

Optimal Dividend Strategies for Two Collaborating Insurance Companies

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Abstract

We consider a two-dimensional optimal dividend problem in the context of two insurance companies with compound Poisson surplus processes, who collaborate by paying each other's deficit when possible. We study the stochastic control problem of maximizing the weighted sum of expected discounted dividend payments (among all admissible dividend strategies) until ruin of both companies, by extending results of univariate optimal control theory. In the case that the dividends paid by the two companies are equally weighted, the value function of this problem compares favorably with the one of merging the two companies completely. We identify the optimal value function as the smallest viscosity supersolution of the respective Hamilton-Jacobi-Bellman equation and provide an iterative approach to approximate it numerically. Curve strategies are identified as the natural analogue of barrier strategies in this two-dimensional context. A numerical example is given for which such a curve strategy is indeed optimal among all admissible dividend strategies, and for which this collaboration mechanism also outperforms the suitably weighted optimal dividend strategies of the two stand-alone companies.

1. Introduction

Ever since de Finetti [14] proposed in 1957 to measure the value of an insurance portfolio by the expected discounted sum of dividends paid during the lifetime of the portfolio, it has been of particular interest to determine the optimal dividend payment strategy which maximizes this quantity. More than that, this field of research over the years turned out to be a challenging and fascinating area, combining tools from analysis, probability and stochastic control. In 1969, Gerber [15] showed that if the free surplus of an insurance portfolio is modelled by a compound Poisson risk model, it is optimal to pay dividends according to a so-called band strategy, which collapses to a barrier strategy for exponentially distributed claim amounts. Whereas Gerber found this result by taking a limit of an associated discrete problem, this optimal dividend problem was studied with techniques of modern stochastic control theory in Azcue and Muler [7], see e.g. Schmidli [22] for a detailed overview. Since then the optimal dividend problem was studied for many different model setups, objective functions and side constraints (we refer to Albrecher and Thonhauser [2] and Avanzi [4] for surveys on the subject). A barrier strategy with barrier b pays out dividends whenever the surplus level of the portfolio is above b , so that the surplus level stays at b , and pays no dividends below that barrier b . The most general criteria currently available for barrier strategies to be optimal can be found in Loeffen and Renaud [20]. The optimality of barrier strategies when including the time value of ruin was studied in [24], and when including capital injections by shareholders

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in Kulenko and Schmidli [18].

All these control problems have been formulated and studied in the one-dimensional framework. However, in recent years there has been an increased interest in risk theory in considering the dynamics of several connected insurance portfolios simultaneously, see e.g. Asmussen and Albrecher [3, Ch.XIII.9] for an overview. Ruin probability expressions for a two-dimensional risk process are studied in Avram et al. [5, 6] for simultaneous claim arrivals and proportional claim sizes and recently in Badila et al. [11] and Ivanovs and Boxma [17] in a more general framework. In Azcue and Muler [8], the problem of optimally transferring capital between two portfolios in the presence of transaction costs was considered, see also Badescu et al. [10]. Czarna and Palmowski [13] study the dividend problem and impulse control for two insurance companies who share claim payments and premiums in some specified proportion for a particular dividend strategy. It turns out that these multi-dimensional problems, albeit practically highly relevant, quickly become very intricate and explicit solutions can typically not be obtained without very strong assumptions.

In this paper, we would like to extend the optimal dividend problem from univariate risk theory to a two-dimensional setup of two collaborating companies. The collaboration consists of paying the deficit ('bailing out') of the partner company if its surplus is negative and if this financial help can be afforded with the current own surplus level. We study the problem of maximizing the weighted sum of expected discounted dividend payments until ruin of both companies. A natural question in this context is whether such a collaboration procedure can be advantageous over merging the two companies completely; we will show that this is the case when the dividends paid by the two companies are equally weighted. For criteria of a merger being an advantage over keeping two stand-alone companies under pre-defined barrier strategies and marginal diffusion processes, see e.g. Gerber and Shiu [16], for the performance of another pre-defined risk and profit sharing arrangement, see e.g. Albrecher and Loutscham [1]. Our goal here is, however, to address the general problem of identifying the optimal dividend strategy (among all admissible dividend strategies) for each company under this collaboration framework. This leads to a fully two-dimensional stochastic control problem, and to the question what the natural analogues of the optimal univariate barrier strategies are in two dimensions. The particular structure of the collaboration implemented in this paper will turn out not to be essential, so the techniques may be applicable to other risk-sharing mechanisms as well. Yet, the concrete specification allows to carry through the necessary analysis of the stochastic control problem explicitly by way of example.

The rest of the paper is organized as follows. In Section 2 we introduce the model and the stochastic control problem in detail and derive some simple properties of the corresponding value function V . In Section 3 we prove that V is a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation for independent surplus processes, and in Section 4 we show that V is in fact its smallest viscosity supersolution. Section 5 provides an iterative approach to approximate the value function V , together with the analogous verification steps at each iteration step. Section 6 discusses the stationary dividend strategies that appear in our model, and in Section 7 we establish *curve strategies* as the appropriate analogues of the univariate barrier strategies. Section 8 shows how to constructively search for optimal curve strategies and in Section 9 an explicit numerical example for the symmetric (and equally weighted) case with exponentially distributed claim sizes is worked out for which such a curve strategy is indeed optimal among all admissible bivariate dividend strategies. It is then also illustrated that for this case the proposed type of collaboration is preferable to adding the best-possible stand-alone profits. Finally, in Section 10 we show that there are examples where the stand-alone case is preferable to collaboration and we address the case of positive proportional transaction cost.

2. Model

We consider two insurance companies, Company One and Company Two, which have an agreement to collaborate. Let us call X_t the free surplus of Company One and Y_t the one of Company Two. We assume that the free surplus of each of the companies follows a Cramér-Lundberg process, i.e. a compound Poisson process with drift given by

$$(2.1) \quad \begin{cases} X_t = x + p_1 t - \sum_{i=1}^{N_t^1} U_i^1 \\ Y_t = y + p_2 t - \sum_{i=1}^{N_t^2} U_i^2, \end{cases}$$

where x and y are the respective initial surplus levels; p_1 and p_2 are the respective premium rates; U_i^k is the size of the i -th claim of Company k , which are i.i.d. random variables with continuous distribution F^k for $k = 1, 2$; N_t^1 and N_t^2 are Poisson processes with intensity λ_1 and λ_2 , respectively. We assume here that the processes N_t^1 , N_t^2 and the random variables U_i^1 , U_i^2 are all independent of each other.

There is a rule of collaboration signed by the two companies: if the current surplus of Company One becomes negative, Company Two should cover the exact deficit of Company One as long as it does not ruin itself, and vice versa. Ruin of a company hence occurs when its surplus becomes negative and the other company cannot cover this deficit.

A simulated surplus trajectory under this collaboration rule is shown in Figure 1.1. In this trajectory the surplus of Company Two becomes negative after its first claim, so Company One covers the deficit. Afterwards, a claim of Company One arrives and since Company Two cannot cover this deficit, Company One gets ruined.

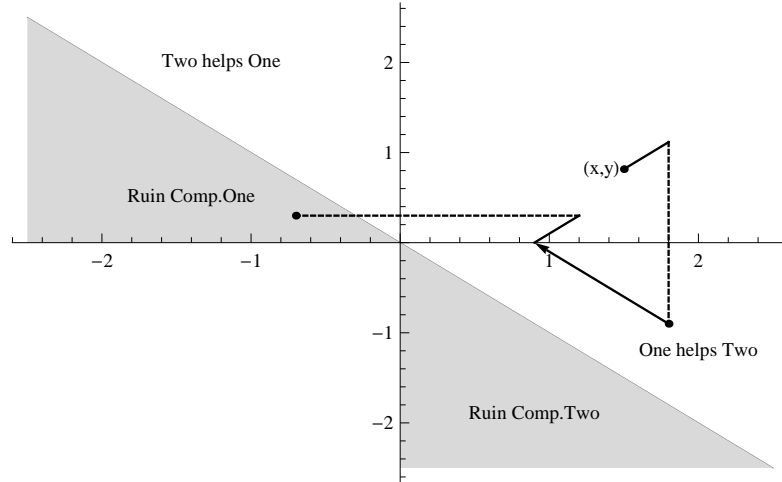


Fig. 1.1: Surplus process under the collaboration rules.

Both companies use part of their surplus to pay dividends to their shareholders. The dividend payment strategy $\bar{L} = (L_t^1, L_t^2)$ is the total amount of dividends paid by the two companies up to time t . Let us call τ_i^k the arrival time of the i -th claim of company k , with $k = 1, 2$. We define the associated controlled process $(X_t^{\bar{L}}, Y_t^{\bar{L}})$ with initial surplus levels (x, y) as

$$(2.2) \quad \begin{cases} X_t^{\bar{L}} = X_t - L_t^1 + C_t^{2,1} - C_t^{1,2} \\ Y_t^{\bar{L}} = Y_t - L_t^2 + C_t^{1,2} - C_t^{2,1}, \end{cases}$$

where

$$C_t^{2,1} = \sum_{i=1}^{N_t^1} I \left\{ X_{\tau_i}^{\bar{L}} < 0, Y_{\tau_i}^{\bar{L}} + X_{\tau_i}^{\bar{L}} \geq 0 \right\} \left| X_{\tau_i}^{\bar{L}} \right|$$

corresponds to the cumulative amount transferred from Company Two to Company One up to time t in order to cover the deficit of Company One and

$$C_t^{1,2} = \sum_{i=1}^{N_t^2} I \left\{ Y_{\tau_i}^{\bar{L}} < 0, Y_{\tau_i}^{\bar{L}} + X_{\tau_i}^{\bar{L}} \geq 0 \right\} \left| Y_{\tau_i}^{\bar{L}} \right|$$

corresponds to the cumulative amount transferred from Company One to Company Two up to time t in order to cover the deficit of Company Two.

Let us call $\bar{\tau}$ the time at which only one company remains (because it cannot cover the deficit of the other), more precisely,

$$(2.3) \quad \bar{\tau} = \inf \left\{ t \geq 0 : X_t^{\bar{L}} + Y_t^{\bar{L}} < 0 \right\}.$$

Note that since the claim arrival processes of the companies are independent, the probability of simultaneous ruin is zero.

The process $(X_t^{\bar{L}}, Y_t^{\bar{L}})$ is defined for $t \leq \bar{\tau}$. We say that the dividend payment strategy $\bar{L} = (L_t^1, L_t^2)_{t \leq \bar{\tau}}$ is *admissible* if it is non-decreasing, càglàd (left continuous with right limits), predictable with respect to the filtration generated by the bivariate process (X_t, Y_t) , and satisfies

$$\begin{cases} L_t^1 \leq X_t + C_t^{2,1} - C_t^{1,2}, \\ L_t^2 \leq Y_t + C_t^{1,2} - C_t^{2,1}. \end{cases}$$

This last condition means that the companies are not allowed to pay more dividends than their current surplus. Let us call \mathbf{R}_+^2 the first quadrant. We denote by $\Pi_{x,y}$ the set of admissible dividend strategies with initial surplus levels $(x, y) \in \mathbf{R}_+^2$. Our objective is to maximize the weighted average of the expected discounted dividends paid by the two companies until ruin of both companies. Note that after time $\bar{\tau}$, the surviving company can continue to pay dividends up to its own ruin. Let us define V_k^0 ($k = 1, 2$) as the optimal value function of the one-dimensional problem of maximizing the expected discounted dividends until ruin of Company k alone. So, for any initial surplus levels $(x, y) \in \mathbf{R}_+^2$, we can write down the optimal value function as

$$(2.4) \quad V(x, y) = \sup_{\bar{L} \in \Pi_{x,y}} V_{\bar{L}}(x, y),$$

where

$$(2.5) \quad V_{\bar{L}}(x, y) = E_{x,y} \left(a_1 \left(\int_0^{\bar{\tau}} e^{-\delta s} dL_s^1 + e^{-\delta \bar{\tau}} V_1^0(X_{\bar{\tau}}^{\bar{L}}) \right) + a_2 \left(\int_0^{\bar{\tau}} e^{-\delta s} dL_s^2 + e^{-\delta \bar{\tau}} V_2^0(Y_{\bar{\tau}}^{\bar{L}}) \right) \right).$$

Here, $\delta > 0$ is a constant discount factor, and $a_1 \in [0, 1]$ and $a_2 = 1 - a_1$ are the weights of the dividends paid by Company One and Company Two respectively. The functions V_k^0 ($k = 1, 2$) are zero in $(-\infty, 0)$ so depending on which company goes to ruin at $\bar{\tau}$, either $V_1^0(X_{\bar{\tau}}^{\bar{L}}) = 0$ or $V_2^0(Y_{\bar{\tau}}^{\bar{L}}) = 0$. The optimal dividend strategy corresponding to (2.4) may be regarded as the best dividend payment strategy from the point of view of a shareholder who owns a proportion ma_1 of the total shares of Company One and a proportion ma_2 of the total shares of Company Two for some $0 < m \leq \min \{1/a_1, 1/a_2\}$. An important particular case is $a_1 = a_2 = 1/2$, in which the dividends paid by the two companies are equally weighted (for an earlier example of weighting separate terms in the objective function in the univariate dividend context, see Radner and Shepp [21]).

Remark 2.1. In case the two companies are owned by the same shareholders, another possibility of collaboration between the two companies is merging, in which case the companies put together all their surplus, pay the claims of both companies and pay dividends up to time τ at which the joined surplus becomes negative (see e.g. Gerber and Shiu [16]). Given the initial surplus levels (x, y) , we can interpret any admissible dividend payment strategy $(L_t)_{t \geq 0}$ for the merger as an admissible collaborating one as follows

$$L_t^1 = \int_0^t \frac{X_s^{\bar{L}}}{X_s^{\bar{L}} + Y_s^{\bar{L}}} dL_s, \quad L_t^2 = \int_0^t \frac{Y_s^{\bar{L}}}{X_s^{\bar{L}} + Y_s^{\bar{L}}} dL_s.$$

Since L_t is constant for $t \geq \tau = \bar{\tau}$, the surviving company does not pay any dividends here. So $V_{\bar{L}}$ defined in (2.5) for $a_1 = a_2 = 1/2$, satisfies

$$2V_{\bar{L}}(x, y) > 2E_{x,y} \left(\frac{1}{2} \int_0^{\bar{\tau}} e^{-\delta s} dL_s^1 + \frac{1}{2} \int_0^{\bar{\tau}} e^{-\delta s} dL_s^2 \right) = E_{x,y} \left(\int_0^{\bar{\tau}} e^{-\delta s} dL_s \right).$$

The last expected value is the value function of the merger dividend strategy $(L_t)_{t \geq 0}$. We conclude that the optimal collaborating strategy for equally weighted dividend payments is better than the optimal merger strategy.

Both optimal value functions V_1^0 and V_2^0 corresponding to the stand-alone companies have an ultimately linear growth with slope one and they are Lipschitz, see for instance Azcue and Muler [7]. Let us state some basic results about regularity and growth at infinity of the optimal value function V defined in (2.4). From now on, let us call $\lambda := \lambda_1 + \lambda_2$ and $p := a_1 p_1 + a_2 p_2$.

Lemma 2.2. *The optimal value function is well defined and satisfies*

$$a_1 x + a_2 y + \frac{p}{\delta + \lambda} \leq V(x, y) \leq a_1 x + a_2 y + \frac{p}{\delta}$$

for all $(x, y) \in \mathbf{R}_+^2$.

Proof.

The proof of the second inequality is similar to the one of Proposition 1.2 of Azcue and Muler [9]: For any initial surplus levels $(x, y) \in \mathbf{R}_+^2$, $x + p_1 t$ and $y + p_2 t$ are upper bounds for the surplus levels of Company One and Company Two at time t , respectively (even after the ruin time $\bar{\tau}$ of one of the companies), so the cumulative dividends paid by Company One and Company Two up to time t are less than or equal to $x + p_1 t$ and $y + p_2 t$, respectively. So, since $e^{-\delta s}$ is a positive and decreasing function,

$$V(x, y) \leq a_1 \left(\int_0^{\infty} e^{-\delta s} d((x + p_1 s) I_{\{s > 0\}}) \right) + a_2 \left(\int_0^{\infty} e^{-\delta s} d((y + p_2 s) I_{\{s > 0\}}) \right) = a_1 x + a_2 y + \frac{p}{\delta}.$$

In order to obtain the first inequality, consider the admissible strategy $\bar{L}_0 = (L_t^1, L_t^2)$ in which each company pays the initial surplus immediately, and then pays the incoming premium up to time $\bar{\tau}$ that coincides with the first claim arrival time $\tau_1 = \tau_1^1 \wedge \tau_1^2$; we have,

$$\begin{aligned} V(x, y) &\geq V_{\bar{L}_0}(x, y) \\ &\geq a_1 x + a_2 y + E_{0,0} \left(\int_0^{\tau_1} e^{-\delta s} p ds \right) \\ &= a_1 x + a_2 y + \frac{p}{\delta + \lambda}. \quad \square \end{aligned}$$

Lemma 2.3. *The optimal value function V is increasing in both x and y , locally Lipschitz and satisfies for any $(x, y) \in \mathbf{R}_+^2$,*

$$a_1 h \leq V(x + h, y) - V(x, y) \leq (e^{(\delta + \lambda)h/p_1} - 1)V(x, y)$$

and

$$a_2 h \leq V(x, y + h) - V(x, y) \leq (e^{(\delta + \lambda)h/p_2} - 1)V(x, y)$$

for any $h > 0$.

Proof.

Let us prove the inequalities at the top, the ones at the bottom are similar. Given any $\varepsilon > 0$, take an admissible strategy $\bar{L} \in \Pi_{x,y}$ such that $V_{\bar{L}}(x, y) \geq V(x, y) - \varepsilon$. We define the strategy $\bar{L}^1 \in \Pi_{x+h,y}$ for $h > 0$ as follows: Pay immediately an amount h of the surplus of Company One as dividends and then follow the strategy \bar{L} . We have that

$$V_{\bar{L}^1}(x+h, y) = V_{\bar{L}}(x, y) + a_1 h$$

and so

$$(2.6) \quad V(x+h, y) \geq V_{\bar{L}}(x, y) + a_1 h > V(x, y) + a_1 h - \varepsilon.$$

Consider also an admissible strategy $\bar{L}^2 \in \Pi_{x+h,y}$ such that $V_{\bar{L}^2}(x+h, y) \geq V(x+h, y) - \varepsilon$ and define the admissible strategy $\bar{L}^3 \in \Pi_{x,y}$ which, starting with surplus (x, y) pays no dividends until

$$\tilde{\tau} = \inf \left\{ t \geq 0 : X_t^{\bar{L}^3} \geq x+h, Y_t^{\bar{L}^3} \geq y \right\},$$

at time $\tilde{\tau}$ pays either $X_t^{\bar{L}^3} - (x+h)$ from the surplus of Company One or $Y_t^{\bar{L}^3} - y$ from the surplus of Company Two, depending on which of these differences is positive, and then follows strategy $\bar{L}^2 \in \Pi_{x+h,y}$. In the event of no claims, $\tilde{\tau} = t_0 := h/p_1$; since the probability of no claims until t_0 is $e^{-\lambda t_0}$, we get

$$(2.7) \quad V(x, y) \geq V_{\bar{L}^3}(x, y) \geq V_{\bar{L}^2}(x+h, y)e^{-(\delta+\lambda)t_0} \geq (V(x+h, y) - \varepsilon)e^{-(\delta+\lambda)t_0}.$$

From (2.6) and (2.7), we get the inequalities at the top. \square

3. Hamilton-Jacobi-Bellman equation

In order to obtain the Hamilton-Jacobi-Bellman (HJB) equation associated to the optimization problem (2.4), we need to state the so called Dynamic Programming Principle (DPP). The proof that this holds is similar to the one given in Lemma 1.2 of Azcue and Muler [9] and uses that V is increasing and continuous in \mathbf{R}_+^2 .

Lemma 3.1. *For any initial surplus (x, y) in \mathbf{R}_+^2 and any stopping time τ , we can write*

$$\begin{aligned} V(x, y) &= \sup_{\bar{L} \in \Pi_{x,y}} \left(E_{x,y} \left(a_1 \int_0^{\tau \wedge \bar{\tau}} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau \wedge \bar{\tau}} e^{-\delta s} dL_s^2 + e^{-\delta(\tau \wedge \bar{\tau})} I_{\{\tau \wedge \bar{\tau} < \bar{\tau}\}} V(X_{\tau \wedge \bar{\tau}}^{\bar{L}}, Y_{\tau \wedge \bar{\tau}}^{\bar{L}}) \right. \right. \\ &\quad \left. \left. + e^{-\delta(\tau \wedge \bar{\tau})} I_{\{\tau \wedge \bar{\tau} = \bar{\tau}\}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}})) \right) \right). \end{aligned}$$

The HJB equation of this optimization problem is

$$(3.1) \quad \max \{ \mathcal{L}(V)(x, y), a_1 - V_x(x, y), a_2 - V_y(x, y) \} = 0,$$

where

$$(3.2) \quad \mathcal{L}(V)(x, y) = V_x(x, y)p_1 + V_y(x, y)p_2 - (\delta + \lambda)V(x, y) + \mathcal{I}(V)(x, y) + U(x, y),$$

$$(3.3) \quad \begin{aligned} \mathcal{I}(V)(x, y) &= \lambda_1 \int_0^x V(x - \alpha, y) dF^1(\alpha) + \lambda_1 \int_x^{x+y} V(0, x + y - \alpha) dF^1(\alpha) \\ &\quad + \lambda_2 \int_0^y V(x, y - \alpha) dF^2(\alpha) + \lambda_2 \int_y^{x+y} V(x + y - \alpha, 0) dF^2(\alpha), \end{aligned}$$

and

$$(3.4) \quad U(x, y) = \lambda_1 a_2 V_2^0(y)(1 - F^1(x + y)) + \lambda_2 a_1 V_1^0(x)(1 - F^2(x + y)).$$

Since the optimal value function V is locally Lipschitz but possibly not differentiable at certain points, we cannot say that V is a solution of the HJB equation, and so we prove instead that V is a viscosity solution of the corresponding HJB equation. Let us define this notion (see Crandall and Lions [12] and Soner [23] for further details).

Definition 3.2. A locally Lipschitz function $\bar{u} : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ is a viscosity supersolution of (3.1) at $(x, y) \in \mathbf{R}_+^2$ if any continuously differentiable function $\varphi : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ with $\varphi(x, y) = \bar{u}(x, y)$ such that $\bar{u} - \varphi$ reaches the minimum at (x, y) satisfies

$$\max \{ \mathcal{L}(\varphi)(x, y), a_1 - \varphi_x(x, y), a_2 - \varphi_y(x, y) \} \leq 0.$$

A function $\underline{u} : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ is a viscosity subsolution of (3.1) at $(x, y) \in \mathbf{R}_+^2$ if any continuously differentiable function $\psi : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ with $\psi(x, y) = \underline{u}(x, y)$ such that $\underline{u} - \psi$ reaches the maximum at (x, y) satisfies

$$\max \{ \mathcal{L}(\psi)(x, y), a_1 - \psi_x(x, y), a_2 - \psi_y(x, y) \} \geq 0.$$

A function $u : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ which is both a supersolution and subsolution at $(x, y) \in \mathbf{R}_+^2$ is called a viscosity solution of (3.1) at $(x, y) \in \mathbf{R}_+^2$.

Proposition 3.3. V is a viscosity supersolution of the HJB equation (3.1) at any (x, y) with $x > 0$ and $y > 0$.

Proof. Given initial surplus levels $x > 0$, $y > 0$ and any $l_1 \geq 0$, $l_2 \geq 0$, let us consider the admissible strategy \bar{L} where Company One and Two pay dividends with constant rates l_1 and l_2 respectively and $\bar{\tau}$ is defined as in (2.3). Let φ be a test function for the supersolution of (3.1) at (x, y) with $x > 0$ and $y > 0$. As before, denote τ_1^1 and τ_1^2 as the arrival time of the first claim of Company One and Two respectively, and $\tau_1 = \tau_1^1 \wedge \tau_1^2$. We have for $t < \tau_1$,

$$\begin{cases} X_t^{\bar{L}} = x + (p_1 - l_1)t, \\ Y_t^{\bar{L}} = y + (p_2 - l_2)t. \end{cases}$$

Note that $N_t^1 + N_t^2$ is a Poisson process with intensity λ , because the arrival times of the two companies are independent. We have from Lemma 3.1 that

$$\begin{aligned} \varphi(x, y) &= V(x, y) \\ &\geq E_{x,y} \left(a_1 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_1 ds + a_2 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_2 ds \right) \\ &\quad + E_{x,y} \left(e^{-\delta(\tau_1 \wedge t)} I_{\{\tau_1 \wedge t < \bar{\tau}\}} V(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) \right) \\ &\quad + E_{x,y} \left(e^{-\delta \bar{\tau}} I_{\{\tau_1 \wedge t = \bar{\tau}\}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}})) \right) \\ &\geq E_{x,y} \left(a_1 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_1 ds + a_2 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_2 ds \right) \\ &\quad + E_{x,y} \left(e^{-\delta(\tau_1 \wedge t)} \varphi(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) I_{\{\tau_1 \wedge t < \bar{\tau}\}} \right) \\ &\quad + E_{x,y} \left(e^{-\delta \bar{\tau}} I_{\{\tau_1 \wedge t = \bar{\tau}\}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}})) \right). \end{aligned}$$

We can write

$$\begin{aligned} &E_{x,y} \left(e^{-\delta(\tau \wedge t)} \varphi(X_{\tau \wedge t}^{\bar{L}}, Y_{\tau \wedge t}^{\bar{L}}) I_{\{\tau \wedge t < \bar{\tau}\}} \right) \\ &= E_{x,y} \left(I_{\{t < \tau_1\}} e^{-\delta t} \varphi(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) \right) \\ &\quad + E_{x,y} \left(I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^1\}} e^{-\delta \tau_1^1} \varphi(X_{\tau_1^1}^{\bar{L}}, Y_{\tau_1^1}^{\bar{L}}) \right) \\ &\quad + E_{x,y} \left(I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^2\}} e^{-\delta \tau_1^2} \varphi(X_{\tau_1^2}^{\bar{L}}, Y_{\tau_1^2}^{\bar{L}}) \right). \end{aligned}$$

So we obtain

$$\begin{aligned}
0 &\geq \lim_{t \rightarrow 0^+} E_{x,y} \left(\frac{a_1 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_1 ds + a_2 \int_0^{\tau_1 \wedge t} e^{-\delta s} l_2 ds}{t} \right) \\
&\quad + \lim_{t \rightarrow 0^+} \frac{e^{-(\lambda+\delta)t} \varphi(x+(p_1-l_1)t, y+(p_2-l_2)t) - \varphi(x,y)}{t} \\
&\quad + \lim_{t \rightarrow 0^+} E_{x,y} \left(\frac{I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^1\}} e^{-\delta \tau_1^1} \varphi(X_{\tau_1^1}^{\bar{L}}, Y_{\tau_1^1}^{\bar{L}})}{t} \right) \\
&\quad + \lim_{t \rightarrow 0^+} E_{x,y} \left(\frac{I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^2\}} e^{-\delta \tau_1^2} \varphi(X_{\tau_1^2}^{\bar{L}}, Y_{\tau_1^2}^{\bar{L}})}{t} \right) \\
&\quad + \lim_{t \rightarrow 0^+} E_{x,y} \left(\frac{e^{-\delta \bar{\tau}} I_{\{\tau_1 \wedge t = \bar{\tau}\}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}}))}{t} \right) \\
&= a_1 l_1 + a_2 l_2 - (\delta + \lambda) \varphi(x, y) \\
&\quad + (p_1 - l_1) \varphi_x(x, y) + (p_2 - l_2) \varphi_y(x, y) + \mathcal{I}(\varphi)(x, y) \\
&\quad + U(x, y).
\end{aligned}$$

Therefore,

$$0 \geq \mathcal{L}(\varphi)(x, y) + l_1 (a_1 - \varphi_x(x, y)) + l_2 (a_2 - \varphi_y(x, y)).$$

Taking $l_1 = l_2 = 0$, $l_1 \rightarrow \infty$ with $l_2 = 0$, and $l_2 \rightarrow \infty$ with $l_1 = 0$, we obtain

$$\max \{ \mathcal{L}(\varphi)(x, y), a_1 - \varphi_x(x, y), a_2 - \varphi_y(x, y) \} \leq 0. \square$$

Proposition 3.4. V is a viscosity subsolution of the HJB equation (3.1).

Proof. Arguing by contradiction, we assume that V is not a subsolution of (3.1) at (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$. With a similar proof to the one of Proposition 3.1 of Azcue and Muler [9], but extending the definitions to two variables, we first show that there exist $\varepsilon > 0$, $h \in (0, \min\{x_0/2, y_0/2\})$ and a continuously differentiable function $\psi : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ such that ψ is a test function for the subsolution of Equation (3.1) at (x_0, y_0) and satisfies

$$(3.5) \quad \psi_x(x, y) \geq a_1, \psi_y(x, y) \geq a_2$$

for $(x, y) \in [0, x_0 + h] \times [0, y_0 + h]$,

$$(3.6) \quad \mathcal{L}(\psi)(x, y) \leq -2\varepsilon\delta$$

for $(x, y) \in [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$, and

$$(3.7) \quad V(x, y) \leq \psi(x, y) - 2\varepsilon$$

for $(x, y) \in \mathbf{R}_+^2 \setminus (x_0 - h/2, x_0 + h/2) \times (y_0 - h/2, y_0 + h/2)$.

Since ψ is continuously differentiable, we can find a positive constant C such that

$$(3.8) \quad \mathcal{L}(\psi)(x, y) \leq C$$

for all $(x, y) \in [0, x_0 + 2h] \times [0, y_0 + 2h]$.

Consider

$$0 < \theta < \left\{ \frac{h}{2 \max\{p_1, p_2\}}, \frac{\lambda}{4\delta(\delta + \lambda)}, \frac{\varepsilon\lambda}{2C(\delta + \lambda)} \right\},$$

and let us take any admissible strategy $\bar{L} \in \Pi_{x_0, y_0}$. Consider the corresponding controlled risk process (X_t, Y_t) starting at (x_0, y_0) , and define the stopping times

$$\begin{aligned}
\tau^b &= \inf\{t > 0 : (X_t, Y_t) \in \partial([x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h])\}, \\
\bar{\tau} &= \inf\{t > 0 : (X_t, Y_t) \in \mathbf{R}_+^2 - [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]\}
\end{aligned}$$

and $\tau^* = \tau^b \wedge (\underline{\tau} + \theta) \wedge \bar{\tau}$. Note that τ^* is finite for h small enough and that it is necessary to introduce θ because before a lump sum dividend payment, $(X_{\underline{\tau}}, Y_{\underline{\tau}})$ can be in $[x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$ and $(X_{\underline{\tau}^+}, Y_{\underline{\tau}^+}) \in \mathbf{R}_+^2 - [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$.

Let us show that

$$(3.9) \quad V(X_{\tau^*}, Y_{\tau^*}) \leq \psi(X_{\tau^*}, Y_{\tau^*}) - 2\varepsilon$$

if $\tau^* = \tau^b \wedge (\underline{\tau} + \theta) < \bar{\tau}$. There are two possibilities:

(1) If $\tau^* = \tau^b$, $(X_{\tau^*}, Y_{\tau^*}) \in \partial([x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h])$ and so, from (3.7), we obtain $V(X_{\tau^*}, Y_{\tau^*}) \leq \psi(X_{\tau^*}, Y_{\tau^*}) - 2\varepsilon$,

(2) If $\tau^* = \underline{\tau} + \theta$, the distance from (X_{τ^*}, Y_{τ^*}) to (x_0, y_0) is at least $h/2 \leq h - \max\{p_1, p_2\}\theta$, so from (3.7), we get (3.9).

Note that $(X_{s^-}, Y_{s^-}) \in [0, x_0 + h + p_1\theta] \times [0, y_0 + h + p_2\theta] \subset [0, x_0 + 2h] \times [0, y_0 + 2h]$ for $s \leq \tau^*$, so we have that

$$\mathcal{L}(\psi)(X_{s^-}, Y_{s^-}) \leq C \text{ for } s \leq \tau^*.$$

Since L_t^1 and L_t^2 are non-decreasing and left continuous, they can be written as

$$(3.10) \quad L_t^1 = \int_0^t dL_s^{1,c} + \sum_{\substack{X_{s^+} \neq X_s \\ s < t}} (L_{s^+}^1 - L_s^1), \quad L_t^2 = \int_0^t dL_s^{2,c} + \sum_{\substack{Y_{s^+} \neq Y_s \\ s < t}} (L_{s^+}^2 - L_s^2)$$

where $L_s^{1,c}$ and $L_s^{2,c}$ are continuous and non-decreasing processes. Since the function ψ is continuously differentiable in \mathbf{R}_+^2 , using the expression (3.10) and the change of variables formula for finite variation processes, we can write

$$\begin{aligned} & \psi(X_{\tau^*}, Y_{\tau^*}) e^{-\delta\tau^*} - \psi(x_0, y_0) \\ &= \int_0^{\tau^*} (p_1 \psi_x(X_{s^-}, Y_{s^-}) + p_2 \psi_y(X_{s^-}, Y_{s^-})) e^{-\delta s} ds \\ & \quad + \sum_{\substack{X_{s^-} \neq X_s \\ s \leq \tau^*}} (\psi(X_s, Y_s) - \psi(X_{s^-}, Y_{s^-})) e^{-\delta s} + \sum_{\substack{Y_{s^-} \neq Y_s \\ s \leq \tau^*}} (\psi(X_s, Y_s) - \psi(X_{s^-}, Y_{s^-})) e^{-\delta s} \\ & \quad - \int_0^{\tau^*} \psi_x(X_{s^-}, Y_{s^-}) e^{-\delta s} dL_s^{1,c} + \sum_{\substack{X_{s^+} \neq X_s \\ s < \tau^*}} (\psi(X_{s^+}, Y_{s^+}) - \psi(X_s, Y_s)) e^{-\delta s} \\ & \quad - \int_0^{\tau^*} \psi_y(X_{s^-}, Y_{s^-}) e^{-\delta s} dL_s^{2,c} + \sum_{\substack{Y_{s^+} \neq Y_s \\ s < \tau^*}} (\psi(X_{s^+}, Y_{s^+}) - \psi(X_s, Y_s)) e^{-\delta s} \\ & \quad - \delta \int_0^{\tau^*} \psi(X_{s^-}, Y_{s^-}) e^{-\delta s} ds. \end{aligned}$$

Note that $(X_s, Y_s) \in \mathbf{R}_+^2$ for $s \leq \tau^*$ except in the case that $\tau^* = \bar{\tau}$, where $X_{\tau^*} + Y_{\tau^*} < 0$. Here we are extending the definition of ψ as

$$\psi(x, y) = a_1 V_1^0(x) I_{\{x \geq 0\}} + a_2 V_2^0(y) I_{\{y \geq 0\}}$$

for $x + y < 0$. We have that $X_{s^+} \neq X_s$ only at the jumps of L_s^1 , and in this case $X_{s^+} - X_s = -(L_{s^+}^1 - L_s^1)$. Since \bar{L} is admissible we have that $X_{s^+} = X_s - (L_{s^+}^1 - L_s^1) \geq 0$. We can write

$$\begin{aligned} & - \int_0^{\tau^*} \psi_x(X_{s^-}, Y_{s^-}) e^{-\delta s} dL_s^{1,c} + \sum_{\substack{X_{s^+} \neq X_s \\ s < \tau^*}} (\psi(X_{s^+}, Y_{s^+}) - \psi(X_s, Y_s)) e^{-\delta s} \\ &= - \int_0^{\tau^*} \psi_x(X_{s^-}, Y_{s^-}) e^{-\delta s} dL_s^{1,c} + \sum_{\substack{X_{s^+} \neq X_s \\ s < \tau^*}} \left(\int_0^{L_{s^+}^1 - L_s^1} \psi_x(X_s - \alpha, Y_s) d\alpha \right) e^{-\delta s} \\ (3.11) \quad & \leq - \int_0^{\tau^*} a_1 e^{-\delta s} dL_s^{1,c} - a_1 \sum_{\substack{L_{s^+}^1 \neq L_s^1 \\ s < \tau^*}} \left(\int_0^{L_{s^+}^1 - L_s^1} d\alpha \right) e^{-\delta s} \\ &= -a_1 \int_0^{\tau^*} e^{-\delta s} dL_s^1. \end{aligned}$$

Similarly,

$$\begin{aligned}
& - \int_0^{\tau^*} \psi_x(X_{s-}, Y_{s-}) e^{-\delta s} dL_s^{2,c} + \sum_{\substack{X_{s+} \neq X_s \\ s < \tau^*}} (\psi(X_{s+}, Y_{s+}) - \psi(X_s, Y_s)) e^{-\delta s} \\
(3.12) \quad & = - \int_0^{\tau^*} \psi_x(X_{s-}, Y_{s-}) e^{-\delta s} dL_s^{2,c} - \sum_{\substack{L_{s+}^2 \neq L_s^2 \\ s < \tau^*}} \left(\int_0^{L_{s+}^2 - L_s^2} \psi_x(X_s, Y_s - \alpha) d\alpha \right) e^{-\delta s} \\
& \leq -a_2 \int_0^{\tau^*} e^{-\delta s} dL_s^2.
\end{aligned}$$

On the other hand, $X_s \neq X_{s-}$ only at the arrival of a claim for Company One, so

$$\begin{aligned}
M_t^1 & = \sum_{\substack{X_s \neq X_{s-} \\ s \leq t}} (\psi(X_s, Y_s) - \psi(X_{s-}, Y_{s-})) e^{-\delta s} \\
(3.13) \quad & - \lambda_1 \int_0^t e^{-\delta s} \int_0^{X_{s-}} (\psi(X_{s-} - \alpha, Y_{s-}) - \psi(X_{s-}, Y_{s-})) dF^1(\alpha) ds \\
& - \lambda_1 \int_0^t e^{-\delta s} \int_{X_{s-} + Y_{s-}}^{X_{s-} + Y_{s-}} (\psi(0, X_{s-} + Y_{s-} - \alpha) - \psi(X_{s-}, Y_{s-})) dF^1(\alpha) ds \\
& - \lambda_1 \int_0^t e^{-\delta s} \int_{X_{s-} + Y_{s-}}^{\infty} (a_2 V_2^0(Y_{s-}) - \psi(X_{s-}, Y_{s-})) dF^1(\alpha) ds
\end{aligned}$$

is a martingale with zero expectation for $t \leq \bar{\tau}$. Analogously,

$$\begin{aligned}
M_t^2 & = \sum_{\substack{Y_s \neq Y_{s-} \\ s \leq t}} (\psi(X_s, Y_s) - \psi(X_{s-}, Y_{s-})) e^{-\delta s} \\
(3.14) \quad & - \lambda_2 \int_0^t e^{-\delta s} \int_0^{Y_{s-}} (\psi(X_{s-}, Y_{s-} - \alpha) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds \\
& - \lambda_2 \int_0^t e^{-\delta s} \int_{X_{s-} + Y_{s-}}^{X_{s-} + Y_{s-}} (\psi(X_{s-} + Y_{s-} - \alpha, 0) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds \\
& - \lambda_2 \int_0^t e^{-\delta s} \int_{X_{s-} + Y_{s-}}^{\infty} (a_1 V_1^0(Y_{s-}) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds
\end{aligned}$$

is also a martingale with zero expectation for $t \leq \bar{\tau}$. So we get

$$(3.15) \quad \psi(X_{\tau^*}, Y_{\tau^*}) e^{-\delta \tau^*} - \psi(x_0, y_0) \leq \int_0^{\tau^*} \mathcal{L}(\psi)(X_{s-}, Y_{s-}) e^{-\delta s} + M_{\tau^*}^1 + M_{\tau^*}^2 - a_1 \int_0^{\tau^*} e^{-\delta s} dL_s^1 - a_2 \int_0^{\tau^*} e^{-\delta s} dL_s^2.$$

Using (3.6), (3.8) and the definition of θ we get

$$\begin{aligned}
(3.16) \quad & \int_0^{\tau^*} \mathcal{L}(\psi)(X_{s-}, Y_{s-}) e^{-\delta s} ds \leq \int_0^{\tau^b \wedge \bar{\tau}} \mathcal{L}(\psi)(X_{s-}, Y_{s-}) e^{-\delta s} ds + C\theta \\
& \leq -2\varepsilon\delta \int_0^{\tau^b \wedge \bar{\tau}} e^{-\delta s} ds + C\theta \\
& \leq -2\varepsilon\delta \int_0^{\tau^*} e^{-\delta s} ds + I_{\{\tau^b \wedge \bar{\tau} < \tau^*\}} 2\varepsilon\delta \int_{\tau^b \wedge \bar{\tau}}^{\tau^*} e^{-\delta s} ds + C\theta \\
& \leq -2\varepsilon(1 - e^{-\delta \tau^*}) + \varepsilon\lambda / (\delta + \lambda).
\end{aligned}$$

From (3.7), Lemma 3.1, (3.9), (3.15) and (3.16), it follows that

$$\begin{aligned}
(3.17) \quad & V(x_0, y_0) \\
&= \sup_{\bar{L}}(E_{x_0, y_0}(a_1 \int_0^{\tau^*} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau^*} e^{-\delta s} dL_s^2 + e^{-\delta \tau^*} V(X_{\tau^*}, Y_{\tau^*}) I_{\{\tau^* < \bar{\tau}\}} \\
&\quad + (a_1 V_1^0(X_{\bar{\tau}}) + a_2 V_2^0(Y_{\bar{\tau}})) e^{-\delta \tau^*} I_{\{\tau^* = \bar{\tau}\}})) \\
&\leq \sup_{\bar{L}}(E_{x_0, y_0}(a_1 \int_0^{\tau^*} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau^*} e^{-\delta s} dL_s^2 + e^{-\delta \tau^*} (\psi(X_{\tau^*}, Y_{\tau^*}) - 2\varepsilon) I_{\{\tau^* < \bar{\tau}\}} \\
&\quad + (a_1 V_1^0(X_{\bar{\tau}}) + a_2 V_2^0(Y_{\bar{\tau}})) e^{-\delta \tau^*} I_{\{\tau^* = \bar{\tau}\}})) \\
&\leq \sup_{\bar{L}} E_{x_0, y_0} \left(\int_0^{\tau^*} \mathcal{L}(\psi)(X_{s-}, Y_{s-}) e^{-\delta s} ds + M_{\tau^*}^1 + M_{\tau^*}^2 - 2\varepsilon e^{-\delta \tau^*} I_{\{\tau^* < \bar{\tau}\}} + \psi(x_0, y_0) \right) \\
&\leq \psi(x_0, y_0) + \sup_{\bar{L}} E_{x_0, y_0} \left(-2\varepsilon(1 - e^{-\delta \tau^*}) + \varepsilon\lambda / (\delta + \lambda) - 2\varepsilon e^{-c\tau^*} I_{\{\tau^* < \bar{\tau}\}} \right) \\
&= \psi(x_0, y_0) - 2\varepsilon + \sup_{\bar{L}} E_{x_0, y_0} \left(2\varepsilon e^{-\delta \tau^*} I_{\{\tau^* = \bar{\tau}\}} \right) + \varepsilon\lambda / (\delta + \lambda) \\
&\leq \psi(x_0, y_0) - 2\varepsilon + 2\varepsilon E(e^{-\delta \tau_1}) + \varepsilon\lambda / (\delta + \lambda) \\
&= \psi(x_0, y_0) - \varepsilon\lambda / (\delta + \lambda) \\
&< \psi(x_0, y_0)
\end{aligned}$$

and this contradicts the assumption that $V(x_0, y_0) = \psi(x_0, y_0)$. \square

From the above two propositions we get the following result.

Corollary 3.5. V is a viscosity solution of the HJB equation (3.1).

4. Smallest Viscosity Solution

Let us prove now that the optimal value function V is the smallest viscosity supersolution of (3.1).

We say that the function $u : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ satisfies the growth condition A.1, if

$$u(x, y) \leq K + a_1 x + a_2 y \text{ for all } (x, y) \in \mathbf{R}_+^2$$

for some positive K .

The following Lemma is technical and will be used to prove Proposition 4.2.

Lemma 4.1. Fix $x_0 > 0$ and $y_0 > 0$ and let \bar{u} be a non-negative supersolution of (3.1) satisfying the growth condition A.1. We can find a sequence of positive functions $\bar{u}_m : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ such that:

- (a) \bar{u}_m is continuously differentiable.
- (b) \bar{u}_m satisfies the growth condition A.1.
- (c) $\bar{u}_{m,x} \geq a_1$, $\bar{u}_{m,y} \geq a_2$ and $p_1 \bar{u}_{m,x} + p_2 \bar{u}_{m,y} \leq (\delta + \lambda) \bar{u}_m$ in \mathbf{R}_+^2 .
- (d) $\bar{u}_m \searrow \bar{u}$ uniformly on compact sets in \mathbf{R}_+^2 and $\nabla \bar{u}_m$ converges to $\nabla \bar{u}$ a.e. in \mathbf{R}_+^2 .
- (e) There exists a sequence c_m with $\lim_{m \rightarrow \infty} c_m = 0$ such that

$$\sup_{(x,y) \in A_0} \mathcal{L}(\bar{u}_m)(x, y) \leq c_m, \text{ where } A_0 = [0, x_0] \times [0, y_0].$$

Proof. The proof follows by standard convolution arguments and is the extension to two variables of Lemma 4.1 in Azcue and Muler [9]. \square

Proposition 4.2. The optimal value function V is the smallest viscosity supersolution of (3.1) satisfying growth condition A.1.

Proof. Let \bar{u} be a non-negative supersolution of (3.1) satisfying the growth condition A.1 and let $\bar{L} \in \Pi_{x,y}$; define (X_t, Y_t) as the corresponding controlled risk process starting at (x, y) . Consider the function \bar{u}_m of Lemma 4.1 in \mathbf{R}_+^2 ; we extend this function as

$$\bar{u}_m(x, y) = a_1 V_1^0(x) I_{\{x \geq 0\}} + a_2 V_2^0(y) I_{\{y \geq 0\}} \text{ for } x + y < 0.$$

As in the proof of Proposition 3.4, we get

$$(4.1) \quad \begin{aligned} & \bar{u}_m(X_{t \wedge \bar{\tau}}, Y_{t \wedge \bar{\tau}}) e^{-\delta(t \wedge \bar{\tau})} - \bar{u}_m(x, y) \\ & \leq \int_0^{t \wedge \bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds - a_1 \int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^1 - a_2 \int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^2 + M_{t \wedge \bar{\tau}}^1 + M_{t \wedge \bar{\tau}}^2, \end{aligned}$$

where M_t^1 and M_t^2 are zero-expectation martingales. So we obtain that

$$\begin{aligned} & \bar{u}_m(X_{t \wedge \bar{\tau}}, Y_{t \wedge \bar{\tau}}) e^{-\delta(t \wedge \bar{\tau})} I_{\{\bar{\tau} > t\}} - \bar{u}_m(x, y) \\ & \leq \int_0^{t \wedge \bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds + M_{t \wedge \bar{\tau}}^1 + M_{t \wedge \bar{\tau}}^2 \\ & \quad - a_1 \left(\int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^1 + e^{-\delta(t \wedge \bar{\tau})} V_1^0(X_{t \wedge \bar{\tau}}) I_{\{\bar{\tau} \leq t\}} \right) \\ & \quad - a_2 \left(\int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^2 + e^{-\delta(t \wedge \bar{\tau})} V_2^0(Y_{t \wedge \bar{\tau}}) I_{\{\bar{\tau} \leq t\}} \right). \end{aligned}$$

Using that both L_t^1 and L_t^2 are non-decreasing processes, from the monotone convergence theorem we get

$$(4.2) \quad \begin{aligned} & \lim_{t \rightarrow \infty} (E_{x,y} (a_1 \left(\int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^1 + e^{-\delta(t \wedge \bar{\tau})} V_1^0(X_{t \wedge \bar{\tau}}) I_{\{\bar{\tau} \leq t\}} \right) \\ & + a_2 \left(\int_0^{t \wedge \bar{\tau}} e^{-\delta s} dL_s^2 + e^{-\delta(t \wedge \bar{\tau})} V_2^0(Y_{t \wedge \bar{\tau}}) I_{\{\bar{\tau} \leq t\}} \right))) \\ & = V_{\bar{L}}(x, y). \end{aligned}$$

From Lemma 4.1(c), we have that $p_1 \bar{u}_{m,x} + p_2 \bar{u}_{m,y} \leq (\delta + \lambda) \bar{u}_m$ and that \bar{u}_m is increasing. So $\mathcal{I}(\bar{u}_m)(x, y) \leq \lambda \bar{u}_m(x, y)$ and then

$$-(\delta + \lambda) \bar{u}_m(x, y) \leq \mathcal{L}(\bar{u}_m)(x, y) \leq \lambda \bar{u}_m(x, y) + U(x, y).$$

But using Lemma 4.1(b) and the inequality $X_s \leq x + p_1 s$, $Y_s \leq y + p_2 s$ we get

$$(4.3) \quad \bar{u}_m(X_s, Y_s) \leq K + a_1 X_s + a_2 Y_s \leq K + a_1 x + a_2 y + ps.$$

So, using the bounded convergence theorem, we obtain

$$(4.4) \quad \lim_{t \rightarrow \infty} E_{x,y} \left(\int_0^{t \wedge \bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds \right) = E_{x,y} \left(\int_0^{\bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds \right).$$

From (4.1), (4.2) and (4.4), we get

$$(4.5) \quad \lim_{t \rightarrow \infty} E_{x,y} \left(\bar{u}_m(X_{t \wedge \bar{\tau}}, Y_{t \wedge \bar{\tau}}) e^{-\delta(t \wedge \bar{\tau})} I_{\{\bar{\tau} < t\}} \right) - \bar{u}_m(x, y) \leq E_{x,y} \left(\int_0^{\bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds \right) - V_{\bar{L}}(x, y).$$

Next, we show that

$$(4.6) \quad \lim_{t \rightarrow \infty} E_{x,y} \left(\bar{u}_m(X_{t \wedge \bar{\tau}}, Y_{t \wedge \bar{\tau}}) e^{-\delta(t \wedge \bar{\tau})} I_{\{\bar{\tau} > t\}} \right) = 0.$$

From (4.3), there exists a \bar{K} such that

$$E_{x,y} \left(\bar{u}_m(X_{t \wedge \bar{\tau}}, Y_{t \wedge \bar{\tau}}) e^{-\delta(t \wedge \bar{\tau})} I_{\{\bar{\tau} > t\}} \right) \leq (\bar{K} + a_1 x + a_2 y + pt) e^{-\delta t}.$$

Since the last expression goes to 0 as t goes to infinity, we have (4.6). Let us prove now that

$$(4.7) \quad \limsup_{m \rightarrow \infty} E_{x,y} \left(\int_0^{\bar{\tau}} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds \right) \leq 0.$$

Given any $\varepsilon > 0$, we can find T such that

$$(4.8) \quad \int_T^\infty \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds < \frac{\varepsilon}{2}$$

for any $m \geq 1$, as by virtue of (4.3), growth condition A.1, Lemma 4.1(b) and Lemma 4.1(c), and the growth property of V_1^0 and V_2^0 , there exist positive constants k_0, k_1, k_2 and \bar{p} such that

$$\begin{aligned} \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) &\leq \lambda \bar{u}_m(X_{s-}, Y_{s-}) + U(X_{s-}, Y_{s-}) \\ &\leq k_0 + k_1 x + k_2 y + \bar{p}s. \end{aligned}$$

Note that for $s \leq T$, $X_{s-} \leq x_0 := x + p_1 T$, $Y_{s-} \leq y_0 := y + p_2 T$. From Lemma 4.1(e) we can find m_0 large enough such that for any $m \geq m_0$

$$\int_0^T \mathcal{L}(\bar{u}_m)(X_{s-}, Y_{s-}) e^{-\delta s} ds \leq c_m \int_0^T e^{-\delta s} ds \leq \frac{c_m}{\delta} \leq \frac{\varepsilon}{2}$$

and so we get (4.7). Then, from (4.5) and using (4.6) and (4.7), we obtain

$$(4.9) \quad \bar{u}(x, y) = \lim_{m \rightarrow \infty} \bar{u}_m(x, y) \geq V_{\bar{L}}(x, y).$$

Since V is a viscosity solution of (3.1), the result follows. \square

From the previous proposition we can deduce the usual viscosity verification result.

Corollary 4.3. *Consider a family of admissible strategies $\{\bar{L}^{x,y} \in \Pi_{x,y} : (x, y) \in \mathbf{R}_+^2\}$. If the function $V_{\bar{L}^{x,y}}(x, y)$ is a viscosity supersolution of (3.1) for all $(x, y) \in \mathbf{R}_+^2$, then $V_{\bar{L}^{x,y}}(x, y)$ is the optimal value function (2.4).*

5. Iterative Approach

In this section, we approximate the optimal value function V defined in (2.4) by an increasing sequence of value functions of strategies which pay dividends (and collaborate if it is necessary) up to the n -th claim (regardless from which company) and then follow the take-the-money-and-run strategy. This iterative approach will be used in Section 9 to find the optimal strategy numerically.

Given initial surplus levels (x, y) , the take-the-money-and-run admissible strategy \bar{L}^0 pays immediately the entire surplus x and y as dividends (that is $X_{0+} = Y_{0+} = 0$), and then pays the incoming premium as dividends until the first claim, where the company facing that claim gets ruined. Note that under this strategy the companies can not help each other.

Consider τ_n as the time of arrival of the n -th claim regardless from which company, that is the n -th point of the Poisson process $N_t = N_t^1 + N_t^2$. We define the set $\Pi_{x,y}^n$ of all the admissible strategies in $\Pi_{x,y}$ which follow \bar{L}^0 right after τ_n . Let us define

$$(5.1) \quad V^n(x, y) = \sup_{\bar{L} \in \Pi_{x,y}^n} V_{\bar{L}}(x, y)$$

for $n \geq 1$, we also define $V^0 = V_{\bar{L}^0}$. We can write

$$(5.2) \quad \begin{aligned} V^0(x, y) &= a_1 x + a_2 y + \frac{\lambda_1}{\delta + \lambda} \left(\frac{p}{\lambda} + a_1 V_2^0(0) \right) \\ &\quad + \frac{\lambda_2}{\delta + \lambda} \left(\frac{p}{\lambda} + a_2 V_1^0(0) \right). \end{aligned}$$

Note that, for $n \geq 1$, the functions V^n can be constructed iteratively as

$$(5.3) \quad V^n(x, y) = \sup_{\bar{L} \in \Pi_{x,y}^n} V_{\bar{L}}^n(x, y),$$

where

$$(5.4) \quad \begin{aligned} V_{\bar{L}}^n(x, y) &= E_{x,y} \left(a_1 \int_0^{\tau_1} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau_1} e^{-\delta s} dL_s^2 + e^{-\delta \tau_1} V^{n-1}(X_{\tau_1}^{\bar{L}}, Y_{\tau_1}^{\bar{L}}) I_{\{\tau_1 < \bar{\tau}\}} \right. \\ &\quad \left. + e^{-\delta \bar{\tau}} \left(a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}}) \right) I_{\{\tau_1 = \bar{\tau}\}} \right). \end{aligned}$$

In this expression, we only consider the admissible strategy $\bar{L} \in \Pi_{x,y}$ for $t \leq \tau_1$. The following DPP holds.

Lemma 5.1. For any initial surplus (x, y) in \mathbf{R}_+^2 and any stopping time τ , we can write

$$\begin{aligned} V^n(x, y) = & \sup_{\bar{L} \in \Pi_{x,y}} (E_{x,y}(a_1 \int_0^{\tau_1 \wedge \bar{\tau}} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau_1 \wedge \bar{\tau}} e^{-\delta s} dL_s^2 \\ & + e^{-\delta \tau} V^n(X_{\bar{\tau}}^{\bar{L}}, Y_{\bar{\tau}}^{\bar{L}}) I_{\{\tau < \tau_1 \wedge \bar{\tau}\}} + e^{-\delta \tau_1} V^{n-1}(X_{\tau_1}^{\bar{L}}, Y_{\tau_1}^{\bar{L}}) I_{\{\tau_1 \wedge \bar{\tau} = \tau_1 < \bar{\tau}\}} \\ & + e^{-\delta \bar{\tau}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}})) I_{\{\tau_1 \wedge \bar{\tau} = \tau_1 = \bar{\tau}\}}). \end{aligned}$$

Proposition 5.2. We have that $V^1 \leq V^2 \leq \dots \leq V$.

Proof. We prove the result by induction:

- (a) $V^0 \leq V^1$, because the strategy $\bar{L}^0 \in \Pi_{x,y}^1$.
- (b) Assume that $V^{n-2} \leq V^{n-1}$. By (5.3), we have

$$\begin{aligned} V^n(x, y) & \geq \sup_{\bar{L} \in \Pi_{x,y}} E_{x,y}(a_1 \int_0^{\tau_1} e^{-\delta s} dL_s^1 + a_2 \int_0^{\tau_1} e^{-\delta s} dL_s^2 + e^{-\delta \tau_1} V^{n-2}(X_{\tau_1}^{\bar{L}}, Y_{\tau_1}^{\bar{L}}) I_{\{\tau_1 < \bar{\tau}\}} \\ & \quad + e^{-\delta \bar{\tau}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}})) I_{\{\tau_1 = \bar{\tau}\}} \\ & = V^{n-1}(x, y). \quad \square \end{aligned}$$

The HJB equation for V^n is given by

$$(5.5) \quad \max \{ \mathcal{L}^n(V^n)(x, y), a_1 - V_x^n(x, y), a_2 - V_y^n(x, y) \} = 0,$$

where

$$(5.6) \quad \begin{aligned} \mathcal{L}^n(V^n)(x, y) & = p_1 V_x^n(x, y) + p_2 V_y^n(x, y) - (\delta + \lambda) V^n(x, y) + \mathcal{I}(V^{n-1})(x, y) + U(x, y). \end{aligned}$$

The following basic results about regularity and growth at infinity of V^n , $n \geq 1$ are similar to those of Lemmas 2.2 and 2.3.

Lemma 5.3. The optimal value function V^n satisfies growth condition A.1, it is increasing and locally Lipschitz in \mathbf{R}_+^2 with

$$\begin{aligned} a_1 h & \leq V^n(x+h, y) - V^n(x, y) \leq (e^{(\delta+\lambda)h/p_1} - 1) V^n(x, y) \\ a_2 h & \leq V^n(x, y+h) - V^n(x, y) \leq (e^{(\delta+\lambda)h/p_2} - 1) V^n(x, y) \end{aligned}$$

for any $h > 0$ and for any (x, y) in \mathbf{R}_+^2 .

In the next two propositions, we see that V^n is a viscosity solution of the corresponding HJB equation.

Proposition 5.4. V^n is a viscosity supersolution of the HJB equation (5.5) for $x > 0$ and $y > 0$.

Proof. Similar to the one given in Proposition 3.3. □

Proposition 5.5. V^n is a viscosity subsolution of the corresponding HJB equation (5.5).

Proof. The proof of this proposition is similar to the one of Proposition 3.4, but using as martingales with zero expectation

$$(5.7) \quad \begin{aligned} M_t^3 & = \sum_{\substack{X_{s^-} \neq X_s \\ s \leq t}} (V^{n-1}(X_s, Y_s) - \psi(X_{s^-}, Y_{s^-})) e^{-\delta s} \\ & \quad - \lambda_1 \int_0^t e^{-\delta s} \int_0^{X_{s^-}} (V^{n-1}(X_{s^-} - \alpha, Y_{s^-}) - \psi(X_{s^-}, Y_{s^-})) dF^1(\alpha) ds \\ & \quad - \lambda_1 \int_0^t e^{-\delta s} \int_{X_{s^-}^- + Y_{s^-}}^{X_{s^-} + Y_{s^-}} (V^{n-1}(0, X_{s^-} + Y_{s^-} - \alpha) - \psi(X_{s^-}, Y_{s^-})) dF^1(\alpha) ds \\ & \quad - \lambda_1 \int_0^t e^{-\delta s} \int_{X_{s^-} + Y_{s^-}}^{\infty} (a_2 V_2^0(Y_{s^-}) - \psi(X_{s^-}, Y_{s^-})) dF^1(\alpha) ds \end{aligned}$$

and

$$\begin{aligned}
(5.8) \quad M_t^4 &= \sum_{\substack{Y_{s-} \neq Y_s \\ s \leq t}} (V^{n-1}(X_s, Y_s) - \psi(X_{s-}, Y_{s-})) e^{-\delta s} \\
&\quad - \lambda_2 \int_0^t e^{-\delta s} \int_0^{Y_{s-}} (V^{n-1}(X_{s-}, Y_{s-} - \alpha) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds \\
&\quad - \lambda_2 \int_0^t e^{-\delta s} \int_{X_{s-}}^{X_{s-} + Y_{s-}} (V^{n-1}(X_{s-} + Y_{s-} - \alpha, 0) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds \\
&\quad - \lambda_2 \int_0^t e^{-\delta s} \int_{X_{s-} + Y_{s-}}^{\infty} (a_1 V_1^0(Y_{s-}) - \psi(X_{s-}, Y_{s-})) dF^2(\alpha) ds.
\end{aligned}$$

instead of the martingales M_t^1 and M_t^2 defined in (3.13) and (3.14) respectively. \square

In the next proposition we state that V^n is the smallest viscosity solution of the corresponding HJB equation.

Proposition 5.6. *The optimal value function V^n is the smallest viscosity supersolution of (5.5) satisfying growth condition A.1.*

Proof. The proof of this proposition is similar to the one of Proposition 4.2, but using as martingales with zero expectation (5.7) and (5.8). \square

Remark 5.7. *From the above proposition we deduce the usual viscosity verification result for the n -step: Consider a family of admissible strategies $\{\bar{L}^{x,y} \in \Pi_{x,y} : (x,y) \in \mathbf{R}_+^2\}$. If the function $V_{\bar{L}^{x,y}}^n(x,y)$ is a viscosity supersolution of (5.5) then $V_{\bar{L}^{x,y}}^n = V^n$.*

Finally, we have the convergence result to the optimal value function (2.4).

Proposition 5.8. $V^n \nearrow V$ as n goes to infinity.

Proof. By Lemma 2.3 and Lemma 2.2, V is increasing and satisfies property A.1, so there exists a $T > 0$ such that

$$(5.9) \quad e^{-\delta t} V(x + p_1 t, y + p_2 t) < \frac{\varepsilon}{3}$$

for $t \geq T$. Let us define $\kappa = V(x + p_1 T, y + p_2 T) > 0$ and take $n_0 > 0$ such that

$$(5.10) \quad P(\tau_{n_0} \geq T) \geq 1 - \frac{\varepsilon}{3\kappa}.$$

There exists an admissible strategy $\bar{L} \in \Pi_{x,y}$ such that

$$(5.11) \quad V(x, y) - V_{\bar{L}}(x, y) \leq \frac{\varepsilon}{3}.$$

We define the strategy $\bar{L}^{n_0} \in \Pi_{x,y}^{n_0}$ as $\bar{L}_t^{n_0} = \bar{L}_t$ for $t \leq \tau_{n_0} \wedge \bar{\tau}$ and $\bar{L}_t^{n_0} = \bar{L}_{t-\tau_{n_0}}^0$ for $t \geq \tau_{n_0}$ if $\tau_{n_0} < \bar{\tau}$. From (5.9), (5.10) and Lemma 2.3, we have

$$\begin{aligned}
&V_{\bar{L}}(x, y) - V_{\bar{L}^{n_0}}(x, y) \\
&\leq E_{x,y} \left(a_1 \left(\int_{\bar{\tau} \wedge \tau_{n_0}}^{\bar{\tau}} e^{-\delta s} dL_s^1 + e^{-\delta \bar{\tau}} V_1^0(X_{\bar{\tau}}^{\bar{L}}) \right) + a_2 \left(\int_{\bar{\tau} \wedge \tau_{n_0}}^{\bar{\tau}} e^{-\delta s} dL_s^2 + e^{-\delta \bar{\tau}} V_2^0(Y_{\bar{\tau}}^{\bar{L}}) \right) \right) \\
&\leq E_{x,y} (e^{-\delta(\bar{\tau} \wedge \tau_{n_0})} V(X_{\bar{\tau} \wedge \tau_{n_0}}^{\bar{L}}, Y_{\bar{\tau} \wedge \tau_{n_0}}^{\bar{L}})) \\
&\leq E_{x,y} (I_{\{\bar{\tau} \wedge \tau_{n_0} \geq T\}} e^{-\delta(\bar{\tau} \wedge \tau_{n_0})} V(x + p_1(\bar{\tau} \wedge \tau_{n_0}), y + p_2(\bar{\tau} \wedge \tau_{n_0})) \\
&\quad + I_{\{\bar{\tau} \wedge \tau_{n_0} < T\}} e^{-\delta(\bar{\tau} \wedge \tau_{n_0})} V(x + p_1(\bar{\tau} \wedge \tau_{n_0}), y + p_2(\bar{\tau} \wedge \tau_{n_0}))) \\
&\leq E_{x,y} (I_{\{\bar{\tau} \wedge \tau_{n_0} \geq T\}} e^{-\delta(\bar{\tau} \wedge \tau_{n_0})} V(x + p_1(\bar{\tau} \wedge \tau_{n_0}), y + p_2(\bar{\tau} \wedge \tau_{n_0}))) + \kappa P(\tau_{n_0} < T) \\
&\leq \frac{2\varepsilon}{3}.
\end{aligned}$$

Then we obtain from (5.11)

$$V(x, y) \leq V_{\bar{L}}(x, y) + \frac{\varepsilon}{3} \leq V_{\bar{L}^{n_0}}(x) + \varepsilon \leq V^n(x, y) + \varepsilon$$

for any $n \geq n_0$. \square

6. Stationary dividend strategies

As in the one-dimensional case (see for instance Azcue and Muler [9]) our aim is to find a stationary dividend strategy whose value function is the optimal value function V . A dividend strategy is *stationary* when the decision on the dividend payment depends on the current surplus only, and not on the full history of the controlled process. More precisely, a stationary dividend strategy is given by four positive and Borel-measurable functions $l_i : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+$ and $\Delta_{L^i} : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+$, $i = 1, 2$. A stationary dividend strategy generates a family of admissible strategies $\{\bar{L}^{x,y} = (L^1, L^2) \in \Pi_{x,y} \text{ for any } (x, y) \in \mathbf{R}_+^2\}$ which are the solutions of the following stochastic differential equations

$$L_t^i = \int_0^t l_i \left(X_s^{\bar{L}}, Y_s^{\bar{L}} \right) ds + \sum_{s \leq t} \Delta_{L^i} \left(X_s^{\bar{L}}, Y_s^{\bar{L}} \right)$$

with initial surplus levels $X_0^{\bar{L}} = x, Y_0^{\bar{L}} = y$. $\Delta_{L^i}(x, y)$ corresponds to lump sum dividend payments and $l_i(x, y)$ corresponds to the continuous dividend payment rate of the i -th company when the current surplus level is (x, y) .

We are not proving in this paper the existence of an optimal strategy. However, if there exist optimal admissible strategies for every $(x, y) \in \mathbf{R}_+^2$, then there exists an optimal stationary strategy. Given the optimal admissible strategy $(\tilde{L}^1, \tilde{L}^2) \in \Pi_{x,y}$, we define

$$\Delta_{L^i}(x, y) = \tilde{L}_{0+}^i \text{ and } l_i(x, y) = d\tilde{L}_0^{i,c} \text{ for } i = 1, 2.$$

It can be proved, using the Dynamic Programming Principle and a stair-step approximation on the time variable, that these functions are Borel-measurable and that the stationary dividend strategy generated by them are optimal for any $(x, y) \in \mathbf{R}_+^2$.

If the optimal value function V is differentiable (and this fact has to be checked for every concrete case, see Section 8), the form in which the optimal value function solves the HJB equation at any $(x, y) \in \mathbf{R}_+^2$ suggests how the dividends should be paid when the current surplus is (x, y) ; here we are using the result given in Corollary 3.5. There are only seven possibilities:

- (i) If the current surplus is in the open set

$$\mathcal{C}^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) = 0, V_x(x, y) > a_1, V_y(x, y) > a_2\},$$

no dividends are paid. The set \mathcal{C}^* is called the *non-action* set.

- (ii) If the current surplus is in the open set

$$\mathcal{B}_1^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) < 0, V_x(x, y) = a_1, V_y(x, y) > a_2\},$$

Company One pays a lump sum as dividends. This lump sum should be $\min\{b > 0 : (x - b, y) \notin \mathcal{B}_1^*\}$.

- (iii) If the current surplus is in the open set

$$\mathcal{B}_2^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) < 0, V_x(x, y) > a_1, V_y(x, y) = a_2\},$$

Company Two pays a lump sum as dividends. This lump sum should be $\min\{b > 0 : (x, y - b) \notin \mathcal{B}_2^*\}$.

- (iv) If the current surplus is in the set

$$\mathcal{B}_0^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) < 0, V_x(x, y) = a_1, V_y(x, y) = a_2\},$$

but not in the closure of $\mathcal{B}_1^* \cup \mathcal{B}_2^*$, both companies pay a lump sum as dividends.

- (v) If the current surplus is in the closed set

$$\mathcal{A}_0^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) = 0, V_x(x, y) = a_1, V_y(x, y) = a_2\},$$

both companies pay their incoming premiums as dividends.

(vi) If the current surplus is in the set

$$\mathcal{A}_1^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) = 0, V_x(x, y) = a_1, V_y(x, y) > a_2\},$$

Company One pays dividends at some special rate so that the surplus remains in $\mathcal{A}_1^* \cup \mathcal{A}_0^*$.

(vii) If the current surplus is in the set

$$\mathcal{A}_2^* = \{(x, y) \in \mathbf{R}_+^2 : \mathcal{L}(V)(x, y) = 0, V_x(x, y) > a_1, V_y(x, y) = a_2\},$$

Company Two pays dividends at some special rate so that the surplus remains in $\mathcal{A}_2^* \cup \mathcal{A}_0^*$.

Note that if V is a continuously differentiable solution of (3.1), then $\mathcal{A}^* = \mathcal{A}_0^* \cup \mathcal{A}_1^* \cup \mathcal{A}_2^*$ is closed, $\mathcal{B}^* = \mathcal{B}_1^* \cup \mathcal{B}_2^* \cup \mathcal{B}_0^*$ is open, and any segment which connects a point of \mathcal{B}^* with a point of the open set \mathcal{C}^* should contain a point of \mathcal{A}^* .

Remark 6.1. *These strategies are the generalizations of the band strategies in the two-dimensional case. The number of bands corresponds to the number of connected components of the set \mathcal{A}^* . As in the one-dimensional case, we do not expect the sets \mathcal{A}^* , \mathcal{B}^* and \mathcal{C}^* to be connected in general; see for instance Chapters 5 and 6 of [9].*

Remark 6.2. *Let us consider the simplest case of identical and independent Cramér-Lundberg processes in (2.1); that is $p_1 = p_2 = p$; $\lambda_1 = \lambda_2$; and U_i^1, U_i^2 have the same distribution F . We also choose the dividends paid by both companies to be equally weighted, i.e. $a_1 = a_2 = 1/2$. Under these assumptions, the optimal value function will be symmetric, that is $V(x, y) = V(y, x)$, and so the sets introduced above satisfy the following properties: the line $y = x$ is an axis of symmetry of the sets \mathcal{C}^* , \mathcal{B}_0^* and \mathcal{A}_0^* ; the sets \mathcal{B}_1^* and \mathcal{A}_1^* are the reflection with respect to the line $y = x$ of the sets \mathcal{B}_2^* and \mathcal{A}_2^* respectively.*

7. Curve strategies

We introduce a family of stationary dividend strategies, called *curve strategies*, in which dividends are paid in the seven ways mentioned in the previous section, having a simple structure: here the boundary between the action and non-action region is given by a curve. These strategies can be seen as the natural analogues of the one-dimensional barrier strategies in this two-dimensional case.

It is reasonable to think that if the optimal strategy is a curve strategy it should satisfy the following properties: If $(x_0, y_0) \in \mathcal{B}_1^* \cup \mathcal{A}_1^*$ (that is only Company One pays dividends), then (x, y_0) should be in \mathcal{B}_1^* for all $x > x_0$; analogously if $(x_0, y_0) \in \mathcal{B}_2^* \cup \mathcal{A}_2^*$ (that is only Company Two pays dividends), then (x_0, y) should be in \mathcal{B}_2^* for all $y > y_0$. Finally, the set \mathcal{C}^* should be bounded because, as in the one-dimensional case, the surplus of each company under the optimal strategy should be bounded for $t > 0$.

Let us define the curve strategies satisfying the properties mentioned above. For these strategies, \mathbf{R}_+^2 is partitioned into seven sets \mathcal{C} , \mathcal{A}_0 , \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{B}_0 , \mathcal{B}_1 and \mathcal{B}_2 where $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ is a curve which intersects both coordinate axes.

- $\mathcal{A}_0 = \{(\bar{x}, \bar{y})\}$ with $(\bar{x}, \bar{y}) \in \mathbf{R}_+^2$. If the current surplus is (\bar{x}, \bar{y}) , both companies pay their incoming premium as dividends. Let us call \bar{u} the x -intercept and \bar{v} the y -intercept of the line with slope p_2/p_1 passing through (\bar{x}, \bar{y}) ; let us denote $\mathcal{O}_1^{(\bar{x}, \bar{y})}$ and $\mathcal{O}_2^{(\bar{x}, \bar{y})}$ the regions in the first quadrant bounded above and below by this line, respectively.
- $\mathcal{B}_0 = [\bar{x}, \infty) \times [\bar{y}, \infty) - \mathcal{A}_0$. If the current surplus is $(x, y) \in \mathcal{B}_0$, Company One and Company Two pay $x - \bar{x}$ and $y - \bar{y}$ as dividends, respectively.
- The set \mathcal{A}_1 is a curve in $\mathcal{O}_1^{(\bar{x}, \bar{y})}$ parametrized by

$$\mathcal{A}_1 = \left\{ \left(u + \frac{p_1}{p_2} \xi_1(u), \xi_1(u) \right) \text{ with } \bar{u} < u \leq M_{\xi_1} \right\},$$

where $\xi_1 : [\bar{u}, M_{\xi_1}] \rightarrow \mathbf{R}$ is a continuously differentiable function with $\xi_1(\bar{u}) = \bar{y}$, $\xi_1(M_{\xi_1}) = 0$ and negative derivative. If the current surplus $(x, y) \in \mathcal{A}_1$, Company

Two does not pay dividends and Company One pays dividends at some special rate for which the bivariate surplus remains in the curve \mathcal{A}_1 . By basic calculus, it can be shown that this rate is given by

$$l_1(x, y) = -\frac{p_2}{\xi_1'(x - (p_1/p_2)y)}.$$

- The set \mathcal{B}_1 is the set to the right of \mathcal{A}_1 in $\mathcal{O}_1^{(\bar{x}, \bar{y})}$, that is

$$\mathcal{B}_1 = \left\{ (x, y) \in \mathbf{R}_+^2 : y \leq \bar{y} \text{ and } x > \xi_1^{-1}(y) + \frac{p_1}{p_2}y \right\}.$$

If the current surplus $(x, y) \in \mathcal{B}_1$, Company Two does not pay dividends and Company One pays the lump sum

$$\min\{b > 0 : (x - b, y) \in \mathcal{A}_1\} = x - (p_1/p_2)y - \xi_1^{-1}(y).$$

- The sets \mathcal{A}_2 and \mathcal{B}_2 in $\mathcal{O}_2^{(\bar{x}, \bar{y})}$ are defined analogously to \mathcal{A}_1 and \mathcal{B}_1 with the roles of Company One and Two interchanged; that is

$$\mathcal{A}_2 = \left\{ (\xi_2(v), v + \frac{p_2}{p_1}\xi_2(v)) \text{ with } \bar{v} < v \leq M_{\xi_2} \right\},$$

and

$$\mathcal{B}_2 = \left\{ (x, y) \in \mathbf{R}_+^2 : x \leq \bar{x} \text{ and } y > \xi_2^{-1}(x) + \frac{p_2}{p_1}x \right\},$$

where $\xi_2 : [\bar{v}, M_{\xi_2}] \rightarrow \mathbf{R}$ is a continuously differentiable function with $\xi_2(\bar{v}) = \bar{x}$, $\xi_2(M_{\xi_2}) = 0$ and negative derivative. If the current surplus $(x, y) \in \mathcal{A}_2$, Company One does not pay dividends and Company Two pays dividends at some special rate for which the bivariate surplus remains in the curve \mathcal{A}_2 . Here this rate is

$$l_2(x, y) = -\frac{p_1}{\xi_2'(y - (p_2/p_1)x)}.$$

- If the current surplus $(x, y) \in \mathcal{B}_2$, Company One does not pay dividends and Company Two pays the lump sum

$$\min\{b > 0 : (x, y - b) \in \mathcal{A}_2\} = y - (p_2/p_1)x - \xi_2^{-1}(x).$$

- The no-action region \mathcal{C} is the open set delimited by the curve \mathcal{A} and the axes. If the current surplus $(x, y) \in \mathcal{C}$, no dividends are paid.

The set partition of the curve strategy corresponding to $(\bar{x}, \bar{y}) = (1, 2)$ and the functions

$$\xi_1(u) = \frac{2(u-4)(u-6)}{35} \text{ for } u \in [-1, 4] \text{ and } \xi_2(v) = \frac{(u-3)(u-6)}{10} \text{ for } v \in [1, 3]$$

are illustrated in Figure 7.1.

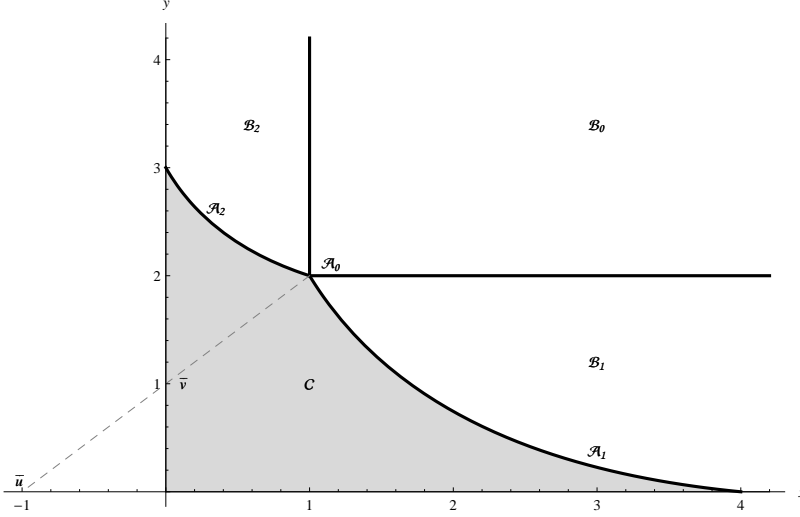


Fig. 6.1: Example of Curve Strategy

For any $w \in \mathbf{R}$, let us define the set

$$(7.1) \quad \Phi^w = \{ \xi : [w, M_\xi] \rightarrow \mathbf{R}_+, \xi(M_\xi) = 0, M_\xi \geq 0, \xi' < 0 \text{ and } \xi' \text{ continuous} \}.$$

Note that the curve strategies depend only on the point $(\bar{x}, \bar{y}) \in \mathbf{R}_+^2$ and the functions $\xi_1 \in \Phi^{\bar{w}}$ and $\xi_2 \in \Phi^{\bar{v}}$, used in the parametrization of the curve \mathcal{A} . We associate to any $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$ and any $(x, y) \in \mathbf{R}_+^2$ the admissible strategy $\bar{L}^{\bar{\xi}} = (L_t^{1, \bar{\xi}}, L_t^{2, \bar{\xi}}) \in \Pi_{x, y}$. Let us define the value function $V^{\bar{\xi}}$ of this curve strategy as

$$(7.2) \quad V^{\bar{\xi}}(x, y) = V_{\bar{L}^{\bar{\xi}}}(x, y).$$

We will look for $\bar{\xi}^*$ such that the associated value function $V^{\bar{\xi}^*}$ is the optimal value function defined in (2.4).

Remark 7.1. In the case that \mathcal{A} is the segment $x + y = K$ for some $K > 0$ in \mathbf{R}_+^2 , the sum of the dividend rates paid by Company One and Two is $p_1 + p_2$ for any current surplus in this line. The point $\mathcal{A}_0 = (\bar{x}, K - \bar{x})$ indicates how this dividend payment is splitted among the two companies in \mathcal{A} : At \mathcal{A}_0 , Company One pays p_1 and Company Two pays p_2 , to the right of this point (\mathcal{A}_1) Company One pays the total rate $p_1 + p_2$ and to the left of this point (\mathcal{A}_2) it is Company Two which pays $p_1 + p_2$.

8. Search for the Optimal Curve Strategy

The goal of this section is to see whether the optimal value function V is the value function of a curve strategy as defined in the previous section. We do not have a theoretical result on the existence and structure of the optimal stationary strategy, however we can use the iterative approach introduced in Section 5 and Proposition 5.8 as a verification result. This verification result is stated in Proposition 8.8; it will be used in the next Section to obtain the optimal curve strategy and establish its optimality among all admissible dividend strategies in a particular example.

We first define an auxiliary function. For any $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$, where $(\bar{x}, \bar{y}) \in \mathbf{R}_+^2$, $\xi_1 \in \Phi^{\bar{w}}$, $\xi_2 \in \Phi^{\bar{v}}$ and any continuous function $W_0 : \mathbf{R}_+^2 \rightarrow [0, +\infty)$, let

$$(8.1) \quad W^{\bar{\xi}}(x, y) := E_{x,y} \left(\int_0^{\tau_1} e^{-\delta s} \left(a_1 dL_s^{1,\bar{\xi}} + a_2 dL_s^{2,\bar{\xi}} \right) + e^{-\delta \tau_1} W_0(X_{\tau_1}^{\bar{\xi}}, Y_{\tau_1}^{\bar{\xi}}) I_{\{\tau_1 < \bar{\tau}\}} \right. \\ \left. + e^{-\delta \bar{\tau}} \left(a_1 V_1^0(X_{\bar{\tau}}^{\bar{\xi}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{\xi}}) \right) I_{\{\tau_1 = \bar{\tau}\}} \right).$$

If W_0 is the value function of a family of admissible strategies $\bar{L} = (\bar{L}_{x,y} \in \Pi_{x,y})_{(x,y) \in \mathbb{R}_+^2}$, and $\mathcal{C}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_0, \mathcal{B}_1$ and \mathcal{B}_2 are the sets associated to $\bar{\xi}$ (as defined in the previous section), then $W^{\bar{\xi}}$ would be the value function of the strategy which pays dividends according to the curve strategy $\bar{L}^{\bar{\xi}}$ up to the first claim and according to \bar{L} afterwards. We call this kind of strategy a *one-step curve strategy*.

Define

$$(8.2) \quad H(x, y) := \mathcal{I}(W_0)(x, y) + U(x, y).$$

In the next proposition, we find an explicit formula for the function $W^{\bar{\xi}}$ in terms of W_0 and ξ_1 for $(x, y) \in \mathcal{O}_1^{(\bar{x}, \bar{y})}$; the formula for the value function for $(x, y) \in \mathcal{O}_2^{(\bar{x}, \bar{y})}$ follows in an analogous way and depends only on ξ_2 .

In order to obtain this formula, we use the fact that $W^{\bar{\xi}}$ satisfies the integro-differential equation $\mathcal{L}(W^{\bar{\xi}}) = 0$ in $\mathcal{C} \cup \mathcal{A}$ and that $W_x^{\bar{\xi}} = a_1$ in $\mathcal{A}_1 \cup \mathcal{A}_0 \cup \mathcal{B}_1 \cup \mathcal{B}$.

Proposition 8.1. *Given $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$ and a continuous function W_0 , we have that*

$$W^{\bar{\xi}}(x, y) = e^{-(\delta+\lambda) \frac{\xi_1(x - \frac{p_1}{p_2}y) - y}{p_2}} k(x - \frac{p_1}{p_2}y) I_{\{(y-\bar{u}) \frac{p_1}{p_2} \leq x \leq \xi_1^{-1}(y) + \frac{p_1}{p_2}y, y \leq \bar{y}\}} \\ + \left(\int_0^{\frac{\xi_1(x - \frac{p_1}{p_2}y) - y}{p_2}} e^{-(\delta+\lambda)w} H(x + p_1w, y + p_2w) dw \right) I_{\{(y-\bar{u}) \frac{p_1}{p_2} \leq x \leq \xi_1^{-1}(y) + \frac{p_1}{p_2}y, y \leq \bar{y}\}} \\ + \left(a_1(x - \xi_1^{-1}(y) - \frac{p_1}{p_2}y) + k(\xi_1^{-1}(y)) \right) I_{\{x \geq \xi_1^{-1}(y) + \frac{p_1}{p_2}y, y \leq \bar{y}\}} \\ + \left(a_1(x - \bar{x}) + a_2(y - \bar{y}) + k(\bar{x} - \frac{p_1}{p_2}\bar{y}) \right) I_{\{x \geq \bar{x}, y \geq \bar{y}\}},$$

for $(x, y) \in \mathcal{O}_1^{(\bar{x}, \bar{y})}$, where $\bar{u} = \bar{x} - \frac{p_1}{p_2}\bar{y}$, the function H is defined in (8.2) and

$$k(u) = e^{(\delta+\lambda) \frac{\xi_1(u) - \xi_1(\bar{u})}{p_2}} \left(\frac{p}{\delta+\lambda} + \frac{1}{\delta+\lambda} H(\bar{u} + \frac{p_1}{p_2}\xi_1(\bar{u}), \xi_1(\bar{u})) \right) \\ + e^{(\delta+\lambda) \frac{\xi_1(u)}{p_2}} a_1 \int_{\bar{u}}^u e^{-(\delta+\lambda) \frac{\xi_1(w)}{p_2}} dw \\ + \frac{e^{(\delta+\lambda) \frac{\xi_1(u)}{p_2}}}{p_2} \int_{\xi_1(u)}^{\xi_1(\bar{u})} H(\xi_1^{-1}(t) + \frac{p_1}{p_2}t, t) e^{-(\delta+\lambda) \frac{t}{p_2}} dt.$$

Proof. Let us consider first an initial surplus $(x, y) \in \mathcal{C} \cap \mathcal{O}_1^{(\bar{x}, \bar{y})}$. By definition (8.1) we have that the controlled surplus process for $t < \tau_1 \wedge h$ and $h > 0$ small enough is given by

$$(X_t, Y_t) = (x + p_1t, y + p_2t).$$

So we have that

$$W^{\bar{\xi}}(x, y) = E_{x,y} \left(e^{-\delta t} W^{\bar{\xi}}(X_{t \wedge \tau_1}, Y_{t \wedge \tau_1}) I_{\{t \wedge \tau_1 = t\}} + e^{-\delta \tau_1} W_0(X_{\tau_1}, Y_{\tau_1}) I_{\{t \wedge \tau_1 = \tau_1 < \bar{\tau}\}} \right. \\ \left. + e^{-\delta \bar{\tau}} \left(a_1 V_1^0(X_{\bar{\tau}}) + a_2 V_2^0(Y_{\bar{\tau}}) \right) I_{\{t \wedge \tau_1 = \tau_1 = \bar{\tau}\}} \right).$$

We can write

$$\begin{aligned}
& E_{x,y} \left(e^{-\delta(\tau_1 \wedge t)} I_{\{\tau_1 \wedge t = t < \tau_1\}} W^{\bar{\xi}}(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) + e^{-\delta(\tau_1 \wedge t)} I_{\{\tau_1 \wedge t = \tau_1 < \bar{\tau}\}} W_0(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) \right) \\
&= E_{x,y} \left(e^{-\delta(\tau_1 \wedge t)} I_{\{\tau_1 \wedge t = t < \tau_1\}} W^{\bar{\xi}}(X_{\tau_1 \wedge t}^{\bar{L}}, Y_{\tau_1 \wedge t}^{\bar{L}}) \right) \\
&\quad + E_{x,y} \left(I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^1\}} e^{-\delta\tau_1^1} W_0(X_{\tau_1^1}^{\bar{L}}, Y_{\tau_1^1}^{\bar{L}}) \right) \\
&\quad + E_{x,y} \left(I_{\{\tau_1 = \tau_1 \wedge t < \bar{\tau} \text{ and } \tau_1 = \tau_1^2\}} e^{-\delta\tau_1^2} W_0(X_{\tau_1^2}^{\bar{L}}, Y_{\tau_1^2}^{\bar{L}}) \right),
\end{aligned}$$

and so

$$\lim_{t \rightarrow 0^+} \frac{e^{-(\lambda+\delta)t} W^{\bar{\xi}}(x + p_1 t, y + p_2 t) - W^{\bar{\xi}}(x, y)}{t} = -H(x, y).$$

Then $g(t) = W^{\bar{\xi}}(x + p_1 t, y + p_2 t)$ is continuous and differentiable as long as $(x + p_1 t, y + p_2 t) \in \mathcal{C}$ with

$$(8.3) \quad g'(0) = (\lambda + \delta) W^{\bar{\xi}}(x, y) - H(x, y).$$

Let us prove now that the function $W^{\bar{\xi}}$ is continuous in \mathcal{A}_1 and has a continuous derivative in the direction of this curve. In case $(x, y) \in \mathcal{A}_1$, we have that for $t < \tau_1 \wedge h$ and $h > 0$ small enough, the controlled surplus process is

$$(X_t, Y_t) = \left(x + p_1 t + \int_0^t \frac{p_2}{\xi_1'(X_s - (p_1/p_2) Y_s)} ds, y + p_2 t \right) \in \mathcal{A}_1.$$

By (8.1), we have that

$$\begin{aligned}
W^{\bar{\xi}}(x, y) &= E_{x,y} \left(a_1 \int_0^{\tau_1 \wedge t} \frac{-p_2}{\xi_1'(X_s - (p_1/p_2) Y_s)} e^{-\delta s} ds + e^{-\delta t} W^{\bar{\xi}}(X_{t \wedge \tau_1}, Y_{t \wedge \tau_1}) I_{\{t \wedge \tau_1 = t\}} \right) \\
&\quad + E_{x,y} \left(e^{-\delta\tau_1} W_0(X_{\tau_1}, Y_{\tau_1}) I_{\{t \wedge \tau_1 = \tau_1 < \bar{\tau}\}} + e^{-\delta\bar{\tau}} (a_1 V_1^0(X_{\bar{\tau}}) + a_2 V_2^0(Y_{\bar{\tau}})) I_{\{t \wedge \tau_1 = \tau_1 = \bar{\tau}\}} \right).
\end{aligned}$$

Then, with an argument similar to the case of \mathcal{C} , we obtain for any $(x, y) \in \mathcal{A}_1$,

$$\begin{aligned}
& \lim_{t \rightarrow 0} \frac{e^{-(\lambda+\delta)t} W^{\bar{\xi}} \left(x + p_1 t + \int_0^t \frac{p_2}{\xi_1'(X_s - (p_1/p_2) Y_s)} ds, y + p_2 t \right) - W^{\bar{\xi}}(x, y)}{t} \\
&= -H(x, y) + \frac{a_1 p_2}{\xi_1'(x - (p_1/p_2) y)}.
\end{aligned}$$

So

$$g_1(t) := W^{\bar{\xi}} \left(x + p_1 t + \int_0^t \frac{p_2}{\xi_1'(X_s - (p_1/p_2) Y_s)} ds, y + p_2 t \right)$$

is continuous and differentiable at $t = 0$ and satisfies

$$(8.4) \quad g_1'(0) = (\lambda + \delta) W^{\bar{\xi}}(x, y) - H(x, y) + \frac{a_1 p_2}{\xi_1'(x - (p_1/p_2) y)}.$$

Since $(x + p_1 t + \int_0^t p_2 / \xi_1'(X_s - (p_1/p_2) Y_s) ds, y + p_2 t) \in \mathcal{A}_1$ for t small enough, we have that

$$(x + p_1 t + \int_0^t \frac{p_2}{\xi_1'(X_s - (p_1/p_2) Y_s)} ds, y + p_2 t) = (u_0(t) + \frac{p_1}{p_2} \xi_1(u_0(t)), \xi_1(u_0(t)))$$

for $u_0(t) := x - (p_1/p_2) y + \int_0^t p_2 / \xi_1'(X_s - (p_1/p_2) Y_s) ds$; therefore

$$W^{\bar{\xi}}(u_0(t) + \frac{p_1}{p_2} \xi_1(u_0(t)), \xi_1(u_0(t))) = g_1(t).$$

Since $u'_0(t) = p_2/\xi'_1(X_t - (p_1/p_2)Y_t)$ is continuous and negative, u_0^{-1} exists and is continuously differentiable, so

$$k(u) := W^{\bar{\xi}}\left(u + \frac{p_1}{p_2}\xi_1(u), \xi_1(u)\right) = g_1 \circ u_0^{-1}(u)$$

is continuously differentiable.

Defining

$$W(u, s) := W^{\bar{\xi}}\left(u + \frac{p_1}{p_2}\xi_1(u) - p_1s, \xi_1(u) - p_2s\right),$$

we obtain for $(u + (p_1/p_2)\xi_1(u) - p_1s, \xi_1(u) - p_2s) \in \mathcal{C}$ that

$$\begin{aligned} \widehat{\mathcal{L}}(W)(u, s) \\ := -W_s(u, s) - (\delta + \lambda)W(u, s) + H\left(u + \frac{p_1}{p_2}\xi_1(u) - p_1s, \xi_1(u) - p_2s\right) &= 0 \end{aligned}$$

The equation $\widehat{\mathcal{L}}(W)(u, s) = 0$ is a linear ODE in the variable s , so

$$W(u, s)e^{(\delta+\lambda)s} - k(u) = \int_0^s e^{(\delta+\lambda)t} H\left(u + \frac{p_1}{p_2}\xi_1(u) - p_1t, \xi_1(u) - p_2t\right) dt;$$

therefore

$$\begin{aligned} (8.5) \quad W^{\bar{\xi}}\left(u + \frac{p_1}{p_2}\xi_1(u) - p_1s, \xi_1(u) - p_2s\right) \\ = e^{-(\delta+\lambda)s} \left(k(u) + \int_0^s e^{(\delta+\lambda)t} H\left(u + \frac{p_1}{p_2}\xi_1(u) - p_1t, \xi_1(u) - p_2t\right) dt\right), \end{aligned}$$

for $\bar{u} \leq u \leq M_{\xi_1}$ and $0 \leq s \leq \min\{\xi_1(u)/p_2, u/p_1 + \xi_1(u)/p_2\}$. So $W^{\bar{\xi}}$ is continuously differentiable in the intersection of the set $\mathcal{C} \cup \mathcal{A}_1$ with $\mathcal{O}_1^{(\bar{x}, \bar{y})}$. We also have, from (8.3) and (8.4), that for any $(x, y) \in \mathcal{A}_1$,

$$\begin{aligned} \lim_{t \rightarrow 0^-} \frac{W^{\bar{\xi}}(x+p_1t, y+p_2t) - W^{\bar{\xi}}(x, y)}{t} \\ = \lim_{t \rightarrow 0^+} \frac{W^{\bar{\xi}}\left(x+p_1t + \frac{p_2t}{\xi'_1(x - \frac{p_1}{p_2}y)}, y+p_2t\right) - W^{\bar{\xi}}(x, y)}{t} - a_1 \frac{p_2}{\xi'_1(x - \frac{p_1}{p_2}y)}. \end{aligned}$$

Then from

$$\begin{aligned} p_1 W_{x^-}^{\bar{\xi}}(x, y) + p_2 W_{y^-}^{\bar{\xi}}(x, y) \\ = \left(p_1 + \frac{p_2}{\xi'_1(x - \frac{p_1}{p_2}y)}\right) W_{x^-}^{\bar{\xi}}(x, y) + p_2 W_{y^-}^{\bar{\xi}}(x, y) - a_1 \frac{p_2}{\xi'_1(x - \frac{p_1}{p_2}y)}, \end{aligned}$$

we conclude that $W_{x^-}^{\bar{\xi}}(x, y) = a_1$.

By (8.5), and since $(u + (p_1/p_2)\xi_1(u), \xi_1(u)) \in \mathcal{A}_1$,

$$\begin{aligned} W_x^{\bar{\xi}}\left(u + \frac{p_1}{p_2}\xi_1(u), \xi_1(u)\right) &= k'(u) + \left(H\left(u + \frac{p_1}{p_2}\xi_1(u), \xi_1(u)\right) - (\delta + \lambda)k(u)\right) \frac{\xi'_1(u)}{p_2} \\ &= a_1, \end{aligned}$$

and then

$$(8.6) \quad k(u) = k(\bar{u}) e^{(\delta+\lambda)\frac{\xi_1(u) - \xi_1(\bar{u})}{p_2}} + \int_{\bar{u}}^u \left(a_1 - H\left(w + \frac{p_1}{p_2}\xi_1(w), \xi_1(w)\right) \frac{\xi'_1(w)}{p_2}\right) e^{(\delta+\lambda)\frac{\xi_1(u) - \xi_1(w)}{p_2}} dw.$$

At the point $(\bar{u} + (p_1/p_2)\xi_1(\bar{u}), \xi_1(\bar{u})) \in \mathcal{A}_0$ the dividend strategy consists of collecting all the incoming premium as dividends up to the time τ_1 , so

$$k(\bar{u}) = \frac{p}{\delta + \lambda} + \frac{1}{\delta + \lambda} H(\bar{u} + \frac{p_1}{p_2}\xi_1(\bar{u}), \xi_1(\bar{u})).$$

Then we have, from (8.6),

$$\begin{aligned} k(u) &= e^{(\delta+\lambda)\frac{\xi_1(u)-\xi_1(\bar{u})}{p_2}} \left(\frac{p}{\delta+\lambda} + \frac{1}{\delta+\lambda} H(\bar{u} + \frac{p_1}{p_2}\xi_1(\bar{u}), \xi_1(\bar{u})) \right) \\ &\quad + \int_{\bar{u}}^u \left(a_1 - H(w + \frac{p_1}{p_2}\xi_1(w)), \xi_1(w) \right) \frac{\xi_1'(w)}{p_2} e^{(\delta+\lambda)\frac{\xi_1(w)-\xi_1(\bar{u})}{p_2}} dw. \end{aligned}$$

We conclude for (8.5) that for any (x, y) in the intersection of the set \mathcal{C} with $\mathcal{O}_1^{\bar{x}, \bar{y}}$,

$$\begin{aligned} W^{\bar{\xi}}(x, y) &= e^{-(\delta+\lambda)\frac{\xi_1(x-\frac{p_1}{p_2}y)-y}{p_2}} k(x - \frac{p_1}{p_2}y) \\ &\quad + \int_0^{\frac{\xi_1(x-\frac{p_1}{p_2}y)-y}{p_2}} e^{-(\delta+\lambda)w} H(x + p_1w, y + p_2w) dw, \end{aligned}$$

which yields the result. \square

Remark 8.2. The formula of $W^{\bar{\xi}}$ in $\mathcal{O}_2^{\bar{x}, \bar{y}}$ can be obtained using the formula given in Proposition 8.1 by interchanging the role of Company One and Company Two using that $W_y^{\bar{\xi}} = a_2$ in $\mathcal{A}_2 \cup \mathcal{A}_0 \cup \mathcal{B}_2 \cup \mathcal{B}_0$. More precisely, if $(x, y) \in \mathcal{O}_2^{\bar{x}, \bar{y}}$,

$$\begin{aligned} W^{\bar{\xi}}(x, y) &= e^{-(\delta+\lambda)\frac{\xi_2(y-\frac{p_2}{p_1}x)-x}{p_1}} \tilde{k}(y - \frac{p_2}{p_1}x) I_{\{x \leq \bar{x}, y \leq \xi_2^{-1}(x) + \frac{p_2}{p_1}x\}} \\ &\quad + \left(\int_0^{\frac{\xi_2(y-\frac{p_2}{p_1}x)-x}{p_1}} e^{-(\delta+\lambda)w} H(x + p_1w, y + p_2w) dw \right) I_{\{x \leq \bar{x}, y \leq \xi_2^{-1}(x) + \frac{p_2}{p_1}x\}} \\ &\quad + \left(a_2(y - \xi_2^{-1}(x) - \frac{p_2}{p_1}x) + \tilde{k}(\xi_2^{-1}(x)) \right) I_{\{x \leq \bar{x}, y \geq \xi_2^{-1}(x) + \frac{p_2}{p_1}x\}} \\ &\quad + \left(a_1(x - \bar{x}) + a_2(y - \bar{y}) + \tilde{k}(\bar{y} - \frac{p_2}{p_1}\bar{x}) \right) I_{\{x \geq \bar{x}, y \geq \bar{y}\}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{k}(v) &= e^{(\delta+\lambda)\frac{\xi_2(v)-\xi_2(\bar{v})}{p_1}} \left(\frac{p}{\delta+\lambda} + \frac{1}{\delta+\lambda} H(\xi_2(\bar{v}), \bar{v} + \frac{p_2}{p_1}\xi_2(\bar{v})) \right) \\ &\quad + e^{(\delta+\lambda)\frac{\xi_2(v)}{p_1}} a_2 \int_{\bar{v}}^v e^{-(\delta+\lambda)\frac{\xi_2(w)}{p_1}} dw \\ &\quad + \frac{e^{(\delta+\lambda)\frac{\xi_2(v)}{p_1}}}{p_1} \int_{\xi_2(\bar{v})}^{\xi_2(v)} H(t, \xi_2^{-1}(t) + \frac{p_2}{p_1}t) e^{-(\delta+\lambda)\frac{t}{p_1}} dt. \end{aligned}$$

From the formulas obtained in Proposition 8.1 and Remark 8.2 we obtain the following regularity result.

Proposition 8.3. *If the function H defined in (8.2) is continuously differentiable, then $W^{\bar{\xi}}$ is continuously differentiable in \mathbf{R}_+^2 .*

Proof. Since ξ_1 and ξ_2 are continuously differentiable, it is clear that $W^{\bar{\xi}}$ is continuously differentiable except possibly at the points of either the boundary of \mathcal{B}_0 or the segment

$$\mathcal{S} = \left\{ (x, \frac{p_2}{p_1}(x - \bar{x}) + \bar{y}) \in \mathbf{R}_+^2 \text{ with } x \leq \bar{x} \right\}.$$

After some easy calculations and using that $W^{\bar{\xi}}$ satisfies

$$p - (\delta + \lambda)W^{\bar{\xi}}(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) = 0,$$

it can be seen that $W^{\bar{\xi}}$ is continuously differentiable in \mathcal{S} with

$$W_x^{\bar{\xi}}(x, \frac{p_2}{p_1}(x - \bar{x}) + \bar{y}) = \int_0^{\frac{\bar{x}-x}{p_1}} e^{-(\delta+\lambda)w} H_x(x + p_1w, \frac{p_2}{p_1}(x - \bar{x}) + \bar{y} + p_2w)dw + e^{-(\delta+\lambda)\frac{\bar{x}-x}{p_1}} a_1$$

and

$$W_y^{\bar{\xi}}(x, \frac{p_2}{p_1}(x - \bar{u})) = \int_0^{\frac{\bar{x}-x}{p_1}} e^{-(\delta+\lambda)w} H_y(x + p_1w, \frac{p_2}{p_1}(x - \bar{x}) + \bar{y} + p_2w)dw + e^{-(\delta+\lambda)\frac{\bar{x}-x}{p_1}} a_2.$$

Finally, the differentiability at the boundary of \mathcal{B}_0 follows from the differentiability of $W^{\bar{\xi}}$ at (\bar{x}, \bar{y}) of \mathcal{S} . \square

Let us define the set of functions

$$\mathcal{M} = \{w : \mathbf{R}_+^2 \rightarrow [0, +\infty) \text{ continuous with } w(x, y) - a_1x - a_2y \text{ bounded}\}.$$

Proposition 8.4. *The value function $V^{\bar{\xi}}$ of the curve strategy corresponding to $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$ as defined in (7.2), satisfies the formulas given in Proposition 8.1 and Remark 8.2 replacing both W_0 and $W^{\bar{\xi}}$ by $V^{\bar{\xi}}$. Moreover, $V^{\bar{\xi}}$ is the unique function in \mathcal{M} which satisfies this property.*

Proof. \mathcal{M} is a complete metric space with the distance $d(w_1, w_2) = \sup_{\mathbf{R}_+^2} |w_1 - w_2|$. The operator $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ defined as

$$\begin{aligned} \mathcal{T}(w)(x, y) := & E_{x,y}(\int_0^{\tau_1} e^{-\delta s} (a_1 dL_s^{1,\bar{\xi}} + a_2 dL_s^{2,\bar{\xi}}) + e^{-\delta\tau_1} w(X_{\tau_1}^{\bar{L}^{\bar{\xi}}}, Y_{\tau_1}^{\bar{L}^{\bar{\xi}}}) I_{\{\tau_1 < \bar{\tau}\}} \\ & + e^{-\delta\bar{\tau}} (a_1 V_1^0(X_{\bar{\tau}}^{\bar{L}^{\bar{\xi}}}) + a_2 V_2^0(Y_{\bar{\tau}}^{\bar{L}^{\bar{\xi}}})) I_{\{\tau_1 = \bar{\tau}\}}) \end{aligned}$$

is a contraction with contraction factor $\lambda/(\delta + \lambda) < 1$. Then, there exists a unique fixed point and by definition (7.2), $\mathcal{T}(V^{\bar{\xi}}) = V^{\bar{\xi}}$. Taking in Proposition 8.1 and in Remark 8.2 the function W_0 as $V^{\bar{\xi}}$ we obtain from (8.1) that $V^{\bar{\xi}} = W^{\bar{\xi}}$ and so we get the result. \square

This last proposition gives a constructive way to obtain $V^{\bar{\xi}}$. Starting with $w_0(x, y) = a_1x + a_2y \in \mathcal{M}$, we define iteratively $w_{n+1} = \mathcal{T}(w_n)$. Hence, $V^{\bar{\xi}} = \lim_{n \rightarrow \infty} w_n$. Note that at each step w_{n+1} can be obtained from the formulas given in Proposition 8.1 and Remark 8.2 replacing W_0 by w_n .

Consider now the function $V^{n,\bar{\xi}}$ defined in (8.1) taking W_0 as the optimal value function V^{n-1} corresponding to step $n-1$ in (5.1). We try to find $\bar{\xi}_n^*$, which maximizes $V^{n,\bar{\xi}}$ among all the possible $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$. If the function $V^{n,\bar{\xi}_n^*}$ is a viscosity supersolution of (5.5), then by Remark 5.7, we would have that $V^{n,\bar{\xi}_n^*} = V^n$. In the case that one-step curve strategies corresponding to $\bar{\xi}_n^*$ exist for all $n \geq 1$, by Proposition 5.8, $V^{n,\bar{\xi}_n^*} \nearrow V$.

Let us call, as in (8.2),

$$(8.7) \quad H_{n-1}(x, y) := \mathcal{I}(V^{n-1})(x, y) + U(x, y).$$

In order to find the optimal one-step curve strategy corresponding to $\bar{\xi}_n^* = ((\bar{x}_n^*, \bar{y}_n^*), \xi_{1,n}^*, \xi_{2,n}^*)$, we look first for the optimal vertex $(\bar{x}_n^*, \bar{y}_n^*)$. By the formula given in Proposition 8.1,

$$V^{n,\bar{\xi}}(\bar{x}, \bar{y}) = \frac{p}{\delta + \lambda} + \frac{H_{n-1}(\bar{x}, \bar{y})}{\delta + \lambda}$$

and

$$\begin{aligned} & V^{n,\bar{\xi}}(x, y) \\ & = V^{n,\bar{\xi}}(\bar{x}, \bar{y}) + a_1(x - \bar{x}) + a_2(y - \bar{y}). \end{aligned}$$

for x and y large enough. So

$$(\bar{x}_n^*, \bar{y}_n^*) = \arg \max_{(\bar{x}, \bar{y}) \in \mathbf{R}_+^2} \frac{H_{n-1}(\bar{x}, \bar{y})}{\delta + \lambda} - a_1 \bar{x} - a_2 \bar{y}.$$

If this maximum is attained at a critical point (assuming that H_{n-1} is differentiable), we have that $(\bar{x}_n^*, \bar{y}_n^*)$ is a solution of

$$(8.8) \quad \begin{cases} \partial_x H_{n-1}(\bar{x}, \bar{y}) = a_1 (\delta + \lambda) \\ \partial_y H_{n-1}(\bar{x}, \bar{y}) = a_2 (\delta + \lambda). \end{cases}$$

Let us call $\bar{u}_n^* = \bar{x}_n^* - (p_1/p_2) \bar{y}_n^*$ and $\bar{v}_n^* = \bar{y}_n^* - (p_2/p_1) \bar{x}_n^*$. Next, we use Calculus of Variations in order to find two curves $\xi_{1,n}^*$ and $\xi_{2,n}^*$ which maximize $V^{n, \bar{\xi}}(x, y)$, among all $\bar{\xi} = ((\bar{x}_n^*, \bar{y}_n^*), \xi_1, \xi_2)$ for $\xi_1 \in \Phi^{\bar{u}_n^*}$, $\xi_2 \in \Phi^{\bar{v}_n^*}$ and (x, y) large enough. The two curves can be obtained separately and independently.

Proposition 8.5. *Assume that H_{n-1} is differentiable and that there exists $\bar{\xi}_n^* = ((\bar{x}_n^*, \bar{y}_n^*), \xi_{1,n}^*, \xi_{2,n}^*)$ where $\xi_{1,n}^* \in \Phi^{\bar{u}_n^*}$ and $\xi_{2,n}^* \in \Phi^{\bar{v}_n^*}$ such that $V^n = V^{n, \bar{\xi}_n^*}$. Then $\xi_{1,n}^*$ satisfies*

$$\partial_x H_{n-1}(u + \frac{p_1}{p_2} \xi_{1,n}^*(u), \xi_{1,n}^*(u)) = a_1 (\delta + \lambda)$$

for $\bar{u}_n^* \leq u \leq M_{\xi_{1,n}^*}$, and $\xi_{2,n}^*$ satisfies

$$\partial_y H_{n-1}(\xi_{2,n}^*(v), v + \frac{p_2}{p_1} \xi_{2,n}^*(v)) = a_2 (\delta + \lambda)$$

for $\bar{v}_n^* \leq v \leq M_{\xi_{2,n}^*}$.

Proof. We will prove this result for $\xi_{1,n}^*$, the proof for $\xi_{2,n}^*$ is analogous. Given any $\xi_1 \in \Phi^{\bar{u}_n^*}$, we have that

$$V^{n, \bar{\xi}}(x, 0) = V^{n, \bar{\xi}}(M_{\xi_1}, 0) + a_1(x - M_{\xi_1}),$$

for $\bar{\xi} = ((\bar{x}_n^*, \bar{y}_n^*), \xi_1, \xi_2)$ and $x \geq M_{\xi_1}$. Then, if there exists $\xi_{1,n}^* \in \Phi^{\bar{u}_n^*}$ such that $V^n = V^{n, \bar{\xi}_n^*}$,

$$V^{n, \bar{\xi}_n^*}(M_{\xi_{1,n}^*}, 0) - a_1 M_{\xi_{1,n}^*} = \max_{\xi_1 \in \Phi^{\bar{u}_n^*}} \left(V^{n, \bar{\xi}}(M_{\xi_1}, 0) - a_1 M_{\xi_1} \right).$$

Consider non-negative test functions ζ with $\zeta(\bar{u}_n^*) = 0$ and $\zeta(M_{\xi_{1,n}^*}) = 0$. We have that $\xi_{1,n}^* + \varepsilon \zeta \in \Phi^{\bar{u}_n^*}$ for ε small enough. Let us write,

$$\xi_\varepsilon(u) = \xi_{1,n}^*(u) + \varepsilon \zeta(u).$$

We have that $M_{\xi_\varepsilon} = M_{\xi_{1,n}^*}$ and then

$$V^{n, \bar{\xi}_n^*}(M_{\xi_{1,n}^*}, 0) - a_1 M_{\xi_{1,n}^*} = \max_{\xi} \left(V^{n, \bar{\xi}_\varepsilon}(M_{\xi_{1,n}^*}, 0) - a_1 M_{\xi_{1,n}^*} \right),$$

where $\bar{\xi}_\varepsilon = ((\bar{x}_n^*, \bar{y}_n^*), \xi_\varepsilon, \xi_2)$. Denote

$$\vartheta(\varepsilon) := V^{n, \bar{\xi}_\varepsilon}(M_{\xi_{1,n}^*}, 0) - a_1 M_{\xi_{1,n}^*}.$$

We have that $\xi_\varepsilon(M_{\xi_{1,n}^*}) = \xi_{1,n}^*(M_{\xi_{1,n}^*}) = 0$ and $\xi_\varepsilon(\bar{u}_n^*) = \xi_{1,n}^*(\bar{u}_n^*)$ so we can write by Proposition 8.1,

$$\begin{aligned} \vartheta(\varepsilon) &= e^{-\frac{(\delta+\lambda)\xi_{1,n}^*(\bar{u}_n^*)}{p_2}} \left(\frac{p}{\delta + \lambda} + \frac{1}{\delta + \lambda} H_{n-1}(\bar{u}_n^* + \frac{p_1}{p_2} \xi_{1,n}^*(\bar{u}_n^*), \xi_{1,n}^*(\bar{u}_n^*)) \right) - a_1 \bar{u}_n^* \\ &\quad + \int_{\bar{u}_n^*}^{M_{\xi_{1,n}^*}} a_1 \left(e^{-\frac{(\delta+\lambda)\xi_\varepsilon(w)}{p_2}} - 1 \right) dw + \frac{1}{p_2} \int_0^{\xi_{1,n}^*(\bar{u}_n^*)} H_{n-1}(\xi_\varepsilon^{-1}(t) + \frac{p_1}{p_2} t, t) e^{-\frac{(\delta+\lambda)t}{p_2}} dt. \end{aligned}$$

Clearly,

$$\begin{aligned} 0 &= \frac{\partial \vartheta}{\partial \varepsilon}(0) \\ &= \frac{1}{p_2} \int_{\bar{u}_n^*}^{M_{\xi_{1,n}^*}} \left(\partial_x H_{n-1}(w + \frac{p_1}{p_2} \xi_{1,n}^*(w), \xi_{1,n}^*(w)) - a_1(\delta + \lambda) \right) e^{-(\delta + \lambda) \frac{\xi_{1,n}^*(w)}{p_2}} \zeta(w) dw. \end{aligned}$$

So we obtain

$$\left(\partial_x H_{n-1}(u + \frac{p_1}{p_2} \xi_{1,n}^*(u), \xi_{1,n}^*(u)) - a_1(\delta + \lambda) \right) e^{-(\delta + \lambda) \frac{\xi_{1,n}^*(u)}{p_2}} = 0$$

for all $\bar{u}_n^* \leq u \leq M_{\xi_{1,n}^*}$. \square

This last proposition gives us a constructive way to find the candidate for V^n in the case that it comes from a one-step curve strategy. We find numerically, if it exists, the solution $z_1(u)$ of the equation

$$(8.9) \quad \partial_x H_{n-1}(u + \frac{p_1}{p_2} z_1(u), z_1(u)) = a_1(\delta + \lambda)$$

for $\bar{u}_n^* \leq u \leq \min\{u : z_1(u) = 0\}$ and the solution $z_2(v)$ of the equation

$$(8.10) \quad \partial_y H_{n-1}(z_2(v), v + \frac{p_2}{p_1} z_2(v)) = a_2(\delta + \lambda)$$

for $\bar{v}_n^* \leq v \leq \min\{v : z_2(v) = 0\}$. If $z_1(u)$ is in $\Phi^{\bar{u}_n^*}$ and $z_2(v)$ is in $\Phi^{\bar{v}_n^*}$, we define $\xi_{1,n}^*(u) = z_1(u)$ and $\xi_{2,n}^*(v) = z_2(v)$ and we obtain the value function $V^{n, \bar{\xi}_n^*}$ by the formula given in Proposition 8.1; this is our candidate for V^n . Afterwards, we check whether $V^{n, \bar{\xi}_n^*}$ is a viscosity supersolution of (5.5); if this is the case, then $V^n = V^{n, \bar{\xi}_n^*}$.

Remark 8.6. Consider claim size distributions F^i , $i = 1, 2$ such that V_i^0 , and hence also U , are differentiable functions (see for instance [19] for sufficient conditions for this property). Assume that there exists $\bar{\xi}_n^* = ((\bar{x}_n^*, \bar{y}_n^*), \xi_{1,n}^*, \xi_{2,n}^*)$ where $\xi_{1,n}^* \in \Phi^{\bar{u}_n^*}$ and $\xi_{2,n}^* \in \Phi^{\bar{v}_n^*}$ such that $V^n = V^{n, \bar{\xi}_n^*}$ for all $n \geq 1$, then H_n and V^n are differentiable for all $n \geq 1$. To see this, note that V^0 given in (5.2) is differentiable, that the differentiability of V^n implies the differentiability of H_n and that, by Proposition 8.3, the differentiability of H_n implies the differentiability of V^{n+1} . We conclude that the differentiability condition on H_{n-1} in Proposition 8.5 is automatically fulfilled.

In the next Proposition we state some conditions under which the optimal strategy of (2.4) is a curve strategy. This result, together with Propositions 8.1 and 8.5 gives a way to find the optimal curve (if it exists). Let us first define a criterion of convergence for a sequence $(\bar{\xi}_n)_{n \geq 1}$ that will be used in the next proposition.

Definition 8.7. We say that $\bar{\xi}_n = ((\bar{x}_n, \bar{y}_n), \xi_{1,n}, \xi_{2,n})$ converges to $\bar{\xi} = ((\bar{x}, \bar{y}), \xi_1, \xi_2)$ if

$$\begin{aligned} \lim_{n \rightarrow \infty} (\bar{x}_n, \bar{y}_n) &= (\bar{x}, \bar{y}), \quad \lim_{n \rightarrow \infty} M_{\xi_{i,n}} \rightarrow M_{\xi_i} \text{ for } i = 1, 2, \\ \lim_{n \rightarrow \infty} \max_{[\bar{u}_n, M_{\xi_{1,n}}] \cap [\bar{u}, M_{\xi_1}]} |\xi_{1,n}(u) - \xi_1(u)| &= 0, \quad \lim_{n \rightarrow \infty} \max_{[\bar{v}_n, M_{\xi_{2,n}}] \cap [\bar{v}, M_{\xi_2}]} |\xi_{2,n}(v) - \xi_2(v)| = 0, \\ \lim_{n \rightarrow \infty} \max_{[0, \xi_{1,n}(\bar{u}_n)] \cap [0, \xi_1(\bar{u})]} |\xi_{1,n}^{-1}(w) - \xi_1^{-1}(w)| &= 0 \text{ and } \lim_{n \rightarrow \infty} \max_{[0, \xi_{2,n}(\bar{v}_n)] \cap [0, \xi_2(\bar{v})]} |\xi_{2,n}^{-1}(w) - \xi_2^{-1}(w)| = 0. \end{aligned}$$

Proposition 8.8. Assume that there exists a $\bar{\xi}_n^*$ such that $V^n = V^{n, \bar{\xi}_n^*}$ for all $n \geq 1$. If $\bar{\xi}_n^*$ converges to some $\bar{\xi}^*$ in the sense of Definition 8.7, then the optimal value function V is the value function of the curve strategy $V^{\bar{\xi}^*}$ as defined in (7.2).

Proof. From Proposition 5.8, we have that $\lim_{n \rightarrow \infty} V^{n, \bar{\xi}_n^*} = V$. So replacing W_0 by $V^{n, \bar{\xi}_n^*}$ and $\bar{\xi}$ by $\bar{\xi}_n^*$ in the formulas given in Proposition 8.1 and Remark 8.2 and letting n go to infinity, we obtain that V satisfies the formulas given in Proposition 8.1 and Remark 8.2 replacing W_0 by V and $\bar{\xi}$ by $\bar{\xi}^*$. Therefore, by Proposition 8.4 and Lemma 2.2, the functions V and V^{ξ^*} coincide. \square

9. Numerical Example

We present a numerical example in the symmetric and equally weighted case with an exponential claim size distribution. By Remark 6.2, we restrict the search of the optimal curve strategy to $\bar{\xi} = ((\bar{x}, \bar{x}), \xi, \xi)$, $\xi \in \Phi^0$. Using the formulas given in Propositions 8.1 and equations (8.8), (8.9) and (8.10), we obtain $\bar{\xi}_n^* = ((\bar{x}_n^*, \bar{x}_n^*), \xi_n^*, \xi_n^*)$. We check that $\xi_n^* \in \Phi^0$ and that the associated value function $V^{n, \bar{\xi}_n^*}$ is a viscosity solution of (5.5). By Remark 8.6, $V^{n, \bar{\xi}_n^*}$ is differentiable because $\xi_n^* \in \Phi^0$, so in order to check numerically that $V^{n, \bar{\xi}_n^*}$ is a viscosity solution of (5.5) it is enough to check that $V_x^{n, \bar{\xi}_n^*} \geq a_1$ and $V_y^{n, \bar{\xi}_n^*} \geq a_2$ in the set \mathcal{C} and that $L(V^{n, \bar{\xi}_n^*}) \leq 0$ in the set $\mathcal{B}_0 \cup \mathcal{B}_1$, where these sets are the ones defined in Section 7 for $\bar{\xi}_n^*$.

We also obtain numerically the convergence of $\bar{\xi}_n^*$ to $\bar{\xi}$ according to Definition 8.7. Then, using Proposition 8.8, one can conclude that the optimal strategy is a curve strategy with curve ξ^* .

The numerical procedure was done with the *Mathematica* software and the calculation is quite time-consuming. The concrete chosen parameters are: exponential claim size distribution with parameter 3, Poisson intensity $\lambda_1 = \lambda_2 = 20/9$, premium rate $p_1 = p_2 = 1$, and a discount factor $\delta = 0.1$. In this numerical procedure we used step-size $\Delta x = \Delta y = 0.002$ and iterated 60 times. The resulting optimal curve strategy is given in Figure 9.1, and $V(x, y) - (x + y)/2$ (the improvement of the optimal dividend strategy over paying out the initial capital immediately) is depicted as the upper curve in Figure 9.2 (the sets \mathcal{A}_1^* and \mathcal{A}_2^* are not straight lines, even if they appear to be so at first glance).

We also compare for this numerical example the optimal value function $V(x, y)$ with the (comparably weighted) sum of the stand-alone value functions without collaboration:

$$V_S(x, y) = \frac{V^0(x) + V^0(y)}{2}$$

and with $V_M(x + y)/2$, where V_M is the optimal value function for the merger of the two companies. Figure 9.2 depicts the graphics of all three value functions $V(x, y)$, $V_S(x, y)$ and $V_M(x + y)/2$, each of them reduced by $(x + y)/2$. The optimal merger strategy is barrier with barrier $b = 2.77$. By Remark 2.1, $V_M(x + y)/2 < V(x, y)$ for all $(x, y) \in \mathbf{R}_+^2$. One sees that whereas for the comparison between the stand-alone case and the merger the initial surplus levels matter (with the merger case being the lowest of the three value functions in $(0, 0)$), the collaboration case outperforms not only the merger case but also the stand-alone one for all combinations of initial surplus levels (i.e. if one measures the overall dividend payments that can be achieved with either behavior, for this numerical example collaboration is always preferable). Hence we have here an instance where collaboration is beneficial not only for safety aspects, but also with respect to collective profitability.

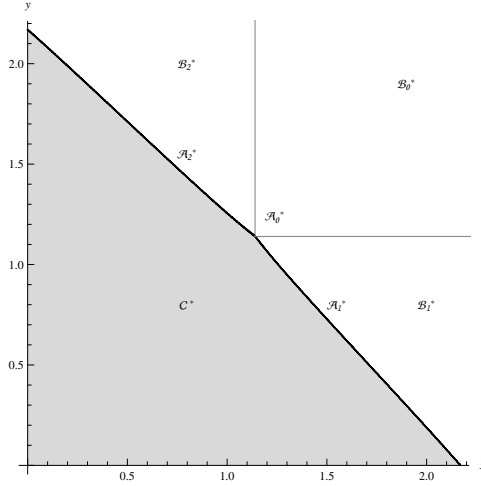


Figure 9.1: Optimal Curve Strategy

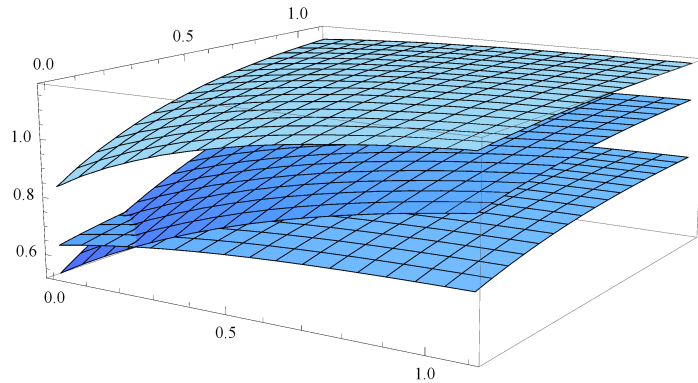


Figure 9.2: $V(x, y) - \frac{x+y}{2}$ vs. $V_S(x, y) - \frac{x+y}{2}$ vs. $\frac{V_M(x+y)}{2} - \frac{x+y}{2}$

10. Concluding Remarks

We finish with some further remarks on collaboration and on transaction costs.

Collaboration vs. stand-alone

In most cases, we expect the optimal value function V corresponding to the collaboration agreement to be greater than the comparably weighted sum of the stand-alone value function without collaboration. However, one can find examples in which this does not hold. For instance, if Company One is so unprofitable that, under the collaboration agreement, the best strategy for Company One is to pay all the current surplus as dividends.

Note that the "extreme" curve strategies where Company One pays all the current surplus as dividends are given by the sets $\mathcal{A}_0 = \{(0, \bar{y})\}$, $\mathcal{A}_1 = \{(0, y) : 0 \leq y < \bar{y}\}$, $\mathcal{B}_0 = \{(x, y) : y \leq \bar{y} \text{ and } x \geq 0\}$, $\mathcal{B}_1 = \{(x, y) : 0 \leq y < \bar{y} \text{ and } x \geq 0\}$ and $\mathcal{C} = \mathcal{A}_2 = \mathcal{B}_2 = \emptyset$, these strategies

depend only on the barrier level $\bar{y} \geq 0$. The value function of these strategies can be written as

$$V^{\bar{y}}(x, y) = W(y) + a_1 x,$$

where $W(y)$ is a viscosity solution of the one-dimensional integro-differential equation

$$\begin{cases} a_1 p_1 + W'(y) p_2 - (\delta + \lambda) W(y) + \lambda_1 \int_0^y W(y - \alpha) dF^1(\alpha) + \lambda_2 \int_0^y W(y - \alpha) dF^2(\alpha) + U(0, y) = 0 \\ W'(\bar{y}) = a_2 \end{cases}$$

in $[0, \bar{y}]$ and $W(y) = a_2(y - \bar{y}) + W(\bar{y})$ for $y \geq \bar{y}$. In the case of exponential claim sizes, this equation can be transformed easily into a non-homogeneous linear ordinary second-order differential equation and the solution W has a closed form.

In the next example, we will find a barrier level $\bar{y}^* > 0$ such that $V^{\bar{y}^*}$ is a classical solution of the HJB equation (3.1), so by Corollary 4.3, $V^{\bar{y}^*}$ is the optimal value function and the extreme curve strategy with barrier level \bar{y}^* is the optimal strategy. Moreover, there are initial surplus values for which the stand-alone value function is larger than V .

Consider the parameters $\delta = 0.1$, $\lambda_1 = 200$, $\lambda_2 = 1$, $p_1 = 10$, $p_2 = 1.3$, $a_1 = a_2 = 0.5$ and claim size distributions $F^1(\alpha) = F^2(\alpha) = 1 - e^{-\alpha}$. Then one obtains $\bar{y}^* = 0.072$. In Figure 10.1, we show the difference between the optimal collaboration value function $V^{\bar{y}^*}$ and the optimal value function $V_S(x, y) = (V_1^0(x) + V_2^0(y)) / 2$ of the stand-alone case. Here the stand-alone case outperforms the collaboration when the initial surplus level y of Company Two is greater than 0.12. Note that $V^{\bar{y}^*} - V_S$ is constant w.r.t. x because $V_1^0(x) = p_1 / (\delta + \lambda_1) + x$.

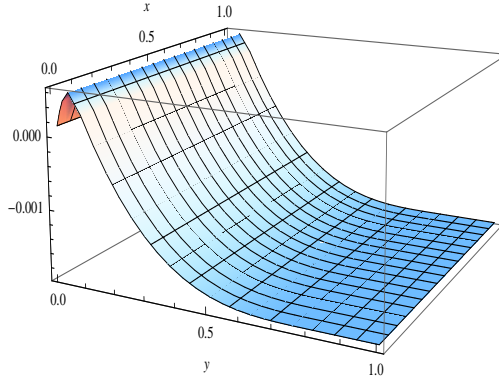


Figure 10.1: $V^{\bar{y}^*} - V_S$

Inclusion of proportional transaction costs

Let us assume that in case of a capital transfer, a unit of capital received by Company One requires $r_1 > 1$ from Company Two and that a unit of capital received by Company Two requires $r_2 > 1$ from Company One.

In this case, we obtain that the associated controlled process $(X_t^{\bar{L}}, Y_t^{\bar{L}})$ and the cumulative amounts transferred between the companies in (2.2) become

$$\begin{cases} X_t^{\bar{L}} = X_t - L_t^1 + C_t^{2,1} - r_2 C_t^{1,2} \\ Y_t^{\bar{L}} = Y_t - L_t^2 + C_t^{1,2} - r_1 C_t^{2,1}, \end{cases}$$

where $C_t^{2,1}$ corresponds to the cumulative amount received by Company One from Company Two up to time t in order to cover the deficit of Company One and $C_t^{1,2}$ corresponds to the cumulative amount received by Company Two from Company One up to time t in order to

cover the deficit of Company Two. In this situation, the time $\bar{\tau}$ at which only one company remains (because it cannot cover the deficit of the other) is given by

$$\bar{\tau} = \inf \left\{ t \geq 0 : \min\{r_1 X_t^{\bar{L}} + Y_t^{\bar{L}} < 0, X_t^{\bar{L}} + r_2 Y_t^{\bar{L}} < 0\} \right\}.$$

The HJB equation for the optimal value function (2.4) in this setting is (3.1) but now the operator \mathcal{I} and U defined in (3.2) and (3.4) become

$$\begin{aligned} \mathcal{I}(V)(x, y) = & \lambda_1 \int_0^x V(x - \alpha, y) dF^1(\alpha) + \lambda_1 \int_x^{x + \frac{y}{r_1}} V(0, y + r_1(x - \alpha)) dF^1(\alpha) \\ & + \lambda_2 \int_0^y V(x, y - \alpha) dF^2(\alpha) + \lambda_2 \int_y^{\frac{x}{r_2} + y} V(x + r_2(y - \alpha), 0) dF^2(\alpha), \end{aligned}$$

and

$$U(x, y) = \lambda_1 a_2 V_2^0(y) (1 - F^1(x + \frac{y}{r_1})) + \lambda_2 a_1 V_1^0(x) (1 - F^2(\frac{x}{r_2} + y)).$$

Using these operators, it is possible to extend all the results of the paper to the case of positive proportional transaction cost.

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