EXTREMES OF RANDOMLY SCALED GUMBEL RISKS

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July 9, 2017

Abstract. We investigate the product Y_1Y_2 of two independent positive risks Y_1 and Y_2. If Y_1 has distribution in the Gumbel max-domain of attraction with some auxiliary function which is regularly varying at infinity and Y_2 is bounded, then we show that Y_1Y_2 has also distribution in the Gumbel max-domain of attraction. If both Y_1, Y_2 have log-Weibullian or Weibullian tail behaviour, we prove that Y_1Y_2 has log-Weibullian or Weibullian asymptotic tail behaviour, respectively. We present here three theoretical applications concerned with a) the limit of point-wise maxima of randomly scaled Gaussian processes, b) extremes of Gaussian processes over random intervals, and c) the tail of supremum of iterated processes.

Keywords and phrases: Gumbel max-domain of attraction; random scaling; log-Weibullian tail behaviour; Weibullian tail behaviour; supremum of Gaussian processes; iteration of random processes.

1. Introduction

Consider $Y_1 \geq 0$ and Y_2 two independent random variables (rvs). If Y_2 is bounded, say $|Y_2| \leq 1$ almost surely, then Y_1Y_2 is commonly referred to as a random contraction. A classical example of this random structure is the case of Y_1Y_2 having an $N(0,1)$ distribution with Y_1^2 a chi-square rv with 1 degree of freedom and Y_2 a symmetric rv around 0 with Y_2^2 having Beta distribution with parameters 1/2,1/2.

In this contribution we are interested also in the case that Y_2 is unbounded. The study of products of rvs is of interest for numerous applications, see e.g., [1–13]. We mention below three recent fields of investigations:

i) An important instance of contraction models is the chaos of random vectors with radial representation. Specifically, if $\boldsymbol{W} = (\mathcal{R}U_1, \ldots, \mathcal{R}U_d)$ is a d-dimensional random vector with $\mathcal{R} > 0$ independent of U_i's, then $X = h(W_1, \ldots, W_d)$ is referred to as the chaos

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of $\boldsymbol{W}$, if further h is a homogeneous function of order $\alpha > 0$, i.e., $h(cw_1, \dots, cw_d) = c^\alpha h(w_1, \dots, w_d)$ for any $c > 0$ and all $w_1, \dots, w_d \in \mathbb{R}$. Consequently,

$$X \stackrel{d}{=} \mathcal{R}^\alpha h(U_1, \dots, U_d) =: Y_1 Y_2, \tag{1.1}$$

with $Y_1 > 0$ being independent of Y_2. In our notation $\stackrel{d}{=}$ means the equality of the distributions. A canonical instance is the Gaussian chaos, where $\boldsymbol{W}$ is a centred Gaussian random vector with $\mathcal{R}^2$ being a chi-square distribution with d degrees of freedom, see [14, 15] for recent results.

ii) In numerous applied problems the study of supremum of a random process $Z(t), t \ge 0$ over a random time interval, say $[0, S]$ with S a positive rv being independent of Z is of particular interest, see e.g., [3, 6, 16, 17]. An interesting instance is $Z = B_H$, with B_H a standard fractional Brownian motion with Hurst index $H \in (0, 1)$. By the self-similarity of B_H we have

$$\sup_{t \in [0,S]} B_H(t) \stackrel{d}{=} \Big(\sup_{t \in [0,1]} B_H(t) \Big) S^H =: Y_1 Y_2. \tag{1.2}$$

iii) Let $X(t), t \in \mathbb{R}$ and $Y(t), t \ge 0$ be two independent random processes. Motivated by [18, 19] several contributions have investigated the basic properties of the iterated process $Z(t) = X(Y(t)), t \ge 0$, see e.g., [20–22] and the references therein. One particular instance is $X = B_H$, and thus by the self-similiarity of fractional Brownian motion we have

$$Z(t) \stackrel{d}{=} B_H(1) |Y(t)|^H =: Y_1 Y_2 \tag{1.3}$$

for any $t > 0$. If X is a more general Gaussian process, for instance X being centred with stationary increments the direct relation in (1.3) does not hold. In view of [5][Theorem 2.1], if both $\mathcal{A}_1 = \sup_{t \in [0,T]} Y(t)$ and $\mathcal{A}_2 = -\inf_{t \in [0,T]} Y(t)$ have for a given $T > 0$ a power-exponential tail behaviour (see the definition given in (1.6) below)), then

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} Z(t) > u \right\} \sim \mathbb{P}\{X(\mathcal{A}_1) > u\} + \mathbb{P}\{X(\mathcal{A}_2) > u\} \tag{1.4}$$

as $u \to \infty$; here $f_1(u) \sim f_2(u)$ means $\lim_{u \to \infty} f_1(u)/f_2(u) = 1$.

Since $X(\mathcal{A}_i) \stackrel{d}{=} \sigma(\mathcal{A}_i) X(1)$ with σ^2 the variance function of X, under tractable assumptions on σ the asymptotics of the right-hand side of (1.4) can be explicitly determined if the tail asymptotic behaviour of the product $\sigma(\mathcal{A}_i) X(1)$ can be obtained. This motivates our investigation of the tail asymptotics of products for unbounded Y_1 and Y_2, see our result in Theorem 2.1 below.

The large values of $Y_1 Y_2$ correspond to large values of both Y_1 and Y_2. However, if Y_2 is bounded (i.e., we have the contraction model), we expect that the asymptotic tail behaviour of $Y_1 Y_2$ will essentially be determined by that of Y_1. This intuition is confirmed in Theorem 1.1 below for the case Y_1 has a distribution with unbounded support, being further in the Gumbel max-domain of attraction (MDA), i.e.,

$$\lim_{u\to\infty} \frac{\mathbb{P}\{Y_1 > u + a(u)t\}}{\mathbb{P}\{Y_1 > u\}} = \exp(-t), \quad \forall t \in \mathbb{R} \tag{1.5}$$

for some positive scaling function $a(\cdot)$. Note in passing that (1.4) implies that $\lim_{u\to\infty} \frac{a(u)}{u} = 0$. The Gumbel MDA consists of many important distributions including Gamma, Normal and log-Normal one. Here we are concerned with a large class of such distributions that have a regularly varying scaling function $a(\cdot)$ at infinity with index $-\tau$ for $\tau \ge -1$, i.e., $\lim_{u\to\infty} a(ux)/a(u) = x^{-\tau}, x > 0$.

We abbreviate (1.5) as $Y_1 \in GMDA(a)$ and refer to, e.g., [23–25] for details on the Gumbel max-domain of attraction and regular variation.

Hereafter, we shall assume without loss of generality that Y_2 is a strictly positive rv.

Theorem 1.1. *If condition* (1.5) *holds with $a(\cdot)$ being regularly varying at infinity with index $-\tau$ for $\tau \ge -1$ and Y_2 has distribution with right endpoint equal to 1, then $Y_1Y_2 \in GMDA(a)$.*

Remark 1.1. *a) If X is the Gaussian chaos, for which* (1.1) *holds, and h is continuous and non-negative, then Theorem 1.1 implies that $X \in GMDA(a)$ with $a(x) = Cx^{1-2/\alpha}, x > 0$ for some C positive, since $\mathcal{R}^2$ is in the Gumbel MDA with scaling function $a(x) = 1/x$. See [14] for further results under additional conditions on h.*

b) From [17][Lemma 3.2] we have that $\sup_{t\in[0,1]} B_H(t) \in MDA(a)$ where $a(x) = 1/x, x > 0$. Consequently, applying Theorem 1.1 to the contraction model given in (1.2) *we obtain that*

$$\sup_{t\in[0,S]} B_H(t) \in GMDA(a), \qquad \text{where } a(x) = 1/x.$$

c) If $Y_1 = e^W$ with W an $N(0,1)$ random variable, then we have that $Y_1 \in GMDA(a)$ with $a(x) = x/\log x$, hence $\tau = -1$ for case. By Theorem 1.1 the contraction Y_1Y_2 is in the Gumbel MDA.

A canonical example for $Y_1 \in GMDA(a)$ is when Y_1 has a power-exponential tail behaviour i.e.,

$$\mathbb{P}\{Y_1 > u\} \sim C_1 u^{\alpha_1} \exp(-L_1 u^{p_1}), \quad u \to \infty, \tag{1.6}$$

where C_1, L_1, p_1 are positive constants and $\alpha_1 \in \mathbb{R}$. Under assumption (1.6) we have $Y_1 \in GMDA(a)$, where $a(x) = x^{1-p_1}/L_1$. Consequently, the assumption of Theorem 1.1 on $a(\cdot)$ holds with $\tau = p_1 - 1$.

If both Y_1 and Y_2 can simultaneously take large values with non-zero probability, then the asymptotic tail behaviour of X is known in few cases. In particular, if also Y_2 satisfies (1.6) with $\alpha_2 \in \mathbb{R}, C_2 > 0, L_2 > 0, p_2 > 0$, then in light of [17][Lemma 2.1]

$$\mathbb{P}\{Y_1Y_2 > u\} \sim C_1C_2 A^{\frac{p_2}{2}+\alpha_2-\alpha_1} \left(\frac{2\pi p_2 L_2}{p_1+p_2}\right)^{1/2} u^{\frac{2p_2\alpha_1+2p_1\alpha_2+p_1p_2}{2(p_1+p_2)}} \exp\left(-Bu^{\frac{p_1p_2}{p_1+p_2}}\right) \tag{1.7}$$

holds as $u \to \infty$, where

$$A = [(p_1 L_1)/(p_2 L_2)]^{\frac{1}{p_1+p_2}} \text{ and } B = L_1 A^{-p_1} + L_2 A^{p_2}. \tag{1.8}$$

Our second result shows that the asymptotic tail behaviour of X can be derived explicitly for a more general case when the power term in the tail expansion of Y_i's in (1.6) is substituted by some regularly varying function, see Theorem 2.1 in Section 2. We refer to, e.g., [1, 2, 4, 10, 11, 15, 26–28] for related results concerned with the asymptotic tail behaviour of the products of rvs.
As an illustration of Theorems 1.1 and 2.1, we shall analyze:

a) limiting behaviour of the maximum of randomly scaled Gaussian processes;

b) exact asymptotic tail behaviour of the supremum of Gaussian processes with stationary increments over a random interval with length which has Weibullian tail behaviour;

c) exact asymptotic tail behaviour of $\sup_{t\in[0,T]} X(Y(t))$ with X a centered Gaussian processes with stationary increments being independent of Y, extending the recent findings of [5].

We organize this paper as follows: Section 2 derives the tail asymptotics of the product of two independent (log-)Weibullian-type rvs. Our applications are presented in Section 3. Proofs of all results are relegated to Section 4, which concludes this article.

2. Log-Weibullian and Weibullian Risks

We say that $Y_i, i = 1,2$ has a *log-Weibullian* tail behaviour (or alternatively Y_i is a log-Weibullian rv), if

$$\lim_{u\to\infty} \frac{\log(\mathbb{P}\{Y_i > u\})}{u^{p_i}} = -L_i \tag{2.1}$$

for some positive constants p_i, L_i. The main result of this section is Theorem 2.1; statement (a) therein shows that if (2.1) holds, then $X = Y_1 Y_2$ has also a log-Weibullian tail behaviour.
The definition of Weibullian tail behaviour is formulated (motivated by (1.6)) by the following condition:

$$\mathbb{P}\{Y_i > u\} \sim g_i(u) \exp(-L_i u^{p_i}), \quad u \to \infty \tag{2.2}$$

for $i = 1,2$, where g_1, g_2 are two given regularly varying at infinity functions and $L_i, p_i, i = 1,2$ are positive constants. We say alternatively that Y_1 and Y_2 are Weibullian-type rvs and abbreviate this fact as

$$Y_i = W(g_i, L_i, \alpha_i).$$

We note that if a rv is of Weibullian-type, then it is log-Weibullian.
For g_1, g_2 being regularly varying and ultimately monotone [29] shows that a similar result to (1.7) is valid. In statement (b) of Theorem 2.1 we remove the assumption that g_1 and g_2 are ultimately monotone.

Theorem 2.1. *Let Y_1, Y_2 be two independent positive rvs, and let $L_i, p_i, i = 1,2$ be positive constants.*

(a) If $Y_i, i = 1,2$ satisfy (2.1) with $p_i, L_i, i = 1,2$, then with B given in (1.8)

$$\lim_{u\to\infty} \frac{\log(\mathbb{P}\{Y_1Y_2 > u\})}{u^{p_1p_2/(p_1+p_2)}} = -B. \tag{2.3}$$

(b) If $Y_i = W(g_i, L_i, p_i), i = 1,2$ and A, B are two constants as in (1.8), then

$$\begin{aligned}\mathbb{P}\{Y_1Y_2 > u\} &\sim Du^{\frac{p_1p_2}{2(p_1+p_2)}} g_1(u/c_u) g_2(c_u) \exp\left(-Bu^{\frac{p_1p_2}{p_1+p_2}}\right)\\ &\sim Du^{\frac{p_1p_2}{2(p_1+p_2)}} \mathbb{P}\{Y_1 > u/c_u\} \mathbb{P}\{Y_2 > c_u\} \end{aligned}\tag{2.4}$$

as $u\to\infty$, where $c_u = Au^{p_1/(p_1+p_2)}$ and $D = \left(\frac{2\pi(p_1L_1)^{\frac{p_2}{p_1+p_2}}(p_2L_2)^{\frac{p_1}{p_1+p_2}}}{p_1+p_2}\right)^{1/2}$.

Remark 2.1. *Theorem 2.1 straightforwardly extends to the case of the product of n rvs. Namely, if $Y_i, i\le n$ are positive independent rvs with tail asymptotics given by (2.2), then $\prod_{i=1}^n Y_i$ also satisfies the condition (2.2) with some g^*, L^* and $p^* = (\sum_{i=1}^n 1/p_i)^{-1}$.*

3. Applications

A direct application of Theorem 1.1 concerns the asymptotics of maxima of products. Specifically, let $(Y_{n1}, Y_{n2}), n\ge 1$ be independent copies of (Y_1, Y_2) and let $F^{\leftarrow}$ and $H^{\leftarrow}$ be the generalized inverse of the distributions of Y_1 and Y_1Y_2, respectively. Define next

$$b_n = F^{\leftarrow}(1-1/n), \quad \widetilde{b}_n = H^{\leftarrow}(1-1/n), \quad n>1.$$

Under the assumptions of Theorem 1.1 for Y_1 and Y_2, we have that (see [30][Eq. (14)])

$$\widetilde{b}_n \sim b_n, \quad n\to\infty. \tag{3.1}$$

Hence by the regular variation of a we get $a(b_n)\sim a(\widetilde{b}_n)$ and further

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\max_{1\le k\le n} Y_{k2} \le a(b_n)x + b_n\right\} - \exp(-\exp(-x))\right| = 0,$$

and

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\max_{1\le k\le n} Y_{k1}Y_{k2} \le a(b_n)x + \widetilde{b}_n\right\} - \exp(-\exp(-x))\right| = 0.$$

These derivations motivate our first application concerned with the investigation of the maximum of randomly scaled Gaussian processes.

The second one, which combines Theorem 2.1 with an interesting finding of [17], derives the asymptotic behaviour of the tail distribution of supremum of Gaussian processes with stationary increments over Weibullian and log-Weibullian random intervals. That result allows us to present a third application concerned with the supremum of an iterated process $Z(t) = X(Y(t)), t\ge 0$ with X Gaussian and Y some general process with continuous sample paths independent of X.

3.1. **Limit law of the maximum of deflated Gaussian processes.** This application is motivated by [31] which studies the point-wise maximum of independent Gaussian processes. Instead of Gaussian processes treated therein, we consider here the point-wise maximum of randomly deflated Gaussian processes. Let therefore $\Gamma(\cdot,\cdot)$ be a negative definite kernel in $\mathbb{R}^2$ and define a Brown-Resnick max-stable process as

$$\eta_{BR}(t) = \max_{i\ge 1}\Big(\Upsilon_i + Z_i(t) - \sigma^2(t)/2\Big), \quad t \in \mathbb{R}, \tag{3.2}$$

where $\{Z_i(t), t \in \mathbb{R}\}, i \ge 1$ are mutually independent centred Gaussian processes with incremental variance function $Var(Z_i(t_1) - Z_i(t_2)) = \Gamma(t_1, t_2), i \ge 1$ and variance function $\sigma^2(\cdot) > 0$, being further independent of the Poisson point process on $\mathbb{R}$ denoted by $\{\Upsilon_i\}_{i\in\mathbb{N}}$ with intensity measure $\exp(-x)\,dx$, see for more details [32, 33].

In the following, for scaling of the Gaussian process, we shall use a generic positive rv S, which has either a distribution with right endpoint 1, or it has a Weibullian tail behaviour satisfying (2.2) with some p, L and g being regularly varying at infinity. Based on the findings of both Theorem 1.1 and Theorem 2.1, our next result generalizes [34][Theorem 5.1], which can be retrieved by setting $S = 1$.

Theorem 3.1. *Let $\{X_{ni}(t), t \in \mathbb{R}\}, 1 \le i \le n, n \ge 1$ be independent Gaussian processes with mean-zero, unit variance function and correlation function $\rho_n(s,t), s,t \in \mathbb{R}$. Let $S_{ni}, i, n \ge 1$ be independent copies of S, and let $H^{\leftarrow}$ be the generalized inverse of the distribution H of $SX_{11}(1)$. Assume that $S_{ni}, X_{ni}(t), t \in \mathbb{R}$ are mutually independent for any $i = 1,\ldots,n$. Suppose further that either S is a bounded positive rv, or $S = W(g, L, p)$. For $d_n = H^{\leftarrow}(1 - 1/n)$ set $c_n = 1/d_n$ if S is bounded, and set $c_n = (2p\log n)/(d_n(2 + p))$ otherwise. If further*

$$\lim_{n\to\infty} \frac{2d_n}{c_n}\Big(1 - \rho_n(t_1,t_2)\Big) \;=\; \Gamma(t_1,t_2) \in (0,\infty), \quad t_1 \ne t_2 \in \mathbb{R}, \tag{3.3}$$

then, as $n \to \infty$

$$c_n\Big(\max_{1\le i\le n} S_{ni}X_{ni}(t) - d_n\Big) \;\Longrightarrow\; \eta_{BR}(t), \quad t \in \mathbb{R}, \tag{3.4}$$

where $\Longrightarrow$ means the weak convergence of the finite-dimensional distributions. Furthermore, $d_n = (1 + o(1))\sqrt{2\log n}$ if S is bounded and $d_n = (1 + o(1))((\log n)/B)^{(2+p)/(2p)}$ otherwise, with $B = LA^{-p} + A^2/2, A = (pL)^{1/(2+p)}$.

3.2. **Supremum over random intervals for Gaussian processes with stationary increments.** The main result of [17] derives the exact asymptotics (as $u \to \infty$) of

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\},$$

where $\{X(t), t \ge 0\}$ with $X(0) = 0$ almost surely is a mean-zero Gaussian process with stationary increments and almost surely continuous trajectories being independent of $\mathcal{T} > 0$, which has tail asymptotics given by (1.6). Based on the statement (a) in Theorem 2.1, the following result extends Theorem 3.1 in the aforementioned paper.

Theorem 3.2. *Let $\mathcal{T}$ be a nonnegative log-Weibullian rv that satisfies (2.1) with some $L, p > 0$ and let $\{X(t), t \geq 0\}$ be, an independent of $\mathcal{T}$, centred Gaussian process with stationary increments and continuously differentiable variance function $\sigma^2(t) = Var(X(t)), t \geq 0$. Suppose that $\sigma^2(t), t \geq 0$ is convex and $\lim_{u\to\infty} \sigma^2(ux)/\sigma^2(u) = x^\alpha, x > 0$ with $\alpha \in (1, 2]$. If further $\sigma^2(t) \leq Kt^\alpha$ holds for any $t > 0$ and some constant $K > 0$, then we have*

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\} \sim \mathbb{P}\left\{\sigma(\mathcal{T})\mathcal{N} > u\right\}, \quad u \to \infty, \tag{3.5}$$

where $\mathcal{N}$ is an $N(0, 1)$ rv independent of $\mathcal{T}$.

A combination of Theorem 2.1 with Theorem 3.2 leads to the following corollary.

Corollary 3.1. *Under the setup of Theorem 3.2 suppose further that $\sigma(t) \sim Ct^{\alpha/2}$ as $t \to \infty$ with $\alpha \in (1, 2]$ and some constant $C > 0$.*
(a) Then $\sigma(\mathcal{T})$ satisfies (2.1) with $\widetilde{p} = \frac{2p}{\alpha}, \widetilde{L} = \frac{L}{C^p}$ and

$$\lim_{u\to\infty} u^{-2\widetilde{p}/(\widetilde{p}+2)} \log\left(\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\}\right) = -\widetilde{L}(\widetilde{L}\widetilde{p})^{\frac{-\widetilde{p}}{\widetilde{p}+2}} - \frac{1}{2}(\widetilde{L}\widetilde{p})^{\frac{2}{\widetilde{p}+2}} =: -\widetilde{B}. \tag{3.6}$$

(b) If $\sigma(\mathcal{T})$ satisfies (2.2) with $\widetilde{p}, \widetilde{L}$ and some regularly varying at infinity function $\widetilde{g}$, then

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\} \sim (\widetilde{p}+2)^{-\frac{1}{2}}\widetilde{g}\left((\widetilde{L}\widetilde{p})^{\frac{-1}{\widetilde{p}+2}} u^{\frac{2}{\widetilde{p}+2}}\right)\exp\left(-\widetilde{B}u^{\frac{2\widetilde{p}}{\widetilde{p}+2}}\right), \quad u \to \infty. \tag{3.7}$$

Remark 3.1. *Clearly, if we specify in the assumptions of Theorem 3.2 that $\sigma^2(x) = Cx^\alpha$ (i.e., X is a fractional Brownian motion with Hurst index $\alpha/2$) and $\mathcal{T}$ is of Weibullian-type, then both $\sigma(\mathcal{T})$ and $\mathcal{N}$ are Weibullian-type rvs, and thus the assumptions of Corollary 3.1 (b) are satisfied. Hence Corollary 3.1 is an extension of [17][Theorem 4.1].*

3.3. **Supremum of iteration of random processes.** In this section $X(t), t \in \mathbb{R}$ is a centred Gaussian process satisfying the assumptions of Theorem 3.2. Let $Y(t), t \geq 0$ be a random process independent of X with continuous sample paths. In view of [5][Theorem 2.1], if both

$$\mathcal{A}_1 = \sup_{t\in[0,T]} Y(t), \text{ and } \mathcal{A}_2 = -\inf_{t\in[0,T]} Y(t)$$

have for a given $T > 0$ a power-exponential tail behaviour specified in (1.6), then as mentioned in the Introduction for $Z(t) = X(Y(t)), t \geq 0$ the approximation (1.4) is valid. If we define $Z(t) = X(SY(t)), t \geq 0$ with $S > 0$ a positive rv independent of X and Y, then in order to apply the aforementioned theorem, we need to have S such that $\bar{\mathcal{A}}_i := S\mathcal{A}_i, i = 1, 2$ has a power-exponential tail behaviour. In view of our Theorem 2.1, statement (b), this is not always the case. In the following we present an extension of Theorem 2.1 in [5], which in particular is applicable for the analysis of $\sup_{t\in[0,T]} X(SY(t))$ when S has a log-Weibullian asymptotic tail behaviour.

Theorem 3.3. *Let $X(t), t \in \mathbb{R}$ satisfy the assumptions of Theorem 3.2 and let $Y(t), t \geq 0$ be a random process independent of X with continuous trajectories. If further $\mathcal{A}_i$'s have a log-Weibullian asymptotic tail behaviour, then we have*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(Y((t)) > u\right\} \sim \mathbb{P}\left\{\sigma(\mathcal{A}_1)\mathcal{N} > u\right\} + \mathbb{P}\left\{\sigma(\mathcal{A}_2)\mathcal{N} > u\right\}, \tag{3.8}$$

where $\mathcal{N}$ is an $N(0,1)$ rv independent of $\mathcal{A}_1, \mathcal{A}_2$.

Remark 3.2. *A canonical example that $\mathcal{A}_i$ has log-Weibullian asymptotic tail behaviour is when Y is centered $\alpha(t)$-locally stationary Gaussian process, see [35–37].*

4. Proofs

It is well-known that for some rv U which has distribution with right endpoint equal to infinity the assumption $U \in GMDA(a)$ implies that the tail of U is rapidly varying at infinity, i.e.,

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{U > \lambda u\right\}}{\mathbb{P}\left\{U > u\right\}} = 0$$

holds for any $\lambda > 1$. First we present a result on random scaling of rvs with rapidly varying tails which is of some interest on its own.

Lemma 4.1. *Let S, Y, Y^* be three independent rvs. Suppose that $S \geq 0$ has distribution G with right endpoint equal to 1. If further Y has a rapidly varying tail and $\mathbb{P}\left\{Y > u\right\} \sim L(u)\mathbb{P}\left\{Y^* > u\right\}$ as $u \to \infty$ for some slowly varying function $L(\cdot)$, then*

$$\mathbb{P}\left\{SY > u\right\} \sim \mathbb{P}\left\{SY > u, S > w\right\} \sim L(u)\mathbb{P}\left\{SY^* > u\right\} \tag{4.1}$$

holds for any $w \in (0,1)$.

Proof of Lemma 4.1 Since S and Y are non-negative, for any $u > 0$ and $w \in (0,1)$, we have

$$\mathbb{P}\left\{SY > u\right\} \leq \mathbb{P}\left\{Y > u/w\right\} + \mathbb{P}\left\{SY > u, S > w\right\}.$$

The assumption that Y has a rapidly varying tail and the independence of S and Y imply for any $t \in (w,1)$

$$\frac{\mathbb{P}\left\{Y > u/w\right\}}{\mathbb{P}\left\{SY > u, S > w\right\}} \leq \frac{\mathbb{P}\left\{Y > u/w\right\}}{\mathbb{P}\left\{SY > u, S > t\right\}} \leq \frac{\mathbb{P}\left\{Y > u/w\right\}}{\mathbb{P}\left\{Y > u/t\right\}\mathbb{P}\left\{S > t\right\}} \to 0, \quad u \to \infty.$$

Hence for any $w \in (0,1)$

$$\mathbb{P}\left\{SY > u\right\} \sim \int_w^1 \mathbb{P}\left\{Y > u/s\right\} dG(s), \quad u \to \infty.$$

By the uniform convergence theorem for regularly varying functions (e.g., [24][Theorem A3.2])

$$\int_w^1 \mathbb{P}\{Y > u/s\}\, dG(s) \sim L(u) \int_w^1 \mathbb{P}\{Y^* > u/s\}\, dG(s), \quad u \to \infty.$$

The assumption $\mathbb{P}\{Y > u\} \sim L(u)\mathbb{P}\{Y^* > u\}$ as $u \to \infty$ yields that Y^* has also a rapidly varying tail at infinity. Hence in view of the above arguments and the fact that S and Y^* are independent, we have that

$$\int_w^1 \mathbb{P}\{Y^* > u/s\}\, dG(s) \sim \mathbb{P}\{SY^* > u\}, \quad u \to \infty$$

establishing the proof. □

Proof of Theorem 1.1 The assumption $Y_1 \in GMDA(a)$ implies that the convergence

$$\frac{\mathbb{P}\{Y_1 > u + xa(u)\}}{\mathbb{P}\{Y_1 > u\}} \to \exp(-x), \quad u \to \infty \tag{4.2}$$

holds uniformly for x on compact sets of $\mathbb{R}$. Since Y_1 has a rapidly varying tail at infinity, then by Lemma 4.1 for any fixed $z \ge 0$ and $w \in (0,1)$

$$\mathbb{P}\{Y_1Y_2 > u + a(u)z\} \sim \int_w^1 \mathbb{P}\{Y_1 > (u + za(u))/s\}\, dG(s), \quad u \to \infty$$

holds with G the distribution of Y_2. By the uniform convergence theorem for regularly varying functions

$$\lim_{u\to\infty} \frac{a(ux)}{a(u)} = x^{-\tau}$$

holds uniformly for $x \in [w, 1]$, with $w \in (0,1)$ some arbitrary constant. Hence

$$z_{u,s} := \frac{z}{s}\frac{a(u)}{a(u/s)} \to \frac{z}{s^{1+\tau}}, \quad u \to \infty$$

uniformly for $s \in [w, 1]$, and thus

$$\frac{\mathbb{P}\{Y_1 > u/s + za(u)/s\}}{\mathbb{P}\{Y_1 > u/s\}} = \frac{\mathbb{P}\{Y_1 > u/s + a(u/s)z_{u,s}\}}{\mathbb{P}\{Y_1 > u/s\}} \to \exp(-z/s^{1+\tau}), \quad u \to \infty$$

uniformly for $s \in [w, 1]$. For any $\varepsilon > 0$ we can find $w \in (0,1)$ such that for all $s \in [w, 1]$

$$(1-\varepsilon)\exp(-z) \le \exp(-z/s^{1+\tau}) < (1+\varepsilon)\exp(-z)$$

implying that as $u \to \infty$

$$\mathbb{P}\{Y_1Y_2 > u + a(u)z\} \sim \int_w^1 \mathbb{P}\{Y_1 > u/s + a(u/s)z_{u,s}\}\, dG(s) \sim \exp(-z)\mathbb{P}\{Y_1Y_2 > u\}.$$

Hence $Y_1Y_2 \in GMDA(a)$ and thus the proof is complete. □

Proof of Theorem 2.1 *Ad.(a).* Since for any $u > 0$

$$\mathbb{P}\{Y_1Y_2 > u\} \ge \mathbb{P}\left\{Y_1 > \left(\frac{p_2L_2}{p_1L_1}\right)^{1/(p_1+p_2)} u^{p_2/(p_1+p_2)}\right\}\mathbb{P}\left\{Y_2 > \left(\frac{p_1L_1}{p_2L_2}\right)^{1/(p_1+p_2)} u^{p_1/(p_1+p_2)}\right\},$$

then we immediately get

$$\liminf_{u\to\infty} \frac{\log(\mathbb{P}\{Y_1Y_2>u\})}{u^{p_1p_2/(p_1+p_2)}} \ge -\left(L_1\left(\frac{p_2L_2}{p_1L_1}\right)^{\frac{p_1}{p_1+p_2}} + L_2\left(\frac{p_1L_1}{p_2L_2}\right)^{\frac{p_2}{p_1+p_2}}\right) =: -B.$$

Next, for all u sufficiently large we have

$$\begin{aligned}\mathbb{P}\{Y_1Y_2>u\} &\le \sum_{k=[u^{p_2/(2(p_1+p_2))}]}^{[u^{(p_2+p_1/2)/(p_1+p_2)}]} \mathbb{P}\{Y_1\in[k,k+1), Y_1Y_2>u\} \\ &\quad + \mathbb{P}\left\{Y_1 < [u^{p_2/(2(p_1+p_2))}], Y_1Y_2>u\right\} + \mathbb{P}\left\{Y_1 > [u^{(p_2+p_1/2)/(p_1+p_2)}], Y_1Y_2>u\right\} \\ &=: \Sigma + P_1 + P_2.\end{aligned}$$

Now observe that, as $u\to\infty$

$$\log(P_1) \le \log\left(\mathbb{P}\left\{Y_2 > u^{1-p_2/(2(p_1+p_2))}\right\}\right) \sim -L_2u^{(p_1+p_2/2)p_2/(p_1+p_2)} \tag{4.3}$$

and

$$\log(P_2) \le \log\left(\mathbb{P}\left\{Y_1 > [u^{(p_2+p_1/2)/(p_1+p_2)}]\right\}\right) \sim -L_1u^{(p_2+p_1/2)p_1/(p_1+p_2)}. \tag{4.4}$$

Moreover, for each $\varepsilon>0$ sufficiently large u and $k\in\left[[u^{p_2/(2(p_1+p_2))}], [u^{(p_2+p_1/2)/(p_1+p_2)}]\right]$

$$\begin{aligned}\log(\mathbb{P}\{Y_1\in[k,k+1), Y_1Y_2>u\}) &\le \log(\mathbb{P}\{Y_1\ge k, Y_2>u/(k+1)\}) \\ &\le -(1-\varepsilon)\left(L_1k^{p_1}+L_2(u/k)^{p_2}\right) \\ &\le -(1-\varepsilon)Bu^{p_1p_2/(p_1+p_2)}, \qquad (4.5)\end{aligned}$$

where (4.5) follows from the fact that $f(x)=L_1x^{p_1}+L_2\left(\frac{u}{x}\right)^{p_2}$ attains its minimum $f(x_u)=Bu^{p_1p_2/(p_1+p_2)}$ at $x_u=\left(\frac{p_2L_2}{p_1L_1}\right)^{1/(p_1+p_2)}u^{p_2/(p_1+p_2)}$ and for any $\delta\in(0,1)$ and all u large $k/(k+1)>1-\delta$. Thus, using the fact that Σ consists of a polynomial (with respect to u) number of elements, we have that

$$\limsup_{u\to\infty} \frac{\log(\Sigma)}{u^{p_1p_2/(p_1+p_2)}} \le -B. \tag{4.6}$$

The combination of (4.3), (4.4) with (4.6) completes the proof of the statement (a).
Ad. (b). Suppose without loss of generality that $L_1=L_2=1$. By [38][Lemma 1] if Y_1^* and Y_2^* are two positive independent rvs tail equivalent to Y_1 and Y_2, respectively, then

$$\mathbb{P}\{Y_1Y_2>u\} \sim \mathbb{P}\{Y_1^*Y_2^*>u\}, \quad u\to\infty.$$

We define next $Y_i^*=S_iZ_i$ where S_i has distribution $G_i, i=1,2$ with right endpoint equal to 1, and Z_1,Z_2 are independent of S_1,S_2. Let α_1^* and α_2^* be the index of the regular variation of g_1 and g_2, respectively. Let $\alpha_i>\alpha_i^*, i=1,2$ be two arbitrary constants. The functions

$\tilde{g}_i(x) = g_i(x)x^{-\alpha_i}$ are regularly varying at infinity with index $\alpha_i^* - \alpha_i < 0$. Hence, we can assume without loss of generality, that

$$\mathbb{P}\{S_i > 1 - a_i(u)/u\} = \frac{1}{\Gamma(\alpha_i - \alpha_i^* + 1)}\tilde{g}_i(u), \quad u \to \infty,$$

where $a_i(u) = u^{1-p_i}, i = 1, 2, u > 0$, and $\Gamma(\cdot)$ is the Euler Gamma function. In view of [39][Theorem A.2] for $i = 1, 2$ we obtain

$$\mathbb{P}\{S_i Z_i > u\} \sim \mathbb{P}\{Y_i > u\}, \quad u \to \infty,$$

where $S_i, Z_i, i = 1, 2$ are independent and positive rvs, and

$$\mathbb{P}\{Z_i > u\} \sim u^{\alpha_i} \exp(-u_i^{p_i}), \quad u \to \infty.$$

Consequently, as $u \to \infty$

$$\mathbb{P}\{Y_1 Y_2 > u\} \quad \sim \quad \mathbb{P}\{S_1 Z_1 S_2 Z_2 > u\} \sim \mathbb{P}\{UW > u\},$$

where $U = S_1 S_2$ and $W = Z_1 Z_2$. The tail asymptotics of U follows by a direct application of [40][Theorem 2.1] whereas the tail asymptotics of W follows from (1.7). Hence, the tail asymptotics of UW follows by applying again [39][Theorem A.2], and thus the proof is complete. □

Proof of Theorem 3.1 The proof follows by the same arguments as the proof of [34][Theorem 5.1]. When S is a bounded rv, then in view (3.1) we have that

$$d_n = (1 + o(1))\sqrt{2 \log n}$$

and since the scaling function $a(x) = 1/x$, then $c_n = 1/d_n$ follows. For the case S has a Weibullian tail behaviour, the relation between c_n and d_n can be established using the same idea as in the proof of the aforementioned theorem. □

Proof of Theorem 3.2 For chosen constants

$$\gamma_1 = 2/(\alpha + 2p), \quad \gamma_2 = 4/(2\alpha + p), \quad \delta = \delta(u) = 2\frac{\sigma^3(u)}{\sigma'(u)}u^{-2}\log^2(u)$$

we have (write $F_{\mathcal{T}}$ for the distribution of $\mathcal{T}$)

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t \in [0,\mathcal{T}]} X(t) > u\right\} \quad &\leq \quad \int_0^{u^{\gamma_1}} \mathbb{P}\left\{\sup_{t \in [0,s]} X(t) > u\right\} dF_{\mathcal{T}}(s) \\
&\quad + \int_{u^{\gamma_1}}^{u^{\gamma_2}} \mathbb{P}\left\{\sup_{t \in [0,s-\delta]} X(t) > u\right\} dF_{\mathcal{T}}(s) \\
&\quad + \int_{u^{\gamma_1}}^{u^{\gamma_2}} \mathbb{P}\left\{\sup_{t \in [s-\delta,s]} X(t) > u\right\} dF_{\mathcal{T}}(s) \\
&\quad + \int_{u^{\gamma_2}}^{\infty} \mathbb{P}\left\{\sup_{t \in [0,s]} X(t) > u\right\} dF_{\mathcal{T}}(s) \\
&=: \quad I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

As in the proof of [17][Theorem 3.1], we conclude that

$$I_1 + I_2 = o(\mathbb{P}\{X(\mathcal{T}) > u\})$$

as $u \to \infty$ and for any $\varepsilon > 0$ and all u large enough

$$I_3 \le (1+\varepsilon)\mathbb{P}\{X(\mathcal{T}) > u\} = (1+\varepsilon)\mathbb{P}\{\sigma(\mathcal{T})\mathcal{N} > u\},$$

where $\mathcal{N}$ is an $N(0,1)$ rv independent of $\mathcal{T}$. Thus it suffices to show that

$$I_4 = o(\mathbb{P}\{X(\mathcal{T}) > u\}) \tag{4.7}$$

as $u \to \infty$. Indeed, since for all large u we have $I_4 \le \mathbb{P}\{\mathcal{T} > u^{\gamma_2}\}$, then

$$\limsup_{u\to\infty} \frac{\log(I_4)}{u^{4p/(2\alpha+p)}} \le -L.$$

On the other hand, for each $\varepsilon \in (0, \alpha/2)$ and sufficiently large u, the assumption that $\sigma(\cdot)$ is regularly varying at ∞ with index $\alpha/2$ implies

$$\mathbb{P}\{\sigma(\mathcal{T}) > u\} \ge \mathbb{P}\left\{\mathcal{T}^{\alpha/2-\varepsilon} > u\right\}.$$

Hence, for some $K > 0$ by statement (a) of Theorem 2.1

$$\liminf_{u\to\infty} \frac{\log(\mathbb{P}\{X(\mathcal{T}) > u\})}{u^{2p/(p+\alpha-2\varepsilon)}} \ge \liminf_{u\to\infty} \frac{\log(\mathbb{P}\left\{\mathcal{T}^{\alpha/2-\varepsilon}\mathcal{N} > u\right\})}{u^{2p/(p+\alpha-2\varepsilon)}} \ge -K.$$

Consequently, since for sufficiently small $\varepsilon > 0$, we have $2p/(p+\alpha-2\varepsilon) < 4p/(p+2\alpha)$, then (4.7) holds. □

Proof of Corollary 3.1 The proof boils down to checking, that for both cases (a) and (b) the conditions imposed on $\sigma(\cdot)$ imply that $\mathcal{T}$ satisfies the assumptions of Theorem 3.2; therefore we omit the details. □

Proof of Theorem 3.3 Without loss of generality we suppose that as $u \to \infty$

$$\mathbb{P}\{\mathcal{A}_1 > u\} \ge \mathbb{P}\{\mathcal{A}_2 > u\}(1+o(1))$$

and $\mathcal{A}_1$ is a log-Weibullian rv with parameters L_1, p_1. Since, by Theorem 3.2

$$\mathbb{P}\left\{\sup_{t\in[-\mathcal{A}_1,0]} X(t) > u\right\} \sim \mathbb{P}\{\sigma(\mathcal{A}_1)\mathcal{N} > u\}$$

and

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{A}_2]} X(t) > u\right\} \sim \mathbb{P}\{\sigma(\mathcal{A}_2)\mathcal{N} > u\},$$

then following the same idea as given in the proof of Theorem 2.1 in [5], it suffices to show that

$$\mathbb{P}\left\{\sup_{t\in[-\mathcal{A}_1,0]} X(t) > u; \sup_{t\in[0,\mathcal{A}_2]} X(t) > u\right\} = o(\mathbb{P}\{\sigma(\mathcal{A}_1)\mathcal{N} > u\}).$$

We present the sketch of the proof, which follows the lines of the proof of Theorem 2.1 in [5]. By Theorem 2.1, for any $\varepsilon > 0$

$$\mathbb{P}\left\{\sigma(\mathcal{A}_1)\mathcal{N} > u\right\} \geq \exp\left(-u^{\frac{2p_1}{\alpha+p_1}+\varepsilon}\right).$$

On the other hand, with $a(u) = u^{\frac{2}{\alpha+2p_1}} A(u) = u^{\frac{4}{2\alpha+p_1}}$,

$$\begin{aligned} &\mathbb{P}\left\{\sup_{t\in[-\mathcal{A}_1,0]} X(t) > u; \sup_{t\in[0,\mathcal{A}_1]} X(t) > u\right\} \leq \\ \leq\ & \left(\int_0^{a(u)} + \int_{a(u)}^{A(u)} + \int_{A(u)}^{\infty}\right) \mathbb{P}\left\{\sup_{(s,t)\in[-x,0]\times[0,x]} (X(s)+X(t)) > 2u\right\} dF_{\mathcal{A}_1}(x) \\ =\ & I_1 + I_2 + I_3. \end{aligned}$$

Then, and analogously to the proof of Theorem 2.1 in [5], for some $\delta > 0$

$$I_1 + I_2 \leq \exp\left(-u^{\frac{2p_1}{\alpha+p_1}+\delta}\right),$$

while $I_3 \leq \varepsilon \mathbb{P}\left\{\sigma(\mathcal{A}_1)\mathcal{N} > u\right\}$ as $u \to \infty$ (observe that the upper bound of I_3 in the proof of Theorem 2.1 in [5] does not depend on the asymptotic behaviour of the tail distribution of $\mathcal{A}_1$). Hence the proof is completed. □

Acknowledgement: We are in debt to the referee for numerous important suggestions. K.D. was partially supported by NCN Grant No 2015/17/B/ST1/01102 (2016-2019) whereas E.H. was partially supported by the Swiss National Science Foundation Grants 200021-166274/1.

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