András Ambrus
Éva Vásárhelyi (Eds.)

Problem Solving in **Math**ematics Education

Proceedings of the 15th ProMath conference
30 August – 1 September, 2013 in Eger

Eötvös Loránd University, Faculty of Science, Institute of Mathematics,
Mathematics Teaching and Education Center
Eszterházi Károly College
Institute of Mathematics and Informatics

Problem Solving in Mathematics Education

Proceedings of the 15th ProMath conference

30 August – 1 September, 2013 in Eger

ISBN

Editors: András Ambrus and Éva Vásárhelyi

Layout: Éva Vásárhelyi

Printing: Haxel nyomda
Hungary
http://www.haxel.hu

Publisher:
Eötvös Loránd University, Faculty of Science, Institute of Mathematics
Mathematics Teaching and Education Center
and
Eszterházi Károly College, Institute of Mathematics and Informatics

Copies available by Mathematics Teaching and Education Center;
Contact: Éva Vásárhelyi vasareva@gmail.com

Content

Preface

ProMath (**Pro**blem Solving in **Math**ematics Education) is a group of didactics of mathematics from all over Europe, who have the common aim of furthering and scientifically exploring problem-solving activity in mathematics among students, exploring the possibilities and pre-conditions of problem-solving orientation in mathematics teaching, and promoting it.

ProMath was founded by Günter Graumann (University of Bielefeld, Germany), Erkki Pehkonen (University of Helsinki, Finland) and Bernd Zimmermann (Friedrich-Schiller-University of Jena, Germany). One of the activities of this group is to organize the annual conferences since 1999.

It is a great pleasure and honour for us to organize and support the 15th ProMath Meeting in Pólya's home country, which took place from August 30 to September 1, 2013 at the Institute of Mathematics and Informatics of Eszterházy Károly College, Eger, Hungary.

"Mathematical Problem Solving Not Only For Talented"

has been chosen as a central theme of the 15th ProMath conference

This choice was justified on the website of ProMath (http://promath.org/meeting2013.html) by the following:

> *"The fostering of talented students in mathematics education has a long tradition in the whole world.*
> *No doubt that the high ability students differ in a great manner form average in their ability, interest, motivation, learning habits, memory, noticing the solution patterns, problem solving strategies without extra teaching them. On the ProMath2013 Eger meeting we are interested for studies and experiences dealing with students not only highly talented concerning mathematical problem solving. "Not only" means that studies about talented or high-achieving pupils are also welcome."*

It is very interesting to see year in year out the evolution of the ProMath conference.

- On one hand there is a dynamic change with respect to the participants. The participants were novice teachers, doctoral students, and retired professors as well, which enables a thought-provoking dialogue between the founders and the young researchers.
- On the other hand one can see dynamic changes in the content of the conference. In addition to presenting interesting problems − with historical aspects and their multi-

colored and varied solutions −, there are more and more examinations regarding to the role of modern teaching tools and methods. (Modelling, graphic calculators, computer simulation, programming, cooperative teaching and learning, ...)

- Thirdly there is a striking expansion of the research aspects, for example the psychological aspects of learning are increasingly prevailing in the system of criteria. You can find this various aspects in school-reality, in didactical researches, and within the topics of this book and of the conference.

This volume contains the papers of the talks given during the meeting. The papers of this volume are peer-reviewed according to the contents, organized by András Ambrus. Each author is responsible for the proper English of his or her paper.

I want to thank all participants for their valuable contributions. Let me mention those talks of the conference, which are not represented by any article in this volume:

A. Ambrus: *The role of working memory in mathematical problem solving teaching,*
E. Árokszállási: *Using different representations in algebraic identities teaching,*
J. Boda: *Problems in symbolic calculation created by adult high school students*,
T. Hodnik Cadez: *Preschool teachers' and children's competences in problem solving*,
L. Naeveri: *Connection between teachers' actions and fifth-graders' performances when solving a non-standard problem.*

Special thanks to Ilona Oláhné Téglási and to the Institute of Mathematics and Informatics of the Eszterházi Károly College for the organization and for their hospitality. The charming little town of Eger also contributed to the success of the conference.

We have to thank the Mathematics Teaching and Education Center of the Eötvös Loránd University as well Eszterházi Károly College for supporting this volume.

The figure on the front page is an illustration of the following problem: *Show that the area of the dotted and the shaded parts are equal.*

Budapest, 2014.

Éva Vásárhelyi

Problem solving and Modelling – Traditions and Possibilities in Hungarian Mathematics Education

Gabriella Ambrus
Eötvös Lóránd University Budapest, Hungary
ambrusg@cs.elte.hu

Abstract

Problem solving and solving modelling tasks are obviously in connection with each other. However, a traditional mathematics problem and a modelling task are different from several points of view so we may assume for example that a traditionally clever problem solver in mathematics is not necessarily good at solving modelling tasks. I am going to discuss some questions in this subject considering the results of modelling tasks solved by secondary school students.

Key words: Problem solving and modelling, modelling tasks, research in mathematics education

ZDM classification: D50, M13, M14, F93, F94, F99

Introduction

The Hungarian education is changing rapidly and these changes with their new contents in several fields are hard to follow. The limit of human flexibility and a pithy reform require (need) a thoughtful consideration of the possibilities which are provided by good teaching traditions.

The application of knowledge for everyday situations has been emphasized for a long time in Hungarian mathematics teaching but for the realisation of this idea only closed word-problems have been used which are based in a lot of cases on a not really real situation. The Hungarian traditions of problem solving may provide a good basis for teaching new or „new like" content, so for teaching modelling as well, which has become one of the main points in the National Curriculum 2013 (NAT 2013).

Theoretical background

There are several definitions for modelling; generally it means the process which describes the solution of modelling tasks. The necessary definitions for the subject can be characterised in the following way:

Modelling means abstraction of reality by mathematical methods (Greefrath, 2007, 15).
Modelling tasks are solved by modelling process (in an ideal case it's a Modelling Cycle, Blum, Leiss 2006), they are problem centred authentic open word problems.
Modelling competency means the competencies, which are necessary for solving modelling tasks (without the competencies while working only mathematically) and these competencies can be given by the Modelling Cycle. To solve modelling tasks based on problem solving, first we should look at the similarity of the four-step model of problem solving to the four-step model of modelling.

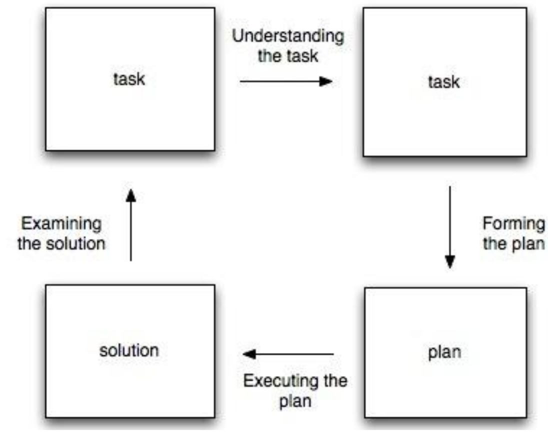


Figure 1: Chart according to Pólya's steps of problem-solving (Greefrath, 2007)

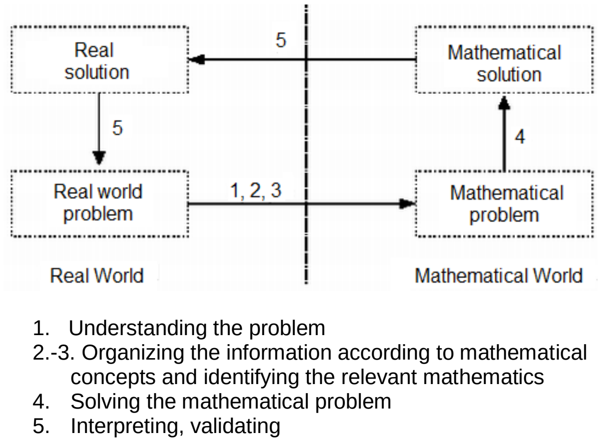


Figure 2: Modelling cycle (PISA 2003)

Based on Pólya's steps we can create another chart, which is more relevant to the modelling cycle (Blum/Leiss, 2006).

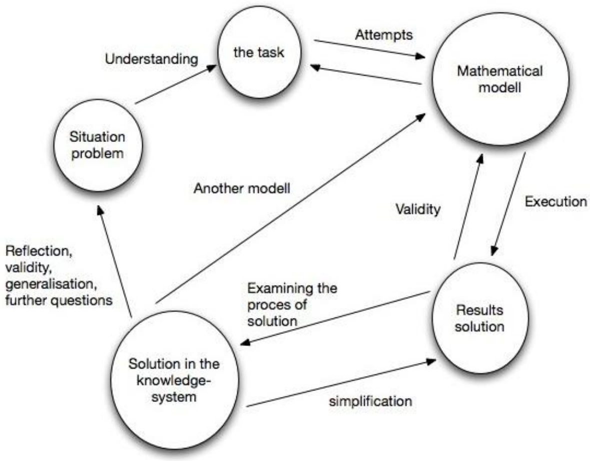


Figure 3: Possible extension of Pólya's steps for a modelling cycle (Ambrus/Vancsó/ Koren, 2012)

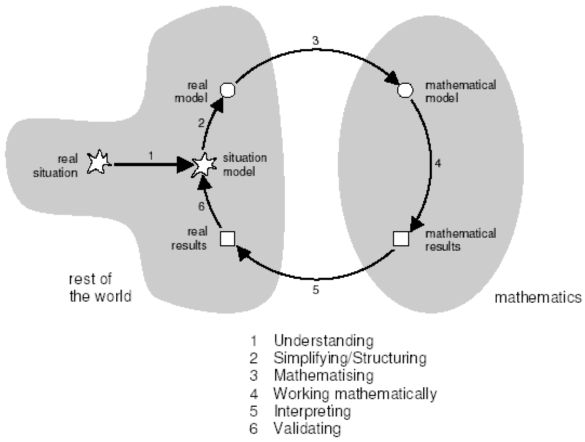


Figure 4: Modelling cycle (Blum/Leiss, 2006)

Analysing the modelling routes of pupils, Borromeo Ferri completed the ideal cycle of Blum/Leiss (2006) with phases referring to individuals' characteristics and she obtained different pictures about the connections among the elements in the cycle in case of pupils with different thinking types (Borromeo Ferri 2010, p. 113).

Greefrath (2010) points out that the Blum/Leiß 2006 cycle theoretically allows the problem solving and the modelling process to be considered as parallel. Each step of the modelling can be interpreted as a separate problem solving process as well. Observing pupils while modelling he concluded, that modelling can be observed through the glasses of problem-solving, however this is not appropriate because of the different character of modelling tasks.

There is a rich tradition of problem solving in Hungarian mathematics teaching, so the question is given: to what extent is it possible to lean on these traditions while solving modelling tasks in schools. Modelling tasks are not used in the everyday practice in Hungarian mathematics lessons nowadays, this way the former question can be formulated as:

How can Hungarian pupils solve modelling tasks without any experience with modelling tasks?

There are several studies about how Hungarian pupils solve word problems (Csíkos et al. 2011). One of them deals with a survey of 4000 students (grades 5-6) using 5 simple modelling tasks. They were asked to choose from three possible answers (solutions) of the three following types: a) routine-based, non-realistic, b) a numerical response that takes realistic elements and considerations into account, c) a realistic response that also refers to the situational complication of the problem but says that therefore the problem cannot be solved for each of the tasks. This last answer can be considered as a modelling solution.
Authors emphasize that in spite of the realistic character of the tasks the majority of students chose a non - realistic answer. However, it is important to mention that in case of some tasks nearly the same number of students chose type b) as type a).
Finally 26,5 % of the students gave for at least 4 tasks a type b answer. The average mark of the pupils in mathematics who gave the answer b is 4,43 (in Hungarian schools students are assessed on a five–level scale, 5 indicating the best and 1 indicating the worst achievement.)

Research questions

Even if a student has a good mark in mathematics it does not necessarily mean that he or she is a good problem solver. However, it seems that pupils with good marks are better at solving modelling tasks.

Considering the connection between problem solving and modelling, in this piece of work the central *research questions* are:

- To what extent are Hungarian students able to work with modelling tasks?
- Which strategies are used by solving modelling tasks and to what extent are strategies known and important for Hungarian students who are good at school mathematics?

Theoretical framework

The *first question* focuses practically on the modelling competency of students. There are several ways (all having their barriers) to measure this competency (Riebel, 2010). Riebel mentions a method where students receive a point for every correct modelling step. The problem with this method is that a student who has constructed an incorrect model automatically loses points on the following steps. It seems more appropriate to modify the method and "give points" for every step that is acceptable.

In this paper we follow this latter method, every step is worth a point if it is correct, because the focus is on the process and not on the result.

For the process analysis the following modelling steps are considered:

1. From the real problem into a mathematical problem
2. Solution of the mathematical problem
3. From the mathematical result to a real-result
4. Interpretation of the real result

For investigating the *second question* the method of „using self-commented strategies" (selbstberichtete Strategienutzung") is used (Schukajlow/Leiss, 2011).

In this research, after solving some modelling tasks, the students received a questionnaire with statements concerning their strategy. They had to rate each strategy from 1 to 5 deciding to what extent they found the strategy useful while working on the given modelling task.

In the research the following theoretically founded statements were used in a modified version (Fig. 5). We used these statements with the modelling task presented in this paper with four students after recording their solutions.

The students' knowledge about strategies was compared to the strategies used while solving the task.

To what extent do you think are the following strategies important for a successful solution? Rate the statements from 1-5.	
While solving this task I would	
1. reread some sentences	
2. highlight (i.e. underline) some important data in the text	
3. make a plan	
4. make sketches and figures	
5. check several times whether my way of solution correct is	
6. look for similar tasks and I would revise their solution before	
7. divide the problem into subtasks before solving it	
8. simplify the task and I would solve this version first	
9. interpret my result whether it is approximately correct	

Figure 5: The statements (strategies) of the students

Methodology and design of the study

For the study I utilised some results of my student B. Tóth (2013), furthermore we recorded and videotaped the students' solutions of a modelling task.

In the first case three versions (for three different age groups) (situation: Olympic Stadium) were created and solved by about 120 pupils (grades 7-11) of a Hungarian secondary school in the autumn of 2012. In every year group two groups (students taking part in normal or in high level course in mathematics) solved it. The versions were different regarding their complexity. The students did not have any former modelling experience. They had about half an hour to solve the problem.

In case of the 120 students we had only their written tests. To get some insight into the thoughts of students while working with a modelling task (without any previous experience) four solutions of the version for grades 9-10 students of the Olympic Stadium were recorded or videotaped in secondary schools in Budapest. This version was chosen for collecting data from different grade students and the complexity of this task fitted this purpose the best. Finally, these four solutions were collected only from students who are interested in mathematics.

The students who were asked to share their thoughts while solving the task were selected by their teachers and they could decide whether to take part in the survey. They are all good at mathematics. The teachers were asked to give a short description of the pupils as well. Among the 4 students 3 were classmates (grade 11) in a secondary school, the last one was a grade 11 student in a bilingual secondary school (grade 11 means here only a grade 10 in terms of mathematical studies because of a preparatory year). The names of the students were changed in the paper.

Tasks

Age group 7-8

This is the Olympic Stadium 2012, and here are 80 000 places for spectators.

The 100m and 400m dash competitions were organized in this stadium.

In the 400m dash the participants run around the track once. Obviously, the on the track the different lanes have different length.

How much is the outermost lane longer than the innermost?

How would you mark the starting point to make the competition fair?

Write down your ideas.
(Task: B. Tóth)

Age group 9-10

This is the Olympic Stadium 2012, and there are 80 000 places for the spectators.

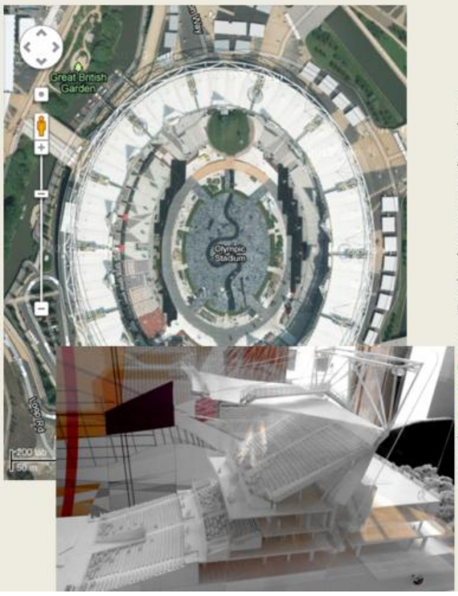
The 100m and 400m dash competitions were organized in this stadium.
In the 400m dash the participants run around the track once.

It must have been great to watch the race from the first row.

How many spectators may sit down in the first row ?

Write down your ideas.
(Task: B. Tóth)

Age group 11



Here is the construction- modell of the Olympic Stadium 2012.

According to the plans the stadium's bottompart with 25 000 places can be used in the future as well. The upper two parts, which are made from reusable elements, are going to taken away after the Olympic Games.

Using this picture ***give the number of spectators, who can sit down in the stadium during the Olympic Games !***

Write down your ideas.
(Task: B. Tóth)

Figure 6: Tasks

Some results of the study

About the students' work that were created in the classes

Among the 120 pupils there were 60 who prepared a model as the first step. In the solutions the occurrence of the 4th step was the rarest. Most of the students who were uncertain in their result wrote an interpretation for their solutions.

Pupils who had some idea for solving the task made 3 steps and got a sort of result as well.

The average number of steps shows an interesting picture. In every age group the number of steps is higher in the high level classes, and older age means generally higher average number of steps, however the tendency isn't linear.

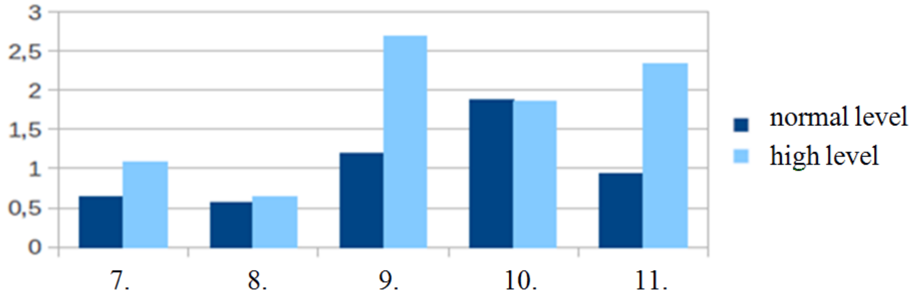


Figure 7: Average number of modelling steps

Although several models were constructed in every age group, these were mostly unacceptable, especially among the works of the pupils in grade 7. Their work was often based on an incorrect assumption regarding the length and width of the runway.

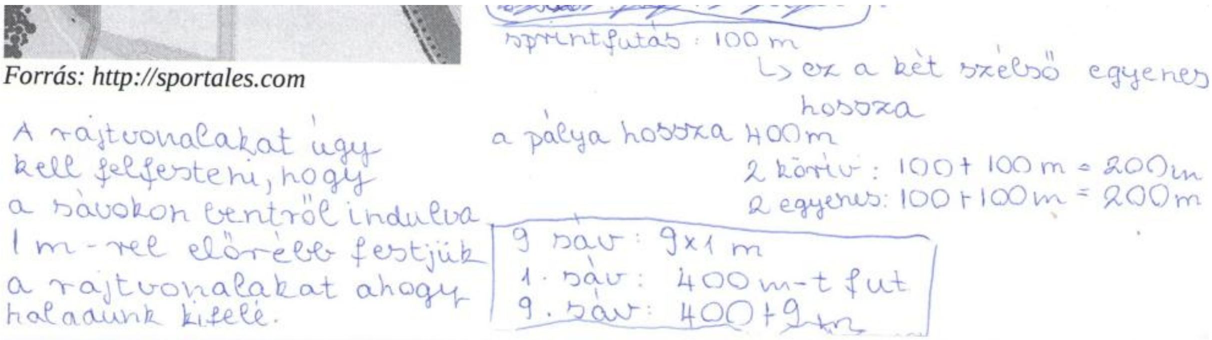


Figure 8: The student assumed that the runway is 1 m wide and so every runway is 1 m longer than the previous one.

Although the starting ideas were right, many students could not use the picture on the sheet but they tried to work only from the text. They prepared a very simple model; they worked out mostly unfounded estimations.

Results of the recorded solutions

The students worked on the version of the "Olympic Stadium" task which was created for grade 9-10. Three students are classmates in a year 11 class; while the fourth is a grade 10 student in a secondary school.

Géza: A boy with an inconsistent behaviour, sometimes he cannot be trusted, but otherwise he is a nice fellow. He always has some ideas. Mathematics is important for him, he wants to be better, but does not always have a 5 mark. He spent the previous year abroad; he has been in the current class since last September.

He started to solve the task coolly and it seemed that he knew how to begin. He prepared a first cast after he threw away his first idea: to work with the number of rows in the picture. He went through the modelling steps neglecting the last interpretation step. In his "final" solution he assumed that the shape of the stadium is a circle and r = 180/2 = 90 m. He obtained an acceptable result, about 700 people in the first row.

He needed about 11 minutes for the solution.

After solving the task he rated the following strategies as the most important ones (4 or 5 marks): *Rereading some sentences, making a plan; checking several times, dividing in sub-tasks, and interpreting the result.* He used these strategies except for the interpretation of the final result.

Zsuzsi: She has a scrupulous character and she usually writes the mathematics tests without mistakes. She is meticulous, and usually does not add anything new to things. She is dutiful, accomplishes what she is asked. She has a steady 5 in mathematics.

She was surprised by the task. She used only the picture for the solution, and made several assumptions for the circumference of the first row. Finally, she took it for 500 m and found that the number of people in the first row is about 880 (60 cm for a person). She didn't interpret the final result, so she used 3 modelling steps.

She needed about 12 minutes for the solution.

After solving the task she rated the following strategies as the most important (with 4 or 5 marks): *Rereading of the text, making sketches and figures, checking several times, dividing into subtasks, simplifying the task, and interpreting the result.* She used some from these strategies while solving the task: rereading of the text, making sketches and figures, checking several times, simplifying the task (she worked only on the picture).

Árpád: He is good at and interested in mathematics. Furthermore, he is creative but often has a problem with accepting novelty. He is a good problem solver and is quick in the uptake. He can't give up things; he often struggles with problems.

It was immediately obvious that he found the task very strange. He said that he had expected a really difficult mathematics problem (not this one). He didn't understand what the expected solution was he tried to find it out (by direct questions) several times while solving the task. He made a sketch, he gave the length of the semicircles and parts by assumption. Afterwards, he gave an average number for the radius of the stadium and he worked with circles (concerning the shape of the runway). For the number of people in the first row his result was 836. He found that it is too many and took (without any explanation referring to the picture) 700 as the answer. He went through the modelling steps neglecting the last interpretation.

He needed about 24 minutes for the solution.

After solving the task he rated the following strategies as the most important (4 or 5 marks): *Making sketches, figures, simplifying, and interpretation of the result.* While solving the task he used: making sketches, simplifying (temptation several times).

It is to mention that from the statements he considered the least important (rated 1-2) he used several strategies: making a plan, looking for a similar task, checking several times (mostly by asking whether his way of solution is correct).

Zoli: He is quick in uptake and good at mathematics. He likes to finish the tasks quickly; he always focuses on the result.

After receiving the task he was thinking for a long time. He tried to work in the picture. He found 72 sectors on the figure so he assumed that about 80 000 : 72 = 1111 people were in one sector. He got stuck here. After a while he went on with another method. He assumed that the runway was a 100 x 100 rectangle (a square) and so calculated 250 people (40 cm for one person) on one side, so 1000 people would sit in the first row. After the question of the teacher who made the videotaping whether this result was realistic, he corrected the place needed for one person to 50 cm obtaining the final result of 800 people. He didn't interpret the final result.

He needed 11 minutes for the solution. He went through the modelling steps neglecting the last interpretation.

After the solution of the task he rated the following strategies as the most important (4 or 5 marks): *Making sketches, checking several times, interpretation of the final result.* He used these while solving the task except for the interpretation - he did it only after the teacher's remark.

Discussion

The samples we used to obtain the results had different size (120 students and 4 students).

The number of modelling steps used by the students in their work shows – although they had not solved modelling tasks before – that many students seem to be able to do an „intuitional modelling”. However, doing several modelling steps does not necessary mean finding an acceptable solution.

Students who are good at problem solving are generally able to execute more modelling steps; it means that they can make a better start with modelling tasks. They consider the task as a problem that they have to solve. Presumably this better result is supported by their positive belief about mathematics as well.

The four observed students rated “making a plan” generally as „unimportant”. It coincides with the result of Schukajlow/Leiss 2011 in which this problem is discussed in detail. Although an interpretation of the final result was missing in almost everyone's work, they checked their solution several times while working on the task. The four good problem solvers mentioned different number of strategies as important when solving this task, so it seems that they know about strategies which can help the problem-solving. The character of the problem solver, i.e. mathematical thinking style (visual thinking style, analytical thinking style, integrated thinking style, Borromeo Ferri, 2010) may have an effect on the person's modelling process and on the quality of the solution as well.

Conclusion and further perspective

Although the questions were investigated only with a few students so the results are not enough for general statements, it is promising that several modelling steps are present in the work of the investigated students without having any previous experience with modelling. This refers to a basically good modelling competency in these cases mostly among good problem solvers. The knowledge of strategies is pretty good among the investigated good problem solvers but there is a discrepancy between this knowledge and the application of strategies in a given situation. An important experience is that during the investigation students must be helped and encouraged while solving unusual tasks. They need (would have needed) more feedback while solving the task. While working on this type of task alone, students may have negative, uncertain feelings which may influence their modelling work and the acceptance of this type of task in a contra productive way as it happened in the case of Árpád. A previous discussion of the modelling process and the character of modelling tasks seem to be necessary especially for low achieving students.

For a more thorough analysis of the research question further investigations are needed. For example:

- Further collection of data among good and normal problem solvers with different types of modelling tasks.
- Investigation among students with different ability in mathematics (with a control group) whether a reassuring feedback from the teacher or the possibility of a discussion with other pupils (i.e. working in groups) has a positive influence on working with modelling tasks.
- In this paper we considered the use of strategies only among good Hungarian problem solvers. A wider investigation could clarify whether a correlation between strategy knowledge and achievement in mathematics exists. In their investigation Schukajlow/Less 2011 couldn't find such a correlation.

Acknowledgments: Thanks to Hegyi Györgyné, Koren Balázs for their help with the realisation of videotaping and recording student's solutions.

References

Ambrus, G., Vancsó Ö., Koren B. (2012). The Application of Modelling Tasks in the Classroom – Why and How? 10/2 231-244 TMCS, Debrecen

Blum, W., Leiss, D. (2006). „Filling up" – The Problem of Independence-Preserving Teacher Interventions in Lessons with Demanding Modelling Tasks. I.: Bosch, M. (Ed.) CERME-4 – Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education, Guixol

Borromeo Ferri, R. (2010). On the Influence of Mathematical Thinking Styles on Learners' Modelling Behaviour 31: 99-118, JMD

Borromeo Ferri, R.: Mathematical thinking styles- an empirical study http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG3/TG3_BorromeoFerri_cerme3.pdf (15.06.2013)

Csíkos, Cs., Kelemen, R., Verschaffel, L. (2011), *Fifth-grade students' approaches to and beliefs of mathematics problem solving: a large sample Hungarian study*, 43: 561-571 ZDM, Springer Verlag

Greefrath, G. (2007). *Modellieren lernen mit offenen realitätsnahen Aufgaben*, Aulis Verlag, Köln

Greefrath, G. (2010). *Problemlösen und Modellieren- zwei Seiten der gleichen Medaille* 2010/3: 44-56 Mathematikunterricht, Friedrich Verlag

Riebel, J. (2010). *Modellierungskompetenzen beim mathematischen Problemlösen. Inventarisierung von Modellierungsprozessen beim Lösen mathematischer Textaufgaben und Entwicklung eines diagnostischen Instrumentariums*. Dissertation, Universität Koblenz-Landau, Fachbereich Psychologie, Landau.

Schukaljov, S., Leiss, D. (2011) . *Selbstberichtete Strategienutzung und mathematische Modellierungskompetenz,* 32: 53-77 JMD, Springer Verlag

Schukaljov, S. (2010). *Schüler-Schwierigkeiten und Schüler-Strategien beim Bearbeiten von Modellierungsaufgaben als Bausteine einer lernprozessorientierten Didaktik, Dissertation* http://kobra.bibliothek.uni-kassel.de/bitstream/urn:nbn:de:hebis:34-2010081133992/7/DissertationSchukajlowWasjutinski.pdf *(18.06.2013)*

How do they solve problems?
Mathematical problem solving of the average and the talented

Krisztina Barczi
University of Debrecen, Neumann János Secondary School, Hungary
bkrixta@gmail.com

Abstract

When it comes to mathematical problem solving probably everybody has a preference in terms of how to handle the task in hand. The choice of solution method depends - among many other things - on the students' level of Maths knowledge. Those who are talented prefer one way while those who have a rather average ability in Maths prefer another one. This article compares and contrasts the problem solving style of a talented and that of an average ability student in different types of lessons. In one lesson frontal teaching was used and in the other one cooperative teaching techniques were applied.

Key words: cooperative learning, talented, average ability, mathematics

ZDM classification: C20, C30, C40, C60, D40

1. Introduction

In Hungarian secondary schools mathematics is mainly taught in heterogeneous classes, which means that talented students work together with the average ability ones and the low-achievers. This scenario requires differentiation from the teacher which is not always easy to accomplish. Furthermore, in this environment the teacher needs to ensure some kind of success for all students regardless of their mathematical ability.

Although more and more alternative teaching methods are appearing in schools the most widely used teaching method is still frontal teaching where differentiation is not impossible but sometimes is really challenging. Even students with similar ability level can have different working paces, or some are slower in grasping concepts and ideas even if they are good at actually solving problems.

This article describes an experiment that was carried out in a secondary school class and whose main aim was to find out how different ability students reacted in different problem solving situations. For this, two students – based on the students' previous mathematical achievement and occasional competition results one of them can be considered as talented in Maths and the other one as an average ability student – were observed in two types of lessons. In one type of the lessons frontal teaching was used while in the other one cooperative teaching techniques were applied. Here, we summarize our experience.

2. Theoretical background

The aims of teaching Mathematics

One of the main aims of teaching mathematics is to make students understand mathematical concepts and their relations and to help them acquire thinking skills that can be used in problem solving regardless of the content, both in the field of mathematics and in other areas as well. (Ambrus, 2004)

As understanding has a primary importance, let us examine first how concepts are formed in our mind. Most everyday concepts are formed with the help of objects or experiences from our everyday life. That is why most of them are easy to remember since we can connect them to something we know very well. The difficulty of understanding and learning mathematical concepts lies in their abstractness. Students cannot learn mathematical concepts from their own experiences only with the help of someone else. In schools teachers try to choose good examples that demonstrate the nature of a concept. Furthermore, according to Skemp (2005) when teaching mathematical concepts the level of the students' knowledge and that of the concept should be in line.

Understanding and learning a mathematical concept is only the beginning of a process. In order to become successful problem solvers students need to be able to apply what they learnt. A concept can be applied if it is fully understood. Two types of transfers can prove that a student had understood a concept. Near transfer means using the concept in a situation that is similar to the ones used in the teaching situation while far or knowledge transfer means that the student can apply the concept in a context that is completely different from the above mentioned one. (Dobi, 2002) For the low-achiever or average ability students achieving near transfer already means success.

Besides helping students understand mathematics and helping them be able to use what they learnt in different problem solving situations, it is important that they experience the joy of thinking. (Szendrei, 2005) For average ability students it is slightly more difficult to show that mathematical thinking can be similar to playing with different thoughts but if the teacher manages to make these students feel that doing maths can be creative and enjoyable than they can become more successful.

Furthermore, for helping all students experience success effective differentiation is needed in the classroom, for which recognizing talented students in mathematics is necessary. As Miller (1990) says students who are talented in mathematics have an ability to understand even complex mathematical ideas and can use mathematical reasoning very well, but they are not necessary the students who are good at arithmetic calculations or have top grades in Maths. Furthermore, maths talents often have emotions related to numbers and they have an interest in the subject in their early years already. They like solving puzzles and they are successful in completing tests that measure memory or visual skills. Moreover, these students often use specific procedures for solving problems. (Gyarmathy, 2006)

Different students have different problem solving preferences which depend on their personality, their mathematical ability, their motivation … etc, so there is no rule which tells us that for talented students we should use a specific approach. For high ability students fast track or acceleration programs might be beneficial especially if talent is matched with motivation. On the other hand, slower or less motivated students might do better if the learning pace is slower and the learning focuses deliberately on the mathematical concepts being taught. (Miller, 1990) In an average classroom using cooperative learning structures might be beneficial for both types of students.

Talent or average ability, all students gain from learning in a mathematically enriched environment. For this social interactions between students should be encouraged, the appearance of emotional connections should be facilitated, the students' individual style should be taken into consideration and the development of students' self-confidence should be emphasized (Caine et al, 2004). Again, when using cooperative learning structures the above mentioned factors appear more easily than in a frontal class.

Cooperative teaching and learning

In cooperative learning students are arranged in groups so that they have to work together in order to achieve a common goal, to solve a problem. For successfully overcoming the possible difficulties the members of the teams must rely on each other, respect and support each other since their success depends on their ability to work together. (Kagan, 2004) In Hungary it was József Benda whose work had an effect on the appearance of cooperative teaching in education as he believed that this teaching method can contribute to the integration, achievement and development of students. (Józsa & Székely, 2004)

Many people think that using cooperative techniques is the same as simply making students work in groups. I want to clarify a difference. In group work students are put together and given a task to solve and it is their job to find out who does which part of the task and share out the responsibilities. This working format often results in the best student doing all the work while the others copy everything while some students might not even listen. To avoid this, cooperative structures were created in a way that the following four principles are always present: **P**ositive interdependence; **I**ndividual accountability; **E**qual participation; **S**imultaneous interactions. These four principles ensure that every member of the group is an active participant and the team's success is everyone's responsibility. (Johnson & Johnson, 1994) To help teachers explain and students to remember these structures they were given catchy names. (Kagan, 2003) Here are some examples for cooperative structures: 1) *Think-Pair-Share*: Students work in groups of four. A problem is presented to the students. They are given time to think on their own about possible answers for a specific amount of time. Students discuss their answers in twos. 2) *Round Robin*: This structure gives an opportunity for each member of the group to speak. Starting with one student, everyone gets 1- 3 minutes to present their point of view. (Kagan, 2004) *3) Jigsaw*: Each team needs to become an expert of a topic/solution of a problem. New groups are formed with one member from each of the original groups. They share their knowledge. Finally, everyone goes back to the original groups (Slavin, 1995).

When using cooperative teaching techniques the issue of composing the groups is important. How the teacher forms groups might depend on the teacher, on the students, on the type of the lesson … etc. However, no matter what the arranging principle is an ideal group consists of four members. The reason for this is that in a group of four two pairs can be formed, furthermore if every member needs to communicate with every member than the number of interactions is 6 and nobody is left out, while if the number of group members is three or five one student might feel neglected (Burns, 1990).

In cooperative teaching and learning not only the classroom setting but the teacher's role is different. First of all, instead of being the person who takes control over the whole teaching and learning process and leads the students through the lesson, the teachers becomes an observer or a coach who consults the groups or with individual students and helps the ones who are left behind. The teacher is still needed in the classroom only the function has changed. Since the teacher can walk around the classroom, he can give an instant feedback for the groups or he can help the groups overcome the obstacles to learning or problem solving. (Crabill, 1990) However, it is still the teacher's responsibility to ensure the right environment for learning. Just because students are working in groups and there are parallel discussions the classroom should not turn into a permissive and chaotic place. (Dees, 1990)

3. Questions

During the experiment and the two particular lessons we focused on the following questions:
1) How do talented/average ability students solve problems in frontal classwork?
2) How do talented/average ability students solve problems in cooperative work?
3) How do they feel about the different scenarios?

4. The experiment

The two lessons where these two specific students were observed were part of a longer experiment. This experiment was an action research, which is usually carried out by practicing teachers who would like to achieve professional development. According to Koshy (2005) action research is a kind of enquiry during which the teacher constantly changes and refines his practice and this process leads to his professional development. Since action research is about the researcher’s own practice, it is participatory and situation – based. Furthermore, action research is a tool that brings the goal of mathematics teachers and researchers of maths education closer. (Zimmermann, 2009)

The present experiment was carried out in a mixed comprehensive secondary school whose students are 12-20 years old. The **school** is considered “a good school” since the students attending here are the ones who did best on the entrance exam.

Sixteen 16 - 17 years old **students** took part in the experiment. In the academic year when the experiment happened they attended a class that specializes in maths and foreign languages. In their preparatory year the number of maths lessons per week was three, in the following two years they had four maths lessons every week. That year the class followed the year 10 scheme of work for Hungarian secondary schools. The teacher of the class was the researcher, as well.

In the first part of the experiment 12 **lessons** were held where 5 mathematical problems were discussed – there were 2-3 lessons planned for each problem. Although these problems were curriculum-based Maths tasks, they were different from the ones that are used in an average Maths class. These problems were open-ended problems or investigations. These 12 lessons were planned with cooperative method only.

In the second part of the experiment, in the rest of the school year the students continued working with the official scheme of work and they had lessons with cooperative method once every 2-3 weeks, depending on the material.

During the experiment for **collecting data** the following methods were used**:** 1) students had so called "*reflection booklets*" which were exercise books where students solved the problems and recorded their feelings and ideas about the methods used; 2) half of the lessons were video recorded; 3) the work of every group was voice recorded; 4) students completed pre-, post and delayed mathematical tests; 5) students completed pre- and post- psychological questionnaires.

5. Focus on these two students

During the above described experiment all 16 students were observed, but in this article the focus will be on two particular students. To obtain a picture on their problem solving style and problem solving preferences their notes from their "*reflection booklets*" were checked, in addition they filled out a questionnaire[1] on cooperative learning at the end of the experiment. The results of these students on the three mathematical tests were analysed as well. Furthermore, these two students were observed in two specific lessons which were planned in the same topic with different teaching methods, one of them with frontal teaching using closed problems, the other with using cooperative techniques and an investigation task. The teacher's notes from these two lessons will be discussed, too.

The students

Let us call the two students Richard who is the talented and Alex who is the average ability student. In the following section their attitude towards mathematics, their behaviour in maths lessons and their maths results are summarized.

Richard's mathematics grades contained mostly 5s[2] with some 4s in the past three years. He often participated in mathematics competitions where he achieved good results, besides he regularly attended maths group study sessions. His attitude towards mathematics was definitely positive and he showed great confidence when it came to sharing ideas with the class. In lessons he took part actively in discussions but worked well individually if necessary. When solving a new task he often contributed with useful ideas or comments.

Alex's mathematics grades contained 2s, 3s, 4s and 5s. His half year mark in 2012/2013 was 3 while his end of year mark was 5. His achievement was rather fluctuating, it often depended

---

[1] Appendix

[2] In the Hungarian system 5 is the best grade and 1 is the worst.

on the topic how well he completed a test. His work was mainly focused on classwork and completing the homework but Alex never attended group study sessions or never applied for competitions in maths in the past 3 years. Especially at the beginning of his first year in this school he had low self-confidence in terms of mathematical knowledge and in a problem solving situation he often relied on the other students' ideas or help. During classwork he often sought assistance, mainly from the teacher, or simply just wanted reassurance. In class discussions he participated only when he was asked.

6. The lessons

Frontal teaching

The topic of the lesson that was taught using frontal teaching was generalizing trigonometric ratios; the type of the lesson was practice. The students had previously learned how to interpret trigonometric ratios for angles bigger than 90º and how to find angles if the trigonometric ratio was given. In this lesson they had to work from the following worksheet:

Complete the following tasks.

1. Without using a protractor mark the following angles in the Cartesian coordinate grid:

a) 120°	*c) – 130°*	*e) – 90°*
b) – 45°	*d) 270°*	*f) 750°*

2. Find the acute angles whose trigonometric ratio is equal to:

a) sin (240°)	*c) sin (210°)*	*e) sin (810°)*
b) cos (315°)	*d) cos (– 45°)*	

3. Find all angles for which the following equations are fulfilled:

a) sin α = 0,47	*c) cos α = 1*	*e) sin α = – 0,6*
b) cos α = – 0,17	*d) sin α = 0*	

In this lesson the students were given the above worksheet and they were asked to solve the tasks. The answers were checked as they proceeded and when we found a question that everybody had difficulty with we had a class discussion. When individual students got stuck on a problem they either helped each other or they asked for the teachers' help.

Cooperative lesson

The topic of this lesson was generalizing trigonometric ratios too, and its type was also practice through an investigation. With cooperative techniques there is a better opportunity to use so called open problems or investigations which can be considered as open problems. (Pehkonen, 1999)

In the lesson students were working in groups of four and they had to solve the following task: *Given two segments and the trigonometric ratio of an angle. Are there any triangles whose sides are the segments and whose angle is the given angle? Try to find and draw all possible triangles.*[3]

The lesson plan was the following:

1. Forming groups
2. Trying to solve the task using the structures "*Think-Pair-Share*" and "*Round Robin*"
3. Group discussion in groups of four to make sure that every member understands the ideas and the group's solution
4. Sharing the different solutions of the groups using the structure "*Jigsaw*"
5. Collecting all the solutions and checking ideas through class discussion

7. Observations and experience

The teacher's perspective

As mentioned before observations were made during the two particular lessons with special attention on the two students, Richard and Alex. Their classwork and participation can be described as the following.

Frontal lesson

Richard worked really well on his own. He used his previous knowledge confidently and when he was ahead of the others in answering the questions he was happy to help them. Richard copied a task first then he solved it and proceeded through the whole worksheet without changing the order of the exercises. During the task solving phase he never asked for help or for reassurance from the teacher. However, he was an active participant in class discussions, when we were checking the solution methods and the answers.

On the other hand, Alex did try to work on his own, but with less success. He often looked puzzled and he was trying to check what his neighbour was doing. When he had no idea how to start he looked back in his exercises book trying to find examples that he could use. In spite of the fact that he did not always know how to answer the questions Alex did not speak to anyone and he hardly asked for help. Even when he decided to ask for clarification of the task or just for some guidelines he chose to ask the teacher. From the worksheet first he copied all

[3] The four groups received four different data to work with. The trigonometric ratio was always the *sin* ratio

the questions and only thereafter did he try to solve them.[4] During class discussions Alex only listened to the discussion instead of taking actively part in the conversation. He checked his answers and copied the missing ones or corrected the incorrect ones but never asked why his solution was incorrect or how to obtain the right answer.

Cooperative lesson

First of all it is important to mention that this lesson was only one of the many where the students used cooperative techniques for mathematical problem solving. By the time this lesson was taught the students had had experience with working in groups.

As mentioned before, Richard preferred working alone and this was obvious from his attitude to cooperative work as well. For him his own success and progress was always more important than that of his group. He was not willing to share his ideas and thoughts if he did not have to. However, when he was talking to his group mates he behaved as if he was superior to the others and played more of a leading role rather than cooperating with the others.

As for Alex the grouping of this lesson was not the most fortunate for him.[5] After receiving the task first he looked puzzled then waited. Following that, all of a sudden he started writing down some ideas then stopped. When he had to share his ideas he was very brief and not really confident. In earlier classes with cooperative learning he was a more active participant.

When using the structure "*Jigsaw*" new groups were formed which had a positive effect on the work of both students being observed.

Richard was more willing to share his work and he was proudly sharing his group's ideas and solutions with the others. However, he still found it hard to listen to the other students and he preferred being the one who tells the others what to do.

This grouping suited Alex better than the previous one which was reflected in his attitude and class work. He became a more active listener and when it was his turn to speak and share ideas he was much more confident, moreover he did not mind making mistakes and was braver in taking the risk of saying something that was not necessary correct.

8. The students' comments on cooperative learning

The following ideas were written in the students' *"reflection booklets"* after the 12 cooperative lessons at the beginning of the school year.

[4] I believe that the reason for this behaviour is that the student wanted the others to think that he was busy working on the problems. I think, in this way he was trying to hide the fact that his knowledge was not stable enough.

[5] Alex and Richard ended up in the same group.

Richard wrote: "*At first I didn't prefer working in groups but often it was useful since we were able to discuss results with each other. We managed to find many solutions that I had never thought of before. Basically now I like cooperative work, although it depends on the members of my group. If my group mates were inactive or just messed about it made progress more difficult.*"
Alex wrote: "*I didn't always like my group mates. There was always someone to help. For me cooperative work was sometimes too much. The tasks were interesting and varied. Since I prefer working alone cooperative work was a negative experience for me. However, in this way we solved special problems. We could come up with good ideas. In the future I would like to avoid working in groups, although it is a good idea.*"

9. Questionnaire about cooperative work

The questionnaire contained 20 statements[6]. For analysing students' answers the statements were organized into three groups on the basis of the following aspects: (1) statements related to cooperative work; (2) statements related to attitude towards Mathematics; (3) statements about relationship to the others (students or the teacher).

Cooperative work

Statement	1 (+)	2 (+)	3 (+)	5 (+)	11 (+)	12 (+)	13 (+)	14 (+)	16 (-)	17 (-)	18 (-)	19 (-)
Richard	3	4	3	1	3	5	5	5	2	5	3	4
Alex	3	2	2	2	3	3	2	2	3	4	5	5

The above table shows how the two students reacted to statements that are related to cooperative work. The statements whose number is marked with (+) are the ones that are in favour of cooperative work while the ones marked with (-) describe some disadvantages of working in groups. The mean average score for Richard on the statements marked with a (+) is 3.625 and the standard deviation is 1.4 while the mean average of Alex's answers is 2.375 and the standard deviation is 0.5. This means that in general Richard had a more positive attitude to group work but his answers have a wider range. Moreover, there are aspects of cooperative work (statement 12, 13, 14) that Richard definitely liked.
On the other hand, Alex's answers clearly show that he did not feel the positive statements true for him. His average score is lower, so is the standard deviation which means that his opinion on group work was quite consistent.

[6] Appendix

Statements 16, 17, 18 and 19 are negative statements about cooperative work and the scores of both students are in line with their opinion on group work.

Maths related feelings

Statement	4	6	7	9	10	20
Richard	2	1	4	3	1	1
Alex	3	3	5	3	2	2

The second table summarizes the answers for statements that refer to the students' attitude to mathematics or a change in their feelings towards the subject. The mean average of Richard's scores is 1.71, the standard deviation is 1.38. The mean average of Alex's scores is 2.57, the standard deviation is 1.51. The data shows that there was no significant change in Richard's attitude towards mathematics – from experience it can be said that it had already been positive.
Alex's answers show that there was less change in his attitude in the positive direction; however, the scores in the last two statements refer to a positive change in his confidence in doing (?) mathematics.

Relationship

Statement	8	15
Richard	1	6
Alex	4	2

Only two statements were related to the relationship with others. The 8th statement asks about the attitude to the teacher. Although the table shows that for Richard being brave enough to ask questions from the teacher is not true, experience proves the opposite. He never had issues with asking for help or further explanation, so he must have misunderstood the statement. On the other hand, Alex said that he became more confident when it comes to asking questions.
Statement 20 is about attitude to fellow students. It can be clearly seen that Richard definitely became more patient with others, so cooperative work improved his interpersonal skills which is supported by the teacher's observation, too. On the other hand, group work did not have much effect on Alex's attitude to his classmates

10. The students' achievement on the mathematical tests

As mentioned before during this action research students had to complete three mathematical tests, one before the 12 lessons, one after the 12 lessons and one test about 8 months after the post-test. All three tests contained 9 mathematical problems which were designed so that they measure the students' knowledge in mathematical areas that will be needed for the discussion of the 5 problems at the beginning of the experiment. Furthermore, they tested how well the students use different problem solving methods (eg.: thinking backwards, trial and improvement … etc. (Ambrus, 2004)

Some of the pre-test tasks were repeated in the post-test, however, the latest contained questions trying to elicit knowledge gained during the 12 lessons of the experiment, too. The delayed test was completely identical to the post-test. When marking the test the individual tasks were equally weighed. All 9 tasks were worth 5 marks. The following diagrams show Richard's and Alex's achievement on the three different tests and the last graph compares the total of their marks on the mathematical tests.

In the pre-test Richard completed 6 tasks fully (see: *Figure 1*), this number remained the same throughout the three tests. It can clearly be seen that his achievement was already good on the pre-test and he managed to perform similarly on the following two tests as well.

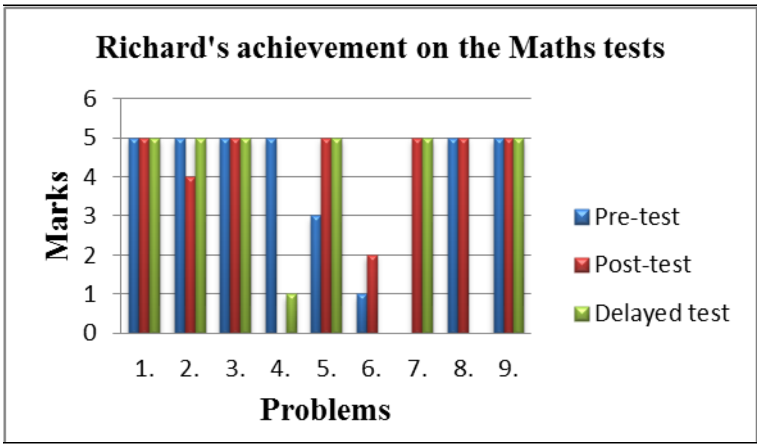


Figure 1

On the pre-test he had difficulties with solving the following problems: 5. permutations; 6. word problem that can be solved with simple equation; 7. calculating area. On the post-test the following types were problematic for him: 2. the justification of a divisibility problem; 4. permutations; 6. justification of a problem including number theory. Finally, on the delayed test the solution of the following problems was incomplete or left out: 4. permutations; 6. justification of a problem including number theory; 8. geometry.

Figure 2 clearly shows that Alex's achievement was not steady. In the pre-test he managed to solve three tasks fully and this number decreased to two in the post-test and increased back to three in the delayed test. On the pre-test he had difficulties with solving the following problems: 3. thinking backwards; 4. and 5. permutations – systematic thinking; 6. word problem that can be solved with simple equation; 7. calculating area; 9. pattern recognition. On the post-test the following types were problematic for Alex: 2. the justification of a divisibility problem; 4. and 5. permutations; 6. justification of a problem including number theory; 7. calculating area; 8. geometric calculation; 9. pattern recognition. Finally, on the delayed test the solution of the following problems was either missing or incomplete: 1. greatest common divisor (gcd) – systematic thinking; 2. justifying a divisibility problem; 4. permutations; 6. justification of a problem including number theory; 7. area calculation; 8. geometry.

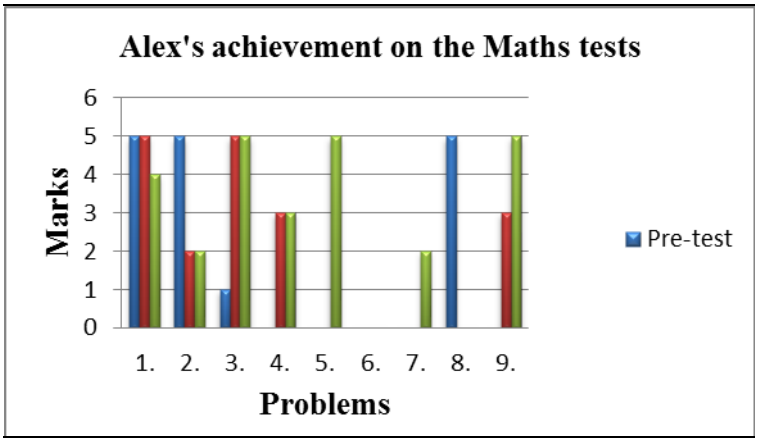


Figure 2

Figure 3 summarizes the marks the students achieved on all three tests. Richard was already at 34 points out of the total 45 and in the post-test he achieved 36 points. There was a slight decrease in the delayed test but his achievement was still satisfactory.

On the other hand Alex's the difference between pre- and delayed test results show a considerable increase from 16 points to the total of 26 points.

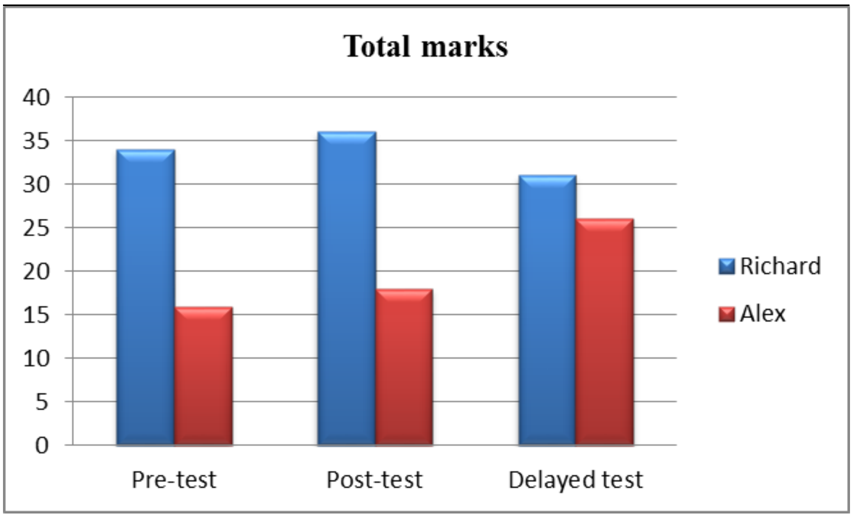


Figure 3

11. Discussion

Taking both the particular lessons and the whole school year into consideration the following observations and thoughts can be phrased.

For Richard, the talented student there was no significant difference between his results in the pre-, post- or delayed Mathematics test. Of course, since his achievement was quite good on the first test already there was not too much area for improvement. However, there were tasks where he showed some improvement, for example in problems that involved permutations his marks were higher in the post- and delayed - tests.

On the other hand there was definitely a positive change in his attitude towards the others. He became more patient with the other students when he had to explain something or when he presented his ideas. From the teacher's observations it is clear that Richard was more and more willing to be a "team player". Moreover, there was a positive change in terms of his classroom activity, too.

Alex's mathematical achievement had never been steady which can be clearly seen from his test result throughout the three years when we worked together. During this experiment he achieved better result in some post-test tasks and his delayed test marks were much better. Comparing the post- and the delayed tests Alex definitely showed an improvement in problems that required pattern recognition (9.), moreover he received more marks for problems where permutations had to be applied (4. and 5.). Besides these tests there was an improvement in his school grades as well which might be a result of his personality becoming more mature, too. His half year mark was 3, while by the end of the school year he managed to achieve 5.

As for Alex's attitude to cooperative work from his comments it is not obvious whether he liked them or not as his opinion is rather inconsistent[7]. However, from the observations it is clear that his self-confidence changed in the positive direction. He became a more confident presenter and he needed less reassurance by the end of the school year.

Since using cooperative techniques definitely had a positive impact on both students, in my future classes I will use this method to encourage the average ability students to share their mathematical ideas and to help talented students to be more willing to share their knowledge with their classmates.

[7] See his comment

12. Future work

The broader action research resulted in a massive amount of data that needs to be organized and analysed. The psychological questionnaires of the other participating students need to be analysed and the results compared to that of the above discussed students. The comments and solutions in the "*reflection booklets*" should be interpreted as well, and the video and voice recordings contain valuable pieces of information, too. After analysing all the tests and questionnaires a new experiment could be designed with another group of students from a different age group with using different problems.

Acknowledgement

This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 'National Excellence Program'.

References

1. Ambrus, A. (2004). *Bevezetés a matematika didaktikába. (Introduction to Mathematical Didactics)* Budapest: ELTE Eötvös kiadó.
2. Burns, M. (1990). Using Groups of Four. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 21 - 46). Boston: Addison-Wesley Publishing Company.
3. Caine, R. N., Caine, G., McClintic, C. L., & Klimek, K. J. (2004). *12 Brain/Mind Learning Principles in Action: Fieldbook for Making Connections, Teaching, and the Human Brain*. Corwin Press.
4. Crabill, C. D. (1990). Small-group Learning in the Secondary Mathematics Classroom. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 201 - 227). Boston: Addison-Wesley Publishing Company.
5. Dees, R. L. (1990) Cooperation in the Mathematics Classroom: A User's Manual. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 160 - 200). Boston: Addison-Wesley Publishing Company.
6. Dobi, J. (2002). Megtanult és megértett matematikatudás (Learnt and understood mathematical knowledge). In Csapó B. (Ed), *Az iskolai tudás (Knowledge acquired in school).* (pp. 177 - 200). Budapest: Osiris Kiadó.
7. Gyarmathy, É. A. (2006). *A tehetség (Talent)*. Budapest: ELTE Eötvös kiasó.
8. Johnson, R. T. and Johnson, D. W. (1994). An Overview of Cooperative Learning. In J. Thousand., A. Villa and A. Nevin (Eds.), *Creativity and Collaborative Learning*. Baltimore: Brookes Press.
9. Johnson, D. W and Johnson, R. T. (2009). An Educational Psychology Success Story: Social Interdependence Theory and Cooperative Learning. *Educational Research, Vol. 38, No. 5*, 365 – 379.

10. Józsa, K., Székely, Gy. (2004). Kísérlet a kooperatív tanulás alkalmazására a matematika tanítása során. (Experiment for Using Cooperative Learning in Teaching Mathematics) *Magyar pedagógia, 104. (3)*, 339 - 362.
11. Kagan, S. (2003). A Brief History of Kagan Structures. *Kagan Online Magazine.* San Clemente, CA: Kagan Publishing.
12. Kagan, S. (2004). *Kooperatív tanulás. (Coopartive Learning)* Budapest: Önkonet.
13. Koshy, V. (2005). *Action Research for Improving Practice*. London: Paul Chapman Publishing.
14. Mécs, A. (2009). *Miben segíti a kooperatív módszer a matematika tananyag megértését? (How do cooperative teachniques help understand Mathematics?)*. (MSc dissertation). ELTE, Budapest.
15. Miller, R. C. (1990). Discovering Mathematical Talent. Retrieved November, 02, 2013, from (http://files.eric.ed.gov/fulltext/ED321487.pdf).
16. Pehkonen, E. (1999). Open-ended Problems: A Method for an Educational Change. In *International Symposium on Elementary Maths Teaching (SEMT 99).* Prague: Charles University
17. Pólya, Gy. (1973). *How to Solve it.* New Jersey: Princeton University Press.
18. Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense-making in Mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.
19. Skemp, R. R. (2005). *A matematikatanulás pszichológiája. (The Psychology of Learning Mathematics)*. Budapest: Edge 2000.
20. Slavin, R. E. (1995). *Cooperative Learning.* Allyn & Bacon.
21. Szendrei, J. (2005). *Gondolod, hogy egyre megy? (Dialogues about teaching Mathematics)*. Budapest: Typotex.
22. Zimmermann, B. (2009). *"Open ended Problem Solving in Mathematics Instruction and some Perspectives on Research Questions" revisited – New Bricks from the Wall?.* In A. Ambrus, & É. Vásárhelyi. Problem Solving in Mathematics Education, proceedings from the 11th ProMath conference, September 2009 (pp. 143-157). Budapest, ELTE.

Appendix

Questionnaire on cooperative learning (Mécs, 2009)

1 - not true for me 6 - totally true for me

Statements

	1	**2**	**3**	**4**	**5**	**6**
1. The other students' explanations helped.						
2. I liked working together.						
3. I enjoy lessons with group work more.						
4. I understood maths better than in the previous year.						
5. I liked talking to people to whom I have never spoken before.						
6. I'm not afraid of maths.						
7. I could explain the solutions to others.						
8. I dare to ask questions from my teacher.						
9. I paid more attention and solved more tasks in maths.						
10. Doing homework is easier now.						
11. I understand my classmates' explanations better.						
12. I prefer sharing my ideas in small groups.						
13. I would like group work in Maths next year.						
14. I am more active when working in groups.						
15. I became more patient with others.						
16. The noise was disruptive.						
17. I would have been faster alone.						
18. I prefer doing maths tasks alone.						
19. I prefer my teacher's explanations.						
20. I don't mind sharing my ideas with the whole class.						

About problem sequences designed for heterogeneous classes in grade seven

Lars Burman

Abo Akademi University Vaasa, Finland

lburman@abo.fi

Abstract

In this article teaching of problem solving in grade seven is addressed. Based on a proposal of teacher-guided practice with problem sequences by Burman and Wallin (2014), the author deals with the challenge that problem solving in mathematics should be designed for all pupils and not only for the talented. Problem sequences may be seen as a supplement to ordinary teaching in mathematics and the article describes pilot-tests in a lower-secondary school in Finland. The tests are situated in a Finnish reality but the design of the teaching process is based on earlier research with a connection to ProMath-conferences and combined with Nordic research. In a problem sequence the pupils work partly in groups and partly individually, they discuss the results with the teacher, they receive new information and guide-lines and then they proceed. The pilot-tests in a heterogeneous class seem promising as two main bases to build a development on have been found.

Key words: Grade seven, heterogeneous classes, problem sequences, problem solving, teacher-guided practice.

ZDM classification: D50, D73

Introduction

In the Finnish Matriculation Examination there is an extremely long tradition to test the students' competencies in mathematics by giving them ten tasks to solve in six hours.[1] Consequently, there is a very strong focus on tasks which can be solved within half an hour whereas tasks of other types, e.g. problems and modeling projects, are not given so much focus. Furthermore, the Matriculation Examination has a great influence, not only on the instruction in upper-secondary school, but also on instruction in grades seven to nine. Accordingly, Pehkonen and Rossi (2007) conclude that the conventional teaching method is still the dominant one although alternative teaching methods (including problem solving and project work, the author's remark) have been delivered to Finnish teachers since more than twenty years.

[1] Nowadays the students have to choose their 10 tasks among 15 tasks.

In comparison to other Nordic countries, Denmark is well-known for the competence perspective on mathematics education, as presented in Danish research by Niss and Jensen (2002). For instance, Blomhøj and Jensen (2007) define the concept competence as someone's insightful readiness to act in response to the challenge of a given situation. In their visual representation of the eight competences in mathematics, four are particularly interesting. These competencies are mathematical thinking competence, reasoning competence, problem tackling competence and modeling competence.

The aim of the research and the aim of this article

The aim of the research behind this article is to design tasks larger than short problem-solving tasks (and the tasks in the Finnish Matriculation Examination) but not as extended as projects. The purpose of the tasks is to develop the instruction in grade seven (and furthermore, in grades eight and nine) and more precisely to strengthen the pupils' skills in mathematical thinking and reasoning, problem solving and readiness to work with applications and modeling. It should be stressed that my purpose is to design some kind of supplement to the courses in mathematics, a supplement that teachers would consider an opportunity and not an obstacle to cover the normal content in the courses of the curriculum. The special focus of this article is to address that fact that problem solving in mathematics should be designed for all pupils and not only for the talented.

Starting point and framework

Almost thirty years ago, Mason, Burton and Stacey (1985) argued that the rapid question/answer format of many mathematics classrooms is the antithesis of the time and space upon which developing mathematical thinking depends. Instead they stated that practice demands ample time for tackling each question independently and the quality of the reflection depends upon the time to review thoughtfully, to consider alternatives and to follow extensions. Furthermore, they urged teachers to choose questions which can provoke thinking and to recognize how essential confidence is and to create a supportive environment where some success comes to each pupil. According to them, working in groups was helpful, choosing of suitable questions was essential and mathematical thinking could be improved by practice with reflection.

In the introduction to 14th ICMI Study Volume, Niss, Blum and Galbraith (2007) outlined a framework that is very useful when the aim is to develop tasks for problem solving. They described the starting point as a space where problem solving meets application and modeling. Accordingly, a *problem* is a task that cannot be solved using only previously known standard methods. Moreover, a method may be standard for one individual at the same time that it is not for another individual. The concept problem is used in a broad sense, including not only practical problems, but also problems of a more intellectual nature that aims at describing, explaining, understanding or even designing parts of the world. Problems taken from the *real world* have to do with nature, society or culture, including everyday life. In the term *modeling* they included the entire process of structuring, generating real world facts and data, mathematizing, working mathematically and interpreting and validating, perhaps several times round (parts of) the loop. Then, an *application* (of mathematics) is a real world problem that has been addressed by means of mathematics. A *competency* is the ability of an individual to perform certain appropriate actions in problem situations where these actions are required and desirable. Consequently, we can speak about *problem-solving competency* as the ability to find the solution to a task not corresponding to the previously known standard methods, and (*mathematical*) *modeling competency* as the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation. As the problem-solving competency and the modeling competency are not sufficient to solve real-world tasks, we also need *other competencies* such as representing mathematical objects involved in an appropriate way and arguing and justifying what is being done when applying mathematical algorithms and procedures. As problem solving and modeling often are used as a group activity, *social competency*, more or less specific for mathematics, is needed for an effective cooperative teamwork and must not be forgotten. Finally, dealing with applications and modeling in mathematics, there are a *challenge*, a dilemma or a problem, and *questions*, which together form an *issue*.

I have found it appropriate to use a framework, very similar to that outlined by Niss, Blum and Galbraith in my research and also to use it in order to reach the aim of this article. But in addition to the definition of problem above, Pehkonen (1997) provides two definitions for my work. He deals with methods for educational change and suggests the use of problem fields as such a method. In this context, he defines a *problem field* as a set of connected problems and the connected problems form a sequence of problems. He also notes that in a problem field, the difficulty of the problems may range from very simple ones that can be solved by the whole class, to more difficult problems which only the more advanced students might be able to solve. I agree with Pehkonen and accept his definition of problem fields but prefer to use the concept *problem sequence* from now on, when I focus on problems formed by the sequence of subproblems more than on problems as separate problems linked to each other or problems as more or less separate extensions from a certain problem.

In an article in the proceedings of the 11^{th} ProMath conference 2009, in Budapest, Henze and Fritzlar (2010) examine model-building processes by the example of Fermi questions and show in a figure that there are several similarities between problem solving and modeling. They also conclude that modeling processes can be understood as one specific type of problem solving. Furthermore, in his doctoral thesis, Ärlebäck (2009) also refers to Fermi problems, as he addresses the issue of how to introduce mathematical modeling to upper-secondary students. He has found that groups of students engaged in solving realistic Fermi problems display problem-solving behavior resembling sub-activities of the modelling process. Consequently, I find that if there is a problem-solving behavior resembling sub-activities of the modeling process, it might be possible to find certain problems or tasks which could be used as good problem-solving tasks but also as an introduction to and exercise for future mathematical modeling tasks. On the other hand, good exercise in problem solving may also include elements resembling modelling processes.

Theoretical conclusions and the design product

Based on the theoretical background above, I conclude that there is a need to improve mathematical thinking and the quality of reflection and thought. This can be done by offering the pupils good questions and challenges, as well as possibilities to work with a somewhat more extended problem. I find an advantage in dividing the work with problems into steps and giving the pupils possibilities to discuss within the whole class and with the teacher between these steps. Apparently, in this case the pupils receive exercise in handling with different kinds of problems and they are not only tested if they are able to solve a certain problem by themselves. If possible, it is preferable that the problems should have a real world connection, but equally important is the need to create a supportive environment, where working in groups and good teamwork are successful.

Consequently, the way of working in a math class should combine the pupils' own problem solving in groups with discussions about the results so far and new information and new directions for the on-going work given by the teacher. I prefer to divide the process in steps, as the teacher then can give new information between the steps and sometimes give the work a more or less new orientation. Depending on the time available, the teacher could also give the pupils one of the steps as an individual homework. As a whole, the way of working, including the kind of problems used, can be described as using

teacher-guided problem sequences.

In this article I have a special focus on obstacles occurring as the instruction is designed to include all pupils and not only talented pupils. When considering this focus, two aspects seem to be very important:

* Doing some of the steps in groups may inspire all pupils to do their very best in order to give their contribution to the group. Furthermore, the positive feeling of having been part of a successful problem solving is more likely to emerge after working in a group than after an individual work.

* The problem sequences or at least some steps in a sequence can be constructed to include problems at different levels of demands. In addition, connected problems may have an increased level of difficulty from (very) easy to (much) more demanding.

Two examples of problem sequences

In the following, I present two examples of problem sequences in order to highlight some of the benefits of the method used.

Example 1 Find the prime factors of the numbers 1 - 50.

Step 1 The teacher gives an introduction to the problem and demonstrates the solution for the numbers 1 - 16. The teacher shows for instance $11 = 11$, $12 = 3 \cdot 2 \cdot 2$, $13 = 13$ and $14 = 7 \cdot 2$ and the pupils make a list. The teacher has told them to reserve space for all the whole numbers up to 50.

Step 2 Pupils are divided into groups and every group receives one paper where the task is to find and write down the prime factors of the numbers 17 - 32. The pupils are also asked to try to find and write down some smart ways of reasoning as they are solving the task.

After step 2 the groups hand in their papers. Then, the teacher notes the prime factors of the numbers 17 - 32 to ensure that every pupil has got a correct list up to the number 32.

Step 3 Pupils are asked to follow the previous examples and the smart ideas from the discussion and are given the numbers 33 - 50 as homework. As a special challenge they receive an optional task to answer the question how it is possible to decide that a number is a prime number or more precisely to know when it is no longer necessary to check any more possibilities to factorize the actual number.

Next lesson, there is a kind of follow-up where the main concern is threefold: to ensure that everybody has got the right prime factors for the numbers up to 50, to repeat the “smart ideas” and to give an answer to the special challenge.

Step 4 For this step, which is the last one in the sequence, there are two options, of which also only one can be used. If there is time, at least for some pupils, it is possible to enlarge the list of numbers with prime factors. Another possibility is to check the pupils’ ability to find prime factors to some number both below and above 50 in some kind of summative assessment.

The task is not new but I have chosen to implement it in steps as a teacher-guided problem sequence. The task can be considered central and important as both the way of working with it and the result of it are important in tasks to come in the future. In the pilot-test, step 1 was done as a teacher-led information and demonstration, but of course, the pupils were constantly asked to participate in the work. Step 2 was carried out as a group work. In the discussion in the pilot group after step 2, most of the pupils found multiplication tables very useful. Some other suggestions were also made, for instance the idea to use the prime factors of 15 when the actual number was 30 and it was possible to write $30 = 15 \cdot 2$. Step 3 was an individual work, but any kind of help and co-work at home was not excluded in step 3. Finally, in step 4, the abilities of the pupils were checked individually.
Concerning the focus on teaching in a heterogeneous group, the step 2 was a work made in groups, and in all the steps 1 - 3, there were numbers which were easy to factorize. In some cases there were also very similar factorizations to imitate. A special notation must be given to the fact that after this problem sequence, all pupils have got a list of their own, where they can find the prime factors of all the numbers from 1 to 50.

Example 2 Find the angle between minute and hour hands of the clock (at different times).

Step 1 The teacher gives an introduction to the problem sequence. After a short discussion with questions and answers, the main information should be clear to everybody: at the same time when the minute hand covers a full angle, 360°, the hour hand covers an angle of 30°.

Step 2 The pupils are divided into groups with approximately three pupils in each group. They are asked to find the angle between minute and hour hands of the clock at the following times: 13.00, 13.30, 13.15 and 13.45. Every group receives a paper on which the pupils are supposed to draw figures and write down their answers.

The groups hand in their papers and the results are discussed with the teacher, who may give explanations to some of the answers.

Step 3 The pupils are asked to solve similar problems individually. Every pupil receives a paper and the given times are: 11.00 , 17.00 , 14.30 , 15.15 , 12.45 , 14.25 and 17.48.

When a certain time has passed (or when everybody is ready), the papers are handed in and the results are discussed. If necessary, the teacher explains some of the results.

In the third step the pupils' papers can be used for a summative assessment. It is also possible to use four steps. In that case the third step may be done as a homework and the fourth step as a summative assessment in class. If three steps are used, it may be possible to do the whole sequence within a lesson. If four steps are used, including a homework-step, it means that the sequence might require more time but there is also the advantage that the problems have been present at three different occasions. As in the example 1, another possibility is to see what the pupils can achieve in some kind of summative assessment later on.

The second problem is also a classic one, but implemented in steps as a teacher-guided problem sequence. The task can be used to train logical work with proportionalities. In this sequence there were steps of teacher-led information and demonstration, group work and individual work. In a heterogeneous group it was important that the work in step 2 was made in groups, and that there was a gradually increasing difficulty level in the steps 2 - 3. Of course, the last time was supposed to be a challenge to most of the pupils in the seventh grade.

Concluding remarks

I have found the use of problem sequences in a heterogeneous class in the seventh grade very useful. It is possible and also desirable to construct and use problem sequences which include group work and problems with an increasing level of difficulty. Then, every pupil is invited to contribute to the solution of the problems in the group and it is more likely that everybody gets to experience the feeling of success in at least some sense. Group activities possess the potential to provide everybody with the feeling of having solved the problems together. Last but not least, with using the problem sequences it was possible to reach a development in the direction towards higher-order thinking, when the pupils had no previously solved tasks to imitate. The results strongly suggest research and development of teacher-guided problem sequences in a larger scale in Finland, as well as in other countries.

References

Blomhøj, M., & Jensen, T. H., (2007). What's all the fuss about competencies? In Blum, W., Galbraith, P. L., Henn, H.-W., & Niss, M. (Eds.). *Modelling and Applications in Mathematics Education* (pp. 45-56). New York. Springer Science Business Media.

Burman, L., & Wallin, S., (2014). Teacher guided practice with problem sequences. In Proceedings of the 30th symposium of the Finnish Mathematics and Science Education Research Association in Vaasa, November 6-8, 2013. (in press)

Henze, J., & Fritzlar, T. (2010). Primary school children's model building processes by the example of Fermi questions. In A. Ambrus, & È. Vásárhelyi (Eds.). *Problem Solving in Mathematics Education* (pp. 60-75). Budapest. Eötvös Loránd University.

Mason, J. with Burton, L., & Stacey, K. (1985). *Thinking mathematically*. Wokingham. Addison-Wesley Publishing Company.

Niss, M., Blum, W., & Galbraith. P., (2007). Introduction. In Blum, W., Galbraith, P. L., Henn, H.-W., & Niss, M. (Eds.). *Modelling and Applications in Mathematics Education* (pp. 3-32). New York. Springer Science Business Media.

Niss, M., Jensen, T. H., (2002). *Kompetencer og matematiklæring* . Utdannelsesstyrelsens temahæfteserie nr 18. København. Undervisningsministeriet.

Pehkonen, E., (1997). Use of problem fields as a method for educational change. In Pehkonen, E. (Ed.). *Use of open-ended problems in mathematics classroom*, (pp. 73-84). Helsinki. University of Helsinki.

Pehkonen, E., & Rossi, M, (2007). Some alternative teaching methods in mathematics. In Pehkonen, E., Ahtee, M., & Lavonen, J. (Eds.). *How Finns Learn Mathematics and Science*, (pp. 143-154). Rotterdam. SensePublishers.

Ärlebäck, J. B., (2009). *Mathematical modelling in upper secondary mathematics education in Sweden.* Linköping. Linköping University.

Kangaroo problems for student teachers in Sweden

Katalin Földesi
Mälardalen University, Eskilstuna-Västerås, Sweden
katalin.foldesi@mdh.se

Abstract

One of the important goals of student teacher education is for the students to develop their problem solving skills.This is the basis of the student learning basic didactic skills within this area. We know from many years of teaching experience that at our university, student teachers have a troubled relationship with mathematics.Thus, we are looking for teaching material that does three things simultaneously: develops the students' problem solving abilities, helps the students improve their knowledge of didactics within the field of problem solving by working through problems with didactics specifically in mind, and helps the students achieve a positive relationship with mathematics. Since the main goal of the international Kangaroo competition is the popularization of mathematics for students, and it allows students of ages 6 to 19 to participate, we chose its problem collection to conduct our research with. The experiences of the last few years have shown us that carefully selected competition problems, and their conscious application helps the achievement of all three goals. Our experiences also show that the problem solving skills of our university's student teachers are weaker than the problem solving skills of the best of the students they, in turn, teach.

Key words: Mathematics education, Kangoroo competition, problemsolving, student teacher education

ZDM Classification: D53, D59, D63, D69

Introduction

The importance of mathematical problem solving is evident from the fact that the concept appears predominantly in the Swedish math syllabus even prior to 2011. In 2011, however, a new syllabus and a new system of evaluation came into effect across the entire Swedish general education, and at the same time, the renewal of Swedish student teaching was kicked off as well.

The prominence of problem solving in the 2011 Swedish math syllabus is emphasized by it being an element of content, an element on the list of skills that needs to be developed, and an element of the evaluation process as well.

Between 2013 and 2016, all Swedish math teachers are required to take a compulsory course called Matematiklyft. Part of this course deals with mathematical problem solving.
The Kangaroo competition (http://www.math-ksf.org) presents an opportunity to grapple with mathematical problems for a wide range of students: from the first year of secondary school all the way to fourth year of high school.
With regard to students teachers, two observations beckon to be made here: the problems require a way of mathematical reasoning that is very unusual for the students, and especially in the beginning feel very difficult. Solving the problems is especially difficult for students who, up to this point, had practiced math in a way that emphasized procedural calculations and the solution of a narrow class of problems that can be easily expressed using algorithms, usually presented by their teacher in the first few minutes of class, after which they practiced solving similar problems using the same methodology.
First I will write about the situation with mathematical problem solving in Sweden, and then about a few of the general and Sweden-specific characteristics of the Kangaroo competition.

The situation of problem solving in Sweden

György Pólya's book "How to solve it?", first published in 1943, is of particular significance when it comes to the didactics of mathematical problem solving. A few years ago it was still mentioned in a couple of high school course books, along with the photo and brief biography of Pólya, as well as five point problem solving method that he had developed - however, none of it got used in Swedish math teaching.
For the last couple of years, the theoretical foundation of mathematical problem solving in Sweden has been a single concept, and later, an American study handling the concept in a special way. This concept is "rich mathematical problem". It appeared in mathematical didactic literature a few years ago. Below is an example of such a problem, from a Swedish mathematical didactics book:
“Linda bought books at the book store. She bought three books for a total of 450 SEK, Kierkegaard, Magorian and Ende. Kierkegaard cost 100 SEK more than Ende. Kierkegaard and Ende together cost 190 SEK more than Magorian.
A) How much did each book cost?
B) Formulate a similar problem and solve it.” (Hagland-Taflin-Hedrén, 2005)
The problem contains the following mathematical concepts: natural numbers, sum, difference, table, equation, matrix and logic. One of the purposes of question B) is to allow the teacher to

evaluate the extent to which the student understands the math behind the problem. This particular problem was aimed at secondary school students, years 7-9.
The rich mathematical problem's strong dependence on the math taught in the secondary schools of a country is immediately obvious, since the same problem is used as a simple but very important practice question in several other countries.
Today in Sweden an American study is used as a model for treating rich mathematical problems in the classroom. (Stein at al., 2008)
The study presents in detail a method for treating problem solving. It comes with the warmest of recommendations for any student teacher. Reading the study, and then trying out the method is warmly recommended to all Swedish math teachers that enroll in the continuation course.
The main points of the method are as follows:

- The teacher presents the problem and makes sure the students understand it linguistically and mathematically
- Each student spends a few minutes trying to solve it individually or in pair
- The students continue to work on the problem in small groups, agree to a solution, the teacher walks between the groups
- Some of the solutions are presented at the blackboard and are discussed by the group or the entire class

According to the method it is extremely important that the teacher selects which solutions to be presented in front of the class based on a mathematically and didactically well thought-through structure and during the discussion labors on the deepening of the mathematical content of the solutions presented by the students.

About the Kangaroo Competition in Sweden

To start with, I would like to summarize some general characteristics of the competition then, I will mention some things that are specific to Sweden.
The competition is individual. It is arranged to take place annually in March. Its goal is the popularization of mathematics, which also means that it does not require extensive technical preparation, making it readily available to a huge number of students. It consists of 12-24 problems, mostly multiple choice questions with five possible choices. Each question has only one correct answer. The time allotted for the competition is 60 minutes. After the competition, the teacher can quickly correct the answers using the provided solution key, and can then send

in the results. A few weeks after the competition, the problems, the correct answers including brief solutions as well as materials that will help the mathematical and didactical write-up of the solutions are presented on the competition web page.

The top results, along with the students' names and schools, are presented on the competition's web page. However, only the top 15 placements are shown on these lists. Thus, students with lower scores lose not only the little bit of publicity in connection with this, but also the opportunity to show development over the past year. Both of these can be incredibly important motivational factors. This way of presenting the top placements obviously reflects some aspects of Swedish culture.

Since 2000, competition materials can be freely downloaded, copied and edited as well, which for practicing student teachers provides an easy way to construct interesting series of problems.

For the practicing student teacher, enrolling in the competition, running the actual competition, marking the answers and sending in the results can be done with a minimum of extra work.

I would like to mention a few characteristics of the problems. I feel that these characteristics make them especially suitable for use in the training of teachers.

Many problems contain visual information, pictures, illustrations, geometrical figures, notations, graphs et cetera. Since most Swedish primary, secondary and high school math is about practical usability and preparation for further studies, when it comes to the selection and discussion of the materials used, analytical students that are good at numbers have an immeasurable advantage, while students who think visually, and mainly think in pictures and have a rich mathematical intuition are disadvantaged.

The problems contain modern mathematical subjects in much greater measure than the normally used course books: graphs, topology, combinatorics, number theory, logic, set theory and so on. This is important, should either the teacher or the student decide to complement the occurring mathematics from other sources.

The problem texts are short, barely a few lines, and are formulated simply, often containing explanatory elements.

Routine problems do not occur at all, which can be somewhat refreshing for students weary of them.

For several years now there have been signs that some students, who traditionally perform poorly at school mathematics, achieve excellent results at the Kangaroo competition.

My own anecdote adding to this is that I have had a student who answered 23 out of 24 questions correctly during the competition, but a week later was able to provide complete, mathematically correct solutions to just a third of the problems.

Currently there is research underway to explore the correlation between students who achieve good results at the Kangaroo competition, and their performance at school mathematics. Could it be that the Kangaroo competition can be used to discover students whose performance is weak, but who have a talent for mathematics?

Now I would like to present the names of the levels of the competition, as well as the corresponding school classes, and high school course names, in Sweden:

- Milou – primary school and secondary school years 1-2
- Ecolier – secondary school years 3-4
- Benjamin – secondary school years 5-7
- Cadet – secondary school years 8-9 as well as Math A (first math course) in high school
- Junior - Math B (second math course and so on) and C in high school
- Student- Math D and E in high school

It can be seen almost immediately that there are two disadvantageous categories for many of the participants: Benjamin and Cadet, which both spam three years and that will not help their mathematical development. Splitting those two categories into several would, from a mathematical didactical point of view, be much more advantageous, even though it would no doubt lead to a considerably increased work load.

Research questions

1. How can we use the Kangaroo competition and its problems in the education of math student teachers?
2. What experience can we gain?

Methodology

I carried out four way of applying Kangaroo competition in the education of students.

Method 1: Student teacher participation in the Kangaroo competition

Most of our student teachers have no experience of mathematical competitions. This is one of the reasons I began to encourage our students to participate in the Kangaroo Competition on the same terms as the other (non-teacher) students. The competition committee found this idea to be good and accepted it. For several of the student teachers, even just one such experience was enough to make them organize competition participation in their own classes in turn.

Method 2: Competition participation, and analyzing the problems

Within my own mathematical and didactical courses, I tried a second method. First, the students participated in the live competition. Immediately afterwards, the complete solution of the competition problems was given to them as homework. Next time the class met, the students worked in threes on the problems. The solution of each problem was then presented at the blackboard by one or more students. Afterwards, both the problems and the solutions were analyzed mathematically and didactically.

Method 3: Problem analysis without competition participation, with group discussion

This method was tried several times, too: the competition problems were given to the students as homework, with a full week to complete it, without knowing the correct answers. After the week had passed, one or more students presented the solutions in front of the class. Naturally, the presented solutions were discussed and analyzed from both a mathematical and a didactical point of view.

Method 4: Competition problem analysis as homework, with complete, written solutions required

I would like to present a fourth method as well, which I tried with several students who were specializing in mathematics. In this case, the competition problems were given to the students without the solutions. Complete, written solutions were requested.

Results and discussion

I had to work hard at convincing the students to participate in the competition. For many students, the experience of the competition was just one more in a row of negative memories. Due to organizational reasons, I did not always have an opportunity to discuss of the competition problems within a sufficiently short time frame after the competition. To me it is obvious that continuous practice of mathematical problem solving using with help from the teacher would be the optimal solution, but making this happen may not be possible in the current organizational frameworks, and the country's cultural traditions present an additional difficulty.

Although we regularly have about a hundred new students on our mathematics and mathematical didactics courses, each year only about 15 participate in the Kangaroo competition.

My experience with the third method has been the following: the students I tried this method on were not specializing in math, were extremely unsure of math in general, and it was for them an added difficulty to stand in front of the class and present their solution. Many of the students heavily opposed this method, and I was not always good at handling the opposition pedagogically. On the other hand, several students distinguished themselves by giving very clear presentations, showing a high degree of mathematical clarity. While this ability should be a very important characteristic of a teacher, it was only present in a small subset of the students.

Students, solved Kangaroo problems by Method 4, ended up having a much more difficult time, than the students using the 2nd method. Often they could not make use of help, or the knowledge of math was lacking, or their knowledge of mathematical problem solving was lacking. My conclusion regarding this method is that oral discussion is indispensable when it comes to the development of the students. One reason for this may be the students' insufficient skill at communicating math in writing.

Next I have collected some important experiences.

- *On the importance of knowledge of basic set theory and mathematical logic*

I would like to mention some particular difficulties I ran into. Knowledge of the basic elements of mathematical logic and set theory are indispensable for our students. In accordance with the previous (pre-2011) math syllabus, these became present only at the end of the students' high school years, and even then only in the classes that studied most maths. Mathematical didactical research supports that these basic elements of mathematics must be present during the whole stretch of secondary and high school studies. I would like to give an example of such a problem.

“The two cats Bill and Bull sometimes run into the two dogs Karo and Lufs. Bill is afraid of both dogs, but Bull is only afraid of Karo and is a good friend with Lufs. Which statement is incorrect?

A: Each cat is afraid of some dog.

B: Some cat is friend with one dog.

C: There is a dog that scares both cats.

D: Each dog scares some cat.

E: There is a dog which is good friend with both cats.” (Problem 18, Ecolier level, 2005.)

The most difficult thing for the students was to understand that of the possible choices, each choice is either true or false, with nothing in between. "All", "each" and "there exist" were all constructs that had to be explained. Because of these mathematical difficulties, many students did not notice the subtle humor of the problem, even though cats and dogs are both favorite pets in Sweden, to such an exists that they are usually included in the answer to the question "Who are in your family?".

- *A piece of good advice: "Do just as the problem text says!"*

Swedish math teaching is still all too focused on numbers, calculations and algorithms, without much formal explanation of the algorithm itself. As a result, most problems that deviate even just a little bit from the usual end up being incredibly difficult for the students.

Even just understanding the problem text often turned out to be very difficult for many students. In these cases, the following simple instruction turned out to be of great help: "Do just as the problem text says, follow it exactly!" The following problem was of this kind.

"There is room for four people around a square table, with one person on each side. For the school prom, the students line up seven such tables next to each other, forming a single long table. How many people is there room for around with long table?" (Problem 6, Ecolier level, 2006)

In this case, following the text of the problem on paper, e.g. using a simple drawing, may not only give the student a key to the solution, but in the process of drawing also show how the problem could be generalized.

- *The mathematical and didactical analysis of the Kangaroo-problems*

It was often very difficult to work through the problem mathematically: How did you solve the problem? What mathematical tools did you use? When? How? Can you expand the problem? Can you rewrite it and make it easier or more difficult?

Then came the didactical analysis of the problem: What mathematical concepts are present in the problem? In what class would you present this problem to your students? Why?

I would like to give an example of such a problem.

"Four house sparrows sit on a fence. They're Leo, Olle, Moa and Ida. Leo is sitting between Olle and Moa. The distance between Olle and Leo is the same as the distance between Moa and Ida. Leo is sitting 4 meters from Ida. What is the distance between Ida and Olle?" (Problem 15, Ecolier level, 2006)

It required lots of assistance from the teacher to make the students realize that the four sparrows represent four points on a line segment. This may be a result of the minimal content of Swedish geometry teaching.

- *Fruitful visual problems*

Some problems have almost no mathematical prerequisites, but the ability to be able to interpret an image, a graph or other visual information and work with it. Here is an example:

“A toy store sells a construction set consisting of black and while blocks arranged in four layers. Each layer consists of blocks of the same color. The image on the right shows the set from above. How many white blocks are there in the construction set? (Figure 1)

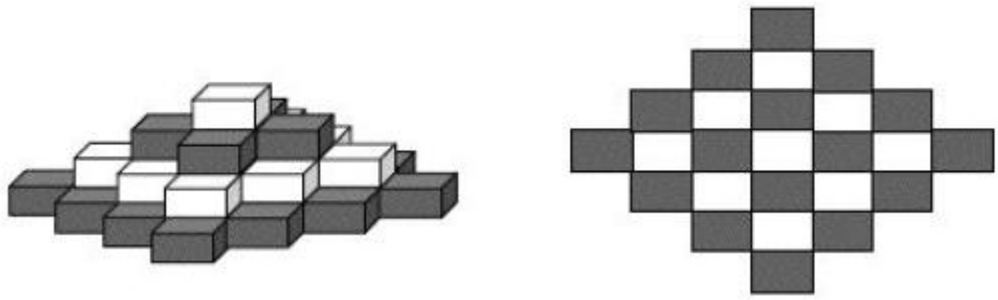


Figure 1

(Problem 4, Benjamin level, 2008)

- *Problems with possible solutions at different levels*

“A class of 28 children stands in line. Ali has twice as many children behind him as in front. Where in the line is Ali?” (Problem 17, Ecolier level, 2002)

It is possible to solve this problem immediately using a drawing, but you can also arrive at an answer by introducing an unknown and setting up a simple equation.

Finally, I would like to compare the different categories of students with regard to how they handled the increasing difficulty levels of the Kangaroo Competition (*Table 1*).

competition categories	**Student teachers**	
	Primary school to secondary school, Grade 5-6, no math	**Secondary school, Grade 7-9 and high school, math**
Milou	yes, but needs to think more than usual	yes
Ecolier	yes, but the difficulties begin here	yes
Benjamin	yes, but even more difficulties	yes
Cadet	no, definitely	yes, but the difficulties begin
Junior	no, some exceptions	yes, but more difficulties
Student	no, some exceptions	yes, but even more difficulties

Table 1

The table clearly shows and reflects my several years of experiences, that in both categories, the problem solving levels of future math teachers is below the problem solving levels of their brightest students.

Summary

Experiences so far show that at our university, the Kangaroo competition and its problems work very well at developing and deepening the problem solving skills of future mathematics teachers. Additional efforts need to be made to increase the number of participations in the competition.

References

Polya, G. (1957). How to solve it? – A new aspect of mathematical method. Princeton, NJ: Princeton University Press.

Kangurun. http://ncm.gu.se/kangaru, (2014-02-26)

“Kangourou sans frontieres” Association. http://www.math-ksf.org, (2014-02-26)

Hagland, K. Taflin, E. Hedrén, R. (2005). Rika matematiska problem (Rich mathematical problems), Liber.

Stein, M. K. Engle, R. A. Smith M. S. Hughes E. K. (2008). Orchestrating Productive Mathematical Discussions: Five Practices forHelping Teachers Move Beyond Show and Tell, Mathematical Thinking and Learning, 10:4, 313-340.

Prospective Teachers' Conceptions of problem oriented Mathematics teaching – explored and challenged

Torsten Fritzlar
University of Halle-Wittenberg, Germany
torsten.fritzlar@paedagogik.uni-halle.de

Abstract

Supporting prospective teachers in developing appropriate conceptions of mathematics, and of teaching and learning mathematics, is an important task of teacher education. In this article, I report on a university seminar in which I tried to explore and challenge the conceptions of (the complexity of) problem solving mathematics teaching of eight teacher students.

Key words: beliefs, problem oriented mathematics teaching, teacher education

ZDM Classification: B50, D50

Introduction

Beliefs, attitudes and conceptions of teachers are a scientific research field that has been frequently addressed by scholars in school research, education sciences, and, in recent years, also in didactics (Forgasz & Leder, 2008; Leder, Pehkonen, & Törner, 2002). However, there is no general agreement on the definition of this research objects (e. g. Forgasz & Leder, 2008; Mason, 2004; Pajares, 1992); it depends largely on the respective specialist discipline or on the research interest in question.

Pehkonen (1994) describes beliefs as relatively stable, experience-based, subjective and often implicit knowledge of a certain object or concern, combined with an affective, meaning-constitutive component. Furthermore, they are of course not free of norms and values (Zimmermann, 1991). The cognitive components are often also referred to as conceptions (Pehkonen & Törner, 1996).

It seems certain that teachers' conceptions have a significant influence on the dynamics in class, on the interaction and communication processes in class and, finally, also on what pupils learn (Forgasz & Leder, 2008; Mayers, 1994; Staub & Stern, 2002; Thompson, 1992). Yet, it cannot be assumed that there is a simple unidirectional or monocausal correlation between teachers' conceptions and their actions (Clarke & Hollingsworth, 2002; Devlin, 2006).

An important task in both teacher education and advanced vocational training thus seems to lie in supporting students and teachers in developing appropriate conceptions (e. g. Mayers, 1994; Rolka, Rösken, & Liljedahl, 2006; Tillema, 2000). If, for instance, one wants to achieve a stronger problem orientation in mathematics classes – as called for by mathematics educationalists all over the world for very good reasons –, it does not suffice to develop a new curriculum and provide teaching materials. Rather, one should (sustainably) influence and change the (prospective) teachers' conceptions of problem solving teaching. E. g., regarding the failure of the problem-solving based reforms of the mathematics curriculum in the USA in the late 1980s (Schoenfeld, 2007), Rösken, Pepin & Törner (2011, p. 451) summarize: "Numerous studies have detected that the reasons for that 'failure' were to some extent the 'inappropriate' beliefs of teachers concerning mathematics in general, and the process of problem solving and characteristics of doing mathematics in particular, in addition to strong teacher convictions concerning students' apparent lack of ability."

However, changing conceptions is not a simple exercise because e. g. already first year students, from their time at school, possess distinctive conceptions of teaching methods, of what defines a good teacher and how they see themselves as teachers (e. g. Kagan, 1992). On the other hand, conceptions that have once been acquired are considered to be stable and difficult to change, also due to their self-stabilising effects that can be explained by both their filter effect (Ambrose, 2004; Pajares, 1992) and by cross-linkages and clustering (Pehkonen, 1994).

Nevertheless, a number of studies indicate that conceptions of (future) teachers can be changed or at least challenged (Fennema et al., 1996; Mayers, 1994). First steps in this direction could be to problematise current conceptions and practices, and to sensitise for new possibilities in mathematics teaching (Kaasila, Hannula, Laine, & Pehkonen, 2006; Ponte & Mercê, 2011; Thompson, 1991).

What could be beneficial conceptions of mathematics teaching?

But which conceptions should teachers actually have of mathematics as a school subject? In search of possible answers, teachers' conceptions of mathematics and mathematics teaching are often divided into different groups (e. g. Thompson, 1992) which are not independent of each other (Felbrich & Müller, 2007).

With regard to *conceptions of mathematics*, it is useful to understand mathematics as an open, dynamically developing system, as a creative, generally insecure construct of research communities that is culturally embedded and historically grown (cf. Ernest, 1991).

With regard to *conceptions of teaching and learning mathematics,* we can, in a first approximation, differentiate between a transmission paradigm which is oriented towards associationistic theories, and a constructivist paradigm which emphasizes the independent learning activities of pupils. In the Scholastik study, Staub and Stern (2002) were able to show that primary school teachers with a cognitive-constructivist view are far more likely to see greater learning progress among their pupils, both with regard to demanding word problems and basic numerical abilities.

Moreover, *conceptions of mathematics classes* seem particularly important to me. In this context, Kießwetter (1994) has called attention to a fundamental consideration in mathematics education. To him, teaching is “problem solving in complex constellations”. Classroom constellations are highly complex – i. e they are characterised by a particularly high comprehensiveness, they are dynamic, somewhat non-transparent and full of insecurities (Fritzlar, 2004). For these reasons, they are generally not designed to result in optimum outcomes, but in acceptable compromises. From my point of view, the complexity of problem oriented mathematics classes even goes beyond this level. This is due to the mathematical comprehensiveness and to the manifold different, simultaneous and highly dynamic working processes of the pupils which should be supervised and supported if necessary. However, these working processes are often surprising, hardly predictable and difficult to understand, also because they are influenced by various classroom conditions and elements of lesson design (Fritzlar, 2004).

If a stronger problem orientation is to be achieved in mathematics classes, teacher students and teachers should also be made aware of this complexity and the associated challenges and limitations. This could be important to enable a reflexive mode of handling and reducing complexities (Helsper, 2003), to initiate a less dogmatic and corrigible manner of dealing with knowledge and apparent certainties, a thinking in possibilities and a curious eye for the uncertain (cf. Blanck, 2007). The availability of routines that reduce complexity, which have played a significant role in research on teachers' expertise for many years (e. g. Bromme, 1992), is important due to the limited capacity of humans to process information. What matters, however, is that these routines and the resulting reduction of complexity can be questioned and given up, and that a willingness, an openness to actually do this is developed. An according sensitization in this context can help to reduce the fear of uncertainties and make unavoidable deficiencies of human action in highly complex situations more bearable (Kießwetter, 1994).

In the following, concept and course of a practical seminar at university are reported, whereby I pursued the aim to problematise students' current conceptions on problem oriented mathematics teaching and to make them aware of its complexity and its effects.

The seminar participants were asked, amongst other things, to collect and reflect on the results of their work in a journal, and occasionally they were explicitly required to submit written thoughts and evaluations relevant to the topic. At the end of the seminar, I conducted a concluding discussion with each of the students, giving the chance to reflect once more on the outcomes and processes of one's work as well as possible learning experiences.

The written and oral comments were examined for expressions indicating conceptions on problem oriented mathematics teaching, especially regarding its complexity, and expressions which imply a questioning or changing of these conceptions. In the following sections of this article, selected search results are presented waiving detailed qualitative and quantitative analysis. First of all, they are gathered to demonstrate that it indeed seems possible to question conceptions on problem oriented mathematics teaching at university within a short period of time.

A practical seminar on the doily-problem

Students at the Martin-Luther-University in Halle-Wittenberg were given the opportunity to exemplarily gather (partly first) experiences on problem oriented mathematics teaching in the context of a practical seminar in 2010. Participants of the seminar were eight first- and second year female teacher students for primary school (which were not selected by the author). Their task was to each have one Year 4 class work on the following problem, which is mathematically rather comprehensive but at the same time approachable even by primary school pupils in a number of different ways (cf. Fritzlar, 2004).

The doily-problem (formulation not for pupils)

A rectangular A4 sheet of paper is halved by folding it parallel to the shorter edge. The resulting double sheet can be halved again by folding parallel to the shorter edge and so on. After n folding procedures the corners of the resulting stack of paper sheets are cut off. By un-folding the paper, we will find that (for $n > 1$) a doily with holes has resulted.

Find and explain a connection between the number n of folding procedures and the number $A(n)$ of holes in the reopened paper.

The uncertainty and complexity of problem oriented mathematics classes were to be made tangible and graspable in a number of different ways in the course of this seminar:

- Students work on a mathematically comprehensive problem and are introduced to a number of different approaches within the group.

- In pairs, students plan a lesson on the doily-problem, exchanging ideas and viewpoints, negotiating different options of designing the lesson, balancing the pros and cons of decisions.
- As a group, the eight students get to know four different lesson designs on the same topic, possibly with very different conceptions, intentions and approaches.
- Each tandem carries out a lesson plan in two fourth grade classes. They can observe which respective steps of the plan are carried out by the teacher, how pupils react, which questions, ideas, approaches they develop, how the teacher in turn reacts to these etc. In the following video-supported reflection of the class run by students in pairs, observations, intentions, considerations, questions, doubts… are addressed.
- Finally, the tandems present, analyse and compare their experiences to and with the rest of the group, especially with regard to similarities and differences in their plans, implementations and results.

The following table gives a summarised overview of the seminar course plan and explicit assignments regarding reflections in the journal.

1st group meeting	Students are introduced to the doily-problem, they can work on it as long as they wish, but are also allowed to “take it home”.
Work in pairs	Further work on the doily-problem if necessary, while students may also use and refer to a text on its mathematical background.
Reflection assignment	*Students are asked to think about the suitability of the doily-problem for a problem oriented mathematics class, which arguments speak for or against its use in a Year 4 class.*
2nd group meeting	Students present their results and their approaches.
Work in pairs	Student tandems each attend lessons in two primary school classes with which they can work on the doily-problem later on. Afterwards, the tandem plans such a lesson.
3rd group meeting	Students present their thoughts regarding the appropriate design of a lesson on the doily-problem to the group and discuss the options.[1]
Work in pairs	Using the discussions during group meetings as a starting point, the teams’ lesson designs are further elaborated and/or modified.
Reflection assignment	*The participants are asked to* • *assess once more the suitability of the doily-problem for a problem oriented mathematics class in Year 4 and mention eventual changes in this regard;* • *think about what is of particular importance to them for their lesson;* • *designate the specific challenges they may possibly see for themselves with regard to the upcoming lesson they have planned.*
Work in pairs	The doily-problem is used in eight fourth grade classes, observed by the respective tandem partner and video-recorded (by the lecturer).

[1] The session is audio-recorded.

	Shortly after both tandem partners have given their planned lesson (often even on the same day), a meeting is held between the tandem and the lecturer during which they watch the video recordings of the lessons. *The students are asked to talk about everything that is going through their heads. Occasionally, the lecturer encourages them to reflect on certain issues. In a final discussion, attention is directed at specific challenges during class, deviations from the original plan, particularly important decisions to be made by the teacher as well as observed similarities and differences between the lessons taught by the tandem partners.*[1]
Work in pairs	*The tandem partners prepare a short presentation for the following group meeting in which they summarise some important impressions and experiences, sections of their lessons, pupils' results etc.*
4th group meeting.	*In the final group meeting, the tandems first of all present their teaching experiences, discuss similarities and differences as well as possible reasons for them.* Afterwards, the lecturer presents further variations of the lesson design. The group discuss their eventual implications on the further progress of the lesson and possible outcomes.[1]
Concluding discussion	*At the end of the course, the lecturer meets with every student individually for about 30 minutes to address and evaluate the newly gained experiences.*[1]

Some Impressions of the practical seminar

Due to the limited scope of this paper, I can only summarise some impressions that derive from my observations and analyses of the classes taught by the participating students, their plans and reflections as well as their journals and interviews. In order to illustrate my argument, I include selected quotations (partly tidied up for easier reading) of comments made by the students in their journals or from the audio-recordings. Some of the students' considerations may be critically questioned; I would however like to refrain from commenting them in detail or even assessing them at this point.

What needs to be said first of all is that all participants were highly motivated and committed to the work at hand. For instance, they worked on the doily-problem for at least 60 minutes during the first meeting, even though it was planned as an open-ended session. All tandems made tables which often also showed the number of folding lines – differentiated by their vertical/horizontal orientation – and the number of resulting sections in the sheet of paper. On this basis, they tried to determine the according arithmetic patterns. That duplications and powers of 2 played a role was frequently identified: three tandems looked for a multiplicative pattern, the other team tried to formulate a general relationship between the number of sections – that had been identified as powers of 2 – and the respective number of holes in the sheet of paper.

U[2] made the following comment on the doily-problem in her journal: *"After all this endless calculating, I started to resign and asked myself: How are Year 4 kids supposed to find a solution to such a complex problem if even I can't do it? … But later on I realized that it isn't all that difficult to make a complicated mathematical problem suitable for children. You just have to simplify the problem and the solution you expect from them accordingly."* The second part of her statement already touches the question of whether the doily-problem is appropriate for mathematics classes in grade 4. Other participants also commented on this issue:

R: "I think this problem is very suitable for use in classroom lessons, especially because it allows for many solutions on different levels and can be approached in a very practical way. … However, I don't believe that a 45-minute period is enough to deal with the problem intensively – particularly considering that even we as students already spent more than an hour on it." (Journal)

L: "When I thought about this question [of suitability], my first thought was that this is too hard for primary school pupils because it was quite tricky even for us to solve, and especially this whole thinking in formulae is only really something for kids that are highly interested and mathematically gifted." (Journal)

S: "… but if, for instance, you provide certain solution approaches or a table or give them a specific introduction into the process, almost every child can really – with appropriate assistance – understand and partly even solve this problem. That's why I find it suitable, because it also demands quite a few own intellectual approaches. As far as I can remember, we never did something like this when I was at primary school." (Concluding discussion)

K: "From my own experience I can say that you tend to underestimate the children's capabilities far too often. You think that they are capable of much less than they actually are. But I think that this will teach us better." (Journal)

In the second group meeting, the tandems presented their approaches to the problem and some suggestions for solutions. The small size of the group contributed to an open and relaxed atmosphere in which also partial solutions and wrong ideas could be presented. As my students found the doily-problem to be very demanding, I had decided to show some selected results that had been developed by pupils in previous classroom experiments: The aim was to demonstrate that the problem can, in fact, also be solved by fourth graders, and in many different ways.

On the basis of these experiences, each team prepared a draft for a teaching unit. The four teaching plans displayed numerous similarities: The first introductory part consisted of a practical exercise in which both pupils and teachers – comparable to their own introduction to the problem – went through several folding-and-cutting-procedures together. Every time before each sheet of paper was unfolded, the pupils were asked to make assumptions on what could

[2] The names of the participating students are abbreviated to their initials: tandem 1: L and R, tandem 2: K and U, tandem 3: G and S, tandem 4: H and N.

have happened. For the subsequent main part of the lesson, pupils were asked to work on the doily-problem in pairs or in teams, and the lesson was concluded by a presentation of the pupils' and partly also the students' results. This points to one of many differences in the design of the teaching plans, which, of course, existed alongside the similarities:

> When formulating the goals and objectives of their lesson plan, for instance, the first tandem also emphasized the geometrical aspects of the doily-problem: *"All pupils realize that new holes appear where two folding lines intersect, that the number of vertical folding lines multiplied by the number of horizontal folding lines equals to the number of holes, that both horizontally and vertically there is always one folding line less than there are sections. Furthermore, pupils can see that the number of sections doubles with each new folding procedure and that this duplication alternates between the horizontal and vertical orientations."* (L & R, Journal)
>
> Also other details of the lesson plan placed special attention on geometrical elements, e.g. in already discussing folding edges, corners and sections during the introductory part of the lesson. Potential assistance for the group work phase was directed at determining the number of folding lines, the location of (new) holes in the sheet of paper or the appearance of the unfolded sheet.
>
> For the first collective exercise of folding and cutting the sheet of paper, it was important to K and U to not only ask their pupils about their assumptions regarding the number of holes, but also for the reasons behind these assumptions. How important a „graspable" result was to the second tandem can also be seen in their first draft for the lesson, in which about one third of the time available was dedicated to the presentation of the results, including a comprehensive presentation of their own solution that they prepared.
>
> The most important goal of the lesson for tandem partners H and N was for their pupils to be able to calculate the number of holes by the end of the lesson. Already their introduction to the problem thus included giving their pupils a prepared table and filling out the first few rows together. They also told the children about a character called *Mathematicus* who already knows from the very beginning that if you fold the sheet of paper seven times and cut the edges accordingly, a total number of 105 holes will appear. The children's task is to get on to his track (H & N, Journal).
>
> G and S set great value upon motivating their pupils. For this reason, they want to work with a stuffed toy called *Billy the Beaver*, who loves to do handicrafts but needs the children's help to solve the doily-problem. They can help him find out "*whether or how the number of foldings is related to the number of holes"* (G & S, Journal).

Other students also found that designing a motivating introduction was an important challenge in the preparation of their lesson, apart from time management and considerations regarding the appropriate assistance of pupils they hardly knew. S's concluding remarks on planning the teaching unit are: *"Well, because that was really complex somehow. You kept remembering one or the other thing that you may have forgotten or that you could also… Then you thought, now we're done, but then we realized that we still needed some little cards, or hadn't decided on what kind of sheets of paper we wanted to use, and how we'd introduce the problem to the kids …"* (Concluding discussion)

Reflections of teacher students

All students perceived the lessons to be particularly demanding; it was, above all, not easy to interpret the frequently different working processes of pupils, react quickly but appropriately to all classroom situations and, in doing so, parting from one's own plans.

> R: "It is extremely difficult in the beginning, you still try really hard to follow it [your teaching plan] exactly the way you wrote it down, … it's my goal to put my plan into action in the best possible way. But sometimes it's actually much better to divert from it and go down a different track, and realize that at this and this point it's necessary to divert from my plan and do something else. But in the beginning, I think, that's still really hard to do." (Concluding discussion)
>
> U: "The lesson I was able to draw from this class overall was that I still have to learn to concentrate on so many kids at the same time. For example, I let a couple of really good ideas presented by the pupils slip through my fingers that I could and should have encouraged and developed further." (Journal)
>
> "I think my greatest challenge was to react to situations that I didn't anticipate, that I didn't prepare myself for, that I didn't imagine possible beforehand." (Concluding discussion)
>
> S: "I guess such a lesson is especially difficult because you have to understand what these kids mean so quickly and if what they say is correct." (Note by the author)
>
> "And then to really respond to each of them… and then also to react so quickly…" (Concluding discussion)

Reviewing the video-recordings of the lessons in tandems and presenting and discussing the students' experiences in the group (4th group meeting) make it possible to compare their plans and implementations as well as pupils' work processes and results. Here, students address a large part of the possible variations; mathematical and cognitive aspects, however, play a minor role.

> N: "And that there are many different ways of making a topic appealing and interesting in various manners, with different goals, different approaches." (4th group meeting)

K: “I wouldn’t have thought that the differences are really that striking, that other classes find out more, progress further… But maybe it also depends on the plans we made and of course also on the classes themselves, they were totally different from each other… That was really fascinating to see.” (Concluding discussion)

The concluding one-on-one conversations clearly showed that the participating students had acquired new and (in my opinion) important experiences in problem oriented mathematics teaching:

R: “What I learned for myself, what I still have to learn, is to hold myself back… You really have to learn to hold your horses a little as well and let the kids go ahead with their ideas.” (Concluding discussion)

“And this practical experience in particular, that it can actually really work in practice …“ (Concluding discussion)

G: “…that you can just do it, that you can trust the children to do it, that you can have the confidence that they are capable of finding their own ways, that you shouldn’t underestimate them.“ (Concluding discussion)

S: “First of all: you should have confidence in these kids… I have learned that this also works, that you can also achieve it that way. Then, the complexity of such a lesson, especially problem oriented, if you have to prepare it and… that sometimes things don’t quite go according to plan or that you don’t always immediately have an answer up your sleeve.” (Concluding discussion)

Finally, the students were asked to comment on the statement *“Teaching is to act and decide in complex and uncertain situations“*:

R: “Well, this is something I really noticed about my lesson: It’s first of all really complex, it’s really difficult to assess, especially because we didn’t know the class beforehand. But I think that even if you do know the class it can be really tricky at times to decide somehow what would be the right move at the right time. And also the dynamics as well… And you constantly have to readjust to the changing situations. So our preparations, in the beginning, they really helped, but at the end of the day, in the concrete situation, you actually do have to deviate from the original plan sometimes and make other decisions.” (Concluding discussion)

K: “… you can make plans and prepare yourself, but at the same time you have to be flexible as well. That is probably the most difficult competency to develop for a teacher, that you’re ultimately flexible concerning your plan and that you bring the plan in line with your pupils. I think this requires the most practice of all.” (Concluding discussion)

U: “Classroom teaching is simply an interaction that you can never be 100 per cent prepared for. You would have to prepare for every kind of possible situation, which, of course, you can’t. There can always be things you aren’t prepared for.” (Concluding discussion)

G: “… sometimes it really comes down to one decision that you make, and the ripple effect from this decision affects the lesson in a way that you perhaps aren’t able to predict beforehand.” (Concluding discussion)

S: “I think that’s one thing you also saw on the video, that when you’re teaching in class it’s like you’re – not under pressure to act, but you have to react at once, …

you really have to do something immediately and decide on the spot. And that is complex, of course, because there are so many different possibilities. And that's why, I think, it's important that you don't always stick to the same pattern and base your decisions on that, but that you always have several alternatives up your sleeve. It is hard, I believe, once you've developed a routine over the years to still think of alternatives, what would have been another way of doing it and not always systematically follow your one fixed plan. And uncertainty is always part of the deal. It also depends on the kids, on the class, on the problems, sometimes maybe even on how you personally feel on the day – or … But I guess that's part of being a teacher … you always deal with situations that are uncertain and indeterminable." (Concluding discussion)

Discussion

The reported seminar seems to be an effective possibility for prompting teacher students to challenge their conceptions on problem oriented mathematics teaching. Nevertheless, several critical questions arise. One could ask, for instance, whether students with very little teaching experience are actually a suitable target group. Teaching primary school children in class definitely placed the students in a special and also psychologically challenging situation. On the other hand, it might just be particularly beneficial to deal with and question one's previously developed conceptions of (problem oriented) mathematics teaching at a very early stage of university education.

Given the students' limited previous experience, integrating a more theory-focused unit into the seminar (on e.g. different movements in the didactics of mathematics) might seem appropriate before commencing work on classroom experiments and teaching plans. This would, however, probably have obscured the students' own conceptions of problem oriented mathematics teaching. In this context, R remarked that *"the lecturer's decision to run the seminar without an introduction to problem oriented mathematics teaching made it more demanding for the students, but also opened up more opportunities to learn."* (Journal)

So what were the learning effects for the students – given that they received no theoretical input and prepared only one single lesson? In my opinion, the students' impressions and reflections that have been compiled in the previous section of this paper clearly illustrate, amongst other things, that they have become aware of a number of different issues: They have been introduced to a new, challenging mathematical problem and have seen that it is (nonetheless) suitable for a regular primary school class, they have realized how many different possibilities there are to design a lesson unit on the same topic without determining an ideal outcome, that a similar approach in different classes can lead to very different results, that problem oriented mathematics teaching requires good preparation but can take very unexpected and surprising turns in class all the same, and that it places specific demands on the teacher. Conceptions on problem oriented mathematics teaching were also challenged, in my opinion. G writes, for instance: *“This seminar has essentially contributed to the fact that I look at the issue of ‘problem oriented teaching’ with much less skepticism than before. Before, I always asked myself whether you should allow pupils to work so independently, but the class I ran on my own, and also the other teams’ classes, clearly proved that it works. Pupils are often underestimated in their ability to work with difficult concepts, and this is why you often use a well-established and well-known pattern that you want them to apply instead of letting them experience for themselves and explore the problem on their own.“* (Journal)

The variety of different plans for the teaching unit, and the corresponding approaches taken by the pupils to solve the problem, would probably have been larger if there had been no collective introduction for the students into the doily-problem (i.e. group session of folding, cutting, assuming, unfolding), or if possible variations of the lesson (e.g. regarding the formulation of problem statements and work assignments) are brought up in group discussions early on in the process. Yet, this would probably also have made it more difficult to comprehend the problem and its stimulative nature would have been less apparent; on the whole, “accessing” the subject would have been much more tedious for the students. Giving them a ready-to-use “toolkit” of various „modules“ they can use for planning their lesson would likely have distracted them from their own ideas, their own conceptions and perhaps even taken too much work out of their hands. Asked about this issue, K replied: *“More input? I don’t really think so. Essentially, it’s not only problem oriented for the kids but also for us, i.e. that we have to figure it out. Actually, I believe that giving us less advice and clues made us think about it more thoroughly. ... This way, we had to think about it ourselves. I think that was actually quite OK.“* (Concluding discussion)

K’s comment suggests a certain self-reference: the seminar on problem oriented mathematics teaching was deliberately designed to be problem oriented for the teacher students. It was also self-referential for the lecturer, though. The deliberations of this section show that already the attempt of making (at least aspects of) the complex reality of problem oriented mathematics teaching tangible to students, encouraging and animating them to reflect their experiences and, in doing so, sensitizing them a little bit for this complexity, is complex in itself.

References

Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education, 7*, 91–119.

Blanck, B. (2007). Unterrichtsqualität entwickeln durch professionellen Umgang mit Ungewissheiten beim Lernen und Lehren. In K. Möller et al. (Eds.), *Qualität von Grundschulunterricht. Entwickeln, erfassen und bewerten* (pp. 111–114). Wiesbaden: VS Verlag für Sozialwissenschaften.

Bromme, R. (1992). *Der Lehrer als Experte*. Bern: Huber.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*, 947–967.

Devlin, M. (2006). Challenging accepted wisdom about the place of conceptions of teaching in university teaching improvement. *International Journal of Teaching and Learning in Higher Education, 18*(2), 112–119.

Ernest, P. (1991). *The philosophy of mathematics*. Basingstoke, UK: Falmer Press

Felbrich, A., & Müller, C. (2007). Erste Ergebnisse aus P-TEDS: Mathematische Weltbilder und Vorstellungen zum Lehren und Lernen von Mathematik. In *Beiträge zum Mathematikunterricht 2007* (pp. 573–576). Hildesheim: Franzbecker.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403–434.

Forgasz, H. J., & Leder, G. C. (2008). Beliefs about mathematics and mathematics teaching. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education. Knowledge and beliefs in mathematics teaching and teaching development* (Vol. 1, pp. 173–192). Rotterdam: Sense Publishers.

Fritzlar, T. (2004). Zur Sensibilität von Studierenden für die Komplexität problemorientierten Mathematikunterrichts. Hamburg: Kovač.

Helsper, W. (2003). Ungewissheit im Lehrerhandeln als Aufgabe der Lehrerbildung. In W. Helsper, R. Hörster, & J. Kade (Eds.), *Ungewissheit. Pädagogische Felder im Modernisierungsprozess* (pp. 142–161). Weilerswist: Velbrück Wissenschaft.

Kaasila, R., Hannula, M. S., Laine, A., & Pehkonen, E. (2006). Facilitators for change of elementary teacher student's view of mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 385–392). Prague: PME.

Kagan, D. M. (1992). Professional growth among preservice and beginning teachers. *Review of Educational Research, 62*(2), 129–169.

Kießwetter, K. (1994). Unterrichtsgestaltung als Problemlösen in komplexen Konstellationen. In F. Padberg (Ed.), *Beiträge zum Lernen und Lehren von Mathematik* (pp. 106–116). Seelze: Kallmeyer.

Leder, G. C., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht: Kluwer Academic Publishers.

Mason, J. (2004). Are beliefs believable? *Mathematical Thinking and Learning, 6*(3), 343–352.

Mayers, C. (1994). Mathematics and mathematics teaching: Changes in pre-service student-teachers' beliefs and attitudes. In G. Bell, B. Wright, N. Leeson, & J. Geeke (Eds.), *Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA-17)* (pp. 419–428). Lismore.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62* (3), 307–332.

Pehkonen, E. (1994). On teachers' beliefs and changing mathematics teaching. *Journal für Mathematikdidaktik, 15* (3/4), 177–209.

Pehkonen, E., & Törner, G. (1996). Mathematical beliefs and different aspects of their meaning. *Zentralblatt für Didaktik der Mathematik, 28* (4), 101–108.

Ponte, J. P. da, & Mercê, C. (2011). A teacher education experiment to challenge conceptions and practices. *ZDM-The International Journal on Mathematics Education, 43* (6-7), 847–859.

Rolka, K., Rösken, B., & Liljedahl, P. (2006). Challenging the mathematical beliefs of preservice elementary school teachers. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 441–448). Prague: PME.

Rösken, B., Pepin, B., & Törner, G. (2011). Beliefs and beyond: affect and the teaching and learning of mathematics. *ZDM-The International Journal on Mathematics Education, 43*(4), 451–455.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: research and theory, practice and politics. *ZDM-The International Journal on Mathematics Education, 39*(5-6), 537–551.

Staub, F. C., & Stern, E. (2002). The nature of teachers' pedagogical content beliefs matters for students' achievement gains: quasi-experimental evidence from elementary mathematics. *Journal of Educational Psychology, 94* (2), 344–355.

Tillema, H. H. (2000). Belief change towards self-directed learning in student teachers: immersion in practice or reflection on action. *Teaching and Teacher Education, 16*, 575–591.

Thompson, A. G. (1991). The development of teachers' conceptions of mathematics teaching. In R. G. Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting. North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2-8–2-14). Blacksburg (VA).

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook Of Research On Mathematics Teaching And Learning* (pp. 127–146). New York: MacMillan.

Zimmermann, B. (1991). *Heuristik als ein Element mathematischer Denk- und Lernprozesse*. Habilitationsschrift, Hamburg.

Problems with the Imagination of Astronomical Measurements

Günter Graumann
University of Bielefeld, Germany
og-graumann@web.de

Abstract

During the whole time of school students are confronted with measurements like monetary value, linear measure, measure of weight, measure of time etc. Usually the size of these measurements are within normal ranges. But since grade five the pupils also come up against with very small as well as very big sizes. One field of huge measurements is that within astronomy. To get an outlook and imagination of such measurements and for better memorization of facts one has to find relations between different sizes, correlations with other types of measurement and special representations. In this presentation besides some facts we will look out for getting relations and representations about linear measures in the near of the earth, within our planetary system and within the whole universe.

Key words: problem field, astronomical measurements, representations of distances

ZDM classification: D53, D59, D83, M53, M59.

Introduction

Since first grade children of primary school are confronted with different measurements like monetary value, linear measure, measure of weight, measure of time, etc. But the size of the measurements range within body measures or measures of normal environment. For example children learn to get an imagination of distances of “1 mm” (e.g. with the thickness of a 1 cent coin as main representative), “1 cm” (thickness of a thumb), “1 dm” (spread of open hand), “1 m” (one big footstep), “1 km” (perhaps the walking distance from home to school) and may be also of “10 km” (spread of a small town) and “100 km” (distance to the next big city). Since grade five the pupils also come up against with smaller and bigger sizes like “1 μm” and “1000 km” or “10 000 km” and computing with translations of scales. But with bigger distances e.g. we all have problems of imagination.

“Man is the measurement of all” Protagoras did say about 2500 years ago. This is true still today for normal life. But we also today have to do with things outside of the all-day experiences. States have to compute with huge amounts of money, rackets have huge velocities and we hear of huge distances in the space.

To get an outlook and imagination of such measurements and for better memorization of facts one has to find relations between different sizes, to detect correlations between different ranges and types of measurement and discover special representations of huge measurements. How this can be done I will discuss on the field of huge measurements – especially linear measures - within astronomy. This is a problem field with a lot of different aspects and in the given frame I can only present some information on main points.

Pupils can find many possibilities for creative work and own decisions in this problem field and can make investigations (within astronomy as well mathematics). They also in the beginning have to structure the whole problem field. Most information the pupils as well as the teachers can find in books and in the internet. But didactical literature in respect to this theme except some special aspects for higher grades you can find hardly.

For this presentation I structured the problem field of linear measures within the astronomy in three domains: The astronomical neighborhood of our earth (the atmosphere and the satellites as well as the moon), our planetary system (sun and the planets from Mercury to Neptune as well as the outer sphere of our sun system with comets etc.) and the whole universe (with some stars nearer to the sun, our galaxy, other galaxies, clusters of galaxies and the end of the visible universe).

The astronomical neighbourhood of our earth

We start with our earth and its satellites. I first present some facts pupils for example can find in the internet or in special books:

- Radius of the earth is 6 370 km so that the mean ***diameter of the earth*** is **12 740 km**
  [polar-diameter 12756 km and equatorial diameter 12714 km]
- Height of atmosphere (from ocean level) is
  for the ***troposphere*** **7 km** to **17 km**,
  for the ***stratosphere*** until about **50 km** and
  for the mesosphere until about **85 km**.
  The ***thermosphere and exosphere*** (were nearly no air exists) is estimated until **10.000 km**.
- ***Low Earth Orbit (LEO) satellites*** circle in a height (measured from ocean level) of **200 km** to **1 500 km**. They need 1.5 – 2 hour for one cycle. The ***ISS*** for example circles in about **400 km** height and needs 90 minutes for one cycle.

- ***Medium Earth Orbit (MEO) satellites*** circle in a height of **6 000 km – 36 000 km**. GPS-satellites or Navigation-satellites like Galileo circle in a height of about 20 000 km respectively 32 000 km.
- ***Geostationary Orbit (GEO) satellites*** circle in a height of **35 790 km**. Their cycle time is exact 24 hours so that they have the same movement like the earth and keep staying above a fixed point of the earth. The telecommunication satellites like Astra, Eutelsat, Immarsat and the metrological satellite Meteosat stand in this height.
- The average radius of the orbit of the ***moon*** is 384 400 km (**363 300 km – 405 500 km**)

To get an imagination of these distances we look out for different scales to put the distances on. The pupils first may try to let the <u>earth look like a coin</u> (e.g. a 1 Euro-coin with diameter of 2.3 cm). This leads to a scale of about 1 : 554 000 000 or better rounded down **1 : 500 Mill.** Then we have the

- diameter of the earth about **2.55 cm**
- mean thickness of the troposphere about **0.0024 cm** (≈ 1/40 mm)
- thickness of the atmosphere up to the mesosphere a bit more than **0.017 cm** (≈ 1/6 mm like the skin of an apple)
- ISS (with orbit radius 400 km above earth's surface) **0.08 cm** (≈ 1 mm)
- geostationary satellites (with 35 790 km above earth's surface) **7.16 cm**
- moon orbit with range from **73 cm** to **81 cm**.

The pupils will find out that this scale is not a good one for the atmosphere and the LEO-satellites but it shows already some relations between the radius of the earth, the thickness of the atmosphere and the height of satellite orbits as well as the moon orbit.

A better scale might be then <u>the earth big like a ball or a terrestrial globe</u> so that we set 25 cm → 12 500 km with the scale **1 : 50 Million**. Then we have the

- diameter of the earth **25,5 cm**
- thickness of troposphere and stratosphere **0.1 cm** (1 mm)
- height of the ISS is **0.8 cm** (8 mm)
- height of GPS-satellites is **40 cm**
- height of geostationary satellites is **71.6 cm**
- radius of the moon orbit between **727 cm** and **811 cm** (≈ 77 m)**.**

With these representations we already can visualize the relations between the named linear measures. To deepen this we can compute some quotients and e.g. get the following representation:

- mean thickness of the troposphere $\xrightarrow{\cdot 4}$ thickness of troposphere plus stratosphere $\xrightarrow{\cdot 200}$ distance from earth's surface to end of the exosphere

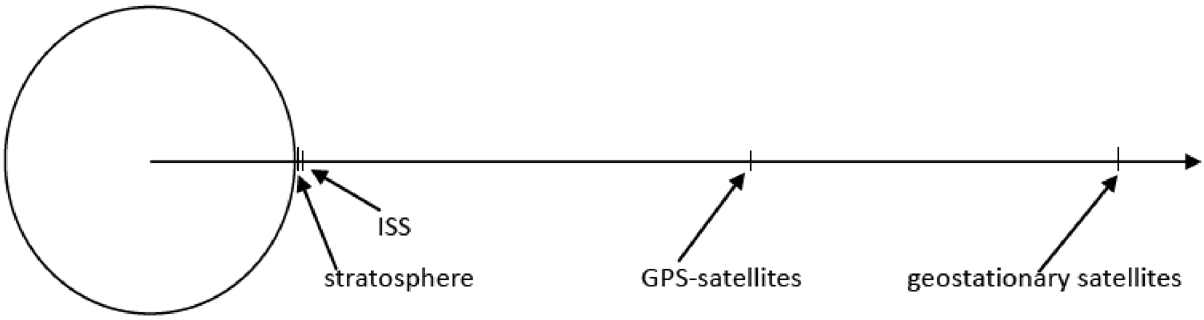
- mean thickness of the troposphere $\xrightarrow{\cdot 1\,000}$ diameter of earth $\xrightarrow{\cdot 30}$ radius of moon orbit
  $\cdot$ 30 000

- height of ISS $\xrightarrow{\cdot 50}$ height of GPS-satellites $\xrightarrow{\cdot 1.8}$ height of geostationary satellites
  $\cdot$ 90

If we want to put the earth until the geostationary satellites (with only half earth circle) on a DIN A4 paper in oblong format then we can use a scale 1 : 150 Mill. whereat the radius of the earth becomes about 4 cm, the thickness of the atmosphere until the stratosphere ⅓ mm and the distance from the midpoint of the earth to a geostationary satellite 28.1 cm.

To get all above distances (devoid of the radius of the moon orbit) in an upright format we can use the scale with **1 : 330 Mill.** so that (as in the following figure) the

- diameter of the earth is **3.86 cm**
- thickness of troposphere and stratosphere is **0.015 cm** (≈ 1/7 mm)
- height of the ISS is **0.121 cm** (≈ 1/9 mm)
- height of GPS-satellites is **6.1 cm**
- height of geostationary satellites is **10.85 cm**.



On the sports field of the school we can represent this with a scale like **1 : 500 000**, i.e.

- the earth is represented as circle with diameter **2548 cm** (25 m 48 cm)
- The mean thickness of the troposphere is **2.4 cm**
- the thickness of troposphere plus stratosphere is **10 cm**
- the distance (from the earth ground level) to the ISS is **80 cm**
- the distance (from the earth ground level) to GPS-satellites is **4000 cm** (40 m)

- the distance (from the earth ground level) to geostationary sat. is **7158 cm** (71.58 m)
- The distance from midpoint of the earth to a geostationary sat. is **8432 cm** (≈ 84.3 m)

Another way of getting an impression of these distances the pupils can find by driving with an "imaginary" ***car with a constant speed of 100 km/h***. This is a speed we know from driving on a motor way. From earth ground we then would need

- to the end of the troposphere not more than **10 minutes**
- to the end of the stratosphere **30 minutes**
- to the ISS **4 hours**
- to GPS-satellites **8 days and 8 hours**
- to a geostationary satellites **14 days and 22 hours**
- to the moon on a straight way about **160 days**.

If you take a supersonic transport (like the Concorde) with 2000 km/h then the above times have to be divided by twenty. For example to get to the stationary satellites you need already a little bit less than a half day.

A ***rocket with constant speed of 20 000 km/h*** would need

- to the end of the troposphere not more than **3 seconds**
- to the end of the stratosphere **9 seconds**
- to the ISS **1 minute and 12 seconds**
- to GPS-satellites **1 hour**
- to a geostationary satellites about **1¾ hours**
- to the moon on a straight way about **19 hours**.

By finding out and putting down all these named representations and by building quotients between the named distances the pupils will get a better imagination of these different distances.

Our planetary system

After the first experience with astronomical distances we turn towards our solar system with its planets. This is a theme that appears very often in school, for instance in grade five at geography. Then (but also at other times) it is good to pick up this theme in mathematics to deepen the imagination of the distances in our planetary system.

Let us start again with data the pupils can find in books or in the internet:

- **Sun**: Radius 696 000 km, weight about $2 \cdot 10^{30}$ kg, density 1.41 g/cm³
- **Earth**: Radius 6370 km, weight about $6 \cdot 10^{24}$ kg, density 5.51 g/cm³
- **Moon**: Radius 1738 km, weight about $7\frac{1}{3} \cdot 10^{22}$ kg, density 3,34 g/cm³

For the planets we e.g. can find the following information:

Planet	Radius	distance to the sun	time of circulation
Mercury	2 439 km	58 Mill. km	87.9 days
Venus	6 052 km	108 Mill. km	224.5 days
Earth	6 370 km	149.6 Mill. km	about 365¼ days
Mars	3 397 km	228 Mill. km	1.88 years
Jupiter	71 398 km	778 Mill. km	11.84 years
Saturn	60 000 km	1 432 Mill. km	29.43 years
Uranus	25 400 km	2 884 Mill. km	84.18 years
Neptune	24 300 km	4 509 Mill. km	164.56 years

We now may start with looking out for scaling the named radii and start with such a scale that the circle of the sun is going on a normal sheet of paper with width of 21 cm. If we take 1cm for 100 000 km the diameter of the sun is 13,92 cm. Multiplying this with 1.5 we get 20.88 cm, i.e. nearly 21 cm. Thus we have the scale 1,5 cm → 100 000 km = 10 000 000 000 cm, i.e. **1 : 6 666 666 667** (= ⅔ · 10 Bill.). With this we can get the circles of the sun and the planets on a DIN 4 paper (without correct distances) by following scaled diameters:

Sun	Mercury	Venus	Earth
208.8 mm	0.7 mm	1.8 mm	1.9 mm

Mars	Jupiter	Saturn	Uranus	Neptune
1.0 mm	21.4 mm	18.0 mm	7.6 mm	7.3 mm

This shows that with this scale Mercury and Mars have a diameter of nearly 1 mm, Venus and Earth of nearly 2 mm, Uranus and Neptune of nearly 7 mm and Jupiter and Saturn lay in the near of 2 cm. All these measures we just can put on our paper.

If we use this scale for the distances to the sun then the planet farthest away from the sun, the Neptune, will lay about 676 m away. This is not a good scale for sheets of paper and even not on the school yard. The sports field of a school often has a length of about 100 m. To get the distance of Neptune – Sun on it we should minimize the length of 676 m by 7. A better scale we will get if we multiply the scale with 7.5 instead of 7. Thus in the next step we will use the scale **1 : 50 Bill.** [Remark: 1 Bill. = 1 000 000 000 and is called in some countries 1 Millard.] Then we have:

	diameter		**distance to the sun**
Sun	27.8 mm		-
Mercury	0.1 mm	(very small point)	1.16 m
Venus	0.2 mm	(small point)	2.16 m
Earth	0.3 mm (normal point)		2.99 m
Mars	0.1 mm (very small point)		4.56 m
Jupiter	2.9 mm (very thick point)		15.56 m
Saturn	2.4 mm (very thick point)		28.64 m
Uranus	1.0 mm (thick point)		57.77 m
Neptune	1.0 mm (thick point)		90.18 m

To get these distance on our paper we have to use the scale **1 : 25 Trill**. ($= 25\cdot 10^{12}$).

Another possibility to get an impression of these distances is to look out for a **planet path** and to walk on such path. In Bielefeld in 2003 for example students of a middle school planed and built such a planet path starting at their school.

You can find several such planet paths in Germany and presumably also in other countries. The scales are mostly **1 : 1 Bill.** or **1 : 1.5 Bill.** or **1 : 2 Bill.** That one in Bielefeld has the scale named in the middle.

For the scale 1 : 1.5 Bill. (1 : 1 500 000 000) we have:

	diameter	**distance to the sun**
Sun	92.67 cm	-
Mercury	3.3 mm	38.7 m
Venus	8.1 mm	72.0 m
Earth	8.5 mm	99.8 m
Mars	4.5 mm	152.0 m
Jupiter	95.2 mm	518.7 m
Saturn	80.0 mm	954.7 m
Uranus	33.9 mm	1 922.7 m
Neptune	32.4 mm	3 006.0 m

In addition: Some larger distances of our solar system ("trans-neptune-objects")

First in the so-called Kupier-Belt we find dwarf planets like **Pluto** (in elderly books it is named a planet but since 2006 it is a dwarf planet which has characteristics of a planet as well as a comet). Pluto's distance to the sun is about **5 900 Mill. km** (in our scale about **3 933 m**).

In the next sphere of our solar system, the so-called Helio sphere - where the magnetic field of the sun begins to interact with interstellar media - we can find **comets** like **Hale-Bopp** with a distance of about **55.6 Bill. km** (in our scale about **37 km**) or the **dwarf planet Sedna** with a distance of about 75 Bill. km (in our scale 50 km) to the sun.

Some people say that the solar system does end here. But latterly some scientist say that the solar system ends much more farer with the so-called **Oort cloud** whose radius in respect to the sun is about **15 Trill. km** (in our scale **1 000 km**).

The distances of Hale-Bopp and Sedna and all the more the distance to the end of the Oort cloud can not be represented in a planet path. But by looking up these in the internet and translating them to our scale we get an imagination about the range of our solar system and the gravitation force of the sun.
Interesting in this connection might be also the time the **Voyager 1 satellite** (the today furthest away satellite which just now is outgoing of the sun system) took to go into these distances:

- Start from Earth Sept. 1977
- Passage of Jupiter 1 ½ year later (March 1979)
- Passage of Saturn again about 3 years after start (Nov. 1980)
- Begin of interstellar mission about 13 year after start (1990)
- Going into Heliosphere 28 years after start (2005)
- Distance today (after 36 years) about *20 Bill. km*.

The present speed of Voyager 1 is 61 380 km/h, so that its distance from sun increases about 540 Mill. km per year.

Up from the distances of the end of the solar system astronomers use another unit of length, the so-called **light year (lj)**. It is the length a light ray (or any electro-magnetic wave) goes in a Julian year of 365.25 days. It is equal to 9 460 730 472 580 800 m $\approx 9.46 \cdot 10^{12}$ km = $9.46 \cdot 10^{15}$ m $\approx 10^{16}$ m). This says that the light from the sun needs about 1 year to get to the end of the solar system.
Sometimes one uses also the units of a light hour, a light minute or a light second at which 1 light hour $\approx 10^{12}$ m, 1 light minute $\approx 1.8 \cdot 10^{10}$ m and 1 light second $\approx 3 \cdot 10^{8}$ m. A good small problem field for the pupils in this connection is that of getting exact computations about theses distances. But we also can compute with our given distances how long light or radio signals do need for the considered distance.

This leads us to another way of comparing the distances within the solar system: We look out for the **time light or radio signals do need** for several distances. So e.g.

- the light from sun to earth needs **8⅓ minutes**
- the light from sun to Neptune needs a little bit more than **4 hours**.
- a laser beam from earth to the moon takes a little bit more than **1 second**

- the radio signal from the satellite on Mars to earth (with distance to the earth between 56 and 400 Mill. km – at present about 250 Mill. km) takes currently about **14 minutes**.
- And a radio signal from Voyager 1 to earth requires about **12 hours**.

With all these given or computed data and descriptions one can get an imagination of the distances within our solar system. Of course there are other possibilities to deepen this (especially by looking out for other scales) or to fill up with new details (for example about asteroids and comets). For instance from the assumedly oldest big asteroid Fester which has a 500 km diameter we just got wonderful pictures from a satellite that circled this asteroid (see: http://de.wikipedia.org/wiki/Dawn_%28Raumsonde%29). More information from many asteroids and comets one can get from the internet.

The whole universe

But now we will leave our solar system and look out for stars and galaxies in the universe. A good introduction to distances in the universe is looking at a movie called "Powers to ten" made by Charles and Ray Eames in 1977 (see: www.youtube.com/watch?v=0fKBhvDjuy0 or google "Powers of ten"). It gives an imaginary journey from the earth to the furthest away quasars with a logarithmic scale using the powers of ten from 1 m to 10^{26} m (≈ 10 Bill. light years).
One has to watch this film several times to get out most of the information and fix them on a paper. Also it is good to complete this information with others in the internet (e.g. http://www. powersof10.com/ or https://en.wikipedia.org/ wiki/Quasar) and in books (like "*Powers of Ten: A Book About the Relative Size of Things in the Universe and the Effect of Adding another Zero"* (1982,1994), by Philip and Phylis Morrison or "*An Introduction to Modern Astrophysics"*, by Carroll and Ostlie, 2007).

But after sampling the information we should work with the information. And at first we can rediscover our measures of our solar system whereat we have to transpose all measures in powers of ten with m as base item, so that we get e.g.

Distance Earth – Moon $3.8 \cdot 10^{\mathbf{8}}$ m (± 6.6 %)

Distance Earth – Sun $1.5 \cdot 10^{\mathbf{11}}$ m (± 1.6 %)

(The exact mean length of 149 597 870 700 m is called **1 au**, i.e. 1 astronomic unit)

Distance Jupiter – Sun $7.8 \cdot 10^{\mathbf{11}}$ m (≈ 5¼ au)

Distance Saturn – Sun $1.4 \cdot 10^{12}$ m (≈ 9⅓ au)

Distance Neptune – Sun $4.5 \cdot 10^{12}$ m (≈ 30 au)

Distance of Voyager 1 - Sun $1.9 \cdot 10^{13}$ m (≈ 125 au)

Distance End of solar system – Sun about $1.5 \cdot 10^{16}$ m (≈ 100 000 au)

A little bit outside of our solar system we find the nearest stars (seen in an approximate distance from the earth).

Proxima Centauri: $4 \cdot 10^{16}$ m (≈ 4.2 light years)

Alpha Centauri: $4 \cdot 10^{16}$ m (≈ 4.3 light years)

Sirius: $8 \cdot 10^{16}$ m (≈ 8.6 light years)

Ross 248: $9.8 \cdot 10^{16}$ m (≈ 10.3 light years)

(Ross 248 is a red dwarf star. Voyager 2 is going in its direction and might arrive there in

40 000 years – see: http://www.marspages.eu/index.php?page=592)

Within the next three powers of ten there are several stars but mostly isolated and in the named movie it was told that the picture always is the same: several sparkling point. The amount of lucent point rises up in the next power of ten when we reach the disk of our galaxy. One also can discuss how long a "prospective rocket" with speed similar to speed of light would need to reach such stars. Some distances within our galaxy we will fix.

Pollux (in Gemini): $3 \cdot 10^{17}$ m (≈ 34 light years)

Schedir (in Cassiopeia): $2 \cdot 10^{18}$ m (≈ 230 light years)

Polar star: $4 \cdot 10^{18}$ m (≈ 430 light years)

Antares (in Skorpius): $5.7 \cdot 10^{18}$ m (≈ 600 light years)

Beteugeuze (in Orion): $6 \cdot 10^{18}$ m (≈ 680 light years)

Canis Majoris (very big): $4.6 \cdot 10^{19}$ m (≈ 4 900 light years)

Super novo from 1604: $1.9 \cdot 10^{20}$ m (≈ 20 000 light years)

Center of our galaxy: $2.5 \cdot 10^{20}$ m (≈ 26 000 light years)

And outside of our milky way galaxy we find other galaxies and clusters of galaxies as well as super clusters of galaxies. Here are only a few examples with their distance to the earth:

Conis major (nearest galaxy): ≈ $4 \cdot 10^{20}$ m (≈ 42 000 light years)

Andomeda galaxy: $\approx 2.4 \cdot 10^{22}$ m (≈ 2.5 Mill. light years)

Centaurus A (nearest radio galaxy): $\approx 1 \cdot 10^{23}$ m (≈ 10 Mill. light years)

Virgo galaxy cluster (in the mean): $\approx 6.5 \cdot 10^{23}$ m (≈ 65 Mill. light years)

Quasar (quasi stellar object) **Mark 509**: $\approx 4.1 \cdot 10^{24}$ m (≈ 440 Mill. light years)

Quasar High Red Shift: $\approx 1.1 \cdot 10^{26}$ m (≈ 12 Bill. light years)

Furthest away galaxy (known today): $\approx 1.2 \cdot 10^{26}$ m (≈ 13 Bill. light years)

[see e.g. http://www.welt.de/wissenschaft/weltraum/article10428102/
Astronomen-erreichen-das-Ende-des-Universums.html]

In addition to these data of astronomic distances metered from the earth or the sun we can take data about diameters from several astronomic objects:

Diameter of the moon: $\approx 3.5 \cdot 10^{6}$ m

Diameter of the earth: $\approx 1.3 \cdot 10^{7}$ m

Diameter of Jupiter: $\approx 1.4 \cdot 10^{8}$ m

Diameter of the sun: $\approx 1.4 \cdot 10^{9}$ m

Diameter of Pollux: $\approx 1.3 \cdot 10^{10}$ m

Diameter of Antares: $\approx 2.5 \cdot 10^{11}$ m

Diameter of Beteigeuze: $\approx 1.4 \cdot 10^{12}$ m

Diameter of Canis Majoris: $\approx 2.0 \cdot 10^{12}$ m

Major Diameter of comet halley: $\approx 5.3 \cdot 10^{12}$ m

Diameter of the heliosphere: $\approx 2.3 \cdot 10^{13}$ m

Diameter of the whole solar system: $\approx 3.0 \cdot 10^{16}$ m (3.2 light years)

Diameter of our galaxy: $\approx 1 \cdot 10^{21}$ m (100 000 light years)

Diameter of a cluster of galaxies: $\approx 1 \cdot 10^{23}$ m (10 Mill. light years)

Diameter of a galaxy-supercluster: $\approx 2 \cdot 10^{24}$ m (200 Mill. light years)

Diameter of the whole universe: $\approx 1.3 \cdot 10^{26}$ m (14 Bill. light years).
(observable farthest object)

Besides looking at the powers of ten the pupils also should calculate quotients between some distances so that they e.g. can find out that the distances to the nearest stars Proxima Centauri and Alpha Centauri is about 100 000 times of the distance from Earth to the sun and that the distance to the center of our galaxy is 10 000 times bigger than the distance to the Centauris and so on.

So we can find e.g. (including also the named diameters):

Sun-Earth $\xrightarrow{\cdot 100\,000}$ Sun-Centauri $\xrightarrow{\cdot 10\,000}$ Sun-Milkyway center $\xrightarrow{\cdot 1 \text{ Mill.}}$ end of the universe

Diameter of moon $\xrightarrow{\cdot 3.6}$ diameter of earth $\xrightarrow{\cdot 11}$ diameter of Jupiter $\xrightarrow{\cdot 9.7}$ diameter of sun

· 109

· 400

Diameter of sun $\xrightarrow{\cdot 11}$ diameter of Pollux $\xrightarrow{\cdot 19}$ diameter of Antares $\xrightarrow{\cdot 8}$ diameter of Canis

· 1500

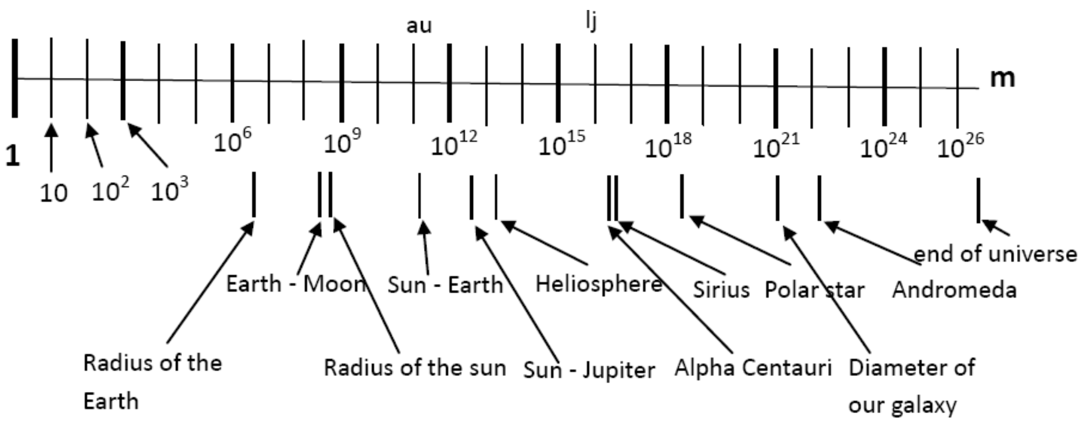
[The star Canis Majoris with its big diameter in the position of our sun would reach beyond Saturn.]

Diameter of the heliosphere $\xrightarrow{\cdot 43 \text{ Mill.}}$ diameter of our galaxy $\xrightarrow{\cdot 70\,000}$ diameter of universe

· 3 Trillion

I think it is very difficult not only for pupils to get an imagination of all these distances. Also a journey with a very fast rocket does it not make better because even with the fastest speed, the speed of light, we have a range from one light year in our solar system to millions of light years. There is no proportional scale we can put on these distances. We only can produce a logarithmic scale in the style of the movie “Powers of Ten”.

In such a scale (as to be seen in the following – but better one uses a scale with a length of about 55 cm) the pupils can put in all the data we got before. This might deepen the recognition of the data.



Final comment

Above all named and possible investigations about astronomic distances one also can look out for numbers of stars one can see in the night without clouds, the estimated number of stars or galaxies in the universe, the amount of visible and dark material in the universe or the mass of different astronomic objects. But here I will end with the given ideas about investigating astronomic measures.

Historical aspects in teaching mathematics

Tünde Kántor
University of Debrecen, Hungary
tkantor@science.unideb.hu

Abstract

In our school practice it is very important to work out mathematical problem fields for secondary school pupils. We have to arouse the pupils' interest in the classroom too. There is growing debate concerning the role of the history of mathematics in mathematics education. Nowadays the new Hungarian mathematics curriculum (NAT) emphasizes the role of history, so we have chosen the topic of the problem field from the history of mathematics, from the geometry of triangles: Viviani's theorem and its generalizations.
We wanted to acquaint our pupils with famous mathematicians (V. Viviani, L. J. Mordell, P. Erdős) and with their mathematical works and results.
On the one hand our main point was to enrich the teaching of mathematics with historical aspects; on the other hand we focused our teaching on the pupils' own heuristic work. We formulated the posed problems as open problems. So pupils can create their own mathematical ideas, similarly to the historical way. They did their problem solving as a research work. We carried out our experiments using the traditional way and by using modern techniques (GeoGebra).
We developed this teaching material for pupils of normal secondary schools (Part I -II.), for students with high ability (Part III- VIII.) and for teachers as guideline. We tested it in five secondary schools (normal and special courses). We applied it on the teacher training track at the university on semester courses (Chapters of Teaching Mathematics), and in-service training of mathematics teachers. Mathematics teachers can benefit from it.

Key words: history of mathematics, teaching of mathematics, material for teaching, Viviani's theorem, problemfield

ZDM classification: A30, C30, G10

Aims by using historical aspects

Viviani's theorem is suitable to illustrate the derivation of a problem field, to build new concepts, to generalize theorems and to investigate the converse problem too. In our teaching material there are easier questions for investigation in classroom not only for the talented pupils, and there are also harder problems aimed at advanced pupils. I intended to improve problem solving and problem posing abilities that are essential for teaching mathematics. It is good to know about mathematical concepts, theorems and proofs, but it is even better to know where they came from and why they are studied.

Which are the benefits of using historical problems?

- We can show the continuity of mathematical concepts and processes over past centuries (Coxeter, 1967; Furinghetti, 2002; Gingyikin, 2003; Honsberger, 1973; Svetz, 2000).
- We motivate learning process in the classroom, because our pupils deal with problems which were objects of investigation centuries ago. These problems allow the pupils to touch distant and recent past (András, 2010; Cofman, 1990, 1991, 1999, 2001, Kántor, 1997; Kronfellner, 1998, 2007).
- Pupils connect mathematics to various cultures and other intellectual developments in science (Gingyikin, 2003; Lévárdi and Sain, 1982; Stewart, 2003).
- We can often learn from the mathematical mistakes and the unsolved problems of the past (Barnett, 2000, Furinghetti and Radford, 2002).
- We have sometimes to include biographies in mathematics classes. The life-stories of mathematicians often encourage talented pupils, and fill them with emotions. They think and believe that if they can solve problems posed by famous mathematicians in their youth - as Paul Erdős - later they may become such great mathematician as Paul Erdős was for example (Erdős, 1993, 1997; Furinghetti and Radford, 2002; Honsberger, 1996; Lévárdi and Sain, 1982).
- Some mathematicians had an interesting life and it is good to know exciting things. It is a good motivation. Galileo Galilei (1564-1642), Vincenzo Viviani (1622-1703), Louis. J. Mordell (1888-1972), Paul Erdős (1913-1996) were such men. We may read passages from books about them, about the *travelling ambassador of mathematics,* in Hoffman's book *Man Who Loved Only Numbers* about P. Erdős or we watch a DVD about the 100th anniversary of his birth. In connection with P. Erdős we can investigate other nice mathematical problems posed by him, or problems from the old series of the Hungarian High School Mathematics and Physics Journal (KöMaL) with the mathematically talented pupils (advanced level). (Babai, 1998; Bollobás 1997; Cofield, 2013; Furinghetti and Radford, 2002; Lévárdi and Sain, 1982).

We followed three stages of problem solving:

1. Encouraging independent investigation.
2. Demonstrating approaches to problem solving.
3. Discussing solutions of famous problems from the past centuries.

We formulated the posed problems as follows: “What happens if…?” “What is your conjecture?” “How can you change the conditions or the way of the proof?” So pupils can create their own mathematical ideas.

The problem field of Viviani’s theorem

Part I. We have chosen the topic of the problem field from the history of mathematics, from the geometry of triangles: Viviani’s theorem and its generalizations.

Viviani’s theorem

For any point *P* inside an equilateral triangle *ABC* the sum *s* of the length of the perpendiculars d_1, d_2, d_3 from the point *P* to the sides is equal to the altitude *h*.

We started with this well known property of the equilateral triangle (Problem 1) and discussed five ways for its proof.

Part II. We changed the position of the point *P*, and we supposed that it lies on a vertex or on a side of the equilateral triangle *ABC*, or outside the triangle, or on the extension of a side. (Problem 2, Problem 3, Problem 4).

Part III. We changed the perpendiculars to other segments (Problem 5, Problem 6 as a complementary problem).

Part IV. We changed the equilateral triangle to an isosceles triangle (Problem 7).

Part V. We generalized the regular triangle to regular *n*-gon (Problem 8, advanced level).

Part VI. We changed the dimension of the equilateral and scalene triangle. We left the plane and went over to space: regular tetrahedron (Problem 9) and its converse theorem.

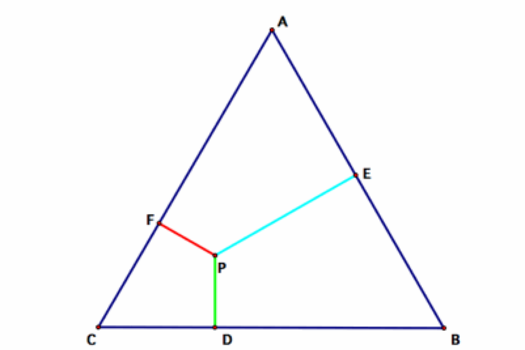
Part VII. Here we collected some theorems only for mathematically gifted students on high level: Generalized inequality of Erdős-Mordell (Problem 10), Barrow’s theorem (Problem 11), generalized Erdős-Mordell inequality for the tetrahedron (Problem 12) and K. Kazarinoff’s inequalities (Problem 13).

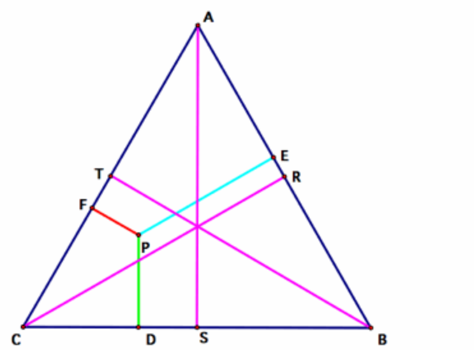
Part VIII. Contest Problems or advanced level Problems (Contest Problems 1- 4). The range of the problems is very wide, from simple exercises (aged 13-14) through contest problems (aged 14-18) to scientific theorems; such as theorem of Erdős-Mordell, Barrow’s theorem, inequality of Kazaniroff (aged 16-19), or on university level Viviani’s four window problem, trisection of an angle by using an equilateral hyperbola, Viviani’s curve, tangent to a cyclois (aged 18-20).

Part I.

Problem 1: Consider an equilateral triangle *ABC* and an arbitrary point *P* inside the triangle *ABC.* Let *PD, PE, PF,* be the segments from *P* perpendicular to the sides *BC, AC, AB,* respectively. Consider the sum $s = PD + PE + PF$. What is your conjecture?

Conjecture: The sum of $PD + PE + PF$ is equal to the length of an altitude of the equilateral triangle**.**





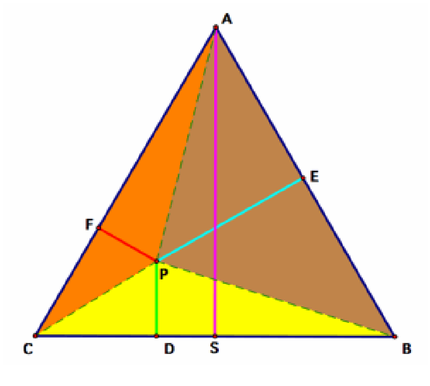
Discussion: First we allowed the pupils to draw, to measure, and to formulate their conjecture and then to prove it theoretically.

We hoped that pupils would recognize that

1. The sum of $PD + PE + PF = s$ is equal to the length of the altitude of the equilateral triangle, i.e., $s = h = \frac{a\sqrt{3}}{2}$, where *h* denotes the length of the altitude of equilateral triangle *ABC* and *a* is the length of its side.
2. The sum *s* is independent from the position of the interior point *P*.

Our pupils found several ways for proving of the conjecture.

Way 1

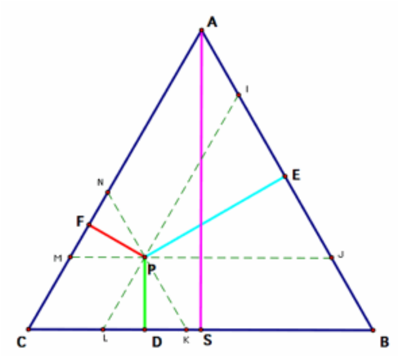


The idea of the proof was that we can write the area of the equilateral triangle *ABC* as a sum of the areas of the triangle *ABP*, triangle *ACP* and triangle *BCP* because each of them has the same base length and the measure of area is additive.

The proof shows that the sum *s* is independent of the position of the interior point *P*.

Discussion: The pedagogical advantage of this method is the fact that the proof uses the concept of area. We should use this proof if we want to enter into the topic of area or if we want to recall or reintroduce the area of an equilateral triangle.

Way 2



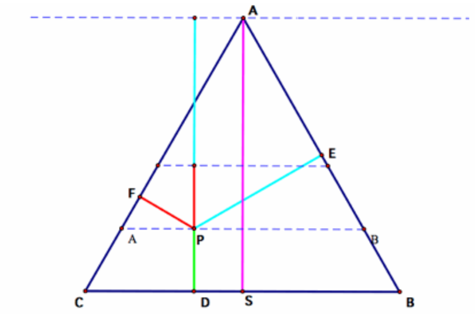
We draw parallel lines to the sides *AC*, *AB*, *CB* through the point *P*.

We get three parallelograms and three small triangles.

We can give the altitude of the triangle *ABC* with the help of the altitudes of small triangles (*PD* + *PF* + *PE* = *AS*).

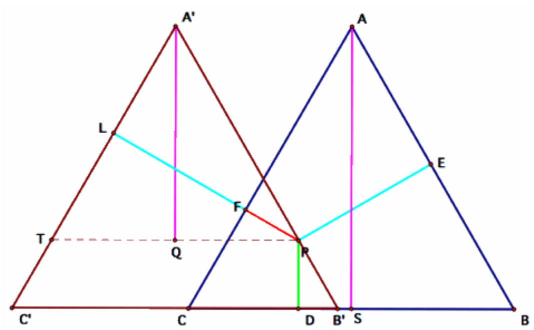
Discussion: Our second way uses two main concepts, similar triangles and their properties, and the properties of parallelograms (*LPK* Δ ≈ *MNP* Δ ≈ *PIJ* Δ ≈ *ABC* Δ*)*. This might be a nice way to review or to remind our pupils of the concepts that they have previously studied.

Way 3



Discussion: The third way utilizes rotation and translation. We rotate segments *PF* and *PE* through 60^0 about point *P* and then we translate the latter vertically by the length of segment *PF*. From this we obtained by aligning that *PD* + *PF* + *PE* = *AS*, which is equal to the length of the altitude of the equilateral triangle *ABC.*

Way 4



We consider a shifted copy *A'B'C'* of triangle *ABC*, such that point *P* lies on *B'A'*. We construct *PT* such that *PT* is parallel to *C'B'*. Let *Q* be the foot of the perpendicular from *A'* onto *PT*, and *L* be the foot of the perpendicular from *P* onto *A'C'*. Then *AS* = *A'Q* + *PD* = *PL* +*PD* = *FL*+ *PF* + *PD* = *PE* + *PF* + *PD.*

(This can be done with GeoGebra too.)

Discussion: Way 3 and **Way 4** use the concept of geometrical transformations. Pupils manipulated with the perpendicular segments, tried forming a straight line, by placing one next to, or on top of, each other, so that they can then be easily compared to the length of our altitude. We think **Way 4** would be the first on we would show a class when discussing the problem, and then ask them to find other ways to prove the same thing. This proof belongs to the category PWW (Proofs Without Words) (Nelsen, Roger B. 1994).

Way 5

Way 5 uses the method of coordinate-geometry and the distance formula to show that the sum of the perpendiculars from *P* is equal to the length of altitude.

Discussion: Way 5 is interesting for lower classes if we use dynamic geometry (GeoGebra). They got immediately the length of these line segments. For higher classes we use the method of coordinate-geometry. In this case the proof requires the knowledge of the distance formula.

Part II.

It is very useful if the secondary school pupils can pose new problems, if they ask new questions after looking over the solved problem (Cofman, 2000; Schupp, 2002).

Viviani's theorem can be generalized and modified in many directions. Our next questions for the discussion were: What happens if in the equilateral triangle *ABC* the point *P* is not located inside the triangle?

From the pupils we got different options for locating the point *P* and formulated new problems. The simplest problems were special extreme cases of Problem 1.

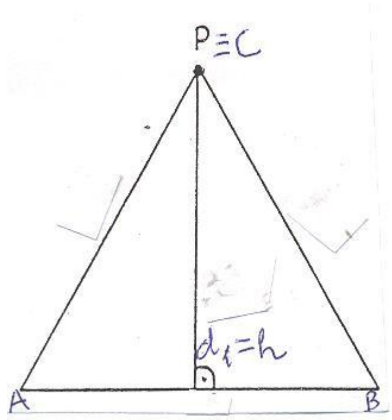
- Point *P* coincides with one of the vertices of the triangle *ABC.*
- Point *P* is placed on one of the sides of the triangle *ABC.*

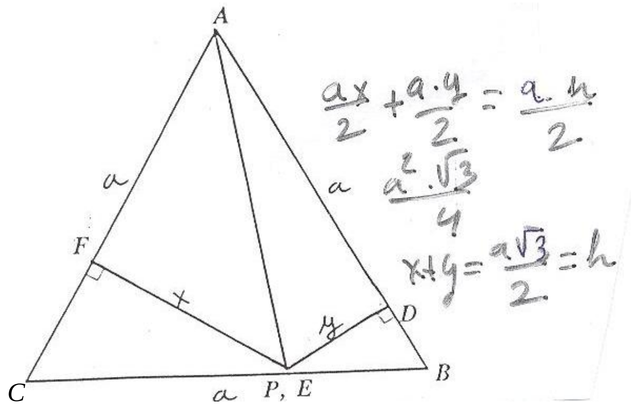
We formulated then following questions as open problems:

Problem 2: What happens with the sum *s* of the distances if in the equilateral triangle *ABC* the point *P* coincides with one of the vertices or it is on one of the sides of the triangle *ABC*?

Discussion: It is obvious that in the first case $s = h$, and in the second case $s = d_1 + d_2 = h$. Both the first and second ways (**Way 1, Way 2**) are suitable in this case.

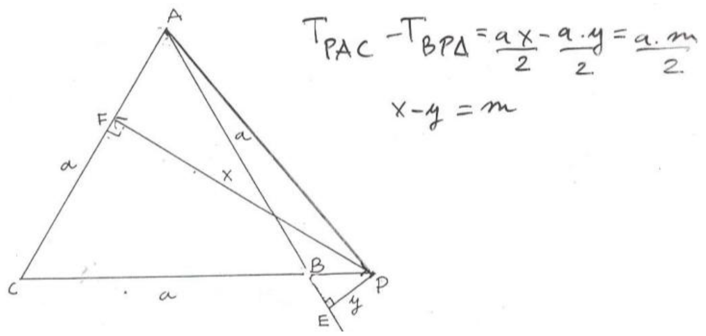
The proofs are obvious from the figures (pupil's work). (They used the notations *x, y* instead of d_1,, d_2.)





Problem 3: What happens if in the equilateral triangle *ABC* point *P* lies on the extension of a side of triangle *ABC*?

Pupil's work: (They used the notations *x*, *y* instead of d_{1},, d_{2}.)



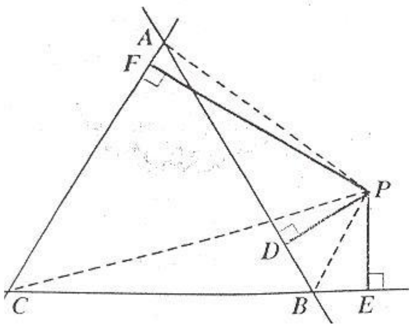
Discussion: They found that $s = d_2 – d_1$ or $s = d_1 - d_2$. The result surprised them, because it depended on the position of the point *P*.

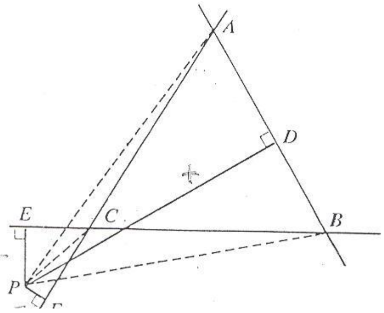
We could unify the two values with the help of the absolute value $s = h = |d_1 – d_2|$.

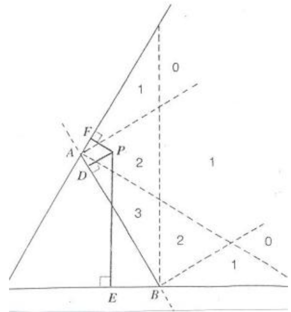
We summarized by $h = k_1d_1 + k_2d_2$ where $k_1 = 1$ and $k_2 = – 1$ or $k_1 = – 1$ and $k_2 = 1$.

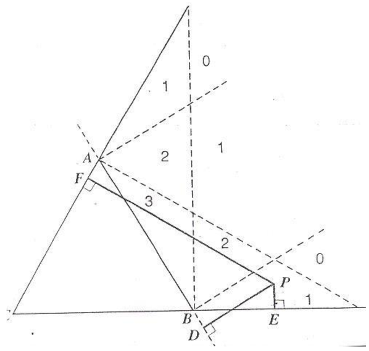
Problem 4: Consider an equilateral triangle *ABC* and a point *P* in the exterior of the triangle *ABC*. How can we express the length of the altitude *h* with the help of the distances from point *P* to the three side-lines?

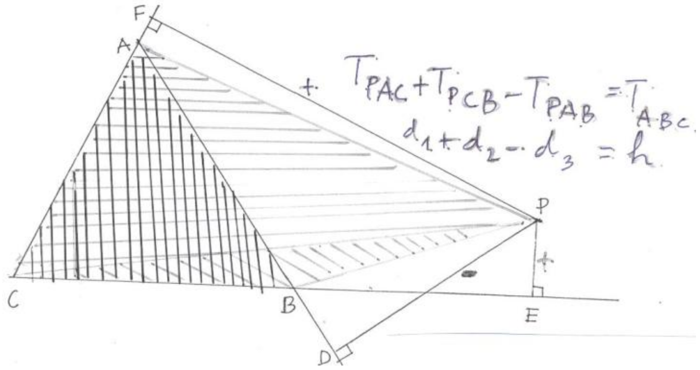
Discussion: Solving Problem 4 seemed to be a bit complicated. The point *P* can be placed outside of the triangle *ABC*. Pupils found different locations of the point *P* and got different results for the linear combinations of the distances. In this case the altitude can be computed by a combination of addition and subtraction of the three distances. Each line containing one of the three sides of the triangle defines two half-planes of the whole plane. One of these half-planes *Π* contains the interior of the given triangle, the other one does not. There must be the sign plus before the front of the distance d_i if *P* belongs to *Π*, otherwise the sign must be minus. We presented the situations on the Figures. For each situation we can do the proof with the help of **Way 1** with computing the areas of the proper triangles.











The pupils recognized that coefficients k_1, k_2, k_3 are equal to ± 1, depending on the location of the point P, but at first they did not know which distances will be positive or negative. Their conjecture was that the decisive factor is whether the foot of the perpendicular intersects a side of the triangle or whether it intersects the extension of the side. But it was easy to show that this conjecture was not true (if the foot points E and D are outside the side CB, and AB respectively, and the foot point F is inside the side AC, then $d_{PF} + d_{PE} + d_{PD} = h$.

By looking at the figures and rethinking the various cases they found that the sign of the distance in the linear combination depends on the relative position of the side to which its perpendicular is drawn and the vertex opposite that side (barycentre coordinates).

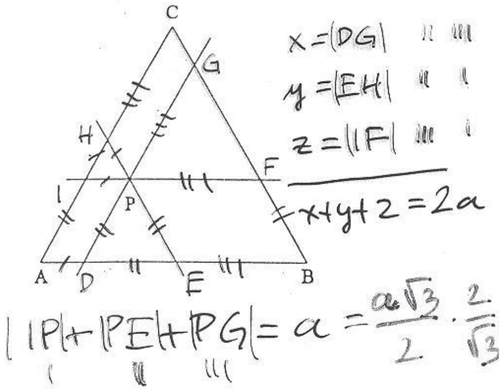
Part III.

If we follow the **Way 2** in the proof of Viviani's theorem we get other variations and new theorems. First we formulated a new open problem.

Problem 5: Consider an equilateral triangle *ABC* and a point *P* inside of the triangle *ABC.* We draw parallels through the point *P* to the sides. These parallels cut out three segments from the triangle *ABC.* What can we say about the sum of these three segments?

Conjecture: The sum of the three segments is constant, $d_1 + d_2 + d_3 = 2a$ (*a* is the length of the side of the equilateral triangle *ABC*.

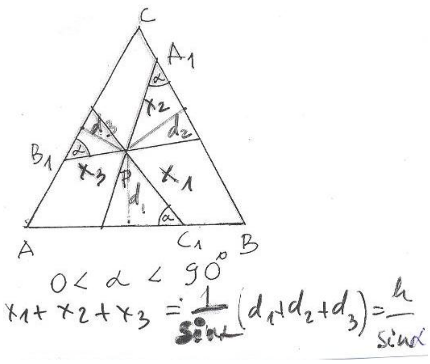
Pupil's solution: (They used the notations *x, y, z* instead of d_1, d_2, d_3).



The segments make an angle of 60^0 to the corresponding side of the equilateral triangle. With variation of the angle we get Problem 6. This examination was homework for the talented pupils.

Problem 6 (complementary problem): Let be given an equilateral triangle *ABC* and a point *P* inside of the triangle. We draw segments through the interior point *P*. These segments make the same angle α ($0 < \alpha < 90^0$) with the corresponding side of the triangle. What can we say about the sum *s* of these segments? Conjecture: s = const.

Pupil's work:



Part IV.

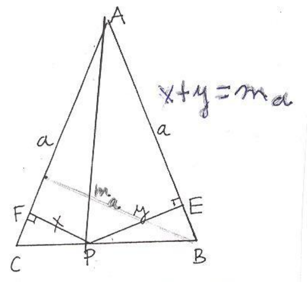
From this point on we made investigations in another direction. We generalised Problem 2.

Problem 7: What can we say about the sum *s* if the point *P* is on the base of an isosceles triangle?

Conjecture: $d_1 + d_2$ = constant.

Discussion: We can prove Problem 7 with the help of **Way 1.** The idea of the proof was that we can write the area of the isosceles triangle *ABC* as a sum of the areas of the triangle *ABP* and triangle *ACP*.

Pupil's work:



Now the question of the converse theorems arises. Which of the discussed problems have a true converse theorem? Can we invert Viviani's theorem? We discussed these problems only with talented pupils of advanced level.

Conjectures: Viviani's theorem has a converse theorem and we can prove it in different ways.

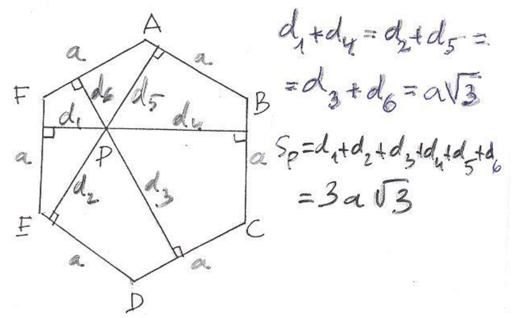
Part V.

We continued our investigations in the direction of regular *n*-gons.

Problem 8: Consider a regular *n*-gon (n > 3) and a point *P* inside of it. What can we say about the sum s of the distances from the point *P* to the sides of the regular *n*-gon?

Conjecture: $s = d_1 + d_2 + \ldots + d_n =$ const.

Pupil's work:



Discussion: The first step was to change the regular triangle to a square, to a regular 5-gon, regular 6-gon (etc.) at finally to a regular *n*-gon. For solving these problems we need the results of Problems 1-4.

We asked the question of converse theorem, but it was easy to find a counter-example in the case of convex *n*-gons a counter-example: if we take a rectangle *ABCD (AB = BC),* we can immediately see that for an arbitrary interior point *P* of the rectangle *ABCD* the sum of the distances from the point *P* to the sides of the rectangle is constant and it is equal to the sum of the length of its two different sides.

At normal high school level we can extend Viviani's theorem to space and we can formulate a similar theorem for a regular tetrahedron.

Part VI.

Problem 9: Consider a regular tetrahedron *ABCD* and a point *P* inside it. Find the sum s of the distances from the point *P* to the four faces. Is the sum $s = d_1 + d_2 + d_3 + d_4$ constant and independent from the position of the interior point *P?*

Discussion: Proof of Problem 9 is similar to the first proof of Problem 1 in the plane. We use the fact that the volume of the regular tetrahedron *ABCD* is equal to the sum of the volumes of the tetrahedra *PABC, PABD, PACD, PCDB* and the value *s* is constant and equal to the length of the tetrahedron's altitude ($s = h = a\sqrt{\frac{2}{3}}$).

We formulated the converse theorem too. The pupils were convinced that it will be true. Their concept was false. The converse theorem of Problem 9 is not true.

From the sum $d_1 + d_2 + d_3 + d_4 = h = \frac{3V}{A^2}$ follows only equality of the areas of triangles *ABC, ABD, ACD* and *BCD,* i.e. the tetrahedron *ABCD* is not necessarily regular, only its faces have equal areas (equipages tetrahedron).

Part VII.

The other parts of the problemfield of Viviani's theorem are difficult; they are only for talented pupils. It is only at the advanced secondary school level or at university in courses of elementary mathematics that we can deal with the theorem of Erdős-Mordell, Barrow's theorem for scalene triangle, with the generalization of Problem 9 to an outside point *P,* with the generalized inequality of Erdős-Mordell for a tetrahedron, Barrow's theorem and Kazarinoff's inequality for a tetrahedron.

Problem 10 (inequality of Erdős–Mordell) (for talented pupils): If *P* is arbitrary point inside or on the boundary of a triangle *ABC* and if d_1, d_2, d_3 are the distances from point *P* to the sides of the triangle then $d_{PA} + d_{PB} + d_{PC} \geq 2\,(d_1 + d_2 + d_3)$, with equality if and only if triangle *ABC* is equilateral and the point *P* is its circumcenter.

Discussion: The first proof of Problem 10 was published in the Hungarian KöMal in1935 by Professor Mordell. A trigonometric solution by *Mordell and Barrow* appeared in the American Monthly in 1937. An elementary proof was published in 1956. In 1957 *Kazarinoff* published an even more elementary proof, based on a theorem of Pappus of Alexandria's Mathematical Collection. It based upon the idea of reflection. We can find *Bankhoff's elementary proof* in the American Monthly (1958, Vol. 65). Barrow proved a generalized form of the

Erdős-Mordell inequality, which follows from Barrow's theorem as a special case. In 2007 *Claudi Alsina and Roger B. Nelsen* gave a visual proof of it. They presented a visual proof of a lemma that reduced the proof to elementary algebra. In our mathematical camps the teachers often discuss the inequality of Erdős-Mordell. (Kubatov, 2012).

Problem 11 (Barrow's theorem) (for talented pupils): Let *P* be an arbitrary interior point of the triangle *ABC*. Prove that $d_{PA} + d_{PB} + d_{PC} \geq 2(d_{PA'} + d_{PB'} + d_{PC'})$, where *A', B', C'* are the intersection point of the bisectors of the angles *BCP, CPA, ABC* with the sides *BC, CA, AB* of the triangle *ABC*.

Discussion: We find the proofs of these problems in several Hungarian mathematical books. The proofs require only elementary knowledge, but the methods are different (principle of reflection, use of congruence transformations, computation of areas, trigonometrical relations). The Erdős-Mordell inequality has a generalization for the tetrahedron.

Problem 12 (D. K. Kazarinoff's inequality for tetrahedra only for talented pupils): Let *P* be an arbitrary interior point of the tetrahedron *ABCD.* Prove that

$$d_{PA} + d_{PB} + d_{PC} + d_{PD} \geq 2\sqrt{2}\,(d_{PA'} + d_{PB'} + d_{PC'} + d_{PD'}),$$

where *A', B', C', D*' are the perpendicular projections of the point *P* to the planes *BCD, CDA, DAB, ABC.*

Discussion: Problem 12 is a generalization of Problem 10. We found that the value of the coefficient in the plane was 2, so we expected that the value of the coefficient in the space will be 3. But it is not true. D. K. Kazarinoff constructed an orthogonal tetrahedron with the coefficient $2\sqrt{2}$.

Problem 13 (for talented pupils): The tetrahedron *ABCD* is equifaced and point *P* its interior point. Prove that

$$d_{PA} + d_{PB} + d_{PC} + d_{PD} \geq 3(d_1 + d_2 + d_3 + d_4)$$

where d_i (i = 1, 2, 3, 4) are distances of *P* to the faces of the tetrahedron. In the case of equality the tetrahedron *ABCD* is regular and the point *P* is the centre of its circumscribed sphere.

Part VIII. Contest Problems

For talented pupils of different age we can find a lot of contest problems. Viviani's theorem and its variants are popular at secondary school contests and in Geometrical Problem books. I collected some of them (aged 13-18). If we prepare our pupils for the contests or for the final examination on advanced level, we have to solve these problems with them.

Contest Problem 1 (aged 13) (Varga Tamás contest, 1994/95): The length of the sides of an equilateral triangle is 5. Let us draw parallel through the interior point *P* to the sides. For which point / points *P* will the sum of these parallel segments be maximal?

Contest Problem 2 (aged 14) **(**Varga Tamás contest, 1997/98): Consider an equilateral triangle and a point *P* inside of the triangle *ABC*. We denote the feet of perpendiculars from the point *P* to the sides *AB, BC, CA* by *D, E, F*. Prove that the value of the fraction $\frac{PD+PE+PF}{BC+AC+AB}$ is independent of the position of the point *P*.

Contest Problem 3 (aged 14) (Hungarian Geometrical Problem Book): Consider an equilateral triangle and a point *P* inside of the triangle. The feet of the perpendiculars to the sides divide each side into two segments. Prove that the sum of the non-joint 3 segments is independent of the location of the point *P*.

Contest Problem 4 (aged 17-18) (final examination problem, 1993): The point *P* is an interior point of a regular tetrahedron. From this point *P* we draw perpendiculars intersect the plane of the face-sides at the points *X, Y,* Z. Prove that the sum $d_{PX} + d_{PY} + d_{PZ}$ is independent of the location of the interior point.

Summary

We carried out experiments using history of mathematics in the classroom. We started with the historical Problem of Viviani, we dealt with an important property of the regular triangle and with variations of the conditions we solved a lot of problems, from the simple ones to the serious scientific problems. So our pupils have been all over the stages of the history of mathematics, from the ancient time to our age.

We observed the same results which Furinghetti and Radford (2002), András (2010) and Lawrence (2010) found. There is a parallel between history and pupils' individual learning process. The development of concepts and proofs followed the historical sequence. Their solutions and mistakes were similar to the old ones.

For the average pupils we made mathematic classes more enjoyable by telling them the life-stories of famous mathematicians (V. Viviani, P. Erdős, L. Mordell).

References

1. Alsina, C., Roger B. Nelsen.: A visual proof of the Erdős-Mordell inequality, Forum Geometricum Vol.7 (2007) 99-102.

2. András, Sz.: Inquiry Based Teaching of Mathematics (Primas Project), (Hungarian), Státus, Miercurea Ciuc 2010.
3. Babai L., Pomerance C., Vértesi P.: The Mathematics of Paul Erdős, Notices, MAS, Vol. 45, No 1 (1998) 19-31.
4. Barnett. J. H.: Anomalies and the Development of Mathematical Understanding, MAA Notes 51, 2000, 77-88.
5. Bollobás B.: Paul Erdős –life and work in: The Mathematics of Paul Erdős, vol. 1(Eds. R.L. Graham et al.) Algorithms and Combinatorics 13, Springer Verlag, Berlin-Heidelberg-New York, 1997, 1- 41.
6. Bos, Henk J. M.: Lectures in the History of Mathematics, American Mathematical Society, London Mathematical Society, 1993.
7. Boyer, B. C.: A History of Mathematics, Wiley, 1968.
8. Cantor, M.: Vorlesungen über Geschichte der Mathematik Vol. 3 (1894-1896).
9. Cofield C.: An Arbitrary Number of Years Since Mathematician Paul Erdős's Birth, Scientific American March, 26, 2013.
10. Cofman, J.: What to solve? Clarendon Press, Oxford University Press, 1990.
11. Cofman, J.: Numbers and shapes revisited. Clarendon Press, Oxford University Press, 1995.
12. Cofman, J.: How to motivate 10-18 years old pupils to work indepently on solving mathematical problems The Teaching of mathematics, Vol. III. 2, 2000, 83-94
13. Cofman, J.: Insight into the history of mathematics I-II. (German: Einblicke in die Geschichte der Mathematik I-II.) Spektrum, Heidelberg, 1999, 2001.
14. Coxeter H.S.M., Greitzer, S.L.: Geometry Revisited Yale University, 1967 (Hungarian)
15. Erdős P.: Some of my favourite problems (Hungarian) Polygon Vol. III. (1993) No 2. 65-67.
16. Erdős P.: Some of my favourite problems and results, in: The Mathematics of Paul Erdős, vol. 1(Eds. R.L. Graham et al.) Algorithms and Combinatorics 13, Springer Verlag, Berlin-Heidelberg-New York, 1997, 47-67.
17. Eves, H.: An introduction to the History of Mathematics, Saunders College Publishing, 1983.
18. Furinghetti F., Radford L.: Historical Conceptual Developments and the Teaching of Mathematics: from Phylogenesis and Onthogenesis, Theory to Classroom Practice, in: Handbook of International Research in Mathematics Education (Ed.: Lyn D. English) Lawrence Erlbaum Assosiates, Publishers, Mahwah, New Jersey, 2002, p. 631-654.
19. Gingyikin, S.G.: Stories about Physicians and Mathematicians (Hungarian), Typotex, Budapest, 2003.
20. Grugnetti, L.: The History of Mathematics and its Influence on Pedagogical Problems, MAA Notes 51, 2000, 29-35.
21. Horvay, K., Reiman I.: Geometrical Problem book I. (Hungarian) Budapest, 2004.
22. Honsberger, R.: Mathematical Gems, MAA, 1973, USA.
23. Honsberger, R.: From Erdős to Kiev, Problems of Olympiad Caliber, MAA 1996, USA.

24. Kántor T.: History of Mathematics in the Secondary School (Hungarian). Matematika tanárképzés-matematika-tanártovábbképzés, Varga Tamás Napok előadásai, Calibra Kiadó 1997, 51-68.

25. Katz, V. (ed.): Using History to Teach Mathematics: An International Perspective, MAA Notes 51, 2000.

26. Kazarinoff, N.D.: Geometric inequalities, MAA 1961, USA.

27. Kronfellner, M.: Historische Aspekte im Mathematikunterricht, Eine didaktische Analyse mit unterrichtspezifieschenh Beispielen. Verlag Hölder-Pichler-Tempsky, Wien, 1998.

28. Kronfellner, M.: Historical Aspects in Calculus Teaching. Matematika v škole dnes a zajtra, Zborník 8. ročníka konferencie s medzinarodnou účast'ou (2007), Ped. Fakulta Katolíckej University v Ružomberku Katedra Matematiky, p 39-48..

29. Kubatov A.: The Erdős-Mordell theorem (Hungarian) http:// matek.fazekas. hu / portal/tanitasianyagok.

30. Lakatos, I.: Proofs and Refutations, Cambridge University Press, 1976 (Hungarian translation, 1998, Typotex, Budapest)

31. Lawrence, S.: History of Mathematics in mathematics classroom. Practical mathematics from long ago- original sources in mathematics classroom. Lecture on History of Mathematics and Teaching of Mathematics Conference, Szeged, 2010.

32. Lévárdi, L., Sain, M.: Problems from History of Mathematics (Hungarian), Tankönyvkiadó, Budapest, 1982.

33. Nelsen, Roger B.: Proofs Without Words, MA, 1994.

34. O' Connor J. J., Robertson E. F.: Louis Joel Mordell, http:// www-history. mcs.st.andrews.ac.uk / Biographies/ Mordell.html

35. Schupp, H.: Thema mit Variationen, Verlag Franzbecker, Hildesheim, Berlin, 2002

36. Shklarsky, D.Q., Chentzow, N.N., Yaglom, I. M.: The USSR Olympiad Problem Book, Freeman, San Francisco and London, 1962.

37. Stewart, J.: Preface in: Mankiewicz: History of Mathematics (Hungarian Translation, HVG Publisher, Budapest, 2003)

38. Svetz, F.J: Problem Solving from the History of Mathematics, MAA Notes 51, 2000, 59-65.

39. Temple G.: 100 Years of Mathematics, Duckworth, 1981.

40. Zimmermann, B.: Interim Report on a Project to identify and foster mathematically gifted students. Problem solving- a World View, Proceeding of a Problem Solving Theme Group, ICME 5, (Ed.: Burkhardt, H., Groves S., Schoenfeld A., Stacey K.) Adelaide, 1984.

41. Zimmermann, B.: History of mathematical thinking and problem solving: possible gain for modern mathematics instruction, Problem Solving in Mathematics Education, Proccedings of the 10th ProMath Conference (Ed.: L. Burman). Publications from the faculty of Education, Åbo University No27/2009, 37-49.

How can high school students solve problems based on the concept of area measurement?

Eszter Herendiné-Kónya
University of Debrecen, Hungary
eszter.konya@science.unideb.hu

Abstract

Mathematical activity like problem solving involves work with concepts. Referring to the work of Usiskin, understanding of a concept in mathematics has different aspects. These aspects are skill-algorithm understanding, property-proof understanding, use-application understanding, representation-metaphor understanding and history-cultural understanding.
The aim of this study is to investigate how these aspects appear in problems based on the concept area measurement, and to look for the reasons of unsuccessful problem solving connected to this topic. The research reports the typical solving strategies and thinking mistakes concerning 5 tasks developed for high school students of class 9.

Key words: area measurement, concept formation, dimensions of understanding, problem solving strategy

ZDM classification: D53, D73, G33

Introduction

The term mathematical understanding includes understanding both of concepts and of problems.
Mathematics as an activity "consists of concepts and problems or questions: mathematicians employ and invent concepts to answer questions and problems; mathematicians pose questions and problems to delineate concepts." (Usiskin, 2012)
We investigate the interaction between concept-formation and problem-solving. Problem-solving requires preliminary knowledge and understanding of concepts connected to the problem: „Do you understand all the words used in stating the problem?" (Polya, 1957) On the other hand the level of understanding of a concept is recognizable through solving problems connected to this concept. The analysis of the process of problem-solving contributes to the exploration of the concept deficiencies.
The aim of this study is to answer the question how the different aspects of understanding the

same object namely the concept of area measurement appear in problems, and to look for the reason of unsuccessful problem solving connected to this topic.

Theoretical background

Skemp (1976) distinguish two meanings of the world “understanding”. The relational understanding means knowing what to do and why, and the instrumental understanding means knowing and using rules. Usiskin (2012) views this meanings as different aspects or dimension of understanding the same subject furthermore he speaks about more than two aspects.

The dimensions of understanding according to Usiskin:

1. Skill-algorithm dimension (instrumental or procedural understanding): Knowing how to get an answer. Obtaining the correct answer in an efficient manner.
2. Property-proof dimension: Knowing why your way of obtaining the answer worked.
3. Use-application dimension (modeling): Knowing when to do something.
4. Representation-metaphor dimension: Knowing represent the concept in some way (with concrete object, picture or metaphor)
5. History-culture dimension (genetic approach, ethno mathematics): Knowing the history of the concept and its treatment in different cultures.

„The dimensions of understanding are relatively independent in the sense that they can be, and are often, learned in isolation from each other, and no particular dimension need precede any of the others … .Ordering ideas or concepts in terms of difficulty is only appropriate if these items are in the same dimension.” (Usiskin, 2012)

In my paper I apply Usiskin’s multidimensional view of understanding, because it helps to clarify the meaning of concepts and broadens options for developing these concepts.

After studying his examples for clarification the meaning of these dimensions, I delineate the first four dimensions of understanding the concept of area measurement.

1. Skill-algorithm understanding is choosing an appropriate algorithm to calculate the area depending on the plane figure and the given sizes.
2. Property-proof understanding includes derivations of the basic formulas for the areas of triangles and other polygons, relations between area and perimeter of the same figure etc.
3. Use-application understanding includes area measurement in everyday life, applications in complex problems etc.
4. Representation-metaphor understanding includes area measurement with congruent tiles, cutting and rearranging polygons, area representation with an array of dots etc.

If we would like to know the level of learners understanding concerning to a concrete concept, it is necessary to pose them problems connected to this concept. In many cases the simple routine tasks can't highlight the gaps and misunderstandings well. So the process of problem solving serves for detection of typical mistakes of the concept-formation too.

In his famous work *How to solve it?* Polya write down the main four steps of problem solving: understanding the problem; devising a plan; carrying out the plan; looking back. If somebody want to devise a plan, it is necessary – after understanding the problem – to activate his or her preliminary knowledge. The main point of the plan is to find the connection between the problem and the appropriate elements of the preliminary knowledge.

A well-stocked and well-organized store of knowledge is an important asset of the problem solver … . In any subject matter there are some key-facts which should be stored somehow in the forefront of your memory. When you are starting a problem, you should have some key facts around you close at hand, just as an expert workman lays out his most frequently used tools around him when he starts working." (Polya, 1962)

Making and carrying out a good plan also means to find the connections between the unknown and the givens. Polya use a graphical representation to demonstrate these connections, where the unknown (what we are looking for); the data (what we have); the auxiliary unknown (what we are looking for to solve an appropriate related problem) are symbolized as points, and the relations connecting the objects are indicated by lines (*Figure 1*).

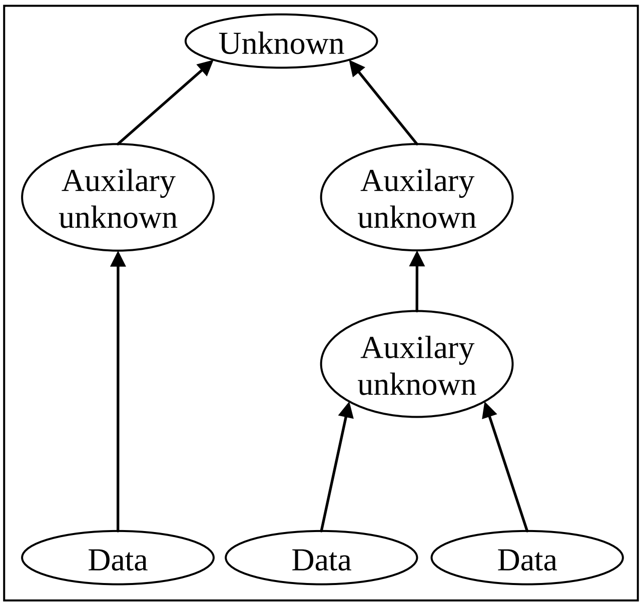


Figure 1

If we can't make direct relation between the unknown and data (we can't solve the proposed problem), we introduce auxiliary unknowns and join these to the original one and so indicate the relation between the quantities. The aim is to establish direct or indirect connection be-

tween the unknown and the data through some auxiliary unknowns. The graphical representation seems to be a suitable tool for investigating learners' different solving strategies.
There are many studies on solving problems with focus on area measurement. The researchers agree that measurement of area is more complex than measurement of length since length is directly measured by ruler while area is indirectly measured through the lengths appearing in the formula for calculating it (Zacharos 2006, Murphy 2009). According to Santi&Sbaragli (2007) the early use of formulas has been criticized on the grounds that it generates misconceptions about area measurement and it could bring children to confuse area and perimeter. Another possible cause of the conflict perimeter vs. area that the intuitive rule "more A, more B" (Stavy&Tirosh 1996) means in case of geometrical shapes that as the perimeter increases so the area increase. Kospentaris at al. (2011) investigates students' strategies in area conservation geometrical tasks and highlights the role of visual estimation.

Research questions

In this research I investigate the understanding of the concept of area measurement through the solution of five associated problems.

1. Which dimensions of understanding the problems refer to? How perform students in class 9 regarding to these dimensions?
2. What kinds of methods use the students if they haven't got the preliminary knowledge necessary to solve the problem?

Methodology

The 27 students participating in this study are 9th graders in a Hungarian high school, in the same class. The class is a special „language-class", they have mainly English lessons in the whole school year and only two mathematics lessons per week. The goal of this year is to preserve their previous mathematical knowledge. The students involved in this study had varied backgrounds in mathematics. In the previous school year they learned in different upper secondary schools. Students don't show up particular motivation, talent and interest in mathematics.
A 45-minute written test was designed in a way that the 5 tasks are built on each other, because we wanted to help students to find a right idea for the solution. For example Task 1 can be understood as an auxiliary problem of Task 2. The solution of the Task 1 may become the part of the solution of the Task 2 and it may suggest the direction in which students should start working. It was considered to be also important to have a real-size picture to every task.

The test was written in March of 2013, earlier in the high school nothing has been taught from the topic of area measurement.

Analysis of students answers

In connection with the first research question I present and identify the area-tasks with dimensions of understanding, then discuss on the students' responses.

Task 1

Determine the area of the following plane figures!

Figure 2

The dimensions of understanding:

Task 1a requires only skill-algorithm understanding in a very simple way.

Task 1b requires three dimensions: representation-metaphor understanding, since the area means covering with tiles; use-application understanding, namely using the formula for the area of the rectangle and skill-algorithm understanding that is breaking down the irregular polygon into pieces that are easier to manage, then add or subtract the areas of the pieces.

Table 1a and *Table 1b* show the distributions of the answers of students:

A = a·a	$A = a^2$	A = a·b	A = 4·4	A = 4·a	A = a·b·2	T = 4·4·a	----
8	7	5	3	1	1	1	1
correct: 23				wrong: 4			

Table 1a

Remark:

26 students used some formula to calculate the area of the square, 23 of them were correct. 1 student couldn't answer.

addition of areas of pieces	subtraction of areas of 2 squares	A=12	A=a·b·2	A=a·b	multiplication of sides	perimeter	A=48 or A=16	----
9	5	1	1	1	1	1	2	6
correct: 15			wrong: 12					

Table 1b

Remarks:

From the 15 correct answers 9 students added and 5 subtracted the areas of known pieces i.e. squares or rectangles and 1 student answered well without reasoning. Between the wrong calculations appeared the multiplication or addition of all the sides (1-1 students).

> *Task 2*
>
> *Determine the area of the colored shape, if the length of the square side is 6 cm!*

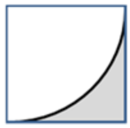


Figure 3

Task 1b is an auxiliary problem of this task especially if somebody solves it applying subtraction.

The dimensions of understanding:

Representation-metaphor us.: the same as in *Task 1b*.

Use-application us.: using the formula for the area of the square and the circle

Skill-algorithm us.: the same as in *Task 1b*.

The expected steps of solution: 1. calculation of the area of the square (A_S); 2. calculation of the area of a quarter of a circle (A_{QC}); 3. the area of the colored shape (A) is A_S-A_{QC}.

Table 2 shows the number of students related to the steps of solution:

subtraction $A=A_S-A_{QC}$	$A_S=a^2$	$A_{QC}=A_C/4$	$A_C=a^2\pi$	$A\approx A_S/4$, $A\approx A_S/3$, $A\approx A_S/2$	-----
3	13	1	0	**7**	12

Table 2

Remarks:

Nobody gave correct answer. 13 students determined the area of the square, but only 3 students indicated the subtraction of the areas. 1 student recognized that the colored shape is a quarter of a circle (*Figure 4*); however he didn't remember the formula of the area of a circle (A_C). There were 7 students who used visual estimation to determine the area *A*. 12 student didn't deal with this task.

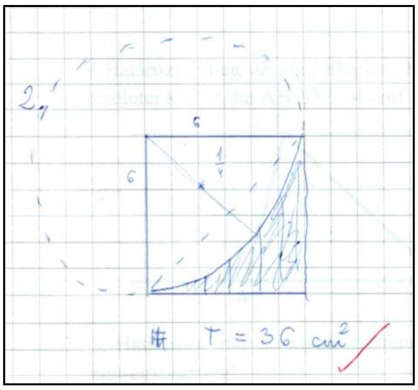


Figure 4

Task 3

Calculate the area of the attached polygon! (Numbers are in cm.)

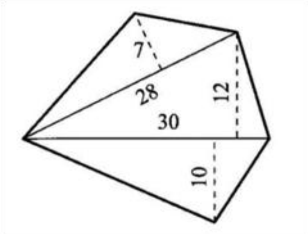


Figure 5

The dimensions of understanding:

Skill-algorithm us.: calculating the area of an irregular polygon (cutting into triangles, calculating the areas, then adding them).

Use-application us.: using the formula for the area of the triangle.

Representation-metaphor us.: recognizing the necessary data (side and the corresponding height) to use the formula.

Table 3 shows the performance of the students.

correct answer	see 3 triangles on the picture	determine sides of pentagon	add all the data	-----
5 students	3 students	3 students	1 student	15

Table 3

Remarks:

15 students (more then the half of them) couldn’t work on this task. There were only 5 correct answer, and further 3 students saw the 3 triangles as useful parts of the irregular pentagon. 3 students determined or tryed to determine the sides of the pentagon, for example *Levente*, who calculated the lenghts of the missing sides with a „formula” $ac=b^2$ (*Figure 6*).

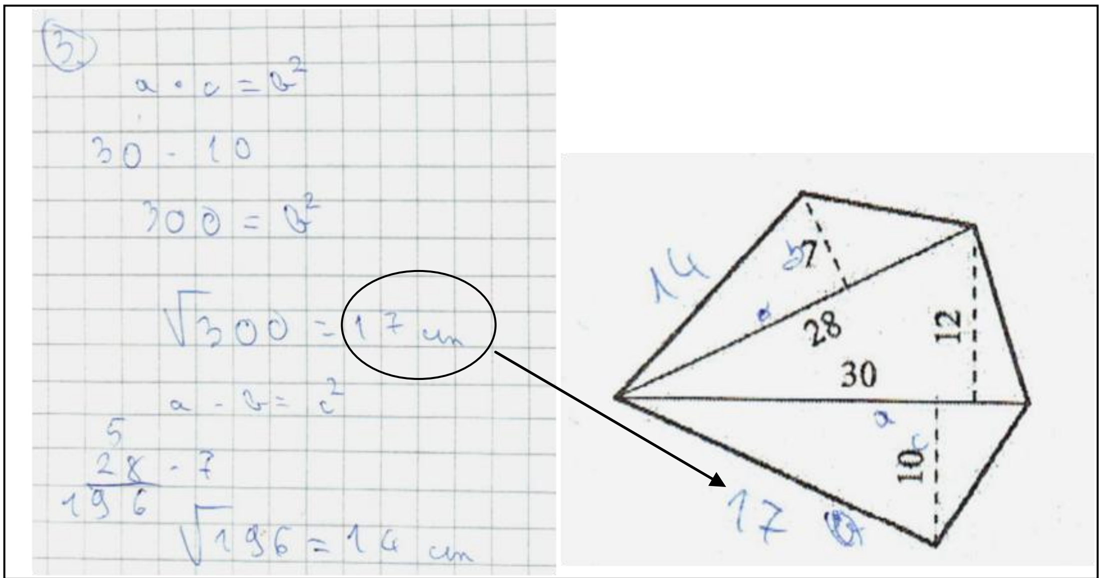


Figure 6

Task 4

Determine the relation between the areas of triangles ABC and DEF if AB=DE=4 cm!

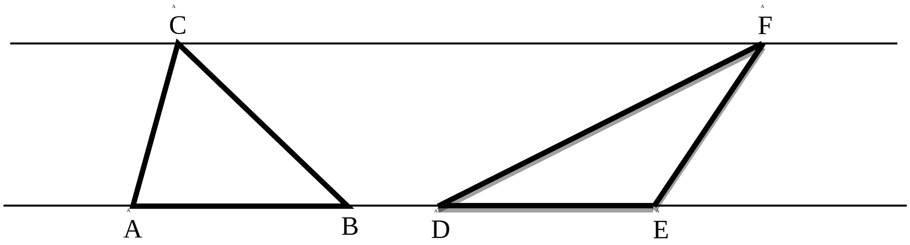


Figure 7

The dimensions of understanding:

Property-proof us.: What is the formula for triangles and why? What are the conditions of area conservation?

Representation-metaphor us.: pictorial understanding of the concepts of area, recognizing base and corresponding height in the picture. In this case cutting and rearranging one triangle into other is not easy.

The skill-algorithm understanding doesn't work, because we know one side exactly, but the height not.

Table 4 shows the performance of our students.

$A_{ABC\Delta}=A_{DEF\Delta}$ (without reasoning)	$A_{ABC\Delta}<A_{DEF\Delta}$ (without reasoning)	determined the length of sides	compared triangles to parallelograms	---
4 students	8 students	10 students	1 student	13

Table 4

Remarks

4 students answered that the areas are equal ($A_{ABC\Delta}=A_{DEF\Delta}$) but nobody gave correct reasoning. Only one solution (*Figure 8*) indicated the property-proof understanding: *Evelin* completed triangles into parallelograms however the statement that the areas are equal was missing.

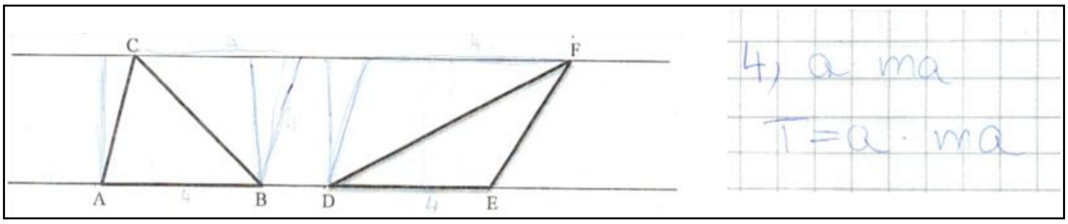


Figure 8

8 students noticed that the area of the triangle *DEF* is larger than the area of the triangle *ABC* ($A_{ABC\Delta}<A_{DEF\Delta}$). 10 students tried to calculate the length of the 3 sides to determine the areas of triangles.

This task is a typical area invariance problem concerning triangles with the same base and equal heights between parallel lines (Kospentaris at all 2011). Kospentaris, Spyrou and Lappas investigated the solutions of 12th graders. 47.3% of the student gave correct answer, but only 8% of them wrote a correct argument. Among the wrong answers 77% noticed that triangle with obtuse vertex has larger area. The explanations, if there were any, corresponded to visual estimations or false reasoning, like "smaller sides, smaller area".

Task 5

Determine the area of the triangle and quadrilateral, if the distance of two adjacent grid-points is 1 unit!

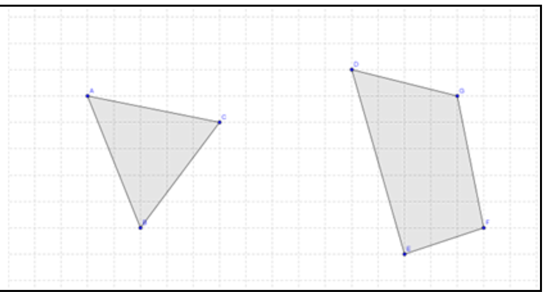


Figure 9

The dimensions of understanding:

Skill-algorithm us.: calculating area with the help of grid; completing the polygon into rectangle then subtracting the calculated area of the right triangles and square.

Representation-metaphor us: the area means covering tiles (squares).

Use-application us.: Using the formula of the area of right triangle and rectangle.

Table 5 shows the performance of the students.

completed into rectangle	determined the number of units (squares)	determined the length of sides	----
5 students	4 students	12 students	8 students

Table 5

Remarks:

Task 1b and *2* are related to this task, but here the additional rectangles aren't drawn, only the grid.

Nobody solved the task correctly. 4 students counted the number of units approximately, so they gave a quite good estimation for the area of the polygons. 5 students drawn the smallest rectangle which enclosed the triangle or quadrilateral, but they didn't continue the work, and didn't determine the area of the complemented triangles. 12 students tried to determine the lengthy of sides again, and gave "formula" to calculate the area from the sides. For example *Anna* thought that the area of every polygon is calculated by multiplication of its sides (*Figure 10*).

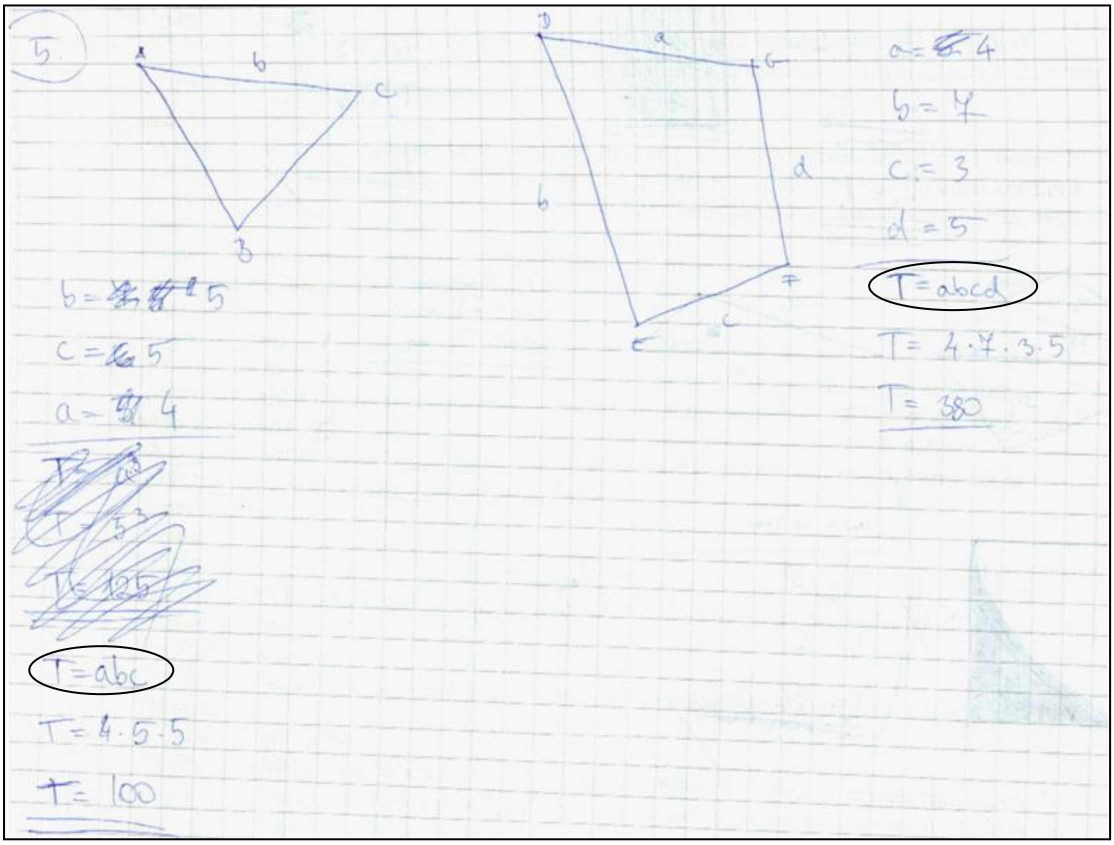


Figure 10

Solving strategies if the preliminary knowledge is missing

“If you are familiar with the domain to which your problem belongs, you know its “key facts”, the facts you had most opportunity to use.” (Polya, 1962) However the appropriate elements of our formerly acquired knowledge are missing, we have to use alternative solving methods. I adopt Polya’s diagram, a graphical tool to describe students solving strategies.

I give two examples to present two different solving strategies in both cases.

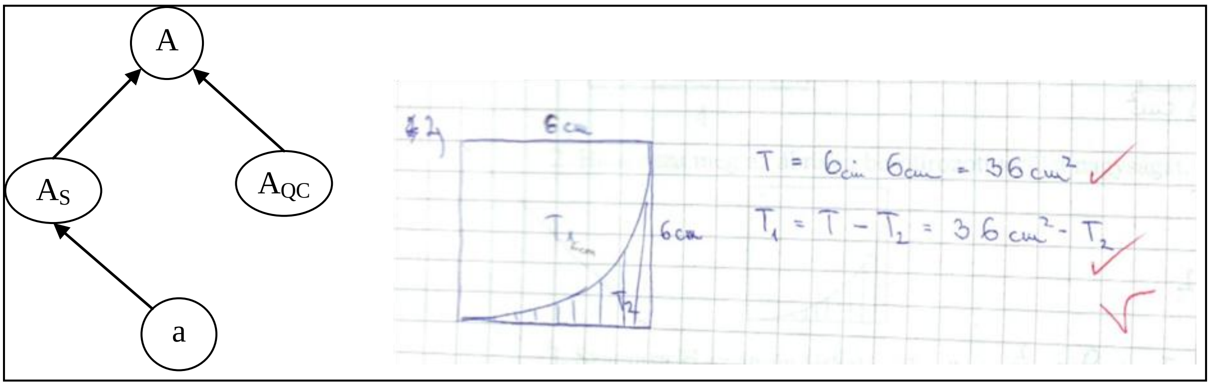


Figure 11

In *Task 2* we have the problem to find the area of the colored figure. The area of an irregular shape can be computed as the difference of areas of shapes we are more familiar with. In this task we have to subtract the area of a quarter of a circle (A_{QC}) from the area of the square (A_S) if the side of the square (a) is given. We can see in *Dominika*'s solution, that the relation between a and A_{QC} is missing, so at first she gave up the work (*Figure 11*).

Dominika tried to solve the problem once again: Instead of the missing connection between a and A_{QC} she used **visual estimation** (*Figure 12*). She recognized that the area of the triangle *DBC* is the quarter of the area of the square ($A_{DBC}=9\ cm^2$). She assumed that the unknown area is the double of the half of this area, so $A=9\ cm^2$. It's an acceptable approximation for A.

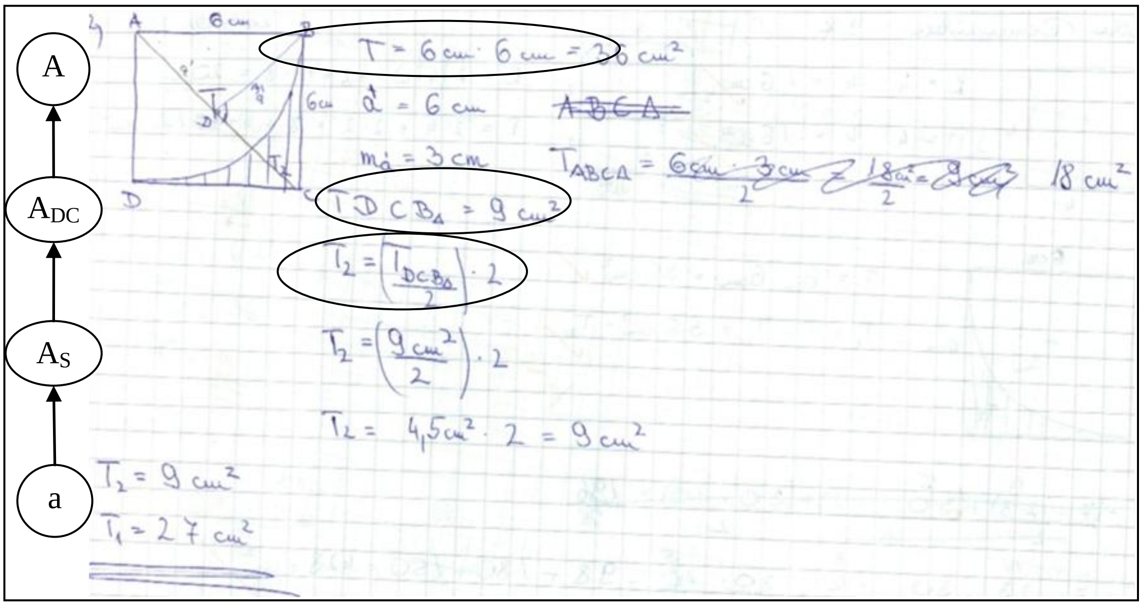


Figure 12

In *Task 4* we have to recognize the equality of the areas of the two triangles. The "key facts" of the solution is to realize that the parallel lines make sure the equality of the heights: $d=h_1=h_2$. *Dominika* couldn't use this visual data, so there are missing lines on the diagram describing her solving strategy. (*Figure 14*)

She tried to overlap the triangles and ***assumed special properties*** incorrectly instead: *ABC* is equilateral triangle (1); *DEF* is isosceles triangle (2). She interpreted the concept of „base and height" not a flexible way, because she wanted to calculate the height starting from the obtuse-vertex (*Figure 13*).

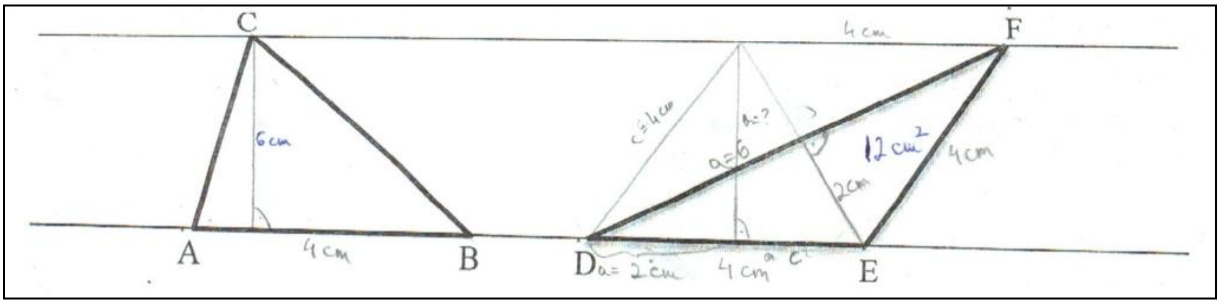


Figure 13

Following *Dominika*’s solution we can see that the diagram is much more complicated, because three new points (*1, 2, DF*) and seven new lines appear (*Figure 15*).

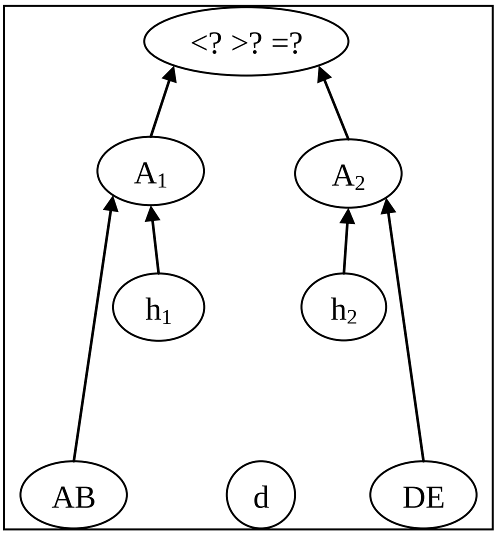


Figure 14

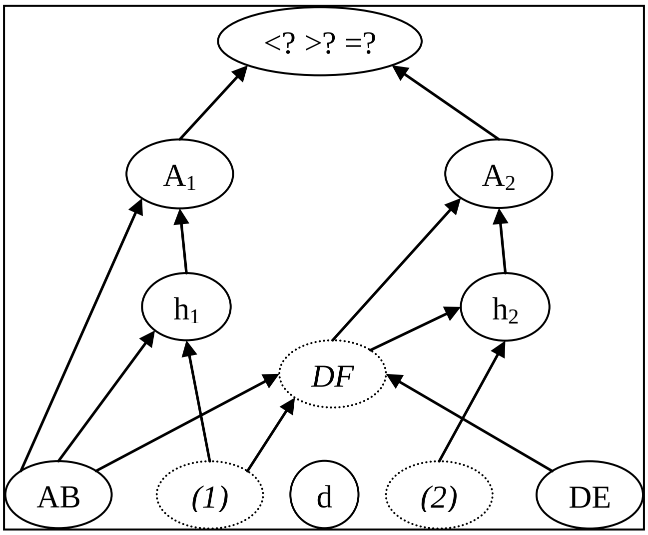


Figure 15

Conclusion

We can examine all the 4 dimensions of the understanding of the concept of area measurement in simple and in complex problems too. We find the representation-metaphor understanding in many problems, which is related to the correct concept image of the area. However the property-proof understanding appears only in a few cases.

Problem-solving is useful to examine the level of the concept formation especially in the cases where the necessary preliminary knowledge is in the long term memory. There are missing preliminary knowledge by most students. They often replace these concepts with false analogy, false assumptions or visual estimation as one can see well from the diagrams based on their solving strategies.

Studied the problem solving process of students we detected some frequent mistakes according to the concept of area:

1) The area of a triangle is connected to the length of sides.
2) Triangles with larger perimeter have larger area.
3) The formula for the area and perimeter of a rectangle is well-known, however the students expend it in an inadequate way, for example multiply all the sides of a polygon to determine its area.
4) In obtuse angled triangle the „base” has to be the side opposite to the vertex of obtuse angle.

The detected mistakes in points 1) and 2) are in keeping with findings of researchers mentioned in the theoretical background section, however they in points 3) and 4) are not. I'm convinced that the analysis of further problems posing for students makes the process of concept formation more clearly.

References

Kospentaris, G. Spyrou, P. Lappas, D. (2011). Exploring students' strategies in area conversation geometrical tasks. Educational Studies 77, 105-127.

Murphy, C. (2010). The role of subject knowledge in primary student teachers' approaches to teaching the topic of area. Proceedings CERME 6, 1821-1830.

Polya, G. (1957). How to solve it? – A new aspect of mathematical method. Princeton, NJ: Princeton University Press.

Polya, G. (1962). Mathematical discovery: On understanding, learning and teaching problem solving. Vol. I and II. John Wiley & Sons, New York.

Skemp, R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching, 77, 20-26.

Stavy, R. Tirosh, D. (1996). Intuitive rules in science and mathematics: the case of 'more of A-more of B'. International Journal of Science Education, 18(6), 653-667.

Tall, D. Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. Educational Studies in Mathematics, 12, 151–169.

Usiskin, Z. (2012). What does it mean to understand some mathematics? paper for the ICME-12, Seoul.

Zacharos, K. (2006). Prevailing educational practices for area measurement and students' failure in measuring areas. Journal of Mathematical Behavior, 25, 224-239.

Talent or Something Else? Preservice Teachers on Mathematical (Problem Solving) Abilities: Implications for Professional Development

Ana Kuzle

University of Paderborn, Germany

akuzle@math.upb.de

Abstract

Problems have an individual component allowing learning opportunities for both low- and high-achieving students. Developing problem solving abilities for all students has become a goal of various professional organizations and reformed curricula (Connected Mathematics Project, Core-Plus Mathematics Project, NCTM, etc.). According to these documents, problem solving encompasses working on given problems individually, posing new problems and solving them using different heuristic strategies, principles and tools, reflecting on the outcome as means of building new mathematical knowledge, and developing reasoning flexibility. However, what abilities one draws on when problem solving? Pólya, though he had reflected throughout his long life on the question of how he and others do mathematics, in his interview with Kilpatrick in 1978 showed, that he had not given much thought as to what abilities are needed. This question was given to a group of preservice teachers who were enrolled in a problem solving content seminar. In this paper, I report on preservice teachers' views on (problem solving) abilities of a mathematically competent person. The data also revealed that held views on problem solving abilities may influence if and how they implement problem solving in their future classroom as well as what type of problems they will use to reach students of different mathematical aptitude.

As a result of research reported here, a concept for a professional development for teachers in problem solving developed within the German Center for Mathematics Teacher Education (DZLM, www.dzlm.de) is presented. Specifically, design of a wide range of learning activities for mathematics teachers is considered in order to achieve the vision of problem solving as culture and standard of mathematics classroom for all.

Key words: problem solving ability, professional development, teacher education, beliefs

ZDM classification: C30, D50

Introduction

Mathematical problems have been central in the mathematics school curriculum since antiquity, but since the 1980s, mathematics educators have agreed upon the idea of developing problem-solving ability, and problem solving has become a focus of mathematics education as a means of teaching curricular material and seeking the goals of education (Stanic & Kilpatrick, 1989). Following Pólya (1945/1973), for whom problem solving was a major theme of doing mathematics in his book, *How to Solve It*, the National Council of Teachers of Mathematics (NCTM, 1989, 2000) has strongly endorsed the inclusion of problem solving in school mathematics for several reasons: (1) to build new mathematical knowledge, (2) to solve problems that arise in mathematics and in other contexts, (3) to apply and adapt a variety of problem-solving strategies, and (4) to monitor and reflect on the mathematical problem-solving processes. Problem solving is considered to be one of the most important aspects of today's mathematics curriculum, which is reflected by new curricula, books (e.g., Fey, Hollenbeck, & Wray, 2010), professional organizations, etc. (NCTM, 1989, 2000). This is quite similar to the objectives stated by the *German Kulturministerkonferenz* ([KMK], 2003, 2012), which represent all government departments of education. Nowadays, topics taught in mathematical classes require more than mere arithmetic or calculation skills, but rather extension and adaptability of previous knowledge, and flexibility in thinking. Problem solving offers students opportunities to learn and do mathematics by the means of posing and solving problems.

Problem solving is an extremely complex human endeavor and for many students is not an easy task. Based on the analysis of both North Rhine-Westphalia curriculum and German standards (KMK, 2003, 2012; NRW, 2004, 2007, 2011), problem solving encompasses working on given and newly posed problems by using heuristics and tools, evaluating the reasonability of the results and negating ideas for problem solving. Thus, it requires numerous cognitive activities and many types of knowledge. However, the abilities one draws on when problem solving is not discussed. Pólya, though he had reflected throughout his long life on the question of how he and others do mathematics, in his interview with Kilpatrick in 1978 showed, that he had not given much thought as to what abilities are needed. Through the conversation with Kilpatrick and his prompts, Pólya outlined and described the different dimensions of ability of someone that is capable in mathematics: good spatial ability, good and organized memory, ask good questions, draw figures, etc. Moreover, both agreed that to some extent mathematicians do not have certain special kinds of abilities, but have ordinary abilities that they then apply to mathematics. With respect to problem solving ability he said, "I think it is not so much 'develop' as it is 'awaken'" (Kilpatrick, 2011, p. 5), but added that some kind of "probability" has to exist. That is, you have to have a genetic predisposition, which then through good teachers can be awakened.

On the other hand, teachers' personal beliefs and theories about mathematics, learners and learning, teaching, subjects or curriculum, learning to teach, and about the self are widely considered to play a significant role in teaching practices (Pajares, 1992; Wilson & Cooney, 2002), task definition and selection (Schoenfeld, 1988), interpretation of content, and comprehension monitoring (Thompson, 1992), and implementation of curriculum reform (Howson, Keitel, & Kilpatrick, 1981). In the rest of the paper a term "view" will be used rather than "beliefs" to allow reporting on non-cognitive dimensions which is then more holistic (Rösken, Hannula, & Pehkonen, 2011).

Taken the above listed considerations in mind, I sought to study prospective teachers' views on (problem solving) abilities of a mathematically competent person and its change as they progresses through a 15-week seminar on problem solving as well as how these held beliefs might influence their teaching practices with respect to problem solving. The following questions were of interest:

- What are prospective teachers' views on the mathematical problem solving abilities?
- How do their views about the mathematical problem solving abilities change at the end of the semester?
- Which elements of their views towards the mathematical problem solving abilities can be identified as relevant factors influencing their future teaching practices?

In addition, based on the results of the study I will present a concept for professional development that would allow the vision of problem solving for all as advocated by new curricula.

Theoretical background

"The preeminence of increased problem-solving ability as a goal of mathematics instruction has long been admitted" (Kilpatrick, 1969, p. 523), whereas this ability is characterized both in terms of content (what mathematics students know) and process (how students go about doing and understanding mathematics). Werdelin (1958) gave a definition for the term:

> The mathematical ability is the ability to understand the nature of mathematical (or similar) problems, symbols, methods and proofs; to learn them, to retain them in memory, and to reproduce them; to combine them with other problems, symbols, methods, and proofs; and to use them when solving mathematical (and similar) tasks. (p. 13)

The notion of mathematical types or disposition has been a topic of interest by both researchers and laymen. It is not rare to hear a student say that his poor mathematical performance is due to the lack of a "mathematical gene". Some researchers, such as a Russian researcher Krutetskii (1969), contended that a *mathematical frame of mind* distinguishes mathematically competent individuals from mathematically not so competent ones.

By examining the relationship between the individual's success in problem solving and his or her cognitive and non-cognitive processes, research studies sought to explain the phenomenon *"ability to solve mathematical problems".* Studies of mathematical abilities of school and university students revealed factors of mathematical ability:

- Deductive reasoning (Blackwell, 1940; Kline, 1960; Werdelin, 1966)
- Inductive reasoning (Werdelin, 1958)
- Numerical ability (Kline, 1960; Martin, 1963; Werdelin, 1958, 1966)
- Spatial-perceptual ability (Blackwelll, 1940; Werdelin, 1966)
- Verbal comprehension (Blackwell, 1940; Kline, 1960; Martin, 1963; Werdelin, 1968, 1966) (as cited in Aiken, 1973, p. 407).

On the other hand, Krutetskii's (1966, 1969) research focused on the components of mathematical ability, that he had isolated from his observations of Russian school children when solving problems. Based on the logical analysis, he identified the following structure of mathematical ability:

1. Obtaining mathematical information
    - the ability for formalized perception of mathematical material and for grasping the formal structures of a given problem.
2. Processing mathematical information
    - the ability for logical thought structure of mathematics (mathematical symbols)
    - the ability for generalization of mathematical material (objects, relations, operations) = generalization
    - the ability for curtailment of thought = condensation
    - the ability to think flexibly during a mathematical activity = flexibility
    - striving for economy of mental forces = being economical
    - the ability for rapid reconstruction of mental processes in mathematical reasoning = reversibility
    - an ability to visualize abstract mathematical relationship and dependencies = visualization
    - an ability for spatial concepts and spatial imagination = spatial reasoning
3. Retaining mathematical information
    - mathematical memory for symbols, numbers and formulas, mathematical relationships, schemes of arguments and proofs, methods of problem solving and principles of approach = structural memory
4. General synthetic component
    - the ability to find mathematical meaning in many aspects of realty that on surface might not seem mathematical with a tendency to categorize the world in terms of

mathematics and logic = mathematical frame of mind

Krutetskii (1976) contended that these components are interrelated and influence one another forming a single integral system of individual's mathematical ability. Hence, mathematical talent is multidimensional and unique; for some it is analytical using verbal-logical reasoning (abstract cast of mind), for some geometrical using visual and spatial reasoning (mathematically pictorial cast of mind), and for some it is harmonic which is a combination of the former too (abstract and image-bearing cast of mind). He said that the harmonic thinker is ingenious in solving various mathematical problems, whereas the two other types are limited to certain mathematical areas. Krutetskii's (1966, 1969) structure of mathematical ability served as a theoretical framework for this study and taxonomy for the measurement instrument to analyze prospective teachers' views about mathematical and problem solving abilities.

Methodology

Settings and Participants

The research was conducted within a problem solving seminar "Problem solving in mathematics" at a large state university in Germany, that was offered both to elementary (grades 1-4) and lower secondary preservice teachers (grades 5-10). The seminar took place once per week for 90 minutes and was organized by both the author of this paper and the students. On the one hand, the course concentrated on learning about problem solving (e.g., problem solving models, heuristics, self-regulated problem solving) done through student presentations, whilst at the same time focusing on solving mathematics problems and how problem solving activities can be implemented in mathematics instruction led mainly by the instructor. The problems came from many sources and contexts in which mathematical ideas were examined with the use of different resources, such as technology, references or colleagues, to engage in genuine mathematics problem solving. The aim of the seminar was to provide the participants a deeper understanding of the problem solving through self-study, inquiry, investigation and exploration.

In total 21 students in their 3rd till 6th semester participated in the study. The cohort included 14 elementary and 7 lower secondary students. The participants had limited, if any, experience in problem solving and knowledge about it. Their school memories of mathematics classroom portrayed a traditional classroom, where the teacher presented learning materials and algorithms. This was then followed by practicing the algorithms on textbook tasks. Neither open-ended problems nor interdisciplinary problems were a part of the instruction. Very few had experienced problem solving – those who did associated problem solving with solving puzzle, modeling and word problems. With respect to their later university experience, some reported being introduced to the topic and engaged in problem solving. Hence, the participants had limited practical experience with problem solving, whereas some got the theoretical experience within methods courses.

Data Collection

Data collection methods included several different written materials: paper on problem solving abilities and its revision at the end of the semester, problem solving homework and reflection on the problem solving process, and final reflection paper. A large amount of data was collected to allow confirming and disconfirming data and to refine the characterization of participants' views as suggested by Philipp (2007). At the beginning of the semester the participants had to write a 2-pages long paper on the problem solving abilities, where the two following aspects were described: (1) what abilities distinguish a mathematical competent person from a less competent one and (2) what competencies are important for mathematical problem solving. At the end of the semester, each participant had to read its paper and revise it. In the paper revision they had to elaborate if and how their views about the listed mathematical problem solving abilities changed. As noted above, the participants during the semester wrote several reflection papers and solved homework with different mathematical problems. These instruments were designed to elicit how their held views affect their future teaching decisions with respect to implementing problem solving. The study design and data resources are summarized in Figure 1.

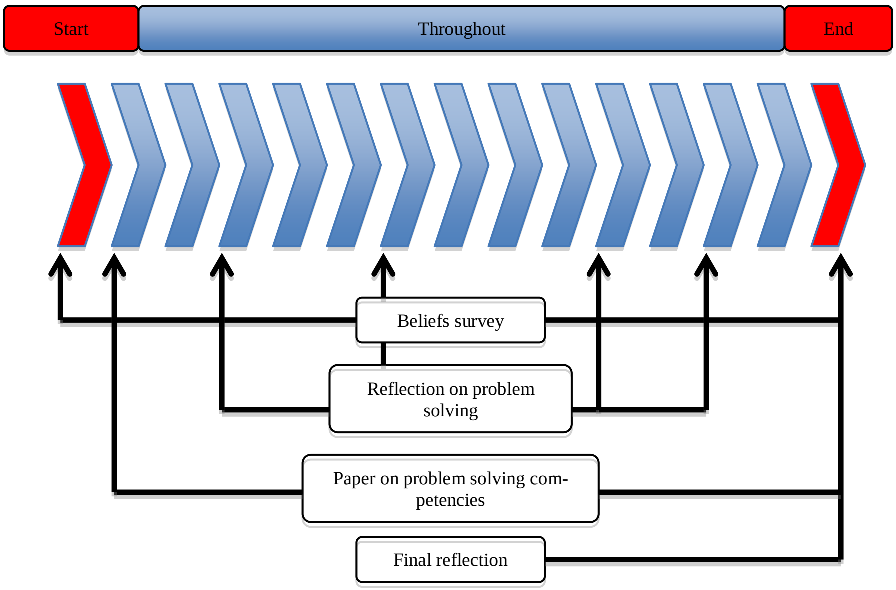


Figure 1: Overview of the study design.

Data Analysis

After data collection was completed, data analysis started including several phases through the data sources (see Figure 1) as suggested by Patton (2002) and Merriam (1998). For each participant, certain survey items relevant to the research questions, reflection papers and paper on problem solving competencies were examined to ascertain initial and ending views about problem solving abilities and their influence on their future professional activities. I conducted a content analysis to determine commonalities and trends in light of the framework introduces earlier in the paper. Nevertheless, in addition I used inductive analytical method (Patton, 2002) to note additional categories not given a priori. Here identification of pertinent passages from the data sources (paper on problem solving competencies, final reflection paper, reflection papers on the problem solving process) for establishing and confirming evidence took place in order to characterize each participant's views. To detect shifts in participant's views similar process was undertaken on reflection papers. This allowed also confirming or disconfirming previously established views. To answer the third question, I examined the final paper to derive factors that may be influential for their future teaching practices. From there, I was able to draw conclusions that are presented in the next section.

Results

In this section, I present participants' responses with respect to the research questions. I first present the initial data on prospective teachers' on mathematical problem abilities, followed by reporting shifts in their views. In the end, I focus on reporting the data relevant for their future teaching practices.

Prospective teachers' views on the mathematical problem solving abilities

The findings from the paper on problem solving abilities were tabulated and compared. The participants (74%) found it difficult to tackle the question of mathematical abilities. Almost all participants (82%) shared the view that a set of particular abilities that differ a more mathematical capable from a less mathematical capable person does not exist. For instance, one student said *"People have different strengths and weaknesses in general und hence also in the area of mathematics. Each possesses a diversity of abilities."* On the other hand, the rest (18%) contended that such a set of abilities exists including: high IQ, mathematical knowledge and relationships taught in school mathematics classes, and the ability for logical and abstract thought. In Table 1 frequency of participants' views on mathematical problem solving abilities is outlined. The abilities belonging to processing information were highly coded (47 times), followed by retaining information (23 times), obtaining information (3 times) and general component (1 time). All participants listed at least one subcategory of processing mathematical information, whereas others (83%) listed one of the retaining mathematical information subcategories. Very few of them (13%) listed abilities to obtain mathematical information. Only one participant indicated general component as a factor.

Table 1

Frequency of Participants' View on (Mathematical) Problem Solving Abilities

Problem solving abilities	Code frequency	Participants (n=23)	%
Obtaining mathematical information	3	3	13
Processing mathematical information	47	23	100
Retaining mathematical information	23	19	83
General component	1	1	4

The abilities for processing mathematical information were with the highest frequency coded (Table 2). Here not all of the categories, as proposed by Krutetskii (1976), were identified, namely, being economical. Very few participants (4%) listed reversibility, reflection and being able to quickly think as a factor, whereas some listed generalization (9%), and condensation and creativity (13%) as a factor. For the rest relevant factors were ability for logical thought (30%), ability to visualize mathematical structures and dependencies and spatial reasoning (39%), and the ability to think flexibly when solving a mathematical problem (48%).

Table 2

Participants' Views on (Mathematical) Problem Solving Abilities with Respect to Processing Mathematical Information

Processing information	Participants (n=23)	%
Flexibility	11	48
Visualization	9	39
Spatial reasoning	9	39
Logical thinking	7	30
Condensation	3	13
Creativity	3	13
Generalization	2	9
Reflection	1	4
Reversibility	1	4
Being economical	0	0

Presented in Table 3 are participants' views on (mathematical) problem solving abilities with respect to retaining information. With respect to retaining information, 19 participants of participants (83%) mentioned it as a factor, whereas 6 participants held memory for mathematics relationships (26%), 5 participants methods of problem solving (22%), and 2 participants principles of approach for important ones (9%). None explicitly mentioned schemes of arguments and proofs.

Table 3

Participants' Views on (Mathematical) Problem Solving Abilities with Respect to Retaining Mathematical Information

Retaining information	Participants (n=23)	%
Mathematical knowledge	6	26
Problem solving methods	5	22
Mathematical approaches	2	9
Proofs	0	0

Given that the inductive methods took place, additional categories were elicited, that were not outlined in the framework from Krutetskii (1966, 1967): ability to convey mathematics to others, IQ and the ability to regulate non-cognitive factors (motivation, enthusiasm, perseverance, volition) as outlined in Table 4.

Table 4

Additional View on (Mathematical) Problem Solving Abilities

Problem solving abilities	Code frequency	Participants (n=23)	%
Nature of phenomenon	17	17	74
Obtaining ability	15	15	65
Affect	22	10	43
IQ	2	2	9
Conveying mathematics	4	3	17

Interestingly, some participants (17%) approached the problem also from a practitioners' perspective. The three participants added that the ability to convey mathematical knowledge, ideas, and thinking on others is a factor that distinguishes a competent person from a less competent one. Two participants (9%) on the other hand held the view that IQ is an important factor. Interestingly, 9 participants (39%) reported the ability to regulate affective domain as an important ability. In addition, 17 participants (74%) argued as to how "mathematical ability" is obtained. The majority (87%) held the view that mathematical ability is genetically predisposed, but that it can be awakened. That is, each child has a good opportunity for it can be awakened through a good teacher. Only two participants also contended that mathematical ability is genetically predisposed, but were not so optimistic as the majority. One participant wrote, *"What is there, can be trained, but what is not there, cannot be developed."*

Shift in prospective teachers' views on the mathematical problem solving abilities

Although the participants entered the seminar with held views on problem solving abilities, the majority by the end of the seminar recognized and came to value different set of mathematical abilities and the need to awaken and support them in their students. Two participants did not submit the assignment, whereas nine participants (39%) reported not having changed their views, but again repeated what they wrote at the beginning of the semester (see Figure 2). These included: visualization, spatial reasoning, organized memory, flexibility, IQ and logical thinking. From the participants (n=2) who changed (4%) or refined (n=10) their views (48%) the following categories were identified: logical thinking, flexibility, reversibility, creativity, reflection, speed, memory of mathematical relationships and problem solving methods and approaches, general component and conveying mathematical knowledge and abilities. The rest explicated in detail how their views either changed or were modified. Only one participant reported having changed his views. Many of them said that participation in the seminar allowed them to think about the topic of problem solving abilities from a different perspective; not only as a learner but also as a future practitioner. Such shift allowed them to broaden their spectrum. The participants mostly commented on the two abilities: the ability to process the information and the ability to retain the information. With respect to the ability to process mathematical information three subcategories were emphasized: flexibility, reversibility and reflection. Having solved many problems within the seminar, they emphasized that it is important to *"to be able to shift thinking, change the approach, consider another idea" and "look at the problems from different perspectives".* In addition, being able to rapidly reconstruct the direction of mental processes alternating from a direct to a reverse train of thought was recognized as an important ability.

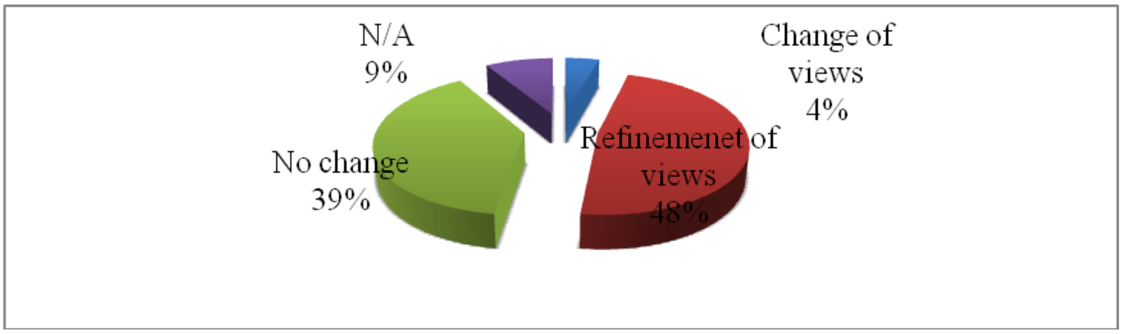


Figure 2: Overview of preservice teachers' shifts in views on the mathematical abilities.

During the seminar the participants had to write a protocol of their problem solving process. The data shed the light on how this activity was an eye-opener. All of the participants noted that *"structured observation of one's own actions"* was an important ability for processing mathematical information. By looking back at the solution, students were able to consolidate their knowledge and develop their ability to solve problems. With respect to the ability to retain mathematical information, the participants emphasized the importance of mathematical memory being organized. Hence, they articulated that the information (mathematical knowledge, relationships between mathematical knowledge, heuristics) has to be *"well-structured",* so that the problem solver can productively use it. Moreover, the participants explicitly listed factors needed for productive problem solving: *"substantial mathematical knowledge"*, *"understand mathematical relationships"*, *"consciously use heuristic methods"* and *"orchestrate the mathematical tools"*. One participant summarized his view nicely,

> However, it become clear to me that mere knowledge is not enough. The elementary aspects, such as, structure, organization, individuality, organized memory, conscious use of heuristics, but also the productivity and creativity, in retrospect through the entire semester have shown to be of importance.

At the beginning of the semester the theme of obtaining mathematical abilities was self-initiated by the participants. With the end of the semester they broached this issue once again. While at the beginning of the semester the participants focused on how problem solving abilities are obtained from a learner's perspective, at the end of the semester a shift was noticed. They reported that students can learn different heuristics as well as when and how to apply them, use different mathematical models and procedures when required by the situation and develop reflective activity and willingness to work hard. For instance, one participant said,

> … students can be 'brought up' into a mathematically competent person. In the seminar I learned that plethora of mathematical abilities can be developed with

a continuous repetition and training, continuous engagement with mathematics. Each kid possesses some potential, which in a right manner must be awakened.

Relevant factors influencing future teaching practices

Through self-engagement in problem solving the participants realized their capacity as practitioners to foster and enhance different mathematical abilities in their students. In order words, shifts in their views about what it means to be a mathematically competent problem solver was a newly found awareness of the complexity of teaching mathematics and problem solving so that their students would develop different problem solving competencies. One participant said, *"I need to activate previous knowledge, create new knowledge through mathematical problems and foster it."* Hence, the view was held that mathematical abilities could be awakened, broadened and fostered. This can be achieved through learning how to problem-solve, using good tasks and implementing phases of practice followed by reflection.

The type of problems they would use was highly coded. The participants talked about routine, open, middle-open, open-ended, puzzles, and Fermi problems, whereas types of problems were tools with which certain goals can be achieved. Fermi problems were seen as good problems as they *"require search for information", "support child's creativity", "allow each child to participate in problem solving", "interdisciplinary", "cognitive potential", "use of different strategies", "motivate kids to do mathematics"*, etc. Similarly, puzzle tasks were favored for logico-mathematical support, and open-ended problems as allowing different problem solving pathways. On the other hand, proofs and *"hard problems"* would not be used, because they could be demotivating. Moreover, geometry tasks were excluded as well because *"geometry is hard"* and *"they can be depressing and lead to frustration"*. Moreover, the participants emphasized the role of reflective activity as a way to *"to improve any solution or the understanding of the solution."* Thus, reflection practices would be fostered so that students can consolidate their knowledge, and develop their ability to solve problems. These few thoughts show that the participants began to see value in teaching differently, than experienced thus far, to meet the needs of their students.

Conclusions and implications for practice

Teacher beliefs and practices are known to be extremely resistant to change. Explicitly making prospective mathematics teachers aware of their views, challenging them and fostering reflection seems to impact their views about teaching and teaching practices. Through reflection one starts to see differently things and hence, challenge their existing beliefs leading to their change (Philipp, 2007). In the present study, this was accomplished through a modest program that relied upon carefully designed sequence of topics, tasks, reflective discussions, and protocols. Moreover, giving prospective teachers time to reflect upon their views and experiences, allowed them to integrated new ideas and thoughts which allowed them to integrate refined or new ideas into existing beliefs structure concerning problem solving abilities and teaching problem solving. However, the question remains how sustainable are these views, taken that other university courses do not follow the same philosophies. In addition, taken that the participants reported not using geometry problems and proofs, future research needs to examine if belief structures are domain-specific and to which extent are they hold.

Implications for professional development

Implementation of problem solving in German mathematics classrooms is now more prominent than ever. Many professional developments are offered with a focus on problem solving and the teaching of it as an attempt to accelerate changes in instructional practices. However, schools fast abandon the newly learned ideas under the pressure to demonstrate quick improvements in student performance, personal views that this will not work in their classroom and school, worries that students would become even more demotivated and disinterested in learning mathematics, etc. Perhaps the abandonment of the new practices is caused by lack of attention in the teacher professional development towards the wide range of instructional practices and beliefs that teachers bring with them into the learning environments. As a result of research reported here a concept for a professional development for teachers in problem solving developed within the German Center for Mathematics Teacher Education (DZLM) is presented.

Taken that problem solving is a relatively a new standard and teachers themselves had few, if any, opportunities to engage in problem solving, the professional development has to address both developing knowledge about problem solving, knowledge for teaching problem solving, and strengthening their existing mathematical content and pedagogical knowledge. Naturally, the goal is that through problem solving students recognize mathematical problems in every day life, learn different heuristics and willingness to work hard, and develop reflective activity for their own actions. To achieve this, an already proven teaching concept from (Bruder, 2003) will be used that is based on a four-phase model of problem solving as a long-term learning by teaching and learning process:

1. developing an intuitive habit of heuristic method and techniques through reflection at the end of problem solving

2. becoming aware of special heuristics on a basis of an example
3. introducing deliberate practice phases on tasks of different difficulty
4. context extension of strategy application

Specifically, this research suggests that when conceptualizing professional development on problem solving following learning opportunities need to be considered:

- allow for teacher's growth of mathematical knowledge for teaching problem solving as well as to shape the forms of the knowledge produced
- encourage teachers to use more challenging examples than they would normally use
- provide teachers with activities that foster differentiation to reach every student's potential
- provide opportunities that challenge teachers' beliefs on student problem solving predispositions through "cognitive conflict" (e.g., through authentic videos).

Adoption of new curriculum standards is not an easy task for teachers. As a community, we know what types of professional development support teachers so that their teaching practices would reflect the ongoing reforms. "Teacher development must be rooted in the ability of the individual teacher to doubt, reflect and reconstruct" (Wilson & Cooney, 2002, p. 132). That said, the reform of mathematics classroom is a hard, but a worthy cause where different facets have to be taken into consideration and addressed. Based on the growing body of knowledge, we hope that our conception of professional development would allow for transformation of teaching and learning as suggested by reformed curricula, organizations and research.

References

Aiken, L. R. Jr. (1973). Ability and creativity in mathematics. *Review of Educational Research, 43*(4), 405–432.

Bruder, R. (2003): *Methoden und Techniken des Problemlöselernens. Material im Rahmen des BLK-Programms „Sinus“ zur „Steigerung der Effizienz des mathematisch-naturwissenschaftlichen Unterrichts.* Kiel: IPN.

Fey, J. T., Hollenbeck, R. M., & Wray, J. A. (2010). Technology and the mathematics curriculum. In B. J. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics curriculum: Issues, trends, and future directions* (pp. 41–49). Reston, VA: National Council of Teachers of Mathematics.

Howson, G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge, UK: Cambridge University Press.

Kilpatrick, J. (1969). Problem solving and creative behavior in mathematics. In J. W. Wilson & L. R. Carey (Eds.), *Review of recent research in mathematics education studies in mathematics series. Studies in Mathematics Series* (Vol. 19, pp. 153–187). Stanford, CA: School Mathematics Study Group.

Kilpatrick, J. (2011). Pólya on mathematical abilities. *The Mathemetics Educator*, *21*(1), 3–8.

Krutetskii, V.A. (1966). Age peculiarities in the development of mathematical abilities in students. *Soviet Education, 8*,15–27.

Krutetskii, V. A. (1969). Mathematical aptitudes. In J. Kilpatrick & I. Wirszup (Eds.), *The structure of mathematical abilities. Soviet studies in the psychology of learning and teaching mathematis* (Vol. 2, pp. 5–57). Standford: School Mathematics Study.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago, IL: University of Chicago Press.

Kultusministerkonferenz (2003). *Bildungsstandards im Fach Mathematik für den mittleren Schulabschluss*. Bonn: KMK.

Kultusministerkonferenz (2012). *Bildungsstandards im Fach Mathematik für die allgemeine Hochschulreife*. Bonn: KMK.

Merriam, S. B. (1998). *Qualitative research and case study applications in education* (2nd ed.). San Francisco: Jossey-Bass.

Ministerium für Schule, Jugend und Kinder des Landes Nordrhein-Westfalen [NRW] (2004). *Kernlehrplan für die Gesamtschule – Sekundarstufe I in Nordrhein-Westfalen*.

Ministerium für Schule, Jugend und Kinder des Landes Nordrhein-Westfalen [NRW] (2007). Kernlehrplan für das Gymnasium – Sekundarstufe I (G8) in Nordrhein-Westfalen.

Ministerium für Schule, Jugend und Kinder des Landes Nordrhein-Westfalen [NRW] (2011). Kernlehrplan und Richtlinien für die Hauptschule in Nordrhein-Westfalen.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks, CA: Sage.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 257–315). Charlotte, NC: Information Age.

Pólya, G. (1973). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press. (Original work published in 1945).

Rösken, B., Hannula, M. S., & Pehkonen, E. (2011). Dimensions of students' views of themselves as learners of mathematics. *ZDM - The International Journal on Mathematics Education, 43* (4), 497–506.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well–taught" mathematics courses. *Educational Psychologist, 23*(2), 145–166.

Stanic, G., & Kilpatrick, J. (1989). Historical perspectives on problem solving in mathematics curriculum. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–31). Reston, VA: National Council of Teachers of Mathematics.

Thompson, A. (1992). Teacher's beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: MacMillan.

Werdelin, I. (1985). *The mathematical ability: Experimental and factorial studies*. Lund, Sweden: C. W. K. Gleerup.

Wilson, M., & Cooney, T. J. (2002). Mathematics teacher change and development. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 127–147). Dordrecht, The Netherlands: Kluwer.

Teachers’ and Fourth graders’ questions during a problem-solving lesson

Anu Laine, Liisa Näveri, Anu Kankaanpää, Maija Ahtee & Erkki Pehkonen
University of Helsinki, Finland
anu.laine@helsinki.fi

Abstract

We have studied the teachers’ and their pupils questioning during a problem-solving lesson when a nonstandard problem was used. Based on the videotaped lessons the teachers’ questions were classified into six categories (Task assignment and marking the solution, Way of working, Progress of working, Asking for justification, Deepening of understanding, and Questions not related to the problem solving) using inductive content analysis. The teachers asked mostly about the task assignment, prompting the pupils to understand the problem and to start solving it. The number of teachers’ questions aiming at understanding varied substantially (0% - 47%), but their number was small on average. Pupils’ questions were classified into six categories. The average proportion of the pupils’questions from all the questions was about 30%. The number of teachers’ and pupils’ questions seems to be interrelated.

Key words: open problem, teachers’ and pupils’ questions, Finnish primary school

ZDM classification: C50, D50

Introduction

At all levels of the Finnish school system, the aim of learning mathematics is, according to the curriculum (NBE 2004), to understand mathematical structures and develop mathematical understanding, not merely the operations of mechanical calculation. Problem solving is considered to be a central tool in the development of mathematical thinking internationally (e.g. Mason, Burton & Stacey 1982, Schoenfeld 1985, Stanic & Kilpatric 1988). Problem solving has even been set as one of the formal objectives for all school subjects in the Finnish national curriculum for comprehensive school (NBE 2004).

This paper considers the use of problem solving – especially open problems – in teaching, and especially teachers’ questioning in guiding pupils during problem-solving lessons. We are also interested in how pupils’ questions are related to the teacher’s questions. Based on the questions it is possible to get a view of the conversation during a problem-solving lesson and thus also about the learning possibilities that the teacher offers during the lesson. With her questions the teacher has an influence on the process of how pupils concentrate and try to solve the given problem. The teacher leads and focuses pupils' thinking, including ways of finding solutions and deepening their understanding with the questions. When posing a question, pupils need to evaluate new information based on their own knowledge and beliefs. Thus the teacher can derive from pupils' questions knowledge about how pupils have understood the topic.

Here we will use a rather widely used characterization for a problem (cf. Kantowski 1980): a task is said to be a problem, if it demands that solvers must connect their earlier knowledge in a new way. If the problem solvers can immediately recognize the procedure needed for solving the task, it is for them a routine task (or a standard problem or an exercise). The concept of 'problem' is thus relative in terms of time and of the person concerned. In an open-ended problem (cf. Pehkonen 2004), the starting point is given, but the end is open, and therefore, there are usually many possible answers. An example of an open-ended task is as follows: "Divide a rectangule into three triangles. Can you find another solution?"

Teacher's questions

The problem-solving literature includes advice on how a teacher could guide students' problem-solving activity. The teacher can scaffold students' problem solving in different ways (e.g., Anghileri 2006) or the teacher may use careful questioning to promote students' reasoning (e.g., Sahin & Kulm 2008; Martino & Maher 1999).

Teacher's questions can be divided into different categories, for example based on their form (Myhill 2006). These types of questions are

1) closed, factual questions, e.g. "What is 5 plus 5?"
2) speculative questions that have no predetermined answer, e.g. questions about opinions
3) process questions which invite children to explain their thinking, e.g. "How do you know that?"
4) procedural questions which are related to the organization of the lesson, e.g. "Can you all see at the table?"

It is important to notice that despite the form of the question its function can vary. Myhill (2006) divides teacher's questions, based on their function, into 11 categories. For example the meaning of the factual question can vary from remembering the fact to pondering the meaning of the fact. Sahin and Kulm (2008) divide questions into three main types: 1) factual questions which check, for example, that the pupil knows the central facts concerning the problem, 2) probing questions which ask for justification or explanation, and 3) guiding questions which help pupils ahead in their problem-solving process. Myhill's process questions and Sahin and Kulm's probing questions require pupils to explain their thinking.

In problem solving, questions that enhance pupils' ability to argue and justify their solutions are central. Martino and Maher (1999) have analysed these type of questions more closely. They can be divided into six categories: questions that 1) estimate pupils' understanding, 2) direct pupils' attention to an incomplete component in their argument, 3) sustain pupils' interest in the problem, 4) encourage mathematical justification, 5) direct pupils to consider justification produced by another pupil, and 6) promote generalization of a solution to similar problem tasks.

Research problems

In this paper our aim is to discover what kind of questions teachers use during a problem-solving lesson when pupils solve an open-ended problem. We are also interested in seeing how the pupils' questions are related to the teacher's questions. Thus the research questions can be formulated as follows:

(1) What kind of questions do the teachers use when guiding their pupils during the problem- solving lesson?

(2) What kind of questions do the pupils pose to the teacher during the problem-solving lesson?

(3) How do the teachers' and the pupils' questions related to problem solving vary in different classrooms?

Method

This study is a part of the three-year follow-up Finland–Chile research project, financed by the Academy of Finland (project number #1135556). In the project, we try to develop a model for improving the level of pupils' mathematical understanding by using open problems in mathematics teaching. Once a month on average, during the mathematics lessons of the experimental group (10 teachers), one open problem will be dealt with and recorded using video equipment. The same problems are also being used in Chile, but the comparison material is not part of this study and comes only later.

Here we consider the results of the task, Gary the Snail, that was implemented in September 2011. Gary the snail is clearly a non-standard problem, and essentially needs creativity in order to be solved:

> *Gary the snail climbs a wall very slowly. Some days the snail ascends 10 cm, some days it ascends 20 cm, some days it sleeps and doesn't move, and other days it is sound asleep and falls 10 cm. The height of the wall is 100 cm. At the end of the tenth day Gary the snail is at the middle of the height of the wall. What could have been happening during the first 10 days? Show as many ways as possible.*

Eight teachers from the experimental group (Anna, Barbara, Cloe, Daisy, Ella, Flora, Grace, Hannah) and their fourth-grade pupils took part in this study. All the teachers work in schools in the Helsinki capital area. All except Daisy have over 10 years of teaching experience. In the mathematics test (max 32 points) carried out at the beginning of the project, the pupils' results varied from 22.3 to 26.7 points. Barbara's and Hannah's pupils had the best results and, respectively, Grace's and Cloe's pupils had the lowest results on the test.

During the lessons (45 minutes/lesson), one of the researchers (LN) recorded the teachers' working. The videos were transcribed. Based on looking at the videos and reading the transcriptions the teachers' questions were categorized. After creating the categories the data was re-read and the categorization was checked. This was done several times. After this some categories which contained similar ideas were joined together and the final categories were decided. When all the questions were categorized the frequencies were counted. One researcher (AK) made the classification and another researcher (MA) made the parallel coding. Uniformity of the teachers' questions scored 97% and uniformity of the pupils' questions scored 94%. The analysis is qualitative and it can be characterized as inductive content analysis (cf. Patton 2002) because we tried to make sense of the situation without imposing any pre-existing expectations. Pearson's correlation and regression analysis were used to find out the relation between the teachers' and the pupils' questions.

The teachers' lesson plans were also used as research data. Noticeable is that Anna and Barbara had already written in their lesson plan that they were not going to guide their pupils. They wanted to see how the pupils would manage to solve the problem by themselves.

Results

In the results we first analyze the teachers' questions, then the questions that the pupils pose to the teachers and, finally, the relationship between these questions.

Teachers' questions

The teachers asked plenty of questions already from the beginning of the lesson. They guided the pupils mostly just by asking questions that were connected to the information in the task assignment. Some of the questions were directed to the whole class, whereas some of them were clearly focused on only one pupil at a time. At the end of the lesson some of the teachers used questioning when they went through the task and its different solutions. The teachers' questions were classified into the following six categories: A) Task assignment and marking the solution, B) Way of working, C) Progress of working, D) Asking for justification, E) Deepening of understanding, and F) The others, i.e. incoherent or unconnected to problem solving. Often the teacher repeated the same question many times during the lesson. She might have presented her question first to several pairs in turn and finally to the whole class in the common discussion. However, here a question consists of a statement that may contain more than one interrogative or the repetition of the same question many times and using different wording.

The questions in category A, **Task assignment and marking the solution**, formed a considerable portion of the teachers' questions. For example, most of Anna's and Flora's questions belong to this category. Many of the teachers' questions pertained to both task assignment and the marking of the solution simultaneously. These questions were asked immediately in the beginning of the lesson when the task was read or when the teacher was walking around helping the pupils. With these questions the teachers encouraged the pupils to return to the task assignment and rethink the conditions in it. With their questions the teachers also guided the pupils to write down their solutions. In part of the questions the teachers helped the pupils to solve the problem by looking at one day at a time.

> *"But, what is written in the instruction? How much could it ascend?" (Barbara)*
>
> *"Here, it is the second day. What is happening during it?" (Grace)*
>
> *"How did you mark them?" (Daisy)*

The questions in category B, **The way of working**, are related to how the pupils started to solve the problem. In their questions the teachers commented on the strategy that the pupils had chosen to solve the problem, and proposed other alternative ways. In this category there are also the questions with which the teacher urges the pupils to think and plan their solution process.

> *"Is there some way how you could draw it?" (Grace)*
>
> *"Yees, how do you think that division could have been used here?" (Barbara)*

"What…, how could you solve this?" (Grace)

Also the questions in category C, **Progress in working**, were presented often, especially when the teacher was walking around and helping the pupils. The teachers used these questions often as openings for discussion by asking the pupils to tell about their working methods and their solutions. In this category there are also the questions with which the teacher discovered whether the pupils had understood how to work with the problem.

"How many different alternatives have you already found?" (Anna)

"Did quite many of you now catch the plot?" (Daisy)

With the questions belonging to category D, **Asking for justification**, the teachers guided the pupils to tell what they were thinking in solving the problem. These questions turned up especially when, near the end of the lesson, the pupils presented their solutions to the other pupils. The teachers asked the pupils also to justify the way they had found their solution. In some cases the teachers tried to help the pupils to find the error in their suggestion for a solution. In the data there were also some questions that were related to the pupils' feelings and to facing the nonstandard problem.

"Yees, so how did you from this first solution ... get this second solution?" (Hannah)

"Explain to me why, if you make the mark here and not to this line, how does it increase the number of your possibilities?" (Cloe)

The teacher used such comments to encourage the pupils to try to approach the problem in some new way or to find more solutions, and these were classified into category E, **Deepening of understanding**. Also included in this category were those questions with which the teacher guided the pupils to ponder in a larger and deeper way in finding a proper solution. For example, Hannah encouraged the pupils to think how they could modify a new solution from the solution that they had already found. The questions in this category have been chosen by looking at the whole event: For example the teacher's question on whether the snail could move up along the wall more than 50 cm on one day and then descend, totally changed one pupil's way of thinking about the problem.

"You found thirteen, but do you mean that you found all the possible alternatives?" (Barbara)

"What was among these different ways, don't think only your own way ... What was the easiest way to find the solution?" (Grace)

The distribution of the question categories among the teachers is shown in Table 1. The teachers were clearly quite different both as to the number of questions they used and also as to what kind of questions they asked. During this problem-solving lesson the total number of questions per teacher varies from 16 to 90. Half of the questions were classified as Task assignment and marking the solution (A). These questions were mostly factual (e.g. "What will happen to the snail during deep sleep?"). Questions that were related to the working process (B & C) were partly factual and partly guiding (e.g. "Is there a way how you could draw it?") (cf. Myhill 2006; Sahin & Kulm 2008).

The teachers differed very much also in the number of questions (D & E) that promote understanding. Some of the questions required justifications and the others aimed to deepen their understanding by generalizing the problem (cf. Martino & Maher). Anna and Daisy did not ask these types of questions at all whereas Hannah asked up to 72 such questions which is almost half of her total amount of questions. Most of the questions that encouraged pupils to justify their thinking were process questions but some of them were also speculative (e.g. "But now I ask again the same question: what is a problem? Is it only in mathematics where there are those problems?").

Table 1. The distribution of the number of the teachers' questions in the categories A) Task assignment and marking the solution, B) Way of working, C) Progress of working, D) Asking for justification, E) Deepening of understanding, and F) The others. The percentage in the brackets is calculated from the teacher's total questions.

Teacher	A	B	C	D	E	F	Total
Anna	11 (69)	0	1 (6)	0	0	4(25)	16 (100)
Barbara	5 (26)	1 (5)	5 (26)	4 (21)	2 (11)	2(11)	19 (100)
Cloe	15 (63)	0	5 (21)	3 (13)	0	1 (4)	24 (100)
Daisy	12 (46)	1 (4)	5 (19)	0	0	8(31)	26 (100)
Ella	19 (54)	4 (11)	4 (11)	2 (6)	2 (6)	4(11)	35 (100)
Flora	43 (74)	3 (5)	4 (7)	3 (5)	0	5 (9)	58 (100)
Grace	47 (68)	3 (4)	2 (3)	9 (13)	5 (7)	4 (5)	69 (100)
Hannah	18 (20)	7 (8)	18 (20)	27 (30)	15 (17)	5 (6)	90 (100)
Total	170 (50)	19 (6)	44(13)	48 (14)	24 (7)	33 (10)	338 (100)

Pupils' questions to the teacher

We were also interested in examining the pupils' questions in order to get a better overwiew of the whole process of questioning in the different classes. The pupils' questions which they posed to their teachers were classified in the following six categories: I) Understanding the task and marking the solution, II) Aim of the task, III) Way of working, IV) Checking of understanding, V) Request for help, and VI) The others, i.e. incoherent or unconnected to open-ended problem solving. In Table 2 we present a summary of pupils' questions in different classrooms.

Table 2. The distribution of pupils' questions in different categories: I) Understanding the task and marking the solution, II) Aim of the task, III) Way of working, IV) Checking of understanding, V) Request for help, and VI) Other questions.

Classroom	I	II	III	IV	V	VI	Total
Anna	1	2	1	0	0	2	6
Barbara	3	1	1	0	0	2	7
Cloe	13	2	7	1	3	2	28
Daisy	7	1	6	0	5	14	33
Ella	4	3	3	3	0	5	18
Flora	22	5	5	4	2	4	42
Grace	4	1	1	1	0	9	16
Hannah	6	2	2	0	1	2	13
Total	60	17	26	9	11	40	163

Most of the pupils' questions were classified in category I) Understanding the task and marking the solution.

"Is the starting point always there?" (Cecilia's pupil)

"Can I put here that it climbs first for example one centimeter and then eight?" (Hannah's pupil)

Questions related to the target of the task (category II) pupils clarified and confirmed that they had understood the idea of the problem. These kinds of questions were also posed later when they were solving the problem.

"Am I supposed to do the same calculation again?" (Grace's pupil)

"So is the purpose here now to invent new ways?" (Hannah's pupil)

Students were figuring out how they should work during the lesson and what they should do next by posing questions classified in the category Way of working (category III). These questions were not related to finding the solution or marking the solution. Instead they were connected to the ways of working, pair work or working equipment.

"Can I do this in a group?" (Grace's pupil)

"So can we have another paper?" (Cloe's pupil)

Pupils asked teachers during the lessons to comment if their solution was correct. These kinds of questions were classified in category IV Checking of understanding.

"Is this wrong?" (Ella's pupil)

"So is this good?" (Flora's pupil)

With questions containing Request for help (category V) pupils were asking the teacher to come and help them.

"Can you come here for a second?" (Cloe's pupil)

In category VI The others, the pupils' questions were incoherent, unconnected to the problem or to solving it, or to anything related to the working at the problem.

"What? Is that a camera?" (Daisy's pupil)

"How did you then translate that from Spanish?"(Grace's pupil)

The relation between teachers' and pupils' questions

The total number of questions asked in the different classrooms varied quite a lot. During one lesson in different classrooms, pupils posed from 6 to 42 questions, whereas the teachers posed from 16 to 90 questions. We wanted to find out the connection between the number of the teachers' and pupils' questions. Would it be possible that teachers' questions also promote pupils to ask more questions? Because we were especially interested in the questions that were related to solving the problem, we eliminated the category 'The others' from both the teachers' and the pupils' questions (see Table 3).

Table 3. Number of questions related to problem solving in different classrooms

Teacher	Teachers' questions	Pupils' questions	The proportion of pupils' questions of all questions
Anna	12	4	25%
Barbara	17	5	23%
Cloe	23	26	53%
Daisy	18	19	51%
Ella	31	13	30%
Flora	53	38	42%
Grace	65	7	10%
Hannah	85	11	11%
Total	304	134	31%

From Table 3 it can be concluded that in most of the classes, when the number of the teacher's questions increases it is also observed that the number of the pupils' questions to the teacher increases. Two classrooms (Grace's and Hannah's) are exceptions. In these classrooms the teacher asks many questions, but the pupils ask only a few. The proportion of the pupils' questions from all the questions is only about 10% when the average is about 30%. Based on the video material, we found that in these classrooms the atmosphere was very authoritarian and therefore the pupils were not so ready to ask questions. It could also be possible that the pupils did not have time to ask questions, because they were kept busy in answering the teachers' questions. However, this explanation was not true in these cases. During the lesson, Hannah asked a lot of questions with which she guided her pupils in the right direction to their solutions, and she was not expecting any answers from them. In the next example Hannah guides one group of pupils: *"Quite good, you have found one solution. Can you find another?"* She continues with another group: *"Could you take advantage of this? You have one good model here. Think how you could advance from it?"* When Hannah moves from one group to another the pupils ponder her questions rather than pose new ones. Whereas Grace prompts her pupils to think by themselves and therefore her pupils do not bother to ask any questions: *"It is wrong then. It has moved a little too much. What do you do? How do you correct it? This is zero…mm…you must take this away… You must probably correct some number. How would you correct it? I don't tell you because you have not counted."*

Anna and Barbara asked very few questions. They had already written in their lesson plans that this time they were going to try seeing how the pupils would manage solving the problem without guidance. On the basis of this plan we may also therefore conclude that the pupils asked very few questions.

In Cloe's, Daisy's and Flora's classrooms the proportion between the teacher's and the pupils' questions seems to be quite the same. Based on the video material, the atmosphere in the classrooms was confidential during all these lessons. It seemed to be easy for the pupils to approach their teacher and ask questions, and the teacher's response to the questions was appropriate.

Ella seems to be an average teacher both as to the total number of questions and to the distribution of the questions between the teacher and the pupils. However, she and her class seem not to represent a typical situation in this research group.

The relation between teachers' and pupils' questions related to problem solving is presented graphically in figure 1. When we leave two teachers (G and H) out of the examination because of their exceptional behavior we get a linear correlation of 0.83 between the number of teachers' and pupils' questions. On this area the model describing the relation between questions is linear (R Square Change =.671[a] , sig. F Change = .046). Coefficient of determination is 67% which means that the number of teachers' questions explain 67% of the number of pupils' questions (or the other way around). It appears that, when teachers' questions increase over some point, the pupils' questions start to decline. If we take into account all teachers, we receive a curved, i.e. parable model ($y = -0.0102x^2+0.9724x$, $R^2=0.304$) and for the coefficient of determination 30%. Because the linear model explains more of the variation than the curved one, we choose the first one. This decision is supported by the observation that Grace's and Hannah's actions differ from the other teachers' actions during the problem-solving lesson.

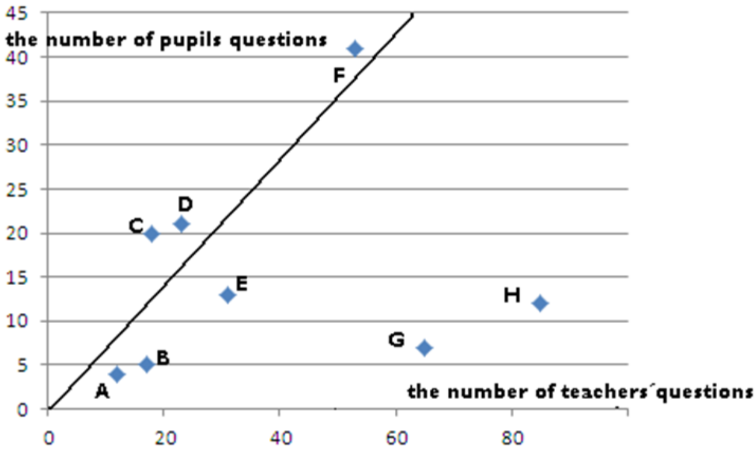


Figure 1. The relation between the teachers' and the pupils' questions.

Discussion

The teachers asked most about the task assignment because the pupils had difficulties to get started on the problem. The teachers asked plenty of good questions with which they prompted the pupils to understand the problem and did not just say what they should do. The teachers asked a very different number of questions during the lesson. The difference between the pupils' knowledge between different classrooms does not explain the number of questions because, based on the mathematics test done at the beginning of the project, there were many (and respectively a few) questions in the classrooms of both good and poor achieving pupils. The number of questions depends therefore evidently on the teachers' philosophy of teaching and on the working culture of the class.

It is also important to pay attention to the quality of the questions and not only to their number. On average there were, however, only a few questions that aimed specifically at understanding. The proportion of these questions varied a lot (0% - 47%) according to the teacher. From the viewpoint of promoting mathematical thinking, it would be important that there were more questions aimed at judging and deepening the solutions. It would be interesting to see whether the teachers' questions will change during the research project, after the teachers have had more guidance and experience in teaching problem solving.

It is also important to find out what kind of questions and answers to pupils' questions encourages the pupils to ask questions that promote the problem-solving process and therefore contribute to a fruitful discussion in a classroom. Later we shall concentrate more explicitly on studying this process with the video material about the communication in the classrooms. Furthermore, in the future we shall also look at the pupils' achievements and compare them with the discussions in the classrooms.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9, 33–52

Kantowski, M.G. (1980). Some Thoughts on Teaching for Problem Solving. In: *Problem Solving in School Mathematics* (eds. S. Krulik & R.E. Reys), 195–203. NCTM Yearbook 1980. Reston (VA): Council.

Martino, A. & Maher, C. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior*, 18(1), 53-78.

Mason, J., Burton, L. & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.

Myhill, D. (2006). Talk, Talk, Talk: Teaching and learning in whole class discourse. *Research Papers in Education*, 21 (1), 19–41.

NBE (2004). National Core Curriculum for Basic Education 2004. http://www.oph.fi/english/publications/2009/national_core_curricula_for basic_education

Patton, M. (2002). *Qualitative research & evaluation methods*. Thousand Oaks: Sage.

Pehkonen, E. (2004). State-of-the-Art in Problem Solving: Focus on Open Problems. In: *ProMath Jena 2003. Problem Solving in Mathematics Education* (eds. H. Rehlich & B. Zimmermann), 93–111. Hildesheim: Verlag Franzbecker.

Sahin, A. & Kulm, G. (2006). Sixth grade mathematics teachers' intentions and use of probing, quiding, and factual questions. *Journal of mathematics teacher education*, 11 (3), 221–241.

Schoenfeld A.H. (1985). *Mathematical Problem Solving.* Orlando (FL): Academic Press.

Stanic, G. & Kilpatrick , J. (1988). Historical perspectives on problem solving in the mathematics curriculum. In: *Teaching and learning mathematical problem solving: Multiple research perspectives* (eds. R. Charles and E. Silver). Reston VA, National Council of Teachers of Mathematics: 1-22.

When do we have to use recursive or asymptotic formulas or Monte Carlo simulation?

Józsefné Libor
Apor Vilmos Catholic University College, Hungary
liborne47@gmail.com

Abstract

If you see or hear the word didactic, usually a primary or secondary school comes to your mind as a place where it is used. But at least as important in higher education too the choice of the most appropriate method to be more effective in educating. My choice of topic partly originates from the continuing decline in the results of college students that can be observed for several years by now, and is a general trend in higher education but is especially true for mastering scientific disciplines. On the other hand because of the continuously decreasing number of class hours and other reasons, there is hardly enough time and opportunity available, especially in terms of teaching of probability, to carry out or at least to demonstrate teaching experiments of acceptable quantity and quality. Therefore those methods and techniques allowing the students to get familiar with more concepts and see their practical application can have a positive effect on mastering the theoretical concepts. For the analysis of this topic my work focuses on a field of probability theory (the analysis of the longest runs), which can be easily understood by the students and can be linked to their everyday experiences and thus provide a way for easy comparison. There is no exact andclosed expression for the distribution function describing the length of the longest runs. Thus for the discussion of this particular topic the differences and the applicability between the various methods becomes more obvious. Observing these differences provides a deeper understanding of the different terms, the applications of the different techniques for the topic (the longest runs) can be demonstrated appropriately.

Key words and phrases: longest run, recursive formula, asymptotic theorem, simulation

ZDM classification: K108, K508, U708

1. Introduction

In my study I present a method that helps to understand the concepts and techniques mentioned in the title, which can be a useful didactic tool for colleagues teaching in academic institutions. The recursive expressions give exact values but they are not closed expressions and calculating the exact values after a greater number of elements can sometimes be problematic even when calculating with a computer. The asymptotic values become more accurate with the increase in the number of constituents (with an increasing n) however they only provide approximate values, whereas repeated studies of simulation experiments provide average results. These terms and definitions can be applied in other fields of mathematics by the students.

Although in various fields of mathematics the basic terminology, definitions, the introduction of computations can be based upon everyday observations and experiences to a great extent, however in the case of probably calculations our situation is a little more difficult. In the introduction of the article by Tibor Nemetz [9] the author writes the following: "Most of the time the students have vague understanding how to interpret an uncertain event, etc.; quite frequently their judgment on random, large scale phenomena is wrong or rather they cannot comprehend the existence or application of such laws." Furthermore, as Alfréd Rényi, [11] the founder of the Hungarian school of probability writes in many cases even the teachers are reluctant to present these experiments. "Statistical principles can be demonstrated by data take from books and newspapers; however it has a greater effect on students if they obtain these results presented to them or, even better if the results originate fromexperiments performed by them. Some teachers do not agree with this as they fear that the experiments will not yield results that they expect as it can happen from the nature of these experiments. I don't think that this fear is justified and if the teacher understands probability calculus well, then he or she cannot end up in an uncomfortable situation. Naturally the teacher has to react quickly as the evaluation of results, which even the teacher could not foresee is more difficult than the analysis ofexamples which were evaluated in advance by the teacher." Of course these experiments will provide further experiences not only to the students but to the instructor as well, which can be a very important point of view especially for the discussion of random events.

In higher education the general opinion of the teachers is that the knowledge of students displays greater insufficiency and the general performance is dropping every year. Many institutions provide preparation courses and additional coaching courses to treat the problem; however I find that it is also just as important to understand the terms, definitions and get proficient with their application during BA education. Everybody knows the obvious fact that there is a direct relationship between theoreticalknowledge and problem solving skills it is justified to expect that results will improve with the better understanding of definitions. Recursive formulae, asymptotic theorems and simulations play an important role in several fields of math-teaching in higher education. But the exact knowledge of these notions leaves a lot to be desired. The study of some illustrative problems may help to understand them. While solving these problems students have an opportunity to compare these definitions.

In this paper I shall use the well-known coin tossing experiment to compare the results of recursive formulae, asymptotic theorems and computer simulations. In connection with coin tossing we examine the length of the longest run. For a sequence of independent coin tosses with p ($0 \le p \le 1$), the longest run of consecutive heads in the first n tosses is a natural object of study. In my work the longest run means the longest sequence of the same objects, for example the longest head-run means the longest sequence among the sequences of homogeneous heads, which is not interrupted by tail. My research includes the examination of the length of the longest head-run and the length of the longest whatever - head or tail - run, studying fare coin. (I have studied the event of biased coin too, so if you are interested in it do not hesitate to write to me to discuss about that.)

The following well-known example by T. Varga can be an interesting introduction to the problem for students. We can read this for example in Révész [12]. A class of school children is divided into two sections. In one of them each child is given a coin which he or she throws two hundred times, recording the resulting head and tail sequence on a piece of paper. In the other section the children do not receive coins but are told instead that they should try to write down a 'random' head and tail sequence of length in 200. Collecting and mixing all the slips of paper, the teacher then tries to subdivide them into their original groups. Most of the time he succeeds quite well. His secret is that he has observed the following. In a randomly produced sequence of length in two hundred, there are, say, head-runs of length seven (knowing Rényi's $\log_2 200$ result). On the other hand, he has also observed that most of those children who had to write down an imaginary random sequence are usually afraid ofwriting down runs of longer than four. Hence, in order to find the slips coming from the coin tossing group, he simply selects the ones which contain runs longer than five. When I had the privilege of meeting professor Révész, he talked about the continuation of Varga's experiment. He made his students acquainted with Varga's experiment and the result of it. Then they made the trial again. Professor Révész was successful in subdividing the slips of paper with quite good accuracy again. His secret was very easy. The students focused on the length of pure head run for example, but they did not pay attention to the length of pure tail run or the length of the head-tail pairs run. These experiments led us to ask the following questions. What is the length ofthe longest head run or the longest whatever run in coin tossing?

For the investigation of the mentioned definitions (recursive formula, asymptotic theorem, simulation) our chosen field - coin tossing - is perfect in many respects. We know that there is no exact and closed formula to give the length of the longest run in case of any n. The recursive formulae are exact, - they give correct results - but using them for large n, is very time- and storage-consuming even if we use very good computer. For large n we can use the asymptotic theorems, but these give only approaching results. And finally, the simulation gives an average value from a lot of repetitions of the trial. Considering these, we can examine the meaning of these definitions by studying the coin tossing trial. Our graphs will show the differences between the values in case of small and large n too and the limits of the scope of the notions.

Consider n independent tosses of a fair coin, and let R_n represent the length of the longest run of heads, and similarly let R_n be the length of the longest whatever (head or tail) run. As we have got a regular (fair) coin, the probability ofhead-tossing equals $p = 0.5$, and of course the probability of tail-tossing is equal to $q = 1 - p = 0.5$. The problem facing us is the following. What is the length of the longest head or whatever run?

2. Recursive formulae

Sometimes it is difficult to define an object explicitly. However, it may be easy to define the object in terms of itself. We define a function or a special series with a recursive formula with defining the starting value (or values) and generally we provide a formula for calculating the subsequent values using the previous value(s) with various operations. Thus the process has two important components. First we need to define the starting value (values) which are the first elements of our recursive series. Then we need to supply the correspondence which demonstrates how the subsequent members are derived from the previous members of the series. It is important to keep in mind that recursive formulas always yield accurate results and they are not average or approximate values. The application of recursive formulas has one disadvantage and barrier at the same time: the calculation of the n^{th} element with an increasing n becomes complicated even with the application of a computer. The recursive functions, which form a class of computable functions, take their name from the process of 'recurrence' or 'recursion'. In its most general numerical form the recursion process consists in defining the value of a function by using other - the previous - valuesof the same function. A recursive sequence is a sequence of numbers $f(n)$ indexed by an integer n and generated by solving a recurrence equation.

And now back to our coin tossing problem.

2.1. Longest head run

Let R_n represent the length of the longest run of heads and let $A_n(x)$ be the number of sequences of length n in which the longest head run does not exceed x. The distribution function of R_n is

$$F_n(x) = P(R_n \le x) = \frac{A_n(x)}{2^n}. \tag{1}$$

First we have to give the initial condition that tells where the sequence starts, then the recursion formula that tells how any term of the sequence relates to the preceding term. Using the results of Schilling [13], we have the following recursive formula for $A_n(x)$

$$A_n(x) = \begin{cases} \sum_{j=0}^{x} A_{n-1-j}(x), & \text{if} \quad n > x, \\ 2^n, & \text{if} \quad 0 \le n \le x. \end{cases} \tag{2}$$

To see how this works, consider the case in which the longest head run consists of at most four heads. If $n \le 4$, then clearly $A_n(4) = 2^n$ since any outcome is a favorable one. For $n > 4$, each favorable sequence begins with either T, HT, HHT, HHHT or HHHHT and is followed by a string having not more than four consecutive heads. Thus,

$A_n(4) = A_{n-1}(4) + A_{n-2}(4) + A_{n-3}(4) + A_{n-4}(4) + A_{n-5}(4)$ for $n > 4$.

From this we can calculate the values of $A_n(4)$:

n	0	1	2	3	4	5	6	7	8	...
$A_n(4)$	1	2	4	8	16	31	61	120	236	...

Remark 1. *For* $n = 1, 2, 3, ...,$ *the number* $A_n(1)$ *of sequences of length* n *where* n *does not continue two consecutive heads is the* $(n+2)$*nd Fibonacci number.*

Remark 2. *The values of* $A_n(k)$ *can be given with the help of* k^{th} *degree Fibonacci numbers. Moreover, we can study the case of biased coins using the application of* k^{th} *degree Fibonacci polynomials. (See Philippou-Makri, [10].)*

2.2 Longest whatever run

Consider n independent tosses of a fair coin, and let R'_n represent the length of the longest run of heads or tails (whatever). If $B_n(x)$ is the number of sequences of length n in which the longest run does not exceed x, then the (cumulative) distribution function is $F_n(x) = P(R_n \le x) = \frac{B_n(x)}{2^n}$. Schilling [13] has proved that $B_n(x) = 2A_{n-1}(x-1)$, for $x \ge 1$.

To prove it consider a sequence of length n consisting of signs H and T (representing 'head' and 'tail', respectively). Write below of this sequence another sequence of signs S and D (representing 'same' and 'different', respectively). We write S if twoconsecutive signs in the first sequence are the same, and write D if they are different. Now a run of length $x-1$ consisting of signs S represents a run of length x in the first sequence. We also see that any sequence containing signs S and D belongs to two sequences of signs H and T. Finally, observe that $A_{n-1}(x-1)$ can be considered as the number of sequences of length $n-1$ and containing signs S and D but not having S-run of length x. (We need a factor 2, because the longest run can occur from heads or from tails same times.)

The example below shows one of the strings that contributes to $B_{13}(4)$:

H H H T H T H T T T T H H
S S D D D D D S S S D S

So we can reduce this case to the case of the longest head run

$$F_n(x) = P(R_n \le x) = \frac{B_n(x)}{2^n} = \frac{2A_{n-1}(x-1)}{2^n} = \frac{A_{n-1}(x-1)}{2^{n-1}} = F_{n-1}(x-1). \qquad (3)$$

The implication of (3) is that for n tosses of a fair coin the longest run tends to be one longer than the longest run of heads alone.

3. Asymptotic theorems

Asymptotic behavior is a property that certain mathematical expressions demonstrate if the subsequent values of a series approach the values of a function closer and closer, but they never actually reach it, even in an infinite number of steps. Therefore in case of large n the asymptotic values are so close to the real values that we can apply them with confidence if the real values cannot be defined accurately, or can only be defined in a difficult and complicated manner.

For the longest head run we have strong limit theorems and results on the limiting behavior of the distribution. A well-known theorem of Rényi says that the length of the longest run in n tosses of a fair coin is about $log_2 n$. This was generalized to excessive blocks by the Erdős-Rényi [4] laws of large numbers.

For the limiting behavior of the distributions there are several known results. Let us see the case of the longest head run. The asymptotic behavior of R_n is described by the following theorem.

Theorem 1. *Földes (1979) [5]. For any integer k we have*

$$P\left(R_n - [\log_2 n] < k\right) = \exp\left(-2^{-(k+1-\{\log_2 n\})}\right) + o(1), \tag{4}$$

where $[a]$ denotes the integer part of a and $\{a\} = a - [a]$.

Similarly we can see the asymptotic behavior of R'_n. Using the above theorem of Földes and (3) we have the following asymptotic result for R'_n.

Theorem 2. *For any integer k we have*

$$P\left(R'_n - [\log_2(n-1)] < k\right) = \exp\left(-2^{-(k-\{\log_2(n-1)\})}\right) + o(1). \tag{5}$$

where $[a]$ denotes the integer part of a and $\{a\} = a - [a]$.

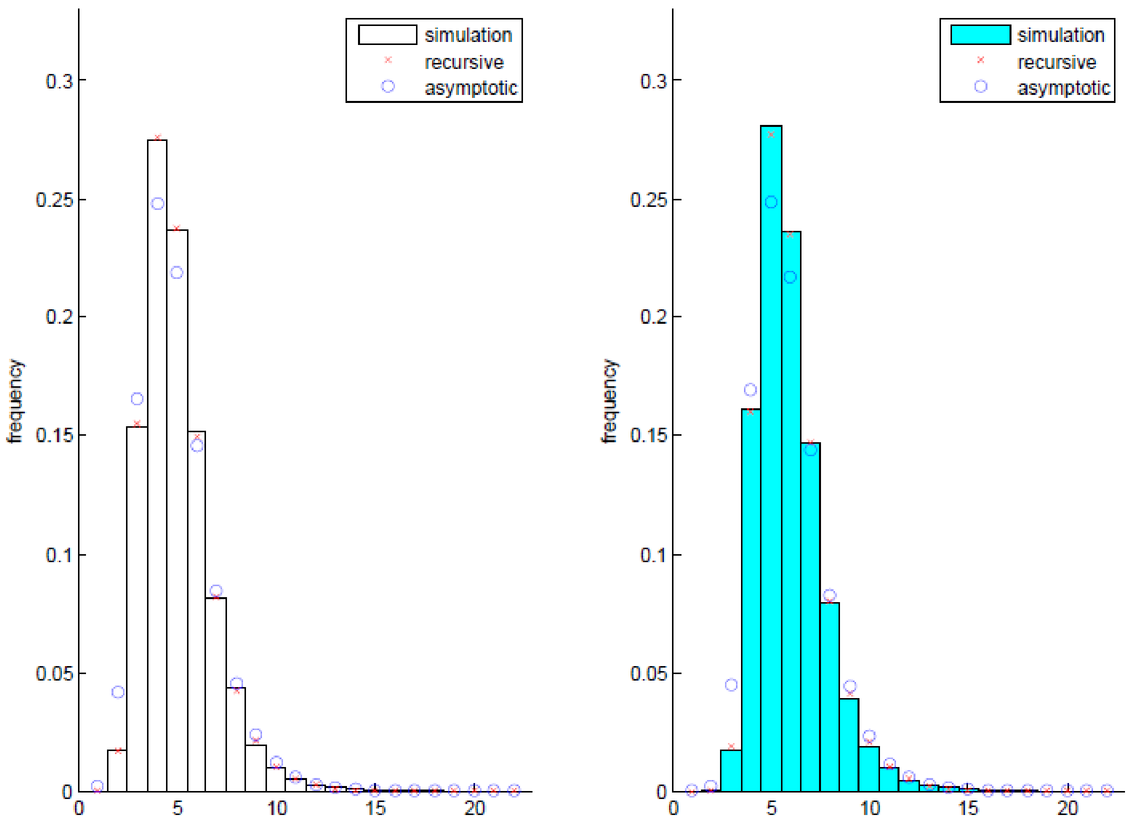
4. Simulation

The mathematical simulation is known as the Monte Carlo method, it is called after the city in the Monaco Principality, because of roulette, a simple random number generator. The name and the systematic development of this method dates from about 1944 (see [7] and [8]). Simulation methods can play an important role in teaching several topics not only mathematics. We can identify the average value of a random variable if we do not know, or if cannot or are not willing to perform complex calculations to define the distribution function. According to the law of large numbers if we supply a sufficient number of samples the resulting average value is going to be close to the unknown real value. Using simulation the quoted law can also be more comprehensible for the students. We can easily show that if we do not perform a sufficient number of repeated measurements we cannot obtain good results. However if we perform a sufficiently large (at least in the order of thousands) repeated measurement (or we perform them using a computer) then the resulting average values will be a really good approximation of the real values.

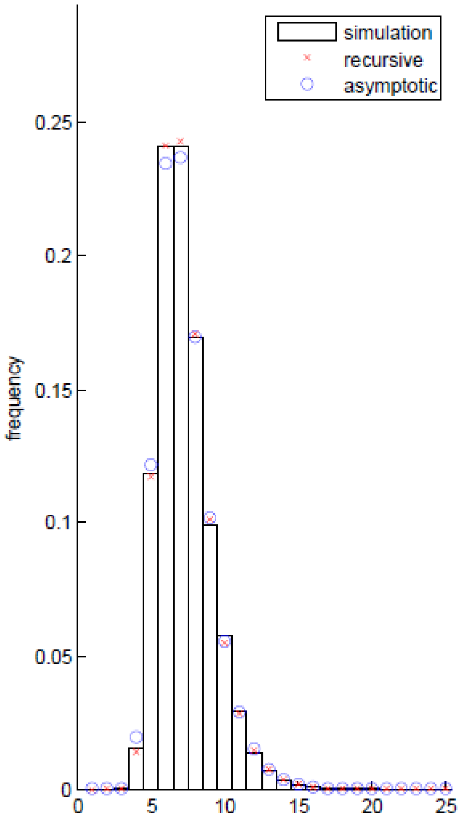
Here we deal with the simplest version of the stochastic simulation. We need the numerical value of a quantity p. Moreover, p can be considered as the probability of an event A: $P(A) = p$. Repeat the experiment N times and calculate the relative frequency of A. Denote it by $\frac{k_A}{N}$. Then, by the Bernoulli law of large numbers, $\frac{k_A}{N} \to p$ almost surely as $N \to \infty$. However, usually we make our experiment on a computer. Using our computer programme, we build the experiment, and we record if A occurs or not. Repeat the computer experiment N times. It means that we run our programme N times with N consecutive random numbers. However, as the coin tossing experiment is a random experiment itself, wedevelop our computer model as follows. Using the first n random numbers, we can perform the coin tossing experiment of length n. Then repeat it N times (N is a large integer, $N = 1000$, say). Therefore, for any fixed n, the distribution of R_n can be obtained as $N \to \infty$. (It is an important issue, because in the previous section we considered the asymptotic distribution of R_n, as $n \to \infty$.)

5. Comparison of numerical results, conclusions

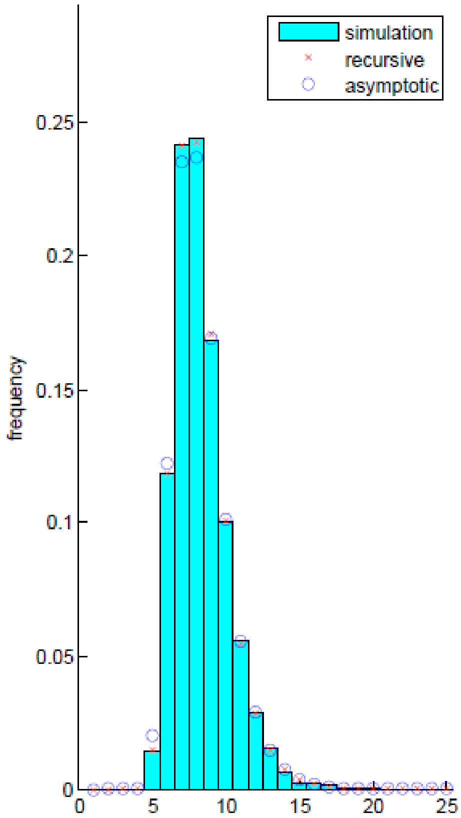
Now let us see our problem. For numerical calculations we used MATLAB software and a high-capacity PC (INTEL Core Quad Q9550 processor, 4Gb. DDR3 memory). We calculated the distribution of R_n and R'_n. We considered the precise values obtained by recursions, the asymptotic values offered by asymptotic theorems, and used simulation with 20,000 repetitions. On the figures below $\times$ denotes the result of the recursion, o belongs to the asymptotic result, while the histogram shows the relative frequencies calculated by simulation. On the left side you can see the case of the longest head run and on the right side you can see the case of the longest whatever run.



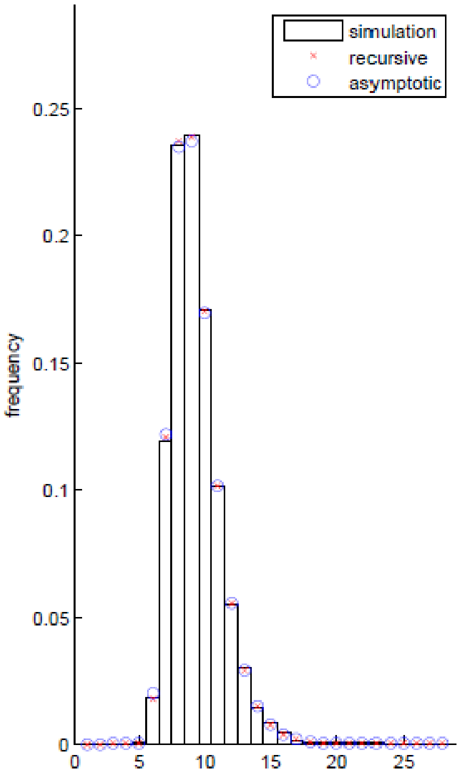
Distribution of the longest head run $p = 0.5, n = 50.$ Distribution of the longest run $p = 0.5, n = 50.$



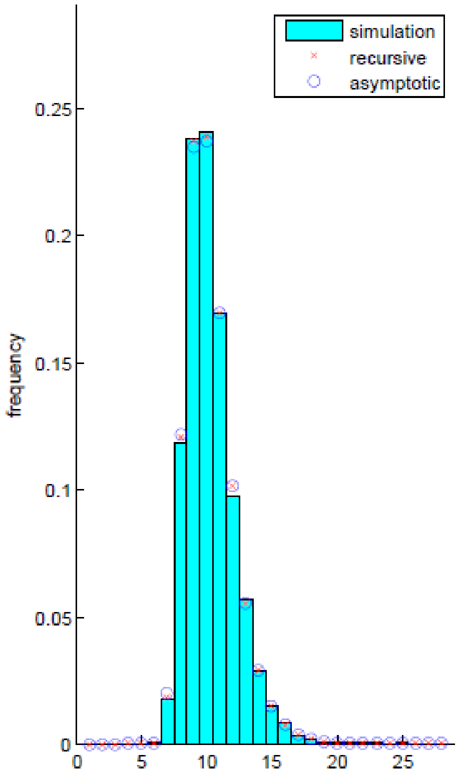
Distribution of the longest head run $p = 0.5, n = 250.$



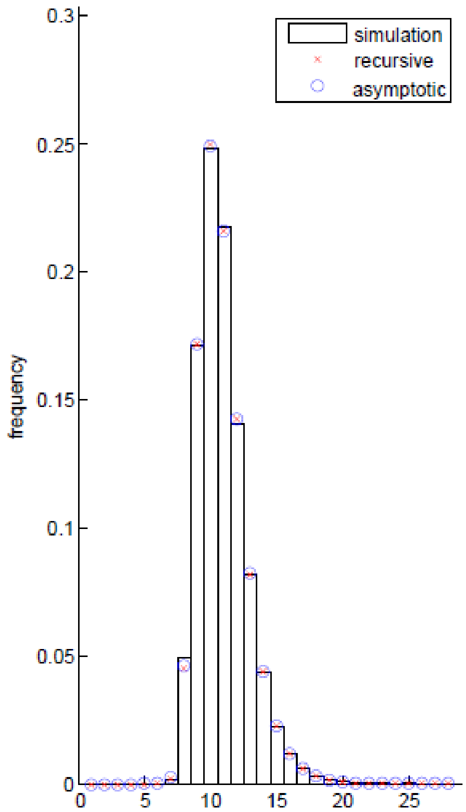
Distribution of the longest run $p = 0.5, n = 250.$



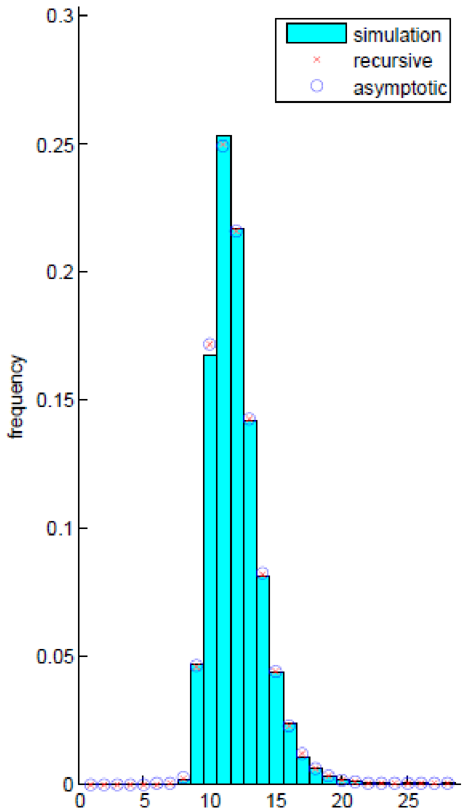
Distribution of the longest head run $p = 0.5, n = 1000.$



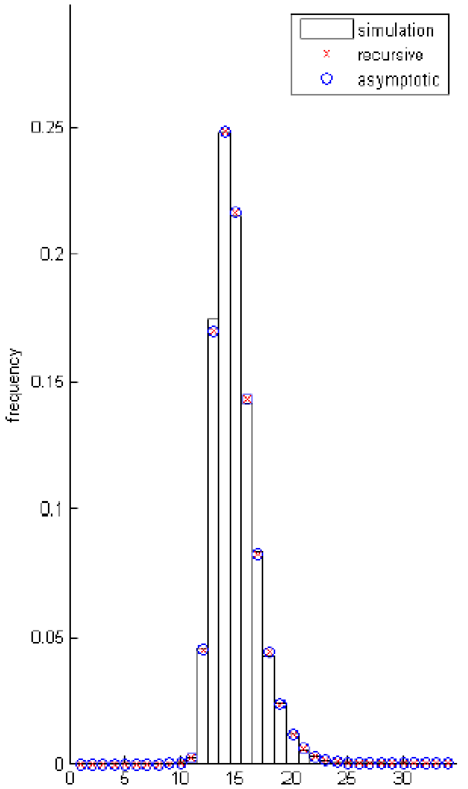
Distribution of the longest run $p = 0.5, n = 1000.$



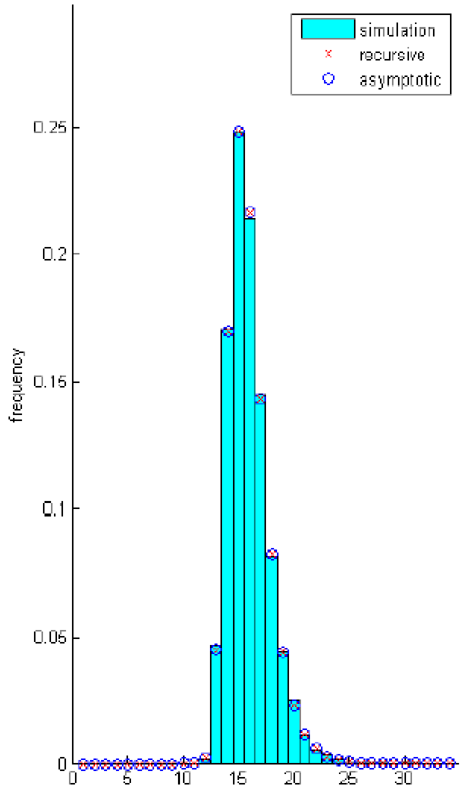
Distribution of the longest head run
$p = 0.5, n = 3100.$



Distribution of the longest run
$p = 0.5, n = 3100.$



Distribution of the longest head run
$p = 0.5, n = 50000.$



Distribution of the longest run
$p = 0.5, n = 50000.$

First we emphasize that the precise values of the distributions are given by the recursive formulae. The practical drawbacks of recursive formulae are the following. They are limited by the capacity of the computer and they can be influenced by numerical errors. On the other hand, asymptotic theorems always offer approximations but they are easy to compute. The approximations are quite bad for a small n but they are almost precise for a large n. Finally, simulation always gives random approximations but it is quite precise if the number of repetitions is large. The above figures show the properties of the procedures mentioned. For a fair coin we studied short trials ($n = 50$), medium size trials ($n = 250$), long trials, ($n = 1000$ and $n = 3100$) and very long trials ($n = 50.000$).

If n is small or medium sized, we can see that the recursive results are closer to the simulated values, than the results given by asymptotic theorems. Although we can say that if n is small, the recursive algorithm is fast, but it slows down if n increases. The next table demonstrates this statement showing some running times.

n	repet.	running time
100,000	20,000	773.575832 s.
10,000	20,000	31.795056 s.
5,000	20,000	14.398654 s.
3,100	20,000	9.009479 s.
1,000	20,000	3.984981 s.
500	20,000	3.240019 s.
250	20,000	2.412824 s.
50	20,000	2.092010 s.
30	20,000	1.935557 s.

For a large n the asymptotic values are closer to the simulated results, so we can use them instead of the recursive values. The asymptotic value is a good approximation if $n \geq 1000$, and it is practically precise if $n \geq 10.000$.

The figures show that the distribution of R'_n can practically be obtained from the distribution of R_n by shifting it to the right by 1, as we have seen in (3).

6. The St. Petersburg Paradox

Almost the most common question of our students is the following. Where and how can I use, apply this or that theorem? What is the economic application of this or that? To show some financial and economic aspects of the studied field is very useful in teaching of Business Mathematics. In connection with the previous part, we should examine a very interesting paradox, the St. Petersburg paradox. What is this paradox? Peter and Paul play a game with a fair coin. (The probability of head-tosses and the probability of tail-tosses are the same: $50-50\%$.) Paul tosses the coin and Peter pays 2 ducats (or euros, or dollars, or what you want) if it shows head on the first toss, 4 ducats if the first head appears on the second toss, 8 ducats if the first head appears on the third toss, and so on. So Peter pays 2^k ducats if the first head appears only on the k^{th} toss. How much should Peter charge Paul as an entrance fee to this game so that the game will be fair? (Fair game means that neither of gamblers wins or loses any money on average.) Surprisingly, the game cannot be made fair, no matter how large the entrance fee is. Paul is always in a wining position, but as Nicolaus Bernoulli wrote in the XVIII. century [3]: "there ought not be asane man who would not happily sell his chance for forty ducats". How can we solve this paradox? Let us see our data in the following table, where x_i means the amount of ducats if the first head appears on the ith toss and p_i means the probability of this event. Even the not excellent students can calculate these values, so this can rewarding ad for them.

Payoff values and probabilities

x_i	2	4	8	16	32	…	2^k	…
p_i	1/2	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$(1/2)^5$	…	$(1/2)^k$	…

For example, let us see the first column. Paul gets 2 ducats if the first tossing is head. The probability of this equals 0.5. On the second column we can see that Paul gets 4 ducats if the first toss is tail and only the second one is head. The probability of this equals 0.5 times 0.5 (because the tosses are independent). Following this, Paul gets 2^k ducats if the first $k-1$ tosses are tails but the k^{th} toss is head. The probability of this equals $(1/2)^{k-1}\cdot(1/2)=(1/2)^k$. And we can continue it to infinite. This is right, because the sum of the probabilities equals: $\sum_{k=1}^{\infty}(1/2)^k=1$. Now we can calculate the expected value of the payoffs, which will be infinite.

$$E(X)=\sum_{i=1}^{\infty}x_i p_i=2\cdot 1/2+4\cdot(1/2)^2+8\cdot(1/2)^3+...+2^k\cdot(1/2)^k+...=\sum_{i=1}^{\infty}1=\infty \qquad (6)$$

So this means that Paul needs to pay an infinite value to Peter as an entrance fee. However, this is a requirement to which almost no rational person would agree to or be able to satisfy. We can see this, because if $x\geq 2$ then the probability of winning at least x value equals the following: $P(X \le x) = \sum_{k:2^k \le x} (1/2)^k = \sum_{k=1}^{[\log_2 x]} (1/2)^k = 1-(1/2)^{[\log_2 x]}$ where $[a]$ means the integer part of a: $[a] = max\{b \epsilon Z, b \le a\}$. Using this, we can give the distribution function:

$$F(x) = P(X \le x) \begin{cases} 0, & \text{if } x < 2, \\ 1-2^{-[log_2 x]}, & \text{if } x \ge 2. \end{cases} \tag{7}$$

So, $P(X > x) = 2^{-[\log_2 x]}$ as for example the probability of winning greater than 40 ducats equals $P(X > 40) = 1/32 \approx 0.03125$, or the probability of winning a "much bigger" value, for example the probability of winning a value which is greater than $32,000$ ducats equals about 0.00006. So Paul does not want to risk a big value (not even 40 ducats!) to enter the game. Although the calculation of Paul's expectation is mathematically correct, the paradoxical conclusion was regarded by many early researchers of probability as unacceptable. It is worth to mention Keynes's words [6]: "We are unwilling to be Paul, partly because we do not believe Peter will pay us if we have good fortune in the tossing, partly because we do not know what we should do with so much money … if we won it, partly because we do not believe we should ever win it, and partly because we do not think would be a rational act to risk an infinite sum or even a very large sum for an infinitely larger one, whose attainment is infinitely unlikely." We can ask students about their opinion of the amount of the entrance fee. For example: Who would like to pay (or to accept) for example 30 ducats for the game? The answers can be very interesting, surprising and varied. If the teachers has got enough time, to talk about the history and some previous solutions of the paradox can be very useful. There are some students who are not good in maths but they are interested in history. We can give some research works for them in this theme. (It has got a huge but very interesting history, so students hopefully will enjoy this work.)

Let us make calculation and simulation for studying the St. Petersburg game. Suppose that we play only finite times, for example $n = 2^{10} = 1024$. As the probability of head-tossing equals 0.5, we expect that half of the games, 512 games complete afterthe first tossing. The amount of payoff equals 2. We can say the same for the remaining games, so the half of the remaining games complete after the next (second) tossing. This means that the quarter of games 216 games complete after the second tossing. The amount of payoff equals 4 in this case. Continuing this line of thought, we can say that only the last game will continue after the 10^{th} tossing. This concrete case can easily be studied with even weaker students also and the general case will be more understandable.

The first head occurs on the k^{th} tossing (k)	1	2	3	4	5	6	7	8	9	10
Number of completed games	512	256	128	64	32	16	8	4	2	1
Payoff (2^k)	2	4	8	16	32	64	128	256	512	1024

Table 1. Playing St. Petersburg game 1024 times

Let us see the average of payoffs: $\frac{512\cdot 2+256\cdot 2^2+...+1\cdot 2^{10}+payoff\ of\ the\ last\ game}{2^{10}}=10(\frac{1024}{1024})+c=10+c$, where $c=\frac{payoff\ of\ the\ last\ game}{1024}$. Now study the value of c. The game would end on the 11^{th} tossing with probability 0.5 and the payoff would be 2^{11}, so $c=\frac{2^{11}}{2^{10}}=2$. But two additional tosses are required to finish the game with 0.25 probability, and in this case c would be $\frac{2^{12}}{2^{10}}=4$ and so on. So the average of payoffs equals $10+2=12$ with 0.5 probability, $10+4=14$ with 0.25 probability and so on.

Generalize the problem, suppose that we play $n=2^k$ games. Our data are the followings:

The first head occurs on the k^{th} tossing (k)	1	2	3	…	k-1	k
Number of completed games	2^{k-1}	2^{k-2}	2^{k-3}	…	2	1
Payoff (2^k)	2	4	8	…	2^{k-1}	2^k

Table 2. St. Petersburg game in general

After the first tossing, the half of the games, 2^{k-1} games complete. After the second tossing 2^{k-2} games complete and so on, after the k^{th} tossing only 1 game completes. The sum of the completed games equals $2^{k-1}+2^{k-2}+...+2+1=2^k-1$, so 1 game would continue after the k^{th} tossing. But this game would complete after the next $(k+1)^{th}$ tossing with 0.5 probability, after the $(k+2)^{th}$ tossing with 0.25 probability and so on. The average of payoffs equals: $\frac{2^{k-1}2+2^{k-2}4+...+2^0 2^k+payoff\ of\ the\ last\ game}{2^k}=k\frac{2^k}{2^k}+c=k+c$, where c equals 2 with $0,5$ probability, 4 with $0,25$ probability and so on. So we can demonstrate the averages of payoffs with a set of lines, exactly $k+2$ line with 0.5 probability, $k+4$ line with 0.25 probability, $k+8$ line with 0.125 probability and so on.

Now simulate the game with MATLAB for example, where $k=20$. Our values are on the broken (red) line, and we can draw our lines in the first four events. Students can easily see that the simulated values are consistent with the calculated results.

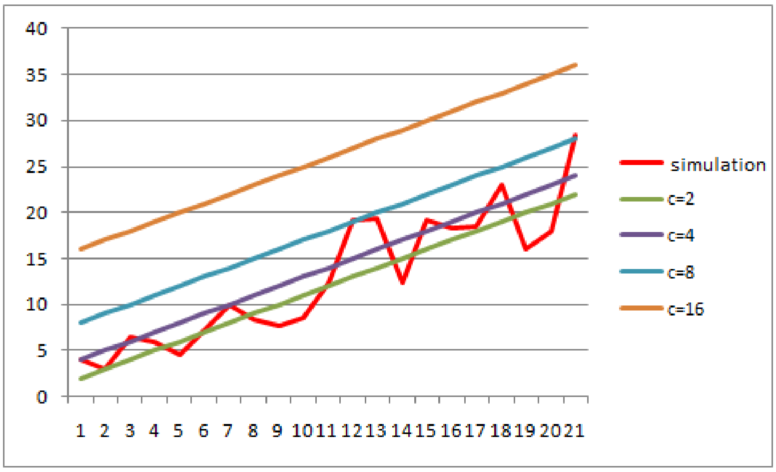


Figure 1. Simulated and calculated values of the game

Buffon and other mathematicians thought that Paul has to pay $n\log_2 n$ ducats for n games, so $\log_2 n$ ducats are spent per game. If Paul would like to play the game 1048575 times, he would have to pay 20 ducats per game. But as these gameswould take almost 30 years, we do not need to study this situation. Therefore, about 10 ducats per game can be an acceptable amount.

Csörgö and Simons existed an extensive literature on asymptotic theory for St. Petersburg games [2]. Csörgö explained why $n\log_2 n$ will not satisfy the banker, and generally no satisfactory solution can be based on laws of large numbers. (There is no asymptotic distribution.) He remarked that in n games $\log_2 n$ fee per game is too little, if Paul passes up his biggest win, but it is too much if he passes up the biggest two wins. The research is still ongoing, many Hungarian mathematicians work in this filed successfully.

Many examples of St. Petersburg games and their generalizations are studied in the statistics, economics, and mathematics literature and in many fields of life. It is well-known in economics that Daniel Bernoulli's work in 1738 was the basis of the formation of the modern utility concept that the economists and psychologists and other researchers use even today [1].

An interesting example of a modified St. Petersburg game is a popular television game show, Who Wants to be a Millionaire? It offers players the chance to win up to one million dollars. Moreover, we can mention the so-called "martingale strategy", that many gamblers like to use. In this game we play against the bank with a 50% chance to win. If we lose in the first game, we double the bet. If we lose in the second game again, we also double the previous bet, and so on, until we win. If our first bet equals A dollar and we lose in the first n-1 game but we win in the nth game, then our winning amount equals $A\cdot 2^n$, but we had to pay out

$A(1+2+4+...+2^{n-1}) = A(2^n - 1)$ dollars. So our benefit is A. It seems to be a good strategy, but thereare some problems. Usually we have not got infinite money (perhaps we win in infinite), and if we have a lot of money then to win A dollars, does not matter. Last but not least, of course the casinos know about this strategy so they limit the amount ofbets. Using the modified paradox Székely and Richards gave an interesting study of economic crisis in [14]. These and many other interesting examples can engage the interest of weaker students either.

References

[1] Bernstein, P. L.: *Against the Gods: The Remarkable Story of Risk* New York: Wiley 1996.

[2] Csörgö, Simons: *Pooling strategies for St. Petersburg gamblers* Bernoulli 12, (6), 2006. pp. 971-1002.

[3] Dutka, J.: *On the St. petersburg Paradox* Archive for History of Exact Sciences, Vol, 39. Number 1. pp.13-39.

[4] Erdős, P., Révész P.: On a new law of large numbers. *J. Analyse Math.* 23 (1970), pp. 103-111.

[5] Földes, A. : The limit distribution of the length of the longest head-run. *Period. Math. Hungar.* 10 (1979), no. 4, pp. 301-310.

[6] Keynes, J. M.: *A treatise on probability* London, Macmillan, 1921.

[7] Metropolis, N.: The Beginning of the Monte Carlo Method. *Los Alamos Science, Special Issue* (1987)

[8] Metropolis, N., Ulam, S.: The Monte Carlo Method. *Journal of the American Statistical Association*, Vol.44, No.247. (Sep. 1949) pp. 335-341.

[9] Nemetz, T.: Egy lehetőség valószínűségszámítási fogalmak bevezető szemléltetésére. *A matematika tanítása* (1970) pp. 143-149.

[10] Philippou, A.N.; Makri, F.S.: Longest success runs and Fibonacci-type polinomials. *The Fibonacci Quarterly* (1985), no. 23

[11] Rényi, A.: Ars Mathematica. *Typotex, Budapest* (2005)

[12] Révész, P.: Regularities and irregularities in a random 0,1 sequence. *Statistical Papers* 31, (1990) pp. 95-101.

[13] Schilling, Mark F.: The longest run of heads. *College Math. J.* 21 (1990) no. 3, pp. 196-207.

[14] Székely, J. G., Richards, P.: *The St. Petersburg Paradox and the Crash of High-Tech Stocks in 2000* The American Statistician, August 2004. Vol 58, No.3. pp. 225-231.

Open problems as means for promoting mathematical thinking and understanding

Erkki Pehkonen
Teacher Education, University of Helsinki, Finland
erkki.pehkonen@helsinki.fi

Abstract

The paper focuses on one method to promote mathematical thinking and understanding: the use of problem fields. They are open problems that are gaining more and more emphasis in all over the world. When using open problems in a proper way, a teacher can improve his mathematics teaching in school. The terms 'open problem' and 'problem field' will be discussed and enlightened with some examples. Furthermore, an example of problems fields is presented and dealt with in detail.

Key words: problem solving, open problems, mathematical thinking

ZDM Classification: D50

Introduction

In Finland the objectives of mathematics teaching for all school forms contain understanding of mathematical structures and development of mathematical thinking, not only the practicing of calculation skills (NBE 2004). The same is valid also for other countries, for example Germany (KMK 2005) and the States (NCTM 2000). The purpose of school teaching in all countries is to develop such citizens that are independent and eigen-initiative as well as are thinking critically. Furthermore, pupils should learn to be motivated and able to meet the situations encountered in their life.

This purpose is not so easy to implement in the conventional mathematics teaching, where the teacher is very eager to take a mathematics book and practice with some calculation tasks. Therefore, one should add new obligatory elements into the school instruction: such tasks where the teacher will practice especially his[1] pupils' problem solving and thinking skills.

On mathematical understanding

Mathematical understanding can be characterized as a continuous process that is fixed to a certain person, a mathematical content domain and a special environment (cf. Hiebert & Car-

[1] If we don't know the gender of the person in question, we will use the male form.

penter 1992). Mathematical understanding answers the question ”Why?” and, in addition, entails, among other factors, the skills required to analyze mathematical statements. In the US-Standards on Mathematics Teaching given by the NCTM about 15 years ago, understanding is one of the important goals of mathematics teaching (cf. NCTM 2000).

In recent decades there have been numerous research projects on pupils' mathematical understanding. Different concepts of understanding were developed that might be important e.g. to curriculum development, evaluation of mathematics teaching or teacher education. In her overview study, Mousley (2005) distinguishes between three types of models for understanding mathematics: understanding as structured progress, understanding as forms of knowing, and understanding as process.

In the first category are, for example, models that are based on ideas of Piaget, or Vygotsky's "zones of development". A well-known model of the second category was developed by Skemp who firstly differentiated between instrumental und relational understanding and later added logical understanding as a third kind (Skemp, 1987). Whereas Pirie and Kieren (1994) presented a model of the third type: understanding as process.

Another aspect of understanding is presented by Leinonen (2011) who discusses different kinds of understanding and their meaning in the learning process of problem solving. According to him understanding has, in this context, four modes: conceptual knowledge, grasping meaning, comprehension and accommodation. The function of those modes is to give the background and conceptual instruments for thinking, to interpret the information, to synthesize the knowledge, to integrate the message into permanent memory, and to reorganize the cognitive structure.

In the reported study we investigate development of pupils' (and teacher') mathematical understanding. Therefore, we will use here the terms of Leinonen when speaking on understanding und trying to understand the term "understanding".

Open approach

Problem solving has a long tradition in school mathematics. Since in the literature there exists no generally accepted characterization for problem solving, we will introduce here the definition we are using, in order to have a common language, when speaking about tasks and problems:

Such a task situation will be named a *problem*, where an individual needs to combine the (for him) known information in a (for him) new way, in order to be able to solve the problem (cf. Kantowski 1980). If he knows the procedures needed, the task is for him a *routine task* (or a practice task or a standard task). Often one uses the term *non-standard task* that means such a task that one cannot, especially earlier, usually find in mathematics school books.

For example, Schröder & Lester (1989) have pointed out that problem solving should not be considered only as a teaching content, but also as a teaching method. In their paper, they introduced three different aspects of teaching problem solving (PS): teaching *about* PS, teaching *for* PS, teaching *via* PS. In the following we concentrate on the last aspect, i.e. using problem solving as a teaching method.

Open problems

The most mathematics problems in the school textbook are *closed*, i.e. the starting and goal position are exactly formulated. Whereas a problem is called *open*, if its starting and/or goal position are not exactly given (cf. Pehkonen 1995a). In such problems, pupils have some free space for solving the problem, since there are some parameters in the problem that the pupils might decide themselves. This means that they can result different but also correct solutions, since the result depends on their additional hypothesis during the solution process. Therefore, open problems have usually many correct answers. When the teacher uses open problems in his mathematics teaching, his pupils have an opportunity to work as creative mathematicians (cf. Brown 1997). Thus, in using open problems in school mathematics pupils can experiences what mathematics really is.

Using only conventional mathematics tasks will limit pupils' understanding of mathematics rather easily to a very narrow one, whereas with the help of open problems this view can be enlarged. Open problems offer on one hand to pupils more discretion in the solving phase, but on the other hand they are compelled to use their knowledge more diverse.

Example 1: *Pupils are given a parallelogram that is cut from a DINA4 paper and a pair of scissors. Their task is to find out, whether it is possible to cut the parallelogram into two pieces in such a way that one may put from the pieces a rectangular together. The next question might be, whether there is another solution to the problem. How many different solutions are there altogether?*

When using open problems we can response challenges of the developing mathematics teaching. Such a teaching will lead almost automatically to problem-oriented teaching and will clearly head to increasing of communication. This is a way to add also openness and pupil-centeredness to teaching.

In the literature, there are some overviews on the situation of open problem solving. About ten years ago Pehkonen (2004) wrote a overview of open problem solving world-wide. And some years later Zimmermann (2010) described the development of open problem solving within the last 20 years in Germany. Both papers deliver a broad picture on the use of open problems.

Problem posing

In the Finnish curriculum (NBE 2004), one of the general goals for all subjects is the development of pupils' creativity. If a teacher aims in his mathematics teaching especially to the development of creativity, the use of problem solving is not enough. It is true that in problem solving there is a creative element always present but not so much freedom, and creativity demands freedom. Thus, a natural solution is to give pupils more freedom, e.g. in such a way that they can solve problems formulated themselves.

About 30 years ago, the open approach method was developed in Japan (cf. Nohda 1991) that would maximize the freedom in problem solving: open-ended tasks. Here the teacher gives to his pupils a problem, but the question is not to solve the problem but the pupils should just find as many different solution ways as possible (cf. Shimada 1997). Investigations developed in 1970's in England form another type of open-ended problems (cf. Cockcroft 1982). In an *investigation*, the teacher gives to his pupils some starting problems in a problem situation, and then the pupils solve the problems and continue on their own.

Problem posing and problem generating have been studied more than 20 years in the mathematics education (cf. Brown & Walter 1983). *Problem posing* means the formulation (or posing) of a problem from a situation, in such a way that it becomes into a solvable form. But behind every such a problem there is a big variety of potential interesting problems, i.e. the variations of the original problem (cf. Schupp 2002). One strategy to grip such unknown problems is to put forward a couple of basic questions on the problem in question: What kind of information do we get from the problem? What kind of information do we have on the unknown (the wanted solution)? What kind of restrictions do we have for the solution? (cf. Moses & al. 1990). And when reformulating this information we will get a big variety of related problems.

The following example (Ulam's spirale) points out how problem posing is functioning. First the situation (i.e. the framework) will be given within which pupils may work. Pupils should generate themselves an interesting problem in the framework and formulate it, and at the end the problem will be solved. If the teacher gives such a freedom for his pupils, the learning atmosphere of the class will be improved (cf. Pehkonen 1995b).

Example 2: *The Ulam's spirale*

The positive integers will be written in a form of a spiral (Fig. 1). Formulate a problem that is connected to this number schema, and try to solve it.

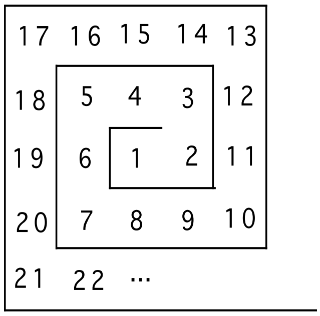


Fig. 1. The starting situation in the Ulam's spirale (e.g. Brown & Walter 1990, 94).

At first, a solver should find a proper problem and formulate it (problem posing). Some examples of possible problems could be the following: "Is there some pattern in the diagonal number sequence 1, 3, 13, …?" and "What would be the n. term in the diagonal sequence?"

It is good to notice that from the Ulam's spirale one can develop very many problems that are proper also for mathematics teaching of secondary schools. Then the pupil in question should think different methods to solve the problem. Therefore, this represents such a situation where professional mathematicians (research mathematicians) are when they try to generate new information (Brown 1997). Nobody will give them a ready-formulated problem, but they must find it themselves and formulate it, and try to solve it.

Problem fields in teaching

A typical investigation is an open-ended problem where pupils have the starting situation, and they should explore further. Investigations can be divided into structured and non-structured ones. The latter ones have been used in England since the 1970's (cf. Cockcroft 1982): A problem situation is given to pupils as well as a couple of starting problems, and then the pupils should work independently further. The structured investigations are called *problem fields* (or problem domains or problem sequences). Here the teacher has a lot of further questions (problems) on the starting situation, and he decides according to the solving activity of the teaching group, which way he will take and how far he will continue with the problem field in question.

An example: Calculation Pyramid

Here we will look at one concrete teaching unit of problem fields. These problem fields are planned for lower secondary mathematics teaching (about 13–15 years old pupils) in Finland. The problem field in question is published earlier in a German teacher journal (cf. Pehkonen 1991).

The objectives for the use of this problem field are two-fold: 1) Pupils practice the calculation with integral numbers. 2) Pupils learn to reason by making different connections. For pupils the important tool here is to make a guess on a general rule, and to check their guess with calculations. Thus pupils begin to see the reasons why their guess is correct or not.

Example 3: *Calculation Pyramid*

The scheme in Fig. 2 will be called a four-step calculation pyramid that has -2, 3, -7 and 5 as its start numbers.

Add always the numbers in the two boxes below and put the sum into the box above. Which number will you get into the uppermost box?

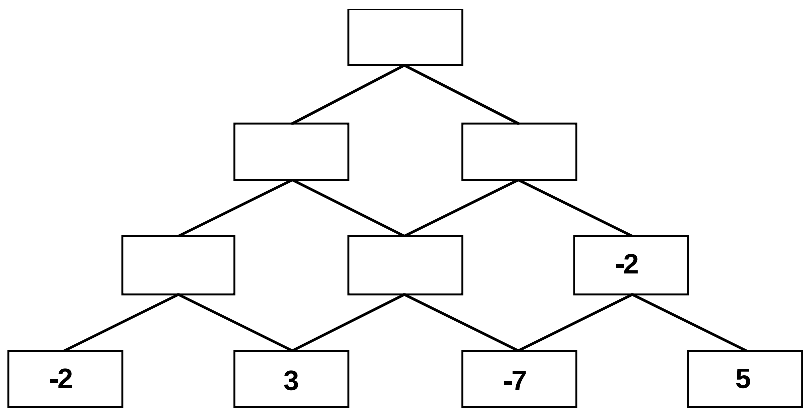


Fig. 2. A four-step calculation pyramid.

<u>Question series 1</u>:

If one changes any of the start numbers (-2, 3, -7, 5), what kind of change has it in the number of the uppermost box?

Try to change only one start number and with only one unit (e.g. -1, 3, -7, 5), and make your suggestion before calculation.

Can you find a pattern, in order you can tell beforehand what kind of change will happen? Check your idea?

Question series 2:

In the example above (Fig. 2) the number -9 was found as the uppermost number.

Investigate, whether it is possible to construct another number pyramid (i.e. with different start numbers than -2, 3, -7 and 5) where the uppermost number is -9.

Can you find still another different number pyramid that also has -9 as its uppermost number?

How many different number pyramids are there altogether that has -9 as its uppermost number? What would you guess? How could you reason your hypotheses?

Question series 3:

Try to construct a number pyramid that has 10 as its uppermost number.

What about a number pyramid that has -100 as its uppermost number? Investigate which numbers are possible as the uppermost number in a number pyramid. Why?

My teaching experiences

This problem field has been probed in several classes in natural settings. It represents such a mathematical problem that we have developed for the classroom use in heterogeneous classes of the Finnish comprehensive school (e.g. Pehkonen 1997). Within every problem field the level of problem difficulty varies from very easy ones that probably the whole class can solve, to more complicated problems that, perhaps, only the most talented pupils can solve.

The problem field is very flexible in the following sense: If we change the negative numbers in the starting situation to positive ones, the problem field is proper to use also in the elementary level. And if we take complex numbers (a + bi), the problem field is good for upper secondary level to practice addition of complex numbers.

On the use of problem fields

Since problem fields are planned to promote in the first place problem solving skills and creativity, they are not connected to a certain grade level (cf. Pehkonen 1989). The same problem field is usually proper for mathematics teaching from elementary level to teacher education. The role of easier problems in a problem field is to strengthen pupils' problem solving persistence, and offer the participating possibility also to low-achievers.

The most important aspect in these problem fields is the way they are presented in class: A problem field should be given to pupils in small pieces, and the continuation with it depends on pupils' solutions. Instead of answers and results the most important in the process of solving a problem field is pupils' independent solving process; therefore, the answers are not given here. The essential task of this method is to apply and develop pupils' creativity. How far the teacher will work in a class with one problem field depends essentially from pupils' answers. When pupils do not find more solutions, the teacher can leave the problem field and, perhaps, come back later on.

During the solution process the teacher's role is, as Schoenfeld (1985) puts it, only to act as a discussion moderator and assistant. I.e. the teacher writes everything offered from the class on the blackboard (also possible errant ways) without any comments, and at the end all offered solution possibilities shall be discussed together. Then the class (pupils together) decide the correctness of the solution paths (and errant ways), and will check their credibility. Of course, the teacher may use pupils' ideas for new problems, and thus develop the problem field further.

The use of problem fields is not meant to substitute conventional mathematics teaching, but to enrich it. Pupils will learn other part of mathematics than plain calculations. When one uses a problem field in the normal school teaching, the probed method is to discuss them in small pieces at the end of the teaching unit (e.g. about 10 min). And to give as much as possible of the rest of the problems to pupils as a homework. Then all pupils have enough time to think about the problems, and therefore, success experiences are more common. The rest of the lessons (beginning) can be used e.g. for conventional mathematics teaching.

What is the meaning of problem fields?

Conventional school teaching has been accused of that it considers as totally separate the action and the context where learning happens. However, psychological investigations have shown that (also mathematics) learning is strongly situation-bound (e.g. Brown & al. 1989, Collins & al. 1989, Bereiter 1990). Learning psychological research has confirmed the earlier hypotheses that learning of facts and processes happens via different mechanisms (Bereiter & Scardamalia 1996). Therefore, we do need new elements into the school learning process, for instance open problems.

Conventional teaching is proper for learning facts, but for processes new elements will be needed, such elements that emphasis pupils' spontaneous studying. The use of open problems offers here an opportunity, since they enable the use of real problems and learning in natural settings. Solving open problems sets pupils into real problem solving environment, and thus it can combine phenomena of real world and classroom.

The use of problem fields brings openness into the teaching. Furthermore, the use of problem fields promotes pupils' thinking skills and creativity which both are important in pupils' future life. The usual understanding of mathematics as a rigid and abstract discipline can be replaced by a wider view of mathematics: mathematics can also be enjoyable. As a result, there are more pupils who enjoy doing mathematical tasks and will have experiences of success.

When pupils are solving problems during mathematics lessons, especially open problems, according to my experiences also the teacher will experience mathematics lessons more interesting. He must think himself on the problems, and not only mechanically do some routine procedure. He should learn to accept the fact that he does not know everything, and not to be afraid to show it. Thus his pupils will consider him more human. Today one should develop richer problem fields and probe them in mathematics teaching. Additionally, it is important to convoy the information to other teachers e.g. in teachers' journals.

If one will develop the use of open problems in teaching of school mathematics, the teacher's own conceptions on good mathematics teaching are very strongly in a leading position: they conduct the implementation of instruction. If the conception of open teaching is not coherent with the teacher's understanding of teaching, the new teaching will not be successful, although the teacher has been trained to use open problems. Therefore, the teacher's own beliefs and conceptions on mathematics teaching are in a key position (cf. Shaw & al. 1991).

End notes

The Finnish curriculum demands that in teaching of mathematics one should develop, besides calculation skills, also skills in problem solving and mathematical thinking, from the very beginning (NBE 2004). This purpose does not seem to be implemented within the conventional teaching where the teacher too eagerly takes the textbook and uses the teaching method of working with calculation tasks. Therefore, one should bring new elements into the instruction: such tasks with which the teacher can develop his pupils problem solving and thinking skills.

In order teachers can implement such teaching, they should have interests in developing their teaching as well as they should be committed to implement new ideas in their classrooms (cf. Shaw & al. 1991).

Literature

Bereiter, C. 1990. Aspects of an Educational Learning Theory. *Review of Educational Research*, 60 (4), 603 - 624.

Bereiter, C. & Scardamalia, M. 1996. Rethinking learning. In: *The handbook of education and learning. New models of learning, teaching and schooling* (eds. D.R. Olson & N. Torrance). Cambridge (MA): Blackwell.

Brown, S. I. 1997. Thinking Like a Mathematician: A Problematic Perspective. *For the Learning of Mathematics* 17 (2), 36–38.

Brown, J.S., Collins, A. & Duguid, P. 1989. Situated cognition and the culture of learning. *Educational Researcher*, 18 (1), 32 - 42.

Brown, S.I. & Walter, M.I. 1983. *The art of problem posing*. Philadelphia (PA): Franklin Institute Press.

Cockcroft Report 1982. *Mathematics counts.* Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. H.M.S.O.

Hiebert, J. & Carpenter, T.P. 1992. Learning and teaching with understanding. In: *Handbook of research on mathematics learning and teaching* (ed. D.A. Grouws), 65–97. New York: Macmillan.

Kantowski, M.G. 1980. Some Thoughts on Teaching for Problem Solving. In: *Problem solving.* NCTM Yearbook 1980, 195-203.

KMK 2005. *Bildungsstandards im Fach Mathematik für den Primarbereich.* München: Luchterhand.

Leinonen, J. 2011. Understanding and Mathematical Problem Solving. In: K. Szücs & B. Zimmermann (eds.), *Proceedings of the ProMath meeting in Jena 2010*. University of Jena.

Moses, B., Bjork, E. & Goldenberg, E.P. 1990. Beyond problem solving: Problem posing. In: *Teaching and learning mathematics in the 1990s* (eds. T.J. Cooney & C.R. Hirsch), 82–91. NCTM 1990 Yearbook. Reston (VA): NCTM.

Mousley, J. 2005. What Does Mathematics Understanding Look Like? In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce (Eds.), *Building connections: Theory, research and practice. Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne.* Sydney: MERGA.

NBE 2004. *Perusopetuksen opetussuunnitelman perusteet 2004* [The basics of the curriculum for the basic instruction 2004]. National Board of Education. Helsinki: Opetushallitus.

NCTM 2000. *Principles and Standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nohda, N. 1991. Paradigm of the ”open-approach” method in mathematics teaching: Focus on mathematical problem solving. *Zentralblatt für Didaktik der Mathematik* 23 (2), 32–37.

Nohda, N. 2000. Teaching by Open-Approach Method in Japanese Mathematics Classroom. In: *Proceedings of the PME-24 Conference* (eds. T. Nakahara & M. Koyama), Vol.1, 39–53. Hiroshima University (Japan).

Pehkonen, E. 1989. Der Umgang mit Problemfeldern im Mathematikunterricht der Sek. I. In: *Beiträge zum Mathematikunterricht 1989*, 290-293. Verlag Franzbecker, Bad Salzdetfurth.

Pehkonen, E. 1991. Die Rechenpyramide: ein Problemfeld zur Entwicklung des Denkens und zur Verbesserung der Rechenfertigkeit. *Mathematische UnterrichtsPraxis* 12 (1), 25–29.

Pehkonen, E. 1995a. Introduction: Use of Open-ended Problems. *Zentralblatt für Didaktik der Mathematik* 27 (2), 55-57.

Pehkonen, E. 1995b. On pupils' reactions to the use of open-ended problems in mathematics. *Nordic Studies in Mathematics Education* 3 (4), 43–57.

Pehkonen, E. 2004. State-of-the-Art in Problem Solving: Focus on Open Problems. In: *ProMath Jena 2003. Problem Solving in Mathematics Education* (eds. H. Rehlich & B. Zimmermann), 93–111. Hildesheim: Verlag Franzbecker.

Schoenfeld, A.H. 1985. *Mathematical problem solving*. Orlando (FL): Academic Press.

Schroeder, T.L. & Lester, F.K. 1989. Developing Understanding in Mathematics via Problem Solving. In: *New Directions for Elementary School Mathematics* (eds. P. R. Trafton & A. P. Shulte), NCTM 1989 Yearbook, Reston (VA): NCTM.

Schupp, H. 2002. *Thema mit Variationen. Aufgabenvariation im Mathematikunterricht*. Verlag Franzbecker: Hildesheim.

Shaw, K.L., Davis, N.T. & McCarty, J. 1991. A cognitive framework for teacher change. In: *Proceedings of PME-NA 13* (ed. R.G. Underhill). Volume 2, 161–167. Blacksburg (VA): Virginia Tech.

Shimada, S. 1997. The Significance of an Open-Ended Approach. In: *The open-ended approach* (eds. J. Becker & L. Shimada), 1–9. Reston (VA): NCTM.

Skemp, R. R. 1987. *The psychology of learning mathematics*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Zimmermann, B. 2010. „Open ended problem solving in mathematics instruction and some perspectives on research question" revisited – new bricks from the wall? In: *Problem Solving in Mathematics Education. Proceedings of the 11th ProMath conference in Budapest* (eds. A. Ambrus & E. Vasarhelyi), 143–157. Eötvös Lorand University.

Solving Diophantine equations with elementary methods and with the help of Gaussian integers in high school mathematics study group sessions

Richárd Rakamazi

Baár-Madas Secondary School Budapest, Hungary

raka82@gmail.com

Abstract

Through solving Diophantine equations, this paper aims to illustrate how we can apply successfully the Gaussian integers in high school study group sessions or even in a math contest. The complex numbers $z = a + bi$, where a and b are integers, are called Gaussian integers. Actually, the complex numbers do not occur in high school curriculum, therefore the Gaussian integers are not mentioned either. However, if we disregard the usual precise discussion of the theory, and refer to the most difficult theorems without proof, then we can solve interesting and beautiful number theory problems without requiring extensive background knowledge. We start the paper with some "usual" elementary solutions of contest problems, showing different methods and also some generalizations. These can help the deeper understanding, and motivate the most talented students. We give also a solution with Gaussian integers to each problem. In the second part of the paper we deal with Diophantine equations, where elementary solution is often very difficult, and it is much more efficient to work with the Gaussian integers. As an illustration, also the author's own elementary solution is presented for the Mordell equation $x^2 + 1 = y^3$.

Key words and phrases: Diophantine equations; Gaussian integer; problem solving situation

ZDM classification: D50, F50, F60

The students learn about Diophantine equations in the $9th$ grade first. The theme usually arises in classes with a stronger mathematical curriculum at the end of an algebraic section. The time given for this topic comprises usually $3-4$ lessons, which proves to be sufficient in most cases for the students to get acquainted with the basic tools and methods. The students can experience how they can use their factoring and other techniques in these new problem-solving situations. The teacher can show some examples for Diophantine equations – especially linear ones –, where we rely on the previously acquired knowledge in number theory. Another typical situation is when we want to prove, that an equation has no integer solutions. In these cases, generally, we have to consider the possible remainders for some modulus of certain types of terms (square numbers, cubes, etc.). Study group sessions help to develop the mathematical problem-solving ability of talented and motivated students. The material is integrally connected to the curriculum of the $9th$ grade and deepens thoroughly the skills and understanding of the students. Fortunately, teachers usually have sufficient freedom to provide some time to deal in depth with Diophantine equations according to their own discretion. There are many excellent problem collections, study group session booklets ([2], [3], [4]) featuring also Diophantine equations. The outward goal of the study group sessions is to promote the successful preparation to math contests. Since in these contests we usually meet Diophantine equations, it is useful to update and extend the relevant knowledge later (in $11th$ and $12th$ grades). It is our personal observation, that the majority of the students participating in the study group sessions like solving equations, in particular if it is a challenge or it has several different approaches. In this paper the author presents a possible extension of the topic which will be "tested" by his own students in a study group session in the near future. The discussion requires the basic concepts of complex numbers leading us to the Gaussian integers. From the theory of the Gaussian integers we shall use without proof the unique prime factorization theorem, the units and the list of Gaussian primes (see e.g. [1]). Armed with this new knowledge we will be able to solve interesting problems which are often very difficult to be handled with elementary tools. We are confident that, using this experience, the students will be more effective in the various national contests later, and also better prepared for the mathematical studies in higher educational institutions.

In the first part of the paper we discuss several solutions of two contest problems. The Diophantine equation $(xy-7)^2 = x^2 + y^2$ will be solved by five different methods the first four of which are elementary, and the last one uses Gaussian integers. For the second problem we present three different solutions, one of which uses Gaussian integers. As an interesting application we solve the Diophantine equation $p^2 + 4 = a^3$ related to the problem. This leads us to the second part of the paper where we examine the so-called Mordell equation $x^2 + k = y^3$ for some values of k. For $k = -7$ we present an elementary solution, whereas for $k = 1$ we give two different solutions, from which the first one is the author's own argument, illustrating that the elementary approach seems to be rather complicated, while the second one is a simpler proof using Gaussian integers. As a byproduct of the elementary solution, we determine also the integer solutions of the equation $n^4 - 6m^2n^2 - 3m^4 = 1$ with infinite descent. Finally we solve the generalized Mordell equation $x^2 + 1 = y^n$, $n \geq 3$, using Gaussian integers.

Problem 1. *Solve the Diophantine equation* $(xy-7)^2 = x^2 + y^2$ *among the nonnegative integers. (Indian National Mathematical Olympiad* 1990*)*

Solution 1

Expanding the expression on the LHS, and adding $2xy$ to both sides, we get

$$(xy-6)^2 + 13 = (x+y)^2.$$

We can rewrite this as

$$\left[xy-6-(x+y)\right]\left[xy-6+x+y\right] = -13.$$

Since the factors are integers and $xy-6-x-y \le xy-6+x+y$, therefore only the following two cases are possible

$$\begin{aligned} xy-6-x-y &= -1 \\ xy-6+x+y &= 13, \end{aligned}$$

$$\begin{aligned} xy-6-x-y &= -13 \\ xy-6+x+y &= 1. \end{aligned}$$

From the first system of equations we get the solutions $x=3$, $y=4$, and $x=4$, $y=3$. From the second system we obtain $x=0$, $y=7$, and $x=7$, $y=0$.

Solution 2

If $x=0$, then $y=7$, and if $y=0$, then $x=7$. If for example $x=1$, then $(y-7)^2 = y^2+1$ has no integer solutions in y. Hence we may assume $x, y \ge 2$. If $xy-7 \ge x+y$, then $x^2+y^2 = (xy-7)^2 \ge (x+y)^2 > x^2+y^2$ yields a contradiction. Therefore, only $xy-7 < x+y$, i.e. $(x-1)(y-1) < 8$ is possible, which can be satisfied only by finitely many integers $x, y \ge 2$. For symmetry reasons it is enough to check $2 \le x \le 8$, and we can get an integer for y only for $x=3$ and $x=4$. The relevant values for y are $y=4$ and $y=3$.

Solution 3

We rewrite the equation as

$$(y^2-1)x^2 - 14yx + 49 - y^2 = 0.$$

If we consider this as a quadratic equation in x, then we can get integer solutions only if the discriminant is a perfect square, i.e.

(1) $$4y^4 - 4y^2 + 196 = d^2$$

for some integer $d \geq 0$, and we get $x = \frac{14y \pm d}{2y^2 - 2}$ from the quadratic formula. We can transform (1) into

$$(2y^2 - 1)^2 + 195 = d^2 \iff (d - 2y^2 + 1)(d + 2y^2 - 1) = 195.$$

If $y = 0$, then $x = 7$ from the original equation. If $y \neq 0$, then the second factor of the LHS of the last form of our equation is greater than the first one. Therefore, only the following cases are possible:

$$\begin{cases} d - 2y^2 + 1 = 1 \\ d + 2y^2 - 1 = 195, \end{cases} \quad \begin{cases} d - 2y^2 + 1 = 3 \\ d + 2y^2 - 1 = 65, \end{cases} \quad \begin{cases} d - 2y^2 + 1 = 5 \\ d + 2y^2 - 1 = 39, \end{cases} \quad \begin{cases} d - 2y^2 + 1 = 13 \\ d + 2y^2 - 1 = 15. \end{cases}$$

We obtain $d = 98, 34, 22$, and 14, resp., and these will yield the same nonnegative integer solutions which we found by the previous methods.

Solution 4

If $x = 0$, then $y = 7$, and if $y = 0$, then $x = 7$. Further, suppose that $x, y > 0$. From $(xy - 7)^2 = x^2 + y^2$ we have $\gcd(x, y, xy - 7) = 1$ or $\gcd(x, y, xy - 7) = 7$. In the first case the well-known formulas for the Pythagorean triples give

$$x = u^2 - v^2$$
$$y = 2uv$$
$$xy - 7 = u^2 + v^2$$

for some integers u, v of opposite parity with $\gcd(u, v) = 1$ and $u > v > 0$. From here $u^2 + v^2 = 2uv(u^2 - v^2) - 7$. Since $v \leq u - 1$, therefore

$$u^2 + (u - 1)^2 \geq 2uv(u - v)(u + v) - 7 \geq 2u(u + 1) - 7,$$

from which we get $2 \geq u$. This is possible only for $u = 2$, $v = 1$, and we will get the solution $x = 3$, $y = 4$. For symmetry reasons, also interchanging the variables provides a solution.

If $\gcd(x, y, xy - 7) = 7$, then $x = 7x'$, $y = 7y'$, $xy - 7 = 7(7x'y' - 1)$, and we have to solve the equation $x'^2 + y'^2 = (7x'y' - 1)^2$, where $x', y', 7x'y' - 1$ are already relatively primes. Thus, there are integers r, s of different parity for which $r > s > 0$, $\gcd(r, s) = 1$, and

$$x' = r^2 - s^2$$
$$y' = 2rs$$
$$x'y' - 1 = r^2 + s^2.$$

This leads to the equation $2rs(r^2 - s^2) - 1 = r^2 + s^2$. A modulo 4 check shows that this has no integer solutions.

Solution 5

From Solution 4, we use that $\gcd(x,y,xy-7)=1$, or $\gcd(x,y,xy-7)=7$.

If $\gcd(x,y,xy-7)=1$, the variables x and y have opposite parities, since otherwise we get a contradiction modulo 4. Assume that x is odd and y is even. The RHS of the equation can be factored among the Gaussian integers:

$$(xy-7)^2=(x+yi)(x-yi).$$

We show that $\gcd(x+yi,x-yi)=1$. If $\gamma\,|\,x+yi$ and $\gamma\,|\,x-yi$, then $\gamma\,|\,2x$ and $\gamma\,|\,2yi$, hence $\gamma\,|\gcd(2x,2yi)=2$. Since $2=(-i)(1+i)^2$ and $1+i$ is a Gaussian prime, therefore γ is a unit or is divisible by $1+i$. However, this latter is not possible, since $\frac{x+yi}{1+i}=\frac{x+y+(y-x)\cdot i}{2}$ is not a Gaussian integer. By the unique prime factorization theorem (UFT) for the Gaussian integers we have $x+yi=\varepsilon(u+vi)^2=\varepsilon(u^2-v^2+2uvi)$, where u and v are integers and ε is a unit. For $\varepsilon=1$, we have $x=u^2-v^2$ and $y=2uv$, by comparing the real and imaginary parts. Let us note, that if ε is another unit, the solutions are similar as above. The handling of the problem can be completed with the formulas for x and y similarly as in Solution 4. Finally, the case $\gcd(x,y,xy-7)=7$ can be treated the same way as described above.

Remark 1. In Solution 1 we used a standard method for Diophantine equations. We formed a product on the LHS side of the equation so it was enough to check the divisors of the integer on the RHS, and solve the appropriate systems of equations. The students were familiar with some similar examples, thus we can expect, that most of the solutions given by the students will follow this line. In Solution 2 we considered the order of magnitude of the two sides: the LHS was greater than the RHS, apart from finitely many integer values of the variables. Here the students can make some nice observations, since this method might already have been introduced in the study group sessions. A typical example is to show that an expression cannot be a square by proving that it falls between two consecutive squares. In Solution 3 we considered one variable as a parameter and solved the quadratic equation for the other variable. This type of application is less familiar to the students, even though they already met with parametric equations in their earlier studies. For Solution 4 one has to know the characterization of the Pythagorean triples, which is discussed only in classes with a higher number of math lessons. And we cannot expect Solution 5 even from them, since it uses Gaussian integers, which is currently not part of any high school curriculum. However, students learning math along a stronger curriculum often get familiar with complex numbers in the study group sessions, and develop a routine for working with these numbers. If the group consists of motivated, interested and outstandingly talented students, then we can talk to them about Gaussian integers, and use certain theorems, usually without rigorous proof, of course. Finally, we note that in the manner seen in Solution 5 we can get the formula for the Pythagorean triples directly.

Remark 2. If we are looking for all integer solutions, not only for the nonnegative ones, then we can argue as follows. If (x,y) is a positive solution, then obviously $(-x,-y)$ and $(-y,-x)$ are integer solutions, as well. The solutions where x and y have opposite signs can be derived from the positive solutions of $(xy+7)^2=x^2+y^2$, and this equation can be treated similarly as our original one. Another generalization is the Diophantine equation $(xy-c)^2=x^2+y^2$, where c is a fixed integer. Our first method shows, that this equation can have only finitely many solutions, since it is equivalent to $\left[xy-(c-1)\right]^2+2c-1=(x+y)^2$.

Problem 2. *Let $3\le p<q$ be primes where both $p+1$ and $q+1$ are divisible by 4. Show, that q^2-p^2 cannot be a square (Dániel Arany Mathematical Contest 1976)*

Solution 1

Proof by contradiction. Assume that for some integer a we have $q^2-p^2=a^2$, i.e. $(q-p)(q+p)=a^2$. If $\gcd(q-p,q+p)=d$, then adding and subtracting relations $d\mid q-p$ and $d\mid q+p$, we obtain $d\mid 2p$ and $d\mid 2q$. This implies $d\mid \gcd(2p,2q)=2$, and since $q-p$ and $q+p$ are even, we have $\gcd(q-p,q+p)=2$. Dividing the equation by 4, we obtain

$$\frac{q-p}{2}\cdot\frac{q+p}{2}=\left(\frac{a}{2}\right)^2.$$

The factors on the LHS are positive and relatively prime, hence each factor must be a square itself, i.e. $\frac{q-p}{2}=x^2$ and $\frac{q+p}{2}=y^2$ for some integers x and y. Adding these we get $q=x^2+y^2$, which is a contradiction, since a number of form $4k-1$ cannot be the sum of two squares. (Actually, we used only condition $4\mid q+1$, and this holds also for the next two solutions.)

Solution 2

Again, assume $q^2-p^2=a^2$, or $a^2+p^2=q^2$. Obviously $\gcd(a,p,q)=1$, therefore a,p,q

form a primitive Pythagorean triple. Thus for some integers u, v, $\gcd(u,v)=1$, $u>v>0$ and $2 \nmid u-v$

$$p = u^2 - v^2$$
$$a = 2uv$$
$$q = u^2 + v^2.$$

The last equation yields a contradiction, because a number of form $4k-1$ cannot be the sum of two squares.

Remark 3. The argument in Solution 1 already shows the way, how we can prove the formulas for the Pythagorean triples. It may well prepare the proof and the better understanding of the theorem about these triples.

Solution 3

This proof uses Gaussian integers. Assuming again $q^2 = a^2 + p^2$, the RHS can be factored among the Gaussian integers: $q^2 = (a+pi)(a-pi)$. Since q is a prime number of form $4k-1$, it is prime also among the Gaussian integers. Hence it can divide the RHS only if it divides (at least) one of the factors. If e.g. $q \mid a+pi$, then $\frac{a}{q} + \frac{p}{q} i$ is a Gaussian integer. However, this contradicts $q \nmid p$.

In the following, we examine an interesting variant of the problem with the help of Gaussian integers. Instead of the difference of two squares of different primes, let us consider their sum. It is easy to see, that the sum cannot be a square, therefore let us investigate, when the sum equals a cube.

Problem 3. *If $0 < p < q$ are primes and a is a positive integer, then the only solution of $p^2 + q^2 = a^3$ is $p = 2$, $q = 11$, $a = 5$.*

Solution

The usual modulo 4 check implies that p and q must have opposite parities, hence $p = 2$ and q is odd. Factoring the LHS of $q^2 + 4 = a^3$ among the Gaussian integers we obtain

(2) $$(q+2i)(q-2i) = a^3.$$

We show that $\gcd(q+2i, q-2i) = \varepsilon$, where ε is a unit. Here

$\varepsilon \mid (q+2i)-(q-2i) = 4i = (-i)\cdot(1+i)^4$, where $1+i$ is a Gaussian prime and $\frac{q+2i}{1+i} = \frac{q+2}{2} + (\frac{2-q}{2})i$

shows, that $1+i$ is not a divisor of $q+2i$, hence ε is a unit, indeed.

Using the UFT and the fact that every unit is the third power of some unit, the factors on the LHS of (2) must be cubes themselves. Therefore $q+2i=(a+bi)^3$ for some integers a and b. Comparing the real and the imaginary parts, we get

$$q=a^3-3ab^2$$
$$2=3a^2b-b^3.$$

The second equality implies $b\,|\,2$, and we will obtain a solution only for $b=-2$, $a=-1$, $q=11$.

Remark 4. The equation $q^2+4=a^3$ just studied is a special case of the equation $x^2+k=y^3$, where $k\in Z$. These are commonly called Mordell equations, since Mordell proved in 1922, that for $k\neq 0$ the equation can have only finitely many solutions in integers, in fact no solutions at all in most cases. First we shall give an elementary solution for $k=-7$ (Problem 4). Then we turn to $k=1$, where we present two proofs (Problem 5). The first one is an elementary but rather complicated argument of the author, involving the elementary solution of another interesting Diophantine equation (Problem 6). The second one, using Gaussian integers, is much simpler. Finally, the generalization of the case $k=1$ for higher powers instead of cubes will be treated using Gaussian integers (Problem 7).

Problem 4. *The Diophantine equation* $x^2-7=y^3$ *has no integer solution.*

Solution

A modulo 4 check shows that y must be odd, moreover of form $4k+1$, and also $y\geq -1$. Now $x^2+1=y^3+8=(y+2)(y^2-2y+4)$, and $y+2\equiv 3 \pmod 4$ imply that the RHS has a prime divisor of form $4k+3$. But using Fermat's little theorem it can be proved, that x^2+1 cannot have such a prime divisor, a contradiction.

Problem 5. *The only solution for the Diophantine equation* $x^2+1=y^3$ *is* $x=0$, $y=1$.

Solution 1

We have to show that no solution exists where $x,y>0$. Write the equation as

$$x^2=(y-1)(y^2+y+1).$$

As $y^2+y+1=(y-1)^2+3(y-1)+3$ indicates, $\gcd(y-1,y^2+y+1)=1$ or 3.

If $\gcd(y-1, y^2+y+1)=1$, then

$$y-1=a^2$$
$$y^2+y+1=b^2.$$

for some integers a and b, but $y^2<y^2+y+1<(y+1)^2$ shows that y^2+y+1 cannot be a square. If $\gcd(y-1, y^2+y+1)=3$, then

$$\left(\frac{x}{3}\right)^2=\frac{y-1}{3}\cdot\frac{y^2+y+1}{3}.$$

The factors on the RHS are relatively prime, hence for some positive integers u,v with $\gcd(u,v)=1$ we have

$$\frac{y-1}{3}=u^2$$
$$\frac{y^2+y+1}{3}=v^2.$$

After substitution and rearrangement we get $3u^4+3u^2+1=v^2$. A modulo 4 check shows that u must be even. Substituting $u=2t$, and forming complete squares, we arrive at

$$3\left(8t^2+1\right)^2+1=4v^2 \Leftrightarrow \left(2v\right)^2-3\left(8t^2+1\right)^2=1.$$

In the decomposition $3(8t^2+1)^2=(2v-1)(2v+1)$ the factors on the RHS are relatively prime, therefore two cases can occur.

Case 1**.** With suitable odd and relatively prime integers $\alpha,\beta>0$, we obtain

$$2v-1=3\alpha^2$$
$$2v+1=\beta^2.$$

Then $\beta^2-3\alpha^2=2$, which has no integer solutions, since the LHS is 0 or 1 modulo 3.

Case 2**.** Now

$$2v-1=\alpha^2$$
$$2v+1=3\beta^2$$

imply $\alpha^2-3\beta^2=-2$ and $\alpha\beta=8t^2+1$. Here α and β are both odd of form $4l+1$, or both of them are of form $4l-1$, because their product is of form $4s+1$. The last two equations imply

$$(\alpha-\beta)(\alpha+3\beta)=16t^2.$$

An easy calculation shows, that the greatest common divisor of the factors on the LHS divides 4, hence according to our comment made for α,β, the greatest common divisor will be exactly 4. Thus $\frac{\alpha-\beta}{4}\cdot\frac{\alpha+3\beta}{4}=t^2$ implies that for suitable relatively prime m,n we have

$$\frac{\alpha-\beta}{4}=m^2$$
$$\frac{\alpha+3\beta}{4}=n^2$$

and $mn=t$. Solving this system of equations we get

$$\alpha=3m^2+n^2, \qquad \beta=n^2-m^2, \qquad \alpha>\beta.$$

Substituting this into equation $\alpha^2-3\beta^2=-2$, we obtain $n^4-6m^2n^2-3m^4=1$. Hence we are done, if we show that this equation has no solution in positive integers.

Problem 6. *The only solution of* $n^4-6m^2n^2-3m^4=1$ *in non-negative integers is* $n=1$, $m=0$.

Solution

We consider the equation as a quadratic equation in variable n^2. To obtain an integer solution, the discriminant must be a perfect square: $36m^4+4(3m^4+1)=z_1^2$ for some integer $z_1>0$. Dividing by 4, we obtain $12m^4+1=z^2$, where $z_1=2z$. This can be rewritten as

$$3m^4=\frac{z-1}{2}\cdot\frac{z+1}{2}.$$

The factors on the RHS are relatively prime, therefore two cases are possible: either $\frac{z-1}{2}=3A^4$ and $\frac{z+1}{2}=B^4$, or $\frac{z-1}{2}=A^4$ and $\frac{z+1}{2}=3B^4$, for some relatively prime integers A and B. These imply $B^4-3A^4=1$ and $3B^4-A^4=1$, resp. The second one is impossible modulo 3, hence we have $B^4-3A^4=1$, where $B=1$, $A=0$ is a trivial solution. We show by infinite descent that no solution exists with $B\geq 2$, $A\geq 2$. Assume the converse, and start with a solution, where A has a minimal value, then construct a solution with $0<t<A$, which yields a contradiction. A modulo 8 check shows that B is odd and A is even. Substituting $A=2C$, we get

$$12C^4=\left(\frac{B^2-1}{2}\right)\left(\frac{B^2+1}{2}\right).$$

The second factor is odd, and is not divisible by 3, thus

$$C^4=\left(\frac{B^2-1}{24}\right)\left(\frac{B^2+1}{2}\right).$$

Then, for some integers r and s we have $B^2-1=24r^4$ and $B^2+1=2s^4$, hence $s^4-12r^4=1$. Similarly to the previous argument, we obtain

$$r^4 = \left(\frac{s^2-1}{6}\right)\left(\frac{s^2+1}{2}\right).$$

From equations $s^2-1=6t^4$ and $s^2+1=2w^4$ we get $w^4-3t^4=1$. Since $B\geq 2$, $s\neq 1$ and $t\neq 0$, also $A=2C$, $C=rs$ and $r=tw$, thus $A=2wst>t$. Hence we obtained a solution of $x^4-3y^4=1$ where $x=w$ and $y=t<A$ which contradicts the minimality of A.

Solution 2 of Problem 5.

Clearly $\gcd(x,y)=1$, and by a modulo 4 check we see that x is even and y is odd. Factoring the LHS among the Gaussian integers we obtain

$$(x+i)(x-i)=y^3.$$

We will prove that $\gcd(x+i,x-i)=\varepsilon$, where ε is a unit. If $\delta \mid x+i$ and $\delta \mid x-i$, then $\delta \mid 2i$. Since $2i=(1+i)^2$ and $1+i$ is a Gaussian prime, it is enough to show that $1+i \nmid x\pm i$:

$$\frac{x+i}{1+i}=\frac{(x+i)(1-i)}{2}=\frac{x+1}{2}+\frac{1-x}{2}\cdot i,$$

and since x is even, the quotient is not a Gaussian integer, i.e. $\gcd(x+i,x-i)$ is a unit indeed. Since every unit is the cube of a unit, and the UFT is valid, therefore $x+i=(u+vi)^3$ for some integers u and v, i.e.

(3) $$x+i=u^3-3uv^2+(3u^2v-v^3)i.$$

Comparing the imaginary parts, we get $3u^2v-v^3=1$. Since v divides the LHS, only $v=1$, or $v=-1$ are possible. We get a solution for u only with $v=-1$, $u=0$. Substituting this into (3), we obtain $x=0$, $y=1$.

Now, let us solve the following generalization of Problem 5 by using Gaussian integers.

Problem 7. *For $n\geq 2$ the Diophantine equation $x^2+1=y^n$ has only trivial solutions, i.e. $x=0$, $y=1$, and for n even also $x=0$, $y=-1$.*

Solution

If $n=2k$, then $x^2+1=(y^k)^2$ means that two consecutive integers are both squares, hence $x=0$, $y^k=\pm 1$, from which we get $x=0$ and $y=\pm 1$, indeed.

For n odd, it is enough to prove the statement for the prime values of n, since a non-trivial solution for n would mean also a non-trivial solution for p using $x^2+1=\left(y^{\frac{n}{p}}\right)^p$ for some prime factor p of n. Thus consider $x^2+1=y^p$, where $p\geq 3$ is a prime. Clearly x is even and y is odd. Assume that we have a solution where $x\geq 2$ and $y\geq 3$. Let us factor the LHS among the Gaussian integers. For the sake of simpler calculations we will use the decomposition $(1-xi)(1+xi)$ instead of the usual $(x+i)(x-i)$ factorization:

$$(1-xi)(1+xi)=y^p.$$

As in the solution of Problem 5, we obtain that $\gcd(1-xi,1+xi)$ is a unit. Again, using the UFT and the fact that every unit is the pth power of a unit ($p\geq 3$ prime), therefore $1+xi=(a+bi)^p$ and $1-xi=(a-bi)^p$ for some integers a and b.

Expanding $1+xi=(a+bi)^p$ by the binomial theorem, and comparing the real parts we get

$$1=a^p-\binom{p}{2}a^{p-2}b^2+\binom{p}{4}a^{p-4}b^4-\ldots\pm\binom{p}{p-1}ab^{p-1}.$$

Since the RHS is divisible by a, therefore $a\,|\,1$, i.e. $a=\pm 1$. At the same time b is even, since otherwise each of the 2^{p-1} terms $\pm a^{p-2j}b^{2j}$ would be odd thus giving an even sum instead of 1. Since b is even, a modulo 4 check gives $a^p\equiv 1 \pmod 4$, hence only $a=1$ is possible. Substituting $a=1$, and dividing by ($b^2\neq 0$) we obtain

$$\binom{p}{2}=\binom{p}{4}b^2-\binom{p}{6}b^4+\ldots\pm\binom{p}{p-1}b^{p-3}.$$

We will show that the LHS is divisible by a smaller power of 2, than the RHS. Write $p-1=2^l s$, where $s\geq 1$ is odd. Then $\binom{p}{2}$ is divisible exactly by the $l-1st$ power of 2. A general term appearing on the RHS is

$$\frac{p(p-1)\cdots(p-2m+1)}{(2m)!}\cdot b^{2m-2},$$

where $2\leq m\leq\frac{p-1}{2}$. In the numerator every second factor is divisible by 2, hence the exponent of 2 is at least $l+(m-1)$, the exponent of 2 is at least $2m-2$ in b^{2m-2}. On the other hand, it is known, that every prime factor of $n!$ can appear at most with exponent $n-1$, hence the exponent of 2 in the denominator is at most $2m-1$. Thus the exponent of 2 in the general term is at least $l+m-1+2m-2-(2m-1)=l+m-2$. Since $m\geq 2$, thus $l+m-2>l-1$, and we have completed the proof.

References

[1] T. Andreescu, D. Andrica, and I. Cucurezeanu *An Introduction to Diophantine Equations*, Springer (2010), 151–162.

[2] L. Faragó *Elements of Number Theory second ed.*, Tankönyvkiadó Budapest, (1967) (in Hungarian)

[3] S. Róka, 2000 *Problems In Elementary Mathematics, fifth ed.*, Typotex Budapest, (2006), 29–33. (in Hungarian)

[4] W. Sierpinski, 250 *Problems In Elementary Number Theory*, American Elsevier Publishing Company, Inc. New York, Warszawa, (1970), 11–17.

Rethinking Heuristics – Characterizations and Examples

Benjamin Rott
Freiburg, Germany
benjamin.rott@ph-freiburg.de

Abstract

The concept of "heuristics" or "heuristic strategies" is central to (mathematical) problem solving and related research; however, there is no generally accepted definition of this term. Trying to clarify the concept might help avoiding misunderstands and difficulties in dealing with studies that use different terms meaning the same concepts or that use the same terms meaning different concepts.
I'm going to discuss differing definitions of the term "heuristics" on a theoretical basis as well as on the basis of judgments by specialists in mathematics education. The goals of this research are a clarification of the term and suggestions for the use of it in future research.

Key words: Heuristics, Problem Solving, Theoretical Foundation

ZDM classification: D 20, D50

Introduction

Heuristics or *heuristic strategies* are central to problem solving and related research. "Once nearly forgotten [before Pólya revived it], heuristics have now become nearly synonymous with mathematical problem solving." (Schoenfeld 1985, p. 23) But what are heuristics? Asked about heuristic techniques, examples like "means-ends-analysis" or "drawing a figure" come to mind quickly. But what does the term mean in general? And how can it be differentiated from routine techniques and algorithms on the one hand and from self- and process-regulatory activities (as parts of metacognition) on the other hand?

Despite its importance for problem solving (research), there is not a generally accepted definition of the term "heuristic". McClintock (1979, p. 174) calls heuristics "the ancient, ill-defined discipline" and other authors approve that there is no agreed upon characterization (e.g., Romanycia & Pelletier 1985).

In most studies the term "heuristic" is not properly defined but only examples of heuristic strategies and techniques are given (like "working backwards" or "examining special cases"), if at all. This is even worse if the term "heuristic" is used in differing meanings.

One could argue that so far research has done fine without a proper definition, so why bother? We should bother, because of several good reasons for proper definitions: Firstly, it is of academic interest to clarify often used terms, especially if they are used as widely as “heuristics”. Secondly, trying to specify this term might help avoiding misunderstandings and difficulties when dealing with studies regarding it. There are studies that use different terms meaning the same concepts and there are studies that use the same terms meaning different concepts. And thirdly, using straight definitions might help judging studies and the extent to which their outcome is due to heuristics or to other activities (cf. Romanycia & Pelletier 1985, p. 47).

Theoretical Background

The word “heuristic” origins in the ancient Greek *heurisko* (ϙυρισκο: “I find out”, “I discover”; also note the famous exclamation “Eureka!” by Archimedes), but a direct translation does not really help to define its meaning. A translation is especially insufficient when characterizing the usages of the term in the differing contexts of mathematics, mathematics education, psychology, or artificial intelligence (AI). To illustrate the differences between usages of the term “heuristics” in various scientific fields, I point out three different concepts:

(1) Heuristics as rules of thumb, as hints, and as recommendations to problem solvers (that are stuck in their process). Examples are: *to better understand the problem, draw a figure* or *add auxiliary elements* or *do you know a similar problem that may help you?* Representatives of this concept are: Pólya (1945); Schoenfeld (1985).

(2) Heuristics as (computer-based) devices (programs, rules, etc.) that operate in a formal problem space; especially for situations in which no algorithm is known or applicable. An example is: *use the Ant Colony Optimization Algorithm* (cf. Dorigo & Stützle 2004) *for your Traveling Salesman Problem, because the brute force approach would take years to perform*. Representatives for this concept are: Feigenbaum & Feldman (1963); Romanycia & Pelletier (1985); Michalewicz & Fogel (2004).

(3) Heuristics as subconscious, automatic answers to difficult questions; shortcuts that can produce efficient decisions (as opposed to conscious, slow, effortful strategic procedures). An example is: instead of answering the question “*How happy are you with your life these days?*” our subconsciousness comes up with an answer to “*What is my mood right now?*” Representatives are: Kahneman (2012); Gigerenzer (1991).

Concept (3) originates in Cognitive Psychology and differs fundamentally from the concepts (1) and (2). In mathematics education, concept (3) is not (widely) used; thus, in this article I will focus on the other two concepts – but one should keep (3) in mind.

Heuristics are an important aspect of solving tasks and problems; therefore, they are closely related to other aspects of task and problem solving like “algorithms” and “metacognition”.

Metacognition “refers to one’s knowledge concerning one’s own cognitive processes and products or anything related to them […] [it] refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear […].“ (Flavell 1976, p. 232)

An *algorithm* “is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems.” (Brousseau 1997, p. 129 f.)

There have been a few approaches at characterizing heuristics, most notably by McClintock (1979) and Romanycia and Pelletier (1985) and to some parts by Koichu et al. (2007). These researchers’ considerations influenced the “deductive approach” of my analysis (see the methodology part for details)

My **research intention** is to help clarify the (ill-defined) term “heuristics” and to provide a better understanding of it. My approach to do so consists of the following steps: (1.a) Looking up definitions, descriptions, and characterizations of the terms “heuristics”, “heuristic strategies”, and “problem solving strategies”. (1.b) Analyzing the selected characterizations, identifying commonalities and differences. (2.a) Asking experts in the research on mathematical problem solving for their opinion on selected characterizations. (2.b) Applying selected characterizations to real and fictional situations.

In this article, due to space reasons, I can only show a part of all the characterizations I looked up (1.a) and of the according analyses (1.b); more precisely, I chose nine characterizations for further discussion (see Table 1; reasons for this selection are presented in the methodology part of this paper). Furthermore, I focus on the opinions of experts (2.a), leaving out the analyses of examples (2.b) for future research.

Methodology

Looking up characterizations. For my Ph.D. thesis (Rott 2013), I obtained a broad overview of the problem solving literature from mathematics education and psychology in general and of heuristics related literature in particular. For this article, I additionally looked up review articles on the topic of heuristics and scanned their reference sections to identify other potentially relevant articles.

I then carefully searched all the books and articles for definitions or characterizations of the terms “heuristics”, “heuristic strategies”, or “problem solving strategies” – which most of them did not provide. This search resulted in a list of more than 20 characterizations, nine of which are presented in Table 1 (the reasons for this selection lie in the fact that certain aspects of these characterizations should be represented in this sample; details are discussed below).

To some extent, the data accumulation took place during analyzing / coding (see the next paragraph for details of the coding process). Whenever I had found a new category I could look for additional characterizations that contributed to it. Therefore, the process of data gathering could be interpreted as a "theoretical sampling" up to "saturation" sensu Grounded Theory (cf. Glaser 2004, paragraph 3.7).

Table 1: Characterizations (C1 – C9) chosen for the questionnaire and the discussion in this article

#	Author(s)	Characterization
C1	Kilpatrick (1967, p. 19)	"A precise definition of **heuristic** in information-processing terms will probably include some procedures that would be better classed as algorithms and will almost necessarily exclude some procedures that aid in problem solving. Let us forego such precision, therefore, and define a **heuristic** as any device, technique, rule of thumb, etc. that improves problem-solving performance. We consider heuristics to be typically provisional, without guarantee of effectiveness, but we do not attempt to contrast them with algorithms. If algorithms are heuristics, they are the least interesting sort for our purposes."
C2	Koichu, Berman, & Moore (2007, p. 101)	"We refer to the concept of *heuristics* as a systematic approach to representation, analysis and transformation of *scholastic mathematical problems* that actual (or potential) solvers of those problems use (or can use) in planning and monitoring their solutions. Some heuristics are narrow and domain-specific [...], whereas others are universal and cut across many problem-solving domains [...]. [...] Heuristics at large can be seen as a cognitive tool used to approach problems, effectiveness of which is never known in advance."
C3	Bruder (2000, p. 72 f.) [translated by BR]	"Disciplinary as well as interdisciplinary methods and techniques for solving problems by mathematical means are called **heuristic education**. […] Heuristic methods and techniques differ in multiple ways from the usual mathematical terms, theorems and procedures. Most striking characteristic: By using them there is no guarantee for a solution. […] Heuristic education can be viewed as a way to partly compensate a lack of mental flexibility via a higher awareness of goals and methods."
C4	Tietze, Klika & Wolpers (2000, p. 98) [translated by BR]	"Heuristic rules, principles and aids – in short heuristics – shall support the student (a) to *understand* the problem via analyzing it properly and transforming it into a more accessible representation and (b) to *plan* the process of solution consciously, not to conduct a goalless trial-and-error procedure. […] It is useful to divide heuristics roughly into two groups: those which relate to the whole process of problem solving, for example general planning (*global heuristics*), and those which are *local* in their nature."
C5	Heinze (2007, p. 6) [translated by BR]	"The goal of a problem solver is […] to find a sequence of operators that transform a given initial state into a desired final state. The operators that are activated within the problem solver and the thereby accessible intermediate states are called search space. This search space can be narrowed by heuristics."
C6	Dörner (1979, p. 38) [translated by BR]	"We call the very structure that organizes and controls such a [problem solving] process a heuristic, a procedure to find a solution. The most primitive form of a heuristic is a program for an unsystematic trial-and-error procedure. Therefore, heuristics are in a sense programs for cognitive operations with which problems of a certain type may potentially be solved. The application of a heuristic does not guarantee the solution, otherwise it would be no problem."
C7	Pólya (1945, p. 112, p. 129 f.)	"The aim of heuristic is to study the methods and rules of discovery and invention. […] **Modern heuristic** endeavors to understand the process of solving problems, especially the *mental operations typically useful* in this process."
C8	Schoenfeld (1985, p. 23)	"Heuristic strategies are rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution."
C9	Winter (1989, p. 35) [translated by BR]	"[M]athematical **heuristic** is the information of gaining, finding, discovering, developing new knowledge and of solving problems methodically (Greek: heuriskein = finding, discovering)."

Analysis of the characterizations. I analyzed the selected passages using the *Qualitative Content Analysis* by Mayring (2000). Coding the characterizations, I started with categories from the literature (deductive approach) and added categories out of the material (inductive approach). The finalized system of categories will be described in the following paragraph.
I have to admit that so far I was the only one to rate these characterizations. Thus, I cannot give a coefficient for the interrater agreement of the coding process.

Deductive approach. Romanycia & Pelletier (1985) identified some elements in the characterizations of "heuristics" in the literature on artificial intelligence that most of these characterizations had in common – though some of them missed one element or another. Heuristics…

- do not necessarily lead to successful problem solving (*lack of guarantee*);
- are not algorithmic procedures (*arbitrary device*);
- can reduce the problem space that needs to be searched (*effort reduction*);
- often do not lead to the best solution but to a sufficient one (*satisfactory solution*);
- can sometimes be very special and domain specific, and sometimes be applied to a broad class of problems (*domain dependence*).

For their own analysis of the concept "heuristics", Romanycia and Pelletier (ibid., p. 51 ff.) proposed four "dimensions of meaning":

- (*uncertainty of outcome*) – Heuristics exist in a "context of subjective uncertainty as to the success of their application"; they are "perfectly compatible with algorithms" and only opposed to those algorithms which "guarantee a practical solution to a problem".
- (*basis in incomplete knowledge*) – Heuristics are plausible without being certain; they are neither optimal efficient algorithms nor inefficient algorithms or processes.
- (*improvement of performance*) – In some characterizations, heuristics include performance improvements whereas in others they need merely be useful.
- (*guidance of decision making*) – Heuristics as a group do not consistently influence memory, clarity of vision, creativity, thoroughness, or any other feature of problem solving; they guide decision making.

Inductive approach. To analyze the selected characterizations, I tried to use the elements and dimensions by Romanycia and Pelletier; some of which were included into my set of categories. Nevertheless, there were certain aspects in the characterizations that were not covered by these elements and dimensions. This is the reason why I developed additional categories after going through the characterizations thoroughly.
Table 2 contains the finalized set of category names and descriptions as well as coding

examples; Figure 1 shows a sample coding of characterization C2.

Table 2: Descriptions of the coding categories for analyzing the characterizations of "heuristics"

Category	Definition	Examples
Descrip-tion	According to the author(s), what is the nature of heuristics? Descriptions range from "rules of thumb" to "kinds of information" and "cognitive tools".	"rules of thumb, general suggestions" (C8) "cognitive tools used to approach problems" (C2) "methods and techniques for solving problems" (C3)
Effective-ness	What does the characterization say about the effectiveness of heuristics? Most say they offer "no guarantee for a solution", but heuristics can also be defined as being "helpful for problem solving".	"Let us [...] define a **heuristic** as any device, technique, rule of thumb, etc. that improves problem-solving performance." (C1) "Most striking characteristic: By using them there is no guarantee for a solution." (C3) "Heuristics at large can be seen as […], effectiveness of which is never known in advance." (C2)
Analysis	Does the characterization explicitly mention understanding and analyzing the problem?	"Heuristic strategies […] help an individual to understand a problem better […]." (C8) "approach to representation, analysis and […]" (C2)
Metacog-nition	Does the characterization explicitly mention metacognitive or self-regulatory activities? Are they included into or excluded from "heuristics"?	"We call the very structure that organizes and controls such a [problem solving] process a heuristic […]." (C6) "solvers of those problems use (or can use) in planning and monitoring their solutions" (C2)
Range	Do the authors mention some kind of range of heuristics? To what kind of problems are they applicable? Are there different types of heuristics (e.g., local and global or domain-specific and general ones) with different fields of application?	"Some heuristics are narrow and domain-specific [...], whereas others are universal and cut across many problem-solving domains […]." (C2) "[…] divide heuristics roughly into two groups: those which relate to the whole process of problem solving, for example general planning (*global heuristics*), and those which are *local* in their nature." (C4)
Algorithm	Does the characterization mention algorithms or other "standard procedures"? Are these included into or excluded from the definition of heuristics?	"A precise definition of **heuristic** […] will probably include some procedures that would be better classed as algorithms […]." (C1) "Heuristic methods and techniques differ in multiple ways from the usual mathematical terms, theorems and procedures." (C3)
Awareness	Does the characterization mention whether problem solving techniques have to be executed consciously to be regarded as heuristics? Some characterizations speak of "systematical" or "methodical approaches" which seem to exclude implicit / subconscious / intuitive uses of such techniques.	"We refer to the concept of *heuristics* as a systematic approach to representation, analysis and transformation of *scholastic mathematical problems* […]" (C2) "[M]athematical **heuristic** is the information of […] solving problems methodically" (C9)
Problem Space	Does it refer to the concept of problem space (Newell & Simon 1972)?	"This search space can be narrowed by heuristics." (C5)
Others	Are there any other features that haven't been covered by the other categories?	"can be viewed as a way to partly compensate a lack of mental flexibility" (C3) "The aim of heuristic is to study the methods and rules of discovery and invention." (C7) "gaining, finding, discovering, developing new knowledge and of solving problems" (C9)

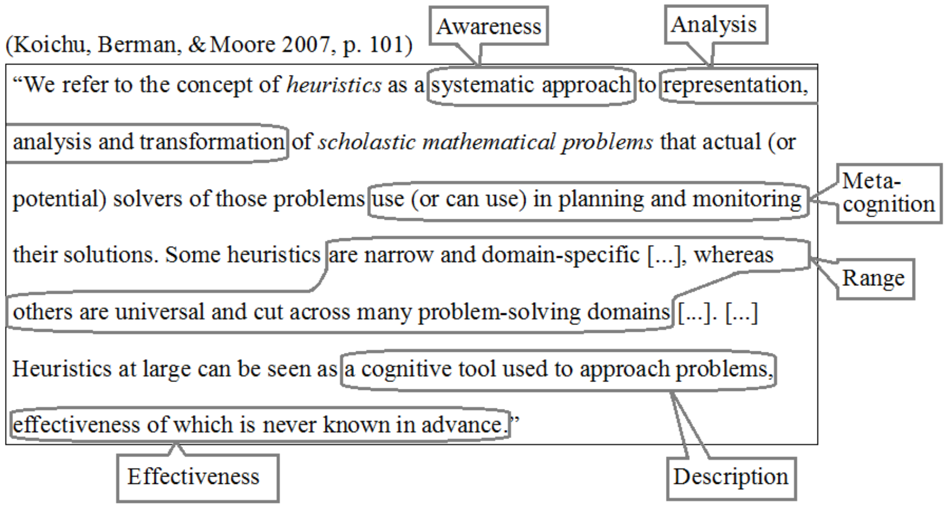


Figure 1: Sample coding of characterization C2

Development of the questionnaire. For the second part of the study, I developed a questionnaire to ask problem solving experts from mathematics education for their opinion on selected characterizations of "heuristics". This idea goes back to a book chapter of Furinghetti and Pehkonen (2002) who analyzed the term "belief" in the very same way. For this questionnaire, the number of characterizations had to be reduced to keep the effort for filling it out within limits. I chose nine of the previously analyzed characterizations (see Table 1) making sure that entries of all the categories from the content analysis were represented in this sample. Also, some "famous" characterizations were included (C7, C8). In the questionnaire, the authors of the characterizations were not indicated. For each characterization, I asked the experts to answer the following two questions (like Furinghetti & Pehkonen 2002, p. 48):

- Do you consider the characterization to be a proper one?
- Please, give reasons for your decision!

The final item was:

- Your characterization: Please write your own characterization for the concept of "heuristic".

Evaluation of the questionnaire. Like Furinghetti and Pehkonen, I sent my questionnaire via e-mail to fellow researchers[1] asking them to judge the given characterizations. In the period before the ProMath 2013 conference (August 30th until September 1st), I got 18 responses from experts from seven different countries.[2] There were six experts from Germany (which is my home country so many of my contacts are German) as well as six experts from Hungary

[1] In July 2013, I wrote e-mails to all 18 problem solving researchers that preregistered for the 2013 ProMath conference in Eger (Hungary). I additionally addressed 20 colleagues I knew from previous conferences on problem solving as well as experts from a list of researchers that Erkki Pehkonen kindly provided me with.

[2] 11 of the 18 preregistered conference participants (not counting myself) answered my call; 7 additional experts filled out the questionnaire. All in all, 47% answered my call.

(the place of the conference). Additionally, there was one expert each from Australia, Finland, Greece, Israel, and Sweden. To determine the experts' nationalities, I used the nationality of the universities they were affiliated with at the time of the ProMath 2013 conference, because I did not gather personal information of the experts (one of them, for example, is originally from Croatia but works in Germany for some years, so s/he was considered German).

Most of the experts judged all nine characterizations, some skipped particular characterizations and many of them wrote down their own characterizations. For a first overview, I coded the responses of the experts on a five-step scale, based on the research by Furinghetti and Pehkonen (2002): Y (= full agreement), P+ (= partial agreement with a positive orientation), P (= partial agreement), P- (= partial agreement with a negative orientation), N (= full disagreement).[3] Sample codings are provided in Table 3 (experts being indicated by a "#").

The experts' reasons for (dis)agreement have been considered carefully and all arguments that have been given individually by at least two experts are stated in the results section.

Table 3: Excerpts of the experts' responses in the questionnaire and examples of my codings

Expert	**Response**	**To**	**Code**
#7	No, I don't. It's too general term for me.	C1	N
#6	I accept partly the characterization #3, because the heuristic tools, principles, strategies help to involve the solution and understand the problem.	C3	P+
#6	This is very important, because in "ad hoc" method could prevent the problem solving.	C4	P+
#11	Yes, I can agree. Students can have a sense of security if they know a heuristic method to solve a problem.	C4	Y
#3	This is the classical way, representatives of the information processing approach deal with problem solving and heuristics. For modern educational/practical use (and after ten years of teaching!) it seems to me too technical and too formal.	C5	P-
#7	No, I don't. It's oversimplification of the concept of heuristic.	C5	N
#11	I can't take a commitment, because I feel, it can be truth, if ever it's proved, that our mind and thinking and our whole world are computable. But if we assume the opposite of this, than I must believe that we can never understand nor the process of problem solving nor other complex mental operations in our mind by examinations of heuristic methods. We can simulate/model something near to this process, but not exactly that. That's why heuristic methods are strange from many people's thinking.	C7	P
#1	This definition has other character than the ones listed above. This is Pólya's characterization of heuretic or heuristic. At that time the term *heuristic*, or heuretic, was used to describe a field of study. The next sentence elaborates on the educational use of the term *heuristic* that has developed historically to describe a process. So these are two distinct things. However, this is a simplistic interpretation of Pólya's writing and no theory developed so far can explain the complexity of his words.	C7	?
#1	Somewhat; Schoenfeld's definition is okay, besides the first part of it (rules of thumb for successful problem solving). I would not use the word rule and thumb because it implies sort of mechanical decision-making. Besides, the usage of it, does not necessarily imply success.	C8	P
#18	Rather not. If one reads "science" for "information" it remains the task to precisely characterize how to gaining, find, discover, develope new knowledge and of solving problems methodically.	C9	P-

[3] Erkki Pehkonen (mail from April 19th 2013) perfectly summarizes the benefits of written responses compared to marks on a Likert-scale: "I would today do just the same, since getting the written responses, you may ponder the answers. If you get only a number from 1…5, you cannot do anything else but calculate the mean value."

Results

The theoretical analyses of the characterizations as well as the evaluation of the experts' opinions both show that there is a wide variety of different – sometimes even contradictory – aspects of the term "heuristics" given and preferred respectively. No characterization covers all aspects of the categories from Table 2 and sometimes the aspects of one characterization conflict those of others. Additionally, there is no characterization favored by all experts.

This section begins with results of the questionnaire evaluation and continues with a theoretical discussion of the characterizations (including some of the experts' comments).

Evaluation of the questionnaires – results. The individual coding of the experts' responses is reported in Table 6 in the appendix; in the last column, the sign "*" means that an own characterization was provided.

In order to get a better overview, the experts' judgments are grouped according to their grade of approval / rejection in Table 4. There are four characterizations that seem to be generally approved (C4, C7, C8, C9), four that are rejected by the majority of experts (C1, C3, C5, C6), and one that is approved / rejected to almost even parts (C2). The reasons given by the experts are presented in Table 5 and are discussed below.

Table 4: Degree of agreement / disagreement with the characterizations (C1 – C9) by the experts

Code	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**C9**
Y = Yes	0	3	0	4	0	2	6	4	4
P+ = Partly Yes	5	4	4	4	2	2	2	6	4
P = Partly	1	2	2	3	2	1	1	4	3
P- = Partly No	1	2	4	3	4	6	2	0	3
N = No	7	3	4	1	6	4	3	2	2
No answer / judgment	4	4	4	3	4	3	4	2	2
(Y&P+) vs. (P-&N)	5/8	7/5	4/8	8/4	2/10	4/10	8/5	10/2	8/5

Reasons for approval. The characterizations C7 and C8 by Pólya and Schoenfeld have been recognized by (nearly) all experts. Reasons for these characterizations' approval might be the authority of their authors or because they are used widely (and everybody got used to them). Or maybe, they just "fit" to what is needed by most researchers. The characterizations C4 and C9 have been translated by me (BR) from German into English; therefore, I highly doubt that being well-known could have been a reason for these two to be approved (like it might have been for C7 and C8). The experts stated that they liked the reference to (Pólya's) problem solving phases and to the etymology of the word "heuristics".

Reasons for rejection. Three of the four characterizations that were rejected by the majority of experts, C3, C5, and C6 (but not C1), have been translated by me; thus, there might be translation errors that led to rejection, but this does not seem to be the case here. Also, being originally in German, these characterizations are not well-known to an international audience. Being unfamiliar with a characterization might lead to its rejection, but there are well-argued reasons: The experts do not agree to heuristics compensating a lack of mental flexibility. They also do not agree with heuristics controlling a process or focusing on only the reduction of the problem space. To some extent, the inclusion of "unsystematic trial-and-error procedures" caused a controversy.

Table 5: Reasons for approval / rejection by the experts (number of experts who say so in brackets)

C.	**Experts' Commentaries to the Characterization**
C1	Most experts (8) stated that heuristics should not include algorithms and would rather contrast these two.
C2	Some experts (3) mentioned that they would not limit heuristics to "scholastic" problems. Also, they would not speak of a "systematic approach" (5).
C3	Some experts (2) were puzzled by the word "education" in this characterization; and they would not restrict heuristics to "mathematical means" (4), but to non-mathematical techniques as well. A good part of the experts (6) does not agree to the claim that heuristics can be viewed as a way to "compensate a lack of mental flexibility".
C4	There were experts (2) that complained about this characterization not being a proper definition, but giving only reasons why to use them. Many experts (6) agreed to highlighting phases in the problem solving process, but on the other hand, some of them missed phases like reflecting the problem or being stuck apart from having a plan.
C5	Most experts (8) located this definition in the computer science / Newell & Simon tradition of problem solving research; though it was published in a pure mathematics education journal. Some experts (5) pointed out that no real definition of the term heuristic is given. Also, some experts (4) did not agree with heuristics being limited to narrowing the search space. They stated that the search space could also be expanded by heuristics etc.
C6	Some experts (4) did not agree with heuristics controlling a process, which belongs to metacognition. Also, the expression "unsystematic trial-and-error procedure" caused some converse responses (4). Some agreed to this procedure being included into heuristics whereas others did not like it that way or thought this to be contradictory to "being a program" (from this very characterization).
C7	Most experts (11) agreed that this characterization differs from the other ones of the list presented. Instead of defining heuristic techniques, it gives aims of the scientific discipline of heuristic research.
C8	Some experts (3) stated the term "rule of thumb" does not really fit to heuristics (if this is the only attribute to characterize them). Most experts (7) did not agree to heuristics being for "successful problem solving" only, because there is no guarantee of success.
C9	Some experts (2) liked the reference to the etymology of the word "heuristic". A few experts (2) pointed out that heuristics do not necessarily belong to mathematics (e.g., working backwards) and that the term "methodically" does not fit the nature of most heuristic techniques (3). Also, some experts (6) stated that the word "information" does not make sense; this could be a translation error, the German original uses the word "Kunde" which could also mean lore, news, or knowledge; I used "information" because Wilson et al. (1993) used this very expression in their characterization.

Analysis of the characterizations – results. In this paragraph, I discuss different aspects of heuristics, structured by my categories for the analysis of the characterizations (Table 2).

► *Effectiveness*. Schoenfeld (C7) and Kilpatrick (C1) speak of "successful problem solving" and "improv[ing] problem-solving performance" respectively, whereas others like Bruder

(C3) speak of "no guarantee for a solution". Actually, I doubt there is a serious discussion whether heuristics do or do not offer a guarantee of success; there is no good reason why they should be able to do so. Even Kilpatrick admits that there is "no guarantee of effectiveness" in the sentence following his initially cited statement. Some experts explicitly stated that lacking such a guarantee is one of the defining aspects of heuristics. For example, expert #3 added to Schoenfeld's characterization: "with a higher rate of probability (than without them)".

► *Analysis*. Some characterizations explicitly mention that heuristics can help to understand a problem (Koichu et al., C2); others do not mention this feature; no authors explicitly exclude analyzing a problem from their characterization of the concept. Again, I doubt that there is – or could be – a serious discussion about this feature. I see no reasons why this should not be part of anyone's concept of heuristics. (Experts #7 and #8 actually vote for also including other Pólya-phases like "Looking Back" into proper characterizations of the term "heuristics".)

► *Awareness*. We have seen that the term "heuristics" can be used for subconscious thoughts and processes (cf. Kahneman 2012) and expert #3 stated that "normally a heuristic method is applied unconsciously". Though, several of the characterizations speak of a "systematic approach" (Koichu et al, C2), or "to plan […] consciously" (Tietze et al., C4). Kahneman (2012, p. 98) himself states that his interpretation of heuristics differs significantly from that of Pólya's ("procedures that are deliberately implemented").

► *Problem Space*. With this category, I wanted to check whether there is a direct reference to the concept of the "problem space" (cf. Newell & Simon 1972). There are characterizations with such a reference (like Heinze, C 5) and there are characterizations without. I see no reason for such a reference being important for the understanding of the term "heuristics", maybe apart from one aspect: References to the "problem space" most often imply an understanding of the problem solving process (and heuristics) as it is used in the AI community. This would come along with a conscious interpretation of the term "heuristics" as programs that affect the problem space. Therefore, this category could be related to the category of "Awareness".

► *Range*. Some characterizations speak of "disciplinary as well as interdisciplinary methods and techniques" (Bruder, C3) and of "narrow and domain-specific [or] universal and cut across many problem-solving domains" (Koichu et al., C2). No characterization opposes this view that heuristics can be used widely, but the question arises whether general strategies like means-ends analysis or situation analysis (often used in psychology) are applicable to problems that are relevant for mathematics education research. On the other hand, several experts stated that heuristics should not be limited to "scholastic" problems or "mathematical means".

► *Algorithm*. Kilpatrick (C1) points out that "[a] precise definition of heuristic […] will probably include some procedures that would better be classified as algorithms"; whereas others like Bruder (C3) oppose the concepts of "heuristics" and "algorithms". This opposition is often grounded in the "Effectiveness" and the "Range" of these two concepts (in a sense of "algorithms can only be applied to a narrow set of tasks but guarantee success, whereas heuristics can be applied to a vast set of tasks and problems but do not offer such a guarantee"). At least five of the experts did not want to include algorithms into the concept of "heuristics".

► *Metacognition*. Authors like Dörner (C6, "the very structure that organizes and controls a process") or Koichu et al. (C2, "in planning and monitoring") explicitly include metacognitive activities, whereas authors like Schoenfeld (1985) differentiate between "heuristics" and "control" as different aspects of problem solving. Some experts (#1, #4, #8 amongst others) do not agree to metacognitive activities being part of "heuristics", whereas others (#2) do not agree in differentiating these two concepts. No characterization explicitly excludes metacognition from "heuristics" but the question remains whether or not this should be done.

► *Description*. What actually are heuristics? In the characterizations, it varies between a "device, technique, rule of thumb" (C1), a "cognitive tool" (C2), a "structure" (C6), or no real description (e.g., C4 and C5). And also in the experts' own characterizations, this varies between "conscious and non-conscious methods" (#8), "kind of things (e.g., information), available to a person in making decisions during a problem solving act" (#1), "strategies that potentially can be applied to any mathematical structure" (#10), or "rules of thumb" (#3). An answer to this question should be related to the question whether heuristics are conscious or subconscious ("Awareness"). It should be obvious, that there has to be a description and not everything that "improves problem solving performance" should be regarded as a heuristic. Expert #13 states that "[t]here are also good habits and general advice for effective problem solving that are NOT heuristics, such as keeping good records or mulling on a problem after a period of intense work, talking to a friend about it, writing up the solution so far etc."

► *Others*. There are additional aspects in some characterizations that need to be discussed further. For examples, for some authors, heuristics might also help in "finding, discovering, developing new knowledge" (Winter, C9) or they might help in the compensation of missing subconscious skills (mental flexibility) (Bruder, C3). For Pólya (C7) and expert #13, "heuristics" also stands for the "study of methods and rules of discovery and invention".

In addition to the experts' comments presented in the theoretical analyses (see above), I present three (of 13) notable characterizations by the experts (with footnotes by me).

- Expert #2: this expert prefers a definition according to Daniel Kahneman ("world famous psychologist and Nobel Price winner in economy"): "Heuristics is a procedure, which helps to find a suitable – many times not a perfect – answer for difficult questions."[4]
- Expert #13: "The subject 'heuristic' is the study of methods of discovery, invention and problem solving.[5] The outcome is a set of 'heuristics'. Heuristics are general approaches to problem solving, discovery and invention that are known to frequently move the solver towards a solution from a state of being 'stuck'. Examples are […]."
- Expert #15: "I like this Wikipedia quote: "*Computer Science*: a procedure for solving problems that is uncertain, inexact or not working for all input values; often used as a substitute for an algorithm that theoretically produces a more precise solution or even the best one, but needs too much time or too much effort."[6]

Discussion and Conclusions

Expert #17 questioned my whole approach of asking experts by saying: "any definition is acceptable or not depending, in particular, on how you want to use it." Of course, this statement is correct and it is hard to argue against, but nonetheless, there are good reasons for comparing characterizations: The evaluation of the questionnaires and the analyses of the selected characterizations show that there is no generally (and implicitly) accepted characterization of the term "heuristics" everyone can refer to. This is the reason why due to significantly different meanings (e.g., subconscious vs. conscious use), a clarification is needed for every study that uses the concept (which is most often not given). With this research, I want to demonstrate the importance of a (good) definition of the term "heuristics"; however, I am not the one to tell anybody which understanding of the term should be the "right one" (actually, I doubt there is a right one). This research also might point out some aspects of the term "heuristics" which are worth a discussion or which are especially arguable.

Suggestions for a unification of concepts. For some of the differences stated above, I propose ideas that might help clarifying the concept of "heuristics" for mathematics education.
► *Algorithm*. I would tie this aspect to the "Effectiveness" in the way Romanycia and Pelle-

[4] Kahneman (2012, p. 98) distinguishes his definition of heuristics from that of Pólya's. Pólya's heuristics are deliberately and consciously implemented, whereas Kahneman's heuristics are a product of the subconscious processes.
[5] This fits Pólya's view of "heuristic" (C7).
[6] Actually, this quote is from the German Wiktionary and was translated into English by me (BR) for this article. This description reminds me of the "ant colony optimization algorithm" (cf. Dorigo & Stützle 2004).

tier (1985, p. 57) did: “any device […] which one is not entirely confident will be useful in providing a practical solution, but which one has reason to believe will be useful […].” According to this characterization, something like the application of the p-q-formula for quadratic expressions[7] would – dependent on the situation – sometimes be classified as a heuristic and sometimes as algorithmic. But this assignment would be done comprehensibly.

► *Awareness*. Whether heuristics are conscious or subconscious (or both?) should be stated by everyone who uses the term. I would plead for “conscious”, because, for example, this decision has direct implications for studies regarding the training of heuristic strategies.

► *Metacognition*. The intermixture of the concepts of heuristics and metacognition might be unproblematic from the practitioners’ perspective (e.g., teachers). But for researchers, it is very important to differentiate between these two concepts (as hard as it might be). There are studies that would be impossible without such a differentiation like the one by Collet (2009) who compared a training of heuristics with a training of self-regulation as well as a combined training.

As far as I know, there are two main reasons for the two concepts not often being differentiated in the literature: (1) In problem solving processes, it is very hard to distinguish between recognizing the need for heuristics and choosing an appropriate strategy, remembering and applying such a strategy, and evaluating its use afterward. (2) When Pólya (1945) wrote about problem solving in the 1940ies, about controlling the process as well as posing heuristic questions, the term “metacognition” had not been defined properly. When Flavell (1976) did so in the 1970ies, he mentioned several techniques that have also been described by Pólya.

My own definition of the term “heuristics”. I hesitate to give a “final definition”, especially not before having conducted all the planned research (analyzing more definitions, applying them to examples, etc.). So far, I can present the following characterization which might be somewhat “clunky”, trying to cover many of the categories discussed above:

> “*Heuristics* is a collective term for devices, methods, or (cognitive) tools, often based on experience. They are used under the assumption of being helpful when solving a problem (but do not guarantee a solution). There are general (e.g., “working backwards”) as well as domain-specific (e.g., “reduce fractions first”) heuristics. Heuristics being helpful regards all stages of working on a problem, the analysis of its initial state, its transformation as well as its evaluation. Heuristics foster problem solving by

[7] The p-q-formula is an example used by expert #18 as a criterion for judging the characterizations – any characterizations of the term “heuristics” that also included the use of this algorithm was a bad one for this expert.

reducing effort (e.g., by narrowing the search space), by generating new ideas (e.g., by changing the problem's way of representation or by widening the search space), or by structuring (e.g., by ordering the search space or by providing strategies for working on or evaluating a problem). Though their nature is cognitive, the application and evaluation of heuristics is operated by *metacognition*.

Future prospects. As stated before, I'm planning on analyzing more characterizations and applying them to examples. Expert #17 had the same idea: "For instance, "trial and error" is a heuristic strategy by some of the definitions in the questionnaire, and by some others it is not. Is it good or bad? How can we decide whether or not "trial and error" should be tagged heuristics? What are the consequences of either decision for research or practice?"

I would like to come up with a better characterization of the term "heuristics" and to clarify related concepts like "mental flexibility". Also, the way heuristics work, why they are sometimes helpful in problem solving, needs to be discussed further. The characterizations give some ideas, it might be due to reducing effort (narrowing search space), or due to generating ideas (widening search space, compensating a lack of mental flexibility), or a combination of these and other effects.

References

Brousseau, G. (1997). *Theory of didactical situations in mathematics: didactique des mathématiques, 1970 – 1990*. Translated from French by Cooper, M. et al. Springer.

Bruder, R. (2000). Akzentuierte Aufgaben und heuristische Erfahrungen. In Herget, W. & Flade, L. (Eds.). *Lehren und Lernen nach TIMSS*. Berlin: Volk & Wissen, pp. 69 – 78.

Collet, C. (2009). *Förderung von Problemlösekompetenzen in Verbindung mit Selbstregulation – Wirkungsanalysen von Lehrerfortbildungen*. Münster: Waxmann.

Dorigo, M. & Stützle, T. (2004). *Ant Colony Optimization*. MIT Press.

Dörner, D. (1979). *Problemlösen als Informationsverarbeitung*. Stuttgart: Kohlhammer, 2nd ed.

Feigenbaum, E. A. & Feldman, J. (1963). *Computers and Thought*. McGraw-Hill Book.

Flavell, J. H. (1976). Metacognitive Aspects of Problem Solving. In Resnick, L. B. (Ed.). *The Nature of Intelligence*. Hillsdale, NJ: Lawrence Erlbaum, pp. 231 – 235.

Furinghetti, F. & Pehkonen, E. (2002). Rethinking Characterizations of Beliefs. In Leder, G.; Pehkonen, E., & Törner, G. (Eds.). *Beliefs: A hidden variable in mathematics education?* Dordrecht: Kluwer Academic Publishers, pp. 39 – 57.

Gigerenzer, G. (1991). How to Make Cognitive Illusions Disappear: Beyond "Heuristics and Biases". In Stroebe, W. & Hewstone, M. (Eds.). *European Review of Social Psychology (Vol. 2)*. Chichester: Wiley, pp. 83 – 115.

Glaser, B. G. with the assistance of Holton, J. (2004). *Remodeling Grounded Theory* [80 paragraphs]. Forum Qualitative Sozialforschung / Forum: Qualitative Social Research, 5(2), Art. 4. URL (20.08.2013): http://www.qualitative-research.net/index.php/fqs/article/view/607/1316

Heinze, A. (2007). Problemlösen im mathematischen und außermathematischen Kontext. In *Journal für Mathematik-Didaktik, Jg. 28(1)*, pp. 3 – 30.

Kahneman, D. (2012). *Thinking, Fast and Slow*. London: Penguin Books.

Kilpatrick, J. (1967). *Analyzing the solutions of word problems in mathematics: An exploratory study*. Unpublished doctoral dissertation, Stanford University. Dissertation Abstracts International, 1968, 28, 4380-A. (University Microfilms, 68-5, 442).

Koichu, B.; Berman, A.; & Moore, M. (2007). Heuristic literacy development and its relation to mathematical achievements of middle school students. In *Instructional Science, 35*, pp. 99 – 139.

Mayring, P. (2000). *Qualitative Content Analysis* [28 paragraphs]. Forum Qualitative Sozialforschung / Forum: Qualitative Social Research, 1(2), Art. 20. URL (26.07.2013): http://www.qualitative-research.net/index.php/fqs/article/view/1089

Michalewicz, Z. & Fogel, D. B. (2004). *How to Solve It: Modern Heuristics*. Berlin: Springer. 2nd ed.

McClintock, C. E. (1979). Heuristic Processes as Task Variables. In Goldin, G. A. & McClintock, C. E. (Eds.). *Task Variables in Mathematical Problem Solving*. Philadelphia, PA: Franklin Institute Press, pp. 171 – 234.

Newell, A. & Simon, H. (1972). *Human Problem Solving*. E. Cliffs, NJ: Prentice Hall.

Pólya, G. (1945). *How to Solve It*. Princeton, University Press.

Romanycia, M. H. J. & Pelletier, F. J. (1985). What is a heuristic? In *Computational Intelligence, Volume 1, Issue 1, January 1985*, pp. 47–58.

Rott, B. (2013). *Mathematisches Problemlösen – Ergebnisse einer empirischen Studie*. Münster:WTM

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando: Academic Press.

Tietze, U.-P.; Klika, M.; & Wolpers, H. (2000). *Mathematikunterricht in der Sekundarstufe II – Band 1*. Braunschweig: Vieweg. 2nd ed.

Wilson, J. W.; Fernandez, M. L.; & Hadaway, N. (1993). Mathematical problem solving. In Wilson, P. S. (Ed.). *Research ideas for the classroom: High school mathematics. ch. 4*, pp. 57 – 77.

Winter, H. (1989). *Entdeckendes Lernen im Mathematikunterricht – Einblicke in die Ideengeschichte und ihre Bedeutung für die Pädagogik*. Braunschweig: Vieweg.

Appendix

Table 6: The codings of the experts' responses in the questionnaires

Expert	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**C9**	**Def**
#1	N	N	N	P+	N	N	?	P	N	*
#2	-	-	-	-	-	-	P+	P+	P+	-
#3	P+	P	P-	P	P-	P-	Y	P+	P-	*
#4	N	N	P-	P-	N	P-	N	N	P	-
#5	N	P+	P-	N	N	P+	N	P+	P+	*
#6	P+	Y	P+	P+	P	P	Y	P	Y	*
#7	N	N	N	P	N	N	Y	P	P+	-
#8	N	P-	N	P-	P-	N	P-	P+	N	*
#9	P+	P+	P	Y	P+	Y	P+	Y	P	*
#10	P	P	P	P	P-	P-	Y	Y	P	*
#11	P-	Y	P+	Y	?	P-	P	Y	Y	*
#12	-	-	-	-	-	-	-	-	-	*
#13	N	P+	P+	P+	P	P-	Y	Y	P+	*
#14	P+	P-	P+	P+	P-	P-	?	P+	Y	*
#15	P+	Y	P-	Y	P+	Y	Y	P+	Y	*
#16	-	-	-	P-	N	N	P-	P	P-	-
#17	-	-	-	-	-	-	-	-	-	-
#18	N	P+	N	Y	N	P+	N	N	P-	*

Solving word problems by computer programming

Takácsné Bubnó Katalin and Viktor László Takács

University of Debrecen, Hungary

kbubno@lib.unideb.hu, viktor.takacs@econ.unideb.hu

Abstract

In every age and in every school word problems pose difficulties. Furthermore, we can state the same about learning computer programming in computer science.

We draw parallels between the two domains, searching for similarities as well as differences. We introduce a graphical programming environment called Blockly [10], which can help us overcome obstacles and develope the abilities in solving word problems and programming.

Blockly is the first block programming language available by a simple Internet browser. It is a user friendly, web-based (cloud-based), graphical programming editor developed by Google Inc. (based on the theoretical foundations of MIT's block language projects ([17], [18], [20]). There is no need to type programming codes or to learn the language's syntactical or semantical rules, so it is suitable for children's first programming language and for those teachers who are not qualified in information technology. Last but not least, it is free, available via web-browsers, we do not need to install anything to our computer device or to our interactive whiteboard in the classroom, or even our own tabletPC or smartphone. It is available from everywhere via Internet, therefore we can use it in the school or at home.

In this paper we present an introduction to Blockly and some of its uses in mathematics: we solve word problems with Blockly showing several exercises from different classes and ages of pupils to illustrate the analogy of the two fields. We had experiments both in primary and secondary school. With the conclusions of the primary school testing we started further experiment in secondary school. Our other testing group was in the Grammar Secondary School of University of Debrecen. We plan to teach computer programming and algorithmization topics in Computer Science lessons in the secondary school from the October, 2013 to January 2014. So, we could not show the final results from all of pupils' works in this paper. We plan it in another paper. Now, we can report the first conclusions and a short comparison between the primary and the secondary students' works, attitudes and difficulties with this type of problem solving method.

Blockly is a new, cloud-based programming language, and we have not found publications about previous experiments where it was used for solving various kind of mathematical word problems. We was wandered if Blockly was suitable programming tool as first programming language and mathematical word problems were suitable as first programming tasks that a child met in computer programming lessons.

Key words: word problems, problem solving, computer programming, Blockly

ZDM classification: D53, Q63, R23

1. About word problems

Our modern world is strongly based upon the applied sciences including information technology. The first application of mathematics a child meets is word problem. Children do not like this type of exercises. Many publications investigate why are word problems so difficult for them and what kind of strategies can help in this topic of mathematical education [1], [5], [7], [9], [11], [14], [16], [19], [23].

The list below is a collection what didactic literature enumerate as the most common difficulties when we teach word problems [11]:

- comprehension;
- recognizing relevant data;
- modelling the problem;
- formalizing open sentences (translating into mathematical notations);
- counting difficulties;
- monitoring and answering the question (translating back to the situation).

2. About computer programming

By computer programming we solve some kind of ’word problems’ even they seem much more difficult that we have met before in mathematical lessons in the school. Our goal is to teach computer-aided mathematical problem solving. However, the ’tool’ we use is not a prefabricated mathematical software, but computer programming language, which we think the essence of computer science teaching. By the Hungarian National Curricula the basic elements of algorithmization are compulsorily acquired for everyone, but it is not specified any concrete programming language. So if we are computer science teachers we have quite a free choice. We can choose, for example, that we teach only as the tools of algorithmization like blockdiagrams, or describing algorithmization languages, so called pseudo-codes. We think, it could be better to teach algoritmization in practice with a children-friendly programming language, and it could be more successful if we teach this topics through well-known problems, like mathematical word problems.

3. Analogies and differencies

We explored some analogies between mathematical problem solving and computer programming. The first we have to state that mathematics is never about just counting, as programming is never about only coding.

The next analogies are based upon George Polya’s famous problem solving model [19].

Word problems in mathematics	Computer programming
1. Comprehension, understanding the problem	
2. Devising a plan	
Determinantion of unknown variable	Declaration of variables
Mathematical model (plan)	Algorithms and their implementation (plan)
3. Carrying the plan	
Solving equations	Compiling and running
Answering to the questions of the starting problem	
4. Discussion (Looking back)	
changing initial conditions	testing
generalizing the problem	debugging
looking for other methods for solution	optimizing code

We have to mention also in the second phase of computer programming the algorithm part of the ‘plan’ (we can also call a *pseudocode*) is could be the mathematical model (equations, diagrams, graphs, etc.).

Obviously we could find also many differences between the two area. The most interesting for us is the problem of different notations in mathematics and in computer languages.

Most of the standard notations and methods in mathematics are well-known all around the world. This fact provides us more generality and public understanding. But in computer science lessons we can choose several kind of algorithmization tool or dozens of programming language to teach. These languages have various kind of syntactical and semantical rules. Furthermore, this question will be more complicated because of the orientation of language which are suitable for implementation. Different syntaxes of different languages use various kind of mathematical notations, rare the ones we have learnt before in mathematical lessons. Children can hardly to accomodate to the many kind of notations and rules.

So in the algorithmic and implementation phase we translate our model to mathematical notations first, and after that we translate again into the chosen programming language.

The second we have to mention about ‘translating’ is the method of nominating our variables and other parameters. In mathematics we try to create short one-letter names for variables and we can give general names like *x* or *y* as unknown. In computer programming we must not do that, because later, if we have a long (many hundreds of rows) program code, even the most experienced programmer can not read a code with too general names. So the variables have to refer the thing what we want to use for.

We suggest for the nomination should be used similar as Hungarian notation, what Charles Simonyi introduced for computer programming in C language [22].

4. Blockly. A visual programming editor

Now we introduce Blockly, a free, open, graphical, web-based, interactive developing tool for young people and laymen to learn computer programming. The Blockly is one of the Google Inc's projects; it started August, 2012 under the guidance of Neil Fraser's engineering team [10]. Let us see its main characteristics.

- It is free: there is no costs overloading families' and our school's budget if we wish to use it in our mathematical lessons.
- Open: It can be used by anyone, without registration.
- Graphical: We don't need to type programming code, we just 'puzzle' the code.
- Web-based and interactive: Most of interactive, web-based called tool are only an interactive Flash-animation, nothing else. Blockly is not Flash-based tool. Blockly is really web-based, it is run in computing cloud. So it is device-free, and we don't have to install anything special to our computing device (smartphone, computer, tabletPC, interactive whiteboard, etc.), we just need a web-browser. To tell the truth it is optimized for Chrome browser, but it have become one of the most popular browsers in the last two year. When we finish our program we get a link of our project what is stored in Google cloud, and we can refer and look back our work everytime when we open this link.

Our meeting with Blockly was, when – as usual – Google the 'outsource' translating, naturalizing and testing of their beta version software product. One of the authors undertake this testing and translated into Hungarian as a hobby activity. He made own applications as well. Later we decided to teach this simple programming tool for children in Hungarian primary and secondary schools.

5. About block languages

Blockly is one of the newest representative of the big family of block languages. There are other projects too (e.g., Scratch [20], MIT App Inventor [17], Code.org [6]), what are very interesting and motivative for learning computer programming, and, of course, there is a long history of this kind of languages and the idea of 'democratization' of computer programming.

In the 1970'ies when personal computers have become more popular there was a thought that children from their early years should have to introducing computers and computer programming [18]. And not only children but everybody needs some knowledge about computer programming, because in our everyday life we can realize that computers can help in our work if we are able to control them, but they can embitter if we are not. Some of computer scientists told it from the very beginning, and they worked for solving this question [21]. The timeline below shows the milestones of this process:

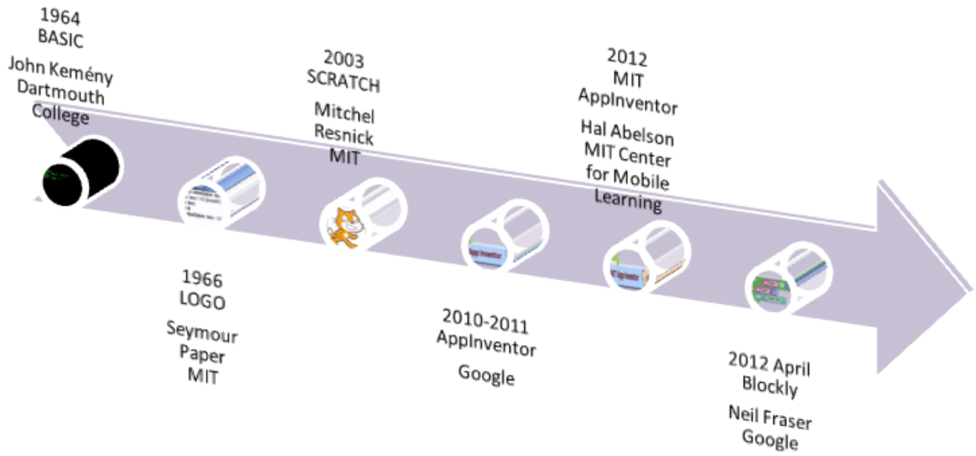


Figure 1: Milestones in the History of Block Languages

The latest remarkable event in the history of block languages was in December, 2013. Google, Apple, MIT, Microsoft and many other information technology companies campaigned for teaching computer programming in schools. The name of the event was the 'Hour of Code'[6]. All over the world several educational institutions, pupils, teachers joint the event, and finally more than 27 million students have done the Hour of Code and written more than 993 million lines of code. The most popular languages were block languages, and well-known personalities from great information technology firms advocate the event and specially the teaching of computer programming prefered block languages.

6. What kind of problems can we use and what teaching for?

The varieties of usage are wide. It is a general programming language, but it has several ready applications beside the programming environment. In these applications we get a problem, for example a maze, where we have to walk out a man from. We can solve this problem, if we 'write' a program with given programming elements from the set of Blockly's elements. These applications provide us some kind of introduction into the using of the language before we start to create an own program. Some of the applications are mathematical applications, for example there is a function plotter application. There is 'turtle graphics' what is similar to

LOGO programming language. In this we can navigate a turtle, we can draw with it and we can transform our drawings. Plane application is a 3 level application for creating mathematical formulas for a given problem. With Block editor we can create own blocks, if we miss something from the set of elements, or we need some special, or we want to make own application (for example our Pursuit application in Figure 3).

What we used was the 'Code' page, where we can make own computer programs like in any other developing environments. The word problems we solved with the children were from Hungarian textbooks ([3], [4], [8], [12], [13], [15]), and the solutions were pupils' solutions, obviously in Hungarian. The translating of the problems[2] attached before the screenshots. Let us see some examples.

– To practice declaration of variables – problems of type 'I think of a number'.

'I think of a number. If I add 5 to it, then I divide the getting number by ten, then I subtract two I get zero. Which number I have thought?'

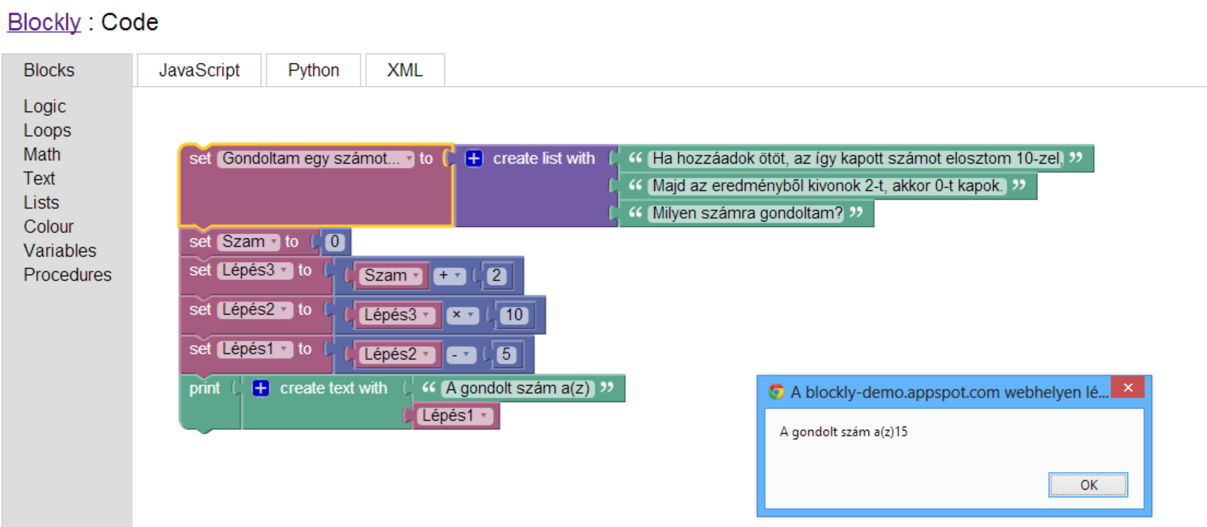


Figure 2: Pupils' solution

With this type of problems – perhaps it is not a classical word problem – we want to show, how we could prepare the students to solve equations by Blockly. This could be the first step, when we show children how to solve the problem in details and how to use opposite (inverse) operations.

– To use variables and simple mathematical operations – several types of word problems, algebraic or early diophantine word problems, joint work problems, problems from physics or chemistry which lead to linear equations or inequalities.

Pursuit application is our own application. We made it with Blockly's block editor. It is based on a classical Diophantine problem for young children.

'I have 35Fts in my pursuit. How many 2Ft and how many 5Ft coins I have, if I have maximum 21 from each type of coins. Write the plan with the program blocks how could you count!'

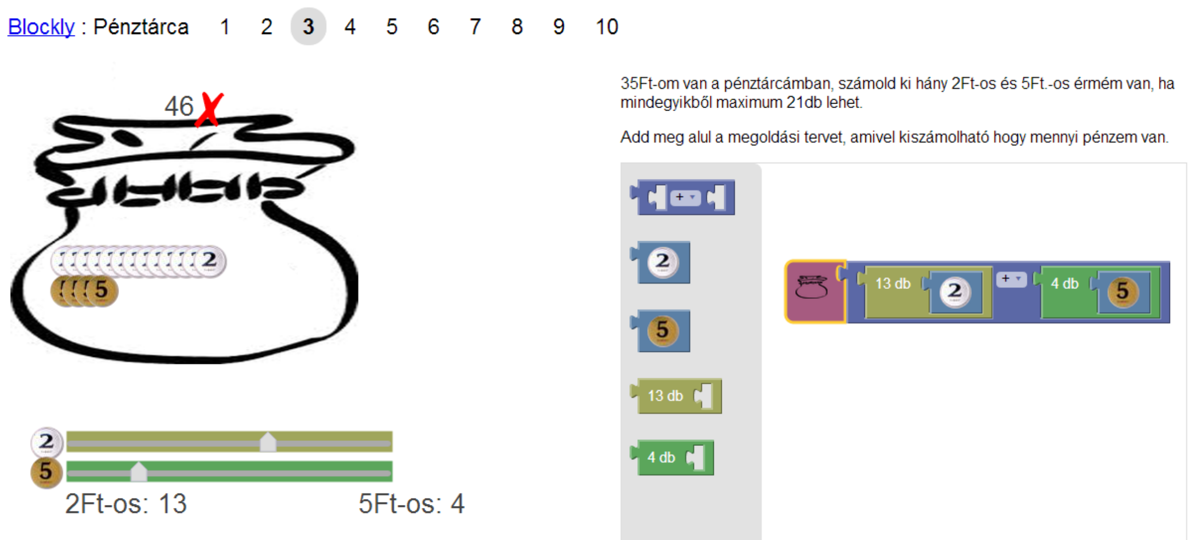


Figure 3: Our own application

The other classical problem to illustrate these simple problems is Santa Claus's pocket-problem.

'There are some peanuts in Santa Claus's pocket. He says, he has 44 peanuts in his two pockets. If he put 12 peanuts from his left pocket into his right pocket, he has equal number of peanuts in his two pockets. How many peanuts are there in Santa's left and right pocket?'

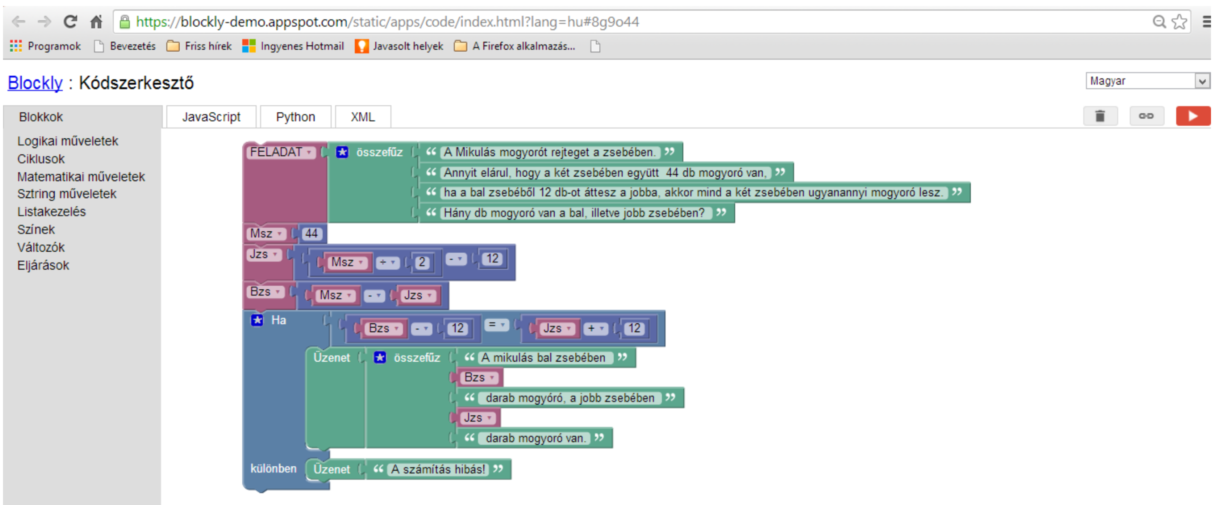


Figure 4: Pupils' solution

– To use loops – sequences (arithmetic, geometric)

– To use conditional statements – early heuristic problems

’David’s mother is older than 24 but younger than 51. If we summarize the digits of the year she was born we get 22. How old is she now?’

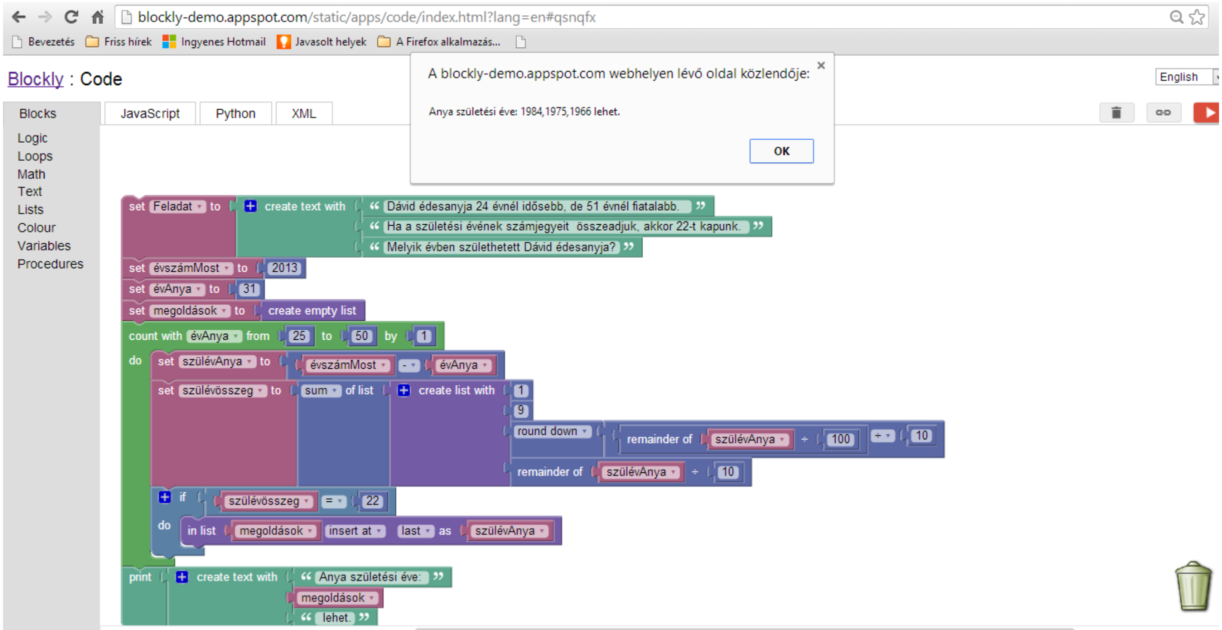


Figure 5: Pupils’ solution

In these examples we can see a heuristic problem solving by computer programming. Most children think about heuristic problem solving is just aimless trial and error experience what we do, and they hardly see the systematic way of thinking in it. But with computer programming we can show children this ’systematic trial’ (we can use program control structures for it, like loops, and ’if…then’ structures, and we think this is why children realized that this type of problem solving as serious and concious mathematically as solving of equations.

– Introducing list, as a data structure – special algebraic problems

In the following example we can see the using of program structures in a simple arithmetic inequalitiy problem.

(As we can see the editor of Blockly has an opportunity to comments on our blocks.)

'Write a program what counts which numbers from the list make the inequalities true.'

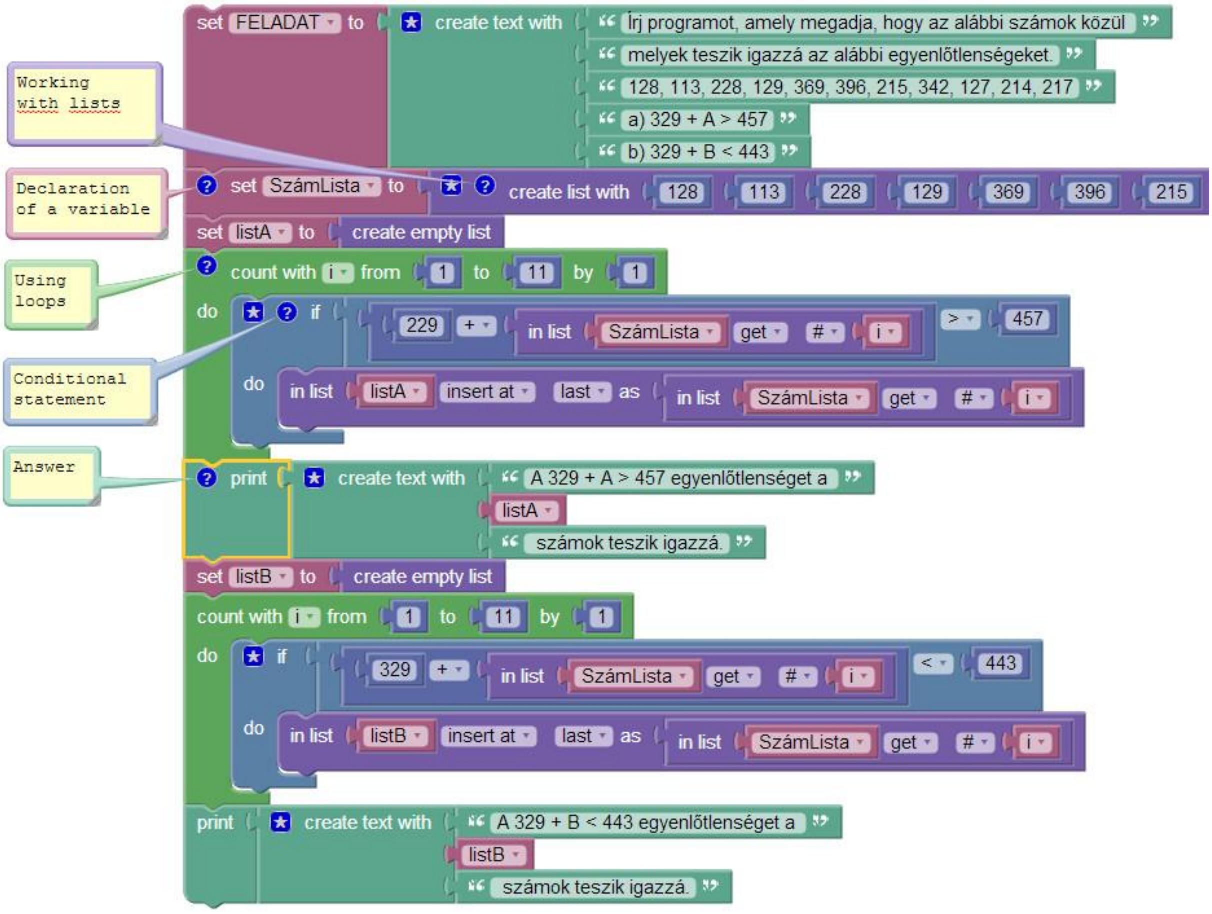


Figure 6: Program structures

7. About our experiment

First we tested our idea in small groups of primary school children. We had two small study groups – 6 and 8 pupils from different classes and ages from 8–14. First group is a regular AppInventor and mobile device study group (we taught it in an elementary school in Debrecen; mainly boys were interested in the topics. The other group is a random group, in a summer day care camp organized by a cultural centre in Bátonyterenye, Nógrád County).

We let children 'play' with Blockly, to develop it, and we waited for the result. First, we introduced ready applications (airplane, pursuit, Turtle Graphics, etc.), later we gave them word problems and asked them to solve it by Blockly. What we must mention first, that no one of children missed paper! They started to solve the problem by their computer device at once, and they did not have difficulties with the use of the tool at all. But to tell the truth there were very simple problems what lead to simple equations.

In this ages we stated the following:

- children have difficulties with formalizing something into mathematical notation;

– the opposite operation is not obvious at all for them;
– they had difficulties to state a model;
– they also had difficulties when they should talk about a modelled or formalized problem, and say by their own words what is it about.

So we suggest to practice it more often both on mathematics and computer science lessons, to develop the meaning of formalization. In our experiences it is a common weakness of mathematics and computer science teaching.

In Blockly's syntax we don't have parantheses, but we must embedding one expression into another when we want to demonstrate the brackets or when we want to make operations with more than two variables. It posed some difficulties to children to think about the right sequence of the expressions and variables, but we think it is another benefit of the Blockly method, because the usage of parantheses could be more conscious.

By Blockly we can inspire children to make problems by their own, what is important to get a familiarity in discussing of problems.

No one of children like when realize, that his/her solution is false. But we saw children accept the criticism of a machine easier than that of a person, and we saw that programming was the best tool to teach children to accept their own mistakes and at this point not to leave the problem but to encourage correcting errors.

'The area of a city is from 5 coincident squares. The layout of the city is L-shaped. The stone wall, bordering the city is the same km long, as the area of the city in km^2. How long is the stone wall?'

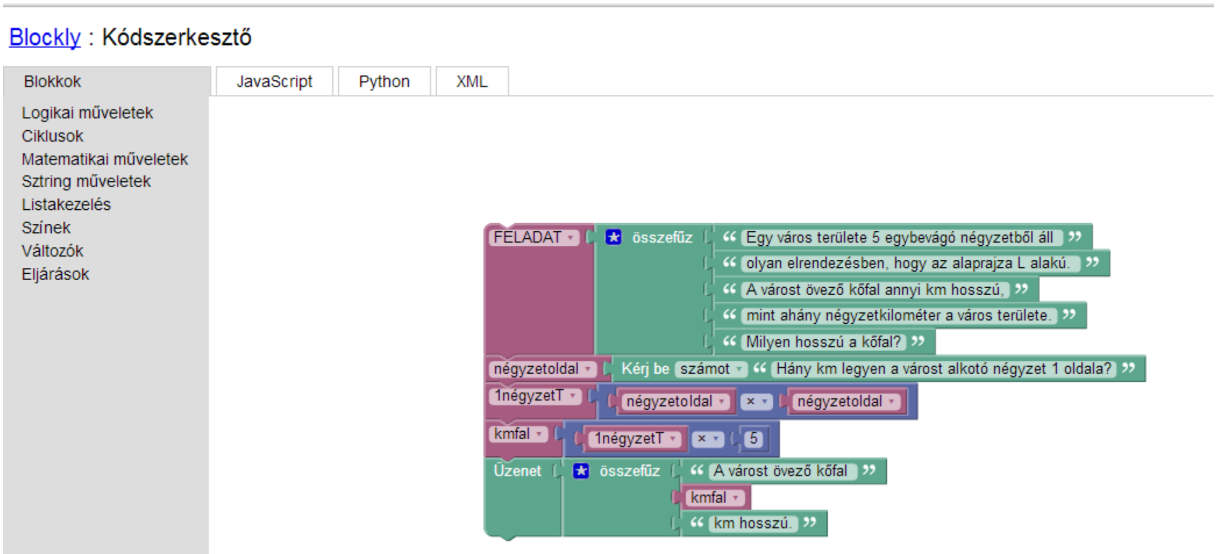


Figure 7: Pupils' wrong solution

After testing, when we broke their programs with wrong testing data children realized what did they wrong, and corrected their thinking about the problem.

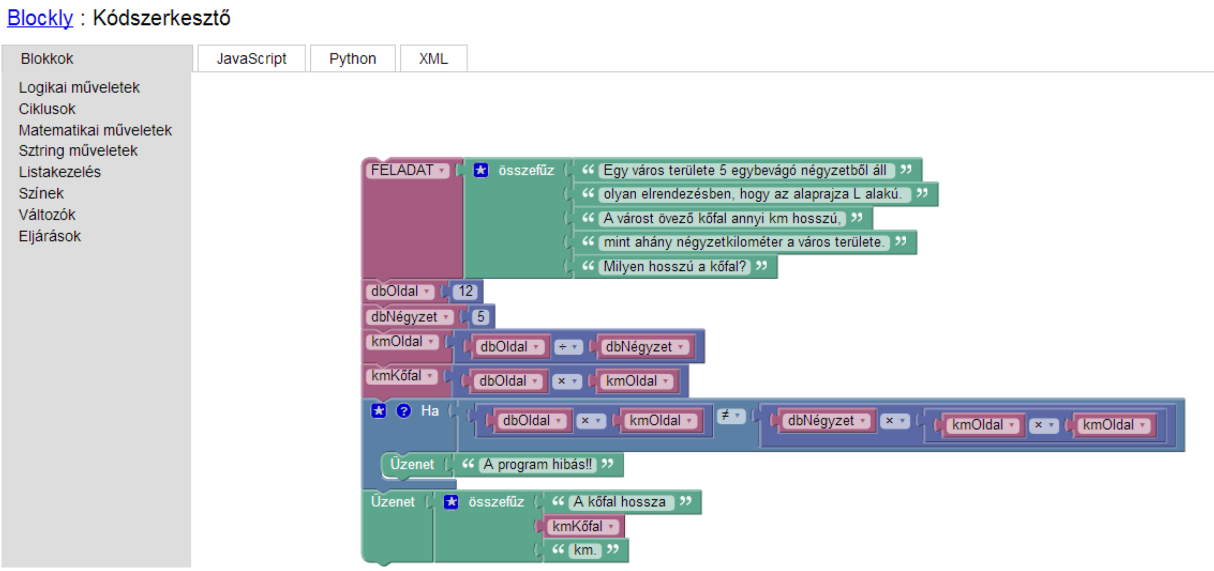


Figure 8: Pupils' good solution

Most of children monitore their result in paper, but they did not want to do that in computer! They said, that computer counted well, so they did not have to control. (So we can see children think, monitoring is about developing miscounts only!). So we suggested to build the answer and the monitoring into the program (as we can saw it at the examples).

The typical problems we found were:

- *Declaration of variables.* One of children's typical mistake was that they wanted to declare too much variables, or they often made rotary declaration, what computer could not understand. So we had to draw their attention to express one variable from another instead of declaring always a new one, but we should do it carefully because of the rotary declaration what did not lead anywhere nor in mathematical problem solving nor in computer programming, as computer indicated for us with an error message.
- *Mathematical modelling problems*. We met similar problems as in the former test group.
- *Distinguish between the knowledge of the user and the computer*. It was a really interesting and instructive thing that children believe that computer is more intelligent than it is, and computer posseses the knowledge what they have. But, obviously, computer doesn not have that knowledge so they realized that programmer has to build into the programcode any small thing what we want to

be as the ’knowledge of the computer’. We think, this is a very important experience for children which make them more responsible about using technology.

- *Finding connection between the elements of mathematical model and the elements of algorithmization*. It was not a serious problem; we found difficulties only in the beginning of the studying of programming. Pupils in secondary school correspond the elements of mathematical model and the appropiate algoritmization tool, as primary school children. It seemed children had to be some maturity to this process.
- *Using controling structures*. Secondary school children in our test group were different orientation groups. We can state that every groups like the Blockly as programming tool, but there were more difficulties about the computer programming those groups whose orientation was not mathematical or engineering. At the same time we can state that the enthusiasm for Blockly was much more definite in the language orientation group as in the others including engineering team. Language orientation team like very much that they can control the computer. Engineering team has already met computer programming topic before, so it was not new to them.

We have not detected problems with the following:

- *Comprehension - understanding problem*. We can state that children in this secondary school have no difficulties in comprehension or the meaning of the problem.
- *Mathematical formalizing and notations*. Children can use mathematical formalizing elements well.
- *Using Blockly’s graphical interface and syntax*. Using of drag-and-drop based graphical interface was natural for pupils. The syntactical elements were also clear for them. Difficulties of using the tool did not divert their attention from the problem and did not take to discourage from solving the basic problem.
- *Speed of solving problems*. The time they solved the problem with Blockly was not slower than with paper and pencil.
- Understanding the importance of *built-in monitoring and answering and discussing the problems*. It was the first moment that monitoring, answering and discussing becomes children’s natural offer. This processes started so naturally in

the first time that later pupils initiated the analysis of the problems by their own and made tests with interesting data. So they wanted their solution to be reliable and they checked it.

If someone could not finish his work in the lesson, he could continue it at home, if he saved into the cloud storage and sended the link himself about his program for example via e-mail or other way. For each five study group we created a closed group on Facebook. (All of our children were more than 13 years old, and they had Facebook-accounts.) They liked this method because they could share their solutions with each other or with us. We could discussed if someone need a help in homework as well by this way.

The pupils in our test group were different orientation, class, grade and gender. They studied in five groups. We introduce them in Figure 9 infographics which is made with the data visualization tool of Microsoft Excel 2013 with Power Pivot 2013 business intelligence and data visualization software. The final analysis of the results for another paper will be processed with this program as well.

An important benefit of the work was in the secondary school that students started to inquire computer programming. In December of 2013, the school and students joint the national event called the ’Hour of Code’, where we can show Blockly and AppInventor for Android block languages for other pupils in the school, and we made own moble device applications with them.

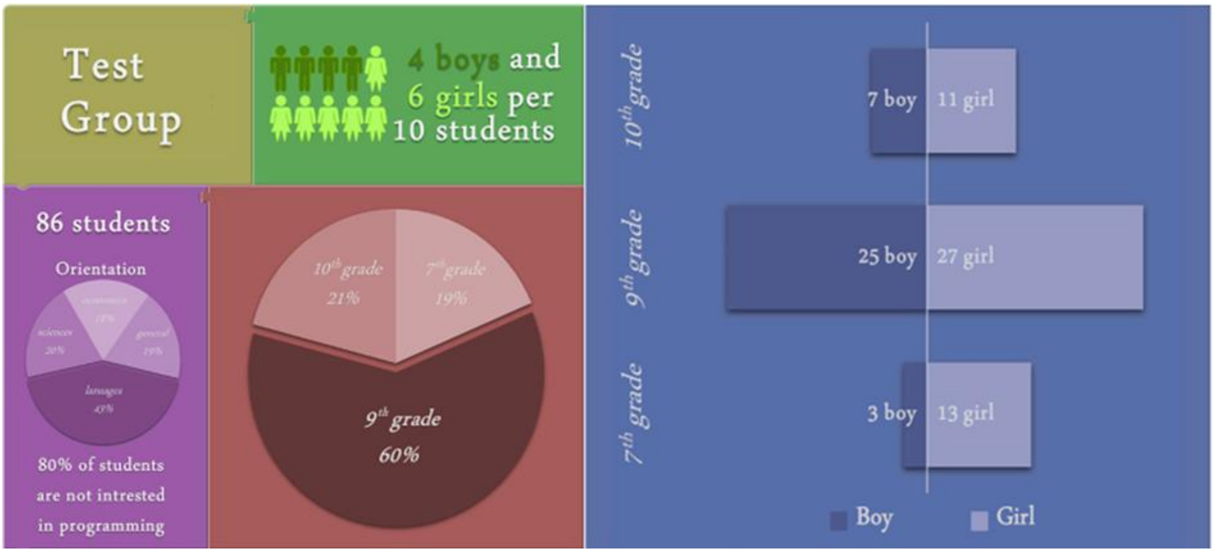


Figure 9: Composition of test group

8. Conclusions

Using Blockly we obtain a promising new method to make mathematical lessons more interesting and more interactive for children. Furthermore, children in early scool year can see that computer is not just a toy but a valuable tool to work, and they can see the relationship between mathematics and computer science in practice.

We think, most of mathematics teacher feel the problem that they have to renew their methods. But they do not know, how could be modern, and, at the same time, how could they preserve the natural worths of mathematics. We think Bockly is a suitable tool for this purpose. Using Blockly requires creativity and autonomy from teachers and also from children. With the usage of Blockly in mathematical lessons an innovative mathematics teacher will not depend neither on ready exercise books nor on ready computer programs. Teachers can create own problems to their lessons, and they can also improvise with discussing problems, changing initial conditions etc.

Blockly’s benefits for mathematical and computer literacy could be its:

- simplicity,
- quickness,
- minimal syntactical errors,
- similarity to mathematical knowledge,
- accessibility,
- easier using to any mathematical editor,
- no accomodation of rules of programming languages,
- built-in automated translator,
- speed of solving problems.

It is worth to explain the last two statement in details.

Blockly has a built-in automated translator into other programming languages. So it can be suitable for talented children who are interested in computer programming seriously to start introducing other programming languages, like Java or Python.

We examed the speed of solving problems by paper and by Blockly. We can say, when children are well read up on themselves in Blockly there was no difference between the duration of the two solving methods. The speed of Blockly method in problem solving is not slower at all as paper and pecil method, so teachers should not have to calculate with time loss when they plan their lessons with Blockly. At the same time, with Blockly we can solve those type of problems, specially heuristic problems, what we would not make in mathematical lessons because of their time-consuming counting periods. We write a program, so we make the model (the plan) of the solution, but we do not have to waste the time for long counting.

The application of Blockly did not posed difficulties for children at all. They used it naturally. But as difficulties or problems we have to mention the rapid developing of the programming environment. It is independent from us and we are providing the changes of the device, what we have to respond quickly. It is not favoured in mathematics teaching, but, in cloud computing technology it is natural, so, from the point of computer science teaching it has some benefit that we can demonstrate it to our pupils.

It could be weakness from mathematical didactical point of view, that there is only mathematical operations with two variables in Blockly language. So, if we want for example add three or more variables, we have to embed more blocks into each other, and too much embeddings can result complicated formulas. We had to state, that it was difficult for primary school children.

We think that mathematical and computer science program solving are similar intellectual processes, so they require analog didactical, pedagogical approach and methodological strategies. Mathematical didactics have a very large literature to make the problem solving process easier and help teacher how to teach it. We suggest for computer science teacher to 'hire' these methods and implement it into their own subject and lessons. First meeting with computer programming could be by solving mathematical word problems.

Finally we can state that the main advantage of this method that children consider much more seriously the solving methods when they solved problems by programming. First they modelled the problem mathematically, and then code to Blockly what is fitting to mathematical thinking and problem solving methods and it is a trendy latest technology (cloud computing) tool what is an important motivation aspect for the digital generation. So we can say that Blockly is an excellent tool of supporting subjects contact, in our case mathematical exercises what help computer science teaching, and computer programming that helps teaching mathematical problem solving.

Bibliography

[1] Ambrus, A. (1995). *Introduction to mathematical didactics.* Budapest: ELTE Eötvös Kiadó, (Hungarian)

[2] Aufman, R. (1985). *Introductory algebra.* Houghton Mifflin Co.

[3] Balassa, L. (2007). 4th *textbook in mathematics.* Apáczai Kiadó, (Hungarian)

[4] Balassa, L. (2009). 3rd *textbook in mathematics.* Apáczai Kiadó, (Hungarian)

[5] C. Neményi, E. (1999). *Teaching of counting; Word problems.* Budapesti Tanítóképző Főiskola, (Hungarian)

[6] *Code.org*. (2013). Source: http://learn.code.org/

[7] Csíkos, C. (2010). The role of visual representations in mathematical problem solving. *Magyar Pedagógia*, 110(2), 149-166. (Hungarian)

[8] Esztergályos, J. (2007). *My mathematics, 2nd grade.* Apáczai Kiadó, (Hungarian)

[9] Faragó, L. (1963). *Solving word problems with equation.* Tankönyvkiadó, (Hungarian)

[10] Fraser, N. (2012). *Blockly. A visual programming editor*. Source: https://code.google.com/p/blockly

[11] Gooding, S. (2009). Children's difficulties with mathematical word problems. *Proceedings of the British Society for Research into Learning Mathematics*.

[12] Hajdu, S. (2012). *Mahematics 7. B version*. Műszaki Kiadó, (Hungarian)

[13] Hajdu, S. (2012). *Mahematics 8. B version*. Műszaki Kiadó, (Hungarian)

[14] Lants-Andersson, A. (2009). *Framing in Education Practices. Learning activity, digital technology and the logic of situated action.* Göteborgs Universität.

[15] Kuruczné Borbély, M. (2007). *My mathematics.* Apáczai Kiadó, (Hungarian)

[16] Michalewicz, Z. (2008). *Puzzle-based learning.* Hybrid Publishers.

[17] *MIT App Inventor*. (2012). Source: http://appinventor.mit.edu/

[18] Papert, S. (1993). *Mindstorms. Children, computers and powerful ideas.* Basic Books.

[19] Polya, G. (1957). *How to solve it.* Doubleday.

[20] Scratch. (s.a.). Source: http://scratched.mit.edu/

[21] Sebesta, R. (2010). *Concepts of programming languages.* Pearson.

[22] Simonyi, C. (1999). *Hungarian notation.* Microsoft Corp.

[23] Török, T. (2009). *Word problems, and solutions, teachers' manual.* Nemzeti Tankönyvkiadó, (Hungarian)

Solving differential calculus problems with graphic calculators in a secondary grammar school

Ferenc Várady
Budapest Business School, CCCT, Hungary
varady.ferenc@kvifk.bgf.hu

Abstract

Use of graphic calculators is officially not allowed in Hungarian middle and secondary schools. But in the last years more and more mathematics teachers may use the smart boards. Numerous functions of graphic calculators can be realized with help of smart boards. This makes necessary to think more consciously about the use of graphic calculators. I am teaching in a German school in Budapest, where the use of graphic calculators based on German curriculums is allowed. In two groups of students Grade 10 I made an experiment about the introducing the differential calculus with help of graphic calculators. We concentrated for two aspects, how much should we teach without graphic calculators at the concept formation and problem solving, and at what point is desirable to allow their use in the teaching process. At the use of graphic calculators the visual representations of mathematical concepts comes in the foreground. Many students have problems with different representations of the function. They do not see the connection between them and cannot transform one representation into another. In our experiment we emphasized these activities. In the brain-based mathematics teaching use of different representations is a base idea. In our experiment we tried to take into consideration of some new results of brain research.

Key words: difference quotient; differential quotient; graphing calculator; representations; secondary grammar school

ZDM Subject Classification: D43; D53; I43; R23

Introduction

Recently the usage of graphing calculators has been introduced in a lot of countries, including Germany, but unfortunately not in Hungary. Many studies showed (Ambrus, 1995; Dunham, 2000; Lichtemberg, 1997) how graphing calculators can help pupils develop their knowledge and skills e.g. in the following areas: concept development, problem solving and computation skills. "*Using graphing calculators in mathematics education bring also new methods of work - especially the possibility of exploration and modelling of mathematical problems, multiple representation of mathematical problems (numerical, algebraic, graphic, algorithmic repre-*

sentation) and graphic support of the results obtained by algebraic procedures" (Robova, 2002).

In my paper there are two mathematical concepts in the centre: the concept of the difference and differential quotient and the increase and decrease of a function in a given interval. Concerning of these concepts representation has a very important role. The appropriate concept image of the difference quotient can be developed with this dual representation (Tall, Vinner, 1981). Pupils have to learn to change between the enactive, iconic and after that the symbolic meaning of representation and applying them at problem solving. In this process the GC can provide much help, according to Kamarulhaili and Sim (2005): "*The cognitive gain in number sense, conceptual development, and visualisation can empower and motivate students to engage the true mathematical problem solving at a level preciously denied to all but the most talented. The calculator is an essential tool to all students in mathematics.*"

In Germany in the last decade teaching has had the following two significant characteristics: 1. They have started to support the content of the curriculum with real life problems, and today on them based mathematics is still present in great volume. 2. In secondary education it is compulsory to use a graphing calculator. Naturally, questions arise regarding both points: Are not the often dominant exercises used at the expense of theoretical knowledge? Are pupils able to apply mathematical relations in different contexts if theory is not appropriately established? While using the opportunities provided by the graphing calculator do not they forget basic mathematical operations? How can they be trained in teaching to do basic exercises securely also without a calculator?

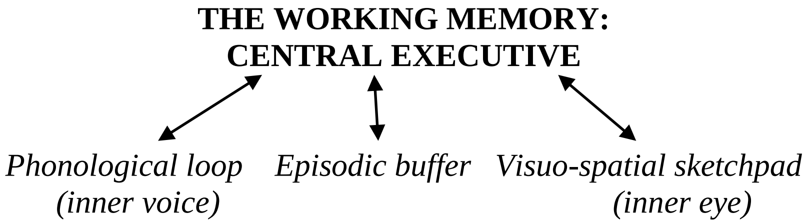
My answers to the first two questions point to the same direction. In my experience, if pupils do not get sufficient theoretical background, most of them do not see beyond relations drilled in practical exercises, they cannot generalise. In the case of questions asked in altered contexts they do not recognise relations, often they are not able to use the rules learned if the question is asked in other words. This is why I consider the five phases of learning defined by Meir Ben-Hur essential: Practice, De-contextualisation, Encapsulating a generalization in words, Re-contextualisation, Realisation (Meir Ben-Hur, 2006). I would like to point out the third phase, the encapsulating process, where generalisation has to be done and deepened in words and mathematically at the same time. I am going to answer the other two questions in this work.

In the school where I teach and I made the experiment it is required to use a GC from class 9 (14-15 years old pupils). The experimental classes were both 10th classes. In these classes are the main concepts of the difference calculus introduced. During this work great emphasis was placed on methodology, I tried to use the advantages of teaching with and without a GC in parallel.

Theoretical background

Memory structures

In the cognitive psychology we differentiate three forms of memories: sensory (perceptual), working memory, long term memory. The role of working memory is decisive at learning, problem solving.



*Executive functions***:** goal setting, planning, organizing, prioritizing, initiating, holding information, inhibiting irrelevant information, self-monitoring, memorizing, self-regulating, representing, problem solving

*Limits of WM***:** very limited capacity holding 7 ± 2 info units,

time limit: 18 – 20 sec, goal maintenance, inhibit irrelevant information

Overcoming the limits of Working Memory

Chunking. It means the compression of information into lager units. For example to remember the number 238465197 is easier by building groups: 238 – 465 – 197.

Another possibility is *connecting the information to previous knowledge*. We can build a concrete metaphor to the abstract situation.

Automatism: if a student can handle a mathematical concept or apply an algorithm automatically, there is no need extra capacity for them in working memory, more free space remains for other mental activities.

To write down the relevant data, using graphs, tables.

Brain-compatible guidelines for mathematics instruction (Bender, 2009)

We analyse in detail only the role of representations as a relevant question in our topic.

Bender writes about the use of representations. "New concepts in math should be presented at three levels: concrete (e. g. manipulative), pictorial or visual and abstract. Visualizing of math problems through the use of concrete examples and /or representational examples assists many students in mastery at almost every grade level. Manipulatives should be used across the grade levels for students with learning problems. Teachers should have students visualize problems using manipulatives and then explain the result to each other. Have other students to develop a representation of a problem, while still others consider it abstractly. Have students consider the question: *„How can we make mathematics representational?"* This will assist learners who have strength in spatial intelligence."

The graphic calculators are god tools to realize the change of representations. As for the memory it is better to have 3 different memory traces for the same concept. In this case the chance for retrieval will be much higher.

Basic instructional strategies to enhance learning.

1. Gaining students' attention. No attention, no engagement, no learning!
2. Activating students' prior knowledge and experiences. Having prior knowledge and experience that relate to current learning enhances memory and visa versa. In our case this was a difficult phase.
3. Actively involving students in the learning process. Students will be able to retain 10% of what they read, 20% of what they hear, 30% of what they see, 50% of what they see and hear, 70% of what they say, 90% of what they say and do
4. Facilitate the ability of students to construct meaning. Mind tends to remember content that is meaningful and well structured.
5. Students demonstrate their learning. It needs a careful preparation: organizing, separating essential vs. nonessential, connecting to previous knowledge, constructing meaning, chunking (compressing information into blocks), summarizing, teach to take tests. The aim should be the development of self evaluation of students, ability to reflect on himself (herself) as a learner: How do I plan to learn? How do I monitor my learning? How well did I do? Do I need to make changes? (Banikowksi, A. 1999)

In our experiment we concentrated to realize these five ideas, at the constructing the meaning first of all the visual representations played dominant role. The visual representations of functions gave a holistic picture about the whole concept.

About the use of graphic calculators

Horton (2004) thinks that GCs can help make connections among representations in the mathematics education and so they can help "*permit realism through the use of authentic data*". In 2000 the National Council of Teachers of Mathematics summarised the results of many resources (e.g Ruthven, 1990; Smith & Shotsberger, 1997; Tolias, 1993) about the positive effects of using a GC:

- Speed: After students have mastered a skill, teachers allow the usage of graphing calculators to compute, graph, or create a table of values quickly.
- Leaping Hurdles: Without technology, it was nearly impossible for students who had few skills and little understanding of fractions and integers to study algebra in a meaningful way. Consequently, lower level high school courses often became arithmetic remediation courses. With technology, all students now have the opportunity to study rich mathematics. They can use their calculators to perform the skills that they are unable to do themselves.
- Connections: A sophisticated use of graphing calculators is to help students make connections among different representations of mathematical models. Users can quickly manoeuvre among tabular, graphical, and algebraic forms.
- Realism: No longer are teachers restricted to using contrived data that lead to only whole number or other simplistic solutions. Graphing calculators permit the creation of several types of best-fitting regression models. This capability allows data analysis to become integrated within the traditional curriculum; the tedium and difficulty of calculating a best-fit model are no longer factors in introducing data analysis into the curriculum. (pp. 24-32)

Tiwari (2007) proved in his study too that the connection between the algebraic and the geometric representations by GC can be deeper in calculus; the GC can support the understanding "when it is used as a supplementary instructional tool in achieving conceptual understanding and enhancing problem solving abilities of students in learning differential calculus". Van Streun, Harskamp and Suhre (2000) showed in a study that the use of GCs can lead to changes in students' approaches in problem solving. These positive changes have affected student achievement. Jones (2005) got similar results. He thinks that using graphing calculators, pupils can approach problems graphically, numerically and algebraically. Ng Wee Leng (2009) found that the use of various problem-solving approaches can support students' visualization in finding the solution and allow them to explore problem situations which they might otherwise not be able to handle.

The technological advancement makes the usage of the tools of information technology and the media possible. Today the range of these tools is rather diverse. We may talk about different computer applications, software, educational videos, and interactive boards. In recent times easily portable tools have come to the foreground. These might be „simple" or graphing calculators, voting-machines, and we can download several useful mathematical and scientific applications even on smart phones. Timo Leuders also points this out in his work. In his opinion, applying these new technologies may be especially important in mathematics: „*Neue Technologien und neue Medien (gemeint ist meist: Computer) bieten für den Mathematikunterricht – mehr noch als die meisten anderen Schulfächer – die Chance zu einer grundlegenden inhaltlichen und methodischen Reform. Sie ermöglichen eine Entlastung von Routinearbeiten und bahnen daher exploratives und kreatives Arbeiten, ebenso die Behandlung realistischer Anwendungssituationen und das Vernetzen von Inhalten*."[1] (Leuders, 2010)

The question is, to what extent it is necessary and possible to use these new opportunities in teaching. According to Tulodziecky, the different tools of information technology have to be used as support and encouragement in school education, if the teaching process is problem, decision and organisation oriented (Tulodziecky, 2007). Tulodziecky considers the application of the media especially important in five cases: 1. Difficult exercises with adequate degree of complexity as initial conditions; 2. If we want to exemplify the goal or the route to the solution; 3. If the individual or cooperative work form comes to the foreground while solving a difficult task; 4. When comparing different modes of solution; 5. During the application of the theory learned and reflections to them. In my experience, pupils use the graphing calculator with pleasure, and they even use such applications that they do not need.[2] Finally, I would like to mention the viewpoints of Erwin Abfalterer, which have to be considered by all means when planning a lesson with the computer (graphing calculator), so that the lesson flows with the greatest efficiency in the time available: 1. The software has to be prepared and tested; 2. The flow of the lesson is planned and the goals are set; 3. The exercises are given, the role of the teacher is clear; 4. A short feedback always needs to be possible.

Research question

How much supportive can be a graphing calculator in problem solving related to tangent line and the increase and decrease of functions?

Research methodology

In my school, in the Deutsche Schule Budapest the curriculum of the German Baden-

[1] *„New technologies and new media (is usually meant: computer) provide for teaching mathematics - more so than most other school subjects - the opportunity for a fundamental substantive and methodological reform. They provide relief from routine tasks and thus pave explorative and creative work, as well as the treatment of realistic situations and use the cross-linking of content."*

[2] Namely, in the 11th class it occurred in the case of a simple linear equation that a pupil could not solve it without a calculator, because the calculator had been doing it for him using the appropriate command for years.

Württenberg province is used. Pupils first meet the notion of functions shortly at the end of the sixth grade. However, this time they do not use the abstract signs yet that they are going to use later, but they examine the simultaneous changing of related phenomena (direct, inverse proportionality). The real concept of functions appears in the following year, also in connection with direct and inverse proportionality and the linear function. Here the mathematical meaning of the function concept already appears together with basic functional definitions (domain, range, zero of a function), representing functions (plotting), and the textbook and its publisher (Klett Verlag) provides the opportunity for the application of the graphing calculator. The quadratic function is introduced in the seventh grade, when exercises requiring deep mathematical knowledge and understanding are also integrated. Plenty of practical applications help pupils understand the forth and back relations between the real activity (enactive mode), the visual representation (iconic mode) and the abstract mathematical language (symbolic mode). The concept of the function is then extended, the parametric functions are introduced. We have the opportunity to plan lessons with the application of graphing calculators, but pupils only get these in the ninth grade. In my opinion, as long as they do not work securely with functions, it is not either advisable to give them such directions. During class work we use the software GeoGebra that they also download at home, and they have to do homework with the help of the software as well. However, at this grade the spine line of the learning process is provided by the traditional method. The theme of functions does not appear in the ninth grade, it is present indirectly in the exponential, logarithmic part.

Fortunately, all of the classes are provided with smart boards. They can help a lot during the teaching-learning process. The lessons are saved by me and they are uploaded on the internet so they can be downloaded by the pupils. During the lesson I can scroll to the earlier board pictures every time when I want to, so I can refer to the material learned earlier. Pictures made with GeoGebra can be saved as well. During the lessons I always have the opportunity to communicate with pupils and to help them. Furthermore, if some of them have problems, they can help each other as well.

We had altogether 28 lessons for the examined topic – 7 weeks, four lessons a week. During this period the following chapters were covered:

Lessons 1-4.: Reviewing functional elements. Domain, range, zero point, function value. Practice.

Lesson 5-8.: Introducing the difference quotient. Practice.

Lessons 9-12.: The differential quotient. Practice.

Lessons 13-16.: Calculation of the differential quotient. Practice.

Lessons 17-20.: Derivative function. Practice.

Lessons 21-24.: Derivative rules. Practice.

Lessons 25-26.: Summary, practice.

Lessons 27-28.: Test

Discussion of Experiment

I am teaching the two examined tenth classes. The 10a class consisted of 18, the 10b consisted of 19 pupils in the observation period. The mathematical knowledge of the pupils is considered to be general, the averages not considering this test were 2.73(standard deviation: 1.46) and 2.89 (1.24) in the whole of the classes (according to the German system, where 1 is the best and 6 is the worst.) The test was written by 12 and 13 pupils in the classes -due to illness. The averages of these pupils in the year apart from this test were 2.50 (standard deviation: 1.26) and 2.57 (1.17).

The pupils bought the TI-*n*spire CAS type graphing calculator (GC) that is used constantly in class and at home as well. The tenth grade mathematics curriculum starts with the functions. As a first step, we reviewed the most important definitions, and then five other lessons followed. On these lessons we introduced the notions of the difference quotient, the derivate, derivate function, and the derivate rules, together with some simple related definitions. In this chapter I tried to use the graphing calculator and handmade calculation in parallel.

We had altogether 28 lessons for the examined topic – 7 weeks, four lessons a week. During this period the following chapters were covered:

Reviewing basic notions of function

Domain, range, zero point, function value.

In this chapter we discussed general functions, like e.g. quadratic function, square-root function, rational function. Pupils had to recognise them and determine their domain, and they had to test points- if they could be found on the graph of the function or not.

Word problems also appeared in this chapter: the volume of a roller had to be expressed through a radius.

A tin has a volume of 0,5 l. Its radius is *r*, its height is *h*.

a) How is the function equation $f: r \to h$?
b) Graph the function with GC and calculate the volume by $r = 5$!
c) A tin has a radius *r* and a height *h*. Its surface is 1000 cm^2.
   Give the function equation $g: r \to h$!
   Give the maximum domain of the function *g*!

This task was not easy for most pupils, nobody could have solved the part a) alone and they actively needed teacher's support. Two children from the group could have solved the problem helping them, with others we gave together the equation. We graphed the function with a GC and – by reading it – we determined the approximate radius. After that they calculated approximately the same value. The pupils had no problem with this part, they could have helped each other. In the question c) they had a same problem like in part one but three of them could have given the equation. With the others we solved the part together. Only few pupils could not give the maximum domain after the equation.

Introducing the difference quotient

The German textbook uses the phrase „Mittlere Änderungsrate – Main rate of change" besides the previously mentioned phenomena. After some practical introduction, e.g.: a ball is rolling down a slope and pupils had to determine its average speed, there were some elementary calculation tasks and word problems among the exercises. Among the word problems there was the growth of a plant in a given interval described in a table, where the task asked about several intervals.

The height of a plant was measured during nine days.

Day (d)	1	2	3	4	5	6	7	8	9
Height (mm)	0	0	0	0	1	2	4	6	7

How is the main rate of change of the function $Day \rightarrow Height$:

a) In the whole 9 days; b) In the first 3 days;

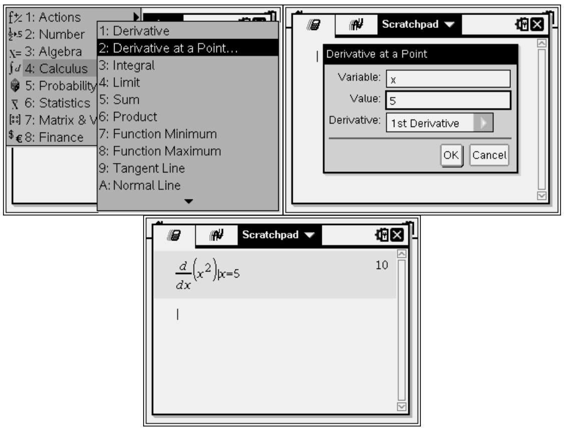
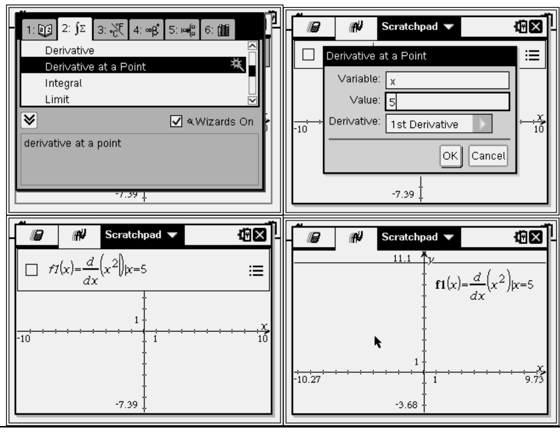
c) In the last 3 days; d) In the 3 days in the middle?

In this task the pupils worked in pairs. All of the pairs could have solved the problem correctly, they found out the right answers without my support. In the next exercises the pupils solved all day problems. Here pupils had to see the degree of growth ($f(x) - f(x_0)$) and the length of the interval punctually ($x - x_0$). There were tasks for the approximately determined average flying speed that could be read from a graph and naturally, the exact average speed of a process given by the function term. Thus pupils met the most various problems. They had to examine the tasks accurately, they needed to filter out the important information and solve the exercise. The chapter could not be lack of the geometric discussion of the difference quotient, which was the slope of secant line. When solving exercises I always put great emphasis on using opportunities of doing a task with and without a GC parallel, so as to be able to improve the skills supported by both parts the best possible way. Later in the test I checked both opportunities through appropriate tasks.

Lessons 9-12.: *The differential quotient*

The introduction of the differential quotient in this chapter began with the same ball exercise as in the previous topic, but the speed measurement interval was shorter $\mathrm{h} = 0{,}1;\ 0{,}01;\ 0{,}001$ etc. Pupils faced the same various group of problems in the tasks as in the previous chapter. The calculator played a more important role here than before, as it could count the value of differential quotient a specific point. In this chapter the phenomena of the limit of difference quotient appeared $(h \to 0)$, however, we counted with an appropriately low *h* value. This way we could only determine an approximate value, but the punctual GC solution was very similar to this. When using a GC it caused some difficulty that the formula for counting the differential quotient had to be given a little bit differently in the algebraic form than in the geometric form. This still caused problems for a few pupils later, after several weeks of practice.

Differences between algebraic and geometric usage of GC

Algebraic process	Geometric process
A: Calculate→→Calculus → Derivative at a Point 	B: Graph→→ 2: $\int\Sigma$ →Derivative at a Point 

Classes began with concrete counting exercises, and then we determined the approximate value of the differential quotient through a graph in given tasks. These exercises were followed by word problems, like the actual speed of a consistently slowing car at specific moments, or counting the amount of water flowing in a reservoir at a specific moment with the help of the derivative of the given function. In this chapter we determined the term of tangent line at a given point with a GC. The difficulty in this process for many pupils was that with and without a GC algebraic and geometric processes were different.

Calculation of the differential quotient

The limit of difference quotient appeared here, which meant that the value of differential quotient was already calculated with the help of the limit $h \to 0$, but still not at specific points.

After the calculation of functions term we solved two word problems again, where the topic was consistently accelerating movement. The problems were finally solved correctly with a lot of teacher's help explanation. Fortunately, pupils were discussing the same topic in Physics at the same time, so it was easy to draw a connection between the two subjects. At the end of the chapter we deducted the equation of the tangent line and solved exercises. Later, in task 6 of the test I asked pupils to: "*Determine the tangent term of the function using GC:* $f: f(x) = 2x^4 - 3x^2 + x - 5$ *at* $x_0 = 2$*! Describe briefly how you proceeded! Sketch the function and the tangent line using the GC on the sheet.*" I wanted to know if there were pupils who were going to deduce the term besides the GC solution and compare the two solutions. One pupil did it.

Derivative function

In this chapter we derived a few derivative functions of polynomial functions (in their largest real domain), like:

$$f(x) = c \ \ (c \in \mathbb{R}); f(x) = x; f(x) = x^2; f(x) = x^3; f(x) = \frac{1}{x} = x^{-1}; f(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

Pupils had to memorize these; they knew in advance that I was going to check one of them in the test. The graph of a function and the graph of a derivative function also had important roles here. Pupils faced such problems where the two graphs had to be paired and such where the derivative function had to be drawn in the given graph of function and vice versa. They had to explain the connection between the graphs in each case.

Derivative rules

Three simple derivative rules were introduced in this chapter –without proof. These were the following: 1. Constant multiple rule; 2. Sum rule; 3. The polynomial or elementary power rule. With these rules we solved a lot of exercises, we derived functions.

During the lessons and in the homework I encouraged pupils to do the exercises with and also without the graphing calculator. There were some situations where they were not allowed to use it, and there were some, where the function describing the process was such that they were not able to do the task without a calculator. Naturally, at the end of the chapter pupils wrote a test, where they were not allowed to use the GC for some exercises, one task could be done with and also without the calculator, and the last two exercises could only be done with the help of the calculator. I tried to combine a variety of tasks and it was very important for me that the pupils could also explain what they had done. In the first part there was an exercise for a theorem and its proof, one exercise for the application of the definition of the difference quotient and derivate, one exercise for the visual representation of the the increase and decrease of the function graph and its tangent lines at special points, and at the end an exercise where the pupils had to find the pairs of functions and their derivate functions – they had to explain why they had made the pairs. In the second part the pupils used their GC. In the first task of this part they had to give the derivate function and the slope of the tangent line at a special point. The pupils could choose between solving this with derivate rules or with a GC. In the second task they had to give the term of the tangent line with a GC and to plot the function with their tangent line. In the third phase of this part the pupils had to plot a polynomial function in a given interval and its derivate function and they had to explain the relations between them. There were two groups in which the tasks were similar to each other. One of the task sheets looked like the following:

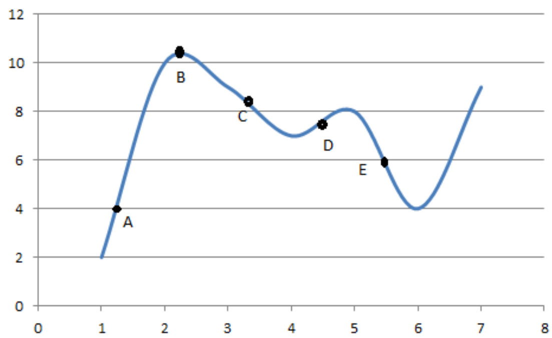
The function test

Group ♣

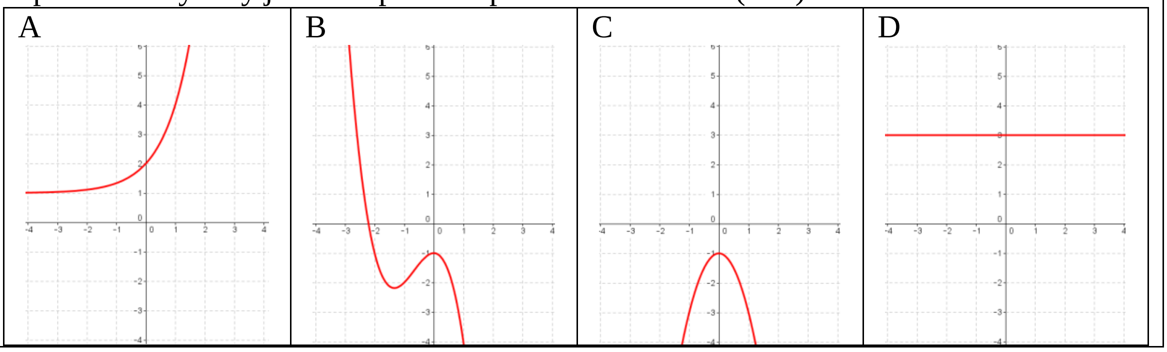
Part A

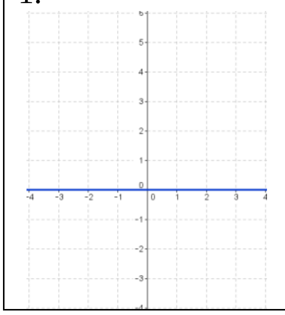
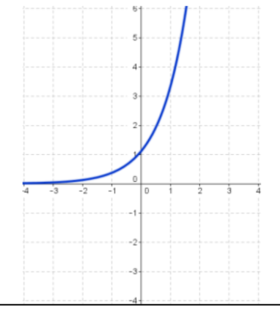
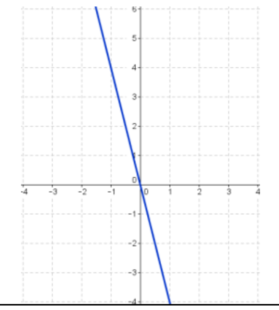
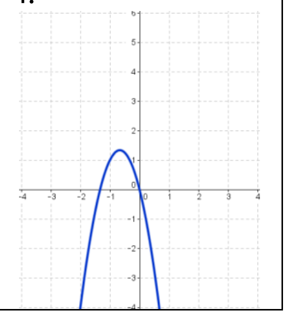
It is NOT ALLOWED to use a GC!

1. Given the function $f: f(x) = \sqrt{x}$ $(x \in \mathbb{R}^+)$. Give the derivate function and prove it. (1+3)
2. Given the function $f: f(x) = \frac{1}{2}x^2 + 1$.
   a) Give the difference quotient in interval [0; 2]. (3)
   b) Give the geometric meaning of the result. (1)
   c) Give the differential of *f* at $x_0 = 2$ with $h \to 0$. (3)
   d) Give the derivate of function. (2)
   e) Give the geometric meaning of the result in c). (1)
3. See the function graph in interval $(D = [1; 7])$ and answer the questions.
   a) Give the intervals when the slope of the function (of the tangent line) is positive. (3)
   b) Arrange the slopes of the tangents at the points marked in a growing order. (4)
   c) Draw the approximate tangent lines at the points B, C and D, and evaluate their slopes. (1,5+1,5)



4. Couple the graphs of the functions (red) with the derivative functions (blue) and explain shortly why just the specified pairs were chosen. (4+4)



1.	2.	3.	4.
			

Group ♣

Part B

It is ALLOWED to use a GC!

5. Give the derivative functions of the following functions, and calculate the slope at the point $x_0 = 3$! *Suggestion: calculate the derivative function and the value function without a GC, and then you can check them with a GC.*
   a) $f(x) = 3x^2 - 2x + 3$ (3+2)
   b) $g(x) = \frac{2}{x} + x^{-3}$ (4+2)
6. Determine the tangent term of the function using GC:
   $f: f(x) = 2x^4 - 3x^2 + x - 5$ at $x_0 = 2$! Describe briefly how you proceeded! Sketch the function and the tangent line using the GC on the sheet. (3+2+3)
7. Given the function $f: f(x) = -2x^3 + 5x + 4$.
   a) Plot the function AND its derivative function with a GC, and sketch them appropriately on the sheet. (3+3)
   b) Explain briefly why the derivative function is just seen at the zero points, or why the derivative function is only positive / negative where you see it. (3)

The test was written by 25 pupils on time, and 12 pupils wrote a similar one two weeks later because of illness. There were no such big differences between the two groups (group ♣ and group ♥) and between the two classes, therefore I assessed the results of the two classes together. These were the following (regarding the 25 tests):

Part A

1st task:

Given the function $f: f(x) = \sqrt{x}$ $(x \in \mathbb{R}^+)$. *Give the derivate function and prove it.*

The theorem was perfectly written by 22 pupils, and 10 pupils gave the proof correctly. 3 children began to write the proof but they could not end it. Unfortunately, 12 of them did not start to write the proof, however they should have learnt it and it was not necessary to find out new ideas. The results of this part were 84% and 46.7%.

2nd task:

Given the function $f: f(x) = \frac{1}{2}x^2 + 1$.

It was a complex exercise but the different parts of them could be solved without the other parts. In this task I tried to find out how deeply the pupils understood the meanings of the difference quotient, differential, derivate function – and their geometric meaning. This theoretical background is very important e.g. for curve sketching.

a) Give the difference quotient in interval [0; 2].

14 pupils answered well, only 4 could not answer the question. The others had the typical problem that they did not correctly substitute the points in the function. The result of this part was 69.3%.

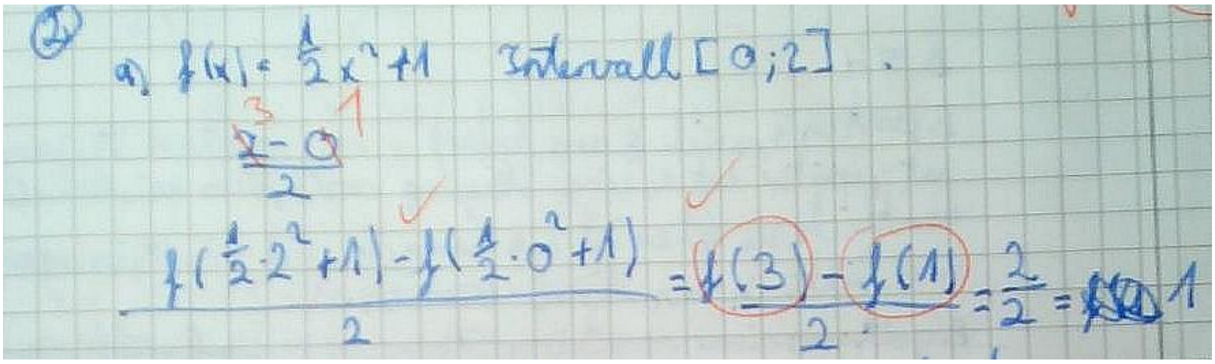


Figure 1: Pupil’s solutions: wrong substitution of points

b) *Give the geometric meaning of the result.*

18 pupils gave the geometric meaning of the difference quotient well while the others did not give anything or gave false explanations. The result of this part was 68%.

c) *Give the differential of f at* $x_0 = 2$ with $h \to 0$.

This task was correctly solved by only 6 children but just 5 did not solve it. Some pupils solved the problem with $h = 0{,}001$ so they got an approximated value for the differential. The result of this part was 53.3%.

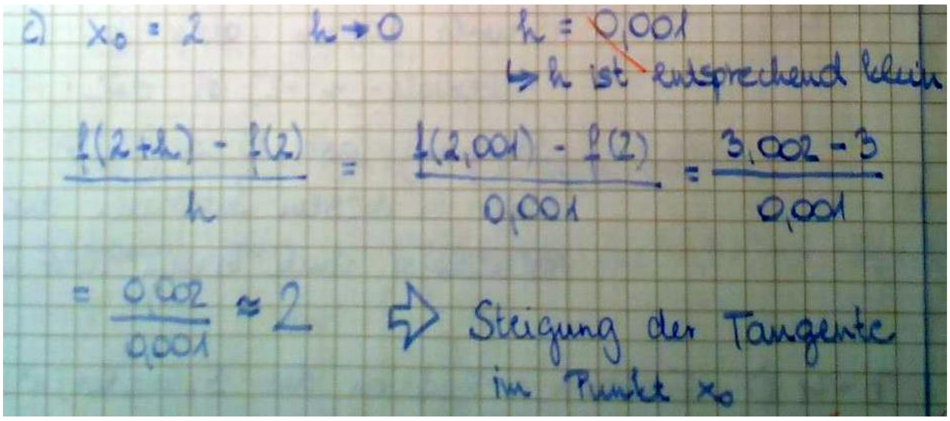


Figure 2: Pupil’s solution: for *h* = 0,001

d) *Give the derivate of function f.*

In this part it was interesting for me which method was chosen by pupils for giving the derivate function. Two opportunities were given: 1. with definition of derivate function 2. with derivate rules. Altogether 15 right solutions were given, and all of the pupils solved the problem with derivate rules. 3 children were not able to do it and the others made some mistakes. The result of this part was 72%.

e) *Give the geometric meaning of the result in c)* .

17 pupils gave the geometric meaning of the difference quotient well; the others did not give anything or gave false explanations. The result of this part was 68%.

3rd task:

See the function graph in interval $(D = [1;7])$*and answer the questions.*

It was important for me in this task that the pupils did not only know the theoretical geometric meaning of the difference quotient, but they were able to apply it as well. If they could draw the tangent line, they could give its slope approximately.

a) *Give the intervals when the slope of the function (of the tangent line) is positive.*

13 of them found all of the three intervals and only 3 could not find any. The others gave one or two right intervals. The result of this part was 73.3%.

b) *Arrange the slopes of the tangents at the points marked in a growing order.*

14 pupils gave the right growing order for all points so they knew when the slope was positive or negative, too. Only three pupils could not understand the exercise, the others solved the problem partly right. The result of this part was 69%.

c) Draw the approximate tangent lines at the points B, C and D, and evaluate their slopes.

Seven pupils could draw the tangent lines correctly and could give their slopes too. Two children drew only the tangent lines – without the slopes, and one of them could not begin this part. The others made some mistakes. The result of this part was 71.3%.

4th task:

Couple the graphs of the functions (red) with the derivative functions (blue) and explain shortly why just the specified pairs were chosen.

The first part of this task was the most successful of all parts in this test. All pupils began this task and only two children made a little mistake. The result of this part was 95%. In the examination part there were some false pieces of information but except for two of them, pupils could give partly correct interpretations. The result of this part was 78%.

In this first part of the test it was not allowed to use a graphing calculator. The result of this part (A) was 71.1%. Here the pupils had to use the theoretical background of differential calculus and they had to see the geometric meaning, too. In my opinion, they understood it well, thus the result of this part was quite good.

In the second part it was necessary to use the GC.

Part B

5th task:

Give the derivative functions of the following functions, and calculate the slope at the point $x_0 = 3$. Suggestion: calculate the derivative function and the value function without a GC, and then you can control it with a GC.

a) $f(x) = 3x^2 - 2x + 3$

In this part the derivate function was given by 21 pupils correctly. 4 children could not give it nor with rules neither with a GC. The same happened with the substitution of $x_0 = 3$, but one of the 21 pupils calculated wrong. The results of this part were 82.7% and 80%.

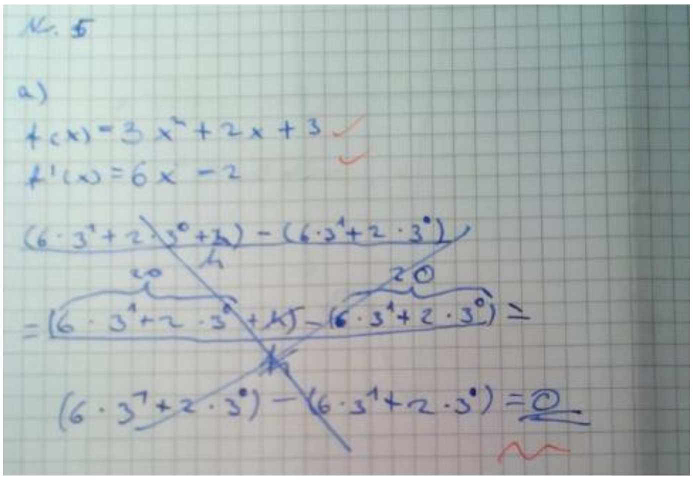


Figure 3: Pupil's solution with wrong substitutions

b) $g(x) = \frac{2}{x} + x^{-3}$

In this exercise the parts of the function are more difficult than in a), consequently, the number of correct answers was lower. 15 pupils gave the correct derivate function and all of them could calculate the substitution of x_0 =3. Unfortunately, 6 pupils could not begin this task. The results of this part were 68% and 60%. In this task I could not recognise if the GC was used for checking the derivate functions or for substitutions or not.

However, the GC was necessary for the solution of the next two tasks.

6th task:

Determine the tangent term of the function using GC:

$f: f(x) = 2x^4 - 3x^2 + x - 5$

at $x_0 = 2$. Describe briefly how you proceeded. Sketch the function and the tangent line using the GC on the sheet.

Six pupils could not solve the first part, one child solved it with both methods: with a GC and without a GC too, but he did not get the same term. The others solved it correctly, and the other two questions they could answer correctly. It is interesting that one pupil who could not solve the first part answered the other two questions correctly. The results of this part were 74.7% 76% and 85.3%.

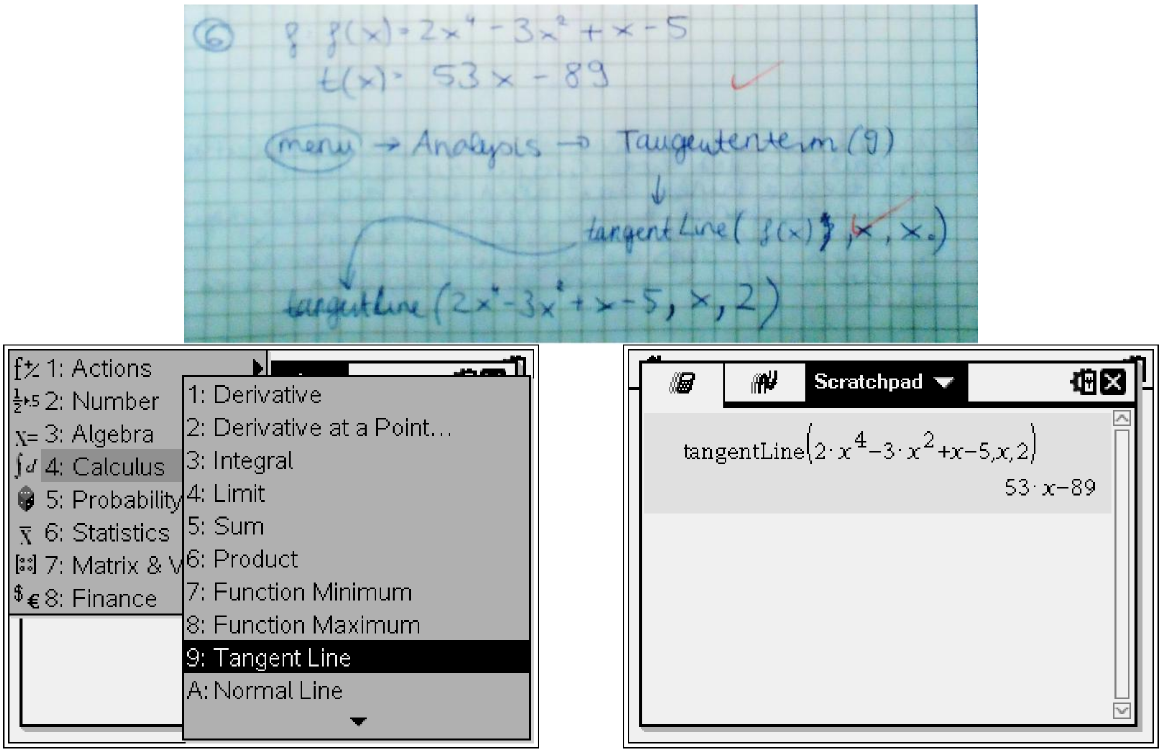


Figure 4: Pupil's and GC solution for tangent line

In plotting the function and its tangent line at$x_0 = 2$the following result was proceeded .e.g.:

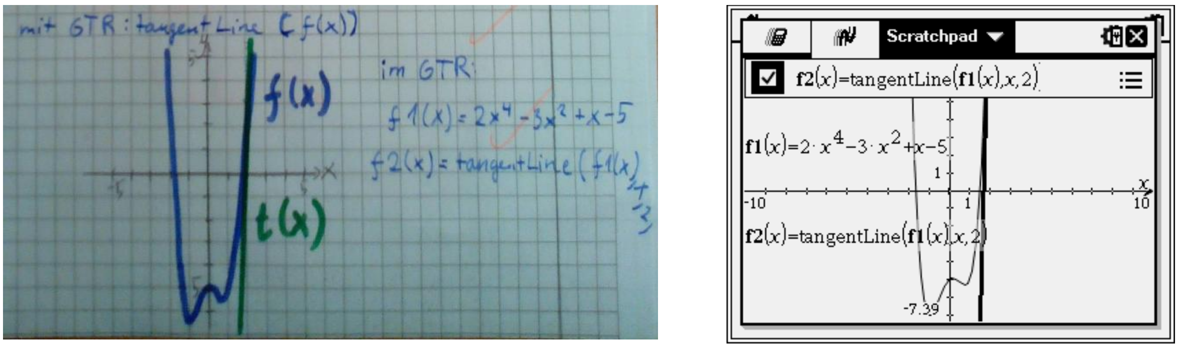


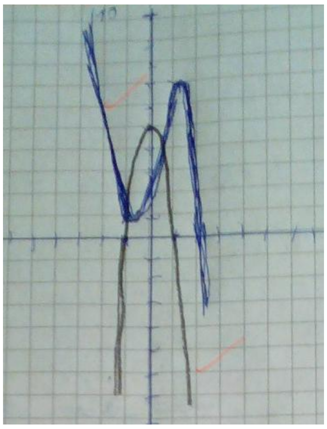
Figure 5: Pupil's and GC plotting of tangent line

7th task:

Given the function $f: f(x) = -2x^3 + 5x + 4$.

a) *Plot the function AND its derivative function with a GC, and sketch them appropriately on the sheet.*

In this exercise the pupils had to plot the functions graph and its derivate function with a GC. The difficulty with their TI-*n*spire GC is that there is another way to calculate the derivate function than to plot it. In the homework and in the lessons a few pupils always had problems because of the different ways of calculation. Finally, only two children could not plot the function correctly (92%), and two others could not either plot the derivate function (84%).



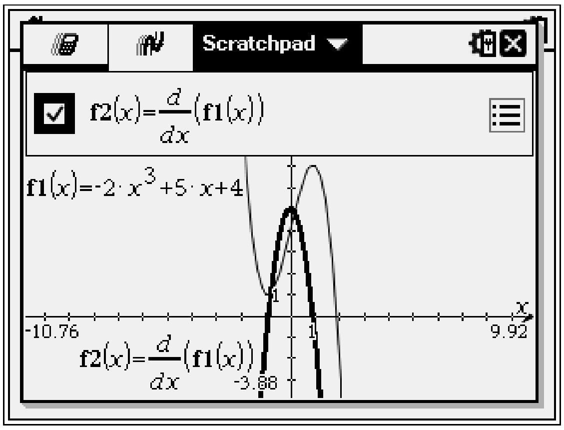


Figure 6: Pupil's and GC plotting of function and derivate function

b) *Explain briefly why the derivative function is just seen at the zero points, or why the derivative function is only positive / negative where you see it.*

In this part the pupils had to explain why the derivate function is such in the picture. These descriptions were mostly correct by those who could plot both of them. There were just little mistakes or inaccuracies. The result of part b) was 74.7%.

While I was writing the article we also discussed the topic of function analysis with the pupils. During this chapter I strived to the above described methodological duality, which means that we solved exercises partly with a GC and partly without it. The test for assessing this part also showed this duality, and the results of the two parts were quite similar. In the last exercise a word problem appeared here, where pupils not only had to solve the task with a GC, but they also had to choose the calibration of the function window of the GC well, and they had to answer and interpret the results based on the words of the exercise. Although this caused smaller or bigger difficulties for a few pupils, I still think that compared to the opportunities and the difficulty of the problem results were good:

The next function gives the profile of a landscape in interval [1; 7]. The value *x* is km, the value of *h(x)* is 10 meters.

$$h(x) = 0{,}3x^4 - 5{,}1x^3 + 29{,}5x^2 - 65{,}1x + 45{,}3$$

a) Sketch the function with GC and draw the curve on the sheet.
b) Calculate the maxima and minima of the landscape and their values with GC.
c) Give the intervals of increase and decrease.

The difficulties for some pupils were caused by the fact that the GC did not sketch the function in such an interval that would have been appropriate for them. First the division of the axes should have been changed and then they would have been able to sketch it well (Figure 5). Those who did not do this got partly wrong solutions. Still, 18 pupils did the sketching well, which I consider a very good result. The maxima, minima were well determined by all the 18 pupils within the interval with the help of the calculator, but only eight of them paid attention to examining the end points of the closed interval. Because of the above reasons, these eight pupils determined the intervals of increase and decrease perfectly.

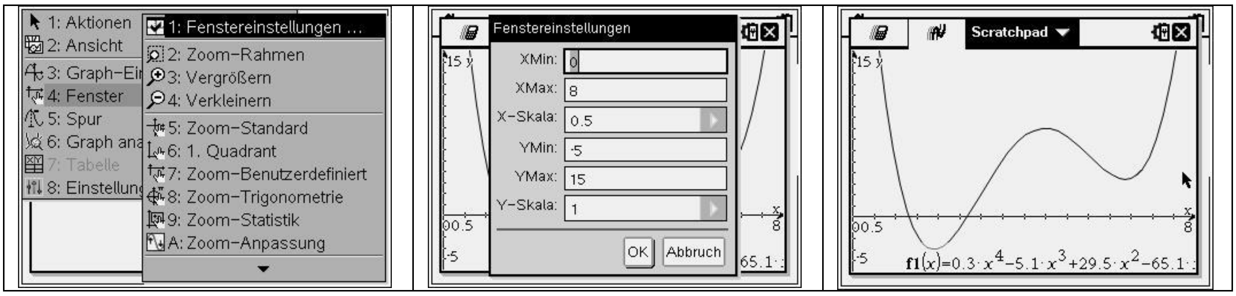


Figure 7: Sketching of functioning exercise 4 with intervals

Conclusions

The application of GCs in teaching is ambiguous. On the one hand, they provide help with exercises that need a lot of calculation but have easy algorithms. Pupils use them with pleasure, though the appropriate use of each application requires a lot of attention. On the other hand, such exercises can be solved with the help of GCs the mathematical background of which pupils know and are able to use, but with their familiar mathematical tools these tasks are not, or only difficultly solvable.

However, we have to make sure that pupils are aware of the mathematical background as well and that they are able to do the learned processes without a GC where possible, so as to avoid the situation that I described in the introduction. I placed especially high emphasis on this idea when planning the lessons, in homework and in assessment. In this test I tried to find balance between theory and praxis and I wanted that pupils not only perfectly knew how to use a GC but they were able to understand the theoretical background, too.

My research question was: "*How much supportive can be a graphing calculator in problem solving related to tangent line and the increase and decrease of functions?*"

Pupils have to use GC in lot of problems in the curriculum of Germany. It is necessary to understand the meaning of tangent line and decrease of functions but without a lot of representations it is impossible. While the understanding of these mathematical concepts without GC is possible but in my opinion the GC is very usefully during this work. The pupils could have plotted the function of a mathematical problem; with the iconic representation of GC they could have started it with the solution. They were able to reflect immediately when the solutions are different with and without GC. During the lessons I was talking a lot with pupils about the exercises and of course about using GC. According to the children's opinion they could have understood the problems of tangent line and slope of function better because they immediately saw the representations of this with GC. Firstly they had problems with programing the calculators but they had possibility to help each other and of course, I supported them, too if nobody was able to solve the problem.

According to my opinion, these were the reasons why pupils could have written the test with better results, what I had not hoped before: the average was in the theoretical part (Part A) 71.1%. In this part there were lots of questions about the meaning of tangent line and about the slope of functions without GC. In the GC part (Part B) the result was 78% where the pupils got questions, among others about the meaning of tangent line and the slope of functions. Both of them are good rates in an average class and they are also balanced.

The tasks in the first part, where the pupils had to use the concept of the difference and differential quotient and the increase and decrease of a function, were:

2.					3.			6.			**Average**
a)	b)	c)	d)	e)	a)	b)	c)	a)	b)	c)	
69,3	68	53,3	72	68	73,3	69	71,3	74,7	76	85,3	**70,9%**

However, the average of these tasks is a little bit lower than the average of the whole test but the difference is not significant. This can represent that the pupils were able to understand the theoretical background of the difference and differential quotient and the increase and decrease of a function as well.

Unfortunately in Hungarian public and tertiary education it is still forbidden to use graphing calculators, which is regulated in the maturity examination system in the following way: „*In solving the problems, you are allowed to use a calculator that cannot store and display verbal information. You are also allowed to use any book of four-digit data tables. The use of any other electronic devices or printed or written material is forbidden.*" Based on my research experiences it would be desirable to introduce the use of GC-s in Hungarian secondary schools too.

References

Abfalterer E.: Foren Wikis Weblogs und Chats im Unterricht, Fachverlag für Medientechnik und –wirtschaft, p. 108 (2007)

Ambrus A.: Some Experiences with Graphing calculator in a Secondary School in Hungary. In.: Proceedings of the seminar „Graphing calculators in High School Mathematics Teaching and in Education of Mathematics Teachers", MFF UK Prague. (1995)

Banikowks A.: Strategies to enhance memory based on brain-research. Focus on exceptional children. 32(2). 1999.

Ben-Hur M.: Concept-Rich Mathematics Instruction Building a Strong Foundation for Reasoning and Problem Solving, Association for Supervision and Curriculum Development (USA) (2006)

Dunham P. H.: Hand-held Calculators in Mathematics Education: A Research Perspective. Hand-Held Technology in Mathematics and Science Education: A Collection of Papers. The Ohio State University. (2000)

Haju S., Czeglédi I., Hajdu S., Kovács Z.: Matematika 11. évfolyam – Gondolkodnijó, p 76-80. (2012)

Horton R. M., Storm J., Leonard W.H.: The graphing calculator as an aid to teaching algebra. Contemporary Issues in Technology and Teacher Education [Online serial], 4(2). Available: http://www.citejournal.org/vol4/iss2/mathematics/article1.cfm(2004)

Jones, K.: Graphing calculators in the teaching and learning of mathematics: A research bibliography, Micromath, 21(2), 31-33.(2005)

Kamarulhaili H, Sim, L, B: Graphing Calculator as a teaching and learning aid for secondary school students and teachers. The 2nd International Conference on Research and Education in Mthematics (ICREM 2). (2005)

Leuders T.: Mathematikdidaktik - Praxishandbuchfür die Sekundarstufe I und II., Berlin: Cornelsen. p.199 (2010)

Lichtenberg W.: EinsatzgraphikfähigerTaschenrechnerimgymnasialenMathematikunterricht (1991-1996). LISA, Halle. (1997)

National Council of Teachers of Mathematics: Principles and standards for school mathematics. Reston, VA: Author.(2000)

Leng, N., Chuen, T., & Ng Meow Leng, N.: Teaching and Learning Calculus with the TI-Nspire: A Design Experiment. Paper presented at the Asian Technology Conference in Mathematics, Beijing, China. (2009).

Robova, J: Graphing calculator as a tool for enhancing the efficacy of mathematics teaching.2nd International Conference on the Teaching of Mathematics. 2002

Ruthven, K.: The influence of graphing calculator use on translation from graphic to symbolic forms. Educational Studies in Mathematics, 21, 431-450.(1990)

Smith, K. B., Shotsberger, P. G.: Assessing the use of graphing calculators in college algebra: reflecting on dimensions of teaching and learning. School Science and Mathematics, 97(7), 368-373.(1997)

Tall, D. - Vinner, S.: Concept image and concept definition in mathematics with particular reference to limits and continuity. Journal Educational Studies in Mathematics, 12 no. 2, (1981)

Tiwari, T. K.: Computer graphics as an instructional aid in an introductory differential calculus course, International Electronic Journal of Mathematics Education, 2(1), 35-48.(2007)

Tolias, G.: The effects of using graphing technology in college pre calculus. Dissertation Abstracts International, 50(04), 1274A-1275A.(1993)

Tulodziecky G.: Handlungs- und entwicklungsorientierte Medienpädagogiktheoretische Grundlagen, Umsetzung und Forschung. In W. Sesnik, M. Kerres, & H. Moser, Jahrbuch Medienpädagogik 6. Wiesbaden: Verlag für Sozialwissenschaften. p. 110 – 111. (2007)

VanStreun, A., Harskamp, E. &Suhre, C.: The effect of the graphing calculator on students' solution approaches: A secondary analysis, Hiroshima Journal of Mathematics Education, 8, 27-39. (2000)

Annex

The detailed results of the test (1)

			A♥	A♣	A♣	B♥	B♣	B	♥	♣	AB
1.	thesis	***1***	100,0	85,7	91,7	85,7	66,7	76,9	91,7	85,7	84,0
	proof	***3***	66,7	42,9	52,8	38,1	44,4	41,0	50,0	43,6	46,7
2.	a)	***3***	100,0	76,2	86,1	47,6	61,1	53,8	69,4	69,2	69,3
	b)	***1***	80,0	57,1	66,7	71,4	66,7	69,2	75,0	61,5	68,0
	c)	***3***	60,0	47,6	52,8	57,1	50,0	53,8	58,3	48,7	53,3
	d)	***2***	90,0	71,4	79,2	64,3	66,7	65,4	75,0	69,2	72,0
	e)	***1***	60,0	57,1	58,3	71,4	83,3	76,9	66,7	69,2	68,0
3.	a)	***3***	73,3	76,2	75,0	71,4	72,2	71,8	72,2	74,4	73,3
	b)	***4***	70,0	57,1	62,5	75,0	75,0	75,0	72,9	65,4	69,0
	c)	***3***	76,7	66,7	70,8	66,7	77,8	71,8	70,8	71,8	71,3
4.	a)	***4***	100,0	82,1	89,6	100,0	100,0	100,0	100,0	90,4	95,0
	b)	***4***	100,0	71,4	83,3	71,4	75,0	73,1	83,3	73,1	78,0
5.	a1)	***3***	100,0	85,7	91,7	71,4	77,8	74,4	83,3	82,1	82,7
	a2)	***2***	100,0	71,4	83,3	71,4	83,3	76,9	83,3	76,9	80,0
	b1)	***4***	100,0	57,1	75,0	57,1	66,7	61,5	75,0	61,5	68,0
	b2)	***2***	100,0	57,1	75,0	42,9	50,0	46,2	66,7	53,8	60,0
6.	a)	***3***	80,0	85,7	83,3	81,0	50,0	66,7	80,6	69,2	74,7
	b)	***2***	100,0	85,7	91,7	71,4	50,0	61,5	83,3	69,2	76,0
	c)	***3***	93,3	85,7	88,9	76,2	88,9	82,1	83,3	87,2	85,3
7.	a1)	***3***	100,0	85,7	91,7	100,0	83,3	92,3	100,0	84,6	92,0
	a2)	***3***	100,0	57,1	75,0	100,0	83,3	92,3	100,0	69,2	84,0
	b)	***3***	100,0	66,7	80,6	61,9	77,8	69,2	77,8	71,8	74,7
Sum:		60	53,5	41,7	46,6	42,6	42,8	42,7	47,1	42,2	44,6
Average of part 1		32	82,2	66,1	72,8	68,3	70,8	69,5	74,1	68,3	71,1
Average of part 2		28	97,1	73,5	83,3	74,0	72,0	73,1	83,6	72,8	78,0
Average			89,2	69,5	77,7	71,0	71,4	71,2	78,5	70,4	74,3

The detailed results of the test (2)

			Class 10a												Class 10b												
			♥					♣							♥							♣					
1.		*1*	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	0	1	1	0
		3	1	0	3	3	3	3	0	0	3	0	3	0	3	3	0	0	2	0	0	0	3	0	2	3	0
2.	a)	*3*	3	3	3	3	3	3	3	1	3	0	3	3	3	2	0	3	1	1	0	1	3	2	2	3	0
	b)	*1*	1	1	1	0	1	1	1	1	1	0	0	0	1	1	0	1	1	0	1	0	1	1	1	1	0
	c)	*3*	2	3	1	3	0	2	2	1	2	0	3	0	1	3	1	2	2	1	2	2	3	0	1	3	0
	d)	*2*	2	2	2	2	1	2	2	2	2	0	2	0	1	1	1	2	2	0	2	2	2	1	1	2	0
	e)	*1*	0	1	1	0	1	1	1	1	1	0	0	0	1	1	0	1	1	0	1	1	1	1	1	1	0
3.	a)	*3*	2	2	2	3	2	3	3	3	3	0	3	1	3	3	3	2	0	2	2	3	3	1	3	3	0
	b)	*4*	4	3	0	3	4	0	4	0	4	0	4	4	4	4	1	4	0	4	4	4	4	3	3	4	0
	c)	*3*	1,5	1,5	3	2,5	3	3	1	2	3	0	3	2	2	2,5	2	2,5	0,5	2,5	2	2,5	3	2	3	3	0,5
4.	a)	*4*	4	4	4	4	4	4	2	4	4	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	b)	*4*	4	4	4	4	4	4	2	4	4	0	4	2	4	4	2	4	0	2	4	2	4	3	4	4	1
5.	a1)	*3*	3	3	3	3	3	3	3	3	3	0	3	3	3	3	0	3	3	0	3	3	3	0	3	3	2
	a2)	*2*	2	2	2	2	2	2	2	2	2	0	2	0	2	2	0	2	2	0	2	2	2	0	2	2	2
	b1)	*4*	4	4	4	4	4	4	1	3	4	0	4	0	4	2	0	2	4	0	4	4	4	0	4	4	0
	b2)	*2*	2	2	2	2	2	2	0	2	2	0	2	0	2	0	0	0	2	0	2	0	2	0	2	2	0
6.	a)	*3*	3	3	3	3	0	3	3	3	3	0	3	3	3	2	0	3	3	3	3	0	3	0	3	3	0
	b)	*2*	2	2	2	2	2	2	2	2	2	0	2	2	2	0	0	2	2	2	2	0	2	0	2	2	0
	c)	*3*	3	2	3	3	3	3	3	3	3	0	3	3	3	3	0	3	3	1	3	3	3	3	3	3	1
7.	a1)	*3*	3	3	3	3	3	3	3	3	3	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
	a2)	*3*	3	3	3	3	3	3	3	0	3	0	3	0	3	3	3	3	3	3	3	3	3	3	3	3	0
	b)	*3*	3	3	3	3	3	3	3	0	3	0	3	2	3	3	0	1	3	0	3	2	3	3	3	3	0
Sum		60	53,5	53	53	56,5	52	55	45	41	59	1	58	33	56	51	21	49	42	30	51	43	60	30	54	60	11
Av. Part 1		32	79,7	79,7	78,1	89,1	84,4	84,4	68,8	62,5	96,9	3,1	93,8	53,1	87,5	92,2	46,9	82,8	42,2	54,7	71,9	70,3	100	56,3	81,3	100	17,2
Av. Part 2		28	100,0	96,4	100,0	100,0	89,3	100,0	82,1	75,0	100,0	0,0	100,0	57,1	100	75,0	21,4	78,6	100	42,9	100	71,4	100	42,9	100	100	17,9
Av.			89,2	87,5	88,3	94,2	86,7	91,7	75,0	68,3	98,3	1,7	96,7	55,0	93,3	84,2	35,0	80,8	69,2	49,2	85,0	70,8	100	50,0	90,0	100	17,5
Note			1	1-	1-	1	1-	1	2	3+	1+	6	1+	4+	1	2+	5+	2+	2-	4	1-	2-	1+	4	1	1+	6

The Role of affective Components in mathematical Problem Solving

Bernd Zimmerman

Friedrich-Schiller-University of Jena, Germany

bernd.zimmermann@uni-jena.de

Abstract

This paper is based on the dissertation „Analyzing problem solving processes involved in problems from incidence-geometry“. First we give a summary of the goals, design and results of this work. We focus on the part of affective behavior which was analyzed too, but has not been published until now. The main affects which were detected in videotaped think-aloud sessions were reactions of the subjects which were interpreted as feelings close to the goal, reactions which were interpreted as feelings distant from the goal and reactions which could not interpreted clearly (hesitations).

Keywords: problem solving, problem solving processes, affect, incidence geometry, category systems, sequence-analysis

ZDM-classification: C 34, C 24, D 54, E 45

Introduction

While empirical studies on the cognitive aspect of mathematical problem solving have already a long tradition (cf. e.g. Duncker 19, Wertheimer 1945), the affective aspect of problem solving seemed to be neglected for sometime (Silver 1985, 254). This component gained increasing attendance during the last years (cf., e.g., Goldin/Hannula 2013). For that reason we decided to present some results, which had been gained in connection with a PhD study (Zimmermann 1977), but had not been published until now. It might contribute to the present discussion of this issue.

First a summary of the foregoing study will be presented to highlight the background (including the theoretical one) of that specific part we publish here for the first time.

Background

An exploratory study of mathematical problem solving processes was carried out with student teachers and some pupils from the upper grade of a gymnasium (n = 22; cf. Zimmermann 1977). Starting points were - except the well-known books of Pólya - the habilitations of the cognitive psychologists Lüer 1973 and Dörner 1974, who analyzed think aloud protocols from students, who solved problems from the "Principia Mathematics" (Whithead/Russel 1964). One outcome was a "fundamental rhythm of thinking". Our original question was: What could be the specific role of different representations (esp. of perception; cf. Bruner 1966) during problem-solving processes? As we started this study in 1973, it was still the time of "New Maths" and "Boubakism". So logic, formalism and abstract structures were still dominating the discussion in mathematics education of that days (as it is still up to date in mathematics university teaching, especially in eastern European states). The logical problems fit into this environment. We decided therefore to take problems from the domain of incidence geometry. Besides logical inferences being also involved, there was the advantage, that different representations of "lines"[1] and "points" might have some impact on problem solving processes. Two questions evolved: 1. Could we observe a similar fundamental rhythm of thinking as Lüer/Dörner? 2. What would be the role of perception in this case?

The theoretical background was set up by the "information processing approach" mainly (cf. e. g. Newell/Simon 1972 and Klix 1971).

Main *results* were as follows:

- A classification system of mathematical problems referring to
  - formal (what is the problem?),
  - personal (for whom it is a problem?) and
  - normative (who is posing the problem?) aspects.
- A method to construct categories for describing problem solving processes in a very flexible way.
- A method for analyzing sequences of action which occurred more often than it could be expected by random.
- Following hypotheses were generated:
  - There is a fundamental rhythm "weaker" than that found by Lüer.
  - Higher achievers solve problems in a more systematic way.
  - There is no difference between males and females in this context.
  - Reflective thinking (metacognition) has a positive effect on problem solving success.

[1] Experience in a pilot study proved that students' intuitive perception of "lines" hampered their concentration on the logical content of the axioms. E. g. it was very hard for the students to prove by referring to the axioms only, that if a line intersects one of two parallel lines, then it will intersect also the second one. As to their intuitive experience, this is a complete obvious fact, where nothing is to be proved. This "functional fixedness" to perception was loosened to some extent, when we replaced "lines" by "curves". In this way changes between different representations (writing, drawing, and "laying" by wool-threads) were facilitated.

- Possible reasons for changes of representations are systematical one, critical situations, notations of ideas for possible solutions, writing down the result of a thought process.
- There are two types of problem solving types: an iconic and an enactive one. The former represents objects by drawings, is more reflective, systematic, refers less to own writings and is a little bit more successful while the latter represents curves only by threads and points by rings, is impulsive, less systematic, refers more to own writings and is a little bit less successful.

We sketch now the *problem domain* of incidence geometry, to give some ideas about the mathematical content:

The subjects had to do their proofs by referring to the following Axioms exclusively:

(1) Exactly one curve passes through two points

(2) Parallel-axiom

(3) Existence of at least 3 points not all on the same curve.

They had to prove three theorems as follows:

Theorem: There exist at least 4 different points, from which no 3 points are located on the same curve.

More details can be also found in Zimmermann 1980, 1982.

Goal of the study

What is the role of affective behavior when solving problems from incidence geometry?

Method

We used the following three *categories* concerning affect:

Affective expressions of *feeling closer* to a solution of the problem were detected by reactions of the subject like:

- being motivated ("now I am on the right track!")
- praising her/himself
- being positive surprised (aha-experience)
- expressing (subjective) idea of solution

Affective expressions of *feeling more distant* from a solution of the problem were detected by reactions like being insecure, helpless, depressed, aggressive or embarrassed

Finally, there we observed *hesitations* which could not be attributed clearly to such feelings.

The problem solving sessions were videotaped and - using this and the comprehensive and very flexible category-system - written protocols were created.

The protocols had the following structure (X denots different arbitrary actions):

focus on \ moment/ time	20 sec.	40	60	80	100
verbal utterances	X (*close*)	X (ob. *idea*)	X (*drawing*)	X (*ref. draw.*)	X
writing		X			X
drawing	X		drawing		
manipulating					
reference to writings			X		
reference to drawings		X		reference drawing	
reference to manipulations					
logical operations	X	X		X	X
mistakes	X		X	X	X
"subjective" solution idea		Idea			
affective behavior	**close**				
"objective" solution idea		OI			

Table 1. Parallel sequences of actions referring to different foci. After setting priorities, these were "compressed" into one (shaded) sequence to be analyzed

When *transcribing* the videotapes, we looked for twelve different aspects where we noted actions (from left to right) which could also happen simultaneously. We came from one "moment" (micro-episode) to the next, if at least one activity in the very column changed (average time for one such "moment": about 20 seconds). After transcribing all videotapes of all sessions, we made a validity check by two additional researchers (Prof. Hefendehl and Prof. Bedürftig). The rate of correspondence was between 70% and 80%.

Because we were interested in specific actions (e.g., solution ideas, mistakes, but especially now: affective behavior) and possible reasons and consequences of their occurrence, we made a sequence analysis of their “pre-“ and “post-history”, i. e. we formed single sequences of actions were we looked for sequences of two up to 5 consecutive actions (including the specific one), which occurred more often than it could be expected by random.

To form one out of twelve sequences, we implemented all protocols into a computer and put all protocols of all subjects together to a “super-protocol”. Then, we started with the line “verbal”, because this proved to be the most objective and reliable information, set priorities and substituted all actions in “verbal” by actions of the other lines, if they had a higher priority.

In case of occurrence in the same line, content of a cell in the first line (“verbal”) was to be replaced in the given order by (cf. Table 1)

(1) Affective behavior

(2) (objective) solution idea (if there is no (1) in the same column)

(3) Change of representation (if there is neither (1) nor (2) in the same column; writing (w), drawing (d), manipulating (m))

(4) Reference to writing, drawing, manipulating (no (1, 2, 3) in the same column).

After this process was finished, we analyzed new single sequences, created in this way, like (real sequences were much longer):

…	verbal utterances:	***close***	obj. *idea*	*drawing*	*ref. draw.*	X	…

Table 2. Super-sequences of actions to be analyzed, including expressions of affective behavior and other actions of high priorities

Of course, other substitutions by other priorities are possible.

First we detected the absolute and relative number of all such actions with respect to the total number of all “micro-episodes” of all subjects. The computer detected 4477 actions, corresponding to a little bit more than on average one hour of problem solving time of every subject.

Action		Frequency abs. (total 4477)	Frequency rel. %
affective behavior	**Close**	273	6.1
	Distant	252	5.6
	Hesitating	381	8.5
(objective) idea		115	1.5
writing		192	4.2
drawing		112	2.5
manipulating		46	1.0
reference to writing		147	3.2
reference to drawing		243	5.4
reference to manipulating		25	0.6

Table 3. Absolute and relative frequencies of affective actions and others of high priority

For carrying out the sequence-analysis, we estimated the probability of each action by its relative frequency. Assuming random distribution and independence of the variables, the probability of the occurrence of the sequence of the three consecutive actions close→idea→drawing (cid) can be estimated by P(cid) = 0.061*0.015*0.025=0.000024.

We checked now by binomial-test, which of all such 3-sequences (n=4357) occurred more often than it could be expected by random. Taking into account the accumulation of the error probability when applying such test repeatedly, we set the error-probability for a single test p=0.00001, to get an error probability for all tests of $\alpha=1-(1-0.00001)^{4357}\approx 0.04$.

Results

The results for all significant sequences with three actions, containing "*feeling close*" to the goal, are as follows:

Frequency absolute	**Action 1**	**Action 2**	**Action 3**
20	reading/interpreting text →	***close*** →	reading/interpreting text
7	writing →	***close*** →	ref. to written
6	referring to written →	writing →	***close***
3	idea →	***close*** →	intention to write
2	intention to write →	ref. to written →	close
2	I feel close →	***close*** →	I go for goal
2	referring to written →	***close*** →	new start
2	intention to write →	ref. to written →	***close***
2	short break →	idea →	***close***
46 total ≈ 1%			

Table 4. All significant sequences of three consecutive actions which include "feeling ***close*** to the goal" (abs. fr. ≥ 2; α≈0.04)

We can see that the feeling of being "close to the goal" seems to be mainly related with reading, writing or referring to text.

The results for all significant sequences with three actions, containing "*feeling distant"* from the goal, are the following:

Frequency absolute	Action 1	Action 2	Action 3
9	reading/interpreting text	→ ***distant*** →	intervention
7	***distant*** →	intervention →	***distant***
7	intervention	***distant***	intervention
6	referring to drawing →	intervention →	***distant***
4	short break →	reading/interpreting text →	distant
2	***distant*** →	intervention →	checking goal-distance
2	***distant*** →	intends to search an "operator" →	operator not appropriate
Total 37 ≈0.8%			

Table 5. All significant sequences of three consecutive actions which include "feeling ***distant*** from the goal" (abs. fr. ≥ 2; α≈0.04)

There is the impression that "feeling distant from the goal" is mainly related to interventions of the observer of the problem solving session, whose only task was to keep the subjects thinking aloud. So it seems plausible that in situations where the subject feels stuck, the leader of the session has to motivate to continue to talk. Furthermore; referring to (given) text drawings seem to be also somewhat tied to this feeling of distant.

The "local context" of "hesitation" was as follows:

Frequency absolute	Action 1	Action 2	Action 3
19	reading/interpreting text →	***hesitating*** →	reading/interpreting text
15	reading/interpreting actual problem →	***hesitating*** →	reading/interpreting actual problem
4	***hesitating*** →	reading/interpreting the actual problem →	checking goal-distance
2	memorizing text →	***hesitating*** →	constructing connections
2	intends to draw →	***hesitating*** →	drawing
2	checking goal-distance →	***hesitating*** →	makes additional assumption
Total 44 ≈ 1%			

Table 6. All significant sequences of three consecutive actions which include "hesitating" (abs. fr. ≥ 2; α≈0.04)

"Hesitating" seems to occur mainly in situations where the subjects is reading or interpreting text, especially that of the actual problem. This might be interpreted as attempt to understand better important "givens". Also the other significant sequences might express somewhat insecurity or attempts to compare the very status with requirements still to be fulfilled.
We carried out additional tests with sequences of 4 consecutive actions, including at least one of these three feelings. But we gained not new insight, as can be see form Table 7:

Frequency absolute	Action 1	Action 2	Action 3	Action 4
11	reading/interpreting text (prob.) →	***hesitating*** →	reading/interpreting text →	***hesitating***
6	reading/interpreting text (prob.) →	break →	reading/interpreting text →	***distant***
3	referring to drawing →	reading/interpreting text →	***hesitating*** →	***distant***
3	operator not appropriate →	reading/interpreting text →	referring to drawing →	***close***
3	referring to written →	writing →	***close*** →	referring to written
Abs. frequ. ≥ 3; total 26 ≈ 0.6%				

Table 7. All significant sequences of four consecutive actions which include "close", "distant" and "hesitating" (abs. fr. ≥ 3; $\alpha \approx 0.04$)

Summary

- ***All*** registered ***affects*** were mainly connected with reading and interpreting ***text***.
- ***Positive affects*** were additionally related to ***own productions***.
- *Hesitation* and negative ***affects*** were sometimes additionally related to reference to drawings.

Discussion

- Appropriate reading- and writing competences (cf. PISA-outcomes) seem to be important also in this "classical" domain.
- The type of problems was quite normal at that time - it was the time of "New Maths" also at school. Incidence geometry is still a usual esp. at university-curriculum, esp. in Eastern Europe.
- The normal habit of trusting your perception and the formal, axiomatic approach, required here, were often conflicting.
- The methodology presented here can be applied also in settings of today.

References:

Bruner, J. S. 1966. *Toward a Theory of Instruction*, Cambridge, Mass.: Belkapp Press.

Dörner, D. 1974. Die kognitive Organisation beim Problemlösen (Cognitive organization in problem solving). Versuche zu einer kybernetischen Theorie der elementaren Informationsverarbeitungsprozesse beim Denken. Bern-Stuttgart-Wien: Huber.

Duncker, K. 1974³. *Zur Psychologie des produktiven Denkens*. (*Psychology of Productive Thinking*). Dritter Neudruck von 1935. New York, Berlin, Heidelberg: Springer.

Goldin, G. A.; Hannula, M. S. 2013. Emotions in the psychology of mathematics teaching and learning. In: Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 216. Kiel, Germany: PME.

Klix, F. 1971. *Information und Verhalten*. Bern-Stuttgart-Wien: Huber.

Lüer, Gerd 1973. *Gesetzmäßige Denkabläufe beim Problemlösen*. (Regular patterns of thought processes in problem solving). Habilitation. Beltz: Weinheim und Basel.

Newell, A.; Simon, H. A. 1972. *Human Problem Solving*. Englewood Cliffs: Prentice Hall.

Silver, E. A. 1985. Research on Teaching Mathematical Problem Solving: Some underrepresented Themes and Directions. In: Silver, E. A. (ed. 1985): *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives.* Hillsdale: Lawrence Erlbaum Associates.

Wertheimer, M. 1945. *Productive Thinking*. New York: Harper.

Whitehead, A. N.; Russel, B. 1964. *Principia Mathematica*. London, Ibadan, New York: Cambridge University Press.

Zimmermann, B. 1977. *Analyse von Problemlöseprozessen bei Aufgaben aus der Inzidenzgeometrie*. Paderborn: Dissertation.

Zimmermann, B. 1980. Analyzing Problem Solving Processes involved with Problems from Incidence Geometry - An Exploratory Study. *ERIC* No. ED 191 685, Columbus, Ohio.

Zimmermann, B. 1982. Denkprozesse beim Lösen mathematischer Deduktionsaufgaben - Bericht über eine exploratorische Untersuchung. In: *Journal für Mathematikdidaktik* 3/4, 175 - 206.