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Tiivistelmä — Referat — Abstract

In this thesis we extend topological model of planar robotic hands emerging in the field of topological robotics. This research elaborates further recent works of Robert Ghrist and others. The main purpose of this thesis is to classify configuration spaces in terms of topological and algebraic invariants, which among others provides complexity estimates for potential optimization algorithms. The thesis is split into two parts. In the first part we investigate a robotic system consisting of a single hand which can occupy any position as long as it doesn't self-intersect. Using a new innovative representation of positions we are able to treat two basic movements of the robotic arm: the 'claw' and the 'swap' movements separately. The main appliance of this part is the nerve theorem, which helps to establish that under some restrictions the configuration space of such robotic hand has the homotopy type of S^1.

In the second part we investigate systems consisting of multiple hands. This time we are dealing with hands limited to length one whose positions satisfy the two conditions: each pairwise hand trace intersection is contractible and the hand intersection graph is a forest. As the local main result we prove that the fundamental group of such robotic system is isomorphic to the Artin right-angeled group, where the set of generators is in bijection with the set of all hands and relations are determined by the intersection graph. The main tool exploited in this chapter is the Seifert-van Kampen theorem.

Although the results are proven only for some special cases, the thesis introduces methodology that can drive their generalization further. In the final chapter we give a few sophisticated research directions.

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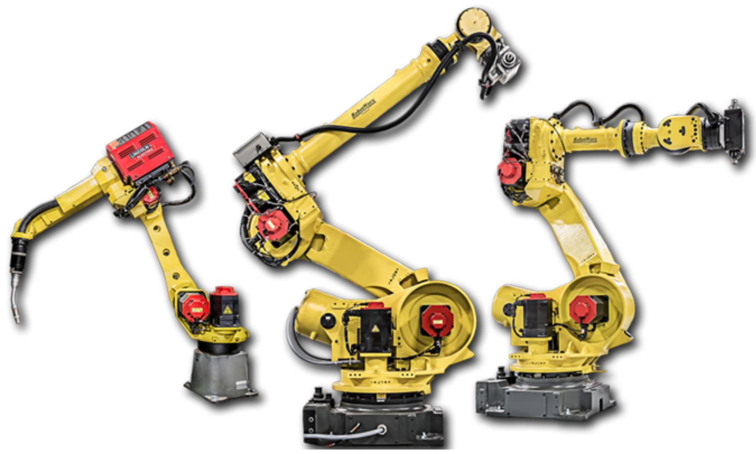
Configuration spaces of robotic hands

Yury Elkin



Contents

Preface

In order to get this work accepted the preface wont be published in this official version.

Chapter 1

Introduction

The topic of this thesis is 'configuration spaces of robotic hands'. We are considering a discrete model of a robotic hand, where every part of the arm can take only a finite number of positions. In literature there are many continuous models of robotic hands, as an example we mention [8]. However, because we would like to keep things simple and be able to use some methods coming from combinatorics, we will be dealing only with discrete hands. It should also be apparent that the continuous hands can be thought as infinite refinements of the corresponding discrete models.

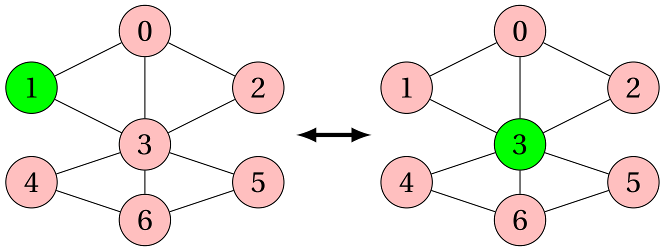
In this thesis we will study the topic mainly from the viewpoint of algebraic topology. Prerequisites for reading this thesis include concepts like CW-complexes and fundamental groups as well as the Seifert-van Kampen theorem and knowledge how to use notation of presentation of a group using generators and relations effectively. Having basic understanding of homology theory and construction of the $K(\pi, 1)$ spaces is recommended, but not required as we will use such theory only in the examples to break down some of the difficult parts of the thesis to the reader.

This thesis also contains new research results. We begin with necessary definitions and theory required for complete understanding of the way they are gained. Later in the work we present the two new results. Their main significance is to help researches with classification of the robotic hand system in terms of algebraic and topological invariants of configuration space. In the final chapter we present some further research directions and a few ideas of how one could approach them.

To provide some insight, let's consider two examples. The first example gives the reader a historical perspective and the second example introduces the reader to the main topic of this thesis. Topics presented in the latter will be presented in more general and precise way in the following chapters.

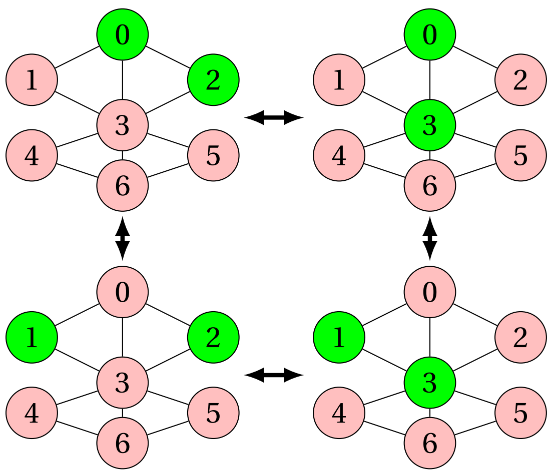
Example 1.1. Cars in the city example:

Let G be a graph corresponding to a topographical map of some city. This city has cars which can move in the city, and we further assume that the cars can stop only at the vertex points. Because the roads of the city are one-sided and narrow, we define that any two cars cannot use the same road or stay in the same vertex at the same time. One of the possible movements of a car is illustrated in the picture below:



We would like to figure out how the cars should be moving from their initial positions to some final positions in an optimal way. To solve this problem we define the following complex which will later be called a configuration space.

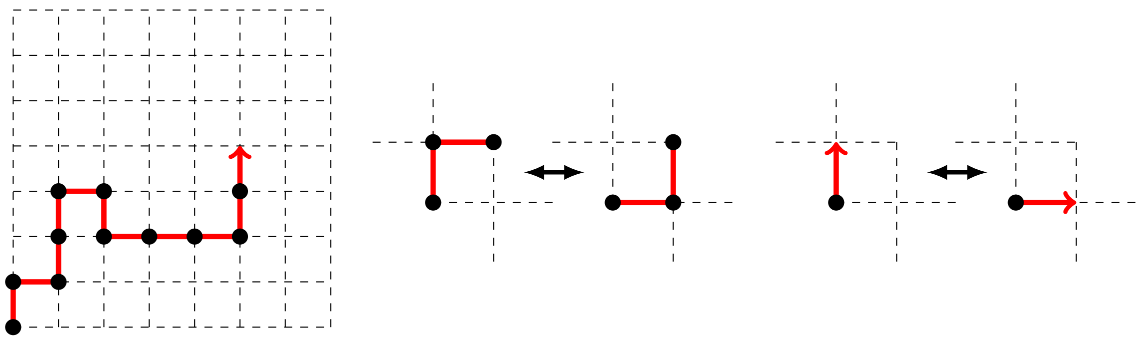
The 0-skeleton of the complex will consist of all possible configurations. In the case there are 2 cars in the city, there will be mathstuff different positions as we do not identify the cars. Two positions in configuration space are connected by an edge, when one state can be achieved from another by moving one car between vertex along some edge. Higher dimensional cells will be formed by executing pairwise commutative actions for the cars illustrated in the picture below:



Configuration spaces of graphs were first introduced by Aaron Abrams in his Ph.D thesis [2] and the theory was later generalized by Robert Ghrist for general robots in his article [3]. The spaces of this type have two important properties. The fundamental group of a configuration space can be embedded in the Artins right-angeled group or so-called graph group. The complex is locally CAT(0), meaning that the complex has a non-positive curvature. Those results are of great importance as they give a way to compute the shortest path in the configuration space in an effective way.

Example 1.2. Robotic hands in plane example:

Consider the following labelling of a lattice, together with the operations as pictured below:



One can modify the position of a hand by performing certain operations on a hand at any point where it is possible. We would like to figure out how to move time-optimally from one position to another if we are allowed to perform the defined operations on different parts of the hand at the same time. To solve this problem we can construct a complex similar to the one we had in the previous example. The 0-skeleton will consist of all the positions which are achievable from the initial position. The 1-skeleton will be the transaction graph of the system. Generally there will be k-cube connecting the two configurations if one configuration can be obtained from other by performing k-number of independent actions on different parts of the arm at the same time.

It was proved by Robert Ghrist in his article [4] that a configuration space of a robotic arm that is limited to the first quarter of $\mathbb{R}^2$-lattice is always contractible. Later a similar result regarding robotic hands in a box was proved by F. Ardila & al. in their article [5] using arguments of combinatorics. The latter article provides an algorithm for finding the optimal way of moving hand from one position to another.

Chapter 2

Toolbox

In this chapter we define the methods and tools of this study.

2.1 Methodology related to complexes

In this section we will discuss methodology directly related to complexes. It is in order to recall the definition of a simplicial complex. Intuitively such complex is constructed as follows: Let X be some set and let $\mathscr{Q}$ be a subset of the power set $\mathscr{P}(X)$, satisfying the condition

$$\forall A \in \mathscr{Q} : \forall B \subset A : B \in \mathscr{Q} \tag{2.1}$$

Then define 0-skeleton to be simply the set X and for every set A of cardinality n in A we draw a n-dimensional simplex that has the elements of A as its edges. It should be noted that from the condition 2.1 it follows that face of any simplex belongs to the collections of simplexes in the simplicial complex. This shows that the simplicial complex defined using the construction above is well-defined. In the definition below we define important special case of the abstract simplicial complex:

Definition 2.2. Let $\mathscr{U}$ be an open cover of a space X. We define the nerve of the covering $\mathscr{N}(\mathscr{U})$ as follows: Let the base space be $\mathscr{U}$ and define the subcollection of $\mathscr{P}(\mathscr{U})$ be the set:

$$\{\{U_{i_1}, U_{i_2}, ..., U_{i_n}\} \mid U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_n} \neq \emptyset \text{ where } U_{i_j} \in \mathscr{U} \text{ for all } i \in [n]\}.$$

It is trivial that the subcollection of the power set in the definition satisfies the required condition as $U_1 \cap ... \cap U_n \neq \emptyset$ implies that $U_{i_1} \cap ... \cap U_{i_m} \neq \emptyset$ for any subcollection of $\{U_i\}$. We present the nerve lemma in a more general way than it is usually presented, as it is necessary for one of the main results of this study.

Theorem 2.3. *Let X be a triangulable space, i.e. space which is homeomorphic to some simplicial complex, and let $\{A_i\}$ be a finite closed cover of X. Now if every intersection*

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n}$$

is either empty or contractible, then X and nerve $\mathcal{N}(A_i)$ are homotopy equivalent.

A proof of this theorem can be found in the survey by Björner, [6], Theorem 10.7.

Remark 2.4. Let $\triangle$ be some fixed cubical complex. It is important to note that subcomplexes of a cubical complex $\triangle$ are closed in the cubical complex. It follows that the theorem 2.3 can be applied to a cubical complex by defining sets A_i to be some contractible subcomplexes of $\triangle$.

Definition 2.5. Let $\triangle$ be a cubical complex[1], we denote the set of all its vertices as $\triangle^0$. Let $\{\triangle_i^0\}$ be a cover of vertex set $\triangle^0$. For every i define the following set:

$$\triangle_i' = \{s \in \text{Cells}(\triangle) \mid s \cap \triangle^0 \neq \emptyset\}$$

Denote by $\triangle_i$ the cubical complex which is spanned by $\triangle_i'$. Then cubical cover of $\triangle$ over $\{\triangle_i^0\}$ is the collection $\{\triangle_i\}$.

2.2 Algebraic methods

It is necessary to use the notation given in the following definition in order to proof the main result 4.23 of this thesis using Seifert-van Kampen theorem [2.8]:

Definition 2.6. Let G be a group and let e be its neutral element. The presentation of G using generators and relations is notation of the form

$$R = < a_1, \ldots, a_n \mid w_1, \ldots, w_k >$$

where the symbols in the notation above have the following meaning:

- symbols of the type a_i are formally known as generators. We denote the free group spanned by symbols of type a_i as T;
- each of w_i corresponds to some relation of the form $r_i = e$, where r_i is some element in the group T. Denote the subgroup of T generated by the elements r_i as U.

We say that R is presentation of group G, if $G \backsimeq T/U$.

[1]Note that in this thesis we use the $\triangle$-symbol to denote a cubical complex instead of the usual $\Box$-symbol.

Let X and Y be groups presented by generators and relations. In the following lemma we give a presentation of the product $X \times Y$ using generators and relations.

Lemma 2.7. *Let $X = \{a_1, .., a_n \mid r_1, .., r_q\}$ and $Y = \{b_1, ..., b_m \mid t_1, ..., t_p\}$ be groups. Denote relation $a_i b_j a_i^{-1} b_j^{-1} = e$ by $s_{i,j}$. Then $X \times Y$ can be represented using generators and relations as follows:*

$$X \times Y = \{a_1, ..., a_n, b_1, ..., b_m \mid r_1, ..., r_q, t_1, ..., t_p, s_{1,1} ..., s_{n,m}\}.$$

Proof. Denote latter group by Z. Define $f : X \times Y \to Z$ by setting $f(a,b) = ab$. We will first show that the map satisfies the homomorphism property. Let (a,b) and (c,d) be elements of $X \times Y$. The claim follows from the equation:

$$f(ac, bd) = acbd = abcd = f(a,c)f(b,d)$$

where the second equation follows from the relations, as we have assumed that all elements of X commute over elements of Y.

We will now prove that the map is well-defined. Denote H, K and U to be the groups generated by relations of X, Y and Z respectively. Let $a \in A$ and $b \in B$ be arbitrary elements. Assume that $(c_1, c_2) = (ah, bk)$, for some $(h,k) \in H \times K$. We will show that $f(c_1, c_2) = f(a,b)$. The claim follows from the following equations:

$$f(c_1, c_2) = f(ah, bk) = f(a,b)f(h,k) = f(a,b)$$

where the second equation follows from the homomorphism property.

To prove that f is an injection we need to show that if $f((x,y)) = e_Z$ for some (x,y) then $(x,y) = e_{X\times Y}$. Let t be an arbitrary element spanned by the relations of Z. By definition of the map f and the elements in the kernel of Z, the element t has to be of the following form:

$$r_{i(1)} ... r_{i(n)} t_{j(1)} ... t_{j(m)}$$

The claim follows from the fact that $r_{i(1)} ... r_{i(n)}$ belongs to the kernel of X and $t_{j(1)} ... t_{j(m)}$ respectively to the kernel of Y.

Consider some element $z \in Z$. By assumption $z = z' + U$ for some z', which is element of $X \cup Y$. Now, by using the relations of the type $s_{i,j}$ we can decompose this element to be of form $X + Y$. It follows that f is a surjection and thus f is an isomorphism.

$\square$

2.3 Topological methods

In this section we discuss methods releated to topology. We begin by introducing the Seifert-van Kampen theorem, which is the most important tool in the second part of this thesis.

Theorem 2.8. ***Seifert-van Kampen Theorem (Hatcher edition)***. *Let (X, x_0) be a base pointed topological space and let $\{A_\alpha\}_{\alpha\in I}$ be an open cover, which satisfies the following conditions:*

- *$x_0 \in A_\alpha$ for all A_α;*
- *every intersection of form $A_\alpha \cap A_\beta \cap A_\gamma$ is either empty or path-connected.*

Now denote the fundamental groups $\pi_1(A_\alpha)$ using generators and relations as follows:

$$< g_{\alpha,1}..., g_{\alpha,n_\alpha} \mid r_{\alpha,1}, ..., r_{\alpha,m_\alpha} >$$

and denote by $i_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ the map induced by inclusion of corresponding spaces. Now the fundamental group has the form: $\pi_1(X) =< G, R >$, where G and R are defined as follows:

$$G = \{g_{\alpha,k} \mid g_{\alpha,k} \in A_\alpha \textit{ for some } \alpha \in I\}$$

$$R = \{r_{\alpha,p} \mid r_{\alpha,p} \in A_\alpha \textit{ for some } \alpha \in I\} \cup \{i_\alpha(w) * i_\beta(w)^{-1} = e \mid w \in A_\alpha \cap A_\beta \textit{ for some } \alpha, \beta \in I\}$$

The proof of the theorem above can be found in Hatchers [1], section 1.2, page 43. Next we introduce definitions of bolding. It is important to note that when using the Seifert-van Kampen theorem we require every element of the cover to be open.

Definition 2.9. Let (X, d) be some metric space and let $p : [0, 1] \to X$ be some path. Then an $\epsilon-$bolding of the path is the following subspace of X:

$$\text{Bold}_\epsilon(p) = \{x \in X \mid d(x, \text{tr}(p)) < \epsilon\}.$$

Definition 2.10. Let A be some subspace of X. We denote its *interior points* by:

$$\text{int}(A) = \{x \in X \mid x \in U \subset X \text{ for some open set } U\}.$$

2.4 Graph section

In this section we introduce two definitions releated to graphs that will be used in the robotic systems specification.

Definition 2.11. A *labeling function c*, or just *labeling* of a graph $\mathcal{G}$ over a set A is a mapping

$$c : \text{Vertices}(\mathcal{G}) \cup \text{Edges}(\mathcal{G}) \to A.$$

Definition 2.12. Let $\mathcal{G}$ be a graph and let f be its labeling over X. The trace over $x \in X$ is the following set:

$$\text{Trace}(f, x) = \{z \in \text{Vertexes}(X) \cup \text{Edges}(X) \mid f(z) = x\}.$$

When using $\{0, 1\}$ as our labeling set, we denote simply $\text{Trace}(f)$ the trace of color 1.

Chapter 3

General robotic hand systems

In this chapter we will first present the definition of a robotic arm system in the plane, and further as the main result we prove that under some limitations the configuration space of a single hand robot has the homotopy type of S^1. Let's begin with a precise definition for the robotic hand configuration space.

Definition 3.1. The lattice graph $L(\mathbb{R}^2)$ on real plane $\mathbb{R}^2$ is an infinite graph, defined as follows:

- vertex set: $\{(n,m) \mid (n,m) \in \mathbb{Z}^2\}$;
- edge set: $\{\{n\} \times [m, m+1] \ \mid (n,m) \in \mathbb{Z}^2\} \cup \{[n, n+1] \times \{m\} \ \mid (n,m) \in \mathbb{Z}^2\}$.

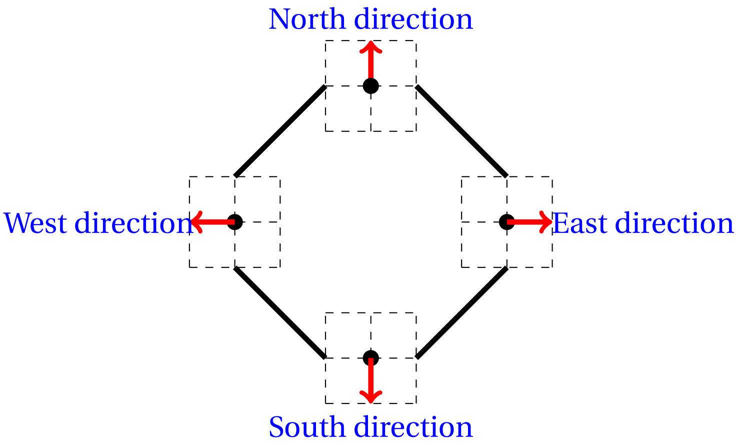
We say that a path is *non-self-intersecting* if it visits each vertex at most once.

Definition 3.2. Let (i,j) and (u,v) be some points in $\mathbb{R}^2$. We define path $(i,j) \to (u,v)$ to be a straight line from (i,j) to (u,v).

Definition 3.3. Let $\mathscr{C} = \{n, w, e, s\}$ be the set of cardinal directions, where the letters have the following meaning:

- $n(i,j) : (i,j) \to (i, j+1)$, $s(i,j) : (i,j) \to (i, j-1)$;
- $e(i,j) : (i,j) \to (i+1, j)$, $w(i,j) : (i,j) \to (i-1, j)$.

When the start point of the system is obvious we denote such path simply by the first letter. Let a_i be some elements of set $\mathscr{C}$ and let x_0 be some point in $\mathbb{Z}^2$. We denote by $(a_1, ..., a_n)$ a path, where we begin from the base point x_0 and then follow the directions a_i.



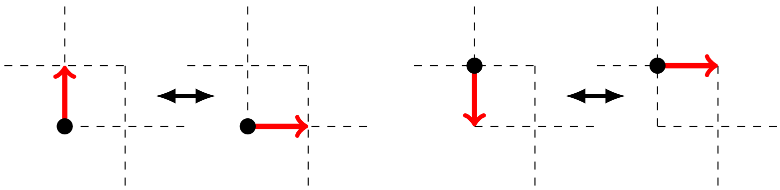
We say that directions are adjacent if they are connected by an edge in the graph above.

Definition 3.4. Let $\mathscr{A} = \{\alpha_i\}$ be a collection of non-self-intersecting paths in $L(\mathbb{R}^2)$. We say that $\mathscr{A}$ forms an *admissible position* in the robotic space if α_i doesn't intersect α_j at any point for all i and j in the collection. Given some path $a_1, ..., a_n$ one is potentially allowed to perform the following actions:

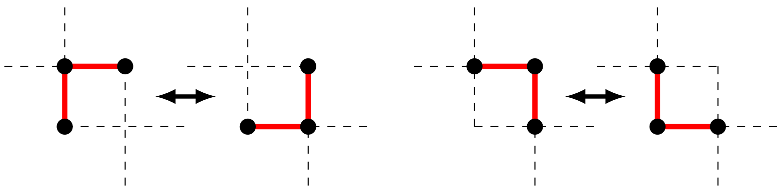
- claw movement: Modify the end of the path in the following way:

$$a_1, ...a_k, a_{k+1}, ..., a_n \leftrightarrow a_1, ...a_k, a_{k+1}, ..., a'_n,$$

  where a'_n is one of the adjacent direction of a_n;

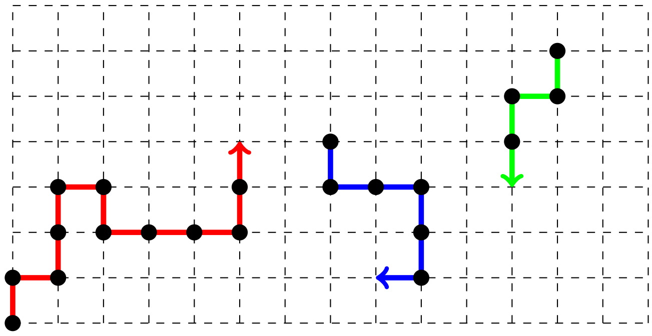


- swap movement: $a_1, ...a_k, a_{k+1}, ..., a_n \leftrightarrow a_1, ..., a_{k+1}, a_k, ..., a_n$.



It is allowed to make an action on the robot if the modified collection of paths forms an admissible position.

Example 3.5. Consider the following example. We have 3 hands with lengths 11, 6, 4, respectively, and base points as illustrated in the image below:



Definition 3.6. Let $\mathscr{A}$ be a collection of robots r_i with base points b_i and lengths l_i. Let G be a collection of all admissible paths spanned by the robotic system $\mathscr{A}$. Configuration space of such robotic system is an abstract cubical complex formed in the following way:

- take all the admissible position from collection G as 0-skeleton;
- any two position which differ by just one action will be connected by an edge and will generally form 1-skeleton of the cubical complex;
- the k-skeleton will consist of the following k-cubes: For every two admissible positions a and b which can be obtained from each other by performing k operations simultaneously on distinct parts of the system we draw k-cube, which has the following set as its vertex set:

$$\{r \mid N(a,r) \leq k\}$$

  where $N(x,y) \leq c$ means that position x can be obtained from position y by performing c or less operations.

3.1 Classification of positions

In the current and the following section we assume that our robotic system consists only of one hand.

We start by introducing some new notations. We do that with intent to break elementary changes in position into the claw and the swap movements which are investigated separately.

Definition 3.7. Let $\mathscr{M}$ be a collection of strings over an alphabet $\mathscr{A}$. Denote by $C_k^m\{a_1,...,a_n\}$ all the substrings of the elements in $\mathscr{M}$ which start from the position k and end in the position m consisting only of letters a_i for $i \in [n]$. In case where the bounds m and k are known we remove the subscripts to denote such set as $C\{a_1,...,a_n\}$.

In our use case $\mathscr{M}$ will be the set consisting of all the admissible positions of some robot and the alphabet $\mathscr{A}$ will be the set of cardinal directions $\mathscr{S}$. It is not hard to see that every position of $\mathscr{S}_n$ can be represented by alternating concatenation of strings of type $C_{k_i}^{m_i}(n,e,s)$ and $C_{k_j}^{m_j}(n,w,s)$.

Definition 3.8. Define the following reduced representation of robotic systems:

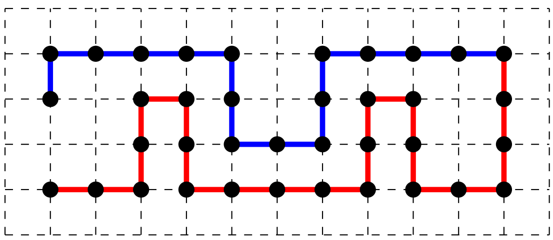
- for every position of the type $C_{k_i}^{m_i}(n,e,s)$ we will remove the "e" letters and write the leftover simply as the sequence $a_1a_2,...,a_n$, where each $a_i \in \{n,s\}$. The construction for the set $C_{k_j}^{m_j}(n,w,s)$ will be done in a similar manner, but instead of removing the letter "e" we remove the letter "w";
- recall that every position can be written as an alternating string concatenation

  $$C_{k_1}^{m_1}(n,e,s) * C_{k_2}^{m_2}(n,w,s) * ... * C_{k_n}^{m_n}(n,w,s)$$

  Using the construction above and letter "|" as concatenation symbol we obtain the following notation:

  $$a_1,...,a_{n_1}|a_{n_1+1},...,a_{n_2}|...|a_{n_{m-1}},...,a_{n_m}$$

Example 3.9. Consider the following example:



The hand has clearly form of $C(n,e,s) * C(n,w,s)$. The part $C(n,e,s)$ is colored with red color and part $C(n,w,s)$ with blue color respectively.

We should note the following trivial facts about the interplay of the notation with the claw movement. In the following two lemma we denote the types of robotic system simply by

$$\alpha = \alpha_1,...,\alpha_k, \text{ where } \alpha_i \in \{s,n,|\}$$

Lemma 3.10. *The swap movement preserves the form of the sequence with the following exceptions. Assume that α_t = "|" for some index t.*

- *In case α_{t-1} has the same type as a_{t+1}, we may make the following exchanges: $\alpha_{t-1} \leftrightarrow \alpha_t$ and $\alpha_t \leftrightarrow \alpha_{t+1}$;*
- *in case α_{t-1} has the same type as α_{t-2}, we may swap α_t with α_{t-1};*
- *in case α_{t+1} has the same type as α_{t+2}, we may swap α_t with α_{t+1}.*

Respectively the notation synchronizes with the claw movement in the following way:

Lemma 3.11. *Every time claw movement is applied the type will change in one of the following ways:*

- *if $\alpha_k \neq |$ we may: remove α_k, add any of the symbols "s","n" to the end of sequence and in case $\alpha_{k-1} = \alpha_k$ we may replace α_k with symbol $|$;*
- *else if $\alpha_k = |$ we may: add any of the symbols "s","n" to the end of the sequence and in case $\alpha_{k-1} = \alpha_{k-2}$ we may replace $|$ with copy of symbol α_{k-1}.*

3.2 Theorems

In the following two sections we will be proving results related to the claw and the swap movement.

3.2.1 Claw movement

Definition 3.12. Let $\mathscr{G}$ be a n-colored forest, i.e a graph which has no cycles, via function $c : \mathrm{Edges}(X) \to [n]$. Define the collection $\mathscr{T}$ as:

$$\{\{x, y, z\} \mid x, y, z \in \mathrm{Edges}(\mathscr{G}), c(x) \neq c(y), c(x) \neq c(z) \text{ and } c(y) \neq c(z)\}$$

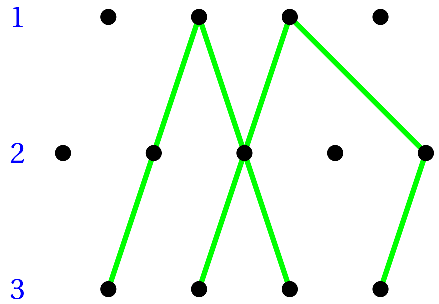
Let $\mathscr{A}$ be a subcollection of $\mathscr{T}$. Define the simplicial complex $\mathscr{S}(\mathscr{G})$ as follows:

(1) 1-skeleton will be exactly the graph $\mathscr{G}$ considered as simplicial complex;

(2) for every triple in the collection $\mathscr{A}$ we draw edges between all the points in the set and construct the 2-cell spanned by the points.

We say that a simplicial complex is of type (T) if it is obtained from Definition 3.12.

Example 3.13. As an example of such graph $\mathscr{G}$ consider a picture of some 3 coloring:



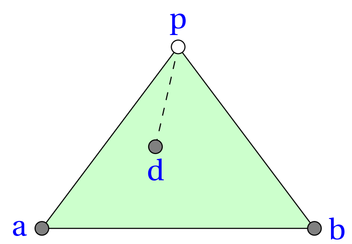
Lemma 3.14. *Every path component of the* (T)*-type simplicial complex is always contractible.*

Proof. To make proof more intuitive we will proceed with this lemma using explicit deformation maps. Because we assumed that the graph $\mathscr{G}$ is a forest, each of its connected components is a tree. We pick an arbitrary vertex $r \in \text{vertices}(C)$ and make it the root of our tree. Assume first that our tree has only one layer. In this case the complex consists only of a single vertex and thus is trivially contractible.

Assume then that a complex spanned by a tree having $n-1$ layers is contractible. Let $\mathscr{R}$ be a complex spanned by a tree having n layers. We show that $\mathscr{R}$ is contractible by reducing number of layers inductively. Let p be some vertex on the layer $n-1$. Let $\mathscr{W}_p$ be a simplicial complex of type (T) spanned by p and its children. Define map

$$H : \mathscr{W}_p \times [0,1] \to \mathscr{W}_p \mid H(d,t) = t * d + (1-t) * p$$

We lift every leaf vertex $a \in \text{Vertices}(\mathscr{W}_p) \setminus \{p\}$ upwards to the root p by using map H. By Definition 3.12 for any children edge (a,b) there exists triangle as pictured below:

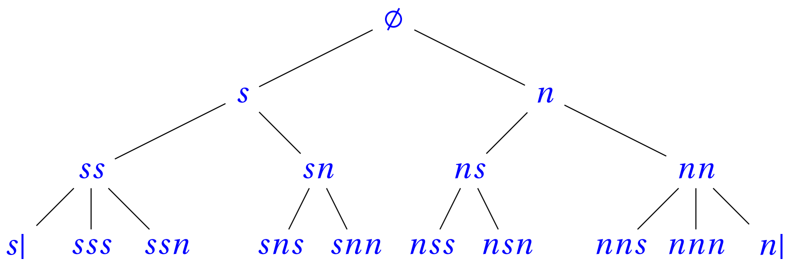


The triangle is contracted into its root by the map H as pictured above. Because any triangle is closed and the intersection of two triangles of Definition 3.12 is a boundary which is closed as well the map is well-defined due to the gluing lemma.

By repeating such operation on all the points in layer $n-1$ the bottom layer is completely contracted resulting in a graph having $n-1$ layers. Such graph yields a contractible complex by induction. □

Definition 3.15. Recall the claw movement on canonical representation of robotic hand as described in Lemma 3.11. Define the graph $\mathscr{X}$ to be a graph whose each vertex corresponds to some string spanned by symbols "n", "s", "$|$", with restriction that there is one of the symbols of "n" or "s" before every symbol of type $|$. For any two strings which can be obtained from each other by performing the claw movement we draw an edge. It is relatively easy to see that $\mathscr{X}$ is a tree. We assign $\emptyset$ to be its root.

The next picture represents the tree's first layers:



One should note that every vertex has at most three children.

Lemma 3.16. *Technical detail: the cubical cover of positions of two types:*

- $a_1, \ldots, a_k, s, |$ *and* $a_1, \ldots, a_k, s, s, s$;
- $a_1, \ldots, a_k, n, |$ *and* $a_1, \ldots, a_k, n, n, n$.

has always empty intersection.

Proof. The claim holds true by observing simple geometry of the robotic hand. It can be seen that the distance between the two position is at least 3. The cubical cover detects all the positions within the distance 1 and thus they will not intersect. $\square$

Theorem 3.17. ***Fundamental theorem of the claw movement**: let $\mathfrak{U}$ be a finite connected subgraph of $\mathscr{X}$. Define the root of the tree to be the element on the lowest layer in respect to the orientation inherited from the original graph (earlier we defined its root to be $\emptyset$). We form the cubical cover over the vertex set of $\mathfrak{U}$ and assume that the following claims are satisfied:*

(1) the cubical element corresponding to each vertex in the graph is a disjoint union of contractible spaces;

(2) for every vertex $v \in \mathrm{Vertices}(\mathfrak{U})$ let $\mathscr{B}_v$ be set of its children. Let graph $\mathscr{G}_v$ be obtained as follows. As the vertex set we pick all the path components (in the cubical representation) belonging to either $\mathscr{B}_v$ or the vertex v. We draw an edge between two vertices if they have a non-empty intersection in their cubical representation;

(3) *assume that every $\mathscr{G}_v$ is a forest and intersection of cubical elements corresponding to any two vertices a and b which are connected by an edge is contractible.*

Then every path component of the configuration space spanned by all the vertices of $\mathfrak{U}$ is contractible.

Proof. This claim will be proven by induction over the length of the tree. Clearly when the graph has zero length we are dealing only with the root which by definition consists of contractible components.

Assuming that a graph of $n-1$ length satisfies the claim we will show that every graph having length n does so as well. Let $\mathscr{Y}$ be a graph having n layers. Let v be its root. By the induction assumption a subgraph rooted from every $a \in \mathscr{B}_v$ is disjoint union of contractible spaces. We will now apply the nerve lemma to all the path components of the cubical complexes corresponding to vertices in $\mathscr{B}_v$. It should be noted that the lemma can be applied, because all the essential intersections are contractible.

Now the goal is to show that the nerve complex will be of type (T), which is contractible. Define the collection $\mathscr{A}$ to be all the triples of the vertex set which have a common non-empty intersection in their cubical representation. It should be noted at this point that following from Lemma 3.16 and the fact that at most 4 distinct vertices may have non-empty common intersection there are no larger cells than those which are of dimension 2. The claim holds true by the assumptions (2) and (3) of the theorem.

□

3.2.2 Swap movement

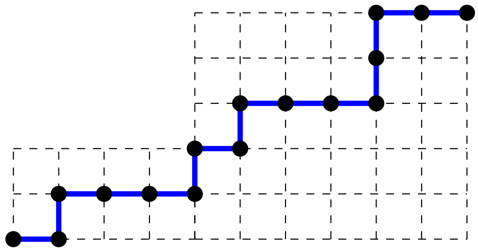
Using the notation developed in this chapter we separated claw movement from the swap movement. In the sections above we prove that the complex is contractible as long as the swap and the claw movements satisfy certain properties. In this section we will ascertain that some subgraphs of $\mathscr{X}$ satisfy required conditions.

Definition 3.18. The cubical interpretation of subgraph $\mathscr{G}$ of $L(\mathbb{R}^2)$ is the following cubical complex:

- the 1-skeleton is determined by the graph structure of $\mathscr{G}$;
- for every cycle of type $(i, j)-(i, j+1)-(i+1, j+1)-(i+1, j)$ we fill the interior.

Lemma 3.19. *Let $\mathscr{G}$ be a connected subgraph of $\mathscr{L}(\mathbb{R}^2)$, with condition that is is contractible in cubical sense . Let p be a path between points $a, b \in$ Vertices$(\mathscr{G})$, which consists only of two cardinal directions. Let $\mathscr{R}$ be the robotic system spanned by the position p and the swap movement. Then the configuration space of $\mathscr{R}$ is contractible.*

Proof. The proof done by induction. The picture below displays a path p and some subgraph which satisfy the assumption in the lemma.



By symmetry we may assume that our hand consists of movements e and n as in the picture above. Induction is done over the distance from the end point. At step 1 there are at most two possible ways of archiving the endpoint. There are no actions, as we defined our only action to be the swap action (which has a natural requirement on the hand length to be at least of 2). Assume now that the configuration space of the hand having m vertices is contractible. We will prove that a hand having $m+1$ vertices is contractible. Let the start point be $m+1$ links away from the end point. The next four possible cases needs to be considered:

- there exists position of the form $n, a_1, ..., a_m$ and of the form $e, b_1, ..., b_m$;
- there exists position only of the form $n, a_1, ..., a_m$;
- there exists position only of the form $e, b_1, ..., b_m$;
- there are no connections.

In the last case the complex will be empty and the two intermediate cases are trivially homotopy equivalent to the configuration space of a robot having n links. Thus the only interesting case is the first one. We form cubical cover $\{C_1, C_2\}$ over the position types $n, a_1, ..., a_m$ and of $e, b_1, ..., b_m$. Because we assumed that the graph is contractible in cubical sense, the intersection $C_1 \cap C_2$ is not empty and it has a form of $e, n, c_2, ..., c_m$, which is contractible by the induction assumption. □

Definition 3.20. For a robotic system of type

$$\mathscr{R} = a_1, ..., a_m | a_{m+1}, ..., a_n.$$

We fix robotic system and define the following concepts:

- the letters e and w in the system will be called resources of the system. One should recall that the resource e is being used between symbols a_i when $i < m$ and respectively the resource w is used when $i \geq m$;

- denote by $[w]$ and $[e]$ the indexation over all the letters w and e. Define total order on the set $[w] \cup [e]$ as $\forall x, y \in [w] \cup [e] : x < y$ iff x comes before y in the standard notation. Let $f : [w] \cup [e] \to \mathbb{Z}$ function defined as follows: for every $x \in [w] \cup [e]$ let $s(x)$ be a set containing all the letters a_i which are before x. Then

$$f(x) = \sum_{t \in s(x)} \mathscr{X}(t, "n") - \mathscr{X}(t, "s")$$

  where $\mathscr{X}$ is a characteristic function retrieving 1 when the characters equal and 0 otherwise;

- for every link $x \in [r] \cup [l]$ define the following operators:
  - $\text{next}(x) = \min\{t \in [r] \cup [l] \mid t > x \text{ and } f(t) \neq f(x)\}$;
  - $\text{prev}(x) = \max\{t \in [r] \cup [l] \mid t < x \text{ and } f(t) \neq f(x)\}$;

- we say that a resource $x \in [r] \cup [l]$ is in locally maximum position if

$$f(\text{prev}(x)) < f(x) \text{ and } f(\text{next}(x)) < f(x).$$

  Conversely a resource is in a locally minimum position if the inequalities of the equation hold in other way;

- let $\mathscr{T}$ be a collection of all the critical positions. Define a function $h : \mathscr{T} \to \mathbb{Z}$, $h(z) = f(a_z)$ where a_z is the first resource in the critical position;

- we say that a configuration is *normalized* if every resource is in critical position. We say that a configuration is *reduced* if every maximal position in the first half of the sequence contains only a single resource and every minimal position in the second half of the sequence also contains only a single resource;

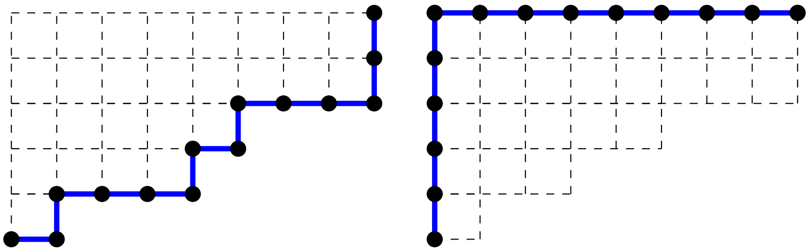
- we call the part $a_1, ..., a_m$ which comes before "|" the first half of the sequence. The part $a_{m+1}, ..., a_n$ after "|" will be called the second half of the sequence.

The definitions below are written for the case where the second part is on top of the first part. The case where the second part is under the first part is symmetric. From now on we will always make such assumption.

Lemma 3.21. *Every path component of the robotic system of type $\mathscr{R}$ is determined by normalized configurations.*

Proof. Assume that the local maximums of the first part are indexed by $\{i_1,...,i_p\}$ and the local minimum of the second part by $\{j_1,...,j_q\}$. By the definition every critical position contains atleast one resource. For every index i_x or j_y we pick some resource in the position determined by index and denote either by e_x or w_y.

Pick arbitrary two adjacent resources w_v and w_{v+1}. Our goal is to show that it is possible to turn up and straighten the path as below:



This is almost a direct implication of Lemma 3.19. Denote by $\mathscr{G} \subset L(\mathbb{R}^2)$ the graph consisting of all the possible routes through north. It is clear that the graph $\mathscr{G}$ is contractible in cubical sense. Following from Lemma 3.19 we know now that such positions are in the same path component. By the same argument we are able to turn down the path between e_u and e_{u+1}. The claim holds now, because we are able to perform such operation separately on each interval. □

Definition 3.22. Let p be some normalized configuration. Hill notation of p is sequence

$$q_1,...,q_n,p_1,...,p_m$$

where elements q_i are indexed by critical positions of the first half and p_i by the critical positions of the second half respectively. Each of the sequence elements $x_i \in \mathbb{N}\setminus\{0\}$ denotes the number of resources in that particular position.

Definition 3.23. Define the normalized robotic system $\mathscr{Q}$ corresponding to robotic system $\mathscr{R}$ in the following way:

- all the admissible states are the normalized states of $\mathscr{R}$;
- let $(q_1,...,q_n,p_1,...,p_m)$ be some state denoted by hill notation. Define first actions for the first part of the sequence:
  - when $q_i > 1$: $(q_1,...q_i,q_{i+1},...,q_n) \leftrightarrow (q_1,...q_i-1,q_{i+1}+1,...,q_n)$;
  - when $q_i > 1$: $(q_1,...q_{i-1},q_i,...,q_n) \leftrightarrow (q_1,...q_{i-1}+1,q_i-1,...,q_n)$;

  For the second part of the sequence the actions are defined correspondingly;

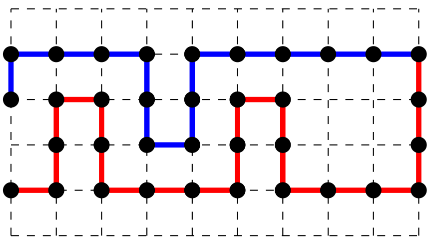
- the configuration space will be defined as in the case of the general robotic hand system.

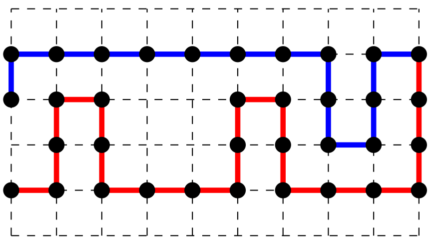
Definition 3.24. The choice of a legal hill component of the system $\mathscr{R}$ is the following subspace of the configuration space of such system: denote by $\mathscr{A}$ the set of all the critical positions of the first part and by $\mathscr{B}$ the set of all the critical positions of the second part accordingly. We index elements in both sets by their lowest x coordinate in the $\mathbb{R}^2$ plane. Denote their indexations by $I = \{i_0, i_1, ..., i_\alpha\}$ and $J = \{j_1, ..., j_\beta\}$ respectively, where i_0 is just dummy element having x coordinate -1. Let h^* be a modified function in a way where $h^*(x) = \infty$ for $x \in \{i_0, i_\alpha\}$ and $h^*(x) = h(x)$ for all the other x. An interval sequence x_t is defined as follows: for every j_t pick some pair $i_u < i_v$ for which following claims hold true:

- $\forall i_z (i_u < i_z < i_v) \Rightarrow h^*(i_z) \leq h^*(j_t)$
- $h^*(i_u) > h^*(j_t)$
- $h^*(i_v) > h^*(j_t)$.

We say that an interval sequence (x_t) is a legal hill component of system $\mathscr{R}$ if there exists a position in a configuration space of the system $\mathscr{R}$ which satisfies the condition that every critical point j_t is in the interval defined by sequence element x_t.

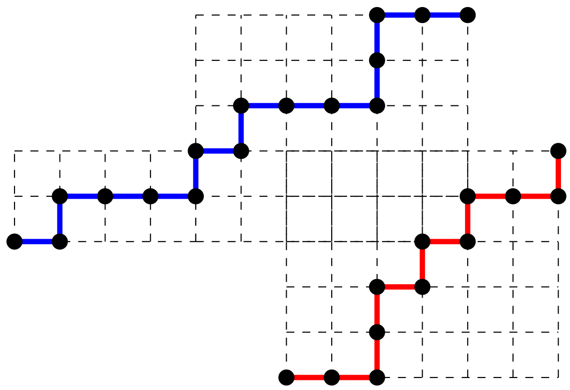
Example 3.25. Consider the following two configurations





Clearly one cannot transform one position into another through swap movements.

Example 3.26. Imagine a situation where two hands which are only allowed swap movements have a non-empty intersection in their trace as pictured below:



It can be proved that such configuration is contractible by showing that the trace of the blue hand is contractible for any choice of position of the red hand. Intuitively the deformation retraction will be defined to deform the red hand to minimal position (position where it has no intersection with trace of the blue hand) over all the positions of the blue hand and afterwards deforming the blue hand to some specific position.

Methodology applied in the proof of the following theorem is similar to Example 3.26. Since the proof is too long for the thesis and it will not be referred from elsewhere of the work we give only an implicit variation of it (and deduce its simplified version later):

Theorem 3.27. *Every choice of a legal hill component of system of the type $\mathscr{R}$ is contractible.*

Idea of proof.

The theorem will be proved by repeated application of the nerve lemma. Fix some legal hill component (x_i) and fix the position of the "|" letter in $a_1, ..., a_n | b_1, .., b_m$. Afterwards proceed as follows:

(1) denote by $\mathscr{T}$ set of all the configurations of the first part which satisfy the property that there exists some legal configuration of the second part satisfying the interval assumption. We will show that $\mathscr{T}$ is contractible when considered as a subspace of the normalized system $\mathscr{Q}$. This will be done by induction:

   (a) in the first step we prove that when we fix all the other columns expect the column corresponding to the local maximum with the lowest x coordinate we obtain a contractible configuration space;

   (b) we prove the general claim using induction. We assume that the configuration space of the system when every other column except $n-1$ first columns are fixed is contractible and prove that the claim holds for n free columns;

(2) fix some configuration of the first part $c \in \mathscr{T}$;

(a) we show that the set of all the available positions of the second part $\mathscr{H}$ is contractible when the first part is in position c;

(b) we show that for every fixed $d \in \mathscr{H}$ all the available configurations of the first part comprise a contractible set;

(3) finally we show that by gluing complexes corresponding to adjacent locations (when ordered by its y coordinate) of the symbol "|" we obtain a contractible complex.

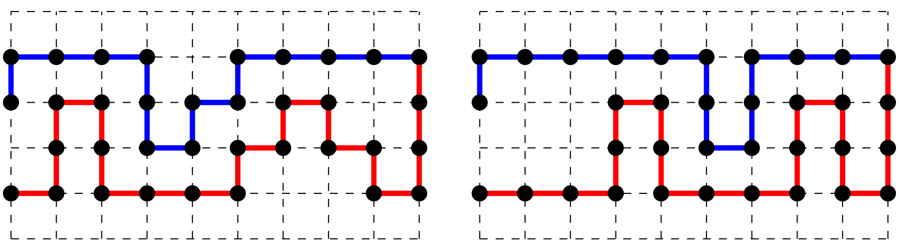
Alexander Engström once said that usually when something is contractible it can be intuitively described with how the deformation map would look like. In an example below we give illustration to the deformation used in the Theorem 3.27.

Example 3.28. Intuitively we perform the following operations:

(1) contract red and blue parts to the normalized configuration;

(2) order all the maximums of the first part and the minimum of second part of the sequence by their x coordinate;

(3) move columns corresponding to the critical points one at the time in order defined in (2) to their rightmost position.



We will prove the simplified case of the theorem above.

Corollary 3.29. *A system of the type $a_1, ..., a_n$ without "|" symbols is contractible.*

Proof. Assume first that the system consists of two critical points, starting with a single local minimum and ending with a local maximum. The claim follows directly from Lemma 3.19. We assume now that the claim holds for the system consisting of n critical points. By symmetry assume that we start in a local minimum and are going towards a local maximum. Assume that we start at coordinate 0 in x-axis. The path reaches its maximum at $x \in [0, m]$ for some fixed m. We form the cubical cover $\{C(i)\}$ over the following sets:

$$S(i) = \{\text{Hand reaches a local maximum first time at coordinate } i\}$$

Every set $C(i)$ consists of two parts: the first part which is a ladder and the rest. The ladder part is contractible by the Lemma 3.19 and the second part is contractible by the induction assumption. Since the parts are independent the set $C(i)$ is contractible for all i. It is easy to see that $C(i)$ has at most two non-empty intersections with sets $C(i-1)$ and $C(i+1)$. By similar argument as used in proving that $C(i)$ is always contractible we prove that $C(i) \cap C(j)$ is contractible as well. Thus by using the nerve lemma we gain that the whole complex is contractible. $\square$

Remark 3.30. In the next section we will use some facts about configuration space of robotic system of type $a_1,...,a_n$ where we have requirements for last letter a_n. The state complex of a system when a_n is allowed to take only cardinal directions which are adjacent to each other is contractible which is proved identically to Corollary 3.29.

3.2.3 Consequences

We will now summarize the main results of this chapter. Our goal is to prove that a robotic arm which can move freely around itself has homotopy type of S^1. However, we were able to prove this only in some special casese. Recall that in this chapter we classified the positions of robotic hands using alternating concatenations of $C(n,e,s)$ and $C(n,w,s)$. Its vertical variation can be defined as follows:

Definition 3.31. The vertical variation of the standard notation is an alternating concatenation using $C(e,n,w)$ and $C(e,s,w)$. Instead of using e and w as resources we use s and n. Respectively, instead of making hills over s and n we make hills over e and w.

Definition 3.32. We define system of type $\mathscr{T}$ to be a system for which we allow positions of type $C(n,e,s)$, $C(e,s,w)$, $C(n,e,s)$ and $C(n,w,s)$.

Lemma 3.33. *Every system of type $C(a_1,a_2,a_3)$ is contractible when both the claw and the swap are allowed.*

Proof. We provide proof only for the case $C(n,e,s)$ as all the other cases are symmetric. The complex generated by the swap movement is contractible by Corollary 3.29. For claw movement consider subgraph $\mathscr{H}$ of $\mathscr{X}$ spanned by positions $C(n,e,s)$. Clearly the graph is connected and every position corresponds to a contractible space. Let $a_1,...,a_z$ be an arbitrary position, where $a_i \in \{n,e,s\}$. All the interesting intersections are of the following types:

(1) $a_1...a_{z-1}n \leftrightarrow a_1...a_{z-1}e$ is possible when $a_{z-1} \neq s$;

(2) $a_1...a_{z-1}e \leftrightarrow a_1...a_{z-1}s$ is possible when $a_{z-1} \neq n$;

(3) $a_1...a_{z-1}n \leftrightarrow a_1...a_{z-1}s$ is possible only when $a_{z-1} = e$.

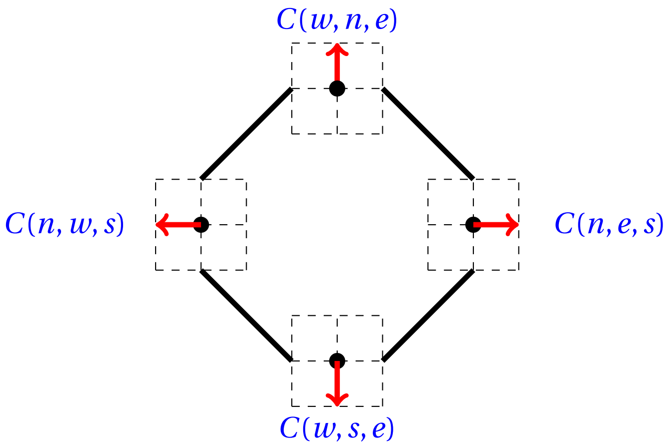
Following from the argumentation of Remark 3.30 all the configuration spaces spanned by the elements described above are contractible. Hence Theorem 3.17 establishes the claim. □

Corollary 3.34. *System of type $C(a_1, a_2)$ is contractible.*

Proof. This claim is proved almost exactly as Lemma 3.33. We will instead use Lemma 3.19 to show that a complex spanned only by the swap movement is contractible. Using the same argumentation as in the Lemma we are able to apply Theorem 3.17 which establishes the claim. □

Theorem 3.35. *Robotic system of type $\mathscr{T}$ has homotopy type of S^1.*

Proof. Our goal is to prove that we obtain the following complex:



First of all by Lemma 3.33 we know that each of the systems having type $C(a_1, a_2, a_3)$ is contractible. An intersection between two such spaces $C(a_1, a_2, a_3) \cap C(b_1, b_2, b_3)$ equals to a space $C(d_1, d_2)$ which is contractible by Corollary 3.34. Thus the claim follows from the nerve lemma.

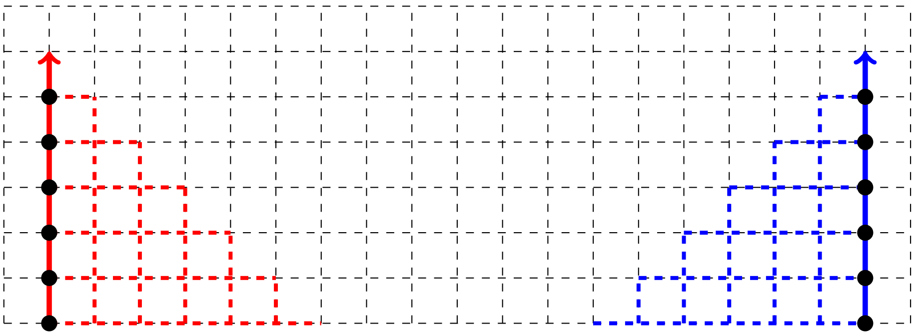
□

Remark 3.36. Using machinery developed in sections 3.2.1 and 3.2.2 similarly to Theorem 3.35 it can be proven that space consisting of an union of the following four parts has homotopy type of S^1:

- Space $C(w, n, e) * C(w, s, e)$;
- Space $C(w, s, e) * C(w, n, e)$;
- Space $C(n, e, s) * C(n, w, s)$;
- Space $C(n, w, s) * C(n, e, s)$.

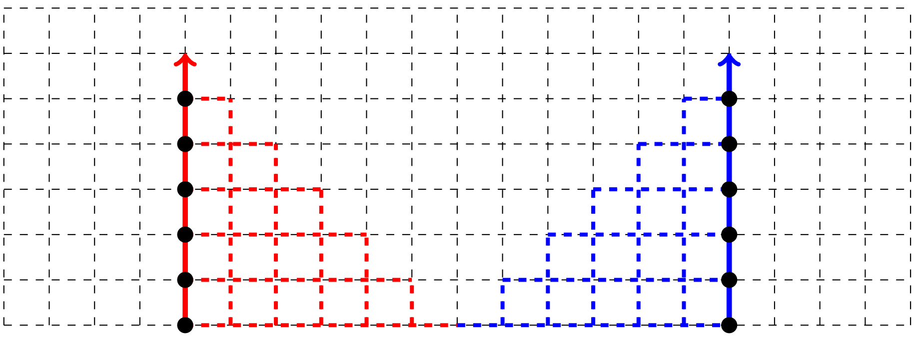
3.3 Motivational example

Example 3.37. Motivational Example: Let $\mathscr{R}$ be a robotic system consisting of two hands of length 6. First consider the situation where the hands have a significant from each other as illustrated below:



It is clear that the hands do not have an intersection point. Thus the state complex will be simply product of the two state complexes corresponding to the hands. By the previous theorem we know that a single hand which is moving freely around its axis has homotopy type of S^1. It follows that their product will be homotopy equivalent to torus $S^1 \times S^1$.

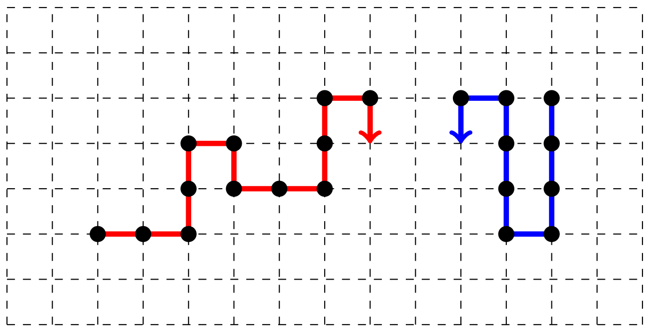
In the picture below we illustrate the situation where we bring the hands closer to each other, so that they have a single intersection point.



The configuration space of the situation can be thought as a torus where we have removed a contractible set corresponding to the intersection point. It can be observed that such complex is homotopy equivalent to one point union $S^1 \vee S^1$.

However, this is not always the case, as by removing the edge corresponding to the intersection point we lose connection to some branch in the tree. In this case the intersection point will create a hole in the space, which will result into new generators in the fundamental group of the space.

One can make the configuration space arbitrarily complicated, by moving the hands even closer, as illustrated below:



We can observe that both robotic systems, marked blue and red, respectively, have done U-turns. The intersection point of the described configuration will create a hole in the space. We obtain a non-contractible path by first rotating both heads of the red hand and the blue hand, respectively, to north. Then by rotating the hands back to their initial positions we observe that this movement creates a singularity, which is detectable by the fundamental group operator.

Chapter 4

Robotic system of length 1

In this chapter we give an alternative approach to this research area. Throughout the both sections we discuss robotic systems that have no predefined geometrical structure, but have only a single moving arm each. The whole Section 4.2 is devoted to the second main theorem of this thesis, which states that the fundamental group of configuration space of the robotic system defined in Section 4.1 has under special conditions discussed later isomorphism type of Artins right-angeled group.

4.1 Theory

First we need to recall the definition of a torus considered as a cubbed complex and then proceed by giving some motivational examples.

Definition 4.1. *Cubbed structure of n-torus* is the following space:

$$[0,1]^n/R$$

where R is a relation spanned by all the elements of the form

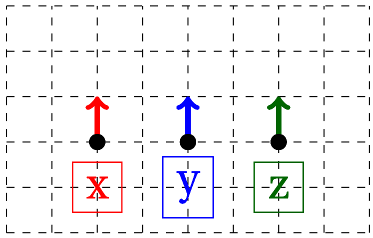
$$(a_1,...,a_m = 0,...,a_n) \backsim (a_1,...,a_{m-1},1,a_{m+1},...,a_n)$$

In other words, we are identifying all the opposite faces of the n-cube.

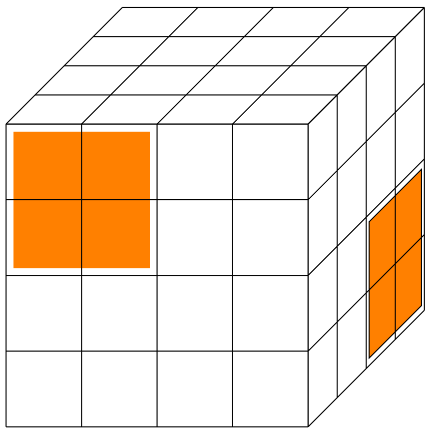
Definition 4.2. Let $<$ be a total order on a finite set X. We define the following two functions:

- $\mathrm{Succ}: X \to X$: $\mathrm{Succ}(x)$ returns the minimum element of the set $\{y \in X \mid y > x\}$. In case the set is empty ,the function $\mathrm{Succ}(x)$ returns the minimum element of X;
- $\mathrm{Decc}: X \to X$: : $\mathrm{Decc}(x)$ returns the maximum element of the set $\{y \in X \mid y < x\}$. In case the set is empty, the function $\mathrm{Decc}(x)$ returns the maximum element of X.

Example 4.3. Consider the following robotic system of 3 hands:



We consider the configuration space of this robotic as a subspace of a cubed n-torus, where each axis of the cube corresponds to the rotation of the respective hand. Because there are exactly four different positions for each hand, each axis will be refined into four parts split by vertices each corresponding to one of the cardinal directions. By removing the disabled positions, i.e. the positions where the hands can intersect, we obtain the following subspace of a cube. The area removed is colored with orange color as illustrated in the picture below:



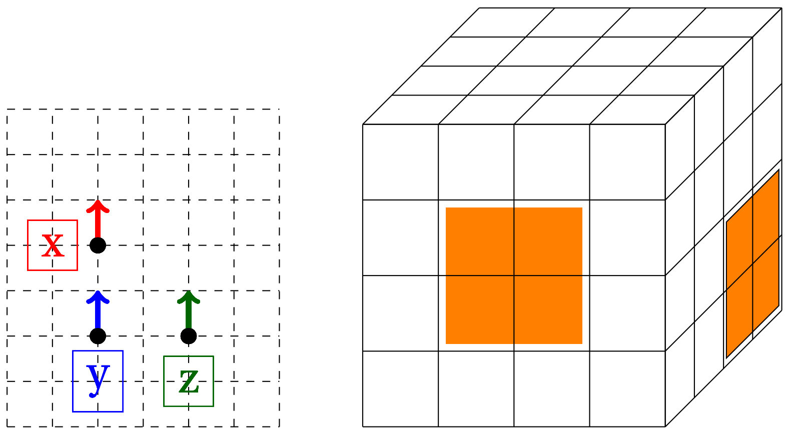
It should be noted that unlike the coloring might suggest, the boxes which are being removed are 3-dimensional, meaning that they are of the form $[a_1, a_2] \times [b_1, b_2] \times [0, 1]$.

We will proceed by describing the fundamental group of the configuration space in an intuitive manner. We will present the explicit calculations later in the text, as they are not necessary for the understanding of these motivational examples. We will fix the base point to be the state where every robot is facing north. Because the space is embedded in the torus, we find three generators, each corresponding to the rotation of a single hand. It is later proved that there are no other generators, and that the only relations in the group will be relations of type $ab = ba$, each occurring for generators a and b which have no intersection points. It is observed that in this case hands x and z can move freely when y is stationary. Thus we will have commutativity relation between x and z. The group of this type is commonly referred in the literature as the Artin's right-angled group.

Unlike in the case of a torus, the homology groups of the space are not determined by the fundamental group. We denote the configuration space by X. By using computer or Mayer-Vietoris sequences we obtain that the homology groups of the space are the following:

$$H_0(X) = \mathbb{Z} \, , \, H_1(X) = \mathbb{Z}^3 \, , \, H_2(X) = \mathbb{Z}^2$$

Let Y be a configuration space of the following robotic system:



It will be proved later that the configuration space where the hands were positioned in the row has exactly the same fundamental group as space X. However, unlike in the previous case the area which is being removed is connected. It is now possible to deform the space to $K(\pi,1)$ space of Artins group. The homology type of such space is calculated in [7] and it has the following type:

$$H_0(Y) = \mathbb{Z},\, H_1(Y) = \mathbb{Z}^3,\, H_2(Y) = \mathbb{Z}$$

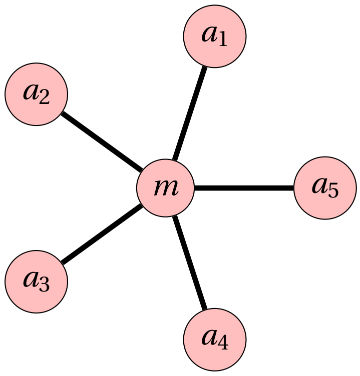
It follows directly from this argument that spaces X and Y cannot be homotopy equivalent, even thought they have the same fundamental group. In this chapter we will mainly focus on computing the fundamental group of robotic hand systems which only have hands of length one.

Definition 4.4. Let $n \in \mathbb{N}$, then star graph of n elements S_n consists of the following ingredients:

- vertex set $\{m, a_1, ..., a_n\}$;
- total order $<$ on set $\{a_1, ..., a_n\}$, i.e. orientation;
- edge set, which has the form of $\{ma_2, ..., ma_n\}$.

We define operator M to be an operator which retrieves the middle link m of the star graph complex.

Example 4.5. For intuitive example, consider the following star graph consisting of 6 vertices together with canonical ordering $a_1 < a_2 < a_3 < a_4 < a_5$. The graph can be pictured as done below:



Definition 4.6. Regular star complex over m cells is the following graph construction:

- let $\mathscr{A} = \{S^1_{n_1}, \ldots, S^m_{n_m}\}$ be collection of some star graphs. Denote graph obtained by disjoint union $\sqcup_{i\in[m]} S^i_{n_i}$ as $\mathscr{G}'$;

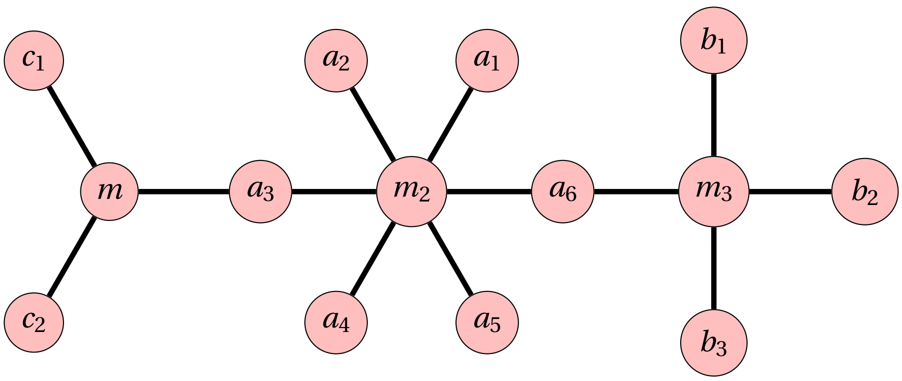
- for every $S^i_{n_i}$ we have a set $\mathscr{B}_i \subset \mathrm{Vertices}(S_i) \setminus M(S_i)$ and a symmetric relation R on $\mathscr{B} = \bigcup_{i\in[m]} \mathscr{B}_i$, satisfying the following condition: For every star graph i and every element $x \in \mathscr{B}_i$ there exists unique t belonging to one of the elements of a set $\{\mathscr{B}_j\}_{j\neq i}$ for which $(x, t) \in R$;

- we define the star graph complex to be

  $$\mathscr{G} = \mathscr{G}'/R.$$

  In other words, we take $\mathscr{G}'$ and make identifications on it as defined by relation R.

Example 4.7. Let $\mathscr{G}$ be the graph as illustrated in the picture below:



Define the vertex set of star graphs in the following way:

$$S_3^1 = \{m, c_1, c_2, c_3\},\ S_6^2 = \{m_2, a_1, a_2, a_3, a_4, a_5, a_6\} \text{ and } S_4^3 = \{m_3, b_1, b_2, b_3, b_4\}$$

and define sets $\mathscr{B}_1 = \{c_3\}$, $\mathscr{B}_2 = \{a_3, a_6\}$ and $\mathscr{B}_3 = \{b_4\}$ and make the following identifications: $a_3 \backsim c_3$ and $a_6 \backsim b_4$. Clearly, graph $\mathscr{G}$ decomposes into star graph complex spanned by sets S^i and relations spanned by $\mathscr{B}_i$.

Definition 4.8. We say that the star complex has rest point, if $\mathscr{B}_i \neq \text{Vertices}(S_i) \setminus M(S_i)$ for all i or, in other words, there exists a distinct collection of vertices, each belonging to unique $S^i_{n_i}$

It is clear that in the previous example 4.7 we had a rest point. For example, one can pick (c_1, a_1, b_1) as a rest point.

Definition 4.9. Let $\mathscr{G}$ be a graph of a n-star complex. A general one-handed robotic system over $\mathscr{G}$ consists of the following ingredients:

- a graph which is a star graph complex with some decomposition $\{S^1_{n_1}, ..., S^m_{n_m}\}$;
- admissible set $\mathscr{A}$ of $\{0,1\}$-labellings of the graph, which is spanned by the following elements: For every star graph $S^i_{n_i}$ we pick one boundary vertex x_i in such way that for all i, j we have $(x_i, x_j) \notin R$. We label all the x_i together with the edges $\overline{x_i M(S^i_{n_i})}$ with the color 1 and everything else with the color 0;
- actions defined on set $\mathscr{A}$: For every star graph $S^i_{n_i}$ we have two actions which can be performed as long as the result is an element of the set $\mathscr{A}$. Action r_i on a vertex x is spanned by function succ acting on the orientation of vertex set of robot $S^i_{n_i}$, in the way where we color x and $\overline{xM(S^i_{n_i})}$ to 0, as well as $\text{succ}(x)$ and $\overline{\text{succ}(x)M(S^i_{n_i})}$ to 1. Respectively we have action l_i with is defined in the similar way, with exception that instead of using function succ we use function decc, which does exactly the same thing but in the opposite direction.

Such robotic system with n number of robots will be denoted simply by T_n.

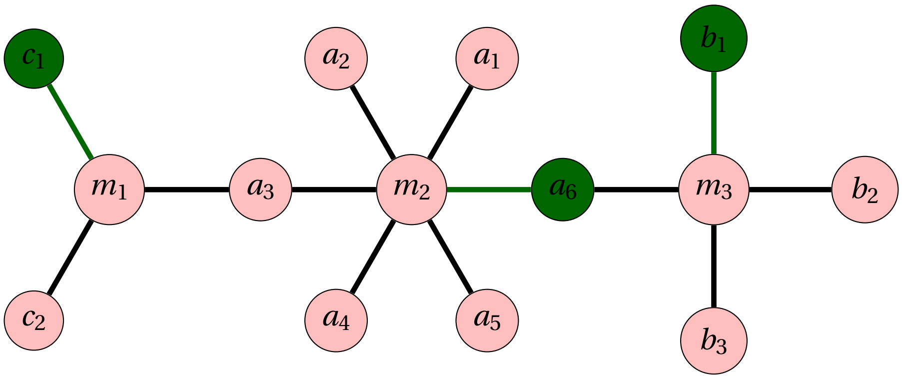
For this system we define a state complex in a similar manner as we did in the previous section for the case where the hands existed in $\mathbb{R}^2$ plane.

Definition 4.10. Configuration space of a robotic system T_n is the following cubical complex:

- vertex set is defined to be the set of all admissible labellings of the robotic system;

- 1-skeleton is just transaction graph spanned by the actions;
- n-cubes are spanned by consecutive actions $a_{i_1}^{j_1} a_{i_2}^{j_2} ... a_{i_n}^{j_n}$ of length n, where each of a^{j_i} is either a functor r or l and every index j_i corresponds to a distinct hand.

Example 4.11. At this point the reader should recall how the graph $\mathscr{G}$ defined in Example 4.7 looked like. In this example we will construct a concrete robotic system on graph $\mathscr{G}$. Consider a following position of the robotic system:



In the picture above we are using pink coloring to denote nodes which are labeled with the color 0 and dark green to denote the nodes labeled with the color 1. At this point we recall that the orientation corresponding to each star graph complex was spanned by the index set. Denote by r_i the r functor corresponding to hand having m_i as middle point and respectively denote by l_i the l functor. As can be seen from picture, the generators r_1,l_1 and r_2,l_2 corresponding to hands m_1 and m_2 respectively, are admissible. In the case of the hand m_3, the action r_3 is clearly admissible, but because vertex a_6 is already occupied l_3 is not. It follows that in this particular state we have five admissible actions in total. One should also note that as there are three hands the dimension of maximal cell of the state complex of the robotic system is 3. For example, action $r_1 r_2 r_3$ generates such cube.

The following example shows that the original robotic hand system of length 1 in a plane is a special case of the new abstract robotic system defined in this chapter.

Example 4.12. It is easy to embed the original robotic hand system, where each hand has length one in the Cartesian plane, into the abstract system by defining the new generalized system as follows: For every hand i define a star graph consisting of five vertices (one in the middle and four on the boundary) and form a star graph complex by gluing all those vertices together where any two hands could geometrically intersect.

Definition 4.13. Let T_n be some a robotic system and let $\mathscr{O}$ be a $\{0,1\}$-labeling of the star graph complex corresponding to the robotic system. The robotic system over obstacles $T_n'(\mathscr{O})$ is defined as follows:

- let $\mathscr{A}$ be a set of admissible states of the complex T_n. The set of admissible states of $T'_n(\mathscr{O})$ will be defined as follows:

$$\mathscr{A}' = \{q \in \mathscr{A} \mid \mathrm{trace}(q) \cap \mathrm{trace}(\mathscr{O}) = \emptyset\};$$

- the action set and the state complex of the system is defined as in the case of the system T_n by using the new set of admissible states $\mathscr{A}'$.

Remark 4.14. Let T_n be a robotic system and let $a_1, ..., a_n$ be a set of vertices that we would like to disable. We denote simply by $T'_n(a_1, ..., a_n)$ the system obtained by defining labeling $\mathscr{O}$ to be the labeling which labels vertices a_i and all their links with color 1 and everything else with color 0.

Definition 4.15. Let a_i be a hand with a disabled link l_i of some robotic system with restrictions. The balanced total order of a_i is the following $(X, <)$ total order:

- define X to be all the available links of the hand a_i;
- for every $l \in X$, which has property that $\mathrm{succ}(l) \neq l_i$ we define $l < \mathrm{succ}(l)$. We extend the obtained ordering by transitivity.

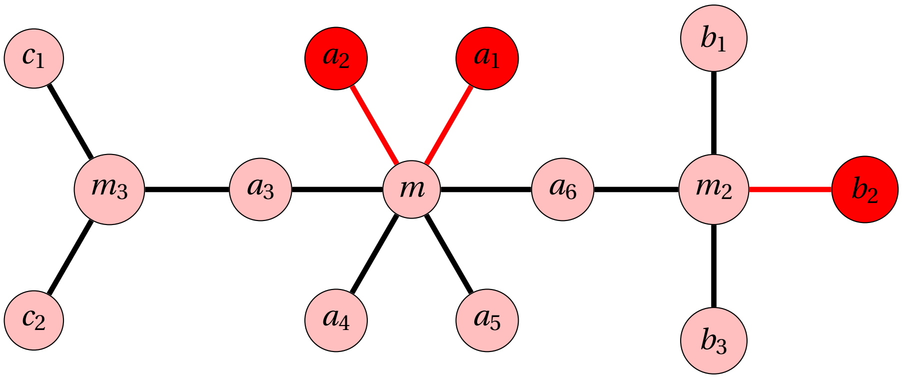
We will simply denote the maximum link with respect of the balanced ordering defined above as $\max(a_i)$ and the minimum link as $\min(a_i)$ respectively when it is clear from the context.

Definition 4.16. Let T_n be a robotic system and let $T'_n(\mathscr{O})$ be its modified system by some obstacle set $\mathscr{O}$. We say that the vertex a_i corresponding to robot i is *critical*, if for all admissible labellings where robot i is facing a_i, one of the actions in the set $\{r_i, l_i\}$ is never admissible. We say that the vertex a_i is *critically connected*, if it is critical and it is connected to some other robot.

The definition above can be also restated by simply stating that a position of a single robot is critical if it has some disabled position as a neighbor.

Definition 4.17. We say that robot i in the disabled robotic hand system $T'_n(\mathscr{O})$ is connected, if the state complex of a robotic hand system consisting of a single robot $\{S^i_{n_i}\}$ with the obstacle set $\mathscr{O}$ restricted to this robot has a path connected configuration space.

Example 4.18. Recall the examples 4.7 and 4.11. In the picture below we use red color to denote the disabled position:



The critical links are $\overline{ma_3}$,$\overline{ma_6}$,$\overline{m_2b_1}$ and $\overline{m_2b_3}$. The critically connected links are exactly $\overline{ma_3}$ and $\overline{ma_6}$, respectively. As an example one should see that the balanced total order of hand 2 is the following order: $b_3 < b_4 < b_1$

4.2 Fundamental theorem of robotic hand systems

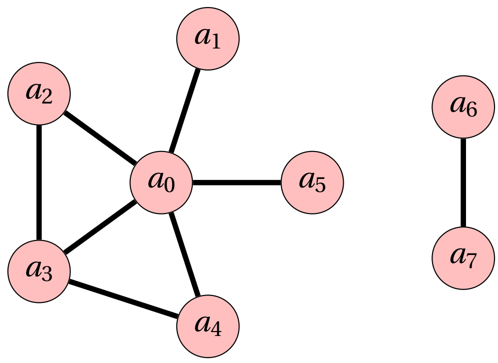
In this section we will prove the second main theorem of this thesis. For the sake of clarity, we will first present some definitions and important examples which will contain the information about proof strategy in intuitive form. To simplify the situation, we will make some restrictions to our system. However, it is widely believed that more general robotic systems will not only have similar fundamental groups, but also that the proof of the theorem can be done in a similar manner.

Definition 4.19. Let G be a group and let a, b be its elements. We say that elements a and b commute, if $aba^{-1}b^{-1} = e$. In this section we denote such relation simply by $a \backsim b$.

Definition 4.20. Let $\mathscr{G}$ be a graph, then Artins right-angled group over graph $\mathscr{G}$ denoted as $\mathrm{A}(\mathscr{G})$ is the group having the following representation using generators and relations:

- generator set consists of vertices of the graph $\mathscr{G}$;
- for every edge $\overline{xy}$, we have a commutativity relation $x \backsim y$ between the corresponding generators.

Example 4.21. Consider the following graph:



clearly group $A(\mathscr{G})$ has the following form:

- generators: $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$;
- relations: $a_0 \backsim a_1, a_0 \backsim a_2, a_0 \backsim a_3, a_0 \backsim a_4, a_0 \backsim a_5, a_2 \backsim a_3, a_3 \backsim a_4, a_6 \backsim a_7$.

Definition 4.22. Let T_n be a robotic system. We define the intersection graph $\mathscr{G}(T_n)$ of the robotic system as follows:

- for every robot $S^i_{n_i}$ we draw an edge in the graph;
- we draw an edge between any two vertices if the corresponding robots have a non-empty intersection.

Intuitively, we are compressing data of our robotic structure to a graph which keeps track of hands which do not have any common intersection point. In the following theorem we will prove that at least in some special cases the fundamental group of the state complex of the robotic system is completely determined by the intersection graph. However, one should recall that this is not the case for homology type, as two robotic hand systems with exactly the same intersection graph can have different homology groups.

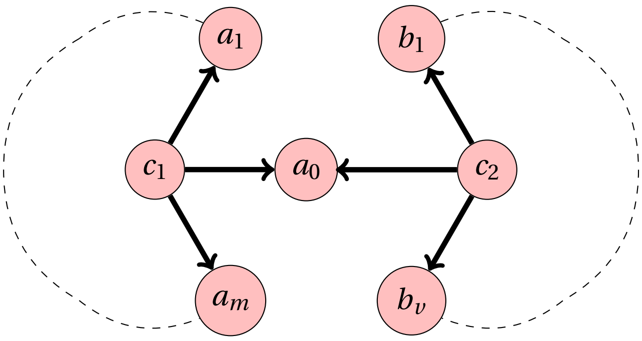
Theorem 4.23. *Let T_n be a robotic system consisting of n robots, which satisfies the following assumptions:*

- *T_n satisfies* regularity assumption*: Any two robots which are adjacent to each other have at most one vertex in common in the star graph decomposition;*
- *T_n satisfies the rest point assumption;*
- *every cell in the star graph complex corresponding to T_n has at least four vertices;*
- *the intersection graph $\mathscr{G}$ of T_n is a tree.*

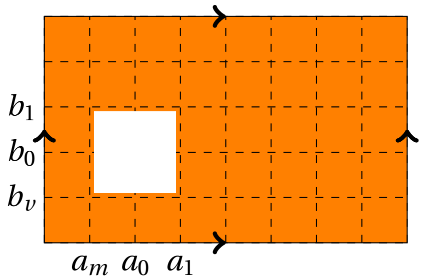
Let x_0 be the position where every hand is in its rest point. Then the fundamental group $\pi_1(T_n, x_0)$ is isomorphic to the Artin's right-angled group $A(c\mathscr{G})$ where $c\mathscr{G}$ is complement graph of $\mathscr{G}$.

Lemma 4.24. *Let T_2 be robotic system consisting of two robots $S^1_{n_1}$ and $S^2_{n_2}$. Then if $S^1_{n_1}$ and $S^2_{n_2}$ intersect in a single point, the state complex of T_2 has the homotopy type of $S^1 \vee S^1$.*

Proof. By symmetry we may assume that the robots are as in the picture below



One can draw the configuration space as the subspace of $S^1 \times S^1$ in a way where we make refinements on cubical structure of the torus as illustrated in the following picture and remove the cells corresponding to the intersection of the hands.



The white area illustrated in the picture above can be extended to cover all the non-boundary points of the square by using deformations. Because we have identified the opposite faces in the square, we are left with only two circles S^1 which intersect only in a single point. It follows that the space has the homotopy type of $S^1 \vee S^1$ $\square$

Definition 4.25. Let T_n be a regular robotic system satisfying the rest point assumption and let a_i be a hand in the system T_n. We define the following two important types of paths:

- let x be some position where the hand a_i is in its rest point. Define path $p_i(x)$ to be the path in the state complex of T_n which corresponds to a single rotation of the hand a_i around its axis in the positive way in respect to its defined orientation, keeping every other hand stationary at the same time;

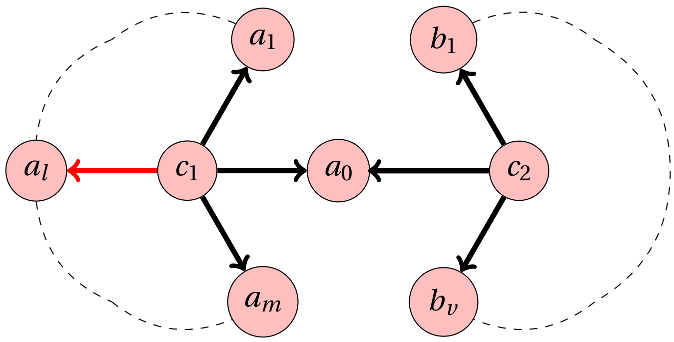
- let x be some position in the configuration space where a_i is facing link l_{w_1}. Let l_{w_2} be some other link of a_i. Path $m_{w_2}^{w_1}(x)$ is the path where we move the hand from position w_1 to w_2 in positive way in respect to its defined orientation, keeping every other hand stationary.

To get some idea on how to compute fundamental group in cases which are more complex and to familiarize ourselves with the notation, one should read the following two examples before proceeding to the complete proof.

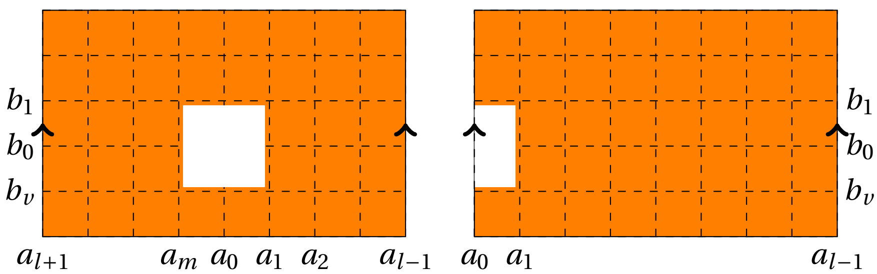
Example 4.26. Let $T_2'(l)$ be the restricted robotic system consisting of two robots $S_{n_1}^1$ and $S_{n_2}^2$. We assume that the robot $S_{n_1}^1$ contains the link l. Assume that $S_{n_1}^1$ and $S_{n_2}^2$ intersect in a single point. Then either of the claim holds:

- if l is next to the intersection point, then $\pi_1(T_2'(l))$ is isomorphic to $\mathbb{Z}$;
- else the group $\pi_1(T_2'(l))$ is isomorphic to $\mathbb{Z} * \mathbb{Z}$.

We will first present things in an intuitive manner and then proceed by providing the proof using Seifert-van Kampen theorem. This claim will be proved in general way using the machinery developed in this section. Denote the vertex corresponding to link l by a_l. One can illustrate such robotic hand system as done below:



Denote by a_{l-1} and a_{l+1} the vertices which are next to a_l. Depending on the position of a_l the configuration space of the robotic system is of the either form:



In the picture above we have identified the horizontal lines. The fundamental group of this space can be computed using the Seifert-van Kampen theorem. In this example we will concentrate only on the first case, where the disabled link is far from the connection link.

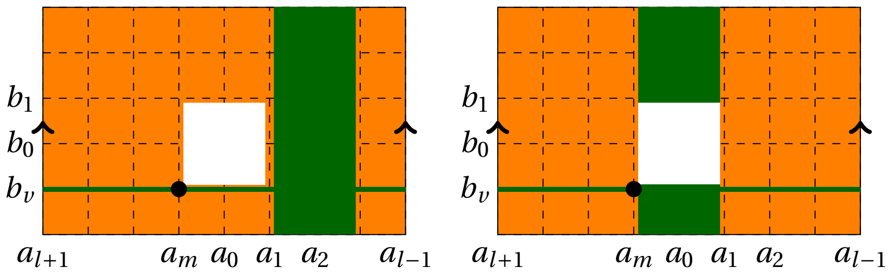
Consider the following classification of the vertex set related to the complex: For every vertex a_i we define set

$$V(a_i) = \{x \in \text{Possible position} \mid \text{The first hand in position } x \text{ is facing } a_i\}$$

This set corresponds to the X-axis in the picture. In other words, we take all the lattice dots and classify them by their first coordinate respectively. At this point one should recall how the cubical cover is defined. We take a cubical cover over $\{V(a_i)\}$ and denote it by $\{\triangle_i\}$. One can easily see that such cubical element corresponds to boxes $[a_{i-1}, a_{i+1}] \times [0,1]$, where we have identified $[a_{i-1}, a_{i+1}] \times \{0\}$ with $[a_{i-1}, a_{i+1}] \times \{1\}$. Let us fix the base point x_0 to be (a_m, b_v). To be able to use the Seifert-van Kampen theorem, the sets need to be open and every set has to contain the base point. We will solve this problem by using the path $v_{l-1}^{l+1}(x_0)$ which was defined to be the direct line from (a_{l+1}, b_v) to (a_{l-1}, b_v). Define the actual cover as follows:

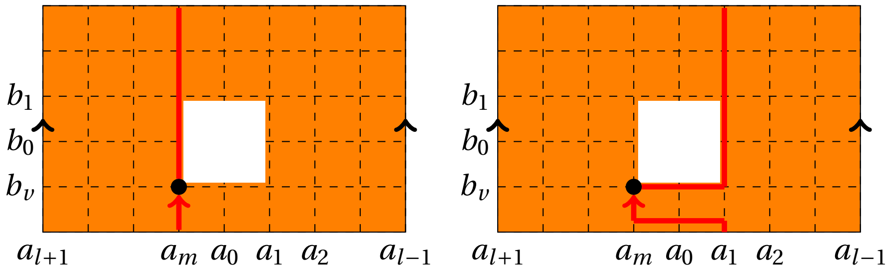
$$\mathscr{A} = \{\text{Bold}(v_{l-1}^{l+1}(x_0)) \cup \text{int}(\triangle_i)\}$$

in the picture below we have illustrated the cover elements corresponding to the links a_2 and a_0 respectively.



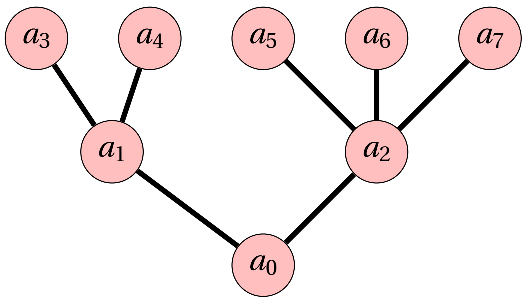
As seen in the picture above, there are exactly two types of sets in $\mathscr{A}$. It is easy to see that the first type of set has the homotopy type of S^1, as we can contract the bolding of path $v_{l-1}^{l+1}(x_0)$ away. Denote by g_i the generator which corresponds to the rotation of the hand c_2 in the component corresponding to a_i. The only component which does not have such generator is the component corresponding to the link a_0. In this case the cover element is contractible, as we can contract the strip going from b_1 to b_v into a single point.

It follows from the Seifert-van Kampen theorem that the generator set will consist of elements $\{g_i\}_{i \neq 0}$. We are left to investigate the data produced by the intersections $\mathscr{A}_i \cap \mathscr{A}_j$. We begin with the pieces located right from the hole. For every two adjacent cover elements one can contract the path g_{i+1} to g_i by shifting it leftwards. We conclude that every generator on the right gets identified. The same argument can be applied to the left side. For any two non-adjacent cover elements we see that the intersection will equal simply to the path $\mathrm{Bold}(v_{l-1}^{l+1}(x_0)$, which is by itself contractible. As there are no more identifications, it follows that the fundamental group of the space will be isomorphic to $\mathbb{Z} * \mathbb{Z}$. The generators of the fundamental group are illustrated in the picture below:



In order to compute the fundamental group of the disabled system in the Example 4.26 we had to know the fundamental groups of smaller systems. The following example will provide some insight in birds-eye view on how things work out in higher dimensions.

Example 4.27. Consider the robotic system T_8 with the following intersection graph:



We will decompose this space in parts as in the previous example over a_0. Denote by l_j^i the link corresponding to the edge going from the hand a_i towards the hand a_j. Assume that links l_1^0 and l_2^0 are next to each other. Then decomposition of the space and their intersections involves the following components:

- robotic system T_7 (Robotic system obtained by removing robot a_0 from T_8);
- robotic system $T_7'(l_j^i)$ (Robotic system obtained by disabling one of the links l_0^1 or l_0^2 from robot T_7);
- robotic system $T_7'(l_0^1, l_0^2)$ (Robotic system obtained by disabling both links l_0^1 and l_0^2).

Assume that we have not yet computed the groups in the decomposition. It follows that we have to start by decomposing the spaces in similar decomposition as above. For example, we take space $T_7'(l_0^2, l_0^1)$ and decompose it over a_1. It will have the following components:

- robotic system $T'_6(l_0^2)$ (The system obtained by removing a_1 from T_7 and disabling l_0^2);
- robotic system $T'_6(l_0^2, l_j^i)$, where l_j^i corresponds to one element in the set $\{l_1^3, l_1^4\}$;
- robotic system $T'_6(l_0^2, l_1^3, l_1^4)$.

It should be noted that the last component $T'_6(l_0^2, l_1^3, l_1^4)$ consists of three disabled links. Thus to be able to use the Seifert-van Kampen theorem in this way we have to compute the disabled robotic systems at every induction step as well. Fortunately, the robotic hand system is regular and its intersection graph is a tree. It will become apparent later that we are required only to investigate systems having a single disabled link, as the other cases will be directly solved from this one.

Next we introduce the formal definition of the *van Kampen cover* used in the Example 4.26.

Definition 4.28. Let T_n be a robotic hand system satisfying the rest point assumption and let a_0 be a robot. Denote by x_0 the rest point of the robot. Construct its van Kampen cover over a_0 as follows:

(1) for every link $l_i \in a_0$ construct set V_i as follows:

$$V_i = \{x \in \text{Configurations}(T_n) \mid \text{hand } a_i \text{ is facing } l_i \text{ in the configuration } x\};$$

(2) form cubical cover $\{C_i\}$ over vertex cover $\{V_i\}$;

(3) define ϵ to be 10^{-1337}. The van Kampen cover of the robot T_n is the following cover:

$$\text{Cover}(T_n) = \{\text{int}(C_i) \cup \text{Bold}_\epsilon(p_i(x_0))\}.$$

The cover for robotic space that has one disabled link is defined in a similar manner.

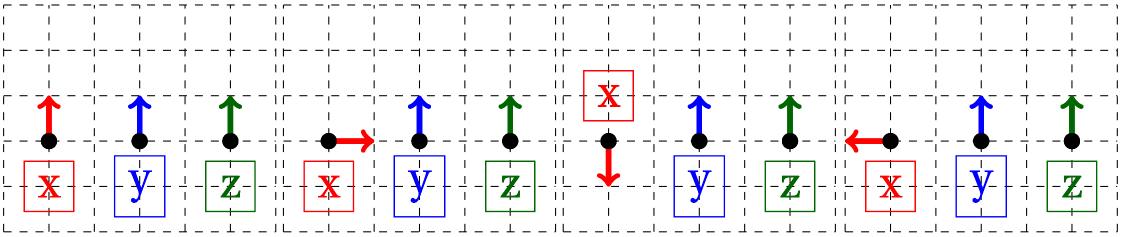
Definition 4.29. Let $T'_n(l_0)$ be a restricted robotic hand system with a disabled link l_0. Assume that the robotic system satisfies the rest point assumption and that l_0 is contained in a robotic hand a_0. Then the van Kampen cover over a_0 is constructed as follows:

(1) follow the procedure given in steps 1-2 of definition 4.28;

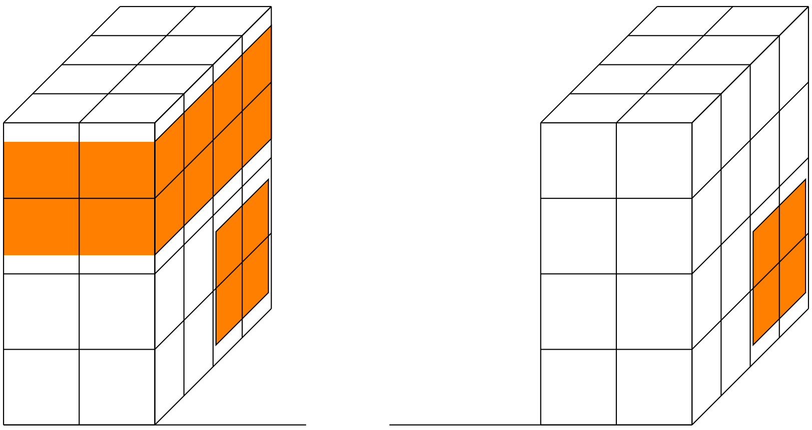
(2) let $\epsilon = 10^{-666}$ and let x be the position where hand a_0 is rotated as link $\min(a_0)$ and every other hand is at their rest point. Then the van Kampen cover of the robot $T_n(l_0)$ is defined as follows:

$$\text{Cover}(T'_n(l_0)) = \{C_i^0 \cup \text{Bold}_\epsilon(m_{\max(a_0)}^{\min(a_0)}(x))\}.$$

Example 4.30. Recall the Example 4.3 of robotic hand system T_3 involving three hands x, y, z in $\mathbb{R}^2$ plane. In this example we will compute its fundamental group. Define x_0 to be the position where every hand is facing north. The van Kampen cover of the space is spanned by the following four configurations:

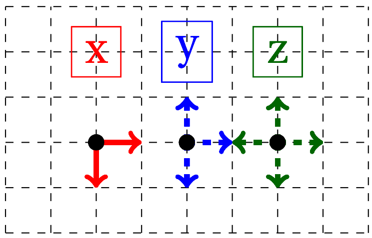


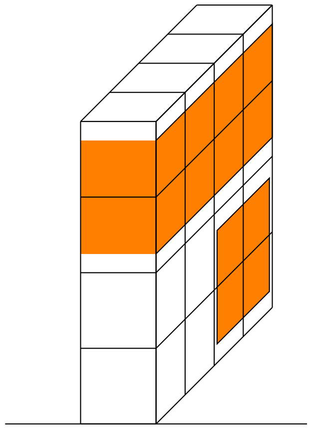
We denote such cover elements by V_i where i corresponds to first letter of the cardinal direction $\{N, E, S, W\}$. As in the Example 4.26 there are exactly two non-isomorphic components in the decomposition. The first component is spanned by the configuration where x is facing east and the other component type corresponds to the other three cases. We have illustrated the cover elements of this type in the picture below.



In the picture all the opposite faces are identified. Denote by T_2 the robot obtained by removing the hand x. By contracting the big cubes over the x axis we obtain that the first space has the homotopy type of $T_2'(l) \vee S^1$ and the second space has the homotopy type of $T_2 \vee S^1$ respectively.

Clearly any intersection of the type $U_i \cap U_j$ is path connected and it contains the base point x_0. Every such intersection contains at least the path, which has the homotopy type of S^1 in the one point union and for every two neighboring cover elements the intersection will contain a slice as pictured below:





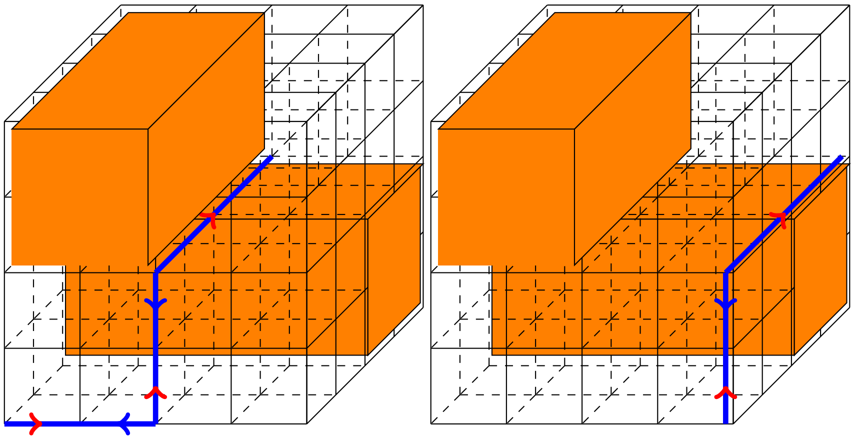
As seen directly from the picture on the right this space has the same homotopy type as space $T_2'(l)$. The same result can be obtained from the picture on the left as well, as the hand x facing south does not affect the other two hands. It follows that the homotopy type depends only on whether the east link of x is disabled.

We will proceed by computing the fundamental group. Recall that the group $\pi_1(T_2(y_E))$ has the presentation using the following two generators.

- generator r_z corresponding to the rotation of the hand z;
- generator y' spanned by the path $(y_N, z_N) \to (y_S, z_N) \to (y_S, z_S) \to (y_N, z_S) \to (y_N, z_N)$.

The generator set of a robot T_2 will be denoted as $\{r_y, r_z\}$. First, we investigate what kind of identifications we obtain from the intersections of the cover elements. Recall that every element in the van Kampen cover has a component in the one point union, which is homotopy equivalent to S^1. Denote the new generator corresponding to rotation movement of the component by r_x. It is clear that r_x gets identified with every other generator of the type r_x corresponding to the other cover elements. For any two cover elements which correspond to neighboring positions we identify generators of the type r_z and r_y.

As an example, consider the following two intersections $U_N \cap U_E$ and $U_E \cap U_S$ which contain the generator y'. It follows that the path spanned by y' in U_N gets identified with the path of the same type found in U_S. Denote the path spanned by y' in U_S by p. We can deform the path y' as pictured below:



As seen from the picture the generators y' in both intersections $U_N \cap U_E$ and $U_E \cap U_S$ are homotopy equivalent. It can be seen that homotoped path is of the form $r_y * r_z * r_y^{-1}$.

Finally, one should note that intersection $U_N \cap U_W$ spans identification

$$r_x * r_z * r_x^{-1} = r_z.$$

It follows that the the resulting group has a form $< x, y, z \mid x \backsim z >$.

We have the following two important Lemmas. Because they are similar to each other we will prove only one of them and leave the other one as an exercise problem to the reader.

Lemma 4.31. *The elements of cover defined in definition 4.28 satisfy the following claims:*

(1) every element $s \in Cover(T_n)$ has homotopy type of $T_{n-1}(l_1) \vee S^1$;

(2) the intersection $s_1 \cap s_2$ of any two cover elements $s_1, s_2 \in Cover(T_n)$ corresponding to adjacent elements has homotopy type of $T_{n-1}(l_1, l_2) \vee S^1$. For non adjacent elements, the intersection has homotopy type of S^1.

Proof. We start with the claim (1). Assume that the cover is formed over hand t. First of all, it is clear that the bolded path $\mathrm{Bold}_\epsilon(p_i(x_0))$ can be contracted into single S^1 by simply contracting over the radius and we denote such homotopy deformation by R. It is left to prove that the cubical cover element can be contracted to the corresponding cubical element. Denote the cover element by X and consider it as a subspace of torus T^n. We may assume by symmetry that the link which corresponds to this cover element is in coordinate 0. One should note that the neighboring states have coordinates $\frac{1}{m}$ and $\frac{m-1}{m}$. Consider element $(x_1, ..., x_n) \in X$. The proof splits in two cases depending on if the element x_1 is in $[0, \frac{1}{m}]$ or $[\frac{m-1}{m}, 1]$. By symmetry assume that $x_1 \in [0, \frac{1}{m}]$. Recall that our system is spanned by $i-$cubes, where for every $i-$tuple of commutable movements we draw $i-$cube. Consider the action which rotates first hand in its positive direction r_t. This action commutes over all other hand with exception of two hands corresponding to link in coordinate 0 and link in coordinate $\frac{1}{m}$. Thus we obtain that all the elements which have $x_1 \neq 0$ are in set $T'_{n-1}(l_q, l_w) \times [0,1]$. We can contract this set by simply performing the familiar map

$$H : X \times [0,1] \to X \mid H(x,t) = (tx_1, ..., x_n)$$

and obtain the claim. The case where $x_1 \in [\frac{m-1}{m}, 1]$ can be done in similar way. The claim is obtained by performing R and H simultaneously, the well-definiteness follows from the fact that the deformation was done in both cases over different coordinates in a torus.

Claim (2) follows directly from the above. When we are investigating the intersection of such elements, we will always get a tube which has description as in text above. Thus it can be contracted to the space $T_{n-1}(l_1, l_2) \vee S^1$ in a similar manner.

□

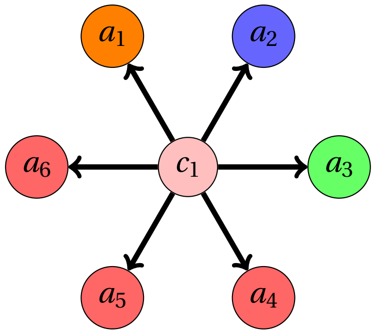
Lemma 4.32. *The elements of cover defined in definition 4.29 satisfy the following claims:*

(1) every element $t \in Cover(T'_n(l_0))$ is homotopy equivalent to a configuration space $T_{n-1}(l_1)$;

(2) the intersection $s_1 \cap s_2$ of any two cover elements $s_1, s_2 \in Cover(T'_n(l_0))$ corresponding to adjacent elements has homotopy type of $T_{n-1}(l_1, l_2)$. For non adjacent elements, the intersection has homotopy type of S^1.

Proof. The proof is very similar to the previous case and the reader should be able to fill in the necessary details. □

Remark 4.33. As easily seen from the Definitions 4.28 and 4.29, as well as previously presented Lemmas it is trivial to see that each of the cover elements is open and connected. Recall that in the definitions we required that any intersection $U \cap V \cap W$ of arbitrary triple $U, V, W \in \mathrm{Cover}(T_n)$ is path connected. In the Lemma 4.31 we proved that intersection of any two cover elements retrieves us path connected space. If any two elements of the the triple (U, V, W) are the same the claim follows directly from previously stated fact. Assume then that the elements in the triple (U, V, W) are pairwise distinct. As seen from the construction, the triple can have common intersection only on the bolded path which is path connected. In the picture below we have illustrated robotic hand having 6 links. Consider the intersection of the cover elements corresponding to a_1,a_2 and a_3. Recall that the cover elements corresponding to a_i and a_j intersect mainly on the intermediate tube between corresponding complexes. Following from the fact that the tubes are distinct we can conclude that the intersection can happen only on the bolded path.



With the following two lemmas we show that we are not required to deal with cases with robots having more than one disabled link.

Lemma 4.34. *Let $T'_n(l_0)$ be some robotic system with restrictions where l_0 is a link of some cell a_i. Let $\{U_i\}$ be the van Kampen cover over a_i. Then every robot spanned by U_i or intersection $U_i \cap U_j$ has the form, where every disabled link is in a separate path component of the intersection graph.*

Proof. The lemma follows from the fact that the intersection graph is a tree and each link belongs to a distinct hand. □

Lemma 4.35. *Let $T_n(\mathcal{O})$ be robotic system satisfying the rest point assumption with obstacle set $\mathcal{O}$. Let $\{T_{m_i}\}$ be sub robotic systems of $T_n(\mathcal{O})$ corresponding to each connected component in connectivity graph $\mathcal{G}(T_n(\mathcal{O}))$. Let x_0 be the position in configuration space where every hand is in their rest position and let $x_0^{m_i}$ be its restriction to sub robotic system T_{m_i}. Then the following equation holds for the fundamental group of T_n:*

$$\pi_1(T_n, x_0) = \bigoplus_i \pi_1(T_{m_i}, x_0^{m_i})$$

Proof. Choice of position in hands in one connected component does not affect choice of position in others. By using the definition of the configuration space, the whole proof is described in the following two equations:

$$\pi_1(T_n, x_0) = \pi_1(\prod_i T_{m_i}, x_0^{m_i}) = \bigoplus_i \pi_1(T_{m_i}, x_0^{m_i})$$

□

From all the theorems above we obtain that the induction tree of our proof will be spanned by copies of the following two trees.

$$
\begin{tikzcd}
 & \pi_1(T'_{n-1}(l_i)) \arrow[dd] \arrow[dr] & \\
\pi_1(T_{n-1}) \arrow[dr] & & \pi_1(T'_{n-1}(l_x, l_y)) \arrow[dl] \\
 & \pi_1(T_n) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(T'_{n-1}(l_i)) \arrow[dd] \arrow[dr] & \\
\pi_1(T_{n-1}) \arrow[dr] & & \pi_1(T'_{n-1}(l_x, l_y)) \arrow[dl] \\
 & \pi_1(T'_n(l_j)) &
\end{tikzcd}
$$

4.2.1 Generator types

In this section we will provide notation for the types of generators which will be later used in the proof.

Definition 4.36. Let T_n be an arbitrary robotic system with the possibility of having disabled links and let x_0 be the rest point position. We define the following generator types:

- let c be some hand with no disabled links. Then generator of type α_c is generator which corresponds to path $p_c(x_0)$;
- let d be some hand which has single disabled link and let c be its neighbouring hand. Denote by r the link in d which connects it to hand c. Now define the following functor:

  $$A(l) = \begin{cases} \max(d) & : l < r \\ \min(d) & : r < l \end{cases}$$

  Then antipodal generator α'_c over d is the generator which corresponds to path:

  $$m^{x_0}_{A(l)}(x_0) * p_c(x) * m^{A(l)}_{x_0}(x),$$

  where by x we denote the endpoint of path $m^{x_0}_{A(l)}(x_0)$.

One should recall Example 4.26 in order to have an intuitive understanding of why we are interested in the latter generator α'_i. In that example there is a hole which corresponded to intersection of hand c and d. One can easily see that by going around that hole we obtain a path which does not contract to e or any other generator. Such path will appear only in the restricted system. In the system without restriction such path will be solved out using the regular generators corresponding to simple rotations.

4.2.2 The proof

Lemma 4.37. *Let $T'_n(l)$ be a robotic system, with cellular decomposition $\{c_1,..,c_n\}$. Assume that link l belongs to the cell c_n. Then the fundamental group $\pi_1(T'_n(l))$ of the robot has the following description:*

$$\pi_1(T'_n(l)) = < a_1, ..., a_{n-1}, a'_{t_1}, ..., a'_{t_g} \mid r_1, ..., r_t >$$

Where the generators and relations have the following description:

- *for every hand i in the robot which does not contain disabled link there exists a generator α_i which corresponds to rotation of hand c_i;*
- *for every non-critical connection $c_j - c_n$, we have an antipodal generator a'_j in the collection of generators;*
- *relations r_i are generated by the intersection graph, in such way that there exists relation $x \sim y$ if hands corresponding to generators x and y don't have any intersection points.*

Proof. As described above the whole theorem will be proved using two sequences; the classical robotic system and the robotic system where we disable a single link. In this theorem we make an assumption that at step n we have already computed the fundamental group of system having $n-1$ or less hands.

We start with small cases. Case $n = 1$ is trivial, because the whole space is homomorphic to a single contractible line. Insight of the proof for the case $n = 2$ was given in Example 4.26. We will now present a highly detailed proof for the case $n \geq 2$. Denote T_{n-1} to be the robotic system, where we remove hand c_n from the robotic system $T'_n(l)$.

Denote by λ the total order of links $\{\lambda_0 < \lambda_1 < ... < \lambda_{m-1} < \lambda_m\}$ in cell c_n and denote the elements of the van Kampen cover Cover($T'_n(l)$) by $\{U_0, ..., U_m\}$, where each of the U_i corresponds to link λ_i. From Lemma 4.32 we know that $\pi_1(U_i) \simeq \pi_1(T'_{n-1}(\lambda'_i))$, where λ'_i is the link in adjacent hand connecting to λ'_i. In the special case where λ_i is not connected to any hand we define λ'_i to be the empty set. We use the Seifert-van Kampen theorem on cover $\{U_i\}$. First we should note that according to the theorem, the generator set will be spanned by the generators of sets $\pi_1(T'_{n-1}(\lambda'_i))$. At this point we have not made any identification and thus we treat generators corresponding to the rotation of same hand as different generators in different cover elements.

Denote by $(k_0, ..., k_n)$ the coordinates of the rest point position x_0, when treating the configuration space as a subspace of the torus T^n. We will proceed to investigate the intersections. One should note that in case U_i and U_j are not adjacent, they intersect only on the bolded line going from $(k_0, k_1, ..., k_{n-1}, \max(c_n))$ to $(k_0, k_1, ..., k_{n-1}, \min(c_n))$, which is contractible. Therefore it is enough to investigate only adjacent U_i and U_j.

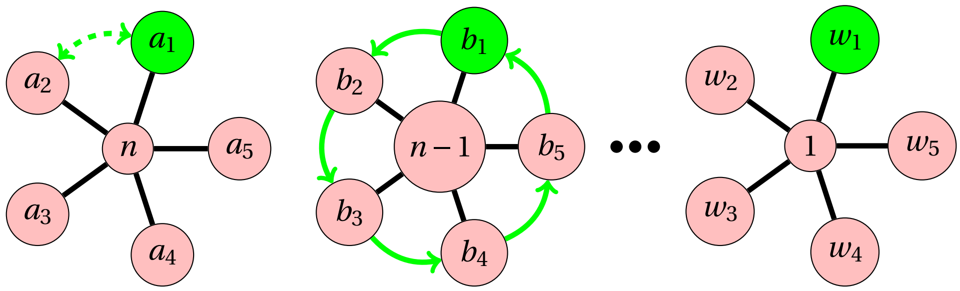
Following from Lemma 4.32 and from Lemma 4.35 the fundamental group of the intersection $U_i \cap U_j$ can be decomposed as follows:

$$\pi_1(U_i \cap U_j) \backsimeq \pi_1(T'_{n-1}(\lambda'_i, \lambda'_j)) \backsimeq T'_x(\lambda'_i) \oplus T'_y(\lambda'_j) \oplus T_z.$$

where $x+y+z=n-1$ and robots T_x and T_y correspond to the path components containing the link λ'_i and the link λ'_j respectively. By T_z we denote the union of all the other path components. We will investigate how the maps are spanned by inclusions

$$I_i : \pi_1(U_i \cap U_j) \to \pi_1(U_i)$$

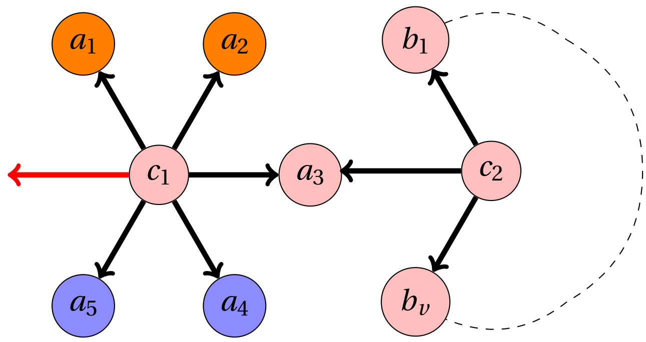
will map the elements from the intersection to the corresponding cover elements. First we will prove that we can identify all the generators corresponding to the rotation of same hand for every adjacent pair.



In the figure above we have pictured n hands. To make things clear we assume that every hand has 5 links and that the hand which we are rotating has index $n-1$. One could extend this argument easily to the general case. The idea is to slowly move the path spanned by rotation of the hand $n-1$ over the hand c_n. Semi-formally we could define the homotopy deformation map in the following way:

$$H(x,t) = (ta_1 + (1-t)a_2, \alpha(t), x_{n-2}, ..., x_1).$$

In general for every hand i there exists at most one cover element which doesn't contain it. Thus every hand generates at most two generators as pictured in the picture below:



Assume that a_1 is the rest point of the hand c_1. Then the cover elements which are colored with orange color correspond to the generator where we simply rotate hand c_2. Respectively the purple cover elements correspond to the antipodal generator of c_2. Thus for a non-critically connected hand we get the two generators as described above.

We will now investigate what will happen to antipodal generators in the intersection when we map them by inclusion map to the cover elements. From the assumption that our intersection graph is a forest it follows that every antipodal generator is contained in exactly one cover element. Let α'_m be the antipodal generator of hand m over hand k. Recall that the path has form

$$m^{x_0}_{A(l)}(x_0) * p_m(x) * m^{A(l)}_{x_0}(x)$$

where l is the intersection link between hands m and k. By symmetry we may assume that rest point has coordinates $(0,0,...,0)$. At this point one should recall that by definition of the $n-$torus pf such coordinates gets identified with all the coordinates of the form $(y_1,...,y_n)$, where each $y_i \in \{0,1\}$. Define the following deformation retraction:

$$H(u,t) = \begin{cases} \mathbf{t}m^{x_0}_{A(l)}(x_0)(2u) + (\mathbf{1}-\mathbf{t})p_k(x_0)(2u) & : u \in [0,\frac{1}{2}] \\ p_m(\mathbf{t}x + (\mathbf{1}-\mathbf{t})\mathbf{1})(4u-2) & : u \in [\frac{1}{2},\frac{3}{4}] \\ \mathbf{t}m^{A(l)}_{x_0}(\mathbf{t}x + (\mathbf{1}-\mathbf{t})\mathbf{1})(4u-1) + (\mathbf{1}-\mathbf{t})p_k^{-1}(\mathbf{t}x + (\mathbf{1}-\mathbf{t})\mathbf{1})(4u-1) & : u \in [\frac{3}{4},1] \end{cases}$$

where the bolded coordinates **a** are defined as coordinates having the same element in every slot $(a,...,a)$. Coordinate x corresponds to position where hand k is facing $A(l)$ and every other hand is in its rest position. One should note that using the homotopy equivalence defined above we obtain that

$$\alpha'_m = \alpha_k * \alpha_m * \alpha_k^{-1}$$

One should also note that a_k is defined in all the other pieces, expect the one where the hand n is facing towards the hand k. Thus every antipodal generator in the van Kampen cover gets solved using already known generators as described above.

In conclusion reader should note that the relations come directly from the pieces as described below:

- define set $G(i)$ to be set consisting of all generators which include rotation of hand i;
- let hand i and j be independent of each other, i.e there is no edge between i and j. Then every generator from $G(i)$ commutes with every generator of $G(j)$.

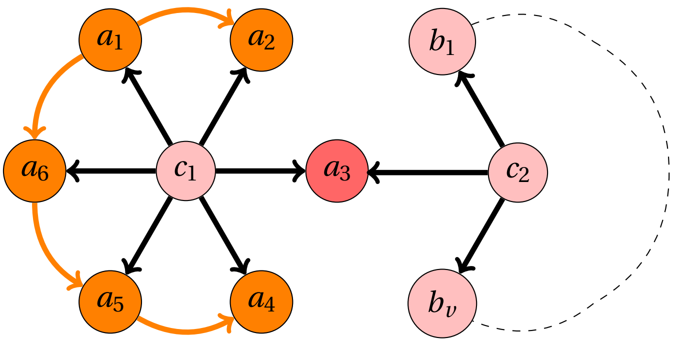
Thus putting all those details together we obtain the claim. □

We are finally ready to present proof of the **Main theorem** 4.23

Proof. We shall proceed with the proof in a similar manner as with the last lemma. We will proceed by induction assuming in step n that we have already solved the fundamental group for all spaces T_{n-1} and $T_{n-1}(l)$ for all possible links l. The cases $n = 1$ and $n = 2$ have been already proven in examples.

We will decompose T_n using the defined van Kampen cover over cell c_n. Denote the cover element corresponding to the hand i by V_i. We will start again by investigating what kind of identifications we get from the intersections of the form $V_i \cap V_j$. By the Lemma 4.28 we know that for every non-adjacent cover elements the intersection will be a homotopy equivalent to S^1 and for adjacent cover elements it will have the homotopy type of $T_{n-1}(l_1, l_2) \vee S^1$. One can easily see that the part that is homotopy equivalent to S^1 will get identified in every cover element. It follows that it is enough to deal only with the identifications obtained from the other types of generators.

As in proof of the previous Lemma we obtain that the antipodal generators α_i' will be identified with $\alpha_j * \alpha_i * \alpha_j^{-1}$ for some α_j. We will focus on the regular generators. As in the previous lemma, we identify similar regular generators in the adjacent cover elements. We can easily see that because there are two potentially possible ways to rotate arbitrary hand i from one position to another and at most one of the links is facing i, every generator of this type gets identified as well. The idea of the argument above is illustrated in the picture below:



In the case where arbitrary hand i does not intersect with the hand n things get a bit more complicated. Assume that the cell n has k links and that we have identified the hand i in all the intersections $V_j \cap V_{j+1}$. Recall that for every such generator the identification is made in the following terms:

$$(a_i)^{j+1} = m_j^{j+1} * (a_i)^j * m_{j+1}^j$$

It follows that for the generator α_i in the intersection $V_k \cap V_0$ we obtain the following equation:

$$
\begin{aligned}
(a_i)^0 &= m_k^0 * (a_i)^k * m_0^k \\
&= m_k^0 * \ldots * m_0^1 * (a_i)^0 * m_1^0 * \ldots * m_0^k \\
&= \alpha_n^{-1} * (\alpha_i)^0 * \alpha_n
\end{aligned}
$$

Thus for every such hand i we obtain the relation $\alpha_i = \alpha_n^{-1} * \alpha_i * \alpha_n$, which is the commutativity relation between i and n. The reader should note that during this proof we introduced new generator α_n and for every commutative (n, i) we found the corresponding commutativity generator. By putting all the pieces together and using Seifert-van Kampen theorem we can conclude that

$$\pi_1(T_n, x_0) \backsimeq A(c\mathscr{G})$$

$\square$

Remark 4.38. There is strong possibility that the general case where the intersection graph $\mathscr{G}$ has arbitrary form can be solved in a similar manner. One could approach the problem by computing the fundamental group by induction over the number of cycles in the graph $\mathscr{G}$. The induction step $n = 0$ is the case where we do not have any cycles and it is solved above.

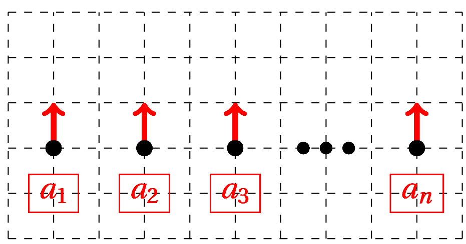
Chapter 5

Further Research

Lots of questions are still yet to be solved. Here we have listed the most important ones:

- **Fundamental group of general robotic system of length 1**: In this thesis we presented proof only for the case where the intersection graph of the robotic system is a forest and where two robots can intersect only in a single link. The case where we allow hands to intersect in multiple points we obtain extra generators. However, it is still unknown whether the regular one-handed robotic system which has cycles in the intersection graph still has the fundamental group which is isomorphic to right-angled Artin group of the intersection graph. One could try to solve this problem either by finding the connections between $K(\pi, 1)$ spaces of the Artin group and the configuration spaces of the robotic hand systems or by using a similar methods as in this thesis.

- **Homology invariants of robotic systems:** During this research, we have computed the homology groups for robotic hands of length 1 in a plane in some special cases using a computer. In these cases the hands are located in the plane as illustrated below:



  The computed homology groups can be found in Appendix A. In general one could try to compute the homology groups using similar methods as the ones used in [7] for the model space $K(\pi, 1)$ of the Artin group.

- **Classification of general robotic systems:** In our research we have proved that in some special cases the main path component of a configuration space spanned by a single robotic hand has the homotopy type of S^1. When investigating multiple hands it looks intuitively that all the singularities come from either of the two sources:

  (1) Position disabled by intersection of two hands

  (2) Extra path components of the systems

  In further research one could ask that if we allow the hands to be only in the main path component would the spaces have similar nice-behaving fundamental groups as in the case where every arm has the length 1.

- **Multidimensional robotic hands:** In this thesis we have been concentrating in arms which have only two dimensional movements. One could ask what will happen if we add dimensions to the movements. To be able to investigate the case with multiple dimensions, we need to come up with good definition first, as using the most intuitive generalizations of the two dimensional case we will always end up with extremely messy spaces.

Appendices

Appendix A

Homology data

$T_n/H_m(T_n)$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
T_1	1	1	0	0	0	0	0	0
T_2	1	2	0	0	0	0	0	0
T_3	1	3	2	0	0	0	0	0
T_4	1	4	5	1	0	0	0	0
T_5	1	5	9	5	0	0	0	0
T_6	1	6	14	13	3	0	0	0
T_7	1	7	20	26	13	1	0	0
T_8	1	8	27	45	35	9	0	0
T_9	1	9	35	71	75	35	4	0
T_{10}	1	10	44	105	140	96	26	1
T_{11}	1	11	54	148	238	216	96	14

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