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Network Interdiction through Length-Bounded Critical Disruption Paths: a Bi-Objective Approach

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Abstract

In this paper the Bi-Objective k-Length-Bounded Critical Disruption Path (BO-kLB-CDP) optimization problem is proposed, aimed at maximizing the interdiction effects provided on a network by removing a simple path connecting a given source and destination whose length does not exceed a certain threshold. The BO-kLB-CDP problem extends the Critical Disruption Path (CDP) problem introduced by Granata et al. in [1]. Several real applications of this class of optimization problems arise in the field of security, surveillance, transportation and evacuation operations. In order to overcome some limits of the original CDP problem and increase its suitability for practical purposes, first we consider a length limitation for Critical Disruption Paths. Second, we generalize the concept of network interdiction considered in the CDP: beside minimizing the cardinality of the maximal connected component after the removal of the CDP, now we are also interested in maximizing the number of connected components in the residual graph. A Mixed Integer Programming formulation for the BO-kLB-CDP problem is therefore proposed and discussed, presenting the results of a multiple objective analysis performed through a computational experience on a large set of instances.

Keywords: Network interdiction, Multi-Objective, Mixed-Integer Linear Programming, Critical Disruption Path, Connected Components

1 Introduction

Many real-world systems can accurately be represented as networks. In recent years, the goal is also to predict and evaluate the behavior of the system, via measuring the rules governing individual vertices, edges, or associated substructures. Network interdiction models explore techniques to inhibit network operations. The problem of removing a node or a set of nodes has been broadly studied by [2] and [3]. In this paper the Bi-Objective k-Length-Bounded Critical Disruption Path (BO-kLB-CDP) optimization problem is proposed, aimed at maximizing the interdiction effects provided on a network by removing a simple path connecting a given source and destination whose length does not exceed a given threshold k.
The BO-kLB-CDP problem extends the Critical Disruption Path (CDP) concept introduced by Granata et al. in [1], in which, given source and destination nodes on a connected graph, any simple path can be selected whose removal gives rise to the most severe among the possible disconnection damages on the considered graph, measured by minimizing the maximal cardinality of the largest connected component. Although many real applications of this class of optimization problems arise in the field of security, surveillance, transportation and evacuation operations, the solutions provided by the CDP as formulated in [1] suffer from two main drawbacks. First, being the original CDP problem closely connected to the longest path problem and the Hamiltonian path problem, the mean length of the selected path in the optimal solutions of the CDP problem is by far too large in general to be considered for real applications purposes. Second, minimizing the cardinality of the largest connected component does not always lead to the maximum possible number of isolated parts in the network, which can be an interesting measure of the interdiction damages after the removal of a path in the network. Therefore, the BO-kLB-CPD problem introduced in this paper is a compound of these two necessities: on one hand we consider the problem of imposing an upper bound on the maximum length of a Critical Disruption Path, based on the size of the considered network, in order to increase the practical suitability within interdiction operations. On the other hand, as a measure of network fragmentation, we take into account the number of connected components arising in the residual graph after the disruption beside the cardinality of the largest connected component. The remainder of the paper is organized as follows. In Section 2, the Bi-Objective k-Length-Bounded Critical Disruption Path is formally introduced, together

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with the notation adopted in this paper, and a Bi-Objective Mixed Integer Programming formulation is presented and discussed. In Section 3, we propose a wide set of computational experiments, based on a large set of instances solved at the optimum by means of a state-of-art commercial solver. An analysis of the results of the experiments is provided, then proposals for further lines of research conclude the paper.

2 Notation and Problem Definition

Let $\mathcal{G} = (V, E, s, t)$ be a symmetric directed graph with vertex and edge sets V and E respectively and two special vertices: a *source* $s \in V$ and a *destination* $t \in V$. Let $l_{ij} \geq 0$ be the length of edge $(i, j) \in E$. We denote the number of vertices (or nodes) in the graph by n while m is the number of edges. In order to formally introduce the Bi-Objective k-Length-Bounded Critical Disruption Path (BO-kLB-CDP), we start from introducing some basic definitions as follows.

Definition 2.1 A *Component-Size-optimal k-Length-Bounded Critical Disruption Path (CS-kLB-CDP)*, p, in $\mathcal{G} = (V, E, s, t)$ is a simple path from s to t, among all the s-t paths whose length does not exceed a given threshold $k \geq 0$, such that the largest connected component of the induced subgraph $\mathcal{G}^p := (V^p, E^p)$ where $V^p := V \setminus V(p)$, $E^p := E \cap (V^p \times V^p)$, obtained via deletion of p, contains the smallest number of vertices.

Definition 2.2 A *Component-Number-optimal k-Length-Bounded Critical Disruption Path (CN-kLB-CDP)*, p, in $\mathcal{G} = (V, E, s, t)$ is a simple path from s to t, among all the s-t paths whose length does not exceed a given threshold $k \geq 0$, such that the number of connected components in the induced subgraph $\mathcal{G}^p := (V^p, E^p)$ where $V^p := V \setminus V(p)$, $E^p := E \cap (V^p \times V^p)$, obtained via deletion of p, is the maximum.

With this notation and definitions in mind, the Bi-Objective Length-Bounded Critical Disruption Path problem can be formulated as the following bicriteria mixed integer programming formulation.

$$\begin{aligned}
& \mathtt{z}^1 := \min \psi \\
& \mathtt{z}^2 := \max \sum_{i \in V \setminus \{s,t\}} d_i \\
\text{s.t.} \quad & \sum_{j:(i,j) \in E} x_{ij} = \sum_{j:(j,i) \in E} x_{ji} \qquad \forall i \in V \setminus \{s,t\} && (1) \\
& \sum_{i:(s,i) \in E} x_{si} = 1 && (2) \\
& \sum_{i:(i,t) \in E} x_{it} = 1 && (3) \\
& \sum_{i,j:(i,j) \in E} l_{ij} x_{ij} \leq k && (4) \\
& x_{ij} + x_{ji} \leq 1 \qquad \forall (i,j) \in E \ \wedge \ j > i && (5) \\
& \sum_{i,j \in S:\ (i,j) \in E} x_{ij} \leq |S| - 1 \qquad \forall S \subset V, \quad |S| \geq 2 && (6) \\
& y_{ii} = 1 - \sum_{l:(l,i) \in E} x_{li} \qquad \forall i \in V \setminus \{s,t\} && (7) \\
& y_{ij} \geq y_{ih} - \sum_{l:(l,j) \in E} x_{lj} \\
& \qquad\qquad \forall h, i, j \in V \setminus \{s,t\} : \ j \neq i \ \wedge \ (h,j) \in E && (8) \\
& \psi \geq \sum_{j \in V \setminus \{s,t\} \wedge \ j \geq i} y_{ij} \qquad \forall i \in V \setminus \{s,t\} && (9) \\
& n \cdot d_i \leq n \cdot y_{ii} - \sum_{j \in V \setminus \{s,t\} : j > i} y_{ji} \qquad \forall i \in V \setminus \{s,t\} && (10) \\
& \psi \geq 0 && (11) \\
& d_i \in \{0,1\} \qquad \forall i \in V \setminus \{s,t\} && (12) \\
& x_{ij} \in \{0,1\} \qquad \forall (i,j) \in E && (13) \\
& y_{ij} \in \{0,1\} \qquad \forall i,j \in V \setminus \{s,t\} && (14)
\end{aligned}$$

The first objective function z^1 is related to the search for the CS-kLB-CDP, as introduced in the Definition 2.1: here, we want to minimize the size of the largest connected component size, indicated by ψ in the formulation. The second objective function z^2 concerns the CN-kLB-CDP, as introduced in the Definition 2.2: the number of connected components to be minimized is represented in this case by the sum of the d_i binary variables. The proposed model is an arc-based formulation requiring the selection of one simple path from vertex s to vertex t in the network, in the following indicated as CDP.

The decision on the choice of the CDP is encoded by the binary variables x_{ij}. Binary decision variables y_{ij} take value 1 if and only if both vertices $i \in V$ and $j \in V$ are contained in the same connected component after the removal of the CDP and 0 otherwise; y_{ii} has value 1 if vertex $i \in V$ is not contained in the CDP and 0 otherwise. A non-negative variable ψ is used to indicate the cardinality of the largest connected component arising in the residual graph after the removal of the CDP. Binary variables d_i are used to select among all the vertices of each connected component in the residual graph the one with the highest index as representative of the connected component: d_i equals 1 if and only if node i has the maximum index among those belonging to its component. Constraints (1)-(6) ensure that variables x_{ij} define a k-length-bounded simple $s-t$ path. In particular, balance constraints (1) verify that every vertex in the CDP has one associated edge in and out. Constraints (2) and (3) ensure exactly one edge exists from the source vertex and to the destination vertex, respectively, in the CDP. The length of the CDP is limited by constraint (4). The complexity of sub-tour elimination constraints (6) is reduced by a separation mechanism and by constraints (5). Connected components in the network are identified by constraints (7) and (8): vertex $j \in V$ is forced to be in the same connected component as vertex $i \in V$, if there exists a vertex $h \in V$ in the connected component which is connected via an edge to vertex j and vertex j is not in the CDP. Constraints (9) assign the largest connected component size to variable ψ. Finally, the number of connected components is computed by constraints (10).

3 Multi-Objective Computational Experiments

A large set of instances was generated at pseudo-random according to two main parameters: the number of nodes n and the number of edges m. Seven groups of instances were created, based on the n parameter and ranging in $[20, ..., 80]$ step 10. The number m of edges in each network was fixed as equal to about 4 times the number of nodes, and 50 different networks were produced for each group, setting the l_{ij} values as equal to 1 for each edge in the network. The proposed model was implemented using ANSI C++ and all the instances were solved by means of IBM ILOG CPLEX 12.6 running on an Intel(R) Core(TM) i7-2600K CPU @ 3.40GHz with 14 GB of RAM. A time limit of 3600 seconds was considered for each single experiment. The Bi-Criteria nature of the optimization problem proposed in this paper was treated in two ways: first, the two objective functions were considered in lexicographical order (z^1,z^2) and (z^2,z^1), and the different results obtained by the related two sets of experiments are presented in Table 1. Second, a scalarization technique approach was considered to explore the pareto front of the optimal solutions. All experiments were conducted at varying the k parameter to analyze the impact of the maximum length of the selected CDP on the disruption performance. The values considered for the k parameters were: $\{0.10 \cdot n,\ 0.15 \cdot n,\ 0.20 \cdot n,\ 0.25 \cdot n,\ 0.50 \cdot n,\ 1.00 \cdot n\}$.

Lexicographic Ordering analysis.

Table 1 reports on the results obtained by adopting the lexicographic ordering approach on the Bi-Objective problem. Each line in the table presents the mean values obtained by solving to optimality within the time limit the set of 50 instances for each group. The characteristics of each group are summarized in the first three columns of the table: the number of vertices n, the number of edges m, and the parameter k limiting the CDP length. Then the following output values are presented for each lexicographic ordering (z^1,z^2) and (z^2,z^1): cardinality of the largest connected component (ψ) and number of connected components (nc) in the residual graph after the removal of the selected CDP, CPU times (t) and length of the CDP (L). Finally, the last two columns present two ratio indexes defined to evaluate the effect of the objective function ordering in the Lexicographic approach, namely $RT(\psi) := \frac{\psi(z^1)}{\psi(z^2)}$ and $RT(nc) := \frac{nc(z^1)}{nc(z^2)}$. Computational times increase with the size of the network. Moreover, the presence of length bounds on selected paths influences the computational effort: when path length constraints are less tight (e.g. $k = 0.50 \cdot n$ or $k = 1.00 \cdot n$) the time required to reach the optimum is much lower. This effect is particularly evident on larger instances. The mean length of the Critical Disruption Paths seems to increase when the size of the largest connected components is the first goal to be minimized. The results confirm the suitability of the model to properly represent the considered Bi-Criteria optimization problem. From observing the $RT(\psi)$ and $RT(nc)$ ratios, it turns out the size of the largest component is largely influenced by the changes in the k parameter and in the ordering considered in the lexicographic approach.

Scalarization Technique Analysis.

In order to study pareto optimal solutions for the BO-kLB-CDP problem, the convex combination of z^1 and z^2 was considered obtaining one single objective function as follows: $z^3 := \min \lambda\psi - (1-\lambda)\sum_{i \in V \setminus \{s,t\}} d_i$. Experiments were conducted at varying the λ parameter in the range (0,1) and the results were computed for an instance with $n = 50$, $m = 196$ and two different values of the k parameter, namely $k = 0.20 \cdot n$ and $k = 0.25 \cdot n$. The results of the analysis, depicted in Figure 1, show the possible conflicts arising between the two considered objective functions and confirm the relevance of the Bi-Objective optimization approach proposed in this paper to cope effectively with Network Interdiction applications. The role of the path length is also important to find a proper trade-off between different and conflicting goals.

Conclusions and further lines of research.

The presented computational experiments provide a valid Proof-of-concept for the Mixed Integer Programming formulation of the Bi-Objective k-Length-Bounded Critical Disruption Path problem introduced in this paper. Furthermore, the analysis of the results showed how the proposed multicriteria optimization problem provides valuable information to increase the effectiveness of the interdiction action, depending on the specific field of application. The computational times required to solve at the optimum the considered in-

			Lexicographic ordering (z^1,z^2)				Lexicographic ordering (z^2,z^1)				Analysis	
n	m	k	ψ	nc	t	L	nc	ψ	t	L	RT(ψ)	RT(nc)
50	196	$0.10 \cdot n$	33.29	9.96	22.60	5.00	11.17	37.98	4.34	4.98	0.88	0.89
50	196	$0.15 \cdot n$	24.62	14.44	77.94	7.00	16.74	38.08	18.67	6.90	0.65	0.86
50	196	$0.20 \cdot n$	13.16	20.74	155.23	10.00	24.66	28.32	16.98	9.96	0.46	0.84
50	196	$0.25 \cdot n$	8.02	25.12	98.58	12.00	28.26	17.30	9.10	11.90	0.46	0.89
50	196	$0.50 \cdot n$	1.00	27.48	2.44	21.52	30.04	7.90	9.06	18.54	0.13	0.91
50	196	$1.00 \cdot n$	1.00	27.36	2.69	21.64	30.04	5.08	8.35	18.64	0.20	0.91
60	247	$0.10 \cdot n$	45.22	5.30	490.03	6.00	6.43	55.00	49.60	5.92	0.82	0.82
60	247	$0.15 \cdot n$	37.02	8.07	915.76	9.00	9.64	50.64	159.56	8.92	0.73	0.84
60	247	$0.20 \cdot n$	21.05	13.46	1339.26	12.00	14.70	42.86	290.98	11.94	0.49	0.92
60	247	$0.25 \cdot n$	11.71	18.53	1429.74	14.98	20.12	29.42	226.38	14.96	0.40	0.92
60	247	$0.50 \cdot n$	1.47	28.16	187.33	29.94	28.82	7.02	9.22	27.74	0.21	0.98
60	247	$1.00 \cdot n$	1.00	22.66	9.47	36.34	28.82	6.94	7.61	27.72	0.14	0.79
70	289	$0.10 \cdot n$	34.86	16.71	272.18	7.00	25.06	51.73	27.49	7.00	0.67	0.67
70	289	$0.15 \cdot n$	20.58	20.42	636.94	10.00	32.64	42.10	59.21	9.94	0.49	0.63
70	289	$0.20 \cdot n$	7.72	38.44	279.92	13.88	40.62	19.56	24.28	13.98	0.39	0.95
70	289	$0.25 \cdot n$	6.20	39.48	188.40	16.86	42.90	15.06	22.67	16.92	0.41	0.92
70	289	$0.50 \cdot n$	1.00	42.04	8.86	26.96	43.00	8.48	11.49	22.32	0.12	0.98
70	289	$1.00 \cdot n$	1.00	41.94	7.04	27.06	43.00	8.50	10.90	23.02	0.12	0.98
80	316	$0.10 \cdot n$	52.79	11.97	1824.18	8.00	14.44	69.14	753.03	7.98	0.76	0.83
80	316	$0.15 \cdot n$	19.38	24.25	1517.71	12.00	24.81	51.54	2228.76	11.98	0.38	0.98
80	316	$0.20 \cdot n$	9.43	31.93	2582.34	16.00	36.02	33.86	777.36	16.00	0.28	0.89
80	316	$0.25 \cdot n$	6.23	41.00	2749.92	20.00	43.36	22.50	267.60	20.00	0.28	0.95
80	316	$0.50 \cdot n$	1.00	45.08	29.53	33.92	51.52	12.24	30.18	26.98	0.08	0.88
80	316	$1.00 \cdot n$	1.00	44.94	27.86	34.06	51.52	10.26	28.36	26.78	0.10	0.87

Table 1

Computational results of the Lexicographic Ordering experiments: instance groups from n=50 to n=80.

stances show a dependence from the size of the network and from the bounds on the length of the disruption path, suggesting as further lines of research in this field the development of efficient algorithms to find pareto optimal solutions on dense and large scale networks.

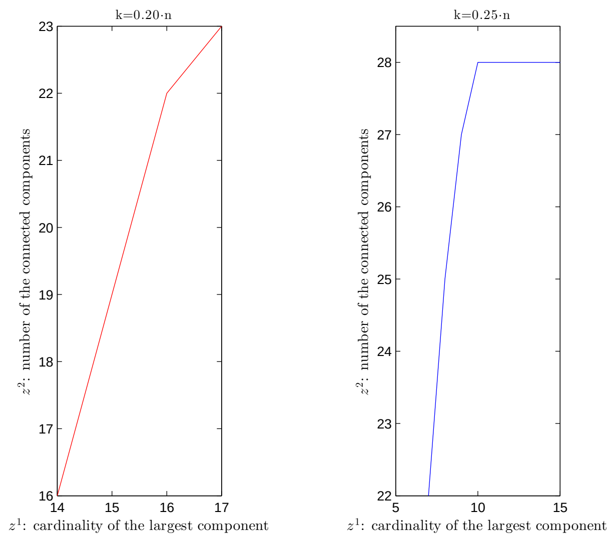

Fig. 1. *Representation of the pareto front at varying λ for different values of the k parameter.*

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