PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 10 · 2017 · Pages 15–22

Schur type inequalities for multivariate polynomials on convex bodies

András Kroó [a]

Communicated by Stefano De Marchi

AMS Subject classification: 41A17, 41A63.

Key words and phrases: convex bodies, multivariate polynomials, Schur type inequalities, Jacobi weights, norms of polynomial factors

Abstract

In this note we give sharp Schur type inequalities for multivariate polynomials with generalized Jacobi weights on arbitrary convex domains. In particular, these results yield estimates for norms of factors of multivariate polynomials.

1 Introduction and main new results

The classical Schur inequality states that for any univariate polynomial p_n of degree at most $n-1$ we have on the interval $I := [-1, 1]$

$$||p_n||_I \leq n||\sqrt{1-x^2}p_n||_I.$$

Throughout this paper $||p||_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ stands for the usual sup norm on a compact set $K \subset \mathbb{R}^d$. The meaning of the above inequality consists in estimating the uniform norm of the polynomial by the uniform norm of the weighted polynomial with the Chebyshev weight $\sqrt{1-x^2}$. This upper bound is known to be sharp as it is attained for the Chebyshev polynomial of the second kind. Schur type inequalities proved to be rather useful in verifying Markov type inequalities for the derivatives of algebraic polynomials. (See [1], p.233 for the basic facts on the classical Schur inequality.) They can be also applied for estimating norms of factors of polynomials. Schur type inequalities have been generalized in two directions: by replacing the uniform norm by a weighted uniform norm and by using instead of Chebyshev weight more general weighted polynomials. Mastroianni and Totik [6] proved a Schur type inequality with generalized Jacobi weights instead of $\sqrt{1-x^2}$ in case when the sup norm is endowed with a nonnegative weight w which satisfies the so called A^* property on I. This means that is there is a constant c_w depending only on w such that for any interval $J \subset I$ and any $x \in J$

$$w(x) \leq \frac{c_w}{\lambda(J)} \int_J w(t)dt. \tag{1}$$

Here and in what follows λ stands for the Lebesgue measure.

Then as shown in [6] given a generalized Jacobi type weight

$$h(x) := \prod_{1\leq j\leq k} |x - x_j|^{\gamma_j}, \gamma_j > 0$$

it follows that for any univariate algebraic polynomial p_n of degree at most n

$$||wp_n||_I \leq cn^{\gamma}||whp_n||_I, \quad \gamma := \max \gamma_j^*, \tag{2}$$

where w is any weight satisfying the A^* property and $\gamma_j^* = \gamma_j$ if $x_j \in (-1, 1)$ and $\gamma_j^* = 2\gamma_j$ if $x_j = 1$ or -1.

In the present note we will give the multivariate analogue of the above Schur type inequality for generalized Jacobi weights. These weights play an important role in the theory of multivariate orthogonal polynomials, see e.g., Y.Xu [7]. It turns out that in the multivariate setting this question is considerably more delicate since the corresponding upper bounds depend on the geometry of the zero sets (algebraic varieties) of the polynomial factors appearing in the weight.

[a]Alfréd Rényi Institute of Mathematics Hungarian Academy of Sciences and Budapest University of Technology and Economics, Department of Analysis, Budapest (HUNGARY)

Let us introduce now the generalized multivariate Jacobi type weights which will play a central part in our considerations. As usual, P_n^d will denote the space of real algebraic polynomials of d variables and degree at most n. Then given any integers $m_j \in \mathbb{N}$, positive real numbers $\alpha_j > 0$ and algebraic polynomials $p_j \in P_{m_j}^d$ of exact degree $m_j, 1 \le j \le s$ the generalized multivariate Jacobi weight on a convex body $K \subset \mathbb{R}^d$ is defined by

$$\phi(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial K)^\alpha \prod_{1\le j\le s} |p_j(\mathbf{x})|^{\alpha_j}, \quad \alpha \ge 0. \tag{3}$$

F. Dai [2] verified a Schur type inequality with generalized Jacobi weights for multivariate polynomials on the unit sphere $S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ in case when $m_j = 1, 1 \le j \le s$ and $\alpha = 0$ in the above definition. Then it is verified in [2] that for a certain class of weights w and any $p_n \in P_n^d$

$$\|wp_n\|_{S^{d-1}} \le cn^{\xi}\|w\phi p_n\|_{S^{d-1}}, \quad \xi := \sum_{1\le j\le s} \alpha_j. \tag{4}$$

If one compares the univariate estimate (2) with the multivariate upper bound (4) then a surprising difference can immediately be noticed: in (2) the maximum of exponents controls the estimate while in (4) the considerably larger sum of the exponents appears in the bound. This raises the natural question: what is the sharp form of the multivariate Schur inequality with generalized Jacobi weights (3) on arbitrary convex bodies $K \subset \mathbb{R}^d$?

It turns out that in order to give a sharp multivariate Schur inequality for a generalized Jacobi weight (3) one has to take into account possible intersections of the algebraic varieties

$$H_j^* := \{\mathbf{x} \in \mathbb{R}^d : p_j(\mathbf{x}) = 0\}, 1 \le j \le s$$

which makes the problem quite delicate. We will need to study their intersection inside the given domain, so we also set $H_j := H_j^* \cap K, 1 \le j \le s$. In order to handle this difficulty and provide a measure of how fast the weight ϕ given by (3) may vanish at a given $\mathbf{x} \in K$ we will introduce now its *zero index* $z(\mathbf{x})$ as

$$z(\mathbf{x}) := \sum_{\{j:\mathbf{x}\in H_j\}} \alpha_j m_j, \quad \mathbf{x} \in \operatorname{Int}K, \quad z(\mathbf{x}) := 2\alpha + 2\sum_{\{j:\mathbf{x}\in H_j\}} \alpha_j m_j, \quad \mathbf{x} \in \partial K. \tag{5}$$

Hence the zero index equals the sum of all properly weighted exponents α_j corresponding to algebraic varieties intersecting at the given point. In addition, it is important that these exponents are doubled for boundary points. Now we introduce the zero index of the Jacobi type weight as the maximal point wise zero index

$$z_\phi := \max_{\mathbf{x}\in K} z(\mathbf{x}). \tag{6}$$

It turns out that this zero index leads to sharp Schur type inequalities for generalized Jacobi weights on any convex body. We will verify this for weighted uniform norms with A^* weights w satisfying (1) for *any segment* $J \subset K$.

Theorem 1. *Let $K \subset \mathbb{R}^d, d \ge 2$ be a convex body and ϕ a generalized Jacobi weight (3) with zero index z_ϕ. Then for any $p_n \in P_n^d$ and A^* weight w we have*

$$\|wp_n\|_K \le cn^{z_\phi}\|w\phi p_n\|_K \tag{7}$$

with some constant $c > 0$ depending only on K and ϕ.

When all sets H_j^* have *pair wise empty intersection* in K then the zero index z_ϕ will equal the maximum of corresponding weighted exponents $\alpha_j m_j$ leading to an estimate similar to the univariate case (2). If all H_j^*-s *intersect at a single point* in K then $z_\phi = \sum_{1\le j\le s} \alpha_j m_j$ which is similar to the estimate (4). In Goetgheluck [3] a multivariate Schur type inequality was given under the condition that the weight ϕ has rather high order of smoothness. Since a generalized Jacobi weight (3) in general is not smooth the analytic method used in [3] does not work for general Jacobi weights. Another version of multivariate Schur type inequality can be found in [4], where the author gives some general upper bounds. However, the implicit estimates given in [4] are not sharp in general for Jacobi type weights, since they do not distinguish between interior and boundary points of the domain. A crucial point of our approach consists in providing precise bounds which are closely related to the intersection of the algebraic varieties H_j. The approach used in the present paper is a mix of refining some polynomial inequalities combined with geometric considerations.

The Schur type estimate (7) naturally leads to the question wether the zero index z_ϕ gives the correct rate of increase. Our next result addresses this question. Let $\mathbf{x}_0 \in K$ be a point in the domain where the zero index is attained, i.e., $z_\phi = z(\mathbf{x}_0)$. It will be shown below that if $\mathbf{x}_0 \in \operatorname{Int}K$ or $\mathbf{x}_0 \in \partial K$ is a so called *vertex* point then estimate (7) can be reversed. Recall that $\mathbf{x}_0 \in \partial K$ is called a vertex point if for some vector $\mathbf{h} \in S^{d-1}$ normal to ∂K at $\mathbf{x}_0$ we have with a $\gamma > 0$ for any $\mathbf{x} \in K$

$$|\mathbf{x} - \mathbf{x}_0| \le \gamma|\langle \mathbf{x} - \mathbf{x}_0, \mathbf{h}\rangle|. \tag{8}$$

We will give the converse to Theorem 1 in the model case when $w = 1$ and the polynomials in (3) are linear.

Theorem 2. *Consider a convex body $K \subset \mathbb{R}^d, d \ge 2$ and Jacobi weight (3) with $\deg p_j = 1, \forall j$. Assume that one of the following conditions hold:*

(i) *the zero index z_ϕ is attained at an interior point of K*

(ii) *the zero index is attained at a boundary vertex point.*

Then there exist polynomials $g_n \in P_n^d$ *such that with some* $c > 0$ *independent of* n

$$\|g_n\|_K \geq cn^{z_\phi}\|\phi g_n\|_K, \;\; n \in \mathbb{N}.$$

Theorem 2 provides nearly a complete converse to Theorem 1 except for the case when the zero index is attained only at a *smooth boundary point*. It turns out that there is an essential reason behind this fact because for convex bodies with smooth boundary the upper bound of Theorem 1 can be improved further.

Let us consider C^2 compact sets D satisfying the so called *uniform interior ball condition* which means that every $\mathbf{x} \in D$ is contained in a ball $B(\mathbf{a}, r_D) \subset D$ with $r_D > 0$ depending only on the domain. In what follows $B(\mathbf{a}, r)$ will denote the closed ball centered at $\mathbf{a}$ and radius r. Now we consider a generalized Jacobi weight with zero index defined as

$$\nu(\mathbf{x}) := \prod_{1\leq j\leq s} |p_j(\mathbf{x})|^{\alpha_j}, \;\; z_\nu = \max_{\mathbf{x}\in D} \sum_{\{j:\mathbf{x}\in H_j\}} \alpha_j m_j. \tag{9}$$

Note that in contrast to (5)-(6) now we do not need to double the exponents at the boundary points. This is related to the fact that for C^2 compacts with the uniform interior ball condition the boundary points can be treated in the same manner as the interior points. However, we need to modify the definition (1) of A^* weights by assuming that $J \subset D$ is any circular arc contained in the domain. With this modifications we have the following analogue of Theorem 1.

Theorem 3. *Let* $D \subset \mathbb{R}^d, d \geq 2$ *be a* C^2 *compact satisfying the uniform interior ball condition. Then for any* $p_n \in P_n^d$ *and* A^* *weight* w *we have*

$$\|wp_n\|_D \leq cn^{z_\nu}\|w\nu p_n\|_D$$

where ν *is a Jacobi type weight (9) and constant* $c > 0$ *depends only on* D *and* ϕ.

Similarly to Theorem 2 the upper bound of Theorem 3 can be reversed in case when $w = 1$ and the polynomials in (9) are linear.

Above results also have interesting implications for estimating norms of factors of multivariate polynomials. Let us formulate a typical corollary of this type.

Given a multivariate polynomial q which factors into the product of polynomials $q = q_1 \cdot \ldots \cdot q_s$ with zero sets $H_j := \{\mathbf{x} \in K : q_j(\mathbf{x}) = 0\}$ let us consider the zero index of q relative to the convex body $K \subset \mathbb{R}^d$ given by

$$z_q := \max_{\mathbf{x}\in K}(\tau_{\mathbf{x}} \sum_{\{j:\mathbf{x}\in H_j\}} \deg q_j),$$

where $\tau_{\mathbf{x}} = 1$ if $\mathbf{x} \in \mathrm{Int}K$ and $\tau_{\mathbf{x}} = 2$ if $\mathbf{x} \in \partial K$. It should be noted that the zero index z_q of polynomial q can be considerably smaller than its degree. Then for any $p_n \in P_n^d$ we have

$$\|p_n\|_K \leq c_q n^{z_q}\|qp_n\|_K.$$

In the next section we will verify the auxiliary analytic and geometric results needed in the sequel. First two lemmas provide some refinements of the univariate Schur and Polya inequalities. Lemma 3 presents some crucial information on intersection of algebraic varieties, while Lemma 4 is related to the geometry of convex bodies. Then in Section 3 the proof of the main new results will be given.

2 Auxiliary results

First we will need a refinement of the univariate Schur type inequality. This refinement will involve the *monotone rearrangement* of the weight ϕ with respect to the *Chebyshev measure* defined as

$$\mu(E) := \int_E \frac{dx}{\sqrt{1-x^2}}.$$

Let $\phi(x) \leq 1$ be positive a.e. on $[-1, 1]$ and for any $\delta > 0$ denote

$$\psi(\delta) := \sup\{c > 0 : \mu(\{x \in [-1, 1] : \phi(x) \leq c\}) \leq \delta\}.$$

In the above definition of monotone rearrangement Chebyshev measure is needed in order to handle properly the end points of the interval.

Lemma 1. *For any* A^* *weight* w *and* $p_n \in P_n^1$

$$\|wp_n\|_I \leq \frac{c}{\psi(1/n)}\|w\phi p_n\|_I.$$

Proof. Set $g(t) := p_n(\cos t)$. The weight $\omega(t) := w(\cos t)$ is also an A^* weight (see [6], p.68 for details). Furthermore, by the above definition of function ψ for any $0 < a < \psi(1/n)$

$$\lambda(t \in [0, \pi] : \phi(\cos t) \leq a) = \mu(x \in [-1, 1] : \phi(x) \leq a) \leq \frac{1}{n}.$$

We will use now a Remez type inequality for A^* weights proved in [6], p.60 which states that for any even trigonometric polynomial t_n of degree n and any $E \subset [0,\pi]$ with Lebesque measure $\lambda(E) \geq \pi - \frac{1}{n}$, we have

$$\|\omega t_n\|_{[0,\pi]} \leq c\|\omega t_n\|_E,$$

where $c >$ is an absolute constant.

Hence setting

$$E := \{t \in [0,\pi] : \phi(\cos t) > a\}, \;\; \lambda(E) \geq \pi - \frac{1}{n}$$

yields

$$\|wp\|_{[-1,1]} = \|\omega g\|_{[0,\pi]} \leq c\|\omega g\|_E \leq \frac{c}{a}\|\phi(\cos t)\omega(t)g(t)\|_E \leq \frac{c}{a}\|w\phi p\|_{[-1,1]}. \;\square$$

The next lemma is a refinement of the well known Polya inequality stating that for any univariate monic algebraic polynomial q_m of degree m

$$\lambda\{t \in [0,1] : |q_m(t)| \leq \delta\} \leq 4\delta^{\frac{1}{m}}, \;\; 0 < \delta < 1.$$

We will need an analogue of this inequality for the Chebyshev measure defined above.

Lemma 2. *Let* $q(t) = |t^2-1|^{\alpha}\prod_{1\leq j\leq s}|t-t_j|^{\alpha_j}, \alpha_j > 0, \alpha \geq 0, \gamma := \alpha + \sum_{1\leq j\leq s}\alpha_j$. *Then*

$$\mu\{t \in [-1,1] : q(t) \leq \delta\} \leq 8\delta^{\frac{1}{2\gamma}}. \tag{10}$$

Moreover, if $t_j \in [-1+a, 1-a]$ *with some* $0 < a < 1, 1 \leq j \leq s$ *then setting* $\gamma_1 := \max\{2\alpha, \sum_{1\leq j\leq s}\alpha_j\}$ *we have*

$$\mu\{t \in [-1,1] : q(t) \leq \delta\} = O(\delta^{1/\gamma_1}), \tag{11}$$

with a constant in the $O(.)$ *term depending only on* a, α, *and* α_j*-s.*

Proof. Assume first that each α_j is an integer. Then setting

$$q_1(t) := (t+1)^{\alpha}\prod_{1\leq j\leq s}(t-t_j)^{\alpha_j}, \;\; q_2(t) := (t-1)^{\alpha}\prod_{1\leq j\leq s}(t-t_j)^{\alpha_j}$$

it follows that both q_1 and q_2 are monic algebraic polynomials of degree γ. Thus applying the Polya inequality separately on intervals $[-1,0]$ and $[0,1]$ to q_1 and q_2, respectively we have

$$\mu\{t \in [-1,1] : q(t) \leq \delta\} \leq 2\lambda\{t \in [-1,1] : q(t) \leq \delta\}^{\frac{1}{2}}$$

$$\leq 2\lambda\{t \in [0,1] : |q_2(t)| \leq \delta\}^{\frac{1}{2}} + 2\lambda\{t \in [-1,0] : |q_1(t)| \leq \delta\}^{\frac{1}{2}} \leq 8\delta^{\frac{1}{2\gamma}}$$

which is the first estimate of the lemma in case of integer exponents.

If $\alpha_j \in \mathbb{Q}$ we can consider $q^m(t)$ with a proper integer m in order to have only integer exponents. Then using the above estimate with $q^m(t)$ and δ^m instead of δ leads to the same upper bound. Finally, the case of real positive exponents then follows from rational case by continuity.

In order to prove the second estimate we split the interval $[-1,1]$ into three parts

$$I_1 := [1-a/2, 1], I_2 := [-1, -1+a/2], I_3 := [-1+a/2, 1-a/2],$$

and consider $q_3(t) := \prod_{1\leq j\leq s}(t-t_j)^{\alpha_j}$, $\gamma_2 := \sum_{1\leq j\leq s}\alpha_j$. As above it suffices to verify the case for integer exponents. Now applying the Polya inequality to q_3 on the interval I_3 yields with some constants c_j depending only on a, α, and α_j-s

$$\mu\{t \in [-1,1] : q(t) \leq \delta\} \leq 2\mu\{t \in I_1 : |t-1|^{\alpha} \leq c_1\delta\} + c_2\lambda\{t \in I_3 : |q_3(t)| \leq c_3\delta\} \leq c_4\delta^{1/\gamma_1}. \;\square$$

Next we present an auxiliary geometric proposition related to intersections of the algebraic varieties H_j^* which will be crucial in the proof of Theorem 1.

Lemma 3. *Let* K *and* ϕ *be as in Theorem 1. Then there exists a* $\delta_0 = \delta_0(K,\phi)$ *so that for every* $0 < \delta \leq \delta_0$, *and* $\mathbf{x} \in K$ *such that*

$$\Omega(\mathbf{x},\delta) := \{j : |p_j(\mathbf{x})| < \delta\} \neq \emptyset$$

we have

$$\cap_{j\in\Omega(\mathbf{x},\delta)}H_j \neq \emptyset.$$

Proof. Assume the contrary. Then $\exists\delta_k \to 0^+$ and $\mathbf{x}_k \in K$ such that $\Omega(\mathbf{x}_k,\delta_k) \neq \emptyset, k \in \mathbb{N}$ but at the same time

$$\cap_{j\in\Omega(\mathbf{x}_k,\delta_k)}H_j = \emptyset.$$

Since K is compact we can assume without the loss of generality that $\mathbf{x}_k \to \mathbf{x}^* \in K$. Furthermore, since $\Omega(\mathbf{x}_k,\delta_k) \neq \emptyset$ there exist $j_k \in \Omega(\mathbf{x}_k,\delta_k), 1 \leq j_k \leq s, k \in \mathbb{N}$. Again without the loss of generality it can be assumed that for any $1 \leq j \leq r, 1 \leq r < s$ index j belongs only to finitely many of $\Omega(\mathbf{x}_k,\delta_k)$-s, while for all $r+1 \leq j \leq s$ this index belongs to infinitely many of $\Omega(\mathbf{x}_k,\delta_k)$-s. Hence there exist q_j so that for any $k \geq q_j$

$$j \notin \Omega(\mathbf{x}_k,\delta_k), \;\; 1 \leq j \leq r.$$

Then setting $q := \max\{q_j, 1 \le j \le r\}$ it follows that for any $k \ge q$

$$\Omega(\mathbf{x}_k, \delta_k) \subset \{r+1, r+2, ..., s\}. \tag{12}$$

Moreover, using that whenever $r+1 \le j \le s$ the index j belongs to infinitely many of $\Omega(\mathbf{x}_k, \delta_k)$-s we obtain that relations $|p_j(\mathbf{x}_k)| < \delta_k$ also hold for this j with infinitely many $k \in \mathbb{N}$. Since $\delta_k \to 0$ and $\mathbf{x}_k \to \mathbf{x}^* \in K$ this means that $p_j(\mathbf{x}^*) = 0$, i.e., $\mathbf{x}^* \in H_j, r+1 \le j \le s$. Thus in view of (12) we have

$$\mathbf{x}^* \in \cap_{r+1 \le j \le s} H_j \subset \cap_{j \in \Omega(\mathbf{x}_k, \delta_k)} H_j$$

contradicting our assumption that $\cap_{j \in \Omega(\mathbf{x}_k, \delta_k)} H_j = \emptyset$.□

Let us denote by $S(\mathbf{a}, r) := \{\mathbf{u} \in S^{d-1} : |\mathbf{a} - \mathbf{u}| \le r\}, \mathbf{a} \in S^{d-1}, r > 0$, the sphere cap in $\mathbb{R}^d$ with center $\mathbf{a}$ and radius r.

Lemma 4. *Let $K \subset \mathbb{R}^d$ be a convex body, $B(\mathbf{0}, r) \subset K \subset B(\mathbf{0}, 1), r > 0$. For any $\mathbf{y} \in K$ and $\mathbf{u} \in S(\mathbf{y}_1, r/2), \mathbf{y}_1 := \mathbf{y}/|\mathbf{y}|$ consider the line $l := \{\mathbf{y} + t\mathbf{u} : t \in \mathbb{R}\}$. Then setting $l \cap K := [\mathbf{a}, \mathbf{b}]$*

$$|\mathbf{b} - \mathbf{a}| \ge \sqrt{3}r, \quad \operatorname{dist}(\mathbf{y}, \partial K) \ge \frac{r}{4} \min\{|\mathbf{b} - \mathbf{y}|, |\mathbf{y} - \mathbf{a}|\}. \tag{13}$$

Proof. Since $\mathbf{u} \in S(\mathbf{y}_1, r/2)$ it follows that $\frac{r^2}{4} \ge \left|\mathbf{u} - \mathbf{y}_1\right|^2 = 2\left(1 - \langle \mathbf{u}, \mathbf{y}_1 \rangle\right)$. Since $|\mathbf{y}| \le 1$ this yields

$$\operatorname{dist}(l, \mathbf{0}) = \sqrt{|\mathbf{y}|^2 - \langle \mathbf{u}, \mathbf{y} \rangle^2} \le \sqrt{2(|\mathbf{y}| - \langle \mathbf{u}, \mathbf{y} \rangle)} \le \frac{r}{2}.$$

Clearly this means that $B(\mathbf{0}, r/2) \cap l \ne \emptyset$. Recalling that $B(\mathbf{0}, r) \subset K$ we easily obtain that $|l \cap K| \ge \sqrt{3}r$ which is the first estimate in (13).

Now we proceed by verifying the second estimate in (13). As shown above $B(\mathbf{0}, r/2) \cap l \ne \emptyset$, i.e., we can consider a point $\mathbf{A} \in B(\mathbf{0}, r/2) \cap l$. Since $B(\mathbf{0}, r) \subset K$ it follows that $B(\mathbf{A}, r/2) \subset K$. In addition, $\mathbf{A}, \mathbf{y} \in [\mathbf{a}, \mathbf{b}]$, hence we may assume without the loss of generality that $\mathbf{y} \in [\mathbf{A}, \mathbf{b}]$. Now set $B_0 := B(\mathbf{A}, r/2) \cap \{\mathbf{w} : \mathbf{w} - \mathbf{A} \perp \mathbf{u}\}$. Thus $B_0 \subset K$ is a $d-1$ dimensional ball in the hyper plane $\{\mathbf{w} : \mathbf{w} - \mathbf{A} \perp \mathbf{u}\}$ centered at $\mathbf{A}$ and of radius $r/2$. Then by the convexity of K the circular cone Q with vertex at $\mathbf{b} \in K$ and base B_0 is also contained in K and point $\mathbf{y}$ belongs to the axis of this cone. Therefore using that $\operatorname{diam} K \le 2$ we have $\operatorname{dist}(\mathbf{y}, \partial K) \ge \operatorname{dist}(\mathbf{y}, \partial Q) = \frac{r}{4}|\mathbf{b} - \mathbf{y}|$.□

3 Proof of the main new results

Proof of Theorem 1. We may assume that $B(\mathbf{0}, r) \subset K \subset B(\mathbf{0}, 1), 0 < r \le 1$. Since each polynomial p_j in (3) is of exact degree m_j it follows that its m_j-s homogeneous part denoted by h_j is a homogeneous polynomial of degree $m_j, 1 \le j \le s$. Let $H^* := \cup_{1 \le j \le s} \{\mathbf{x} \in \mathbb{R}^d : h_j(\mathbf{x}) = 0\}$ be the union of all zero sets of these homogeneous polynomials. Clearly H^* is nowhere dense in S^{d-1}. In what follows for any given $\epsilon > 0$ and $D \subset \mathbb{R}^d$ let $D(\epsilon) := \cup_{\mathbf{x} \in D} B(\mathbf{x}, \epsilon)$ denote the ϵ enlargement of D. Then we can choose a proper $a > 0$ so that $S(\mathbf{z}, r/2) \setminus H^*(a) \ne \emptyset$, for any $\mathbf{z} \in S^{d-1}$ and hence

$$|h_j(\mathbf{u})| \ge \xi, \quad \forall \mathbf{u} \in S(\mathbf{z}, r/2) \setminus H^*(a), \quad 1 \le j \le s \tag{14}$$

with some ξ depending only on K and ϕ.

Now take any $p_n \in P_n^d$ and let $\mathbf{y} \in K$ be such that $\|wp_n\|_K = |w(\mathbf{y})p_n(\mathbf{y})|$. For any $\mathbf{u} \in S(\mathbf{y}/|\mathbf{y}|, r/2) \setminus H^*(a)$ consider the line

$$l := \{\mathbf{y} + t\mathbf{u} : t \in \mathbb{R}\}, \quad l \cap K := [\mathbf{y} + a_l \mathbf{u}, \mathbf{y} + b_l \mathbf{u}], a_l \le 0 \le b_l.$$

Then $p_j(\mathbf{y} + t\mathbf{u}), 1 \le j \le s$ are univariate polynomials of variable t of degree m_j. Let $t_{kj} \in \mathbb{C}, 1 \le k \le m_j, 1 \le j \le s$ denote all zeros of $p_j(\mathbf{y} + t\mathbf{u})$.

Then for any fixed $a_l \le t \le b_l$ we have by (3), (14) and (13)

$$\phi(t) := \phi(\mathbf{y} + t\mathbf{u}) = \operatorname{dist}(\mathbf{y} + t\mathbf{u}, \partial K)^\alpha \prod_{1 \le j \le s} |p_j(\mathbf{y} + t\mathbf{u})|^{\alpha_j} \ge c(r, \alpha)|t - a_l|^\alpha |t - b_l|^\alpha \xi^{\sum \alpha_j m_j} \prod_{1 \le j \le s} |g_j(t)|^{\alpha_j},$$

where $g_j \in P_{m_j}^1$ are the *monic* univariate polynomials corresponding to $p_j(\mathbf{y} + t\mathbf{u}), 1 \le j \le s$.

Furthermore, denote by M the maximum of all directional derivatives of $p_j, 1 \le j \le s$ in $K(1)$.

Now with δ_0 from Lemma 3 and any fixed $a_l \le t \le b_l$ set

$$\Omega_t := \{j : \min_{1 \le k \le m_j} |t - t_{kj}| \le M^{-1}\delta_0\}. \tag{15}$$

.

For $j = 1, ..., s$ we denote by t_j^* the zero of g_j closest to t and we have

$$|t - t_j^*| = \min_{1 \le k \le m_j} |t - t_{kj}|.$$

Then by the previous estimate

$$\phi(t) \ge c_1 |t - a_l|^\alpha |t - b_l|^\alpha \prod_{j \in \Omega_t} |t - t_j^*|^{\alpha_j m_j} \prod_{j \notin \Omega_t} |t - t_j^*|^{\alpha_j m_j}.$$

Here and in the remaining part of the proof c_j denote positive constants depending only on K and ϕ. Evidently, it follows from (15) that for any $j \notin \Omega_t$ we have $|t - t_j^*| > M^{-1}\delta_0$. Thus we obtain from the last lower bound with $\Re t_j^*$ being the real part of the corresponding zero

$$\phi(t) \geq c_2|t-a_l|^\alpha|t-b_l|^\alpha \prod_{j\in\Omega_t}|t-t_j^*|^{\alpha_j m_j} \geq c_2|t-a_l|^\alpha|t-b_l|^\alpha \prod_{j\in\Omega_t}|t-\Re t_j^*|^{\alpha_j m_j}. \tag{16}$$

Clearly, we have by the mean value theorem for every $j \in \Omega_t$

$$|p_j(\mathbf{y}+t\mathbf{u})| = |p_j(\mathbf{y}+t\mathbf{u}) - p_j(\mathbf{y}+t_j^*\mathbf{u})| \leq M|t-t_j^*| \leq \delta_0.$$

This means that $\Omega_t \subset \Omega(\mathbf{y}+t\mathbf{u}, \delta_0)$. Thus by Lemma 3

$$\cap_{j\in\Omega_t} H_j \neq \emptyset.$$

We will distinguish between the cases when the set $\cap_{j\in\Omega_t} H_j$ contains some boundary points of K, or is completely embedded into Int$K$.

Case 1. $\cap_{j\in\Omega_t} H_j \cap \partial K \neq \emptyset$, i.e., for some $\mathbf{x} \in \partial K$ we have $\mathbf{x} \in \cap_{j\in\Omega_t} H_j$. Then by the definition of the zero indices (5) and (6) it follows that

$$z_\phi \geq z(\mathbf{x}) = 2\alpha + 2\sum_{\mathbf{x}\in H_j}\alpha_j \geq 2\alpha + 2\sum_{j\in\Omega_t}\alpha_j. \tag{17}$$

Now let us denote by Q the set of all generalized algebraic polynomials q of the form

$$q(t) = |t-a_l|^\alpha|t-b_l|^\alpha \prod_{j=1}^{m}|t-\Re t_j|^{\alpha_j}, \quad m \leq s, \tag{18}$$

with each t_j chosen arbitrarily from the set $\{t_{kj}, 1 \leq k \leq m_j\}$ and α, α_j-s satisfying (17). Clearly Q is a finite set with some cardinality N depending only on $m_j, 1 \leq j \leq s$. Thus in view of (16) we obtain that

$$\mu\{t \in [a_l, b_l] : \phi(t) \leq \delta\} \leq N\mu\{t \in [a_l, b_l] : q(t) \leq \delta/c_2\}, \tag{19}$$

where $q(t)$ is any generalized polynomial of the form (18) with exponents satisfying (17), i.e.,

$$\gamma := \alpha + \sum_{j=1}^{m}\alpha_j \leq z_\phi/2.$$

Now we are going to apply estimate (10) of Lemma 2 to the generalized polynomial $q(t)$ on the interval $[a_l, b_l]$ instead of $[-1,1]$. Note that by Lemma 4 we have $2 \geq |a_l - b_l| \geq \sqrt{3}r$, i.e, the transition to this segment can alter the outcome only by a constant factor. Hence by (19) and (10)

$$\mu\{t \in [a_l, b_l] : \phi(t) \leq \delta\} \leq N\mu\{t \in [a_l, b_l] : q(t) \leq \delta/c_2\} \leq c_3\delta^{\frac{1}{2\gamma}} \leq c_3\delta^{\frac{1}{z_\phi}}. \tag{20}$$

Now consider the univariate algebraic polynomial $g_n(t) := p_n(\mathbf{y}+t\mathbf{u}) \in P_n^1, t \in [a_l, b_l]$ where $a_l \leq 0 \leq b_l, |b_l - a_l| \geq \sqrt{3}r$ and $\|wg_n\|_{[a_l,b_l]} = |w(\mathbf{y})p_n(\mathbf{y})| = \|wp_n\|_K$. Note that by (20) and definition of monotone rearrangement ψ of ϕ we have $\psi(\delta) \geq c_4\delta^{z_\phi}$. Now we can apply Lemma 1 to $g_n \in P_n^1$ (since $|b_l - a_l| \geq \sqrt{3}r$ we can obviously transform $[a_l, b_l]$ to $[-1,1]$) yielding

$$\|wp_n\|_K = \|wg_n\|_{[a_l,b_l]} \leq \frac{c}{\psi(1/n)}\|w\phi g_n\|_{[a_l,b_l]} \leq c(K,\phi)n^{z_\phi}\|w\phi p\|_K$$

which is the needed estimate.

Case 2. $\cap_{j\in\Omega_t} H_j \subset$Int$K$. In this case using the definition of the zero indices (5) and (6) and the obvious relation $z_\phi > 2\alpha$ it follows that

$$z_\phi \geq \max\{2\alpha, \sum_{j\in\Omega_t}\alpha_j\}. \tag{21}$$

Consider the quantity

$$Q_1(\delta) := \min_{H_{i_1}\cap\ldots\cap H_{i_r}\cap\partial K=\emptyset} \text{dist}(\partial K, H_{i_1}^*(\delta)\cap\ldots\cap H_{i_r}^*(\delta)).$$

We clearly have that $Q_1(0) > 0$ and hence using the continuity of functions involved it follows that $Q(\delta) > 0$ for any $\delta > 0$ sufficiently small. Let us choose such a $\delta_1 > 0$, where it can be assumed without the loss of generality that $2\delta_0 < Q_1(\delta_1)$ with δ_0 being the quantity from Lemma 3. Then using that $\cap_{j\in\Omega_t} H_j \cap \partial K = \emptyset$ we obtain that for any $\mathbf{x} \in \cap_{j\in\Omega_t} H_j^*(\delta_1)$

$$\text{dist}(\mathbf{x}, \partial K) \geq Q_1(\delta_1) > 2\delta_0. \tag{22}$$

Consider now $\mathbf{y}+t_j^*\mathbf{u}, \mathbf{y}+t_i^*\mathbf{u}, i, j \in \Omega_t$, where by (15) we have $|t-t_j^*|, |t-t_i^*| \leq M^{-1}\delta_0$. Then

$$|p_i(\mathbf{y}+\Re t_j^*\mathbf{u})| = |p_i(\mathbf{y}+\Re t_j^*\mathbf{u}) - p_i(\mathbf{y}+t_i^*\mathbf{u})| \leq M(|\Re t_j^* - t| + |t_i^* - t|) \leq 2\delta_0.$$

This upper bound clearly means that $\mathbf{y}+\Re t_j^*\mathbf{u}$ must be in the vicinity of H_i^* for δ_0 small enough, i.e., we may assume without the loss of generality that δ_0 is chosen sufficiently small so that $\mathbf{y}+\Re t_j^*\mathbf{u}\in H_i^*(\delta_1), \forall i,j\in\Omega_t$. Then clearly $\mathbf{y}+\Re t_j^*\mathbf{u}\in\cap_{i\in\Omega_t}H_i^*(\delta_1)$, and therefore (22) yields

$$\min\{|\Re t_j^*-a_l|,|\Re t_j^*-b_l|\}\geq \operatorname{dist}(\mathbf{y}+\Re t_j^*\mathbf{u},\partial K)\geq Q_1(\delta_1)>2\delta_0.$$

In addition, by (15) $|\Re t_j^*-t|\leq\delta_0$. Since $t\in[a_l,b_l]$ the last two estimates obviously yield

$$\Re t_j^*\in[a_l+2\delta_0,b_l-2\delta_0],\quad j\in\Omega_t. \tag{23}$$

Now similarly to Case 1 we can obtain the upper bound (19) where q is a generalized algebraic polynomial (18) whose exponents by (21) satisfy relation

$$\max\{2\alpha,\sum_{1\leq j\leq m}\alpha_j\}\leq z_\phi. \tag{24}$$

Another crucial information concerning q consists in the fact that in view of relation (23) all its zeros belong to $[a_l+2\delta_0,b_l-2\delta_0]$. Hence we are in position to apply now the second estimate (11) of Lemma 2 yielding together with (24)

$$\mu\{t\in[a_l,b_l]:\phi(t)\leq\delta\}\leq N\mu\{t\in[a_l,b_l]:q(t)\leq c\delta\}=O(\delta^{\frac{1}{z_\phi}}).$$

Therefore, similarly to Case 1 this yields for the monotone rearrangement ψ of ϕ estimate $\psi(\delta)\geq c\delta^{z_\phi}$. This means that we can finish the proof now analogously to Case 1. □

Proof of Theorem 2. We may assume that diam$K\leq 1$. Consider first the case when the zero index is attained at a boundary vertex point $\mathbf{x}_0\in\partial K$ satisfying (8) for a certain outer normal $\mathbf{h}$ at $\mathbf{x}_0$. Then

$$z_\phi=z(\mathbf{x}_0)=2\alpha+2\sum_{\mathbf{x}_0\in H_j}\alpha_j.$$

Note that $0\leq\langle\mathbf{x}_0-\mathbf{x},\mathbf{h}\rangle\leq$diam$K\leq 1,\forall\mathbf{x}\in K$. Moreover, whenever $\mathbf{x}_0\in H_j$ it follows by (8) that

$$|\langle\mathbf{x}-\mathbf{c}_j,\mathbf{h}_j\rangle|=|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}_j\rangle|\leq|\mathbf{x}-\mathbf{x}_0|\leq\gamma|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle|,$$
$$\operatorname{dist}(\mathbf{x},\partial K)\leq|\mathbf{x}-\mathbf{x}_0|\leq\gamma|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle|.$$

Now we will need a result from [5] (see Proposition 1 on p. 84) according to which given any $\rho>0$ there exist univariate polynomials $q_n\in P_n^1$ such that

$$x^\rho|q_n(x)|\leq 1, x\in[0,1],\ \ |q_n(0)|\geq cn^{2\rho},\ n\in\mathbb{N}. \tag{25}$$

Set $\rho:=\frac{z_\phi}{2}$. Consider the polynomial $p(\mathbf{x}):=q_n(\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle)\in P_n^d$. Then using above estimates together with (25) we obtain for any $\mathbf{x}\in K$

$$\phi(\mathbf{x})|p(\mathbf{x})|=\operatorname{dist}(\mathbf{x},\partial K)^\alpha\prod_{1\leq j\leq s}|\langle\mathbf{x}-\mathbf{c}_j,\mathbf{h}_j\rangle|^{\alpha_j}|p(\mathbf{x})|$$
$$\leq c_1|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle|^\alpha\prod_{\mathbf{x}_0\in H_j}|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle|^{\alpha_j}|p(\mathbf{x})|=c_1|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle|^\rho|q_n(\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}\rangle)|\leq c_1,$$

with $c_1>0$ independent of n. On the other hand (25) also yields

$$|p(\mathbf{x}_0)|=|q_n(0)|\geq cn^{2\rho}=cn^{z_\phi}.$$

This is the required lower bound in the case when the zero index is attained at a boundary vertex point.

Now assume that the zero index is attained at an interior point $\mathbf{x}_0\in$Int$K$. Hence by (5)

$$z_\phi=z(\mathbf{x}_0)=\sum_{\mathbf{x}_0\in H_j}\alpha_j.$$

Set $p(\mathbf{x}):=q_n(|\mathbf{x}-\mathbf{x}_0|^2)\in P_{2n}^d$. Then using again (25) we have for any $\mathbf{x}\in K$

$$\phi(\mathbf{x})|p(\mathbf{x})|=\operatorname{dist}(\mathbf{x},\partial K)^\alpha\prod_{1\leq j\leq s}|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}_j\rangle|^{\alpha_j}|p(\mathbf{x})|\leq\prod_{\mathbf{x}_0\in H_j}|\langle\mathbf{x}-\mathbf{x}_0,\mathbf{h}_j\rangle|^{\alpha_j}|p(\mathbf{x})|$$
$$\leq|\mathbf{x}-\mathbf{x}_0|^{z_\phi}|p(\mathbf{x})|=|\mathbf{x}-\mathbf{x}_0|^{2\rho}|q_n(|\mathbf{x}-\mathbf{x}_0|^2)|\leq 1$$

and

$$|p(\mathbf{x}_0)|=|q_n(0)|\geq cn^{2\rho}=cn^{z_\phi}.$$

This completes the proof of Theorem 2.

The proof of Theorem 3 can be given quite similarly to the proof of Theorems 1, in fact it follows the same arguments with considerable simplifications due to the absence of the boundary difficulties. Since for a C^2 compact D satisfying uniform interior ball condition any point $\mathbf{x}\in D$ lies on a sphere of radius r_D imbedded into the domain it suffices to consider the case when $D=S^{d-1}$ is the unit sphere. Now we can essentially repeat the arguments of Theorem 1 by modifying Lemma 1 for monotone rearrangements based on circular Lebesgue measure instead of the Chebyshev measure, and considering circular arcs passing through $\mathbf{x}$, we omit the details.

Acknowledgments: Supported by the OTKA Grant K111742. Written during the author's visit at the Department of Mathematics of University of Padova.

References

[1] P. Borwein, T. Erdélyi, *Polynomials and polynomial inequalities*, Springer-Verlag, Berlin-New York-Heidelberg, 1991.

[2] F. Dai, *Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights*, J. Funct. Analysis, **235** (2006), 137-170.

[3] P. Goetgheluck, *Une ineégalité polynômiale en plusieurs variables*, J. Approx. Th. **40** (1984), 161-172.

[4] M. Ganzburg, *Polynomial Inequalities on Measurable Sets and Their Applications II. Weighted Measures*, J. Approx. Th., **106**(2000), 77-109.

[5] A. Kroó, J. Szabados, *Markov-Bernstein type Inequalities for Multivariate Polynomials on Sets with Cusps*, J. Approx. Th.,**102**(2000), 72-95.

[6] G. Mastroianni, V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx., **16**(2000), 37-71.

[7] Y. Xu, *Asymptotics of the Christoffel functions on a simplex in $\mathbb{R}^d$*, J. Approx. Theory **99**(1999), 122-133.