*Journal of Applied Mathematics and Stochastic Analysis*, **12**:1 (1999), 85-90.

# LINEAR FILTERING WITH FRACTIONAL BROWNIAN MOTION IN THE SIGNAL AND OBSERVATION PROCESSES

M.L. KLEPTSYNA
*Russian Academy of Sciences*
*Institute of Information Transmission Problems*
*Moscow 101447, Russia*
*marina@sci.lpi.ac.ru*

P.E. KLOEDEN
*Johan Wolfgang Goethe Universität*
*Fachbereich Mathematik*
*D-60054 Frankfurt am Main Germany*
*kloeden@math.uni-frankfurt.de*

V.V. ANH
*Queensland University of Technology*
*School of Mathematical Sciences*
*Brisbane 4001, Australia*
*v.anh@fsc.qut.edu.au*

(Received December, 1997; Revised July, 1998)

Integral equations for the mean-square estimate are obtained for the linear filtering problem, in which the noise generating the signal is a fractional Brownian motion with Hurst index $h \in (3/4, 1)$ and the noise in the observation process includes a fractional Brownian motion as well as a Wiener process.

**Key words:** Linear Filtering, Fractional Brownian Motion, Long-Range Dependence, Optimal Mean-Square Filter.

**AMS subject classifications:** 93E11, 60G20, 60G35.

## 1. Introduction

We consider the linear problem with the signal $\theta_t$ and the observation $\xi_t$ defined by the linear equations

$$\theta_t = \int_0^t a(s)\theta_s ds + B_t^h, \; \xi_t = \int_0^t A(s)\theta_s ds + W_t + B_t^h, \tag{1}$$

 

where the noise generating the signal is a fractional Brownian motion (fBm) $B_t^h$ with Hurst index $h \in (3/4, 1)$ and the noise disturbing the observation of the signal consists of both a standard Wiener process $W_t$ and the fractional Brownian motion $B_t^h$. The coefficients $a(t)$ and $A(t)$ are bounded measurable functions and the noise processes $B_t^h$ and $W_t$ are independent.

Fractional Brownian motion $B_t^h$ with Hurst index $h \in (1/2, 1)$ is often used to model the long-range dependence in random data commonly encountered in many financial and environmental applications [7, 9]. It is a zero mean Gaussian process having the correlation function

$$\Gamma^h(t,s) = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right), \quad 1/2 < h < 1. \tag{2}$$

It is known that $B_t^h$ is not a semimartingale (see e.g. [4, 6]), so neither is the signal process $\theta_t$ nor the observation process $\xi_t$, and the martingale approach to filtering expounded in [6] is not applicable here. In particular, as shown in [8], we cannot uniquely determine an innovation process corresponding to $\xi_t$. Nevertheless, we can derive an explicit expression for the conditional expectation of the signal

$$\widehat{\theta}_t \triangleq \mathbb{E}(\theta_t \mid \xi_s, 0 \le s \le t),$$

using a theorem on normal correlation in [5] provided we restrict the Hurst index $h$ to the interval $(3/4, 1)$. We formulate this results as a theorem in the next section and present its proof in Section 3. Finally, a simple example is provided in Section 4 to illustrate the result.

## 2. The Optimal Filter

Let $\mathcal{F}_t^\xi$ be the $\sigma$-algebra $\sigma(\xi_s, 0 \le s \le t)$ and note that $\widehat{\theta}_t = \mathbb{E}\left(\theta_t \mid \mathcal{F}_t^\xi\right)$. Define

$$K(t,s) \triangleq \mathbb{E}(\theta_t \theta_s), \quad \widetilde{K}(t,s) \triangleq \mathbb{E}(\theta_t B_s^h).$$

Then it follows directly from the first equation of (1) that $K(t,s)$ and $\widetilde{K}(t,s)$ satisfy the system of integral equations

$$K(t,s) = \int_0^s a(l) K(t,l) dl + \widetilde{K}(t,s), \tag{3}$$

$$\widetilde{K}(t,s) = \int_0^t a(l) \widetilde{K}(l,s) ds + \Gamma^h(t,s). \tag{4}$$

With these we can obtain an explicit closed-form representation of the optimal mean-square filter for system (1).

**Theorem 2.1:** *There exists a unique deterministic function* $\Phi \in L^2([0,T]^2, \mathbb{R})$ *satisfying*

$$\Phi(t,s) = -\int_0^s \Phi(t,\tau)[h(2h-1) \mid s-\tau \mid^{2h-2} \tag{5}$$

$$+ A(s)A(\tau)K(\tau,s) + A(\tau)\frac{\partial \widetilde{K}}{\partial s}(\tau,s) + A(s)\frac{\partial \widetilde{K}}{\partial \tau}(s,\tau)]d\tau$$
$$+ A(s)K(t,s) + \frac{\partial \widetilde{K}}{\partial s}(t,s)$$

*such that the optimal mean-square filtering estimate* $\widehat{\theta}_t$ *of the linear system* (1) *satisfies*

$$\widehat{\theta}_t = \int_0^t \Phi(t,s)d\xi_s, \tag{6}$$

*for* $t \in [0,T]$, *where the integral is understood in the mean-square sense.*

It follows from the proof of Theorem 1 that system (5) has a solution. This solution is in fact unique.

**Theorem 2.2:** *The system of integral equations* (3)-(5) *has a unique solution.*

## 3. Proof of Theorem 1

We note that the joint distribution of $(\xi_s, \theta_t)$ for all $0 \le s,t \le T$ is Gaussian, so Theorem 13.1 of [5] on normal correlation holds here. Let $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{2^n}^{(n)} = t$ be the dyadic partition of $[0,t]$, that is, with $t_j^{(n)} = \frac{j}{2^n}t$ for $j = 0,1,\ldots,2^n$, and denote the $\sigma$-algebra $\sigma\left(\xi_{t_0^{(n)}}, \xi_{t_1^{(n)}} - \xi_{t_0^{(n)}}, \ldots, \xi_{t_{2^n}^{(n)}} - \xi_{t_{2^n-1}^{(n)}}\right)$ by $\mathfrak{F}_t^{\xi,n}$. Then $\mathfrak{F}_t^{\xi,n} \uparrow \mathfrak{F}_t^{\xi}$ as $n \to \infty$, so

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi,n}\right) \to \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right), \quad n \to \infty$$

for all $t \in [0,T]$. Furthermore,

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi,n}\right) - \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right)\right]^2 = 0 \tag{7}$$

for all $t \in [0,T]$. Hence using Theorem 13.1 of [5] we obtain

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi,n}\right) = \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right) + \sum_{j=1}^{2^n-1} \Phi_n\left(t, t_j^{(n)}\right)\left(\xi_{t_{j+1}^{(n)}} - \xi_{t_j^{(n)}}\right) \tag{8}$$

for all $t \in [0,T]$, where $\Phi_n: [0,T]^2 \to \mathbb{R}$ is a deterministic function. Denote $\Phi(t,s) = \Phi_n(t, t_j^{(n)})$ for $t_j^{(n)} \le s < t_{j+1}^{(n)}$. Then we can rewrite (8) as

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi,n}\right) = \int_0^t \Phi_n(t,s)d\xi_s. \tag{9}$$

But the processes $W_t$ and $(B_t^h, \theta_t)$ are independent, so

$$\mathbb{E}\left[\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi,n}\right) - \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{x,m}\right)\right]^2 = \int_0^t \mid \Phi_n(t,s) - \Phi_m(t,s) \mid^2 ds$$

$$+\mathbb{E}\left\{\int_0^t \Phi_n(t,s)(dB_s^h + A(s)\theta_s ds) - \int_0^t \Phi_m(t,s)(dB_s^h + A(s)\theta_s ds)\right\}^2.$$

(The integration of a deterministic function with respect to an fBm here is understood in the mean-square sense, cf. [4]). Applying (7) we obtain that

$$\lim_{n,m\to\infty}\int_0^t |\Phi_n(t,s)-\Phi_m(t,s)|^2 ds = 0,$$

so the sequence $\{\Phi_n\}$ is a Cauchy sequence in $L^2[0,t]$. Hence there exist a function $\Phi \in L^2[0,t]$ such that

$$\lim_{n\to\infty}\int_0^t |\Phi_n(t,s)-\Phi(t,s)|^2 ds = 0.$$

It then follows from [1] that

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\int_0^t \Phi_n(t,s)dB_s^h - \int_0^t \Phi(t,s)dB_2^h\right|^2\right) = 0$$

and

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\int_0^t \Phi_n(t,s)A(s)\theta_s ds - \int_0^t \Phi(t,s)A(s)\theta_s ds\right|^2\right) = 0,$$

so we obtain

$$\widehat{\theta}_t = \int_0^t \Phi(t,s)d\xi_s.$$

We shall now show that $\Phi$ satisfies equation (5). Let $f:[0,T]^2 \to \mathbb{R}$ be a bounded and jointly measurable function, so the integral

$$I_t := \int_0^t f(t,s)d\xi_s$$

is well-defined and the process $I_t$ is $\mathcal{F}_t^\xi$-measurable with $\mathbb{E}(I_t^2) < \infty$ and $\mathbb{E}((\theta_t - \widehat{\theta}_t)I_t) = 0$. Consequently

$$\mathbb{E}(\theta_t I_t) = \mathbb{E}(\widehat{\theta}_t I_t)$$

$$= \mathbb{E}\left(\int_0^t \Phi(t,s)d\xi_s \int_0^t f(t,s)d\xi_s\right)$$

$$= \int_0^t \Phi(t,s)f(t,s)ds + \int_0^t\int_0^t \Phi(t,\tau)f(t,s)\frac{\partial^2}{\partial s\partial r}\Gamma^h(\tau,s)d\tau ds$$

$$+ \int_0^t\int_0^t \Phi(t,\tau)f(t,s)A(\tau)A(s)K(\tau,s)d\tau ds$$

$$+ \int_0^t\int_0^t \Phi(t,\tau)f(t,s)A(s)\frac{\partial\widetilde{K}}{\partial\tau}(s,\tau)d\tau ds$$

$$+ \int_0^t\int_0^t \Phi(t,\tau)f(t,s)A(\tau)\frac{\partial\widetilde{K}}{\partial s}(\tau,s)d\tau ds$$

and

$$\mathbb{E}\left(\theta_t\int_0^t f(t,s)d\xi_s\right) = \int_0^t f(t,s)\frac{\partial\widetilde{K}}{\partial s}(t,s)ds + \int_0^t f(t,s)A(s)K(t,s)ds.$$

Now, $f$ is otherwise arbitrary and the function $\frac{\partial^2}{\partial t\partial s}\Gamma^h(t,s) = h(2h-1)\mid t-s\mid^{2h-2}$, so

$$\int_0^T\int_0^T\left(\frac{\partial^2}{\partial t\partial s}\Gamma^h(t,s)\right)^2 dtds < \infty,$$

which means the process $W_t + B_t^h$ is equivalent in an innovation sense [8] to $W_t$. Hence equation (5) is valid. Uniqueness follows from the linearity of the equations under consideration.

This completes the proof of Theorem 2.1.

## 4. Example

Consider the case $a(t) \equiv 0$ and $A(t) \equiv 0$, so system (1) reduces to

$$\theta_t = B_t^h,\ \ \xi_t = W_t + B_t^h. \tag{10}$$

By Theorem 2.1, the filtering estimate $\widehat{\theta}_t = \int_0^t\Phi(t,s)d\xi_s$ and $\Phi$ satisfies the integral equation

$$\Phi(t,s) = -\int_0^t h(2h-1)\Phi(t\tau)\mid s-\tau\mid^{2h-2}d\tau$$

$$+\frac{1}{2}\frac{\partial}{\partial s}\ \{t^{2h}+s^{2h}-\mid t-s\mid^{2h}\}, \tag{11}$$

which is a Fredholm integral equation with a singular kernel. However, for $h > 3/4$ this kernel is square integrable and equation (11) has a unique solution. In this case, the filtering error $\gamma_t = \mathbb{E}\left(\left|\theta_t - \widehat{\theta}_t\right|^2\right)$ is given by

$$\begin{aligned}
\gamma_t &= \mathbb{E}\left(|\theta_t|^2\right) - \mathbb{E}\left(\left|\widehat{\theta}_t\right|^2\right) \\
&= t^{2h} - \mathbb{E}\left(\left|\int_0^t \Phi(t,s)d\xi_s\right|^2\right) \\
&= t^{2h} - \int_0^t \Phi(t,s)\left[\Phi(t,s) + \int_0^t h(2h-1)\Phi(t,\tau)\,|s-\tau|^{2h-2}d\tau\right]ds \\
&= t^{2h} - \frac{1}{2}\int_0^t \Phi(t,s)\frac{\partial}{\partial s}\left[t^{2h} + s^{2h} - |t-s|^{2h}\right]ds
\end{aligned}$$

using (11).

## Acknowledgement

Partially supported by the Australian Research Council Grants A89601825 and C1960019. The authors wish to thank the referee for his many constructive comments.

## References

[1] Dai, *Long-range Dependent Processes and Fractional Brownian Motion*, Ph.D. thesis, Australian National University 1996.

[2] Kleptsyna, M.L., Kloeden, P.E. and Anh, V.V., Linear filtering with fractional Brownian motion, *Stoch. Anal. Appl.* (to appear).

[3] Kleptsyna, M.L., Kloeden, P.E. and Anh, V.V., Nonlinear filtering with fractional Brownian motion, *J. Info. Trans. Problems* (to appear).

[4] Lin, S.J., Stochastic analysis of fractional Brownian motion, *Stoch. and Stochastics Rep.* **55** (1995), 121-140.

[5] Lipster, R.S. and Shiryayev, A.N., *Statistics of Random Processes*, Volume II, Springer-Verlag, Berlin 1978.

[6] Lipster, R.S. and Shiryayev, A.N., *Theory of Martingales*, Springer-Verlag, Berlin 1984.

[7] Mandelbrot, B.B. and Van Ness, J.W., Fractional Brownian motion, fractional noises and applications, *SIAM Review* **10** (1968), 422-437.

[8] Rozovskii, B.L., *Evolutionary Stochastic Systems, Linear Theory with Applications to the Statistics of Random Processes*, Nauka, Moscow 1983.

[9] Samorodnitsky, G. and Taqqu, M..S., *Stable Non-Gaussian Random Processes*, Chapman and Hall, New York 1994.