCD,   ABDC,   ACBD,   ACDB,   ADBC,   ADCB,

BACD,   BADC,   BCAD,   BCDA,   BDAC,   BDCA,

CABD,   CADB,   CBAD,   CBDA,   CDAB,   CDBA,

DABC,   DACB,   DBAC,   DBCA,   DCAB,   DCBA.

For the *geometrical interpretation* of these twenty-four permutations as mappings of the tetrahedron onto itself we first can remind that symmetry mappings of a bounded body in space only can be a plane-reflection, an axial rotation or a combination of plane-reflection and axial rotation. Secondly it is good by identifying the geometrical interpretation to use a system by working through of all these twenty-four permutations. One idea might be the *classification by cyclic sub-permutations* - especially circles of only one point (i.e. fixed points of the corresponding mapping).

1. If all *four vertices are fixed points* then of course we do have the **identity mapping** which can be written down as   ABCD → ABCD.

2. If *three vertices are fixed points* then the forth point has no other image than itself, i. e. all four points are fixed points as before.

3. If *two vertices are fixed points* then on one hand (3a) they can lie on a reflection plane while the two other vertices build a pair point-image and on the other hand (3b) the two fixed points can lie on a rotation axis.

3a) If the two fixed points lie in a reflecting plane (i.e. we have a **plane-symmetry**) then the reflection plane is determined by an edge and the midpoint of the opposite edge (e.g. $\overline{CD}$ and M_{AB}) whereat the opposite edge must be perpendicular to the plane. Looking out for all plane- symmetries that come into question it is clear that such a symmetry is possible for all 6 edges (in combination with the midpoint of the opposite edge). We can write down these six permutations for instance as


P_1:   ABCD → BACD (with fixed edge $\overline{CD}$)   [see figure]

P_2:   ABCD → CBAD (with fixed edge $\overline{BD}$)

P_3:   ABCD → ACBD (with fixed edge $\overline{AD}$)

P_4:   ABCD → DBCA (with fixed edge $\overline{BC}$)

P_5:   ABCD → ADCB (with fixed edge $\overline{AC}$)

P_6:   ABCD → ABDC (with fixed edge $\overline{AB}$)
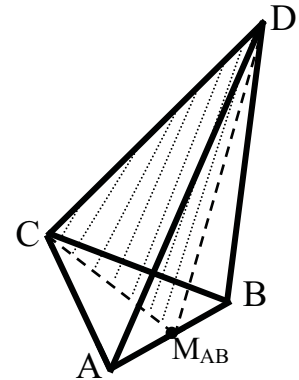
3b) If the two fixed vertices lie on a rotation axis then the other two vertices must build a pair point-image. This is possible only for rotations with 180°. But then all four points lie in a plane. Thus such a 180°-rotation is not possible for a triangular pyramid.

4. If we have only *one fixed vertex* then the other three vertices must build a cyclic permutation of order three. This is possible on two ways (if e.g. D is the fixed vertex then we can have ABC → BCD or ABC → CAB). These two different mappings with one fixed point therefore build an axial rotation with an axis through one vertex and the midpoint of the opposite triangular side (e.g. D and M_{ABC}) and the rotation angle 120° or 240°. That means we have a **rotation-symmetry.**

Looking out for all rotation-symmetries with 120° or 240° that come in question it is clear that such a symmetry is possible with any vertex so that we can get $4 \cdot 2$ rotation-symmetries with angle 120° or 240° . They can be written down for instance as
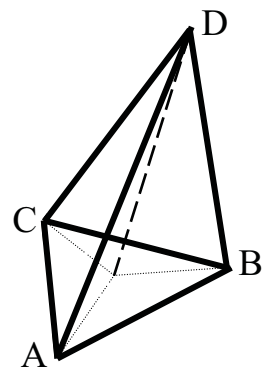
R1:   ABCD → BCAD (with fixed vertex D and 120°) [see figure on the previous page]

R2:   ABCD → CABD (with fixed vertex D and 240°)

R3:   ABCD → BDCA (with fixed vertex C and 120°)

R4:   ABCD → DACB (with fixed vertex C and 240°)

R5:   ABCD → CBDA (with fixed vertex B and 120°)

R6:   ABCD → DBAC (with fixed vertex B and 240°)

R7:   ABCD → ADBC (with fixed vertex A and 120°)

R8:   ABCD → ACDB (with fixed vertex C and 240°)

In any of these eight cases the triangular pyramid has at least one side as an equilateral triangle and the fourth vertex perpendicular to the midpoint of this equilateral triangle. Thus the triangular pyramid also has three plane-reflections as symmetry mappings (in our example – see figure above – with fixed edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$).

5. If we have **no fixed vertices** then we can have two cycles of cardinal number two (5a) or one cycle of cardinal number four (5b).

5a) The first case causes two pairs of vertices (i.e. two edges) which are rotated with 180°. This means we have a 180°-rotation (called *line-reflection*) with an axis (line) through the midpoints of two opposite edges of the triangular pyramid whereat these two edges must be perpendicular to the axis. This means we have a **line-reflection-symmetry**.
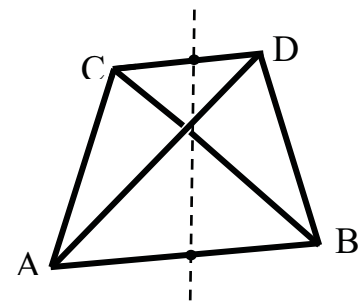
Looking out for all line-reflection-symmetries that come in question it is clear that such line-refection-symmetries are possible on three ways (because any time two opposite edges determine such line-reflection). These three line-reflections are

L1:   ABCD → BADC (with axis through the midpoints of $\overline{AB}$ and $\overline{CD}$) [see figure above]

L2:   ABCD → CDAB (with axis through the midpoints of $\overline{AC}$ and $\overline{BD}$)

L3:   ABCD → DCBA (with axis through the midpoints of $\overline{AD}$ and $\overline{BC}$)

5b) **Permutations with cycles of cardinal number four** we can find out as the remaining six permutations of our above named twenty-four permutations:

C1:   ABCD → BCDA [see figure on the right],
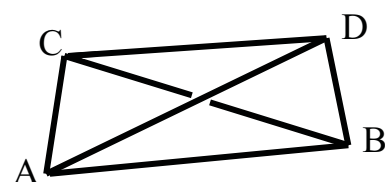
C2: ABCD → BDAC,

C3: ABCD → CADB,   C4: ABCD → CDBA,

C5: ABCD → DCAB,   C6: ABCD → DABC.

As geometrical interpretation we can find different *combinations of reflection and rotation*,

e.g. ABCD → BACD (plane-reflection with CD fixed and then rotation with B fixed and 120°)

ABCD → BACD (plane-reflection with CD fixed and then rotation with B fixed and 240°)

ABCD → CBAD (plane-reflection with BD fixed and then rotation with C fixed and 120°)

ABCD → CBAD (plane-reflection with BD fixed and then rotation with C fixed and 240°)

ABCD → DBCA (plane-reflection with BC fixed and then rotation with D fixed and 120°)

ABCD → DBCA (plane-reflection with BC fixed and then rotation with D fixed and 240°).

In any of these six cases the cyclic permutation causes more symmetries. Because the permutation keeps lengths of edges four edges have the same length and the two others have one (possible other) length so that all four triangular sides are congruent isosceles triangles. This causes two plane-symmetries with one of the two "other" edges as fixed edge (in our example C_1 the reflection-planes with fixed edge $\overline{AC}$ respectively $\overline{BD}$). Applying the given permutation two times we get a line-reflection and applying it three times we get another cyclic permutation which causes a cyclic change in the opposite direction than the given permutation (e.g. applying C_1 two times gives the line-refection L_2 and applying it three times gives the cyclic permutation C_6. Applying C1 four times leads us to the identity mapping). The combination of the two plane-reflections then delivers a second line-reflection and finally the combination of the two line-reflections delivers the third line-reflection. The three line-reflections together with the two plane-reflections and the two cyclic permutations as well as the identity mapping build a group. Thus in the case of one symmetry generated by a cyclic permutation we have three line-reflection-symmetries, two plane-symmetries and one more cyclic symmetrie.

Well! The discussed cases together did give us all twenty-four permutations of the vertices of a triangular pyramid (i.e. all twenty-three symmetries of a regular tetrahedron). And with this we also did get all *possible* twenty-three symmetries of a triangular pyramid.

Moreover, it came out that a symmetric triangular pyramid always does have at least one plane-symmetry or one line-reflection-symmetry because a rotation-symmetry as well as a cyclic symmetry causes plane-symmetry or/and line-reflection-symmetry.

Looking out for symmetric triangular pyramids

We now have the instruments to work out all types of symmetric triangular pyramids. For this we first look out for triangular pyramids with a symmetry of one of the above named possible types. After that we look out for triangular pyramids which have besides a plane-symmetry or a line-reflection-symmetry one more symmetry (plane-symmetry, line-reflection-symmetry, other rotation-symmetry, symmetry with cyclic permutation). Finally we investigate combinations of a plane-symmetry or a line-reflection-symmetry with two or more symmetries in addition.

a) A triangular pyramid can have only <u>one plane-symmetry</u> of the above named ones and there exist triangular pyramids which have only this **one plane-symmetry (type P)** [see figure above].

b) A triangular pyramid can have only <u>one line-reflection-symmetry</u> and there exist triangular pyramids which have only this **one line-reflection-symmetry (type L)** [see figure above].

c) A triangular pyramid that does have <u>a rotation axis with rotation-angles 120°, 240°</u> also has three plane-symmetries as shown above and with the consideration above we find a triangular pyramid with only **one rotation-axis** and **three plane-symmetries** in addition **(type R)**.

d) A triangular pyramid with <u>a symmetry generated by a cyclic permutation</u> has more symmetries as shown above. From the figure above (see 5b) we get a triangular pyramid with **symmetries generated by one cyclic permutation and its inverse permutation** as well as **three line-reflection-symmetries** and **two plane-reflection-symmetries** in addition **(type C)**.

e) If we look out for a triangular pyramid with <u>two plane-symmetries</u> then we have to differentiate whether the two edges which are defining the symmetry planes have one vertex in common (first case) or not (second case).

In the first case (e.g. $\overline{CD}$ and $\overline{BD}$ with D in common are the two edges which determine the two plane-reflections) the combination of the two plane-reflection generates a symmetry-rotation (in our example we have $P_2 \circ P_1 = R_1$) and the combination of this rotation with itself generates the rotation with same axis but different rotation measure (in our example $R_1 \circ R_1 = R_2$). Moreover the combination of this second rotation with the second plane-reflection results in a third plane-reflection (e.g. $R_2 \circ P_2 = P_3$). Thus a triangular pyramid with two plane-symmetries whereat the determining edges have one vertex in common is a triangular pyramid with at least five symmetries we discussed already under **situation c)**.

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In the second case of two plane-symmetries where the two determining edges of the two plane reflections are opposite to each other (e.g. $\overline{CD}$ and $\overline{AB}$ are the determining edges) the combination of these two plane-reflections gives a line-reflection-symmetry (in our example $P_2 \circ P_5 = L_2$). These three mappings together with the identity mapping build a group. Thus we have a triangular pyramid with **two plane-symmetries** and **one line-reflection-symmetry (type PL)**.

f) If we are looking out for a triangular pyramid with <u>two line-reflection-symmetries</u> we can trace back to 5b) and find out that the combination of two line-reflections delivers the third line-reflection. These three line-reflections together with the identity mapping build a group. Thus we have a triangular pyramid with **three line-reflection-symmetries (type LL)**.

g) If we have <u>a plane-symmetry *and* a line-reflection-symmetry</u> then we have to differentiate whether the reflection-line does lie in the reflection-plane (first case) or not (second case).

In the first case the combination of both mappings generates a second plane-reflection (e.g. $P_1 \circ L_1 = P_6$) and we have the **situation of e)**.

In the second case the combination of both mappings generates a cyclic permutation (e.g. $P_5 \circ L_1 = C_1$) and we have the **situation of d)**.

h) If we look out for a triangular pyramid with <u>a plane-symmetry *and* a rotation-symmetry of type R</u> then we have to differentiate whether the rotation axis does lie in the symmetry plane (first case) or not (second case).

In the first case we find the **situation of c)**. (E.g. $P_1 \circ R_1 = R_2$, $R_1 \circ P_1 = P_3$ and the combination of these both gives P_3.)

In the second case (e.g. with P_6 and R_1) we have two sides that are equilateral triangles because by mapping the equilateral triangle of the rotation-basis (in our example ABC) with the plane reflection we get an equilateral triangle (in our example ABD) too. But because the three edges which match with the rotation axis (in our example $\overline{AD}$, $\overline{BD}$, $\overline{CD}$) have the same length all edged must have the same length. Thus we have a **(regular) tetrahedron**.

i) If we investigate a triangular pyramid with <u>a plane-symmetry *and* a symmetry generated by a cyclic permutation</u> then we again have to differentiate between two cases.

In the first case the plane-reflection is already generated by the cyclic permutation as shown in **situation d)** and we have a triangular pyramid described there.

In the second case we have in addition to the seven symmetry mappings which are generated by a cyclic permutation (e.g. $C_1, C_6, L_1, L_2, L_3, P_2, P_3$) – as shown above – another plane-reflection

(e.g. P$_1$). Then not only four edges but also all six edges must have the same length. Thus we have once more a **(regular) tetrahedron**.

j) As next we look out for a triangular pyramid with <u>a line-reflection-symmetry *and* a rotation-symmetry of type R.</u> Because (compare c) with the rotation-symmetry we also get three plane-symmetries (e.g. with R$_1$ we get P$_1$, P$_2$, P$_3$) it always comes out that the given reflection-line does lie in one of these three planes (in our example M$_{AB}$M$_{CD}$ ⊂ plane of P$_1$ and M$_{AC}$M$_{BD}$ ⊂ plane of P$_2$ and M$_{AD}$M$_{BC}$ ⊂ plane of P$_3$). From the situation of i) we can conclude another plane-reflection (e.g. P$_6$) and can deduce as done in situation k) that all edges have the same length. Thus we have a **(regular) tetrahedron** too.

k) If we look out for a triangular pyramid with <u>a line-reflection-symmetry *and* a symmetry generated by a cyclic permutation</u> then we only have to go back to the **situation of d)** because a cyclic permutation already produces all three line-reflections.

l) If we look out for a triangular pyramid with <u>three plane-symmetries</u> then we have to differentiate whether the three planes of symmetry have one vertex in common (first case) or not (second case).

In the first case we have the **situation of c).**

In the second case we can choose two planes with no vertex in common (e.g. P$_1$ and P$_6$). The third plane (e.g. P$_2$) then must have one vertex in common with one of these two planes because we have only four vertices. By combination of each two of these plane-reflections we get symmetry-rotations of type R and a line-reflection (e.g. P$_1$ ∘ P$_2$ = R$_1$ and P$_1$ ∘ P$_6$ = L$_1$). With situation j) then we get that the triangular pyramid must be a **(regular) tetrahedron**.

m) For the discussion of <u>three line-reflection-symmetries</u> we just have the **situation of f)**.

n) For the discussion of <u>three or more symmetries with at least two different types</u> we have to add at least one symmetry to each of the cases c) to k). Ignoring those cases which already induced the (regular) tetrahedron we only have to start with one of the cases c), d), e).

If we have in situation c) one more symmetry then we can find three plane-symmetries whereat the three planes do not have one vertex in common because we already have three planes with one vertex in common and each additional symmetry delivers an additional plane-symmetry [see c) d) or h)]. From l) then it follows that the triangular pyramid is the **(regular) tetrahedron**.

If we have in situation d) one more plane-reflection or rotation or cyclic permutation then we again get at least a new plane-symmetry and can conclude from i) that triangular pyramid is the **(regular) tetrahedron**.

If we have in situation e) with type PL one more symmetry then we easily can find out that all edges must have the same length, thus the triangular pyramid is the **(regular) tetrahedron**.
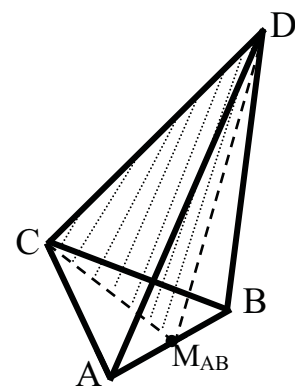
Summing up all situations we get the following types of symmetric triangular pyramids:
- (P)   triangular pyramids with only one plane-symmetry
- (L)   triangular pyramids with only one line-reflection- symmetry
- (PL)   triangular pyramids with two plane-symmetries and one line-reflection-symmetry
- (LL)   triangular pyramids with just three line-reflection-symmetries
- (R)   triangular pyramids with three plane-symmetries and two rotation-symmetries
- (C)   triangular pyramids with two plane-symmetries, three line-reflection-symmetries and two symmetries generated by a cyclic permutation
- (regT) tetrahedrons with twenty-three symmetries.

Each of these seven types of symmetric triangular pyramids has special geometric characteristics.

(P) A triangular pyramid with one plane-symmetry can be constructed in the following way:
Given the symmetry plane we can fix any two different points (e.g. named C, D) on it and furthermore fix any point (e.g. named A) outside of the symmetry plane. As forth vertex we then take the point (e.g. named B) we get as reflection point of this point out of the symmetry plane. These four points (A, B, C, D,) then of course build a triangular pyramid with one plane-symmetry (in respect to our named points with fixed edge $\overline{CD}$). This triangular pyramid has two isosceles sides with a common basis (in our example ABC and ABD) and the two other sides are congruent to each other (in our example ACD and BCD).

(L) A triangular pyramid with one line-reflection- symmetry can be constructed in the following way:
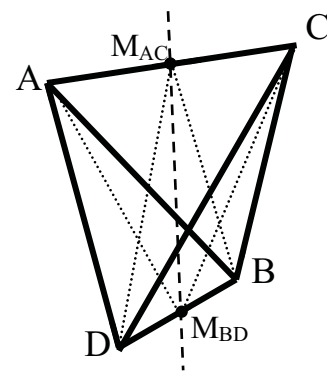Given any straight line and two different segments which have this line as perpendicular

bisector but do not lie in one plane. Then of course the endpoints of these two segments (in the attached figure A, B and C, D) build a triangular pyramid with the first given line as reflection line. Because the line-reflection keeps length in this triangular pyramid we have two pairs of opposite edges with equal length (in our example $|\overline{AC}| = |\overline{BD}|$ and $|\overline{AD}| = |\overline{BC}|$) and two pairs of congruent sides (in our example ABC is congruent to ABD and ACD is congruent to BCD).
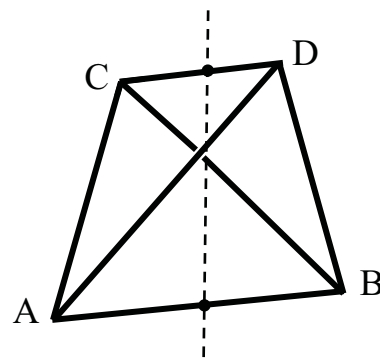
(PL) A triangular pyramid with two plane-symmetries and one line-reflection-symmetry we can construct by starting with two *congruent* isosceles triangles which are matched together at their basis (e.g. ABD and DBC are matched together at $\overline{BD}$). Then the plane determined by the connection of the apexes of the two isosceles triangles and the midpoint of the basis (in our example ACM_{BD}) is perpendicular to the basis and the plane determined by the basis and the midpoint of the connection of the apexes (in our example BDM_{AC}) is perpendicular to the connection of the apexes.

Thus the corresponding plane-reflections deliver plane-symmetries of the triangle pyramid. As consequence we get a line-reflection-symmetry in addition (in our example with $M_{AC}M_{BD}$ as axis). This triangular pyramid has four edges of equal length (e.g. $\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{AD}|$).

(LL) A triangular pyramid with three line-reflection-symmetries we can construct in the following way:

Given any straight line and two segments *of same length* which have this line as perpendicular bisector but do not lie in one plane. One can show that in this construction the endpoints of these two segments build a triangular pyramid with all three line-reflection-symmetries.
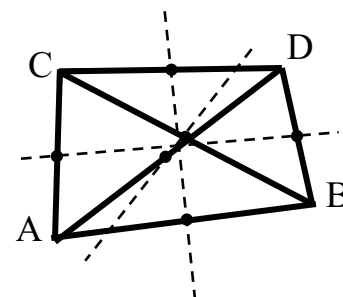
This triangular pyramid can be characterized with equal length of each pair of opposite edged (e.g. $|\overline{AB}| = |\overline{CD}|$, $|\overline{AC}| = |\overline{BD}|$, $|\overline{AD}| = |\overline{BC}|$).

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(R)   A triangular pyramid with rotation-symmetries of type R we can construct e.g. in the following way:

Given an equilateral triangle (e.g. named ABC) and a point (in our example named D) lying on the line which goes through the midpoint of the given triangle and is perpendicular to the plane of the equilateral triangle. This fourth point of course has not to lie in the plane of the triangle.
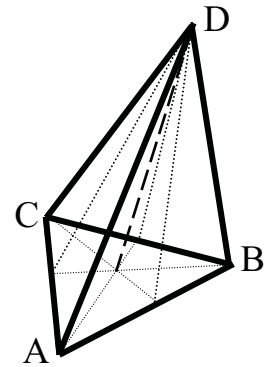
Then these four points build a triangular pyramid with two rotation-symmetries (rotation measures 120° and 240° and one axis) as well as three plane-symmetries. This triangular pyramid has two triples of edges with equal length (in our example $|\overline{AB}| = |\overline{BC}| = |\overline{AC}|$ and $|\overline{AD}| = |\overline{BD}| = |\overline{CD}|$). Thus it has three sides as isosceles triangles which are congruent to each other and the last side as equilateral triangle.

(C)   A triangular pyramid with symmetries generated by cyclic permutations can be constructed in the following way:

We start with two congruent isosceles triangles which are matched together at their basis' of equal length (e.g. ACB and ACD are matched together at $\overline{AC}$) so that the distance of the apexes of the two isosceles triangles has equal length with the basis of the isoscales triangles (in our example $|\overline{AC}| = |\overline{BD}|$). This triangular pyramid has four edges of equal length whereat the last two edge have (a different) equal length (in our example we have $|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{AD}|$ and $|AC| = |\overline{BD}|$). Thus all four sides are congruent to each other.
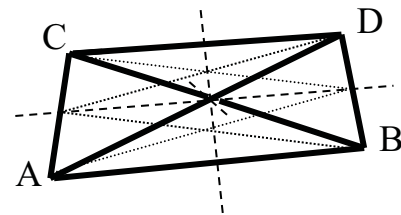
(regT)   A regular tetrahedron we can construct e.g. by starting with two congruent *equilateral* triangles which are matched together at one side of each so that the distance of these apexes is equal to length of the sides. Such a triangular pyramid then does have all edges with same length; thus it is a regular tetrahedron.
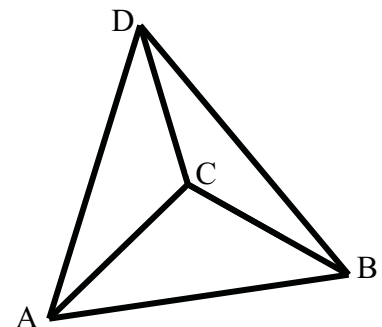
The symmetry groups of the symmetric triangular pyramids (apart from the regular tetrahedron) can be described e.g. in the following way:

Type (P):   Type (L):

1. \ 2.	Id	P_1
Id	Id	P_1
P_1	P_1	Id

1. \ 2.	Id	L_1
Id	Id	L_1
P_1	L_1	Id

Type (PL):

1. \ 2.	Id	P_2	P_5	L_2
Id	Id	P_2	P_5	L_2
P_2	P_2	Id	L_2	P_5
P_5	P_5	L_2	Id	P_2
L_2	L_2	P_5	P_2	Id

Type (LL):

1. \ 2.	Id	L_1	L_2	L_3
Id	Id	L_1	L_2	L_3
L_1	L_1	Id	L_3	L_2
L_2	L_2	L_3	Id	L_1
L_3	L_3	L_2	L_1	Id

Type (R):

1. \ 2.	Id	R_1	R_2	P_1	P_2	P_3
Id	Id	R_1	R_2	P_1	P_2	P_3
R_1	R_1	R_2	Id	P_3	P_1	P_2
R_2	R_2	Id	R_1	P_2	P_3	P_1
P_1	P_1	P_2	P_3	Id	R_1	R_2
P_2	P_2	P_3	P_1	R_2	Id	R_1
P_3	P_3	P_1	P_2	R_1	R_2	Id

Type (C):

1. \ 2.	Id	L_1	L_2	L_3	P_2	P_5	C_1	C_6
Id	Id	L_1	L_2	L_3	P_2	P_5	C_1	C_6
L_1	L_1	Id	L_3	L_2	C_1	C_6	P_2	P_5
L_2	L_2	L_3	Id	L_1	P_5	P_2	C_6	C_1
L_3	L_3	L_2	L_1	Id	C_6	C_1	P_5	P_2
P_2	P_2	C_6	P_5	C_1	Id	L_2	L_3	L_1
P_5	P_5	C_1	P_2	C_6	L_2	Id	L_1	L_3
C_1	C_1	P_5	C_6	P_2	L_1	L_3	L_2	Id
C_6	C_6	P_2	C_1	P_5	L_3	L_1	Id	L_2

Graumann, G. (2012).

These seven different types of symmetric triangular pyramids can be ordered in respect to the relation "is special case of" (e.g. by comparing the symmetry groups or their geometric characteristics) in the following way:

P   L

R   PL   LL

C

reg T
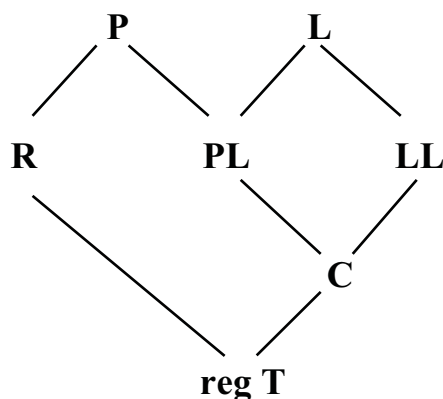
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MODEL FOR TEACHER ASSISTED TECHNOLOGY ENRICHED OPEN PROBLEM SOLVING

Markus Hähkiöniemi[1], Henry Leppäaho[1] & John Francisco[2]

[1]University of Jyväskylä, [2]University of Massachusetts

Abstract: *In this paper, we study ninth grade students' problem-solving process when they are working on an open problem using dynamic geometry software. Open problems are not exactly defined in a sense that the solver can select what aspects to investigate about the problem, several solution methods may be used or there are multiple correct solutions for the problem. Dynamic geometry software, such as GeoGebra, may enhance open problem solving because students can easily explore the problem situation and try out different ideas in practice. There exist several models which describe mathematical problem-solving processes. However, the openness of problems, use of technology and classroom conditions affects problem-solving processes. The aim of this study is to conceptualize students' problem-solving processes when they engage in technology enhanced open problem solving under a teacher's guidance. Data was collected by videotaping a 45 minute lesson with two video cameras and by capturing the screens of the students' computers. The lesson was taught by a teacher trainee and the seven student pairs used GeoGebra. On the basis of the analysis of the students' problem-solving processes, we developed a model for the open problem solving, according to which students cycle through the following four phases: 1) framing the problem, 2) exploring solution, 3) conjecturing, 4) justifying or investigating the conjecture.*

Keywords: Dynamic geometry software, open problem solving, problem solving model
ZDM subject classification: 97- D50

INTRODUCTION

There are several models that describe the mathematical problem-solving process (e.g., Lester, 1978; Mason, Burton & Stacy, 1982; Pólya, 1945; Schoenfeld, 1985). However, the use of technology, classroom conditions, and the nature of the problem might affect the problem-solving process. For example, use of technology enhances exploration of the problem (Healy & Hoyles, 2001), in real classroom implementations of problem solving there are time constraints and use of

open problems emphasizes problem posing and creation of different ideas for solutions (Nohda, 2000). Therefore, it is important to investigate how these factors affect problem-solving processes.

A problem is said to be open if the starting situation or end products are open (Pehkonen, 1997) or if the process is open (Nohda, 2000). Let us illustrate this with the *Amusement Park Problem*:

> *Four towns will build together a magnificent amusement park. Investigate using GeoGebra what would be the most optimal and fair location for the amusement park. (Modified from Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2005.)*

If the starting situation of the problem is not exactly given, *the starting situation is open* (Pehkonen, 1997). This means that the solver has to make selections about what aspects of the problem are to be investigated. For example, in the Amusement Park Problem the solver has to think what does optimal and fair mean and how the towns are located. When *the end products are open,* there are multiple correct answers for the problem (Pehkonen, 1997). For example, there are different reasonable locations for the amusement park. The *process is open* when there are multiple correct ways to solve the problem (Nohda, 2000). For example, in Amusement Park problem, students may use different GeoGebra-tools and ways of reasoning.

Open approach is a teaching method, developed in Japan, in which open problems are used to promote students' mathematical reasoning (Nohda, 2000). Open approach has similar ideas as inquiry mathematics (see, Hähkiöniemi & Leppäaho, 2012, in press) because in both teaching methods learning happens through solving problems and discussing solution methods. According to Stein, Engle, Smith and Hughes (2008), inquiry mathematics lessons typically consist of the following three phases: 1) A *launch phase* where the teacher introduces the problems without giving solution methods or examples. 2) An *explore phase* in which the students work on problems in small groups as the teacher guides them. 3) A *discuss and summarize phase* where the students present and discuss their solution methods and the teacher summarizes the lesson. These phases are similar to those used in open approach (Nohda, 2000) and more generally in Japanese mathematics teaching (Shimizu, 1999). However, according to Nohda (2000), open approach highlights the need for students to participate in a) formulating the problem, b) building different solution methods, and c) posing new more general problems on the basis of the previous solutions.

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The aim of this study is to conceptualize students' technology-enhanced open problem-solving processes with teacher's guidance in an inquiry mathematics lesson involving the Amusement Park problem. The research questions that guide the study are: What kind of phases are there in students' problem-solving process and how can the teacher support students' such processes?

THEORETICAL FRAMEWORK

Problem-solving models

The most known mathematical problem-solving model is Pólya's (1945) model which consists of four phases: *1) Understanding the problem*, *2) Devising a plan*, *3) Carrying out the plan*, and *4) Looking back*. Other researchers have further developed Pólya's model (Lester, 1978; Mason & al., 1982; Schoenfeld, 1985). In all the models there is a phase related to understanding the problem in which the solver figures out what actually is asked in the problem and what conditions are given. Schoenfeld (1985) adds that in what he calls *analysis*-phase, solver may examine special cases, simplify the problem and re-formulate the problem. Most of the changes suggested to Pólya's model concern the phases 2 and 3, which may give a too straightforward image of problem solving. Instead of phases 2 and 3, Lester (1978) suggests that the solver moves back and forth between the following phases: *goal analysis, plan development, plan implementation*, and *procedures evaluation*. Mason et al. (1982) emphasizes the nonlinearity of the problem-solving process, whereby the solver moves back and forth between *entry* and *attack* phases as the solver comes up with ideas, tries to implement them but may get stuck and begins a new entry. Schoenfeld (1985) divides Pólya's second phase into *design* and *exploration* phases and emphasizes a cyclic movement between these phases. Design means explicit planning and controlling of the solution process whereas in exploration phase the solver uses problem-solving heuristics, examines related problems and might go back to the analysis phase (Schoenfeld, 1985). All the models also include a phase where the solution is checked or reflected upon in the end of the solving process.

A different perspective on modelling problem solving is given by Davis and Maher (1990) and Nunokawa (2005). Both of these problem-solving models emphasize cycling through gathering information from the problem situation and comparing that to the solver's existing mathematical knowledge.

Pólya's (1945) straightforward model has been modified to include exploration without a specific aim. Mason et al. (1982) takes this into account in the attack phase. Schoenfeld (1985) instead emphasizes cycling between design and exploration. Thus, a solver does not necessarily develop a plan but rather may try out different ideas without an explicit plan. The models by Davis and Maher (1990) and Nunokawa (2005) fulfil the other models by explaining cognitive processes. In these models, a solver explores the problem situation and tries to connect this to existing mathematical knowledge.

Teacher's guidance of students' problem solving

The problem-solving literature includes advices how a teacher should guide students' problem-solving activity. These advices can be in a form of guidelines such as be interested in your subject, let students learn guessing, let students learn to prove (Pólya, 1965, 116). The teacher can also scaffold students' problem solving in different ways (e.g., Anghileri, 2006) or the teacher may use careful questioning to promote students' reasoning (e.g., Sahin & Kulm, 2008; Martino & Maher, 1999). In open approach and inquiry mathematics, the teacher should support students to engage deeper and deeper in mathematical investigations. Hähkiöniemi and Leppäaho (2012; see also Hähkiöniemi & Leppäaho, in press) have defined three levels of teacher's guidance: a) In *surface-level guidance*, the teacher does not notice a certain essential aspect of the student's solution, b) *Inactivating guidance* means that the teacher reveals the potential investigation to student, and c) In *activating guidance*, the teacher guides the student to investigate the essential aspect.

Use of dynamic geometry software programmes in problem solving

Dynamic geometry software (DGS) enriches students problem solving. For example Hölzl (2001) shows in his case study how students go beyond the use of DGS for verification purposes only. Furthermore, according to a study by Healy and Hoyles (2001), using DGS "can help learners to explore, conjecture, construct and explain geometrical relationships, and can even provide them with a basis from which to build deductive proofs" (p. 251). DGS can promote students' mathematical problem solving in the same way as in experimental mathematics computers are used in 1) gaining insight and intuition, 2) discovering new patterns and relationships, 3) graphing to expose math principles, 4) testing and especially falsifying conjectures, 5) exploring a possible result to see if it merits formal proof, 6) suggesting approaches for formal proof, 7)

computing replacing lengthy hand derivations, and 8) confirming analytically derived results (Borwein & Bailey, 2003).

Arzarello, Olivero, Paola and Robutti (2002) have used ascending and descending processes to describe students' exploration with DGS. In ascending processes students move "from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc." and in descending processes students move "from theory to drawings, in order to validate or refute conjectures, to check properties, etc." (Arzarello & al., 2002, p. 67). These processes describe how students can use DGS to generate and verify conjectures (ascending) and then find reason for why the conjectures are true (descending). According to Jones (2000), this way DGS can be used to promote the need for deductive justifications. However, several studies have pointed out that students need teacher's guidance to transit from verifying to explaining or from empirical work with software to deductive reasoning (Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Jones, 2000; Lew & So, 2008).

METHODS

The data of this study is a part of a larger study on teacher trainee's implementation of inquiry in mathematics led by the first author. In the study, prospective teachers were taught principals of inquiry mathematics. For example, the teacher trainees practiced how to guide students in hypothetical teaching situations (see, Hähkiöniemi & Leppäaho, 2012). Then, each teacher trainee implemented one inquiry mathematics lesson in grades 7–12. One of these lessons was built around the Amusement Park problem. The lesson was implemented in grade 9 (age 15) and it lasted 45 minutes. The students had computers and access to a webpage (http://users.jyu.fi/~mahahkio/huvipuisto) including a GeoGebra applet where a new tool was added to GeoGebra. The students could use the new tool to compute the sum of distances from a point to four other points. In the beginning of the lesson the teacher trainee introduced the students to the use of GeoGebra software with some examples because the students used GeoGebra for the first time. This *launch phase* lasted 11 minutes. In the *explore phase* (23 minutes), the seven pairs of students tried to solve the Amusement Park problem by using GeoGebra and the teacher circulated guiding them. The last 11 minutes were used to the *discuss and summarize phase* in which the teacher trainee presented a review of solutions invented by the students. Different solutions were discussed and evaluated with the whole class.

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Data was collected by videotaping the lesson with two video cameras. One camera followed the teacher trainee who had a wireless microphone. The other camera followed one pair of students who also had a wireless microphone. In addition, the seven student pairs' computer screens were recorded using a screen capture software programme and students' written answers were collected. Altogether, nine videos of the lesson were collected.

Data was analyzed using Atlas.ti video analysis software. First, we coded the lesson for launch, explore, and discuss and summarize phases from each video. Second, we coded the episodes where the teacher discussed with each pair of students. Third, we coded and described the pairs' solutions and solution attempts. Fourth, we analyzed what kind of phases there were in the students' problem solving and defined these phases. Fifth, we coded and described the problem-solving phases from the students' solution processes. Sixth, we described how the teacher guided the pairs and how they used GeoGebra in each stage. Seventh, we designed schematic summaries of the students' movement between problem-solving phases and the teachers' support in changing a phase. Finally, we constructed a model for open problem solving.

RESULTS

In the analysis of the student pairs' problem-solving processes we found the following four phases: *Framing the problem, Exploring the solution, Conjecturing,* and *Justifying or investigating the conjecture*. In the following sections we define these phases and give examples of them.

Framing the problem

If the starting situation of the problem is open, as in the case of the Amusement Park problem, solvers have to make selections about what aspects of the problem they are going to investigate. The selections are not necessarily explicitly stated. We call this phase as framing the problem. Under this phase we coded episodes where students made choices about locations of the four towns or about the criteria of the location of the amusement park. However, students did not necessarily start by framing the problem. For example, Mary and Mark had started to explore solution without thinking how the towns are located. Students' work seemed to be random exploration but the teacher guides them to frame the problem:

Teacher: How is it going Mark and…?

Mark: It's really not going at all.

Teacher:	What do you have? Do you have some idea that you are trying here?
Mark:	No. Just experimenting.
Teacher:	Right. Well, it's not bad. Would it be a good idea to try something a little simpler? Now you have placed the towns a bit randomly. But if you would first look for, for example, a situation where the locations of the towns in relation to each other are simpler?
Mary:	But these (towns) are already (located) quite simply.
Teacher:	Move them so that it would be easier to solve first. Drag the points to such locations. [...] Solve first, for instance, some easy situation. And then change it more difficult.

After the teacher's guidance, the students dragged the towns so that they form a rectangle. They found the location for the amusement park by drawing the intersection point of the diagonals of the rectangle. They justified the solution by writing "Most fair because the distance from the amusement park to each of the towns is the same". In this episode, the teacher returns the students to frame the problem which results in the pairs' first solution to the problem. It should be noted that the teacher did not tell how to frame the problem but let the students to think about this.

Exploring solution

Exploring solution phase includes students' all task-related mathematical work before they come up with a conjecture. However, the exploration does not necessarily result in a conjecture. In one of their solutions, Ian and Irene drew four perpendicular bisectors for four pairs of towns and placed the amusement park to one of the intersection points of these perpendicular bisectors. After they had explained the solution to the teacher, he guided them to explore another solution:

Teacher:	Okay. Good. You are on the right track. Well. Could you find a way, sort of, to always find the point, the certain right point? Like, basically, these can locate, like this could be one centimeter toward this direction. Then, how would you invent the point in a handy way? Could you find a sort of general way of solving the problem?
Irene:	Well, I don't know.
Ian:	Well, if you put the perpendicular bisectors to each and every place.
Irene:	Yeah.
Ian:	And then the intersection point.
Teacher:	Do it. Let's see what it will be. [...] (Ian draws perpendicular bisectors to five pairs of towns.)
Teacher:	Now we have a problem. They all don't intersect in the same point.
Irene:	So, it should be placed there, in the middle (points to the interior of a triangle formed by the three intersection points of the perpendicular bisectors).

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Teacher:	How do you get that?
Ian:	With some circle.
Irene:	Yeah.
Teacher:	Try it. This is. Hey, this is amazing.

In this episode, the teacher tried to guide the students toward a solution which is based on reasoning instead of random locations of the towns. The teacher also pointed out that there is a problem because the perpendicular bisectors do not meet in a single point. However, the teacher did not notice that the students already were close to a reasonable solution because they had chosen an intersection point of two perpendicular bisectors in such a way that there is equal distance from the amusement park to three towns.

Conjecturing

In conjecturing phase, the students suggest an answer, in this case the location of the amusement park. The conjecture is not necessarily written down. Altogether, the students presented 20 conjectures (Table 1).

Conjectures	f
Center of a square (5 solutions) or a rectangle (2 solutions)	7
The intersection point of the diagonals when the towns are placed in the vertices of a non-symmetric quadrangle	1
The point where the total distance to the towns seems to be smallest	1
Midpoint of a segment connecting midpoints of diagonals	1
The towns are dragged in order to get the midpoints of diagonals to overlap and the amusement park is placed in this point	1
Intersection point of perpendicular bisectors of diagonals	2
Intersection point of two perpendicular bisectors (equidistance to three towns)	1
The midpoint of a circle that passes through three intersection points of the perpendicular bisectors of the four towns	2
The towns are on a straight line and the amusement park is placed on the midpoint of the outer most towns (1 solution) or the inner most towns (1 solution)	2
The towns are on a straight line and the amusement park is placed on the perpendicular bisector of the outer most towns outside the segment connecting the towns	2

Table 1. Students' conjectures

The various conjectures illustrate that the problem really was an open problem. The most popular and usually the students' first solution was to place the amusement park to the centre of a square or a rectangle.

Justifying or investigating the conjecture

Justifying means that students try to explain why their conjecture, in this case the location of the amusement park, is reasonable. Students' explanation can be more or less mathematical but for them the explanation justifies the conjecture. For example, Cecilia and Carol had arrived to their second conjecture and explained that to the teacher:

Students:	How do we explain this (their solution)?
Teacher:	(Reads the written answer.) Hey, hey, amazing. So the park would be in? [...] How did you, hmm?
Carol:	We did the segment, the segment and then the perpendicular bisectors for them and there is the intersection point of the perpendicular bisectors (points to the figure).
Cecilia:	And then these have the same distance (points to the two towns connected by a segment) and these have the same distance (points to the other two towns connected by a segment).
Teacher:	Okay. Would you draw them? Let's see. (The students draw their solution, see Fig. 1.)
Carol:	We calculated the distances. To these two it is the same and to these two it is the same. Then it would be a kind of fair to all of the towns.
Teacher:	Yeah, okay. Alright, so in your opinion it would be most optimal location because all of them would have equal distance.
Carol:	No. These two have smaller distance than these. But anyway, none of the towns has a longer distance in its own.
Teacher:	Uhm. Uhm. Yes, I think that I understood. I think that I understood. […]
Cecilia:	But how do we explain this situation?
Teacher:	Just like you explained to me before. (The students write their justification when the teacher asks them to write it.)

In this episode, the Cecilia and Carol have a conjecture and presumably the teacher has difficulties to understand their conjecture. The teacher asks Cecilia and Carol to explain and draw their solution. The teacher even proposes the incorrect justification, but Cecilia and Carol do not accept this and explain again their justification. In this episode, the students and the teacher seem to be equal "mathematicians" and the teacher values the students' explanation by admitting that "I think that I understood". This episode illustrates the teacher's role in asking and listening

students' explanations. In other episodes, the teacher even asked to justify a conjecture even though the students were already constructing another solution.
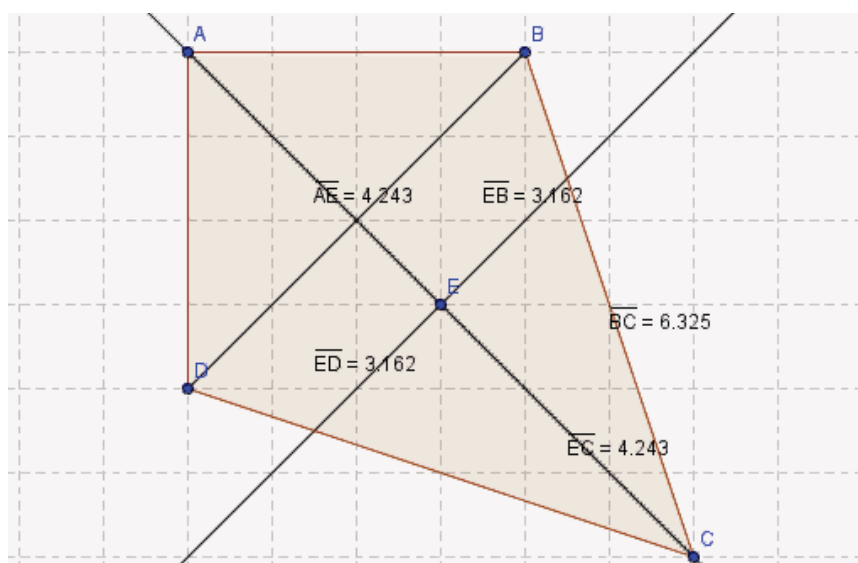

Figure 1. Screen capture of Cecilia's and Carol's solution.

In some cases, the students were not justifying their conjecture but they explained how they arrived to the conjecture or examined whether it is reasonable or not. Thus, we coded these episodes as investigating the conjecture. For example, George's and Gabrielle's first conjecture was that the amusement park is placed to the midpoint of a square. They investigated the conjecture empirically by drawing a circle that passes through the four towns and by measuring the distances from the amusement park to the towns.

DISCUSSION

Based on the analysis of the students' problem-solving processes we constructed a model of open problem solving (Fig. 2). According to the model, a solver cycles through the following four phases: Framing the problem, Exploring the solution, Conjecturing, and Justifying or investigating the conjecture. Problem solving begins with the open problem and a solver becoming aware of the problem and deciding to solve the problem. In the optimal solution process, the solver first makes selections about what to investigate about the problem. Then, the solver explores the solution by developing and trying out ideas. Through the exploration, the solver may build an idea for an answer and formulate a conjecture. Then, the solver may investigate whether the conjecture actually is true or the solver may justify the conjecture. In the optimal solution process, the solver then goes back to the open problem, decides to investigate

some other aspect of the problem and begins a new cycle in the model. The optimal solving process is presented by the full arrows in Figure 2.

However, the problem-solving process does not often proceed so smoothly. Often, the solver has to return to a previous phase. For example, we noticed that the students (e.g., Mary and Mark) returned from exploring the solution to frame the problem differently. Also Rott (in press) found that in some cases fifth graders' problem solving processes proceeded linearly and in some cases non-linearly. It may also happen that a solver does not go through all the phases. For example, we noticed that some students did not justify their conjecture or investigate it but started to build a new solution. We also found that sometimes students do not frame the problem but begin to explore a solution straight away (e.g., Mary and Mark). And in some cases the students' built the conjecture directly after framing the problem. The optional ways of moving are presented by the dash arrows in Figure 2.
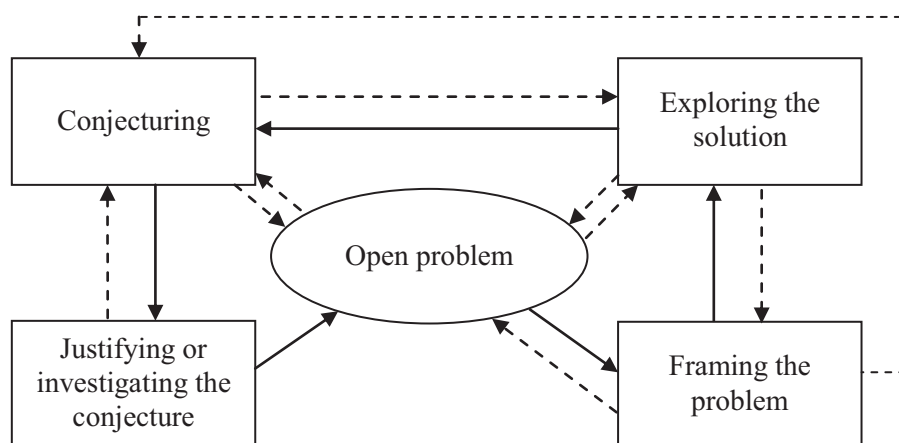
Figure 2. The open problem-solving model

The open problem-solving model takes into account also unstructured exploration as in the models by Schoenfeld (1985) and Mason et al. (1982). However, the aim of our model is to describe students' problem-solving processes in ordinary school lesson where students do not often create a plan before solving a problem. Thus, we do not include at all the phase of devising a plan unlike Pólya (1945) and Schoenfeld (1985). In this respect, our model is similar to the model by Mason et al. (1982). Like Nunokawa (2005), Davis and Maher (1990), and Schoenfeld (1985), we also emphasize cyclic movement between the phases. Clear difference to the other models is that our model is developed especially for open problem solving, particularly to the case where the beginning of the problem is open.

The teacher has a crucial role when open problem solving is implemented in a classroom. We noticed that in addition to guiding students in a certain phase, the teacher also guided the students to change a phase. For example, the teacher guided Mary and Mark to return to frame the problem when the students' work seemed to be based on random experimenting. In addition, the teacher also pushed the students to justify their conjecture (e.g., Cecilia and Carol). Some students were even in exploring another solution when the teacher made them return to justify their previous solution. This emphasizes the teacher's role in activating the students towards more mathematical work (cf. Hähkiöniemi & Leppäaho, 2012).

The open problem-solving model was developed especially for DGS enriched open problem solving. As previous research has shown, DGS makes exploration easy and fast and students can easily create conjectures (Arzarello & al., 2002; Healy & Hoyles, 2001). According to previous studies, the teacher also has to support students to mathematically explain, that is to justify their conjecture (Christou & al., 2004; Jones, 2000; Lew & So, 2008). Our model emphasizes this through the phases of exploring a solution, conjecturing, and justifying or investigating the conjecture. This is also similar to use of the concepts of ascending and descending processes by Arzarello et al. (2002) to describe reasoning with DGS before and after creating the conjecture.

The open problem-solving model can be used as a research tool to follow the flow of students' problem-solving activity and how a teacher guides them to change a phase. Furthermore, the model helps to structure the analysis of teacher support for students problem solving and use of technology in different phases. We believe that the model can be used also as a teaching tool because it helps teachers to conceptualise open problem solving and prepare for helping the students, for example, to return them in framing the problem if they have skipped that phase.

As final words we want to note that in this paper we have focused on presenting the open problem-solving model. However, it should be noted that the lesson was successful in engaging the students to rich creative mathematical reasoning and the students were active and enthusiastic. This was the case even though this was the students' first lesson where GeoGebra was used so intensively, open problems were new to them and the lesson was taught by a teacher trainee in ordinary classroom conditions in 45 minutes. Thus, this also illustrates that the first experiences of technology enriched open problem solving may be very positive.

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TEACHERS' LEVELS OF GUIDING STUDENTS' TECHNOLOGY ENHANCED PROBLEM SOLVING

Markus Hähkiöniemi & Henry Leppäaho

Department of Teacher Education

University of Jyväskylä

Abstract: *The aim of this research is to study how prospective teachers guide students' reasoning in GeoGebra supported inquiry tasks. Twenty prospective mathematics teachers answered a questionnaire. The questionnaire included hypothetical situations where high school students present their GeoGebra supported solutions to a teacher. The prospective teachers were asked how they would react as a teacher in these situations. We found that the respondents had difficulties, for example, in guiding the students to justify their observations and in reacting to "trial and error" solution methods. We found three levels of the prospective teachers' guidance of students: surface level guidance, inactivating guidance and activating guidance.*

Keywords: Inquiry mathematics, teacher guidance, technological pedagogical content knowledge
ZDM subject classification: 97- D40

INTRODUCTION

In this study, inquiry mathematics means mathematics teaching and learning in which students solve non-standard mathematical problems designed to potentially bring forth mathematical ideas related to the topic at hand while the teacher supports students' reasoning and orchestrates classroom discussion. Previous studies have shown the benefits of teaching through inquiry mathematics and integrating technology to teaching (e.g., Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Fennema & al., 1996; Goos, 2004; Jones, 2000; Lew & So, 2008; Marrades & Gutiérrez, 2001; Staples, 2007). However, teachers are documented to have difficulties in implementing this kind of teaching (e.g., Son & Crespo, 2009; Stein, Engle, Smith & Hughes, 2008). Using effectively dynamic mathematics software, such as GeoGebra, requires teachers to guide their students from empirical observations toward deductive reasoning and justifications.

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The aim of this research is to study prospective secondary and post-secondary mathematics teachers' abilities to guide students to reason and justify in GeoGebra enhanced inquiry mathematics. For this purpose, we developed a new kind of questionnaire in which prospective teachers are asked how they would react in hypothetical situations where a student explains his/her solution with GeoGebra. The following research question guided the study: How the prospective teachers try to draw a student's attention to deeper reasoning and justification in the hypothetical situations?

Technology enriched inquiry mathematics

According to Stein et al. (2008), a typical inquiry mathematics lesson has three phases: 1) A launch phase: the teacher introduces the problems without giving solution methods or examples. 2) An exploration phase: the students work in small groups as the teacher guides them. 3) A discussion and summary phase: the teacher orchestrates the whole class discussion on different solutions. In this study, we focus on teachers' actions in the exploration phase. In this phase, challenges for a teacher include listening to and making sense of students' ideas, activating students in mathematical inquiry, helping students to pay attention to the essential aspects of the problems, guiding students to build justifications and proofs, initiating discussions between students, and guiding students to build connections between mathematical ideas.

Several studies have investigated how interactive software support inquiry mathematics (Arzarello, Olivero, Paola & Robutti, 2002; Christou & al., 2004; Jones, 2000; Lew & So, 2008; Marrades & Gutiérrez, 2001). In this study we chose to use GeoGebra (www.geogebra.org) because it is free, easy to use and students can upload it also to their home computers. With GeoGebra students can, for example, investigate connections between graphs and corresponding equations. GeoGebra makes it possible to try different kinds of solution methods which would be too inconvenient with paper and pencil. Thus, GeoGebra encourages students to try out multiple ideas as well as to make conjectures and to test them.

A central idea of inquiry mathematics is that students build ideas which are meaningful for them, evaluate their ideas, explain their ideas, and justify their ideas or solutions (Yackel & Cobb, 1996; Staples, 2007). Goos (2004) found that a teacher can create this kind of culture, for example, withholding his own judgment of students' conjectures and orchestrating discussion assisting to justify solutions. Using dynamic mathematics software changes the role of

justifications from verifying to explaining the reason why a statement is true (Arzarello & al., 2002; Christou & al., 2004; Jones, 2000). Often students first build empirical justifications and then deductive justification (Christou & al., 2004; Jones, 2000; Marrades & Gutiérrez, 2001). Several studies have pointed out that students need teacher's guidance to transit from verifying to explaining or from empirical work with software to deductive reasoning (Christou & al., 2004; Jones, 2000; Lew & So, 2008). This has been noticed also without using ICT. For example, Martino and Maher (1999) found that students may first be satisfied with solutions obtained by trial and error, but by teacher's use of careful questioning, the students can re-examine their solutions and build justifications.

To conceptualize the knowledge needed for teaching mathematics, we use the notion of pedagogical content knowledge (PCK) introduced by Shulman (1986). According to Shulman, PCK includes "the ways of representing and formulating the subject that make it comprehensible to others" and "an understanding of what makes the learning of specific topics easy or difficult" (p. 9). Mishra and Koehler (2006) consider PCK as the intersection of content knowledge and pedagogical knowledge and propose technological knowledge to be a third component (Figure 1). According to them, in productive technology integration these three elements are in complex relationships composing teacher's technological pedagogical content knowledge (TPCK). Akkoc, Bingolbali & Ozmantar, (2008) have applied this framework to a case study of a prospective teacher's TPCK of the derivative concept. Their results illustrate how difficulties in using computer software in teaching cannot be attributed only to technical knowledge.
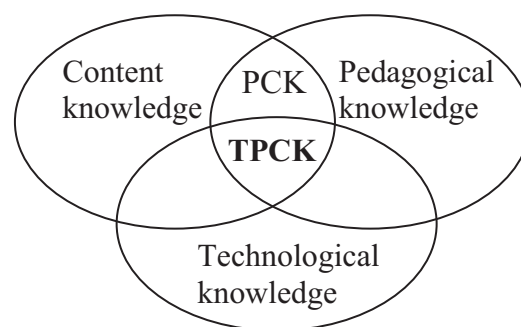

Figure 1. Technological pedagogical content knowledge (Mishra & Koehler, 2006).

One method for studying teachers' PCK and other factors influencing their teaching has been questionnaires in which teachers are asked how they would react in some hypothetical teaching situations. Son and Crespo (2009) implemented this kind of questionnaire and found that

prospective teachers tended to rely on teacher explanation and justification instead of asking the students to do so. In this study, we extend the idea of this kind of questionnaire to include situation with GeoGebra.

METHODS

Data collection

The participants of the study consisted of 20 prospective mathematics teachers from the University of Jyväskylä. At the time of the data collection, they had completed at least 50 credits of mathematics and 15 credits of education. The participants had little experience with supervised teaching practice.

In September 2008, the prospective teachers participated in a 90-minute introduction to basic technical features of GeoGebra. At this point only the technical features of the GeoGebra were discussed because our aim was to find what other forms of support teachers need besides technical advice.

Five days after the introduction to GeoGebra, the group answered the questionnaire which was held in a computer class and lasted 90 minutes. The questionnaire included eight hypothetical situations where a high school student presented his/her GeoGebra supported solution to a teacher. One of the researchers demonstrated these hypothetical solutions with GeoGebra through a data projector. The prospective teachers also opened the solution files with their own computer, examined them and had the opportunity to try their own solutions. Subsequently, the prospective teachers were asked to write down how they would react as a teacher in these hypothetical situations. Altogether, we constructed eight hypothetical situations. In this paper, we focus on three hypothetical situations A, B and C. For example, in hypothetical situation A the prospective teachers were given the following instructions:

A: In the high school course Polynomial Functions, students are given the following task concerning the function $f(x) = ax^2 + bx + c$: *"Use GeoGebra to study how the parameters a, b and c affect the parabola. Use the sliders to change the parameters."*

One high school student proposes the following solution: "The parabola intersects *y*-axis at the point *c*, because, as in the case of the straight line, the constant term tells you the *y*-intercept".

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At the same time he/she shows the figure (Figure 2) in which the parabola intersects the *y*-axis at the point *c*.
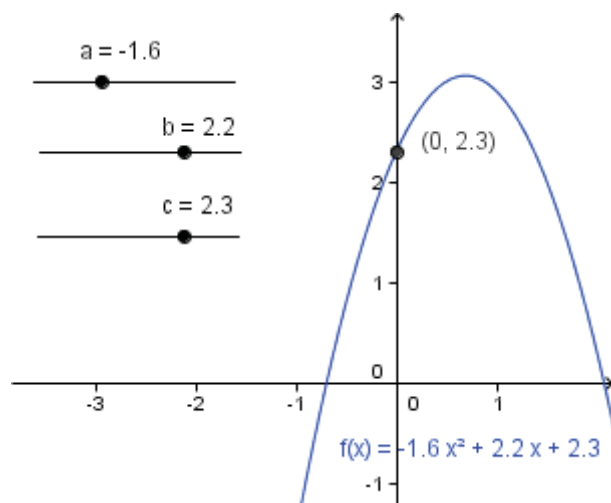

Figure 2. A high school student's solution in hypothetical situation A.

In this paper the hypothetical situations B and C are described only shortly:

B: Students are asked to draw a parabola with zero points -2 and 1. One student solves this by dragging a parabola so that finally GeoGebra shows the asked zero points.

C: Students are given an applet including the circle $(x - a)^2 + (y - b)^2 = c$ and they are asked to change the values of the parameters a, b and c so that the centre will be at the point (3, 2) and it will pass through the point $(1, \frac{1}{2})$. One student solves this by trying out different values for *c* until GeoGebra announces that the point $(1, \frac{1}{2})$ lies on the circle.

Data analysis

In the analysis we applied the principles of qualitative research (e.g., Denzin & Lincoln, 2005). We started by reading the data and familiarizing ourselves with the prospective teachers' responses. First, both researchers analysed independently the collected responses. In this preliminary analysis, we estimated how each of the prospective teachers tries to draw the high school student's attention to a particular essential aspect in the hypothetical situations. After reading the responses couple of times, we realised that they could be categorized using the following three codes:

Code 0: The prospective teacher does not draw the student's attention to the essential aspect in the task or in the solution.

Code 1: The prospective teacher draws the student's attention to the essential aspect in the task or in the solution. However, he/she also reveals the potential investigations of the aspect straight away or asks to find a different solution without any motivation.

Code 2: The prospective teacher draws the student's attention to the essential aspect in the task or in the solution, and he/she guides the student to investigate this aspect or motivates the student to find a different solution.

RESULTS

In this paper we present five classifications [A(1), A(2), B(1), B(2) and C(1)] of the prospective teachers' responses for the hypothetical situations A, B and C. The classifications are presented in Table 1.

Table 1. The classifications of the prospective teachers' responses in hypothetical situations A, B and C.

	code 0	*code 1*	*code 2*
A(1) Does the prospective teacher try to draw the high school student's attention to the reasons why the *y*-intercept equals *c*?	15	2	3
A(2) Does the prospective teacher try to draw the high school student's attention to the reason why both the parabola and the straight line intersect the *y*-axis at the point determined by the constant term?	20	0	0
B(1) Does the prospective teacher try to draw the high school student's attention to the fact that the zero points are not necessarily the required zero points (because of the rounding error of the software)?*	14	5	0
B(2) Does the prospective teacher try to draw the high school student's attention to other (more mathematical) solutions?*	6	10	3
C(1) Does the prospective teacher try to draw the high school student's attention to other (more mathematical) solutions?	3	13	4
Total:	58	30	10

* One respondent did not answer.

It is remarkable that 15 of the 20 prospective teachers did not try to draw the high school student's attention to justifying his/her observation in hypothetical situation A [Table 1, A(1)]. Even these responses were not empty, but the respondents had commented on some other issues. For instance, one response was:

"Good discovery. The parameter c does indeed tell the intersection point of the parabola and the y-axis. But how does changing the parameters a and b affect the parabola?" [code 0, A(1)]

Two respondents noticed the need for justification but also provided the reason for the observer property. For example, one of them wrote:

"The answer supports the observation, but it could be seen more easily by substituting $x = 0$ in the equation, which would immediately give the result $y = c$." [code 1, A(1)]

Only three prospective teachers guided the student to investigate the reason himself. For example, one prospective teacher gave only a hint to use the equation of the parabola in justification:

"Right. Could you justify this using the equation of the parabola? (Substitute $x = 0$:

$f(0) = c$)" [code 2, A(1)]

Even more remarkable is that none of the prospective teachers elaborated the connection between the parabola and the line mentioned by the student [Table 1, A(2)].

In hypothetical situation B, the solution could have even been incorrect because GeoGebra rounds the zero points. Only five prospective teachers suspected the answer given by the software [Table 1, B(1)]. Furthermore, nobody tried to activate the student to investigate whether the answer is correct, for example, by dragging the parabola so that the equation changes but the zero points remain the same.

Hypothetical solutions B and C were based on trial and error use of GeoGebra. We analyzed how the prospective teachers tried to activate the students to notice limitations of their solutions and build more mathematical solution method. According to classifications B(2) and C(1), majority of the prospective teachers would have required a different solution. However, only few gave a reason for the student to search for another solution. For example, one of them asked the student: "Can you discover other parabolas which have the same zero points? [code 2, B(2)]".

Table 1 indicates that the code 0 was most frequent (58/98). This indicates how challenging it is for the prospective teachers to notice the essential guiding points in the students' solutions. According to the amount of codes 2 (10/98), they had difficulties to guide and motivate the students to justifying and reasoning even when they noticed the need for this.

DISCUSSION

The prospective teachers had difficulties in guiding students to justify their findings [classification A(1), A(2), B(1)]. The prospective teachers did not ask for a justification even in a situation where the answer could have been incorrect [classification B(1)]. In this case their technological pedagogical content knowledge was constrained because these kinds of situations where a software gives misleading information by rounding a result are common (see Olivero & Robutti, 2007). Often prospective teachers tended to only praise the students for a correct finding without suggesting justifying it. However, in inquiry mathematics, the aim is to develop the classroom culture towards justifying (Yackel & Cobb, 1996; Staples, 2007). Dynamic mathematics software have potential to make justifying and proving more interesting and meaningful for students, because they can themselves find the property which needs to be justified instead of the teacher demanding they prove some theorem. Only the perspective to justifying needs to be changed. It is not so much finding whether the noticed property is true, but finding the reason why the property holds (see Arzarello & al., 2002; Christou & al., 2004; Jones, 2000).

The prospective teachers had also difficulties in reacting to trial and error solution methods. The majority of the respondents noticed in classifications B(2) and C(1), that the trial and error solutions are not meaningful. However, they suggested finding another solution method without motivating the student. The ease of testing possible solutions is an advantage of dynamic mathematics software, but teachers have to guide students to use this meaningfully (cf. Olivero & Robutti, 2007).

The main idea of inquiry mathematics is that a teacher guides and motivates students to deeper and deeper investigations of the topic in question. Therefore, the most concerning result of this study is that when the prospective teachers noticed the point where the student needed some guidance, they tended to present justifications themselves or demand other kind of solution without motivation. These results corroborate previous findings about the teachers' difficulties in implementing inquiry mathematics (e.g., Son & Crespo, 2009; Stein & al., 2008).

Based on the analysis of the eight hypothetical situations of the questionnaire, we noticed the following three levels of the prospective teachers' responses: a) *Surface level guidance* which means that the teacher does not notice a certain essential aspect of the student's solution or gives

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advice which is not related to the student's solution [see code 0 response]. b) *Inactivating guidance* which means that the teacher notices the essential aspect but reveals the investigations of this aspect to the student or asks for another solution method without motivating the student [see A(1) code 1 response]. c) *Activating guidance* which means that the teacher notices the essential aspect and activates the student to investigate this [see A(1) code 2 response]. These kinds of essential aspects are, for example, justifying the finding, investigating deficiencies of the solution, moving from trial and error towards mathematical reasoning, generalizing and building connections.

We have used the questionnaire and the three levels also as a teaching method. Our subjective observation is that when prospective teachers' responses are discussed and they are asked to classify their colleagues' responses to the three levels, it helps them to reflect the role of teacher in technology enriched inquiry mathematics. Particularly, all components of TPCK are discussed in the same activity. This helps prospective teachers to learn TPCK as integrated knowledge which is the main idea of TPCK (Mishra & Koehler, 2006). Clearly, according to our results, there is a need for this kind of activity as inquiry mathematics and use of dynamic geometry software changes the pedagogical situation. The questionnaire also informs a teacher educator of the prospective teachers' present abilities to apply inquiry mathematics. Then the teacher educator can attend to the difficulties he/she has noticed. When the prospective teachers' challenges in applying technology enriched inquiry mathematics are discussed already in hypothetical situations, they will more probably get positive experiences about inquiry mathematics and technology integration in their actual teaching in training schools.

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Primary teacher students' competences in inductive reasoning

Vida Manfreda Kolar, Adrijana Mastnak & Tatjana Hodnik Čadež

Faculty of Education, University of Ljubljana

Abstract

When we develop mathematical thinking in the early school years, we mainly deal with the situations in which children have to reason inductively. Inductive reasoning is used as a strategy in teaching basic mathematical concepts, as well as in problem solving situations. When educating primary teachers, the emphasis is also on developing problem solving skills based on inductive reasoning. In the paper the results of the study on primary teacher students' competences in inductive reasoning are presented. The students were posed a mathematical problem which provided for the use of inductive reasoning in order to reach the solution and make generalizations. Their results were analysed from different perspectives: from the perspective of understanding the problem situation, from the perspective of the problem solving depth, and from the perspective of the applied strategies. Further, we analysed the relationship between the depth and the strategy of problem solving and established that not all strategies were equally effective at searching for problem generalizations.

Key words: problem solving, inductive reasoning, primary teacher students

ZDM subject classification-number: 97D50

Theoretical background

In many cases the researchers related the inductive reasoning process to the problem solving context (e. g. Christou & Papageorgiou, 2007; Küchemann & Hoyles, 2005; Stacey, 1989). These examinations pay attention to the cognitive process, as well as to the general strategies, that students use to solve the posed problems. Problem solving is considered a highly formative activity in mathematics education fostering various kinds of reasoning, more specifically, inductive reasoning.

In literature terminology of various kinds is used when addressing reasoning in mathematics: deductive reasoning, inductive reasoning, mathematical induction, inductive inferring, reasoning and proving. *Deductive reasoning* is unique in that it is the process of inferring conclusions from the known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information, and there is no need to validate them by experiments (Ayalon & Even, 2008). Although there are also other accepted forms of mathematical proving, a deductive proof is still considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality (Hanna, 1990; Mariotti, 2006; Yackel & Hanna, 2003). On the other hand, *inductive reasoning* is also a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases. Pólya (1967) indicates that inductive reasoning is a method of discovering properties from phenomena and of finding regularities in a logical way, whereby it is crucial to distinguish between inductive reasoning and mathematical induction. *Mathematical induction* (MI) is a formal method of proof based more on deductive than on inductive reasoning. Some processes of inductive reasoning are completed with MI, but this is not always the case (Canadas & Castro, 2007). Stylianides (2008, 2008a) uses the term *reasoning-and-proving* (RP) to describe the overarching activity that encompasses the following major activities that are frequently involved in the process of making sense of and in establishing mathematical knowledge: identifying patterns, making conjectures, providing non-proof arguments, and providing proofs. Given that RP is central for doing mathematics, many researchers and curriculum frameworks in different countries, especially in the United States, noted that a viable school mathematics curriculum should provide for the activities that comprise RP central to all students' mathematical experiences, across all grade levels and content areas (Ball & Bass, 2003; Schoenfeld, 1994; Yackel & Hanna, 2003).

As our research shall be dedicated to inductive reasoning, this will be specified from the perspectives of various theories and practices. Glaser and Pellegrino (1982, p. 200) identified inductive reasoning, as follows: »All inductive reasoning tasks have the same basic form or generic property requiring that the individual induces a rule governing a set of elements.« There is general agreement that tasks such as classifications, analogies, incomplete series, and matrices require inductive reasoning, and that they are widely accepted as typical inductive reasoning tasks (Büchel & Scharnhorst, 1993). It is commonly accepted that these four types of tasks require the detection of a rule or, more generally, of a regularity (Klauer & Phye, 2008).

Inductive reasoning tasks can be solved either by applying the *analytic strategy or the heuristics strategy (*Klauer & Phye, 2008*).* The former enables one to solve every kind of an inductive reasoning problem. Its basic core would be the comparison procedure. The objects (or, in case of correlations, the pairs, triples, etc., of objects) would be checked systematically, predicate by predicate (attribute by attribute or relation by relation), in order to establish commonalities and/or diversities. However, the solution seekers generally tend to resort to the heuristics strategy, at which a participant starts with a more global task inspection and constructs a hypothesis, which can then be tested, so that the solution might be found more quickly, depending of the quality of the hypothesis. We believe that problem solving in mathematics is based on both strategies, with pupils, who learn mathematics, as well with scientists, who can reach new cognitions by applying either the analytic strategy or the heuristics one.

There are various theories as to the detailed identification of the stages of inductive reasoning. Pólya (1967) indicates four steps of the inductive reasoning process: observation of particular cases, conjecture formulation, based on previous particular cases, generalization and conjecture verification with new particular cases. Reid (2002) describes the following stages: observation of a pattern, the conjecturing (with doubt) that this pattern applies generally, the testing of the conjecture, and the generalization of the conjecture. Cañadas and Castro (2007) consider seven stages of the inductive reasoning process: observation of particular cases, organization of particular cases, search and prediction of patterns, conjecture formulation, conjecture validation, conjecture generalization, general conjectures justification. There are some commonalities among the mentioned classifications: Reid (2002) believes the process to complete with generalization, Polya adds the stage of »conjecture verification«, as well as Cañadas and Castro (2007), who name the final stage "general conjectures justification". In their opinions general conjecture is not

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enough to justify the generalization. It is necessary to give reasons that explain the conjecture with the intent to convince another person that the generalization is justified. Cañadas and Castro (2007) divided the Polya's stage of conjecture formulation into two stages: search and prediction of patterns and conjecture formulation. The above stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels are necessarily present, there are a lot of factors involved in their reaching.

Empirical part
Problem Definition and Methodology

In the empirical part of the study conducted with primary teacher students the aim was to explore their competences in inductive reasoning. In the early school years inductive reasoning is often used as a strategy to teach the basic mathematical concepts, as well as to solve problem situations. In the very research the focus was on the use of inductive reasoning at solving a mathematical problem. We believe that in mathematics only teachers who have competences in problem solving can create and deal with the situations in the classroom which contribute to the development of those competences in children.

The empirical study was based on the descriptive, casual and non-experimental method of pedagogical research (Hartas, 2010; Sagadin, 1991). The method allowed us to explore the problem solving strategies in relation to generalisation among primary teacher students.

Research Questions
The aim of the study was to answer the following research questions:

1. Is the posed problem situation perceived as a problem by the students?
2. How much do the students delve into problem solving, i.e. which step in the process of inductive reasoning do they manage to take?
3. Which strategies are used by the students at their search for problem generalizations?
4. Are all the applied strategies equally effective for making generalizations?

Sample Description
The study was conducted at the Faculty of Education, University of Ljubljana, Ljubljana, Slovenia in May 2010. It encompassed 89 third-year students of the Primary Teaching Programme.

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Data Processing Procedure

The students were posed a mathematical problem which provided for the use of inductive reasoning in order to reach a solution and make generalizations. The problem was, as follows:

On the picture below the shaping of the spiral in the square of 4x4 is presented. Explore the problem of the spiral length in squares of different dimensions.
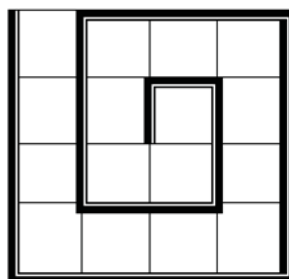
The students were solving the problem individually, they were simultaneously noting down their deliberations and findings, they were also aided with a blank square paper sheet of, so they could delve into the problem by drawing new spirals.

The data gathered from solving the mathematical problem were statistically processed by employing descriptive statistical methods. The students' solutions were analysed from different perspectives: from the perspective of understanding the problem, from the perspective of the problem solving depth, and from the perspective of the applied strategies. As some students tested various problem solving strategies, thus contributing more than one solution to the result analysis, the decision was made to use the number of the received solutions and not the number of the participating students as the basis for the analysis of the problem solving depth and of the strategies of solving. We received 95 solutions, i.e. 6 students contributed two different approaches to problem solving.

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Results and Interpretation

In continuation the results are shown, which are analysed as to various observation aspects.

a) Understanding of the instructions

The instructions to the problem read to explore the length of the spirals. Our first interest was in what way the students understood this particular instruction, i.e. whether they were aware of the fact that the verb »explore« means that one should not only deal with the given example, but also extend the case to similar situations and make generalizations. The corresponding results are shown in Table 1.

Understanding of the instruction »Explore«	Number of responses	Responses in percentage
Solves the isolated example	3	3,4%
Explores more cases	86	96,6%
Total	**89**	**100,0%**

Table 1: Presentation of the results from the perspective of the students' understanding the instructions to the problem

The results show that the majority of the respondents perceived the posed problem situation as a problem, which had also been expected, as during their studies the students encountered with inductive reasoning many a time.

b) The solving problem depth

Those students, who correctly interpreted the instruction of the problem and extended the situation to other cases with spirals of different dimensions, further underwent the examination of their level of depth at dealing with the situation. The received solutions were classified into many levels, which were graded as to the achieved problem solving depth:

Level 1: the record contains only the pictures of the spirals,

Level 2: the record contains the drawn spirals and the corresponding calculations/ measurements of the spiral length (an example: dimension 2x2 → length: 8; dimension 3x3 → length: 15...).

Level 3: the record contains the drawn spirals, the corresponding calculations and the structured record of the lengths of the spirals, but only for those cases, that are graphically presented (an example: dimension 2x2 → length 8 → structured record of the length 2x2 + 2x2; dimension 3x3 → length: 15 → structured record of the length: 3x3 + 2x3)

Level 4: the record contains the drawn spirals, the corresponding calculations as for the level 3 case and the prediction of the result for the case, which is not graphically presented.

Level 5: apart from the pictures and concrete calculations encompassed in the level 3 example, the prediction of the record for the general case is added (an example: generalization to nxn square dimension: $n^2 + 2n$).

As obvious the transformation of the problem from the geometric to the arithmetic one, and consequently operating with numbers and not only with pictures of the spirals is witnessed not until one has reached the level 2. Taking into account the stages in inductive reasoning (Polya, 1967, Reid, 2002, Canadas and Castro, 2007) we can also state that all the students at the levels from 1 to 5 reached the stage »observation of particular cases«, yet they were not equally successful in the process of searching and predicting of patterns. Mere drawings of spirals and calculations of their lengths (the levels 1 and 2) did not provide for a deeper insight into the nature of the problem and for making a generalization for the spiral of any dimension. The level 3 may be considered a transitional stage. These students already knew that mere calculations would not suffice, so they tried to structure them, i.e. they analysed the calculated numbers, and tried to define a certain pattern and a rule, respectively. However, they considered this to be enough and did not try to make a rule for the "n"-number of times-steps. In these cases students were deliberating on a possible pattern just for the cases they were observing. In comparison with them the level 4 students were already thinking about a possible pattern for a non-observing case, but they were still not thinking about applying their pattern to all cases. According to Reid (2002) the students at the level 4 reached the stage of conjecture (with doubt). They were convinced about the right of their conjecture for those specific cases, but not for other ones (see also Canadas and Castro, 2007). Only those students who achieved the level 5 can be considered to have reached the stage called »generalization of the conjecture« according to Reid (2002). In the opinions of Canadas and Castro (2007) generalization is by no means the final stage in the inductive reasoning process. The final stage - general conjectures justification – includes a formal proof that guarantees the veracity of the conjecture, namely. Similar to the research conducted by Canadas and Castro (2007), also in our research none of the students recognised the necessity to justify the results. They interpreted the results as an evident consequence of particular cases, with no need of any additional justification to be convinced of its truth.

Table 2 shows the distribution of responses regarding the achieved problem solving depth.

Depth	Number of responses	Responses in percentage
Level 1	6	6,3%
Level 2	19	20,0%
Level 3	24	25,3%
Level 4	11	11,6%
Level 5	32	33,7%
Other	3	3,1%
Total	**95**	**100,00%**

Table 2: Distribution of the responses regarding the achieved problem solving depth.

As can be inferred from the table, more than one third of the students achieved the stage of making a generalization**,** whereby it should be pointed out that two ways of making generalizations were considered in this group: 30 students did it symbolically with records for »n« number of times-steps, whereas two students generally created rules in a descriptive manner with words, e.g. two lengths of the side are added to the square of the side length. The »Other« group comprises the responses of three students who were eliminated from further analysis of the problem solving procedures due to their non-understanding of the instructions.

During the problem solving procedure also failures were caused, mainly of three types:

- failures at drawing spirals: the inappropriate picture of spirals of larger dimensions (7 responses);
- failures at interpretation of the concept of the length of the spiral: the student equals the concept of the length of the spiral with the number of squares covered by a spiral instead of with the length of the line (2 responses);
- Miscalculations/mismeasurements of the length of the spiral (7 responses).

If we eliminate all the responses containing some failure during the problem solving procedure, 14 responses at the level 2 (14,7%), 18 responses at the level 3 (18,9%) and 11 responses at the level 4 (11,6%) remain; although these students did not make any mistake during solving procedure, they still did not manage to make generalizations. For the students at the level 2 it is assumed that among the collected data they did not notice any structure, which prevented them from further exploration. On the other hand, the students of the levels 3 and 4, who did notice the

structure anyhow (total 30.5%), most likely either did not know how to write their findings in a general form or they did not feel the need to upgrade their concrete findings with a general record. Similar conclusion was made also by Cooper and Sakane (1986) who investigated 8th-grade students' methods of generalising quadratic problems where most of the students could not explicitly recognise that particular cases should be examined for the general rule; some of them claimed that a pattern of numbers was sufficient rule in and of itself. Nevertheless, we think that the percentage of the primary teacher students who reached level 3 or 4 is quite high, and may reflect the orientation of primary teacher education focusing on dealing with concrete situations.

c) Problem solving strategies

The analysis of the modes of reasoning that the students applied at their search for generalizations revealed that it was possible to perceive the posed problem from various perspectives. Various problem perception modes are addressed as various solving strategies in continuation, out of which the ones that were encountered among the students' solutions are presented:

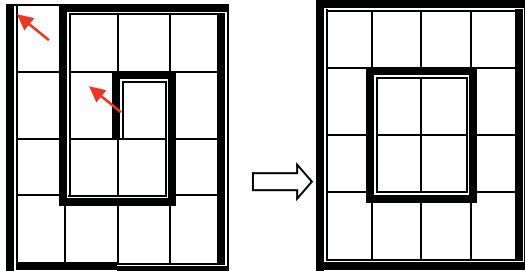| Strategy denotation	Strategy description	Generalization record
1 – »squares« strategy	It is observed that the values of the lengths are obtained by squaring the lengths of the consecutive square (e.g. $15 = 16 - 1$)	$(n+1)^2 - 1$
2 – »product« strategy	It is observed that the length of the spiral is equal to the product of two numbers that differ for 2 (e.g. $15 = 5 \times 3$)	$n(n+2)$
3 – »binomial« strategy	It is observed that the length of the spiral is calculated by adding the double length to the square of the square length (e.g. $15 = 3 \times 3 + 2 \times 3$)	$n^2 + 2n$

4 – »difference« strategy	When observing the differences among the lengths of the spirals, it is obvious that the result is the sequence of odd numbers (e.g. from 1x1 square onwards the lengths of the spirals increase by 5, 7, 9, 11, 13, 15….	The difference between the spiral in the square with nxn dimensions and the consecutive spiral is $2n + 1$ or in a recursive manner: $d_{nxn} = d_{(n-1)x(n-1)} + (d_{(n-1)x(n-1)} - d_{(n-2)x(n-2)} + 2)$, whereby the denotation d_{nxn} stands for the length of the spiral in the square with nxn dimensions.
5 – »sum« strategy	It is observed that the length of the spiral can be presented as the sum of individual even sections of the spiral (e.g. $15 = 1 + 1 + 2 + 2 + 3 + 3 + 3$.	$3n + 2(n-1) - 2(n-2) …+ 2x2 + 2x1$
6 – »transformation strategy«	It is observed that in cases when the dimension of the square is an even number, spirals can be transformed in squares, the perimeters of which can be calculated.	$4n + 4(n-2) + 4(n-4) …+ 4x2$; $n=2k$, $k \in N$

Table3: Description of the applied problem solving strategies

In continuation the students' selection of the strategies is presented. The strategy was evaluated only with the responses, achieving the depth of the levels 3, 4. or 5., i.e. of those students, who noted the length of the spiral in a structured record, as it was possible to define the applied strategy and the mode of reasoning, respectively, only with this record.

Strategy	Number of responses	Responses in percentage
1 – squares	2	2,1%
2 – product	8	8,4%
3 – binomial	12	12,6%
4 – difference	28	29,5%
5 – sum	16	16,8%
6 – transformation	1	1,1%
Other	28	29,5%
Total	**95**	**100,0%**

Table 4: Distribution of the responses as regards the applied problem solving strategy

According to Klauer & Phye (2008) it can be stated that the majority of the students approached problem solving by applying the analytic strategy, including a systematic analysis of individual cases, hence the search for potential patterns and generalizations. In two cases the strategy of insight or the heuristics strategy could be considered. Both students first established the rule for the »n« number of times-case $(3n + 2(n-1) + 2(n-2)+…+2 \times 1$ and in the follow-up they tested their rule on concrete cases with spirals. It can be assumed that as early as at the analysis of the given case the students figured out the spiral structure, the fact which led them directly to the general record. Nevertheless it has to be added that all the students who have chosen the sum strategy performed a generalisation in a recursive form as a sum of the even lengths of the spiral and none of them tried to simplify it by transforming it into some of the records recognised in strategies 1, 2 or 3 (for example: $3n + 2(n-1) – 2(n-2) …+ 2 \times 2 + 2 \times 1 = 3n + 2((n-1) +(n-2)+...+2 +1) = 3n + 2n(n-1)/2 = n^2 + 2n)$.

Let us have a closer look of the results presented in Table 4. As can be inferred the strategy prevails, with which the students focused on the difference between the neighbouring spirals (29,5%). The »sum« and »binomial« strategies are often used; however, only 2 students noticed that there was a correlation between the lengths of the spirals and the squares of the natural numbers. In the »Other« column the responses were placed at which it was not possible to consider the selected strategy (all of the students who did not reach the level 3).

d) Effectiveness of the strategies for generalizing

With the analysis of the problem solving strategies we get better understanding of the problem solvers' strategies what helps us to make conclusions about effectiveness of a particular strategy for creating generalisation. It is very important to know that all strategies are not equally effective for making generalisation and that the context of the problem might support or not support generalisation (Amit in Neria 2008). The following table provides for the answer to the question whether all the applied strategies are equally effective for making generalizations, clarifying the relation between the selected strategy and the problem solving depth:

Strategy	Level 3	Level 4	Level 5	Total	Percentage of responses at the level 5
1 – squares	0	0	2	2	**100,0%**
2 – product	2	1	5	8	**62,5%**
3 - binomial	1	1	10	12	**83,3%**
4 – difference	17	7	4	28	**14,3%**
5 – sum	3	2	11	16	**68,8%**
6 – transformation	1	0	0	1	**0,0%**

Table 5: Problem solving depths in relation to the problem solving strategy

The values in the last column attest to the percentage of the responses pertaining to the selected strategy of those students who managed to reach the final level, i.e. the generalization. According to the results one of the applied strategies was less effective than the other ones, i.e. the strategy 4 - strategy »differences«, but it was the most often used strategy of all (see table 4). Thus, a conclusion can be reached that the percentage of responses including a generalization to a common case was largely influenced by the selected strategy. From this perspective some of them (e.g. strategies 2, 3 and 5) seem to be more useful than the other ones (the strategy 4). Let us examine more in detail the strategy which has been used by most problem solvers and gave least correct generalisations – the 'difference strategy'. The reason for choosing that strategy by a lot of students might be that searching for difference between consecutive numbers is easy and basic strategy for generalisation and it is not difficult to obtain a generalisation if we get a constant difference between consecutive numbers at the first level of differences. On the other hand the generalisation on the basis of the difference between consecutive numbers can be much more difficult if demands generalisation by function of higher order (not linear). In our case the

generalisation of the problem with spirals is expressed as quadratic function and this is in our opinion the main reason for small ration of those who succeed in creating generalisation on the basis of 'difference strategy' (see table 5).

Discussion

In the course of their studies at the Faculty of Education one of the important competences to be developed with primary education students is to qualify them to solve mathematical problems. We are aware of the fact that this field of expertise is often neglected in our primary schools, mostly in favour of consolidating the learning contents by calculations and attending to classical word problems. We believe that students – future primary teachers are the ones, to whom we should start to bring about changes of this mindset, and introduce the role of the problem situations as an indispensable part of mathematics lessons in elementary schools. The presented research provided us with some important responses as to the qualification of students for problem solving by inductive reasoning. It was established that the majority of the students usually perceive the given situation as a problem, however, their abilities to delve into the problem are rather different: based on the stages of inductive reasoning according to Polya (1967), Reid (2002) and Castaneda and Castro (2007) it can be inferred that the students' responses were mainly pertaining to the following three stages: observation of particular cases, searching for pattern and prediction, as well as generalization. We find it important to establish that the stage an individual student manages to reach is largely influenced by his strategy selection. Some strategies in the process solving proved to be more effective than the other ones, from the perspective of making generalizations. According to Steele and Johanning (2004) we could learn that the different quality levels of forming generalisation are the result of different schemas of the learners. In their study they found out that students whose schemas were partially formed could not consistently or clearly articulate the generalizations and had more recursive unclosed forms of symbolic generalizations (e.g. $n+(n-1)-(n-1)+(n-2)$ and not $4n-4$). If we compare their results with ours it could be concluded that students who have chosen the squares, binomial or product strategy achieved a level of well connected schema whereas schemas of students who have chosen difference or sum strategy are only partially formed since they still use recursive forms of generalization.

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 What is the value of these results in terms of primary teacher education research? As researchers we have got a new understanding of students' mathematical competences of problem solving and also the idea how to promote their skills for generalisation. From algebra point of view these results can be used to discuss with the students that one generalisation can be expressed in many different combinations of symbols and the comparison of these symbolic generalisations leads us to the awareness of the equality of different algebraic expressions. Problem solving activities must be organised in the classroom in such a way that the students are enabled to share different strategies, to explain, compare and evaluate them from different perspectives.

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Third-graders' problem solving performance and teachers' actions

Anu Laine, Liisa Näveri, Erkki Pehkonen, Maija Ahtee & Markku S. Hannula

University of Helsinki, Department of Teacher Education

Abstract: In this article the aim is to find the connection between teachers' action (N =7) and third-graders' performances (N = 86) when solving an open non-standard problem. A teaching model developed from Polya's problem solving model is used to perceive the teachers' action by concentrating on three components: presenting the problem, guidance during the solving phase and looking back at the pupils' performances. Pupils' solutions were categorized and classified against the teachers' action. In addition, we analysed how the teachers took up the critical feature of the task and its influence on pupils' performances. As results we noticed that presenting the problem seemed to play a central role in the problem solving lesson. Also the teacher's guidance during the solving phase was significant. In addition, the importance of the planning of the problem-solving lesson became obvious.

Keywords: problem solving, open problem, teaching problem solving, view of mathematics

ZDM subject classification-number: 97D50

Already at the elementary level, the aim of learning mathematics has to be to understand mathematical structures, not merely mechanical calculation. The curriculum for the comprehensive school (NBE 2004) sets problem solving as one of the formal objectives for all school subjects. This paper considers the use of problem solving – especially open problems – in teaching, and especially pupils' skills in solving a non-standard task.

Problem solving

One may say that the base for research on modern problem solving was created in the 1950s by George Polya (cf. Polya 1945). Nowadays problem solving is usually offered as a method to develop mathematical thinking (e.g. Schoenfeld 1985). Here we will use a rather widely used characterization for a problem (cf. Kantowski 1980): a task is said to be a *problem* if the solving demands that the solver must connect his/her earlier knowledge in a new way. If he/she can

immediately recognize the procedure needed for solving the task, it is for him/her a *routine task* (or a standard task or exercise).

The concept of 'problem' is thus relative in terms of time and of the person concerned. Simple addition tasks, such as 3 + 4, could be problems for a school beginner, whereas after some years they are routine tasks. When a teacher offers a new problem task to pupils, it might be familiar (solved before) to some of them, and thus it is no longer a problem. One type of task often mentioned, is a *non-standard* task that differs markedly from those usually presented in mathematics textbooks. Non-standard tasks are often surprising and novel, and demand new kinds of thinking from solvers. These could be problems also for older pupils, and even for teachers. Examples of non-standard tasks can be found e.g. among PISA tasks.

Tasks used in teaching can be divided into open and closed tasks. In a closed task, both the starting and end points are uniquely defined. Mostly the tasks in mathematics textbooks are closed, whereas in *open tasks* there are several alternatives for the starting and end situations. Open tasks will often be offered as a form of problem situation (problem field) that contains many problems with different levels of difficulty. On one hand, open tasks offer more freedom to pupils to think in the solving phase, but on the other hand pupils are compelled to use their knowledge in a more complex way.

From the literature, Pehkonen (1987, 73) has collected a set of reasons why it is important to teach problem solving. These are grouped into four categories: a) problem solving develops general cognitive skills, b) problem solving supports the development of creativity, c) problem solving is a part of mathematical application process, and d) problem solving motivates pupils to learn mathematics. An open problem situation usually contains some easy problems, and thus even a poorly performing pupil will get started and be able to find some solutions, and thus his/her self-confidence and problem solving persistence will develop. The evaluation of solution alternatives will help pupils' meta-cognitive skills to grow and promote higher order thinking skills. Examples of open problems are given e.g. by Pehkonen (1997).

In learning situations, problems should be on such levels that every learner would be able to solve at least some of the problems to some extent, to encourage his/her motivation. This idea supports the argument for the use of open problem tasks. Lester & al. (1989, 75) emphasized that "*any good mathematics teacher would be quick to point out that students' success or failure in*

solving a problem is as much a matter of self-confidence, motivation, perseverance, and many other non-cognitive traits, as of the mathematical knowledge they possess".

About sixty years ago Polya (1945) introduced his 4-step model for problem solving (Understanding the problem. Devising a plan. Carrying out the plan. Looking back). From this model for pupils to solve problems, we have modified a model for teaching problem solving by connecting the second and third steps. This model, which we shall call *a teaching model developed from the Polya model*, is as follows: 1. Understanding the problem – the significance of the task introduction; 2. Devising and carrying out the plan – the significance of guidance; 3. Looking back – feedback on the pupils' solutions.

Such a teaching model will help to structure the teacher's actions, when he/she has a problem-solving situation in class. Thus we can distinguish these three phases in teachers' actions: introduction to the task (understanding the problem), the solving phase, and looking back. And here we will concentrate on these three phases of teachers' actions: Introduction to the task means the way how teachers will present the problem to her class. In the solving phase pupils solve the problem. Looking back phase is usually at the end of the lesson when pupils' achievements are looked through.

Research problems

In this paper our aim is to clarify the kind of connections that exist between a teacher's actions and pupils' performances. The environment of our study is an open problem task, a nonstandard problem that is new to the pupils and to the teachers. We wanted to observe how both pupils and teachers will manage in such a new situation.

To perceive a teacher's actions, we will use the earlier mentioned *teaching model developed* based on Polya's model. In this teaching model, the focus is on three factors of a teacher's actions: introduction to the task, guidance during the solving phase, and the way of looking back over the work done. Pupils' performances will be classified according to these points, and their results will be compared in different teaching groups. Thus, we put forward the following two research problems:

(1) In what manner did pupils solve the open problem task of dividing a square with a line into two exactly similar pieces?

(2) What is the correlation between a teacher's actions and the level of her pupils' performance?

The empirical study

This study is part of the three-year follow-up Finland–Chile research project (2010–13), financed by the Academy of Finland (project number #135556). In the project, we try to develop a model for improving the level of pupils' mathematical understanding by using open problem tasks in mathematics teaching. In order to reach this goal, we must, among other things, help pupils to develop self-confidence, motivation and persistence in solving mathematical problems. All this is needed in order for pupils' problem solving skills to improve and the level of their mathematical understanding to rise.

The experimental group of the research project includes 10 teachers and their third-graders from the surroundings of Helsinki. Once a month on average, during the lessons of the experimental group, one open problem task will be dealt with and recorded using video equipment. The same problem tasks are done also in Chile, but we are getting comparison material not until later because school start in Chile half a year later than in Finland.

Data gathering

Here we consider the results of the task (Divide a square with a line into two exactly similar pieces) that was dealt with in November 2010. In this study, there were seven teachers from the experimental group (Ann, Beatrice, Cecilia, Danielle, Eve, Fatima, Gabrielle) and a total of 86 pupils. In particular, we consider pupils' different solutions, and how these teachers' actions seem to be connected with their pupils' performance.

In order to triangulate research results (cf. Cohen & al. 2000), we collected several data about both pupils' and teachers' actions.: A teacher's lesson plan – from every teacher about half a page with the main points that she gave beforehand to the researchers. The observation of the problem solving lesson (two researchers in the class) and videos of the lesson; one video about the teacher's actions, the other on some target pupils' performances. Furthermore, in the beginning of December 2010, the discussion of the November task by the research personnel of the project and the experimental teachers was also recorded on video.

The open problem task

Here we consider the following open problem task that is clearly a non-standard problem, and essentially needs creativity in order to be solved:

The problem: Divide a square with a line into two exactly similar pieces.

The task has been written in such a form that any third-grader could understand it. The pupil can imagine that the division into two similar pieces could be checked when a paper square is cut along the dividing line and the two resulting pieces are put one on top of the other.

This problem was selected, since it is easy to present and understand, and offers a multiplicity of solutions. Its content (point symmetry) is part of 5th grade curriculum, but the problem is thought to be proper in developing pupils' thinking and creativity.

Analyzing data

Pupils' solutions were evaluated by two researchers (the first two authors) in cooperation in the following way: firstly, the kind of solutions pupils had discovered and the ranking levels of these solutions were decided. Secondly, the two researchers looked together through the pupils' solutions in one class and classified them into different levels. Finally the answers were looked through together. The consensus between the two classifiers was very good (about 95 %).

The two researchers also classified in cooperation the variables connected to the teachers and their actions (introduction of the task, guidance, looking back, introduction of the critical feature of the task). First, existing alternatives were charted, and then the researchers decided the categories for the teachers. In this classification, the lesson plans, the videos recorded during the lessons, and the video from the meeting of the project group were used.

Results

In the results we first discuss pupils' performance in solving the open problem task. The second step is the description of teachers' actions with the three phases of the teaching model *developed from Polya's model*: Introduction, Guidance, Looking back. We also analyse teachers actions in point of view how they deal with the critical feature of the task. . In addition, we clarify to what extent the different areas of the teacher's actions are connected with the pupils' performance.

The solution levels of the task

When reading pupils' solution papers again and again, the following five levels popped-up from the data. Pupils' performances in solving a problem task can be divided into five hierarchical levels (cf. Table 1). The lowest level is No solution (Level 0). The next step is Basic level (Level 1); only the two obvious solutions (with a diagonal into two triangles, and with a straight line parallel to the sides into two rectangulars) were found. The next level is Straight line (Level 2); in addition to the two obvious solutions the square is divided with a straight line that is neither a diagonal nor is parallel to the side of the square. Such a solution needs some amount of creativity, i.e. the solver must be able to see outside of the frame of the basic solutions. There will be an infinite number of different solutions. In the third level Curved line (Level 3) the dividing line can be arbitarily curved, as a fraction line or a curved line composed of arcs; thus the solver breaks away from the barrier of the straight line. The number of these solutions is also infinite, here the cardinality of potential solutions is even greater than in the earlier case. The highest level is Middle point (Level 4); the middle point of the square is seen as the essential part of the solutions, as dividing lines are symmetrical in relation to the middle point.

Table 1. The distribution of pupils on different levels (N = 86).

No solution	Basic level	Straight line	Curved line	Middle point
Level 0	Level 1	Level 2	Level 3	Level 4
1 (1%)	33 (38%)	21 (25%)	18 (21%)	13 (15%)

Most of the pupils reached levels 1–3 in their solutions, but only 13 pupils (i.e. 15%) reached the highest level (Level 4). The mode value in solutions was Level 1.

Introduction to the task

Now we will consider teachers' actions through the first phase (Introduction) in the teaching model *developed from Polya's model*. In a teacher's introduction to the task of the studied problem we could distinguish three ways: No model, Model, Incorrect model. In the first case,

pupils were given only the verbal formulation of the task. The second case Model represents those solutions where the teacher showed (in addition to the text) some concrete model – a square, circle, triangle, etc. – showing what the division "into two exactly similar pieces" means. The third group (Incorrect model) represents a situation, where the teacher used a misleading model, e.g. folding of a napkin that shows symmetry according to a line. Table 2 shows the relative percentages connecting the teacher's introduction to the level of the pupils' performances.

Table 2. The connection of the teacher's introduction to pupils' performances.

	No model	Model	Incorrect model
Teachers	Eve	Ann, Beatrice, Danielle, Cecilia	Gabrielle, Fatima
Level 4	0%	20%	13%
Level 3	12%	27%	7%
Level 2	38%	27%	0%
Level 1	50%	24%	80%
Level 0	0%	2%	0%

Pupils working without a model reached level 1–3, and the mode value of performance was level 1 (50%). Showing a model helped about three fourths of pupils at least to Level 2 (i.e. 74%); here the mode value was Level 2 and 3. Whereas the use of an incorrect model seemed to restrict the level of pupils' solutions: the mode value was Level 1 (80%), the rest of the solutions (20%) were at levels 3–4.

With the help of the video recordings, we could analyse the introduction phase more accurately. Every teacher began the task introduction by explaining the concept of a square. Cecilia also discussed a triangle, since the model she used was connected to it. Fatima examined the properties of a rectangular and a square. All teachers in their task introduction explained to the class that the similarity of the pieces can be checked by cutting the pieces and putting them on

top of each other. They did not discuss the properties of the dividing line yet in the task introduction.

Guidance

Three different levels (Questioning, Commenting, No hints) could be distinguished when the teachers were guiding pupils during the problem solving. During the solution process a questioning teacher asked many questions that helped pupils forward. A commenting teacher just gave a positive comment on a pupil's performance, as "*Well invented*". A no-hints teacher restricted her communication with pupils to brief comments like "*Think for yourself*" or "*I won't give any more advice*". The relative percentages between a teacher's guidance and the level of pupils' performances are shown in Table 3.

Table 3. The connection of a teacher's guidance with pupils' performances.

	Questioning	Commenting	No hints
Teachers	Ann, Beatrice	Danielle, Cecilia, Eve, Gabrielle	Fatima
Level 4	35%	7%	0%
Level 3	23%	21%	14%
Level 2	15%	32%	0%
Level 1	27%	38%	86%
Level 0	0%	2%	0%

More than half of the questioning teachers' pupils (58%) reached levels 3–4. But still 27% of pupils remained at Level 1. One reason for this might be that Ann was not able to guide all her pupils during the lesson. Almost two-thirds (70%) of the pupils of the commenting teachers remained at Levels 1–2. However, one fifth (21%) of the pupils reached Level 3, and less than

one tenth Level 4. The majority of pupils (86%) of the teacher who had given no advice at all during the solution process, were at Level 1, only 14% reached Level 3.

Looking back

In the looking back phase, one could distinguish three different possibilities: Summary of solutions with discussions (Ann, Danielle), Summary of solutions by presenting them at the end of the lesson (Beatrice, Cecilia, Eve, Gabrielle, Fatima), No looking back (none).

In looking back, the teachers used two alternative ways: either the solutions were discussed under the guidance of the teacher – at the end of the lesson or during the lesson – or the teacher let pupils present their own solutions. The presentation of pupils' work was more popular. All the teachers included the looking back phase in their lesson. It is not appropriate to investigate the connection of the looking back phase with pupils' performances here, as its effect could only be seen in the future in similar problem tasks.

In Figure 1, there is an example of one teacher's (Danielle) way of implementing the looking back phase. First, pupils drew one after another their solutions on the blackboard, but when the empty space on the blackboard was almost used up, they began to draw their solutions in the same square. They then suddenly observed that the middle point of the square had a key position: every solution line should go through the middle point.
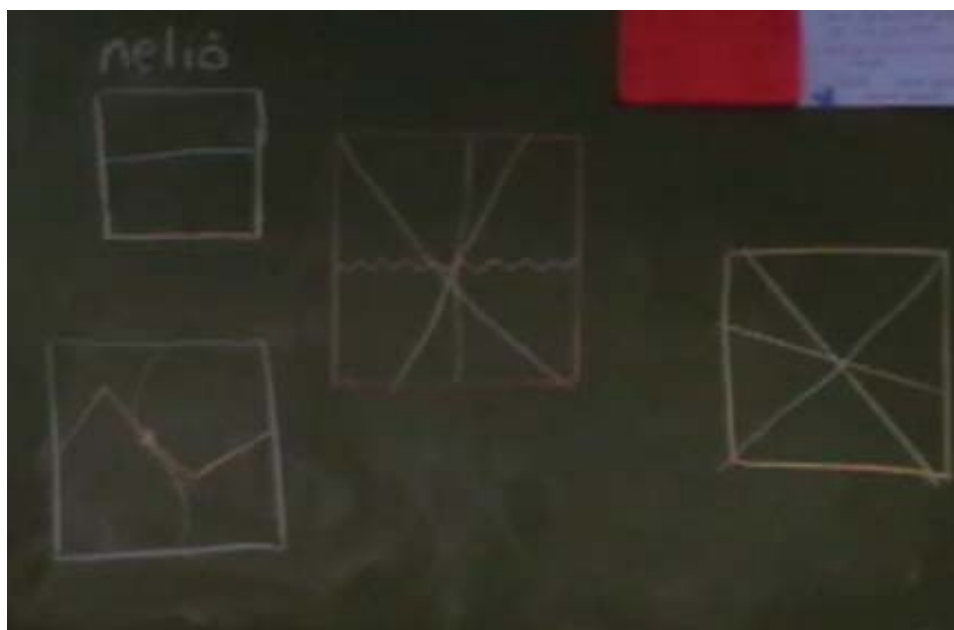

Figure 1. Pupils' solutions drawn on the blackboard in the looking back phase.

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How did the teachers deal with the critical feature of the task?

In order to better perceive the starting points of the teachers' actions, we looked next at how the teachers took up the critical feature of the task (the idea of point symmetry) in their actions. With the help of video recordings and lesson plans we were able to conclude that one teacher (Ann) took up the idea of point symmetry already in her planning of the lesson. This could be seen, among other things, from how the teacher systematically showed those pupils' solutions that were developing in the desired direction with the aid of the document camera. Another teacher (Danielle) took up the meaning of point symmetry just at the end of the lesson, when she was giving the summary of pupils' solutions, as pupils' solutions marked in the same figure seemed to intersect at the same point (cf. Figure 1). But the majority of the teachers did not pay attention to the meaning of the middle point even when they were handling the task. Table 4 provides a summary of pupils' performance levels classified according to a teacher's behaviour.

Table 4. The connection of a teacher's reference to the point symmetry in the square and pupils' solutions.

	The point symmetry in the planning phase	The point symmetry in the summary	No mention of the point symmetry
Teachers	Ann	Danielle	Beatrice, Cecilia, Eve, Gabrielle, Fatima
Level 4	25%	22%	9%
Level 3	33%	19%	19%
Level 2	0%	48%	17%
Level 1	42%	7%	55%
Level 0	0%	4%	0%

It can be very clearly seen in pupils' solutions whether the teacher did or did not take up the critical feature of the task in her teaching: as Ann paid attention to the meaning of the middle

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point in the square already at the planning phase, more than half of her pupils reached Level 3–4 (or 58%), although the mode value of her pupils' solutions was Level 1. This might be due to the fact that Ann was not able to guide all her pupils in the same way. The teacher Danielle became aware of the meaning of the point symmetry at the end of the lesson, and about half of her pupils reached Level 2 (or 48%). This was also the mode value, but one third of her pupils reached Levels 3–4. This result might be connected with her guidance during the lesson. Whether the pupils really understood point symmetry will only be seen in future in similar tasks. When the teachers did not point out the significance of the middle point, more than half (55%) of their pupils stayed at Level 1, while the rest of the pupils reached Levels 2–4, but only less than one tenth reached Level 4.

Discussion

A summary of results

The first research question "How do pupils solve an open non-standard problem?" can be answered as follows: The mode value of pupils' solutions was Level 1 (38%), thus two-fifths of the pupils reached only the basic level. On the other hand, 60% of pupils' solutions showed creativity. As many as 15% of pupils reached midpoint thinking.

 In the case of the second question "What is the connection of a teacher's action to the level of the pupils' solution?" we can say that during the problem-solving lesson the introduction of the task seemed to be in the central position, since in that phase the concept was opened. The introduction of the task with a model seemed to be more successful than the other alternatives. In the solution phase a teacher's guidance was very important. Teachers' questions that directed pupils toward solutions seemed to be especially important. The importance of the looking back phase can only be evaluated later when we have experience of similar tasks.

The importance of the planning phase of the problem-solving lesson

When teaching problem solving a teacher's level of having familiarised herself to the problem seems to be in a key position. Based on the results it seems that pointing out the meaning of the middle point affected pupils' performance in a positive way. If a teacher is able to guide pupils' work by asking good questions and showing solutions that were heading in right direction, it is natural that the results are better. Why did not all the teachers then point out the importance of the middle point? One possible solution is that they had not used enough time to plan their lesson

and, for example, themselves tried to solve the problem in advance. If they had done that, they might have realized the key point of the problem and therefore also acted in a different way during the problem solving lesson. Therefore, we want to stress the importance of the planning phase of the problem solving lesson and improve the teaching model with the level 0 (planning phase), i.e. the teacher's beforehand planning phase she designs her teaching implementation. Thus, we present here an improved *teaching model developed from the Polya model* as follows:

0. **Planning phase**

1. Understanding the problem – the significance of the **task introduction**

2. Devising and carrying out the plan – the significance of **guidance**

3. Looking back – **feedback** on the pupils' solutions

Another reason for the fact that the teachers did not take up the importance of the middle point in their teaching could be teachers' views of mathematics (cf. Pietilä 2002). They could have thought, for example, that in problem solving pupils are supposed to think the problems themselves without teacher's guidance. In future, it would be interesting to compare teachers' actions with different problems and therefore have a broader picture of their views of mathematics.

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A comparative study on elementary teacher students' understanding of division in Finland and Germany

Erkki Pehkonen & Torsten Fritzlar

University of Helsinki & University of Halle-Wittenberg

Abstract

In the paper, Finnish and German elementary teacher students' skills in division are discussed with the help of the following non-standard task: "We know that 498 : 6 = 83. How can you reason with this information (without doing the long division algorithm), what would be the result of the task 491 : 6 = ?" Elementary teacher students (N=74 in both countries) did the task in autumn 2008, the Finnish students in Helsinki and the Germans in Lueneburg. Results of the study show that division does not seem to be fully understood, since only one fourth of the Finnish students (26%) and slightly less than a third of the German students (31%) were able to reason the correct answer and to give sufficient arguments for these. Many of reasoning strategies were somehow insufficient or incorrect. Keywords: division, understanding, comparison Finland–Germany, non-standard problem
ZDM classification numbers: 97C30, 97F90

Introduction

Teacher education programs face a major challenge in trying to influence elementary teacher students' views of mathematics, that is, their beliefs, attitudes and knowledge. In this study we concentrate only on teacher students' knowledge and understanding of division, in order to see how successful the school system and teacher education have been. Since this is a comparative study we will start with a couple of words about school system and teacher pre-service education in both countries (Finland and Germany).

Finland. In Finland there is a nine-year comprehensive school where all children learn in heterogeneous classes. The class size varies about 20 pupils, and therefore, teachers have difficulties in balancing between low-achievers and successful pupils. After the comprehensive school, about half of the age cohort selects to continue in upper secondary school (3-4 years) aiming to the matriculation examination.

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In teacher education we have two lines to follow: elementary teachers and secondary teachers. On elementary teachers' responsibility, there is the teaching of all school subject, and therefore, also mathematics for the six first grades of the comprehensive school. At the university, all elementary teacher students have one basic course on mathematics education that means in Helsinki 7 study points, but in other Finnish universities still less, e.g. in Jyväskylä it is only 4 study points, and some other universities more, e.g. in Lapland 10 study points. About 10% in average of each cohort of elementary teacher students has selected to study more mathematics, corresponding the studies of the first-year mathematic (about 60 study points), but they make their selection until during their third study year. Secondary teachers will teach in the upper grades of the comprehensive school (grades 7–9), and in the upper secondary school. They study at the department of mathematics, and come only for the pedagogical studies (of one year) to the department of teacher education. See more on the Finnish school system and teacher education in the published book Pehkonen, Ahtee & Lavonen (2007).

Germany. In most of all 16 German federal states, children learn four first years in heterogeneous groups. But after the fourth grade (in some states after the sixth grade), the school system is divided into three different school forms: Gymnasium, Realschule, Hauptschule.[1] The first one aims for academic studies and careers, the second one for vocational schools and careers, and the third one is for those youngsters who are not eager to study further and want to get quickly to practical work. Actually there is still an additional school form: Gesamtschule, where pupils study from Year 5 to Year 9, Year 10 or even Year 13 within one school. Therefore, the teacher education in Germany has a different starting point, there is a variety of teacher education programs.

At present there are two lines of teacher education at the university of Lueneburg, where the second author worked at the time of the study: elementary teachers (grades 1-4) and secondary teachers (grades 5-10). However, the specialization to the school form takes place only in the master's programme. All teacher students select two school subjects of equal value. Those who choose mathematics as a school subject (about one third) have extensive studies in mathematics

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[1] Particularly in the federal states on the territory of the former GDR, the latter two are combined in the secondary school.

and mathematics didactics in the extent of 60 study points (1800 hours of study and self-study). However, this made it possible that some future teachers (about two thirds) have no training in mathematics, although it is very likely at least in the elementary school that they have to teach this subject.

Theoretical background

The central concepts of the study are understanding and division that will be discussed briefly in the following.

Understanding

In recent decades there have been numerous research projects on mathematical understanding of pupils or students. Different concepts of understanding were developed that might be important e.g. to curriculum development, evaluation of mathematics teaching or teacher education.

In her overview study, Mousley (2005) distinguishes between three types of models for understanding mathematics: understanding as structured progress, understanding as forms of knowing, and understanding as process. In the first category are, for example, models that are based on ideas of Piaget, or Vygotsky's "zones of development". A well-known model of the second category was developed by Skemp who firstly differentiated between instrumental und relational understanding and later added logical understanding as a third kind (Skemp, 1987, p. 166):

> "*Instrumental understanding* is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> *Relational understanding* is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> *Formal [...] understanding* is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning."[2]

For instance, Pirie and Kieren (1994, p. 166) presented a model of the third type: understanding as process. For them mathematical understanding is a "whole, dynamic, levelled but non-linear,

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[2] Later logical understanding was again differentiated in formal and symbolic understanding (cf. Skemp, 1987).

transcendently recursive process". They described eight potential levels or distinct modes within the growth of understanding (from "primitive knowing" to "inventising") for a specific person on any specific topic. The Pirie and Kieren model is transcendent in that each level, while compatible with prior ones, transcends those levels in sophistication; and it is recursive because one level of understanding can, e.g. in case of a cognitive conflict, call into action a previous understanding (Kastberg, 2002).

Another aspect of understanding is presented by Leinonen (2011) who discusses different kinds of understanding and their meaning in the learning process of problem solving. According to him understanding has, in this context, four modes: conceptual knowledge, grasping meaning, comprehension and accommodation. The function of those modes is to give the background and conceptual instruments for thinking, to interpret the information, to synthesize the knowledge, to integrate the message into permanent memory, and to reorganize the cognitive structure.

In the reported study we investigate teacher students' understanding of division at a single time. Therefore, we will use the terms of Skemp: instrumental understanding, relational understanding and formal understanding.

On division

Division is an important but complex arithmetical operation to consider in elementary teacher education. Pre-service teachers' understandings on division have usually been measured with tasks involving real-world contexts (e.g. Graeber et al., 1989; Simon, 1993) or an abstract context, and in both contexts students have been allowed to use a calculator or long division as an aid (e.g. Simon, 1993; Campbell, 1996; Zazkis & Campbell, 1996).

There are several studies revealing that division is not a well-mastered operation among pre-service teachers. In a study by Graeber, Tirosh & Glover (1989), 129 female pre-service teachers had high scores on all verbal problems involving the partitive model of division. But, they were less successful on the quotitive division problems. The researchers concluded that the reason for these difficulties was the context of the tasks. Primitive models of division derive from early experiences, and these primitive models influence pre-service teachers' choice of operations. (see also Tirosh & Graeber, 1990)

Primitive models seem to reflect an understanding whereby the student spreads out things into equal size groups. The problem is whether pre-service teachers can use this view to make sense of the abstract aspects of division. In Simon's (1993) study of pre-service elementary teachers, the whole-number part of the quotient, the fractional part of the quotient, the remainder, and the products generated in long division did not seem to be connected with a concrete notion of what it means to divide a quantity.

Campbell (1996) studied 21 pre-service elementary teachers' understandings of division with remainder. He conducted clinical interviews with the students, who tried to solve four tasks with abstract contexts. The task we use here has some similarities compared to the following task used by Campbell (1996, p. 179): "*Consider the number 6·147 + 1 which we will refer as A. If you divide A by 6, what would be the remainder? What would be the quotient?*" In Campbell's (1996) study, of the 19 participants who tried to solve this task, 15 calculated the dividend although it entailed additional difficulty. Of those 15 respondents, 9 calculated the dividend and relied upon long division in solving the task. Of the 4 who did not calculate the dividend, only 2 correctly identified the remainder and the quotient, and in this way demonstrated relational thinking.

Zazkis & Campbell (1996) investigated 21 pre-service elementary school teachers' understanding of divisibility and the multiplicative structure of natural numbers in an abstract context. The following is an example of the tasks used: "*Consider the numbers 12358 and 12368. Is there a number between these two numbers that is divisible by 7 or by 12?*" Many pre-service teachers used long division as the instrumental activity, and their responses worked particularly well in revealing the pervasiveness of instrumental thinking. Yet, some degree of relational understanding was evident as well.

In Finland, there has been implemented a research project on pre-service elementary teachers' views of mathematics and its development. The results of the project were described using a survey data (N=255) collected at the beginning of the project (2003) and at the end of a mathematics education course (2004). In the report of Kaasila & al. (2005), it is described the results of the starting stage i.a. in division within the project. Whereas Laine & al. (2012)

reported on students' development in understanding of division, using data collected at the beginning and at the end of the course. Based on these results students do not seem to master division as well as they should as future teachers. At the end of their mathematics studies, differences between students' understanding of division have vanished, but some students still fail to notice wrong solutions in pupils' division tasks.

Objects of research

Understanding is stated in the objectives of the Finnish elementary school mathematics curriculum (cf. NBE, 2004), and also in the German standards for mathematics education in elementary schools (KMK, 2005, p. 6): "*Mathematics learning in elementary school must not be reduced to the acquisition of knowledge and skills. The goal is to develop a firm understanding of mathematical content.*"

Despite of these objectives, it looks like that in school reality teaching is often more emphasized on calculation skills than understanding. Teachers are mainly training children's instrumental skills, and not stressing the relational or later on the formal part. In school, children learn to perform different mathematical operations exactly. But although children learn to perform them, they have, however, usually not really understood. In the study at hand, the research questions are, as follows:

(1) How well do elementary teacher students solve a certain non-standard division task?

(2) What kind of differences are there in the solutions between Finnish and German elementary teacher students?

Answers to these questions are important for the planning of teacher education program at the university. Do students have enough understanding on division or should it be taken extra care of during the teacher education?

Implementation

Participants

The study was implemented in elementary teacher education, both in University of Helsinki (Finland) and in University of Lueneburg (Germany). In both universities, we took a sample of 74 elementary teacher students who were doing their first year of teacher studies. In the sample of

Lueneburg all participants had chosen mathematics as a school subject; thus they are planning to study mathematics education in detail, and we may expect that they have some interest in mathematics. Instead in Helsinki, mathematics is a compulsory subject for all elementary teacher students; they will have the opportunity to deepen their mathematics insight later on, but they are not compelled to it and they have not yet made their selection.

Indicator

We gave to elementary teacher students the following non-standard division task (cf. below) to solve, during their first lesson on mathematics education of the autumn 2008. The students had about 10 minutes time to do the task:

> "*We know that 498 : 6 = 83. How can you reason with this information (without doing the long division algorithm), what would be the result of the task 491 : 6 = ?*"

The same task was used some years earlier in Finland, in order to check high school students' understanding, and the corresponding paper was presented in an earlier ProMath meeting (cf. Hellinen & Pehkonen, 2008). Results of the study show that division do not seem to be fully understood, since only a few students were able to reason the correct answer and to give sufficient arguments for these.

The non-standard division task with an abstract context we will use in this study differs specifically from the tasks used in earlier studies (e.g. Campbell, 1996, Zazkis & Campbell, 1996) in that 1) participants must use a given equation as a starting point for their reasoning, and 2) may not use the long division algorithm nor a calculator when solving the task. They must show their understanding of division by deducing the result of a division task from another given solved division task with the same divisor but another dividend.

Data analysis

The results were classified according to answers into three main categories: (A) not tried, (B) tried but no numerical result, (C) tried and a numerical result was reached. Furthermore, the answers in the two last categories were considered according to the following aspects: (1) art of result, (2) argumentation used, (3) correctness of the method used.

Especially interesting regarding our research questions are results of category (C). When looking

nearer to the aspect (1) – art of result – we could distinguish in this category five arts of presenting the result: an interval, an approximation value, a decimal number, a result with remainder, and a fraction. The three last ones were correct answers. If the student also used the given equation as starting point then we recognised her / his solution as a proof of relational or formal understanding.

Both researchers discussed and agreed together the classification scheme. First discussions were done via e-mail, and then the last one was a face-to-face meeting where we fixed the classification scheme. About a couple of months later we then classified the German data separately. When we compared our classification results, the rate of consensus varied in different categories (see above) from 78% to 95%.

On results

Then we will look at the results in three phases. Firstly, there is the grouping of the responses into three main categories: (A) not tried, (B) tried but no numerical result, (C) tried and a numerical result was reached (Table 1). And secondly, the distribution of the C-answers according to the aspect (1) – art of result – will be looked after (Table 2). Thirdly, the number of the answers proving relational or formal understanding in both countries is considered (Table 3).

Main classification categories

In the following we will present and discuss the three main categories (A–C).

Table 1. The grouping of the responses into three main categories (A not tried, B tried but no numerical result, C tried and a numerical result was reached); there are given firstly the frequencies and then the percentages.

	A	B	C
Finland	6 (8%)	12 (16%)	56 (76%)
Germany	2 (3%)	11 (15%)	61 (82%)

There are a few students who have not at all tried to solve the task. Perhaps, they were too unsure about their ability to solve the task, and therefore, have left an empty paper. In Finland, there were more unsure students than in Germany.

The number of the students who answered without a numerical result was less than one fifth of all students. In many papers, the answers showed that students expected a whole number solution, and therefore gave no numerical answers. For example, a Finnish student writes "*498:6 = 83. In the division 491:6 the dividend is seven smaller and since the divider is 6, each 'part' gets 7:6 more.*" (F42), and a German one "*The division of 491:6 will not end as a whole number. It will be a decimal number.*" (G3).

Some typical C-answers with mistakes or with an insufficient warrant are, as follows: A Finnish student writes "*498–491 = 7, 7–6 = 1, 491:6 = 82 rest 1*" (F39), and a German one *"81 rest 5. The result must be less than 82 because 498–6 = 492."* (G8) And a good model answer was, for example, the following: A Finnish student writes "*498:6 = 83, then 492:6 = 82, and 486:6 = 81. Since 491–486 = 5, we have 491:6 = 81 rest 5.*" (F8), and a German one: "*498–12 = 486, 486:6 = 81, 491–486 = 5, 5:6 = 5/6 ⇒ 491:6 = 81 5/6"* (G22).

In the Finnish answers there are more calculations (also crazy ones), whereas the Germans used more verbal explanations. In the classification of the categories A, B, C, the rate of the consensus between the two researchers was excellent (95%).

The distribution of the C-answers

The share of the answers in both countries was rather similar in the art of results (Table 2).

Table 2. The distribution in the C-answers of the aspect (1) art of result: interval, approximation value, decimal number, result with remainder, and fraction; firstly is given the frequency and then the percentage.

	Interval	Approximation value	Decimal number	Result with remainder	Fraction	Number of C-answers
Finland	3 (5%)	6 (11%)	14 (25%)	10 (18%)	23 (41%)	56
Germany	6 (10%)	14 (23%)	3 (5%)	14 (23%)	24 (39%)	61

Here are some examples of interval answers from the teacher students' responses. A Finnish student's response: *"The difference between the numbers 498 and 491 is 7, thus it is a number*

that is "less". Since 498:6 is 83, the number 491:6 shall be slightly below 82." (F27) And a German student writes: *"Based on the first result we know that 498 is a multiply of 6. Therefore, 491:6 must be smaller than 82 but bigger than 81. "* (G12).

In the aspect (1), art of result, the consensus of the classification is still fairly good (78%).

Relational and formal understanding

Next we looked for the correctness of the given answers, and for the type and completeness of justification. As correct answers we accepted the fraction (81 5/6), the decimal number ($81,8\bar{3}$) or the whole number with remainder (81 and remainder 5). If a student got one of these answers starting from the given equation, than we categorized it as relational understanding. If the student additionally presented a mathematically sound reasoning we accepted the solution as a proof of formal understanding of division.

Table 3. The share of all C-answers, and C-answers proving relational understanding, as well as formal understanding

	C-answers	relational understanding	formal understanding
Finland	56 (76%)	29 (39%)	26 (35%)
Germany	61 (82%)	33 (45%)	31 (42%)

In Finland, fewer students than in Germany tried to provide a numerical result. Of these, approximately the same quota used the given equation as starting point for the solution, and again, approximately the same quota was able to fully justify the given answer. As a result we received that in Finland there were 26 (35%) cases of formal understanding, and in Germany the corresponding number was 31 (42%).

Discussion

Within the framework of our project we diagnosed the understanding of division only selectively and at a certain time. But regarding to Skemp we can differentiate between only instrumental and relational or formal understanding, since the used task demands more than computational skills. In the study only 39% of the Finnish and 45% of the German students achieved relational understanding of division, most of these students were also able to formulate a complete

justification (35% and 42%).

More than one third of all the students in the both samples has either not answered at all or has answered totally wrong; in Finnish sample the share was bigger. Although division is known to be difficult and an operation that has many interpretations, the result is still surprisingly poor. Surprising was also that so many students left his/her response without recognizable reasoning, although it was especially asked for in the task.

Some of the students were answering without warrants, although they have given the correct solution. Thus we may conclude that these students have enough instrumental knowledge. On one hand it can be that those students have no way used in their reasoning the connection given in the task, and therefore, have not written their reasoning. Then they did not show relational understanding of division. On the other hand in such answers one may also see the lack of language skills. Although a student in question could solve the task and receive the correct answer, and although he/she could use the given equation in a proper way, he/she was not able to express the actions needed, i.e. the actions happening in his/her mind. So his/her formal understanding of division was not sufficient.

Number concept restricted to integers is surprisingly common in elementary teacher students (almost one fifth of the answers). According to the Finnish curriculum (NBE, 2004), the number concept is extended to fractions already on the lower grades of the comprehensive school. In Germany, simple fractions as ½ or ¼ are introduced in elementary school already, a detailed extension of the number concept takes place in the fifth and sixth grade.

Concluding note

In the division task at hand, a student should apply his/her mathematical knowledge. To solve the task demands real understanding of division as well as skills to reason and explain the logical chain used. The areas of mathematical knowledge and understanding evaluated in the task were thus similar to those measured e.g. in the PISA comparison (cf. Anon., 2006). Therefore, the percentage of correct solutions in Finland and in Germany was surprising, when one takes into account the Finnish and the German success in the PISA comparisons.

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An explanation to the difference in results between Finns and Germans might be due to the amount of mathematics to be learned during the teacher education program. Since the Germans have selected mathematics as their study subject, they had clearly more interest and motivation in learning mathematics. Whereas the Finns form an unselected population, some of them would not have selected mathematics at all, if possible. It would be interesting to test the whole age cohort in German teacher students, and then compare results. Another interesting comparison would be those Finns who will select mathematics in their third study year, and the German students with mathematics as their school subject.

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Models of the Problem Solving Process – a Discussion Referring to the Processes of Fifth Graders

Benjamin Rott
Leibniz University of Hanover

Abstract

In the 1940s George Pólya (1945) created a model of the problem solving process that is well known and cited to this day. Pólya presents four steps, which are (1) understanding the problem, (2) devising a plan, (3) carrying out the plan, and (4) looking back. Nearly all subsequent models from mathematics educators are based on these steps, for example Schoenfeld (1985, chapter 4), Wilson, Fernandez, & Hadaway (1993) or Mason, Burton, & Stacey (1982). These interpretations and enhancements highlight the importance of guessing and of managerial activities within the problem solving process; they break up with the linear nature of Pólya's model or add new steps. But are these models suited to describe actual problem solving processes of pupils? An analysis of the problem solving attempts of fifth graders (ages 10 to 12) from German secondary schools shall help answering this question. The videotapes which supplied the raw data were coded using an adapted version of the protocol analysis framework from Schoenfeld (1985, chapter 9), with which the processes are parsed into steps called episodes. I'm going to look at the existence and absence of these episodes and the order in which they appear in the pupils' processes.

ZDM classification-number: D53
Keywords: mathematics education, problem solving, process model

1 BACKGROUND

According to Halmos (1980, p. 519) "the mathematician's main reason for existence is to solve problems, [...] what mathematics *really* consists of is problems and solutions." Problem solving is also important for the learning of mathematics beyond memorizing algorithmic procedures (cf. Zimmermann 2003), thus it is part of many school curricula (cf. NCTM 2000, KMK 2003).

There are different definitions for the term "problem solving". Most of them include a starting point, a goal and the way between those two, to which the problem solver – in contrast to algorithmic or routine tasks – has no immediate access (cf. Dörner 1979; Schoenfeld 1985):

> *When you are faced with a problem and you are not aware of any obvious solution method, you must engage in a form of cognitive processing called problem solving. Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver [...]. (Mayer & Wittrock 2006, p. 287)*

It is important to note that the attribute "problem" depends on the solver, not on the task. A difficult problem for one student can be a routine task for another (maybe more experienced) one. Thus, research on problem solving should focus on the problem solving *process* (instead of the *product*). The research on problem solving and problem solving processes was heavily influenced by Pólya, who's seminal work "How to Solve It" (1945) revolutionized our view of problem solving processes and heuristics (cf. Schoenfeld 1985, p. 22 f.; and Fernandez et al. 1994, see below):

> *How to Solve It marked a turning point [...] for problem solving. [...] For mathematics education and for the world of problem solving it marked a line of demarcation between two eras, problem solving before and after Polya. (Schoenfeld 1987, p. 283)*

Models of the Problem Solving Process

There are many models of the problem solving process from psychologists, mathematicians and mathematics educators, serving different purposes.

> *To discuss and investigate the processes involved in problem solving, researchers find it useful to develop frameworks. Similarly, frameworks are useful for discussing general processes and approaches to problem solving with students. Most formulations of a problem-solving framework attribute some relationship to Pólya's ([1945]) problem-solving stages: understanding the problem, making a plan, carrying out the plan, and looking back. (Fernandez, Hadaway & Wilson 1994, p. 196)*

This quote shows that there are two kinds of frameworks: *descriptive* (delineating empirical processes) and *normative* (telling people what to do) frameworks.

In the following I want to discuss Pólya's stages as well as frameworks by mathematics educators, that succeeded him. Whereas most of these models are normative.

The Problem Solving Process According to George Pólya

After working with students and reflecting on his own attempts to solve mathematical problems, Pólya (1945) presented four steps and composed questions and instructions for each of those steps to assist problem solvers achieving a solution (see Figure 1).

Pólya was a mathematician and knew the odds of problem solving, going back and forth while trying to solve a task (cf. Wilson et al. 1993, below). Nonetheless, the four steps seem to be very

linear. My educated guess is, that this is due to his target group: Pólya wrote *How to Solve It* for interested readers, students and teachers, instead of researchers. It is easier to learn with linear steps than follow meandering pathways through the problem space, thus Pólya simplified.

Even though Pólya addressed his work "to teachers who wish to develop their students' ability to solve problems, and to students who are keen on developing their own abilities" (ibid., p. vi), his steps were – and are still – received as a model for the problem solving process by researchers from different professions.

Pólya's four steps resemble those of Dewey's (1910) who was one of the first psychologists that engaged in problem solving (cf. Neuhaus 2002, p. 427 ff.). Like Pólya, Dewey chose a systematic approach to problem solving which differs from those of Wallas (1926), Poincaré (1914) and Hadamard (1945) who emphasize the intuitive aspects of problem solving. Their frameworks all look very similar, with steps for (1) preparation, (2) incubation, (3) illumination, and (4) verification.

Fig 1: Pólya's model

Understanding the Problem
What is the unknown / condition? Draw a figure

Devising a Plan
Do you know a related / an analogous problem? Obtain a plan of the solution.

Carrying out the Plan
Carry out your plan. Check each step.

Looking Back
Can you check the result? Can you derive it differently? Can you use the result, or the method, for some other problem?
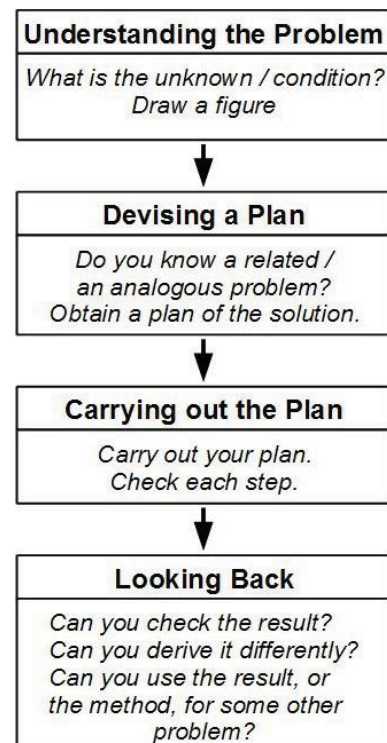
The Problem Solving Process According to Alan H. Schoenfeld

Schoenfeld (1985, ch. 4) describes the course of a problem solving process[3] in five stages by adding *Exploration* to the stages of Pólya. "*Exploration* is the heuristic heart of the strategy, for it is in the exploratory phase that the majority of problem-solving heuristics come into play." (ibid., p. 109)

His view of *Planning / Design* differs from Pólya's "Devising a Plan" by being an element of
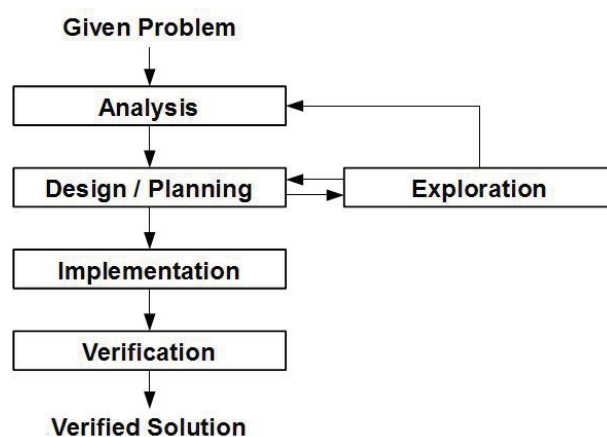
Given Problem → **Analysis** → **Design / Planning** ↔ **Exploration** → **Implementation** → **Verification** → **Verified Solution**

Fig 2: Schoenfeld's model (1985, ch 4)

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3   Similar to Pólya, Schoenfeld developed this framework to help students and not as a framework for research. In his 1985 book, Schoenfeld calls his five stages "strategies", but in the 1979 ancestor of chapter 4, he calls it a "model".

control, "something that pervades the entire solution process; its function is to ensure that you [as problem solvers] are engaged in activities most likely ([…]) to be profitable. Most generally, it means keeping a global perspective on what you are doing and proceeding hierarchically." (ibid., p. 108)

A problem solver, Schoenfeld states, often goes through cycles of (*Analysis*,) *Exploration* and *Planning* until there is a plan to *implement* (see Figure 2).

In chapter 9 of his book, Schoenfeld modifies this framework to become the method of an empiric study (see below for details). He adds *Reading* as part of an observed problem solving process and opens up the possibility to intertwine the *Planning* and *Implementation* parts.

The Problem Solving Process According to James W. Wilson and Colleagues

Wilson and his colleagues present a "dynamic, cyclic interpretation of Polya's stages." (Wilson et al., 1993, p. 61)

> *Clearly, the linear nature of the models used in numerous textbooks does not promote the spirit of Polya's stages and his goal of teaching students to think. (Wilson, Fernandez, & Hadaway 1993, p. 60)*

Therefore, they highlight the possibility to jump from each step to every other possible step. Explicitly, they include *Managerial Decisions* as the control center in their graphic interpretation of the problem solving process (see Figure 3).

> *The arrows [in Figure 3] represent managerial decisions implicit in the movement from one stage to another, and the overall diagram suggests that the process is not necessarily linear. For example, a student may begin by engaging in thought to understand a problem and then move into the planning stage. After some consideration of a plan, the student's self-monitoring of under-standing may indicate the need to understand the problem better and cause the student to return to the understanding-the-problem stage. (Fernandez, Hadaway, & Wilson 1994, p. 196)*
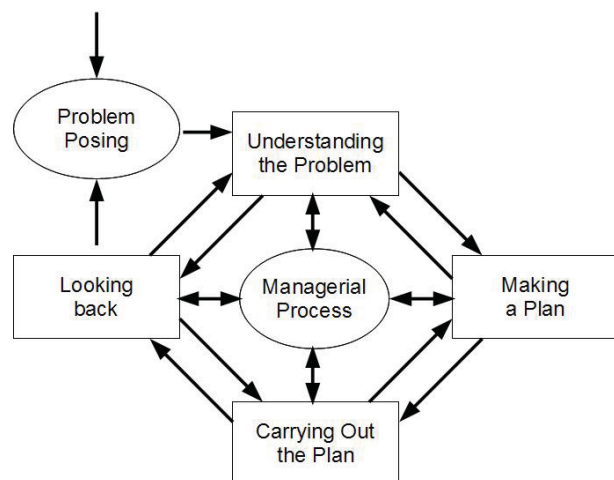

Fig. 3: Wilson et al.'s (1994) model

The Problem Solving Process According to John Mason and Colleagues

Mason, Burton, & Stacey (1982 / 2010) distinguish three steps, *Entry*, *Attack* and *Review*. The Entry phase includes questions like "what do I know", "what do I want", and "what can I introduce" and should help the problem solver in deciding what to do. During the Attack phase, several approaches may be taken and different plans may be formulated on the way to a solution. Once a solution is found, the problem solver should enter a Review phase, which contains elements like "check the resolution", "reflect on the key ideas and key moments", and "extend to a wider context". Mason et al. are aware
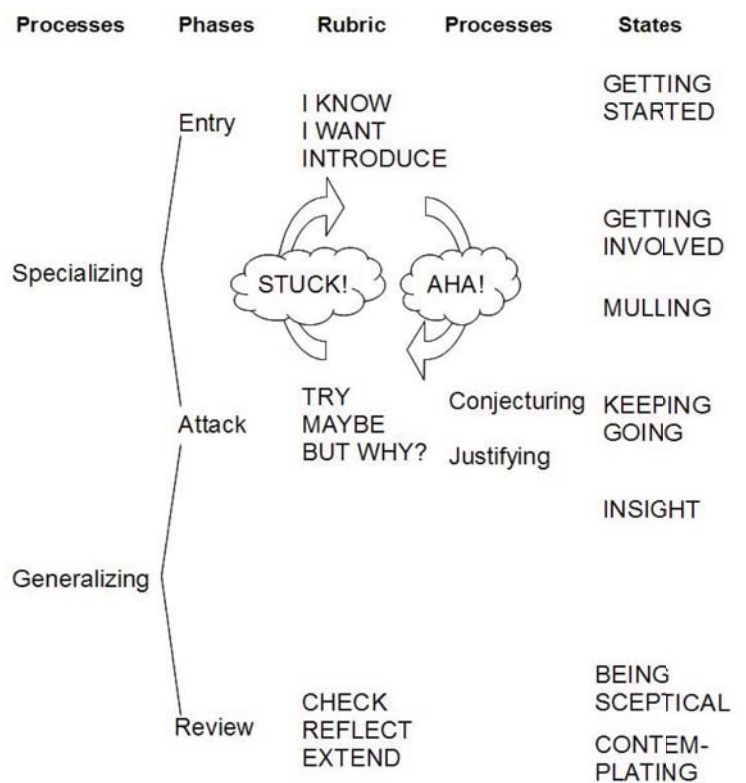
Fig. 2: Mason et al.'s model (1982)

of the fact, that problem solving processes seldom proceed in a linear way and indicate this with arrows (see Figure 4). In chapter 7, they emphasize the impact of metacognition for problem solving.

Short Summary and Comparison

All three frameworks, Schoenfeld's, Wilson's, and Mason's, show clear references to Pólya's steps. They all break with the linear nature of Pólya's phases and highlight the importance of self-regulatory activities.

Comparing these models leads to the following findings: The steps of Schoenfeld (1985, ch. 9) and Pólya
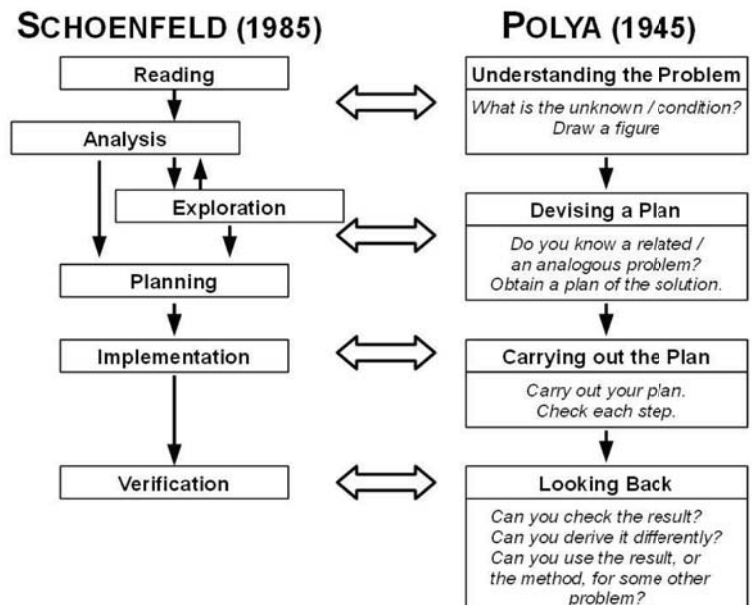

Fig. 1: Analogy between Schoenfeld's and Pólya's models

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can be assigned to each other directly. The only exception being the splitting of Pólya's "Devising a plan" into "Exploration" and "Planning" as shown in Fig. 5.

The same relation is true for the frameworks of Schoenfeld and Wilson, because Wilson's steps are identical to Pólya's (despite their cyclic arrangement).

The steps of Mason mix up Pólya's and don't fit so easily to the others'. Only "Review" equals "Looking Back". The "Entry" phase[4] contains both elements of "Understanding the Problem" and "Devising a Plan", while "Attack" is an assortment of "Devising a Plan" and "Carrying out the Plan".[5] In terms of Schoenfeld, "Entry" includes "Analysis" and "Planning" and "Attack" embodies "Implementation", whereas "Exploration" is part of both of Mason's steps.

Research Questions

➢ Are these (mostly normative) frameworks (see above) suited to describe actual problem solving processes of pupils (fifth graders)?

➢ Which model or which parts of these models are best suited to do so?

The following analysis shall help answering these questions.

2 DESIGN OF THE STUDY Our support and research program MALU[6] was an enrichment project for interested fifth graders (ages 10 to 12) from secondary schools in Hanover. From November 2008 till April 2011 pupils came to our university once a week. A group of 10 – 16 children was formed every new term.

The sessions usually followed this pattern: After some initial games and tasks, the pupils worked in pairs on one to three mathematical problems for about 40 minutes and were videotaped thereby. They eventually presented their results to the whole group. Altogether, we had 45 pupils working on about 30 tasks in the first four terms till June 2010; I concentrate my research on these groups.

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4   The German translation for the *Entry* phase is "Planung" (= Planning) which is rather inapt in my opinion.
5   Mostly, the "Attack" phase combines Pólya's "Devising" and "Carrying out the Plan". But "Entry" includes looking at special cases, which is not part of Pólya's "Understanding the problem".
6   *Mathematik AG an der Leibniz Universität* which means *Mathematics Working Group at Leibniz University*.

Rott, B. (2012).

The pupils worked on the problems without interruptions or hints from the observers, because we wanted to study their uninfluenced problem solving attempts. We decided not to use an interview or a think-aloud method, because this would have interrupted the pupils' mental processes. To get an insight into their thoughts, we let the children work in pairs to interpret their communication. Also, working in pairs made the pupils feel more comfortable being filmed (cf. Schoenfeld 1982, p. 10).

The tasks were selected to represent a wide range of mathematical areas and to allow the use of different heuristics. Here are the four tasks I selected for the analysis that is presented in this paper:

Beverage Coasters
The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.
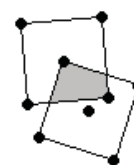
Examine the size of the area covered by **both** coasters.

Idea: Schoenfeld (1985, p. 77)

Marco's Number Series
Marco wants to arrange the numbers from 1 to 15 into the caskets so that the sum of every adjoining pair is a square number:

For instance, if there are the numbers 10, 6, 3 in three consecutive caskets, the 6 adds up to a square number with its left (10+6=16) and its right neighbor (6+3=9).
How could Marco fill-up his 15 caskets?

Source: Fürther Mathematikolympiade, 2005/06, 1. round (www.fuemo.de)

Seven Gates
A man picks up apples. On his way into town he has to go through seven gates. There is a guardian at each gate who claims half of his apples and one apple extra. In the end the man has just one apple left. How many apples did he have first?

Source: Bruder (2003, p. 12)

Squares on a Chessboard
Peter loves playing chess. He likes playing chess so much that he he keeps thinking about it even when he isn't playing. Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!
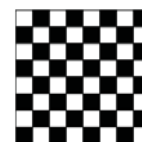
Idea: Mason, Burton, & Stacey (2010, p. 17)

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3 METHODOLOGY

The pupils' behavior – the *processes* – was coded using a framework for the analysis of videotaped problem solving sessions by Schoenfeld (1985, ch. 9). His intention is to "identify major turning points in a solution. This is done by parsing a protocol into macroscopic chunks called episodes [...]." (ibid., p. 314) An episode is "a period of time during which an individual or a problem-solving group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal [...]." (ibid., p. 292) Schoenfeld (1992, p. 189) continues: "We found […] that the episodes fell rather naturally into one of six categories:"

(1) *Reading* or rereading the problem.

(2) *Analyzing* the problem (in a coherent and structured way).

(3) *Exploring* aspects of the problem (in a much less structured way than in Analysis).

(4) *Planning* all or part of a solution.

(5) *Implementing* a plan.

(6) *Verifying* a solution.

As *Planning* and *Implementation* are often very hard to distinguish, Schoenfeld (1985, p. 299 f.) allows for the combined coding of *Planning-Implementation*.

For our study, we adapted the framework with the following modifications:

In the first place, our children – unlike the university students Schoenfeld observed – showed a great deal of non-task related behavior. So we added new categories of episodes comprising acts of *digression*, when our pupils talked about their schools or TV series instead of working on the task, or *writing,* when they needed minutes to write an answer without achieving any new information or making any kind of progress. But this is not important for the results presented in this paper, because I'll focus on the task-related episodes, which are (2) – (6) of Schoenfeld's list. Secondly, we had some problems figuring out the differences between some episodes, especially *Analysis* and *Exploration* (as predicted in Schoenfeld 1992, p. 194). We solved those difficulties by assuming an analogy between Schoenfeld's framework and Pólya's stages, as shown above. Applying Pólya's questions and instructions to the problem solving processes helped us deciding whether to code *Analysis* or *Exploration* (see Figure 5 above for a summary).

The coding of the videotapes was done independently by research assistants and me. Our codes coincided most of the time, but when they didn't, we attained agreement by recoding together (cf. Schoenfeld 1992, p. 194). It is not uncommon to have differing episode-codes within one pair, when the children worked separately sitting next to each other. Therefore, to avoid an imbalanced

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weighting of the data, all processes were counted independently, even if the members of a pair worked together all the time and got the same episode-coding. See the appendix for a sample process.

4 RESULTS AND DISCUSSION

The results are presented in three parts being Schoenfeld's *Exploration* episode, the linearity of the processes, and an empirical framework of the problem solving processes of our fifth graders.

Exploration and Planning

As Schoenfeld splits up Pólya's "Devising a plan" into both *Exploration* and *Planning*, it is interesting to look at this part of the pupils' processes in detail – starting with the latter episode type:

Table 1 shows the number of processes with *Planning* and/or *Implementation* in comparison to those with explicit *Planning* (= no combination of *Planning-Implementation*) episodes.

task	Number of processes			
	total	with *Exploration*	with *Planning* and/or *Implementation*	with explicit *Planning* episodes thereof
Beverage Coasters	32	30	5	1
Marco's Number Series	32	18	22	2
Seven Gates	15	6	12	4
Squares on a Chessboard	19	9	17	2
sum	**98**	**63**	**56**	**9**

Tab. 1: Planning and Exploration within our processes

Even though planning and implementing behavior seems to be task specific (for example, it is hard to plan a guiding hypothesis for the coasters task), explicit planning is rare in every task's processes. Unlike the professional mathematicians Schoenfeld (1985, ch. 9) describes, our fifth graders don't formulate plans prior to implementing them. Their planning is often combined with the carrying out of these steps, leading to episodes of *Planning-Implementation*.

Regarding *Exploration,* Table 1 (column 3) shows it's occurrence within our data – it appears in roughly two-thirds of the processes. Schoenfeld's addition of this episode type is a great gain from a researcher's perspective: It allows the discrimination of structured episodes in comparison to unstructured attempts. This discrimination led Schoenfeld to important research results regarding the importance of metacognition and self-regulation during problem solving (cf. Schoenfeld 1985, chap. 9; 1992). These findings could be reproduced within the data set of the study on hand (cf. Rott 2011a; 2011b). *Exploration* episodes also help in identifying phases with heuristic usage and highlights the importance thereof for problem solving processes.

In my opinion, a problem solving framework appropriate of describing empirical processes, should contain an *Exploration*-like phase for the distinction of structured approaches and "broad tour[s] through the problem space" (Schoenfeld 1985, p. 298).

Linear or cyclic nature of the processes

Are the courses of our pupils' problem solving processes rather linear (Pólya) or cyclic (Schoenfeld, Wilson)? An easy way to answer this question, is to look at the *order* in which the episodes of the process codings appear. (I'm only counting the task-related episodes without *Reading* and those types of episodes we added to adapt the framework to the behavior of fifth graders.)

- A process is considered *linear*, if the order of the episodes follows those of Pólya's steps respectively that of Schoenfeld's: *Analysis → Exploration → Planning → Implementation → Verification*; e.g. [A,E,P,I,V].
- A process is still considered *linear*, if some of these episode types are missing or repeated, as well as if *Planning-Implementation* is coded together; e.g. [A,P-I,V] or [A,E,E,P,I].
- A process is perceived as *non-linear*, if the above mentioned order is broken (regardless of omissions and repetitions); e.g. [E,A], [A,E,P-I,E] or [A,P-I,V,P-I].

Applying this definition gives us the following numbers (see Table 2, columns 3 & 4 for details): 30 out of 98 processes are considered non-linear. Even though the majority of the processes of each task progresses linear, the number of non-linear processes differs significantly from 0.

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Hence a strictly linear framework is not suited to describe our empirical processes adequately.

As a next step, I want to narrow down the question to the nature of the non-linear processes. Where do the cycles take place? Do the pupils change their problem solving behavior (swap episodes respectively) mainly during cycles of *Analysis*, *Exploration*, and *Planning* (as predicted in Schoenfeld's framework)? Or do non-linear junctures between episodes take place anywhere in the processes (as postulated by Wilson et al., which contains Schoenfeld's cycles as a sub-group)?

- A non-linear process is considered *design-cyclic* (as in Schoenfeld's model), if the episode swaps occur between *Analysis*, *Exploration*, and *Planning* only; e.g. [E,A] or [A,P,E].
- It isn't considered *design-cyclic*, if the non-linear juncture occurs after (*Planning*-) *Implementation* or *Verification*; e.g. [A,P-I,A] or [P,I,V,I].

Out of the 30 non-linear processes, 12 are design-cyclic; the task dependent numbers are shown in Table 2 (columns 4 & 5). Therefore a non-linear framework should allow for junctures between any steps of a problem solving process.

task	Number of processes			
	total	*linear*	*non-linear*	*design-cyclic* **thereof**
Beverage Coasters	32	27	5	4
Marco's Number Series	32	17	15	5
Seven Gates	15	11	4	3
Squares on a Chessboard	19	13	6	0
sum	**98**	**68**	**30**	**12**

Tab. 2: Linearity and non-linearity within our processes

Modeling our processes

If I was to develop a framework, suited to describe[7] the problem solving processes of our fifth graders, it would have to have the following properties:

- There should be a distinction between structured and unstructured behavior (*Planning* and *Exploration*) as in Schoenfeld's model.
- It should be possible to intertwine *Planning* and *Implementation*.

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7   Nota bene not as a normative framework to teach problem solving (strategies).

Rott, B. (2012).

- The framework should be able to display both linear and cyclic processes – with the majority of those processes being linear.
- Managerial activities and self-regulatory decisions should be included as a major part as in Wilson's model.

The result of these thoughts is shown in Figure 6. The arrows stand for (explicit or implicit) managerial decisions. The steps of *Planning* and *Implementation* are intertwined but can be passed individually.

Future analyses will aim at possible connections between the order of episodes and success in the problem solving activity. I also plan to take a closer look at some episodes, especially at *digression* – maybe it is possible to identify incubation and illumination in terms of Poincaré (1914) and Hadamard (1945).
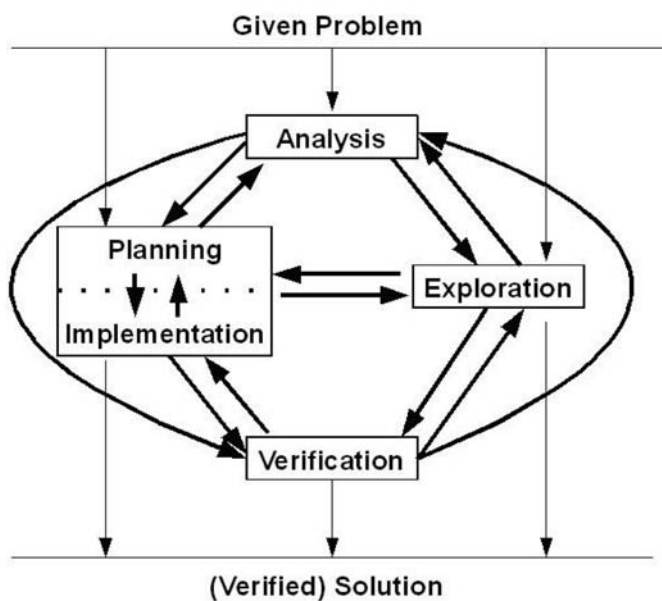
Fig. 3: My descriptive model

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APPENDIX – SAMPLE PROCESS

Beverage Coasters

The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.

Examine the size of the area covered by **both** coasters.
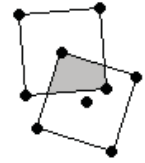
Idea: Schoenfeld (1985, p. 77)

20081211 – V & J – Beverage Coasters

After *reading* the text on the worksheet (00:00 – 00:49), V starts sketching two squares – an imprecise copy of the figure on the sheet. While J asks for a ruler, V expresses that he finds this task difficult and starts to draw a second figure. This episode (00:49 – 02:15) was coded as an *Analysis*, because both of the pupils tried to get a feeling for this task; it ends with V finishing his second sketch – a special case – and exclaiming: "It's always the same size."

Fig. 4: V's 1st sketch
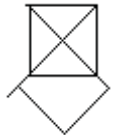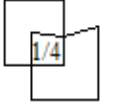
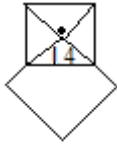| time	V	J	figures
02:16	"it's always the same size" *(Draws subsidiary lines from the centre of the square to it's corners, prolonging the sides of the other square.)*		V's 2nd sketch w. subsidiary lines
02:17		*(Looking up)* "What?"	
02:18	"the area is always the same size"		
02:23		"yes, er" *(looks at the figure of his worksheet)* "no"	
02:24	<silently "it is">		
02:25		*(pointing at V's sketch)* "you drew it wrongly"	
02:27	"look"		
02:28		*(pointing at V's sketch)* <aloud "Look! This is not supposed to meet the corner.">	
02:31	"it is (.) look!" *(he starts a third sketch, the second special case, right next to his other sketches)* "if this is the centre?"		V's 3rd sketch
02:34		"yes?" *(bending forward to look at V's worksheet)*	
02:35	"you could, for example, make ONE square, here"		
02:40	"this is ONE fourth of the whole, you see? (..) you see?"		
02:44		<affirmative "uh-huh">	

time	V	J	figures
02:50	"if I make it this way" *(starts drawing a fourth sketch)* "it's been a bit to big, don't mind" "then it, it, er, covers a fourth, SEE?" *(draws subsidiary lines)*	*(looks at V's worksheet)*	V's 4th sketch w. subsidiary lines
03:11	*(looks at the worksheet)*	"yes, but" "you know, you have to draw it like that to see, if it still fits" *(He points at the sketch on the worksheet, which doesn't show a special case.)*	
03:19		"because, because this is a totally different figure"	
03:23	"NO! Have a look! You can"		
03:25		"but"	
03:26	"you can insert this into this; this has to go there, you see?" *(He rotates his worksheet by 180° and points at the marked area.)*		
03:33	*(He looks at his worksheet and then draws some subsidiary lines. He uses his fingers to cover some areas while looking at other parts of the sketch.)*	"let's try to insert this (.) somewhere else" *(He looks at his worksheet. In the next 20 seconds, he divides the upper square into four parts and hatches them.)*	J divides the figure on his sheet
03:52	"it's always one fourth"		
03:55		"it always has the same size"	
03:57	<*affirmative* "uh-huh">		

This episode (02:15 – 04:00) was labeled as an *Exploration*, because the pupils take good use of heuristic methods (drawing figures, using special cases). There are no signs of *Planning* or *Implementation*.

They continue the process with a *Verification* (04:00 – 04:50), looking for reasons for their conjecture ("it's always one fourth"). J points out, that it wouldn't be one fourth, if the corner [of one square] wouldn't meet with the center [of the other square]. He adds other squares to his first sketch which could be interpreted as "arguing with rotational symmetry".
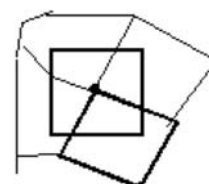
Soon, they agree to *write* (04:50 – 07:10) down their conjecture. They don't come up with any new arguments or ideas, so this is an episode, that isn't task-related. Afterward, they ask for the next task.

Fig. 5: J's sketch

The episode coding of this process is [R,A,E,V,Write]; counting only task-related episodes, it is [A,E,V]. Therefor this process is considered *linear*, because this order conforms to Pólya's steps.

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