Depletion of Intense Fields

S. S. Bulanov[1,a)], D. Seipt[2], T. Heinzl[3] and M. Marklund[4]

[1]*Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
[2]*Helmholtz-Institut Jena, Frobelstieg 3, 07743 Jena, Germany*
[3]*School of Computing, Electronics and Mathematics, Plymouth University, Plymouth PL4 8AA, UK*
[4]*Department of Physics, Chalmers University of Technology, SE-41296 Gothenburg, Sweden*

[a)]Corresponding author: sbulanov@lbl.gov

Abstract. The problem of backreaction of quantum processes on the properties of the background field still remains on the list of outstanding questions of ~~the~~ high intensity particle physics. Usually ~~the~~ photon emission by an electron or positron, photon decay into electron-positron pair s in strong electromagnetic fields, or electron positron pair production by such fields are described in the framework of the external field approximation. It is assumed that the external field has infinite energy and is not affected by these processes. However the above mentioned processes have a multi-photon nature, i.e., they occur with the absorption of a significant number of field photons. As a result an interaction of intense electromagnetic field with either a highly charged electron bunch or a fast growing population of electrons, positrons, and gamma photons (as in the case of an electromagnetic cascade) may lead to a depletion of the field energy, thus making the external field approximation invalid. Taking the multi-photon Compton process as an example, we estimate the threshold of depletion and find it to become significant at field strengths ($a_0 \sim 10^3$) and electron bunch charge of about tens of nC.

Introduction.

Recently there was were a lot of interest in the study of charged particle interactions with ultra-intense electromagnetic (EM) pulses due to the fact that many laser facilities able to deliver such pulses are either being planned, built, or just became operational. These processes are a part of high intensity particle physics, a new branch of physics, born out of quantum electrodynamics (QED) and the theory of strong EM background fields. Typical QED processes, modified by the presence of strong classical fields, demonstrate ~~the~~ effects not encountered in perturbative quantum field theory [1, 2, 3, 4, 5, 6]. Two crucial components necessary for such studies are laser facilities, able to generate EM fields of sufficiently high intensity [2], and particle accelerators, able to produce high energy charged particle beams. The development of compact multi-GeV laser electron accelerators [1, 2, 7, 8] allows for the two components necessary to study the effects of high intensity particle physics to be handled by the same technology.

As we mentioned above the the strong EM field is provided by an ultra-intense laser pulses, the main source for ultra-strong EM fields under laboratory conditions. The laser pulse is usually characterized by the wave vector k, central frequency $\omega = 2\pi/\lambda$ and electric field amplitude E. The interaction of charged particles and photons with strong EM fields in classical and quantum regimes is parametrized in terms of[1] dimensionless amplitude of the EM vector potential, $a_0 = eE/\omega m$, the QED critical field, $E_S = m^2/e$ [9], and two parameters accounting for the energy and momentum of charged particles: $\chi_e^2 = -e^2(F^{\mu\nu}p_\nu)^2/m_e^6$ and $\chi_\gamma^2 = -e^2(F^{\mu\nu}k'_\nu)^2/m_e^6$ [6]. Here, e and m_e are electron charge and mass respectively. The tensor $F_{\mu\nu}$ is the EM field tensor, while p_ν and k'_ν stand for the 4-momenta of electron and photon respectively. The nonlinearity of the particle interaction with EM field in the classical regime is governed by the parameter a_0, which is the energy gain of an electron (in units of its rest energy) over a reduced wavelength, $\lambda\!\!\!^{-} = 1/\omega$, of the field. For $a_0 > 1$ the electron/positron motion becomes relativistic. The quantum nature of the interaction is revealed through the ability to produce new particles. In QED, the field strength, which characterize this ability, is E_S [9]. For example, χ_e is the EM field strength in the electron rest frame in units of E_S. Quantum effects reach their optimal value at $E \approx E_S$ or $\chi_{e,\gamma} \sim 1$.

[1]We set $\hbar = c = 1$ throughout the paper

As it was shown in early papers on QED processes in strong fields, for $a_0 \gg 1$ the interaction of charged particles and photons with strong EM fields is clearly multi-photon, i.e., involves the absorption of a significant number of photons from the background fields. Due the use of the external field approximation this energy loss by background fields is not taken into account. And possible depletion of background field energy is neglected. Though for a single process ti can be quite small, the interaction of a highly charged particle bunch may result in a significant effect. Looking at the results on multi-photon Compton and Breit-Wheeler processes [5, 6, 10], we determine that there is indeed a parameter range, for which depletion of the laser becomes substantial. These processes recently received a lot of attention [11, 12] since their experimental study seems to be a near term goal of some laser facilities [2]. However this attention was drawn mainly to the final states (a frequency shifted photon or electron positron pairs).

In this letter we aim our investigation at the effect the nonlinear Compton scattering on the laser EM field, which is best characterized by the number of laser photons absorbed. The requirement of this number to be small compared to the number of photons in the pulse will allow us to establish a threshold for the validity of the external field approximation and discuss some immediate consequences. They should be taken into account when studying the QED backreaction [13] and EM avalanches[2] [14, 15, 16], since background depletion will significantly alter the energy partitioning of these processes.

Depletion Estimate

Classical Depletion. The external field approximation is valid when the energy absorbed from the field is negligible with respect to the total field energy. This can be reformulated in terms of photon numbers: the number of photons absorbed from the laser, ΔN_A, should be much smaller than the number of photons in the pulse, N_L. Of course the laser field should be macroscopic with number of photons much larger than unity, $N_L \gg 1$. For simplicity we assume that the laser pulse is focused in a volume equal to laser wavelength cubed, $V = \lambda^3$. Then the threshold for depletion can be set as $\Delta N_A = N_L$ ($N_L \approx (2\pi/\alpha)(\lambda\!\!\!^{-2}/\lambda\!\!\!^{-2}_e)a_0^2 \approx 2 \times 10^{14} a_0^2$). Here $\alpha = e^2/4\pi \simeq 1/137$ is the fine structure constant. The number of absorbed photons can be written as $\Delta N_A \simeq (P_{rad}T/\hbar\omega)N_T$, where N_T is the number of electrons in the bunch, P_{rad} is the power radiated by electrons per laser period T, which can be estimated classically by making an analogy with synchrotron radiation [18]. This would allow us to determine the number of absorbed photons from the field, the characteristic energy of an emitted photon, and the angle of emission, which means the full characterization of the processes. In order to carry out the estimate we make a Lorentz transformation to a boosted frame, where the electron is on average at rest. For circularly polarized laser pulse the electron demonstrates circular motion like in a synchrotron, and we can use Larmor's formula,

$$P_{rad} = -(2/3)\,\alpha\dot{u}^2 = (2/3)\,\alpha\,\bar{\omega}^2\,a_0^2(1 + a_0^2)\ . \tag{1}$$

Here u is the electron 4-velocity, $\bar{\omega} = \omega\gamma_e(1 + \beta_e)\,(1 + a_0^2)^{-1/2}$ denotes the laser frequency 'seen' in the average rest frame (ARF) by an electron, which, in the lab, has relativistic parameters β_e and γ_e. In the ARF the radiation is emitted in the plane of electron motion, being perpendicular to the laser axis. In the lab frame this transforms into an emission angle:

$$\tan\theta = 2[(1 + a_0^2)^{1/2} m/p_e^-]/[1 - (a_0 m/p_e^-)^2], \tag{2}$$

which is written in terms of the lightfront momentum [3]. The number of absorbed photons per laser period $\bar{T} = 2\pi/\bar{\omega}$ is then $\Delta N_A = (4\pi/3)\,\alpha\,a_0^2(1 + a_0^2)\,N_T$. As the characteristic frequency of the radiation emitted by an electron in the ARF is $\bar{\omega}_m \approx 0.3\ \bar{\omega}a_0^3$ [21], the number of photons emitted per laser period becomes $\Delta N_E \approx 4\pi\,\alpha a_0 N_T$. Thus, for the electron to emit one high frequency photon, it needs to absorb $s \sim \Delta N_A/\Delta N_E \approx (1/3)\,a_0(1 + a_0^2)$ photons from the EM field, and we obtain the important result that the number s of absorbed photons scales as $s \sim a_0^3$ for large a_0. Equating $\Delta N_A = N_L$, we see that depletion requires

$$a_0^2 N_T \sim 6.5 \times 10^{15}. \tag{3}$$

Thus in the case of an 1 nC electron bunch interaction with a laser pulse, having $a_0 \approx 10^3$, the external field approximation is no longer valid. Also such value of a_0 means the emitted photons energy (ω_m) is of the order of electron energy gain per laser period, and the emission angle (2) significantly deviates from $\sim 1/\gamma_e$. Therefore, one expects

[2] An avalanche is formed when Compton and Breit-Wheeler processes occur subsequently in an EM field of sufficiently high intensity, resulting in an exponential growth of the number of emitted particles.

[3] If ℓ is an arbitrary four-vector its scalar product with the laser momentum can be written as $k\cdot\ell = \omega(\ell^0 - \ell_z) \equiv \omega\ell^-$, which defines the light-front component ℓ^- [19]. For $a_0 \ll 1$ the emission angle is $\sim 1/\gamma_e$, and for $a_0 \gg 1$ it shows a significant emission in the perpendicular direction.

not just significant radiation reaction with ensuing changes of the particle trajectories [22] but also strong recoil of electron momentum, which calls for the quantum description.

Quantum Depletion. In the quantum regime the average number of absorbed photons, s, still follows the classical scaling law $s \sim a_0^3$ for $a_0 \gg 1$ [6], but the classical expression for ΔN_A is no longer valid and should be replaced by a new one, which takes into account the discrete nature of photon emission in multi-photon Compton process: $\Delta N_A = a_0^3 N_T(\lambda/L_C)$. Here, we assume that the electron emits λ/L_C photons per laser wavelength, and L_C is the radiation length of the electron in a strong EM field [6], *i.e.*, or the mean free path of an electron in strong field with respect to radiation. For $\chi_e \gg 1$, the radiation length is $L_C = 0.43\lambda\gamma_e^{1/3}a_0^{-2/3}$ [6]. Thus the threshold of depletion is given by the following expression

$$a_0^{5/3}\gamma_e^{-1/3}N_T \sim 10^{14}\ . \tag{4}$$

Again, an 1nC electron beam interacting with a $a_0 \approx 10^3$ laser mark the applicability threshold for the external field approximation. The threshold field strength, a_0, depends weakly on the initial electron energy ($\sim \gamma_e^{1/5}$). Thus, taking quantum effects into account increases the threshold value of a_0 needed to deplete the laser pulse for a given value of initial electron momentum. Thus we showed that when a sufficiently charged electron bunch collides with an intense laser pulse, depletion of the laser pulse can become significant, with the originally strong EM field turning weak. The required number of electrons is quite typical for an EM avalanche [15], where an intense laser produces a copious amount of high energy photons and subsequently electron-positron pairs.

Multi-Photon Compton Process

Let us calculate the dependence of the multi-photon Compton process probability, $dP^{\gamma,e}/ds$, on the number of absorbed photons in order to have a more accurate estimate of the depletion threshold. The average amount of energy $< \mathcal{E} >$ drawn from the laser field in a single photon emission or pair production is then $< \mathcal{E} >= \hbar\omega < s >$, with the average number of absorbed laser photons given by the expectation value $< s >_{e,\gamma}= Z^{-1}\int ds\, s\,(dP^{e,\gamma}/ds)$ with normalization integral $Z = \int ds\, dP^{e,\gamma}$.

In a monochromatic plane wave laser field, taken to be circularly polarized for simplicity, the variable s is discrete and describes the emission of higher harmonics due to absorption of s laser photons, $e + s\gamma_L \to e' + \gamma$. Energy momentum conservation implies $\chi_e = \chi_{e'} + \chi_\gamma$ with $\chi_e = a_0 k\cdot p/m^2$ and $\chi_\gamma = a_0 k\cdot k'/m^2$. The partial probabilities or rates (i.e. events per time) are [6, 10]

$$P_s^e = \frac{4\alpha\omega s}{1+a_0^2}\int_0^1 \frac{dt}{(1+su_1t)^2}\left\{-J_s^2(s\zeta) + a_0^2\left(1+\frac{s^2u_1^2t^2}{2(1+su_1t)}\right)\left[(\zeta^{-2}(t)-1)J_s^2(s\zeta) + J_s'(s\zeta)^2\right]\right\}, \tag{5}$$

where the various arguments are $u = (k'\cdot k)/(p'\cdot k)$, $t = u/(su_1)$, $u_1 = 2(k\cdot p)/m_*^2$ and $\zeta(t) = 2a_0(m/m_*)[t(1-t)]^{1/2}$, with $m_*^2 = m^2(1+a_0^2)$, the effective mass squared [20]. The total probability for Compton photon emission is the sum over all harmonics: $P^e = \sum_{s=1}^{\infty} P_s^e$. For large values of a_0, the number of harmonics contributing grows like $s \sim a_0^3$ such that the sum may be replaced an integral over s. In this regime the formation length of the Compton process scales like $1/a_0$ [6], and the laser field can be approximated as a locally constant crossed field. Formally, this is achieved by employing the Watson representation of the Bessel functions: $J_s(s\zeta) \approx (2/s)^{1/3}\,\Phi\,(\eta(s,t))$, $\eta(s,t) = (2/s)^{2/3}\left[1-\zeta^2(t)\right]$. Asymptotically, when $a_0 \gg 1$, the probability to absorb s photons from the laser while emitting a single high-frequency photon of momentum k', is found to be

$$\frac{dP^e}{ds} = \frac{4\alpha\omega s}{1+a_0^2}\left(\frac{2}{s}\right)^{2/3}\int_0^1 \frac{dt}{(1+su_1t)^2}\left\{-\Phi^2(\eta) + a_0^2\left(\frac{2}{s}\right)^{2/3}\left(1+\frac{s^2u_1^2t^2}{2(1+su_1t)}\right)\left[\eta\Phi^2(\eta) + \Phi'(\eta)^2\right]\right\}, \tag{6}$$

where $u_1 = 4\omega\gamma_e/m(1+a_0^2)$. Using this result we can calculate the number of electrons needed to deplete the laser pulse, $N_T = 10^{14}a_0^{4/3}\gamma_e^{1/3}/ < s >_e$. For $a_0 = 10^3$ it gives $N_T \approx 10^{11-12}$, which is larger than $N_T \approx 10^{10}$ predicted by the simple estimate (4), but still within reach of EM avalanches [15].

The probabilities dP^e/ds determine the number distribution of absorbed photons. However, we are also interested in the spectral distribution $dP^e/d\chi_\gamma$ of the scattered photon longitudinal momentum (recall $\chi_\gamma \sim k\cdot k'$). The two distributions are obviously related via the chain rule, $dP^e/d\chi_\gamma = (ds(\chi_\gamma)/d\chi_\gamma)(dP^e/ds)$, but we do not know the functional relation $s = s(\chi_\gamma)$. An approximate way to determine the latter is as follows. From the t-integral in (6) we see that the integrand is sharply peaked at $t = 1/2$. Using energy momentum conservation we can solve $t = u/(su_1) = 1/2$ for s with the result

$$s(\chi_\gamma) = \frac{a_0^3}{\chi_e} \frac{\chi_\gamma}{\chi_e - \chi_\gamma} , \tag{7}$$

valid for $a_0 \gg 1$. Thus, when a Compton photon with a certain value of χ_γ is emitted, the number of laser photons drawn from the laser field approximately equals $s(\chi_\gamma)$ as given by (7). We use this result to examine how the most probable emission angle depends on the longitudinal photon momentum in the case of electrons colliding head-on with a laser pulse. To this end, we follow [6] and assume *quasi* momentum conservation, $q + sk = q' + k'$ with quasi momentum $q = p + (m^2 a_0^2/2k \cdot p)k$ and analogously for q', such that $q^2 = q'^2 = m_*^2$. Let us write the scattered photon momentum as $k' = (\omega', \mathbf{k}'_\perp, k'_z)$ where $k'^2 = 0$. We can then find $k'_\perp$ from quasi-momentum conservation. Assuming a head-on collision of electrons and laser ($\mathbf{p}_\perp = 0$) the following answer is obtained:

$$k'^2_\perp = 2sk \cdot k' - \left(\frac{k \cdot k'}{k \cdot p}\right)^2 \left(m_*^2 + 2sk \cdot p\right) . \tag{8}$$

It defines an ellipse in the $(k'_z, k'_\perp)$ plane for given values of γ_e, s and a_0. However, the differential probability $dP/dsdt$ has a sharp maximum at s defined by (7). Thus, only one point from the ellipse (8) effectively contributes to the probability. It can be obtained by rewriting (7) as $s = m^2 a_0^2 (k \cdot k')/[(k \cdot p)(k \cdot p - k \cdot k')]$. Plugging this into (8) yields the surprisingly simple result $k'^2_\perp = m^2 (k \cdot k'/k \cdot p)^2 a_0^2$, when $a_0 \gg 1$, which implies that the transverse scattered momentum, $k'_\perp$, grows linearly with the light-front component[4] $k'^- = k \cdot k'/\omega$. The scattering or emission angle can be defined as $\theta = \tan^{-1} k'_\perp/k'_z$. Denoting by θ_0 the emission angle corresponding to the condition $t(\theta_0) = 1/2$ [12] we get for θ_0 the same expression as for the classical emission angle (2), which is consistent with the results of [12]: if $\gamma_e \gg a_0$, the photons are predominantly emitted in the forward direction, with $\theta_0 \sim a_0/\gamma_e \ll 1$. However, as a_0 increases, significant photon emission takes place in the perpendicular direction. This can be understood classically, in particular in the ARF where $p^- = m_*$ ($a_0 \simeq 2\gamma_e \gg 1$) so that $\theta_0 = \pi/2$ as it must be for circular (synchrotron) motion in the transverse plane. One can show that this emission angle can be obtained classically through $\tan\theta_0 = (\pi_\perp/\pi_z)_{\rm rms}$, the appropriate ratio of the rms values of the classical electron momentum components in the laser field A, $\pi^\mu = p^\mu - eA^\mu + (ep \cdot A - e^2 A^2/2)\, k^\mu/k \cdot p$. In other words the photon is emitted along the direction of the instantaneous electron momentum. However the recoil changes not only the electron total energy, but also the direction of its momentum. This is usually not taken into account in PIC codes with QED modules. The inclusion of this effect might lead to significant changes in final particle distributions, since the cascade development is very sensitive to the changes in electron trajectories.

Conclusion.

We have reconsidered the multi-photon Compton process in strong EM fields from the point of view of laser energy loss due to absorption. If a significant amount of laser energy is absorbed, then the external field approximation, usually utilized for studying QED processes in strong fields, is no longer valid. We found that the threshold of this approximation applicability is reached at sufficiently high values of laser field strength ($a_0 \sim 10^3$), when the laser interacts with electron beam containing a charge of the order of tens of nC. In order to have actual depletion, a lower bound for the number of electrons in a volume of a laser wavelength cubed needs to be exceeded, namely $N_T > 10^{14} \gamma_e^{1/3} a_0^{-5/3}$. Based on the previous results, we expect that this threshold will be overcome in the case of EM avalanches. We have further analyzed the photon emission rates differential in multi-photon number s and discovered that they strongly peak near the value $s = (a_0^3/\chi_e)[\chi_\gamma/(\chi_e - \chi_\gamma)]$. This value of s determines the direction of the photon emission relative to the initial electron momentum direction in terms of an emission or scattering angle, θ_0, which ranges between $\theta_0 \ll 1$ (forward scattering, $a_0 \ll \gamma_e$) and $\theta_0 \approx \pi$ (back scattering, $a_0 \gg \gamma_e$). The latter will dominate in the EM avalanche regime, i.e. in colliding laser pulses or during interactions of laser pulses with solid density foils or plasmas of near-critical density. The classical interpretation of the emission angle θ_0 in terms of averages over trajectories should yield a new test of the PIC codes currently in use.

[4] If ℓ is an arbitrary four-vector its scalar product with the laser momentum can be written as $k \cdot \ell = \omega(\ell^0 - \ell_z) \equiv \omega\ell^-$, which defines the light-front component ℓ^- [19].

ACKNOWLEDGMENTS

We acknowledge support from the Office of Science of the US DOE under Contract No. DE-AC02-05CH11231. MM was supported by the Swedish Research Council grants # 2012-5644 and 2013-4248. MM would like to thank Anton Ilderton for fruitful discussions. The authors acknowledge the hospitality of the Kavli Institute for Theoretical Physics (KITP), where this research was initiated during the Frontiers of Intense Laser Physics Program and so was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

REFERENCES

[1] G. Mourou, T. Tajima, and S. V. Bulanov, Rev. Mod. Phys. **78**, 309 (2006); M. Marklund and P. Shukla, Rev. Mod. Phys. **78**, 591 (2006); Y. I. Salamin, S.X. Hu, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rep. **427**, 41 (2006); F. Ehlotzky, K. Krajewska, and J. Z. Kamiński, Rep. Prog. Phys. **72**, 046401 (2009); A. Di Piazza, C. Muller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. **84**, 1177 (2012).

[2] M. Dunne, Nature Phys. **2**, 2 (2006); *ELI-Extreme Light Infrastructure Science and Technology with Ultra-Intense Lasers WHITEBOOK*, edited by G. A. Mourou, G. Korn, W. Sandner, and J. L. Collier (THOSS Media GmbH, Berlin, 2011).

[3] S. V. Bulanov, *et al.*, Nuclear Instruments and Methods in Physics Research A **660** 31 (2011).

[4] S. S. Bulanov, C. B. Schroeder, E. Esarey, and W. P. Leemans, Phys. Rev. A **87**, 062110 (2013).

[5] H. Reiss, J. Math. Phys. **3**, 59 (1962);

[6] A. I. Nikishov and V. I. Ritus, Sov. Phys. JETP **19**, 529 (1964); *ibid.*, 1191 (1964); V.I. Ritus, J. Sov. Laser Res. 6, 497 (1985).

[7] E. Esarey, C. B. Schroeder, and W. P. Leemans, Rev. Mod. Phys. **81**, 1229 (2009).

[8] W. P. Leemans, *et al.*, Phys. Rev. Lett. **113**, 245002 (2014).

[9] F. Sauter, Z. Phys. **69**, 742 (1931); W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936); J. Schwinger, Phys. Rev. **82**, 664 (1951).

[10] G. Breit and J. A. Wheeler, Phys. Rev. **46**, 1087 (1934); H. R. Reiss, J. Math. Phys. (N. Y.) **3**, 59 (1962); L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964); I. I. Goldman, Sov. Phys. JETP **46**, 1412 (1964); V. P. Yakovlev, Sov. Phys. JETP **22**, 223 (1966).

[11] N. B. Narozhnyi and M. S. Fofanov, J. Exp. Theor. Phys. **83**, 14 (1996); M. Boca and V. Florescu, Phys. Rev. A **80**, 053403 (2009); T. Heinzl, *et al.*, Phys. Lett. B **692**, 250 (2010); T. Heinzl, *et al.*, Phys. Rev. A **81**, 022125 (2010); A. Di Piazza, *et al.*, Phys. Rev. Lett. **105**, 220403 (2010); F. Mackenroth and A. Di Piazza, Phys. Rev. A **83**, 032106 (2011); A. I. Titov, *et al.*, Phys. Rev. Lett. **108**, 240406 (2012); K. Krajewska and J. Z. Kamiński, Phys. Rev. A **85**, 062102 (2012).

[12] C. Harvey, T. Heinzl, and A. Ilderton, Phys. Rev. A **79**, 063407 (2009); see also Ehlotzky *et al.*, Ref. [1].

[13] F. Cooper and E. Mottola, Phys. Rev. D **40**, 456 (1989); Y. Kluger *et al.*, Phys. Rev. Lett. **67**, 2427 (1991) and Phys. Rev. D **45**, 4659 (1992); N. B. Narozhny, *et al.*, Phys. Lett. A **330**, 1 (2004); *ibid.*, JETP Lett. **80**, 382 (2004); *ibid.*, JETP **129**, 14 (2006); S. S. Bulanov, *et al.*, JETP Lett. **80**, 865 (2004); *ibid.*, Phys. Rev. E **71**, 016404 (2005).

[14] A. R. Bell and J. G. Kirk, Phys. Rev. Lett. **101**, 200403 (2008).

[15] A. M. Fedotov,*et al.*, Phys. Rev. Lett. **105**, 080402 (2010); E. N. Nerush, *et al.*, Phys. Rev. Lett. **106** 035001 (2011), B. King, *et al.*, Phys. Rev. A **87**, 042117 (2013); T. Grismayer, et al., Phys. Plasmas **23** 056706 (2016); S. Meuren, *et al.*, Phys. Rev. D **93**, 085028 (2016).

[16] S. S. Bulanov, *et al.*, Phys. Rev. Lett. **105** 220407 (2010).

[17] Y. B. Zeldovich, Sov. Phys. Usp. **18**, 79 (1975).

[18] J. Schwinger, L. L. De Raad, Jr., K. A. Milton and W.-Y. Tsai, *Classical Electrodynamics*, Westview, 1998.

[19] S. J. Brodsky,*et. al.*, Phys. Rept. **301**, 299 (1998); T. Heinzl, Lect. Notes Phys. **572**, 55 (2001);

[20] Brown and Kibble, Ref. [10]; C. Harvey, *et. al.*, Phys. Rev. Lett. **109**, 100402 (2012).

[21] E. Sarachik, G. Schappert, Phys. Rev. D **1**, 2738 (1970); E. Esarey, *et. al.*, Phys. Rev. E **48** 3003 (1993).

[22] A. Di Piazza, Lett. Math. Phys. **83**, 305 (2008); C. Harvey, T. Heinzl and M. Marklund, Phys. Rev. D **84**, 116005 (2011); T. Heinzl *et al.*, Phys. Rev. E **91**, 023207 (2015).

[23] C. P. Ridgers et. al., Journal of Computational Physics **260**, 273 (2014).

[24] A.A. Mironov, N.B. Narozhny, A.M. Fedotov, Phys. Lett. A **378**, 3254 (2014).

[25] A. Gonoskov et al., Phys. Rev. E **92**, 023305 (2015).

[26] M. Jirka, O. Klimo, S. V. Bulanov, *et al.*, Phys. Rev. E **93**, 023207 (2016).