Article:

Iten, Raban and Colbeck, Roger Andrew orcid.org/0000-0003-3591-0576 (2018) Smooth manifold structure for extreme channels. Journal of Mathematical Physics. 012202-1. ISSN 0022-2488

https://doi.org/10.1063/1.5019837

Smooth Manifold Structure for Extreme Channels

Raban Iten[a)]

Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland

Roger Colbeck[b)]

Department of Mathematics, University of York, YO10 5DD, UK

(Dated: 7^{th} December 2017)

A quantum channel from a system A of dimension d_A to a system B of dimension d_B is a completely positive trace-preserving map from complex $d_A \times d_A$ to $d_B \times d_B$ matrices, and the set of all such maps with Kraus rank r has the structure of a smooth manifold. We describe this set in two ways. First, as a quotient space of (a subset of) the $rd_B \times d_A$ dimensional Stiefel manifold. Secondly, as the set of all Choi-states of a fixed rank r. These two descriptions are topologically equivalent. This allows us to show that the set of all Choi-states corresponding to extreme channels from system A to system B of a fixed Kraus rank r is a smooth submanifold of dimension $2rd_Ad_B - d_A^2 - r^2$ of the set of all Choi-states of rank r. As an application, we derive a lower bound on the number of parameters required for a quantum circuit topology to be able to approximate all extreme channels from A to B arbitrarily well.

[a)]Electronic mail: itenr@ethz.ch

[b)]Electronic mail: roger.colbeck@york.ac.uk

I. INTRODUCTION

We describe the differential structure of the set $\mathcal{E}_{s,t,r}$ consisting of all completely positive trace-preserving (CPTP) maps from $\mathbb{C}^{s\times s}$ to $\mathbb{C}^{t\times t}$ (which we refer to as s to t channels) of fixed Kraus rank r. A linear map $\mathcal{E}:\mathbb{C}^{s\times s}\mapsto\mathbb{C}^{t\times t}$ is called positive if it sends positive semi-definite matrices to positive semi-definite matrices. It is called completely positive (CP) if $\mathcal{E}\otimes\mathcal{I}_p$ is positive for all $p\in\mathbb{N}$, where $\mathcal{I}_p:\mathbb{C}^{p\times p}\mapsto\mathbb{C}^{p\times p}$ denotes the identity channel. Choi [1] showed that a map $\mathcal{E}$ is completely positive if and only if it admits an expression $\mathcal{E}(X)=\sum_{i=1}^{r'}A_iXA_i^*$ (for all $X\in\mathbb{C}^{s\times s}$), where the $A_i\in\mathbb{C}^{t\times s}$ are called Kraus operators in quantum information theory [2]. The Kraus representation is not unique in general and the minimum number of Kraus operators r', such that a representation of the form above exists, is called the Kraus rank r of the map $\mathcal{E}$ (and the corresponding representation is called a 'minimal' Kraus representation). Note that, by Remark 4 of [1], a Kraus representation is minimal if and only if the Kraus operators $A_1,A_2,\dots,A_{r'}$ are linearly independent. Finally, a map $\mathcal{E}$ is called trace preserving if $\operatorname{tr}\mathcal{E}(X)=\operatorname{tr}X$ for all $X\in\mathbb{C}^{s\times s}$, which corresponds to the requirement $\sum_{i=1}^{r}A_i^*A_i=I$ on the Kraus operators.

CPTP maps are of interest in physics, because they describe the most general evolution a quantum system can undergo. Since the set $\mathcal{E}_{s,t}$ of all s to t quantum channels is convex, one can investigate the decomposition of a quantum channel into a convex combination of extreme channels. In particular, such decompositions can help to implement quantum channels in a cheaper way [3; 4]. However, there are open questions about the structure of the (closure of the) set of extreme channels and finding convex decompositions into such channels. In particular, a tight bound on the number of generalized extreme channels, i.e., channels which lie in the closure of the set of all extreme channels, required for such a convex decomposition is not known [5]. The set of extreme channels has been described by Friedland and Loewy [6] using the framework of semi-algebraic geometry. In contrast, we consider the set of extreme channels in the framework of differential geometry. In other words, this work focuses on assigning a smooth structure to the set of all extreme channels and we refer to [6] for other interesting properties of this set.

The paper is structured as follows. First we give an overview of the notation used in the paper. In Section III we describe the smooth manifold structure of the set $\mathcal{E}^{\mathrm{e}}_{s,t,r}$ of s to t extreme channels of a fixed Kraus rank r [Note that we do not require a manifold to be connected.]: First, we adapt the characterization of unital extreme channels given by Choi [1] to trace-preserving channels in Section III A. [Recall that a channel $\mathcal{E}$ is unital if $\mathcal{E}(I) = I$, where I denotes the identity.] Then, in Section III B, we describe the set of channels and extreme channels with the smooth structure induced by the standard smooth structure on the Kraus operators. In this picture, we find that $\mathcal{E}^{\mathrm{e}}_{s,t,r} \subset \mathcal{E}_{s,t,r}$ is an open subset and hence a smooth submanifold. In Section III C, we transfer this topological property (founded in the Kraus representation picture) to the Choi-state picture, which will show that $\mathcal{E}^{\mathrm{e}}_{s,t,r}$ can be considered as a smooth submanifold of the set of all Choi-states of fixed rank r. In Section IV, we give a rigorous proof of the known fact [5] that every channel can be decomposed into a finite convex combination of extreme channels. Finally, we look at an application to quantum information theory in Section V, where we derive a lower bound on the number of parameters required for a quantum circuit topology for extreme channels, which we have used in [3].

II. NOTATION AND BACKGROUND

A. Notation

We use the notation $[A, B] \in \mathbb{C}^{t\times(s_A+s_B)}$ to denote the (horizontal) concatenation of two matrices $A \in \mathbb{C}^{t\times s_A}$ and $B \in \mathbb{C}^{t\times s_B}$, i.e., the first s_A columns of $[A, B]$ correspond to the columns of A and the $(s_A + 1)$th column to the $(s_A + s_B)$th column to the columns of B. And we denote the vertical concatenation of the matrices A^T and B^T by $[A^T; B^T] = [A, B]^T$. For arbitrary $s, t, r \in \mathbb{N}$, we define:

- $\mathbb{C}^{s\times t}$: Complex $s \times t$ matrices
- H_s: Hermitian $s \times s$ matrices
- $H_{s,+}$: Positive semi-definite $s \times s$ matrices
- $H^r_{s,+}$: Elements in $H_{s,+}$ of rank r
- $V_{s,t}$: Set of all $V \in \mathbb{C}^{t\times s}$ s.t. $V^*V = I$ (i.e., set of all isometries from an s to a t dimensional system)

- $V_{s,t,r}$: Set of all $V = [A_1; A_2; \ldots; A_r] \in V_{s,rt}$, such that the elements in $\{A_i\}_{i\in\{1,2,\ldots,r\}} \in \mathbb{C}^{t\times s}$ are linearly independent (over $\mathbb{C}$)

- $U(s) = V_{s,s}$: Unitary $s \times s$ matrices

- $\mathcal{E}_{s,t}$: CPTP ($\mathbb{C}$-linear) maps from $\mathbb{C}^{s\times s}$ to $\mathbb{C}^{t\times t}$

- $\mathcal{E}^{\mathrm{v}}_{s,t}$: CP and unital ($\mathbb{C}$-linear) maps from $\mathbb{C}^{s\times s}$ to $\mathbb{C}^{t\times t}$

- $\mathcal{E}^{\mathrm{e}}_{s,t}$: Elements in $\mathcal{E}_{s,t}$ that are extreme

- $\mathcal{E}_{s,t,r}$: Elements in $\mathcal{E}_{s,t}$ of Kraus rank r

- $\mathcal{E}_{s,t,\leqslant r} = \bigcup_{j=1}^{r} \mathcal{E}_{s,t,j}$: Elements in $\mathcal{E}_{s,t}$ with Kraus rank at most r

- $\mathcal{E}^{\mathrm{e}}_{s,t,r} = \mathcal{E}^{\mathrm{e}}_{s,t} \cap \mathcal{E}_{s,t,r}$: Elements in $\mathcal{E}_{s,t,r}$ that are extreme in $\mathcal{E}_{s,t}$

- $\mathcal{C}_{s,t}$: Set of all Choi-states corresponding to channels from an s dimensional system A to a t dimensional system B, i.e., $C_{AB} \in H_{st,+}$, such that $\operatorname{tr}_B C_{AB} = \frac{1}{s} I$

- $\mathcal{C}^{\mathrm{e}}_{s,t}$: Elements in $\mathcal{C}_{s,t}$ that are extreme

- $\mathcal{C}_{s,t,r}$: Elements in $\mathcal{C}_{s,t}$ with rank r

- $\mathcal{C}_{s,t,\leqslant r} = \bigcup_{j=1}^{r} \mathcal{C}_{s,t,j}$: Elements in $\mathcal{C}_{s,t}$ with rank at most r

- $\mathcal{C}^{\mathrm{e}}_{s,t,r}$: Elements in $\mathcal{C}_{s,t,r}$ that are extreme in $\mathcal{C}_{s,t}$

B. Restriction of the Domain or Image of a Smooth Map

The following propositions give sufficient conditions for a map to remain smooth when its domain or image is restricted.

Proposition 1 (Theorem 5.27 of [7])**.** *Let M and N be smooth manifolds (with or without boundary). If $F : M \mapsto N$ is a smooth map and $D \subset M$ is an (immersed or embedded) submanifold, then $F\big|_D : D \mapsto N$ is smooth.*

Proposition 2 (Corollary 5.30 of [7])**.** *Let M and N be smooth manifolds, and $S \subset N$ be an embedded submanifold. Then any smooth map $F : M \mapsto N$ whose image is contained in S is also smooth as a map from M to S.*

III. SMOOTH MANIFOLD STRUCTURE FOR EXTREME CHANNELS

A. Characterization of Extreme Channels

We want to characterize the set of extreme points $\mathcal{E}^{\mathrm{e}}_{s,t} \subset \mathcal{E}_{s,t}$. For this purpose we have to slightly modify Theorem 5 of [1]. This modification was also considered in [6].

Theorem 3 (Characterization of extreme channels)**.** *Let $\mathcal{E} \in \mathcal{E}_{s,t}$ with minimal Kraus representation $\mathcal{E}(X) = \sum_{i=1}^{r} A_i X A_i^*$. Then $\mathcal{E}$ is extreme in $\mathcal{E}_{s,t}$ if and only if all elements of the set $\{A_i^* A_j\}_{i,j\in\{1,2,\ldots,r\}}$ are linearly independent.*

Remark 1. If the Kraus rank r of the channel $\mathcal{E}$ in Theorem 3 is bigger than s, then $\mathcal{E}$ cannot be extreme, since in this case $|\{A_i^* A_j\}_{i,j\in\{1,2,\ldots,r\}}| > \dim_{\mathbb{C}}(\mathbb{C}^{s\times s})$.

Proposition 4. *There exists a bijection $\Psi : \mathcal{E}_{s,t} \mapsto \mathcal{E}^{\mathrm{v}}_{t,s}$ that sends extreme points of $\mathcal{E}_{s,t}$ to extreme points of $\mathcal{E}^{\mathrm{v}}_{t,s}$ and vice versa.*

Proof. Note first that $\mathbb{C}^{l\times l}$ together with the Frobenius inner product is a (finite dimensional) Hilbert space for $l \in \mathbb{N}$. Let $\mathcal{E} \in \mathcal{E}_{s,t}$. Then $\mathcal{E}$ is $\mathbb{C}$-linear by definition and bounded. By the Fréchet-Riesz representation theorem there exists an injective map $\Psi(\mathcal{E}) = \mathcal{E}^* : \mathcal{E}_{s,t} \mapsto \mathrm{B}(\mathbb{C}^{t\times t}, \mathbb{C}^{s\times s})$, where $\mathrm{B}(\mathbb{C}^{t\times t}, \mathbb{C}^{s\times s})$ denotes the set of linear bounded operators from $\mathbb{C}^{t\times t}$ to $\mathbb{C}^{s\times s}$ and $\mathcal{E}^*$ denotes the adjoint map of $\mathcal{E}$, i.e., for all $C \in \mathbb{C}^{s\times s}$ and $D \in \mathbb{C}^{t\times t}$ we have $\langle \mathcal{E}(C), D\rangle = \langle C, \mathcal{E}^*(D)\rangle$. Let $\{A_i\}_{i\in\{1,2,\ldots,r\}}$ be the Kraus operators of $\mathcal{E}$. By a direct computation one finds that the Kraus operators of $\mathcal{E}^*$ are $\{A_i^*\}_{i\in\{1,2,\ldots,r\}}$, and therefore, $\mathcal{E}^* \in \mathcal{E}^{\mathrm{v}}_{t,s}$. Since $(\mathcal{E}^*)^* = \mathcal{E}$, we can set $C = I$ in the adjoint property above to see that Ψ^{-1} sends unital maps to trace-preserving maps. Therefore the map $\Psi : \mathcal{E}_{s,t} \mapsto \mathcal{E}^{\mathrm{v}}_{t,s}$ is a bijection. Assume that $\mathcal{E} \in \mathcal{E}_{s,t}$ is not extreme, i.e., there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}_{s,t}$, $\mathcal{E}_1 \neq \mathcal{E}$, $\mathcal{E}_2 \neq \mathcal{E}$ and $p \in (0,1)$ s.t. $\mathcal{E} = p\mathcal{E}_1 + (1-p)\mathcal{E}_2$. By the linearity of the adjoint map we have $\mathcal{E}^* = p\mathcal{E}_1^* + (1-p)\mathcal{E}_2^*$, which shows that elements of $\mathcal{E}_{s,t}$ that are not extreme cannot be mapped to extreme elements of $\mathcal{E}^{\mathrm{v}}_{t,s}$. The reverse direction follows analogously. $\square$

Proof of Theorem 3. Theorem 5 of [1] shows that $\mathcal{E}^* \in \mathcal{E}^{\mathrm{v}}_{t,s}$ (with linearly independent Kraus operators $\tilde{A}_i \in \mathbb{C}^{s\times t}$) is extreme if and only if the elements in $\{\tilde{A}_i \tilde{A}_j^*\}_{i,j\in\{1,2,\ldots,r\}}$ are linearly independent. By Proposition 4, this leads to a characterization of the extreme points in $\mathcal{E}_{s,t}$. Since the Kraus operators of $\mathcal{E}$ (where $\mathcal{E}$ denotes the adjoint map of $\mathcal{E}^*$) are $A_i := \tilde{A}_i^* \in \mathbb{C}^{t\times s}$

(cf. the proof of Proposition 4), the map $\mathcal{E} \in \mathcal{E}_{s,t}$ is extreme if and only if the elements in $\{A_i^* A_j\}_{i,j\in\{1,2,\ldots,r\}}$ are linearly independent. $\square$

B. Structure of the Set of Extreme Channels in the Kraus Representation

In this section we consider the smooth structure of the set $\mathcal{E}^{\mathrm{e}}_{s,t,r}$ working with the Kraus representation of channels. Our first goal is to describe the set $\mathcal{E}_{s,t,r}$ of s to t channels with Kraus rank r. We can assume that $s \leqslant rt$ and $r \leqslant st$, since $\mathcal{E}_{s,t,r} = \emptyset$ if $s > rt$ (cf. Lemma 6 of [6]) or $r > st$. Let $\{A_i\}_{i\in\{1,2,\ldots,r\}}$ denote a set of (linearly independent) Kraus operators of $\mathcal{E} \in \mathcal{E}_{s,t,r}$. Then we define $V = [A_1; A_2; \ldots ; A_r]$ which lies in $V_{s,t,r}$, because $V^*V = \sum_{i=1}^{r} A_i^* A_i = I$. Since the Kraus representation is not unique, we do not have a one-to-one correspondence between $\mathcal{E}_{s,t,r}$ and $V_{s,t,r}$. However, we can exploit the desired correspondence by taking the quotient of $V_{s,t,r}$ with respect to the unitary freedom of the Kraus operators.

In the following, we always assume that $s \leqslant rt$ and $r \leqslant st$. The next Lemma is generally known (see for example [8]).

Lemma 5 (Stiefel manifold)**.** *Let $s \leqslant t$. Then the Stiefel manifold $V_{s,t}$ is a compact, smooth embedded submanifold of $\mathbb{R}^{2ts}$ of dimension $2st - s^2$.*

Proposition 6. *The set $V_{s,t,r}$ is an open subset of $V_{s,rt}$. In particular, $V_{s,t,r}$ is a smooth embedded submanifold of $V_{s,rt}$.*

Proof. We can write all coefficients of a complex $t \times s$ matrix in a column vector leading to a natural correspondence $\psi : \mathbb{C}^{t\times s} \mapsto \mathbb{C}^{ts}$. Let $l = \binom{ts}{r}$. We define the map $F : V_{s,rt} \to \mathbb{C}^l$ sending $V = [A_1; A_2; \ldots ; A_r]$ to all $r \times r$ minors of the matrix $[\psi(A_1), \psi(A_2), \ldots , \psi(A_r)]$ (ordered in an arbitrary way). [Recall that an $r \times r$ minor of a matrix D is the determinant of an $r \times r$ sub-matrix of D formed by 'deleting' rows (or columns).] Then the condition that the elements of the set $\{A_i\}_{i\in\{1,2,\ldots,r\}}$ are linearly independent reads: $F(V) \neq (0, 0, \ldots , 0)$. Since F is continuous, $F^{-1}(\{(0, 0, \ldots , 0)\}^{\mathrm{c}}) = V_{s,t,r}$ is open. $\square$

We can use Theorem 21.10 of [7] to describe the manifold structure of the orbit space $V_{s,t,r}/U(r)$.

Definition 1. A group G acts *freely* on a set S if the only element of G that fixes any element of S is the identity, i.e., for all $p \in S$ and $g \in G$, $g \cdot p = p$ implies $g = I$.

Definition 2. Let G be a Lie group that acts continuously on a manifold M. The action is said to be *proper* if the map $G \times M \mapsto M \times M$ given by $(g, p) \mapsto (g \cdot p, p)$ is a proper map, i.e., the preimage of a compact set is compact.

The following Proposition gives a sufficient condition for a group action to be proper.

Proposition 7 (Corollary 21.6 of [7])**.** *Any continuous action by a compact Lie group on a manifold is proper.*

Theorem 8 (Quotient Manifold Theorem [7])**.** *Suppose a Lie group G acts smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to $\dim(M) - \dim(G)$, and has a unique smooth structure with the property that the quotient map $\pi : M \mapsto M/G$ is a smooth submersion.*

Lemma 9 (Lemma 21.1 of [7])**.** *For any continuous action of a topological group G on a topological space M, the quotient map $\pi : M \mapsto M/G$ is open.*

Proposition 10. *The Lie group $U(r)$ acts smoothly, freely and properly on the manifold $V_{s,t,r}$ by the action $U \cdot V = (U \otimes I)V$, where $U \in U(r)$, $V \in V_{s,t,r}$ and I denotes the $t \times t$ identity matrix.*

Proof. We first show that $U \cdot V \in V_{s,t,r}$ for all $U \in U(r)$ and $V \in V_{s,t,r}$. Note that $V^*(U^* \otimes I)(U \otimes I)V = I$ and that the linear independence of the matrices A_i (where $V = [A_1; A_2; \ldots; A_r]$) is preserved under the unitary action: Assume $\sum_{i=1}^r \alpha_i \left(\sum_{j=1}^r (U)_{ij} A_j \right) = 0$ for some coefficients $\alpha_i \in \mathbb{C}$. This is equivalent to $\sum_{j=1}^r \left(\sum_{i=1}^r \alpha_i (U)_{ij} \right) A_j = 0$ which implies $\sum_{i=1}^r \alpha_i (U)_{ij} = 0$ for all $j \in \{1, 2, \ldots, r\}$, since the A_j are linearly independent. Since U is unitary, this implies $\alpha_i = 0$ for all $i \in \{1, 2, \ldots, r\}$. We conclude that the group action is well defined. To show that the action is free, choose a $V \in V_{s,t,r}$ and a $U \in U(r)$ and assume that $U \cdot V = V$. Writing $V = [A_1; A_2; \ldots; A_r]$, the last equation becomes $\sum_{j=1}^r (U)_{ij} A_j = A_i$ for all $i \in \{1, 2, \ldots, r\}$. Since $V \in V_{s,t,r}$, the A_j are linearly independent and we conclude that $(U)_{ij} = \delta_{ij}$ or equivalently $U = I$. To see that the action is smooth, consider the map $F(U, V) = (U \otimes I)V$: $\mathbb{C}^{r \times r} \times \mathbb{C}^{rt \times s} \to \mathbb{C}^{rt \times s}$. [Note that we always identify $\mathbb{C} \cong \mathbb{R}^2$, and hence we can treat F as a map from $\mathbb{R}^{2\left(r^2 + rts\right)}$ to $\mathbb{R}^{2rts}$.] Since taking tensor products is a smooth operation, the map F is smooth. By Propositions 1 and 6 and Lemma 5, the map $\tilde{F}(U,V) = (U \otimes I)V\colon\ U(r) \times V_{s,t,r} \mapsto \mathbb{C}^{rt\times s}$ is smooth. Then, by Propositions 2 and 6 and Lemma 5, the map $F'(U,V) = (U \otimes I)V\colon\ U(r) \times V_{s,t,r} \mapsto V_{s,t,r}$ is also smooth. Since the Lie group $U(r)$ is compact, the action is proper by Proposition 7. $\square$

Definition 3. We define the equivalence relation $\sim$ as follows: Let $V_1, V_2 \in V_{s,t,r}$. Then $V_1 \sim V_2$ if there exists a $U \in U(r)$, such that $U \cdot V_1 = V_2$. The orbit space is $V_{s,t,r}/U(r) := \{[V] : V \in V_{s,t,r}\}$ (together with the quotient topology).

Lemma 11. *The orbit space $V_{s,t,r}/U(r)$ is a topological manifold of dimension equal to $\dim(V_{s,t,r}) - \dim(U(r)) = 2srt - s^2 - r^2$ with a unique smooth structure such that the quotient map $\pi : V_{s,t,r} \mapsto V_{s,t,r}/U(r)$ is a smooth submersion. Moreover, π is an open map.*

Proof. The first part of the theorem follows from Theorem 8, where the assumption for the theorem are satisfied because of Proposition 10. The quotient map π is open by Lemma 9. $\square$

Lemma 12. *There is a one-to-one correspondence between the set $\mathcal{E}_{s,t,r}$ of channels of Kraus rank r and the orbit space $V_{s,t,r}/U(r)$.*

Proof. We define the quotient map $\pi(V) = [V]\colon\ V_{s,t,r} \mapsto V_{s,t,r}/U(r)$ and the map $\psi : \mathcal{E}_{s,t,r} \mapsto V_{s,t,r}/U(r)$, by sending a channel $\mathcal{E} \in \mathcal{E}_{s,t,r}$ with (linearly independent) Kraus operators $\{A_i\}_{i\in\{1,2,\ldots,r\}}$ to $\pi([A_1; A_2; \ldots; A_r])$. To show that this map is well defined, we must show that it is independent on the choice of the Kraus operators. By Remark 4 of [1], two Kraus representations $\{A_i\}_{i\in\{1,2,\ldots,r\}}$ and $\{B_i\}_{i\in\{1,2,\ldots,r\}}$ describe the same channel $\mathcal{E} \in \mathcal{E}_{s,t,r}$ if and only if there exist a unitary $U \in U(r)$, such that $B_j = \sum_{i=1}^{r} (U)_{ji} A_i$ for all $j \in \{1, 2, \ldots, r\}$ or equivalently $V_B = (U \otimes I)V_A$, where $V_A = [A_1; A_2; \ldots; A_r] \in V_{s,t,r}$ and $V_B = [B_1; B_2; \ldots; B_r] \in V_{s,t,r}$. Therefore, the two Kraus representation describe the same channel if and only if $V_A \sim V_B$. We conclude that the map ψ is well defined and injective. On the other hand, for all $W \in V_{s,t,r}/U(r)$, we can define a channel $\mathcal{E} \in \mathcal{E}_{s,t,r}$ with $\psi(\mathcal{E}) = W$ by choosing a representative element $V \in \pi^{-1}(W)$, breaking it into blocks $V = [A_1; A_2; \ldots; A_r]$ and treating those as the channel's Kraus operators. This shows that ψ is also surjective. $\square$

We are now ready to study the structure of the set $\mathcal{E}^{e}_{s,t,r}$ of extreme channels. In the following we show that we can identify $\mathcal{E}^{e}_{s,t,r}$ with a smooth manifold.

Proposition 13. *The set* $\tilde{O} := \{[A_1; A_2; \ldots; A_r] \in V_{s,t,r} : \{A_i^* A_j\}_{i,j \in \{1,2,\ldots,r\}}$ *are linearly independent*$\}$ *is an open subset of the manifold* $V_{s,t,r}$.

Proof. Works analogously to the proof of Proposition 6. □

Lemma 14 (Manifold structure for $\mathcal{E}^e_{s,t,r}$)**.** *Let* $O := \pi(\tilde{O}) \subseteq V_{s,t,r}/U(r)$, *where* π *is the quotient map of Lemma 11.* O *is a smooth manifold of dimension* $2srt - r^2 - s^2$ *and there is a one-to-one correspondence between the set* $\mathcal{E}^e_{s,t,r}$ *of extreme channels of Kraus rank* r *and* O. *Moreover,* $\mathcal{E}^e_{s,t,s} \neq \emptyset$.

Proof. Since π is an open map and $\tilde{O}$ is an open subset of $V_{s,t,r}$, O is an open subset of the orbit space $V_{s,t,r}/U(r)$. Together with Lemma 12 and Theorem 3, this implies the first part of the Lemma.

For the second part, we borrow an argument from [6]. Consider a channel $\mathcal{E}$ with Kraus operators $A_i = |\psi\rangle\langle i|$ for $i \in \{1, 2, \ldots, s\}$, where $|\psi\rangle \in \mathbb{C}^t$ is of unit length and $|i\rangle \in \mathbb{C}^s$ denotes the ith standard basis vector. [Note that $\langle\phi| \in \mathbb{C}^{1\times d}$ denotes the conjugate transpose of a d-dimensional vector $|\phi\rangle \in \mathbb{C}^{d\times 1}$.] Note that $\sum_{i=1}^s A_i^* A_i = I$ and that the elements in the set $\{A_i^* A_j\}_{i,j\in\{1,2,\ldots,s\}} = \{|i\rangle\langle j|\}_{i,j\in\{1,2,\ldots,s\}}$ are linearly independent. By Theorem 3, we have $\mathcal{E} \in \mathcal{E}^e_{s,t,s}$ and therefore $\mathcal{E}^e_{s,t,s} \neq \emptyset$. □

Note that the above shows that the channel corresponding to the operation of discarding a system (tracing out) and then generating a new pure state is extremal.

C. Structure of the Set of Extreme Channels in the Choi-State Representation

We found a smooth description of the set of extreme channels $\mathcal{E}^e_{s,t,r}$ in Section III B. This will allow us to transfer the characterization of extreme channels to the Choi-state representation.

Lemma 15 (Manifold structure for $H^r_{s,+}$)**.** *The set* $H^r_{s,+}$ *is a smooth embedded submanifold of* $\mathbb{R}^{2s^2}$ *of dimension* $2sr - r^2$.

The 'real case' of Lemma 15 was shown in [9] (cf. also [10]), where they considered the manifold of real symmetrical $s \times s$ matrices of rank r. Our proof is a straightforward generalization of the proof given in [9] to the complex case. We begin with some preparatory results.

Definition 4. We define E_r to be a square matrix, whose first r diagonal entries are equal to one and all the other entries are equal to zero. [The dimension of the matrix will always be clear from the context.]

Proposition 16. *We have* $H^r_{s,+} = \{AE_rA^* : A \in \mathrm{GL}(\mathbb{C}, s)\}$*.*

Proof. The inclusion "$\supseteq$" is clear. To see the inclusion "$\subseteq$", let $H \in H^r_{s,+}$. By the spectral theorem there exists a $U \in U(s)$ such that $H = UDU^*$, where D is a $s \times s$ matrix with positive diagonal entries $d_1, d_2, \dots, d_r$ and zeros elsewhere. We define $\tilde{D}$ as the matrix D where we replace the zeroes on the diagonal by ones. Then we have $D = \sqrt{\tilde{D}}E_r\sqrt{\tilde{D}}$, and hence $H = U\sqrt{\tilde{D}}E_r\sqrt{\tilde{D}}U^* = AE_rA^*$, where we set $A = U\sqrt{\tilde{D}} \in \mathrm{GL}(\mathbb{C}, s)$. $\square$

A sufficient condition for orbits of Lie group actions to be smooth manifolds was given in [11].

Definition 5. A map $f : D \to \mathbb{R}^t$ with $D \subset \mathbb{R}^s$ is *semialgebraic* if the graph of f is semialgebraic in $\mathbb{R}^s \times \mathbb{R}^t$.

Theorem 17 (Theorem B4 of Appendix B of [11])**.** *Let $\Phi : G \times S \mapsto S$ be a smooth action of a Lie group G on a smooth manifold S. And suppose that the action is semialgebraic. Then all the orbits are smooth embedded submanifolds of S. [Note that "smooth submanifolds" in Theorem B4 of Appendix B of [11] correspond to "smooth embedded submanifolds" in our terminology.]*

Proof of Lemma 15 (part 1). We define the map $\Phi(A, H) = AHA^* : \mathrm{GL}(\mathbb{C}, s) \times \mathbb{C}^{s\times s} \mapsto \mathbb{C}^{s\times s}$. Note that Φ describes a smooth action of the Lie group $\mathrm{GL}(\mathbb{C}, s)$ on the smooth manifold $\mathbb{C}^{s\times s}$. Moreover, Φ is a semialgebraic map: The (complex) graph of Φ corresponds to the set $\{(A, H, \tilde{H}) \in (\mathbb{C}^{s\times s})^{\times 3} : \det(A) \neq 0 \text{ and } AHA^* - \tilde{H} = 0\}$. We can embed the complex space $(\mathbb{C}^{s\times s})^{\times 3}$ into $(\mathbb{R}^{2s\times s})^{\times 3}$ and rewrite the conditions as real polynomial equations. By Theorem 17 and Proposition 16, we conclude that the orbit $H^r_{s,+} = \{AE_rA^* : A \in \mathrm{GL}(\mathbb{C}, s)\}$ is a smooth embedded submanifold of $\mathbb{R}^{2s\times s} \cong \mathbb{R}^{2s^2}$. To determine the dimension of this manifold, we need an additional result. $\square$

Proposition 18. *Let $p \in H^r_{s,+}$ and write $p = A_p E_r A_p^*$ for some $A_p \in \mathrm{GL}(\mathbb{C}, s)$. Then the tangent space at p is given by $\mathrm{T}_p H^r_{s,+} = \{\Delta E_r A_p^* + A_p E_r \Delta^* : \Delta \in \mathbb{C}^{s\times s}\}$.*

Proof. We define the map $\phi(A) = AE_rA^* : \mathrm{GL}(\mathbb{C}, s) \to H^r_{s,+}$ (where we used Proposition 16 to determine the image space). Note that the map ϕ is smooth, since from the first part of the proof of Lemma 15, the set $H^r_{s,+}$ is a smooth embedded submanifold (and hence we can apply Proposition 2 to the smooth map $\phi'(A) = AE_rA^* : \mathrm{GL}(\mathbb{C}, s) \mapsto \mathbb{C}^{s\times s}$). Then the pushforward of ϕ at A_p is given by $\mathrm{D}\phi_{A_p}(\Delta) = \Delta E_r A_p^* + A_p E_r \Delta^* : \mathrm{T}_{A_p}\mathrm{GL}(\mathbb{C}, s) \mapsto \mathrm{T}_p H^r_{s,+}$. The inclusion "$\supseteq$" of the claim in Proposition 18 follows, because $\mathrm{T}_{A_p}\mathrm{GL}(\mathbb{C}, s) \cong \mathbb{C}^{s\times s}$. To see the inclusion "$\subseteq$", we show that ϕ has constant rank. To see this, note that the pushforward of ϕ at an arbitrary $A \in \mathrm{GL}(\mathbb{C}, s)$ is related to the pushforward at the identity in the following way $\mathrm{D}\phi_A(\Delta A) = \Delta A E_r A^* + AE_rA^*\Delta^* = A\left(A^{-1}\Delta A E_r + E_r A^* \Delta^* (A^*)^{-1}\right) A^* = A\,\mathrm{D}\phi_I\left(A^{-1}\Delta A\right) A^*$. This implies that $\tilde{\Delta} \in \ker(\mathrm{D}\phi_A)$ if and only if $A^{-1}\tilde{\Delta} \in \ker(\mathrm{D}\phi_I)$ and hence $\dim(\ker(\mathrm{D}\phi_I)) = \dim(\ker(\mathrm{D}\phi_A))$ for all $A \in \mathrm{GL}(\mathbb{C}, s)$. By the rank-nullity theorem, we conclude that $\mathrm{rank}(\mathrm{D}\phi_I) = \mathrm{rank}(\mathrm{D}\phi_A)$ for all $A \in \mathrm{GL}(\mathbb{C}, s)$, i.e., ϕ has constant rank. Since ϕ is also surjective by Proposition 16, we can apply the global rank theorem (cf. for example Theorem 4.14 of [7]) to see that ϕ is a submersion. In particular, $\mathrm{D}\phi_{A_p}$ is surjective, which shows the inclusion "$\subseteq$". $\square$

We are now ready to determine the dimension of the manifold $H^r_{s,+}$.

Proof of Lemma 15 (part 2). We define the map ϕ as in the proof of Proposition 18. Then $\mathrm{D}\phi_I(\Delta) = \Delta E_r + E_r\Delta^*$. Since ϕ is a submersion (cf. proof of Proposition 18), $\mathrm{D}\phi_I$ is surjective and hence, by the rank-nullity theorem, $\dim\left(\mathrm{T}_{E_r} H^r_{s,+}\right) = \dim(\mathrm{image}(\mathrm{D}\phi_I)) = \dim(\mathrm{T}_I\mathrm{GL}(\mathbb{C}, s)) - \dim(\ker(\mathrm{D}\phi_I))$, where $\dim(\mathrm{T}_I\mathrm{GL}(\mathbb{C}, s)) = 2s^2$. Note that $\Delta \in \ker(\mathrm{D}\phi_I)$ if and only if $\Delta E_r = -E_r\Delta^*$. Writing Δ in block matrix form $\Delta = [\Delta_{11}, \Delta_{1,2}; \Delta_{21}, \Delta_{22}]$, where $\Delta_{11} \in \mathbb{C}^{r\times r}$, the condition above is equivalent to the two conditions $\Delta_{21} = 0$ and $\Delta_{11} = -\Delta_{11}^*$. Therefore $\dim(\ker(\mathrm{D}\phi_I)) = 2s(s-r) + r^2$ and hence $\dim\left(\mathrm{T}_{E_r} H^r_{s,+}\right) = 2s^2 - (2s(s-r) + r^2) = 2sr - r^2$. $\square$

Lemma 15 allows us to show that the set of all Choi-states corresponding to channels from an s-dimensional to a t-dimensional system of Kraus rank r is a smooth manifold.

Lemma 19 (Manifold structure for $\mathcal{C}_{s,t,r}$)**.** *The set $\mathcal{C}_{s,t,r}$ is a smooth embedded submanifold of $\mathbb{R}^{2s^2t^2}$. Its dimension is $2srt - r^2 - s^2$.*

Proof. Define the smooth map $\Psi(H_{AB}) = \operatorname{tr}_B H_{AB} : H^r_{st,+} \mapsto H_s$ (the smoothness follows again from Proposition 1 and 2). By the Regular Level Set Theorem (cf. Corollary 5.14 of [7]) and because $\dim(H_s) = s^2$ and $\dim(H^r_{st,+}) = 2srt - r^2$ (cf. Lemma 15), it suffices to show that $p' := \frac{1}{s} I \in H_s$ is a regular value of Ψ, i.e., that for all $p \in \Psi^{-1}(p')$ the pushforward $D\Psi_p : \mathrm{T}_p H^r_{st,+} \mapsto \mathrm{T}_{p'} H_s$ is surjective. To see this, choose $p \in \Psi^{-1}(p')$ and write $p = A_p E_r A_p^*$ for some $A_p \in \mathrm{GL}(\mathbb{C}, st)$. Choose a tangent vector $X' \in \mathrm{T}_{p'} H_s \cong H_s$ and write $X' = C + C^*$, where $C = \frac{1}{2} X' \in \mathbb{C}^{s \times s}$. Define $\Delta := s(C \otimes I) A_p \in \mathbb{C}^{st \times st}$ and $X = \Delta E_r A_p^* + A_p E_r \Delta^* \in \mathrm{T}_p H^r_{st,+}$ (by Proposition 18). Since the partial trace is a linear (and continuous) map, we have: $D\Psi_p(X) = \Psi(X) = \operatorname{tr}_B \Delta E_r A_p^* + \left(\operatorname{tr}_B \Delta E_r A_p^*\right)^*$. Using the definition of Δ, we have $\operatorname{tr}_B \Delta E_r A_p^* = s \operatorname{tr}_B (C \otimes I) A_p E_r A_p^* = sC \operatorname{tr}_B A_p E_r A_p^* = sC\Psi(p) = sCp' = C$. We conclude that $D\Psi_p(X) = C + C^* = X'$. Since $X' \in \mathrm{T}_{p'} H_s$ was arbitrary, we showed that the pushforward $D\Psi_p$ is surjective for any $p \in \Psi^{-1}(p')$. $\square$

To describe the set of extreme channels in the Choi-state representation, we transfer the description of Lemma 14 to $\mathcal{C}_{s,t,r}$.

Definition 6. We use $|\gamma\rangle_{A'A} = \frac{1}{\sqrt{s}} \sum_i |i\rangle_{A'} \otimes |i\rangle_A \in \mathbb{C}^{s^2}$ to denote the maximally entangled state between the s-dimensional system A and a copy of this system, denoted by A'. We define the *Choi map* $\Gamma(\mathcal{E}) = \mathcal{I}_{A'} \otimes \mathcal{E}(|\gamma\rangle\langle\gamma|_{A'A}) : \mathcal{L}_{s,t} \mapsto \mathbb{C}^{st \times st}$, where $\mathcal{I}_{A'}$ is the identity map on A' and where $\mathcal{L}_{s,t}$ denotes the set of all linear maps from $\mathbb{C}^{s \times s}$ to $\mathbb{C}^{t \times t}$. The Choi map sends a channel to its Choi-state.

Definition 7 (Definition of the map T)**.** Let $V_{A \mapsto CB} \in \mathbb{C}^{rt \times s}$, where the systems A, B and C have the (complex) dimensions s, t and r respectively. We define the linear map $\mathcal{E}_{V_{A \mapsto CB}}(M_A) = \operatorname{tr}_C V_{A \mapsto CB} M_A V^*_{A \mapsto CB} : \mathbb{C}^{s \times s} \mapsto \mathbb{C}^{t \times t}$. This allows us to define the smooth map $T(V_{A \mapsto CB}) = \Gamma\left(\mathcal{E}_{V_{A \mapsto CB}}\right) : \mathbb{C}^{rt \times s} \mapsto \mathbb{C}^{st \times st}$.

Note that the map T sends a Stinespring dilation $V \in V_{s,rt}$ of a channel $\mathcal{E} \in \mathcal{E}_{s,t,\leqslant r}$ to the Choi-state representation of $\mathcal{E}$.

Lemma 20. *The manifolds $V_{s,t,r}/U(r)$ and $\mathcal{C}_{s,t,r}$ are homeomorphic.*

Proof. Since the Kraus rank of a channel is equal to the rank of the corresponding Choi-state [1], we can consider the map T as a map from $V_{s,t,r}$ to $\mathcal{C}_{s,t,r}$. This map (which we still denote by T) is smooth by Proposition 1 and 2. Let $\pi : V_{s,t,r} \mapsto V_{s,t,r}/U(r)$ denote the quotient map introduced in Lemma 11. For all $U \in U(r)$ we have $T((U \otimes I)V_{A\mapsto CB}) = T(V_{A\mapsto CB})$, because the unitary action corresponds to a change of the basis of the system C, which is traced out. [We can also think of the unitary action as exploiting the unitary freedom on the Kraus representation, so the channel itself is unchanged under the unitary action.] In other words, the map T is constant on the fibers of the quotient map π. By Theorem 4.30 of [7], there is a unique smooth map $\phi : V_{s,t,r}/U(r) \mapsto \mathcal{C}_{s,t,r}$, such that the following diagram commutes.

$$\begin{tikzcd}
V_{s,t,r} \arrow[d, "\pi"] \arrow[dr, "T"] & \\
V_{s,t,r}/U(r) \arrow[r, "\phi"] & \mathcal{C}_{s,t,r}
\end{tikzcd}$$

By Lemma 12 we have a one-to-one correspondence between $\mathcal{E}_{s,t,r}$ and $V_{s,t,r}/U(r)$ (which we denote by $\mathcal{E}_{s,t,r} \leftrightarrow V_{s,t,r}/U(r)$) and by the Choi-Jamiolkowski isomorphism we have $\mathcal{E}_{s,t,r} \leftrightarrow \mathcal{C}_{s,t,r}$. Together, this implies that ϕ is a bijection.

We have left to show that ϕ^{-1} is continuous. We would like to use the fact that a bijective continuous map from a compact space to a Hausdorff space has a continuous inverse (cf. Lemma A.52 of [7]). To make our domain $V_{s,t,r}/U(r)$ compact, we enlarge it to $V_{s,rt}/U(r)$. Let $\tilde{\pi} : V_{s,rt} \mapsto V_{s,rt}/U(r)$ denote the quotient map. [The action of the Lie group $U(r)$ on $V_{s,rt}$ is not free in general.] Since $\tilde{\pi}$ is continuous and the Stiefel manifold $V_{s,rt}$ is compact, $\tilde{\pi}(V_{s,rt}) = V_{s,rt}/U(r)$ is compact. We enlarge the domain of the map T and denote this map by $\tilde{T} : V_{s,rt} \mapsto \mathcal{C}_{s,t,\leqslant r}$. Note that $\tilde{T}$ is continuous. Since $\tilde{T}$ is constant on the fibers of $\tilde{\pi}$, we can define a map $\psi : V_{s,rt}/U(r) \mapsto \mathcal{C}_{s,t,\leqslant r}$, such that the following diagram commutes.

$$\begin{tikzcd}
V_{s,rt} \arrow[d, "\tilde{\pi}"] \arrow[dr, "\tilde{T}"] & \\
V_{s,rt}/U(r) \arrow[r, "\psi"] & \mathcal{C}_{s,t,\leqslant r}
\end{tikzcd}$$

Note that ψ is a bijection, because $\mathcal{E}_{s,t,\leqslant r} \leftrightarrow V_{s,rt}/U(r)$ by Remark 4 of [1], and $\mathcal{E}_{s,t,\leqslant r} \leftrightarrow \mathcal{C}_{s,t,\leqslant r}$ by the Choi-Jamiolkowski isomorphism. The map ψ is continuous, because $\tilde{T}$ is continuous (and by the definition of the quotient topology). Since $V_{s,rt}/U(r)$ is compact and $\mathcal{C}_{s,t,\leqslant r}$ is Hausdorff, ψ^{-1} is continuous.

To see that ϕ^{-1} is continuous we restrict ψ^{-1} to $\mathcal{C}_{s,t,r}$. For this purposes, we define the inclusion map $\imath(V) = V : V_{s,t,r} \mapsto V_{s,rt}$. Note that $\imath$ is continuous and open (because $V_{s,t,r}$ is an open subset of $V_{s,rt}$ by Proposition 6). Since the map $\tilde{\pi} \circ \imath$ is constant on the fibers of π, we can define a map $\tilde{\imath}$ such that the following diagram commutes.

$$\begin{tikzcd} V_{s,t,r} \arrow[r, "\imath"] \arrow[d, "\pi"] \arrow[dr, "\tilde{\pi} \circ \imath"] & V_{s,rt} \arrow[d, "\tilde{\pi}"] \\ V_{s,t,r}/U(r) \arrow[r, "\tilde{\imath}"] & V_{s,rt}/U(r) \end{tikzcd}$$

By Lemma 9, $\tilde{\pi}$ is an open map. Then $\tilde{\pi} \circ \imath$ is an open and continuous map and hence, we can conclude that $\tilde{\imath}$ is continuous and open (and injective). We are now ready to show that ϕ^{-1} is continuous. Note that the restriction $\tilde{\psi}^{-1} : \mathcal{C}_{s,t,r} \mapsto V_{s,rt}/U(r)$ of ψ^{-1} is still continuous. Because $\tilde{\imath}$ is injective and $\psi^{-1}(\mathcal{C}_{s,t,r}) = \tilde{\imath}(V_{s,t,r}/U(r))$ we can define a map $\chi : \mathcal{C}_{s,t,r} \mapsto V_{s,t,r}/U(r)$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C}_{s,t,r} \arrow[r, "\tilde{\psi}^{-1}"] \arrow[dr, "\chi"'] & V_{s,rt}/U(r) \\ & V_{s,t,r}/U(r) \arrow[u, "\tilde{\imath}"] \end{tikzcd}$$

The map χ is continuous because $\tilde{\imath}$ is open and note that $\chi = \phi^{-1}$. $\square$

Theorem 21 (Manifold structure for $\mathcal{C}^{\mathrm{e}}_{s,t,r}$)**.** *The set $\mathcal{C}^{\mathrm{e}}_{s,t,r}$ is an open subset of $\mathcal{C}_{s,t,r}$. In particular it is a smooth embedded submanifold of $\mathcal{C}_{s,t,r}$ (and of $\mathbb{R}^{2s^2t^2}$). Its dimension is $2srt - r^2 - s^2$. Moreover, $\mathcal{C}^{\mathrm{e}}_{s,t,s} \neq \emptyset$.*

Proof. Follows from Lemma 14 together with Lemma 20. $\square$

Remark 2. An alternative and more explicit characterization of extremality in the Choi-state representation is given in Theorem 4 in [6].

IV. DECOMPOSITION OF CHANNELS INTO CONVEX COMBINATIONS OF EXTREME CHANNELS

We show that every element $\mathcal{E} \in \mathcal{E}_{s,t}$ can be decomposed into a convex combination of at most $s^2(t^2-1)+1$ extreme channels in $\mathcal{E}^{\mathrm{e}}_{s,t}$.

Theorem 22 (Convex decomposition)**.** *For every channel $\mathcal{E} \in \mathcal{E}_{s,t}$ there exists a set $\{(p_j, \mathcal{E}_j)\}_{j\in\{1,2,\ldots,k\}}$, where $k \leqslant s^2(t^2-1)+1$, $p_j \in [0,1]$, $\sum_{j=1}^k p_j = 1$ and $\mathcal{E}_j \in \mathcal{E}^{\mathrm{e}}_{s,t}$, such that $\mathcal{E} = \sum_{j=1}^k p_j \mathcal{E}_j$.*

Remark 3. It is conjectured by Ruskai and Audenaert [5] that $k \leqslant t$ if we allow convex combinations of channels $\mathcal{E}_j \in \mathcal{E}_{s,t,\leqslant s}$ (note that $\mathcal{E}_{s,t,\leqslant s}$ is equal to the closure of the set of all s to t extreme channels [5]). However, as far as we know, this remains unproven.

Proof of Theorem 22. In the proof of Lemma 20 we saw that $V_{s,st^2}/U(st)$ is compact and homeomorphic to $\mathcal{C}_{s,t,\leqslant st}$. Therefore, $\mathcal{C}_{s,t} = \mathcal{C}_{s,t,\leqslant st}$ is compact. Since $\mathcal{C}_{s,t} \subset \mathbb{R}^{2s^2t^2}$ is also convex, by the Minkowski Theorem (see for example Theorem 2.3.4 of [12]), $\mathcal{C}_{s,t}$ is the convex hull of its extreme points. By Carathéodory's theorem (see for example Theorem 1.3.6 of [12]), we can always find a decomposition of the required form for which $k \leqslant \dim(\mathrm{aff}[\mathcal{C}_{s,t}]) + 1 = s^2(t^2-1)+1$, where $\mathrm{aff}[\mathcal{C}_{s,t}]$ denotes the affine hull of the set $\mathcal{C}_{s,t}$, i.e., $\mathrm{aff}[\mathcal{C}_{s,t}] = \{C_{AB} \in H_{st} : \mathrm{tr}_B\, C_{AB} = \frac{1}{s} I\}$. $\square$

V. APPLICATION: IMPLEMENTATION OF QUANTUM CHANNELS

Methods for implementing quantum channels from a system A to a system B with low experimental cost as a sequence of simple-to-perform operations were considered in [3; 4; 13; 14]. In [3], a lower bound on the number of parameters required for a quantum circuit topology that is able to perform arbitrary extreme channels from m to n qubits (i.e., from a system A of dimension $d_A = 2^m$ to a system B of dimension $d_B = 2^n$) was given. Here, we give a rigorous mathematical proof of this statement and strengthen the result by showing that a circuit topology that has fewer parameters than required by the lower bound is not able to approximate every extreme channel from m to n qubits arbitrarily well.

From a mathematical point of view, a quantum circuit topology can be defined as follows.

Definition 8. A *quantum circuit topology* is a 5-tuple $Z := (d_A, d_B, d_C, p, h)$, where $d_A, d_B, d_C \in \mathbb{N}$, $d_B d_C \geqslant d_A$, $p \in \mathbb{N}_0$ and $h : [0, 2\pi]^p \mapsto V_{d_A, d_B d_C}$ is a smooth function.

The physical interpretation is the following: We consider a quantum system BC of dimension $d_{BC} := d_B d_C$, where an input state for a quantum channel is given on a subsystem A of dimension d_A and where the other part of the system BC starts in a fixed pure state. We think of a fixed sequence of unitary operations performed on the system BC, where the unitaries have p free parameters between them. Since the parameters corresponds to rotational angles in [3], we take them to lie in the interval $[0, 2\pi]$. [We could replace 2π by any positive real number without changing one of the following statements.] Each choice of parameters corresponds to the implementation of a certain isometry from the system A to the system BC. The function h maps each choice of the parameters to the corresponding isometry. After performing the isometry, the system C is discarded (traced out), and we read out the output of the channel on the remaining system B.

Definition 9. The set of quantum channels (in the Choi-state representation) that can be generated by the quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$ is defined by $H(Z) := T(h([0, 2\pi]^p))$, where the map T was introduced in Definition 7 and where we take the partial trace over the first d_C-dimensional system, i.e., the partial trace $\mathrm{tr}_C(\cdot)$ corresponds to $\sum_{i=1}^{d_C} (\langle i| \otimes I) \cdot (|i\rangle \otimes I)$. [Note that this specification does not restrict the physical setting since we can always adapt the map h, such that the output of a channel is read out at the last d_B dimensional system.]

Lemma 23. *Let $r \in \mathbb{N}$ be fixed and $O \subset \mathcal{C}_{d_A, d_B, r}$ open (and non empty). A quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$ with $p < \dim(\mathcal{C}_{d_A, d_B, r}) = 2d_A d_B r - d_A^2 - r^2$ or $d_C < r$ can only generate a set of measure zero in O, i.e., $H(Z) \cap O$ is of measure zero in O.*

Proof. The idea of the proof is based on Sard's theorem (similar to [15; 16]). Let us fix a quantum circuit topology $Z = (d_A, d_B, d_C, p, h)$, $r \in \mathbb{N}$ and an open set $O \subset \mathcal{C}_{d_A, d_B, r}$. We define the map $T : V_{d_A, d_{BC}} \mapsto \mathcal{C}_{d_A, d_B}$ as in Definition 7, and a map $F = T \circ h$, such that the following diagram commutes.

Case 1 ($d_C < r$): In this case, note that $H(Z) = F([0, 2\pi]^p) \subset T(V_{d_A, d_{BC}})$. But $T(V_{d_A, d_{BC}})$ contains only (Choi-states of) channels of Kraus rank at most $d_C < r$. Therefore, $H(Z) \cap O = \emptyset$.

$$\begin{tikzcd} V_{d_A,d_{BC}} \arrow[dr, "T"] & \\ {[0,2\pi]^p} \arrow[u, "h"] \arrow[r, "F"] & \mathcal{C}_{d_A,d_B} \end{tikzcd}$$

Case 2 ($d_C \geqslant r$): Define the set $S := F([0,2\pi]^p) \cap O$. To show that S has measure zero, define the domain $D = F^{-1}(O)$ and the function $\tilde{F} = F\big|_D : D \mapsto O$. By Sard's theorem (see Appendix A for the full technical details) we conclude that $S = \tilde{F}(D)$ is of measure zero in the smooth submanifold O if $\dim(D) \leqslant p < \dim(O) = \dim(\mathcal{C}_{s,t,r})$. □

Theorem 24 (Strong lower bound)**.** *Let $q \in \mathbb{N}$ and consider a set of quantum circuit topologies $R = \{Z_i = (d_A, d_B, d_{C_i}, p_i, h_i)\}_{i\in\{1,2,\ldots,q\}}$ where for each $i \in \{1,2,\ldots,q\}$ either $p_i < 2d_A^2(d_B - 1)$ or $d_{C_i} < d_A$. Then there exists an extreme channel $\mathcal{E}_0 \in \mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A}$ and a neighborhood $B(\mathcal{E}_0) \subset \mathcal{C}_{d_A,d_B}$ of $\mathcal{E}_0$, such that for all $\mathcal{E} \in B(\mathcal{E}_0)$ we have $\mathcal{E} \notin \bigcup_{i=1}^q H(Z_i)$.*

This theorem considers a finite set of circuit topologies each of which either has fewer free parameters than the dimension of the set of extreme channels or discards a system whose dimension is too low to generate channels of the maximal Kraus rank for any extreme channel. The theorem says that there exist extreme channels that cannot be approximated arbitrarily well using circuit topologies from this set.

Proof. By Theorem 21, the set $\mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A} \neq \emptyset$ is an open subset of $\mathcal{C}_{d_A,d_B,d_A}$. Hence, $H(Z_i) \cap \mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A}$ is of measure zero in $\mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A}$ by Lemma 23. Since a finite union of set of measure zero is again of measure zero, we conclude that $S := \bigcup_{i=1}^q \left(H(Z_i) \cap \mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A}\right)$ is of measure zero. Since $\mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A} \neq \emptyset$, we can choose a channel $\mathcal{E}_0 \in \mathcal{C}^{\mathrm{e}}_{d_A,d_B,d_A} \cap S^{\mathrm{c}}$ and hence $\mathcal{E}_0 \in H^{\mathrm{c}}$, where we set $H := \bigcup_{i=1}^q H(Z_i)$. We have left to show that H is closed in $\mathcal{C}_{d_A,d_B}$. To see this, note that the map $T_i : V_{d_A,d_{BC_i}} \mapsto C_{d_A,d_B}$ (as defined in Definition 7) is continuous and hence $F_i := T_i \circ h_i : [0,2\pi]^{p_i} \mapsto C_{d_A,d_B}$ is also continuous. Since $[0,2\pi]^{p_i}$ is compact, $H(Z_i) = F_i([0,2\pi]^{p_i})$ is closed in $\mathcal{C}_{d_A,d_B}$ and therefore H is also closed. □

VI. ACKNOWLEDGEMENTS

We thank Matthias Christandl for useful discussions. R.C. acknowledges support from the EPSRC's Quantum Communications Hub (grant number EP/M013472/1).

Appendix A: Smoothness of $\tilde{F}$ and a version of Sard's theorem

The domain D of the function $\tilde{F}$ in the proof of Lemma 23 might not be open. We therefore first clarify the meaning of smoothness in the case of arbitrary domains.

Definition 10. Let $D \subset \mathbb{R}^s$. A function $F : D \mapsto \mathbb{R}^t$ is *smooth* if for all $p \in D$ there exists a neighborhood $B(p)$ of p in $\mathbb{R}^s$, such that there exists an extension $\hat{F} : B(p) \mapsto \mathbb{R}^t$ of F, with $\hat{F}$ smooth.

Lemma 25 (Measure of the image)**.** *Let $s < t$, $D \subset \mathbb{R}^s$ and $F : D \mapsto \mathbb{R}^t$ be smooth. Then $F(D)$ has measure zero in $\mathbb{R}^t$.*

Proof. We define $D' := D \times \{0\} \times \{0\} \times \cdots \times \{0\} \subset \mathbb{R}^t$. Note that D' lies in a affine subspace of $\mathbb{R}^t$ and is therefore of measure zero (in $\mathbb{R}^t$). Let $F' := F \circ \pi : D' \mapsto \mathbb{R}^t$, where $\pi : \mathbb{R}^t \mapsto \mathbb{R}^s$ denotes the projection map to the first s coordinates. To show that F' is smooth, choose $p' \in D'$ and let $p := \pi(p')$. Because F is smooth by assumption, there exists a neighborhood $B(p) \subset \mathbb{R}^s$ around the point p and a smooth extension of F denoted by $\hat{F} : B(p) \to \mathbb{R}^t$. Hence $\hat{F}' := \hat{F} \circ \pi : B(p) \times \mathbb{R}^{t-s} \mapsto \mathbb{R}^t$ is a smooth extension of F' around p'. Therefore, F' is a smooth map and by Proposition 6.5 of [7], we conclude that $F'(D') = F(D)$ is of measure zero. □

Lemma 26 (Restrict the range of a smooth map)**.** *Let $D \subset \mathbb{R}^s$ be arbitrary and let N be a smooth manifold. Let N' be a smooth embedded submanifold of N and $F : D \mapsto N$ be a smooth map, such that $F(D) \subset N'$. Then $\tilde{F} : D \mapsto N'$ is smooth.*

Proof. The proof works analogously to the proof of Theorem 5.29 of [7] (see also Corollary 5.30 of [7]). □

Proof: $\tilde{F}(D)$ is of measure zero (completes the proof of case 2 of Lemma 23). We use the notation of the proof of Lemma 23. First note that the function $F\big|_D : D \mapsto \mathbb{C}^{d_A d_B \times d_A d_B}$ is smooth. By Lemma 26 the function $\tilde{F} : D \mapsto O$ is also smooth. To show that $S := \tilde{F}(D) \subset O$ is of measure zero, we choose a collection of smooth charts $\{(U_\alpha, \phi_\alpha)\}$ of the submanifold O whose domains cover S. By Lemma 6.6 of [7] we have left to show that for all α the image $\phi_\alpha(S \cap U_\alpha)$ is of measure zero in $\mathbb{R}^d$, where d denotes the dimension of O. Consider the smooth map $\tilde{F}_\alpha := \phi_\alpha \circ \tilde{F} : D_\alpha := \tilde{F}^{-1}(U_\alpha) \to V_\alpha := \phi_\alpha(U_\alpha) \subset \mathbb{R}^d$.

By Lemma 25, the image $\tilde{F}_\alpha(D_\alpha) = \phi_\alpha(S \cap U_\alpha)$ is of measure zero if $p < d$.

REFERENCES

[1]M.-D. Choi, Linear Algebra and its Applications **10**, 285 (1975).

[2]K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Springer Verlag (1983).

[3]R. Iten, R. Colbeck and M. Christandl, *Quantum circuits for quantum channels*, arXiv:1609.08103.

[4]D.-S. Wang, D. W. Berry, M. C. de Oliveira and B. C. Sanders, Phys. Rev. Lett. **111**, 130504 (2013).

[5]M. B. Ruskai, *Some Open Problems in Quantum Information Theory*, arXiv:0708.1902 (2007).

[6]S. Friedland and R. Loewy, *On the extreme points of quantum channels*, Linear Algebra and its Applications **498**, 553-573 (2016).

[7]J. M. Lee, *Introduction to Smooth Manifolds* (2nd edition), Graduate Texts in Mathematics **218**, Springer New York, (DOI 10.1007/978-1-4419-9982-5) (2012).

[8]D. B. Fuchs and O. Ya. Viro, Topology II, Springer-Verlag (2004).

[9]U. Helmke and M. A. Shayman, *Critical points of matrix least squares distance functions*, in Linear Algebra and its Applications **215**, 1–19 (1995).

[10]B. Vandereycken, P.-A. Absil and S. Vandewalle, *Embedded geometry of the set of symmetric positive semidefinite matrices of fixed rank*, SSP09, 389 (2009).

[11]C. G. Gibson, *Singular points of smooth mappings*, Research Notes in Mathematics **25**, Pitman (1979).

[12]J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer-Verlag Berlin Heidelberg (2001).

[13]D.-S. Wang and B. C. Sanders, New J. Phys. **17**, 043004 (2015).

[14]D.-S. Wang, *Convex decomposition of dimension-altering quantum channels*, arXiv:1510.01040 (2015).

[15]V. V. Shende, I. L. Markov, and S. S. Bullock, Phys. Rev. A **69**, 062321 (2004).

[16]V. Shende, I. Markov, and S. Bullock, in *Proceedings of the Design, Automation and Test in Europe Conference and Exhibition*, 2004, p. 980.