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$g(x)$-FULL CLEAN RINGS

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Abstract. Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial of ring $C(R)[x]$. An element $r \in R$ is called "$g(x)$-clean" if $r = s + u$ where $g(s) = 0$ and u is a unit of R and R is $g(x)$-clean if every element is $g(x)$-clean. In this paper, we introduce the concept of $g(x)$-full clean rings and study various properties of them.

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1. INTRODUCTION

Clean rings were introduced by Nicholson [4]. A ring R is called clean if for every element $a \in R$, there exist an idempotent e and a unit u in R such that $a = e + u$. Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r = u + s$ where $g(s) = 0$ and u is a unit of R, and R is $g(x)$-clean if every element is $g(x)$-clean. Moreover, Fan and Yang have studied $g(x)$-clean rings and their generalizations in [3]. Ashrafi and Ahmadi also studied weakly $g(x)$-clean rings [1].
In this paper, we extend $g(x)$-clean rings and introduce the concept of $g(x)$-full clean rings and study various properties of them. Also we prove that $M_n(R)$ is $g(x)$-full clean rings for any $g(x)$-full clean rings R and get a condition under which the definitions of $g(x)$-cleanness and $g(x)$-full cleanness are equivalent.

Throughout this paper all rings are assumed to be associative with identity and modules are unitary. $J(R)$ always stands for the Jacobson radical of a ring R, $U(R)$ is the set of all invertible elements of a ring R, $M_n(R)$ denotes the $n \times n$ matrix ring over the ring R and $\mathbb{T}_n(R)$ stands for $n \times n$ upper triangular matrix ring. Recall that:

Definition 1. Let I be an ideal of a ring R, we say that:

(1) Idempotents can be lifted modulo I if, whenever $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $e - a \in I$.
(2) The root $\bar{s}$ of the polynomial $\bar{g}(x) \in (R/I)[X]$ can be lifted modulo I, if there exists $a \in R$ such that $g(a) = 0$ and $s - a \in I$.

Definition 2. An element $x \in R$ is said to be a full element if there exist $s, t \in R$ such that $sxt = 1$. The set of all full elements of a ring R will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$.

Definition 3. A ring R is called full-clean if every element of R is a sum of a full element and an idempotent.

Definition 4. Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial of ring $C(R)[x]$. An element in R is said to be $g(x)$-full clean if it can be written as the sum a root of $g(x)$ and a full element. A ring R is called a $g(x)$-full clean ring if each element in R is a $g(x)$-full clean element.

2. $g(x)$-FULL CLEAN RINGS

Firstly, we get some basic properties of $g(x)$-full clean rings.

Let R and S be rings and $\theta : C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then θ induces a map θ' from $C(R)[x]$ to $C(S)[x]$ such that for $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i) x^i \in C(S)[x]$. We should note that if $n \in \mathbb{Z}$, then $\theta(n) = \theta(1 + \cdots + 1) = n\theta(1) = n$. So if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then clearly $\theta'(g(x)) = g(x)$.

Here we give some properties of $g(x)$-full clean rings which are similar to those of $g(x)$-clean rings.

Proposition 1. *Let $\theta : R \longrightarrow S$ be a ring epimorphism. If R is $g(x)$-full clean, then S is $\theta'(g(x))$-full clean.*

Proof. Let $g(x) = a_0 + a_1 x + \cdots + a_n x^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in C(S)[x]$. As θ is a ring epimorphism so for any $s \in S$, there exist $r \in R$ such that $\theta(r) = s$. Since R is $g(x)$-full clean, there exist $w \in K(R)$ and $s_0 \in R$ such that $r = w + s_0$ where $g(s_0) = 0$ and $swt = 1$ for some $s, t \in R$. Then $s = \theta(r) = \theta(w + s_0) = \theta(w) + \theta(s_0)$. But as $swt = 1$ we have $\theta(s)\theta(w)\theta(t) = \theta(swt) = \theta(1) = 1$. Therefore $\theta(w) \in K(S)$. But $\theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + \cdots + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1 s_0 + \cdots + a_n s_0^n) = \theta(g(s_0)) = \theta(0) = 0$, so s is $\theta'(g(x))$-full clean. Therefore S is $\theta'(g(x))$-full clean. □

Corollary 1. *If R is $g(x)$-full clean, then for any ideal I of R, R/I is $\bar{g}(x)$-full clean where $\bar{g}(x) \in C(R/I)[x]$.*

Proof. Let θ: $R \longrightarrow R/I$ be the canonical epimorphism. Note that if $a \in C(R)$ then $\bar{a} \in C(R/I)$, so the result follows from previous proposition. □

Proposition 2. *Let $I \leq J(R)$ be an ideal of R, $\eta : R \longrightarrow R/I$ with $\eta(r) = r + I = \bar{r}$, and $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ with $\bar{g}(x) = \sum_{i=0}^{n} \bar{a}_i x^i \in C(R/I)[x]$. If R/I is $\bar{g}(x)$-full clean and roots of $g(x)$ lift modulo I, then R is $g(x)$-full clean.*

Proof. For any $r \in R$, Let $r + I = \bar{r} = \bar{s} + \bar{w}$ be the $\bar{g}(x)$-full clean expression, i.e., $\bar{g}(\bar{s}) = 0$, $\bar{w} \in K(R/I)$ and $\bar{s'}\bar{w}\bar{t} = \bar{1}$ for some $s', t \in R$. Since roots of $\bar{g}(x)$ lift modulo I, there exist $e \in R$ such that $g(e) = 0$ and $\bar{e} = \bar{s}$. So, $r - e - w = i$ for some $i \in I$ and $r = e + (w + i)$. Hence $\bar{s'}\bar{w}\bar{t} = \bar{1}$, we have $s'wt = 1 + h \in 1 + I \subseteq 1 + J(R) \subseteq U(R)$ for some $h \in I$. Therefore, there exist $a \in R$ where $(s'wt)a = 1$ and $s_1, t_1 \in R$ such that $s_1 w t_1 = 1$. Hence $s_1(w + i)t_1 = 1 + s_1 i t_1 \in 1 + J(R) \subseteq U(R)$. We have $s_1(w + i)t_1 u = 1$ for some $u \in U(R)$, hence $w + i$ is a full element. Therefore, r is $g(x)$-full clean, as asserted. □

Proposition 3. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is $g(x)$-full clean if and only if for all $i \in I$, R_i is $g(x)$-full clean.*

Proof. Let $\prod_{i \in I} R_i$ is $g(x)$-full clean. Define $\pi_j : \prod_{i \in I} R_i \longrightarrow R_j$ by $\pi_j(\{a_i\}_{i \in I}) = a_j$. Since for all $j \in I$, π_j is a ring epimorphism, so by Proposition 1, for every $i \in I$, each R_i is $g(x)$-full clean ring.
For the contrary, suppose that for every $i \in I$, Ri is a $g(x)$-full clean ring. For any $x = \{x_i\}_{i \in I} \in \prod_{i \in I} R_i$, we write $xi = s_i + w_i$ with $g(s_i) = 0$ and $s_i w_i t_i = 1$ for some $s_i, t_i \in R$. Then $x = s + w$, where

$$\begin{aligned} g(s = \{s_i\}_{i \in I}) &= a_0\{1_{R_i}\}_{i \in I} + a_1\{s_i\}_{i \in I} + \cdots + a_n\{s_i^n\}_{i \in I} \\ &= \{a_0\}_{i \in I} + \{a_1 s_i\}_{i \in I} + \cdots + \{a_n s_i^n\}_{i \in I} \\ &= \{a_0 + a_1 s_i + \cdots + a_n s_i^n\}_{i \in I} \\ &= \{g(s_i)\}_{i \in I} = \{0\}_{i \in I} \end{aligned}$$

and $w = \{wi\}_{i \in I} \in K(\prod_{i \in I} R_i)$ with $\{s_i\}_{i \in I}\{wi\}_{i \in I}\{ti\}_{i \in I} = \{1\}_{i \in I}$. Hence x is $g(x)$-full clean, as required. □

Recall that for a ring R with a ring endomorphism $\alpha : R \longrightarrow R$, the skew power series ring $R[[x;\alpha]]$ of R is the ring obtained by giving the formal power series ring over R with this property that $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, id_R]]$.

Proposition 4. *Let α be an endomorphism of R and $g(x) \in C(R)[x]$. Then the following statements are equivalent.*

(1) *R is a $g(x)$-full clean ring.*

(2) *The formal power series ring* $R[[x]]$ *of* R *is a* $g(x)$*-full clean ring.*
(3) *The skew power series ring* $R[[x;\alpha]]$ *of* R *is a* $g(x)$*-full clean ring.*

Proof. Being homomorphic image of $R[[x]]$ and $R[[x;\alpha]]$, R is $g(x)$-full clean when $R[[x]]$ or $R[[x;\alpha]]$ is $g(x)$-full clean.

Now, suppose R is a $g(x)$-full clean ring. For any $h = a_0 + a_1x + \ldots \in R[[x,\alpha]]$, write $a_0 = e_0 + u_0$ such that $g(e_0) = 0$ and $u_0 \in K(R)$. Assume that $s_0u_0t_0 = 1$ for some $s_0, t_0 \in R$ and let $h' = h - e_0 = u_0 + a_1x + \ldots$. The equation $u = (s_0 + 0 + \ldots)h'(t_0 + 0 + \ldots) = 1 + s_0a_1\alpha(t_0)x + \ldots$ shows that $u \in U(R[[x,\alpha]])$, since $U(R[[x;\alpha]]) = \{a_0 + a_1x + \ldots : a_0 \in U(R)\}$ without any assumption on the endomorphism α. Hence $h' \in K(R[[x,\alpha]])$ and $h = e_0 + h'$ where $e_0 \in R[[x,\alpha]]$ and $g(e_0) = 0$. so, $R[[x;\alpha]]$ is a $g(x)$-full clean ring.

Since $R[[x]] = R[[x, id_R]]$, the proof is similar to that of ((1) $\Rightarrow$ (3)), as desired. □

Remark 1. Generally, the polynomial ring $R[t]$ is not $g(x)$-clean for an arbitrary nonzero polynomial $g(x) \in C(R)[x]$. For example let R be a commutative ring, then the polynomial ring $R[t]$ is not $g(x)$-clean ring [3]. Full elements and invertible elements are the same when the ring R is commutative, so the concept of $g(x)$-clean and $g(x)$-full clean are equivalent for commutative rings. Now, let $g(x) = x$, we show that t is not $g(x)$-full clean. If $t = w + s$ then it must be that $s = 0$, so $t = w$. As, w is a full element then $fth = 1$ for $f, h \in R$. Since R is commutative, $t \in U(R[t])$. But clearly $t \notin U(R[t])$, therefore $R[t]$ is not $g(x)$ full-clean.

Next we will investigate some cases in which the concept of $g(x)$-full cleanness and $g(x)$-cleanness are equivalence. Yu [5] called a ring R to be a left quasi-duo ring if every maximal left ideal of R is a two-sided ideal. Commutative rings, local rings, rings in which every nonunit has a power that is central are all belong to this class of rings [5].

Theorem 1. *For a left quasi-duo ring* R *and* $g(x) \in C(R)[x]$, *the followings are equivalent:*

(1) *R is a $g(x)$-clean ring;*
(2) *R is a $g(x)$-full clean ring.*

Proof. If R is $g(x)$-clean, then this is trivial that R is $g(x)$-full clean.

Now, let R be a $g(x)$-full clean ring and $r \in R$. So $r = w + s$ such that $g(s) = 0$ and $w \in K(R)$. It suffices to show that $w \in K(R)$ implies that $w \in U(R)$. Let $swt = 1$ for some $s, t \in R$, so s is right invertible. Assume that s is not left invertible. Then $Rs \subsetneqq R$, and there exists a maximal left ideal M of R such that $Rs \subseteq M \subsetneqq R$. But since R is a left quasi-duo ring, so M is a two sided ideal and $s \in M$. Therefore $sR \subseteq M$. But as s is right invertible, so M is not a proper ideal and this is a contradiction. So s should have left inverse as well and therefore s is invertible. Thus $wts = 1$. In a similar way, we get that $w \in U(R)$, and the result follows. □

In Fan and Yang [3], proved that if R is $g(x)$-clean, then so is $M_n(R)$ for all $n \geq 1$. Here we have a similar result for $g(x)$-full clean. Define $\pi_n : C(R) \longrightarrow M_n(R)$ by $a \longmapsto aI_n$ where I_n is the identity matrix of $M_n(R)$ and $a \in C(R)$. Then $M_n(R)$ is a $C(R)$-algebra for all $n \geq 1$.

Theorem 2. *Let R be a ring and $g(x) \in C(R)[x]$. If R is $g(x)$-full clean, then $M_n(R)$ is also $g(x)$-full clean ring for all $n \geq 1$.*

Proof. Suppose that R is $g(x)$-full clean. Given any $x \in R$, there exist $e \in R$ and $w \in K(R)$ such that $x = e + w$ and $g(e) = 0$. We write $swt = 1$ for some $s, t \in R$. Assume that theorem holds for the matrix ring $M_k(R)$, $k \geq 1$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{k+1}(R)$$

with $A_{11} \in R$, $A_{12} \in R^{1\times k}$, $A_{21} \in R^{k\times 1}$ and $A_{22} \in M_k(R)$.
We have $A_{11} = e + w$ where $g(e) = 0$ and $swt = 1$ for some $s, t \in R$. There also exist matrix E and a full matrix W such that $A_{22} - A_{21}tsA_{12} = E + W$ where $g(E) = 0$ by induction. We write $SWT = I_k$ for some $S, T \in M_k(R)$. Therefore, we have $A = diag(e, E) + \begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}$. Let $g(x) = a_0 + a_1x + \cdots + a_mx^m$, then we have:

$$\begin{aligned} g(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix}) &= a_0I_{k+1} + a_1(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix}) + \cdots + a_m(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix})^m \\ &= \begin{pmatrix} a_01_R & 0 \\ 0 & a_0I_k \end{pmatrix} + \begin{pmatrix} a_1e & 0 \\ 0 & a_1E \end{pmatrix} + \cdots + \begin{pmatrix} a_me^m & 0 \\ 0 & a_mE^m \end{pmatrix} \\ &= \begin{pmatrix} g(e) & 0 \\ 0 & g(E) \end{pmatrix} = 0 \end{aligned}$$

Also, let $P = \begin{pmatrix} s & 0 \\ -SA_{21}ts & S \end{pmatrix}, Q = \begin{pmatrix} t & -tsA_{12}T \\ 0 & T \end{pmatrix} \in M_{k+1}(R)$
and the equation

$$P\begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}Q = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix} = I_{k+1}$$

shows that $\begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}$ is a full matrix, hence A is $g(x)$-full clean, as desired. □

Proposition 5. *Let $a \in R$ be a $g(x)$-full clean element and $g(x) \in C(R)[x]$ where $g(1) = 0$, then $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is always $g(x)$-full clean in $M_2(R)$ for any $b \in R$.*

Proof. If $a = e + w$ where $g(e) = 0$ and $swt = 1$ for some $s, t \in R$, then we can write A as

$$A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} w & b \\ 0 & -1 \end{pmatrix}$$

We also have $\begin{pmatrix} s & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} w & b \\ 0 & -1 \end{pmatrix}\begin{pmatrix} t & -tsb \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now if $g(x) = \sum_{i=0}^{n} a_i x^i$ we have,

$$\begin{aligned} g(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}) &= a_0 I_2 + a_1(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}) + \cdots + a_m(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix})^m \\ &= \begin{pmatrix} a_0 1_R & 0 \\ 0 & a_0 1_R \end{pmatrix} + \begin{pmatrix} a_1 e & 0 \\ 0 & a_1 \end{pmatrix} + \cdots + \begin{pmatrix} a_m e^m & 0 \\ 0 & a_m \end{pmatrix} \\ &= \begin{pmatrix} g(e) & 0 \\ 0 & g(1) \end{pmatrix} = 0 \end{aligned}$$

Therefore, A is a $g(x)$-full clean element. □

Theorem 3. *Let* $C = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ *where* A, B *and* ${}_A V_B, {}_B W_A$ *are respectively two rings and bimodules. Also let* $g(x) \in \mathbb{Z}[x]$. *Then* C *is* $g(x)$*-full clean if and only if* A *and* B *are* $g(x)$*-full clean.*

Proof. Assume that C is f-($g(x)$-clean). Let $I = \begin{pmatrix} 0 & V \\ W & B \end{pmatrix}$ and $J = \begin{pmatrix} A & V \\ W & 0 \end{pmatrix}$. One can check that I, J are ideals of C and $C/I \simeq A$, $C/J \simeq B$ (it is enough to consider the epimorphism $\varphi : C \longrightarrow A$ by $\varphi(\begin{pmatrix} a & v \\ w & b \end{pmatrix}) = a$ and the epimorphism $\psi : C \longrightarrow B$ by $\varphi(\begin{pmatrix} a & v \\ w & b \end{pmatrix}) = b$, respectively with kernel I and J). Clearly $g(x)$-full cleanness of A, B follows from Corollary 1.

Conversely, let A and B be both $g(x)$-full clean rings. For any $r = \begin{pmatrix} a & v \\ w & b \end{pmatrix} \in C$, we have $a = e_1 + u_1$ and $b = e_2 + u_2$ for some $e_1, e_2 \in R$ where $g(e_1) = g(e_2) = 0$ and $u_1, u_2 \in K(R)$. Assume that $s_1 u_1 t_1 = 1$, $s_2 u_2 t_2 = 1$ for some $s_1, t_1, s_2, t_2 \in R$. So we have $r = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} u_1 & v \\ w & u_2 \end{pmatrix} = E + U$. Now $g(x) = \sum_{i=0}^{n} a_i x^i$. Hence

$$\begin{aligned} g(E) &= a_0 I_0 + a_1 E + \cdots + a_n E^n \\ &= \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 e_1 & 0 \\ 0 & a_1 e_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_n e_1^n & 0 \\ 0 & a_n e_2^n \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} g(e_1) & 0 \\ 0 & g(e_2) \end{pmatrix} = 0$$

and the equation

$$\begin{pmatrix} s_1 & 0 \\ -s_2 w t_1 s_1 & s_2 \end{pmatrix} \begin{pmatrix} u_1 & v \\ w & u_2 \end{pmatrix} \begin{pmatrix} t_1 & -t_1 s_1 v t_2 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies that U is a full matrix. Hence r is $g(x)$-full clean, as required. □

Proposition 6. *Let R and S be two rings, M be an (R,S)-bimodule and $g(x) \in \mathbb{Z}[x]$.*

(1) *Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the formal triangular matrix ring. Then E is $g(x)$-full clean ring if and only if R and S are $g(x)$-full clean rings.*
(2) *For any $n \geq 1$, R is $g(x)$-full clean if and only if the $n \times n$ upper triangular matrix ring $T_n(R)$ are $g(x)$-full clean.*

Proof. Formal triangular matrix rings are special cases of C in Theorem 3.

Let R be $g(x)$-full clean and $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since R is $g(x)$-full clean, for any $1 \leq i \leq n$, there exist $e_{ii} \in R$ and $w_{ii} \in K(R)$ such that $a_{ii} = w_{ii} + e_{ii}$ with $g(e_{ii}) = 0$. Also assume that $s_{ii} w_{ii} t_{ii} = 1$ for some $s_{ii}, t_{ii} \in R$. So we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} w_{11}+e_{11} & a_{12} & \dots & a_{1n} \\ 0 & w_{22}+e_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{nn}+e_{nn} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} w_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & w_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w_{nn} \end{pmatrix}}_{W} + \underbrace{\begin{pmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{pmatrix}}_{E}.$$

Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$, so we have

$$g(E) = a_0 I_n + a_1 E + \cdots + a_n E^n$$
$$= \begin{pmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 \end{pmatrix} + \begin{pmatrix} a_1 e_{11} & 0 & \dots & 0 \\ 0 & a_1 e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 e_{nn} \end{pmatrix} + \ldots$$

$$+\begin{pmatrix} a_m e_{11}^m & 0 & \dots & 0 \\ 0 & a_m e_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m e_{nn}^m \end{pmatrix}$$

$$=\begin{pmatrix} g(e_{11}) & 0 & \dots & 0 \\ 0 & g(e_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(e_{nn}) \end{pmatrix} = 0.$$

Also, it is straightforward with induction on n, to prove calculate that $W \in K(\mathbb{T}_n(R))$. So $\mathbb{T}_n(R)$ is $g(x)$-full clean.

Now let $\mathbb{T}_n(R)$ is $g(x)$-full clean. Define $\theta : \mathbb{T}_n(R) \longrightarrow R$ by $\theta(A) = a_{11}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

then it is clear that θ is a ring epimorphism. For any $a \in R$, let B be the diagonal matrix diog$(a, ...a)$. Then $a = \theta(B) = \theta(W + S) = \theta(W) + \theta(S)$ where $\theta(W) = w_{11} \in K(R)$ and

$$\begin{aligned} g(\theta(S)) &= a_0 + a_1\theta(S) + \cdots + a_n\theta(S^n) \\ &= \theta(B_0) + \theta(B_1)\theta(S) + \cdots + \theta(B_n)\theta(S^n) \\ &= \theta(B_0 + B_1 S + \cdots + B_n S^n) \\ &= \theta(a_0 I_n + (a_1 I_n)S + \cdots + (a_n I_n)S^n) \\ &= \theta(g(S)) = 0. \end{aligned}$$

Thus a is $g(x)$-full clean, i.e., R is $g(x)$-full clean ring. □

Proposition 7. *Let R be a ring, $n \in \mathbb{N}$ and $2 \in U(R)$. Then the followings are equivalent:*

(1) *R is full-clean;*
(2) *R is $(x^2 - 2^n x)$-full clean;*
(3) *R is $(x^2 + 2^n x)$-full clean;*
(4) *R is $(x^2 - 2x)$-full clean;*
(5) *R is $(x^2 + 2x)$-full clean;*
(6) *R is $(x^2 - 1)$-full clean;*
(7) *every element of R is the sum of a full element and a square root of* 1.

Proof. (1) $\Rightarrow$ (7) Suppose R is full-clean and $x \in R$. Then $(x+1)/2 = e + u$ for some $e^2 = e$ and $u \in K(R)$. So $x = (2e-1) + 2u$ with $(2e-1)^2 = 1$ and $2u \in K(R)$.

(7) $\Rightarrow$ (1) Let every element of R is the sum of a full element and a square root of 1. Then given $x \in R$, we have $2x-1 = t+w$ with $t^2 = 1$ and w a full element in R. So $x = (t+1)/2 + w/2$ with $((t+1)/2)^2 = (t+1)/2$ and $w/2$ is a full element in R, as asserted.

(1) $\Rightarrow$ (2) Since $2 \in U(R)$, $2^n \in U(R)$. Let $a \in R$, then $a/2^n = e+u$ such that $e^2 = e$ and $u \in K(R)$. So, $a = 2^n e + 2^n u$ where $(2^n e)^2 - 2^n(2^n e) = 0$ and $2^n u \in K(R)$. Therefore, R is $(x^2 - 2^n x)$-full clean.

(2) $\Rightarrow$ (1) Let $r \in R$. Since R is f-$((x^2-2^n x)$-clean), $r2^n = s+w$ such that s is a root of $(x^2 - 2^n x)$ and $w \in K(R)$. Thus, $r = s/2^n + w/2^n$ where $w/2^n \in K(R)$ and $(s/2^n)^2 = s(s-2^n+2^n)/(2^n)^2 = s2^n/(2^n)^2 = s/2^n$. So R is f-clean.

Similarly, we can prove (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (5). □

Let R be a ring and ${}_R V_R$ be an $R-R-$bimodule which is a ring possibly without a unity in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. The ideal extension of R by V is defined to be the additive abelian group $I(R,V) = R \bigoplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Proposition 8. *Let R be a ring and ${}_R V_R$ be an $R-R-$bimodule, $g(x) \in \mathbb{Z}[x]$. An ideal-extension $E = I(R,V)$ of R by V is $g(x)$-full clean if R is $g(x)$-full clean and for any $v \in R$, there exists $w \in R$ such that $v + w + wv = 0$.*

Proof. Let $t = (r,v) \in E$. Then $r = s+u$ where $g(s) = 0$ and $u \in K(R)$. Therefore $t = (s,0) + (u,v)$. Let $g(x) = \sum_{i=0}^{n} a_i x^i$, we have

$$\begin{aligned} g((s,0)) &= a_0(1,0) + a_1(s,0) + \cdots + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + \cdots + a_n(s^n,0) \\ &= (a_0,0) + (a_1 s,0) + \cdots + (a_n s^n,0) \\ &= (a_0 + a_1 s + \cdots + a_n s^n, 0) = (g(s),0) = (0,0) \end{aligned}$$

and we will show that $(u,v) \in K(E)$. Assume that $sut = 1$. For $svt \in V$, there exists $w \in V$ such that $svt + w + wsvt = 0$ by assumption. Also, one can check that $(s,ws)(u,v)(t,0) = (1,0)$. Hence $(u,v) \in K(E)$ and E is a $g(x)$-full clean ring. □

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