

**COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF  
ANALYTIC FUNCTIONS**

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*Abstract.* Let  $\mathcal{T}(s, \lambda, \mu, \beta)$  be the class of normalized functions,  $f$  defined in the open unit disk  $\mathbb{U}$  by

$$\operatorname{Re} \left( \frac{\Theta_{\mu}^{\lambda, s+1} f(z)}{\Theta_{\mu}^{\lambda, s} g(z)} \right) > 0 \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0, z \in \mathbb{U})$$

for some  $g \in \mathcal{R}_{\beta}(s, \lambda, \mu, \beta)$ . The authors in [15] introduced the operator  $\Theta_{\mu}^{\lambda, s}$  defined by

$$\Theta_{\mu}^{\lambda, s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \lambda - 1)!(\mu - 1)!}{\lambda!(k + \mu - 2)!} k^s a_k z^k,$$

where  $\mathcal{R}_{\beta}(s, \lambda, \mu, \beta)$  denotes the class of normalized functions  $g$  in the open unit disk  $\mathbb{U}$  defined by

$$\left| \arg \left( \frac{\Theta_{\mu}^{\lambda, s+1} g(z)}{\Theta_{\mu}^{\lambda, s} g(z)} \right) \right| < \frac{\pi}{2} \beta \quad (0 < \beta \leq 1).$$

For  $f \in \mathcal{T}(s, \lambda, \mu, \beta)$  and given by  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ , a sharp upper bound is obtained for  $|a_3 - \tau a_2^2|$  when  $\tau \geq 1$ .

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**1. INTRODUCTION AND DEFINITIONS**

Let  $\mathcal{A}$  denote the family of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z : |z| < 1\}$ . Further, let  $\mathcal{S}$  denote the class of functions which are univalent in  $\mathbb{U}$ . A function  $f(z)$  belonging to  $\mathcal{A}$  is said

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to be strongly starlike of order  $\beta$  in  $\mathbb{U}$ , and denoted by  $\mathcal{S}\mathcal{S}^*(\mathcal{L})$  if it satisfies

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\beta \quad (0 < \beta \leq 1, z \in \mathbb{U}). \quad (1.2)$$

If  $f(z) \in \mathcal{A}$  satisfies

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\beta \quad (0 < \beta \leq 1, z \in \mathbb{U}), \quad (1.3)$$

then we say that  $f(z)$  is strongly convex of order  $\beta$  in  $\mathbb{U}$ , and we denote by  $\mathcal{S}\mathcal{C}(\mathcal{L})$  the class of all such functions.

With the help of the differential operator  $\Theta_{\mu}^{\lambda,s}$ , we say that a function  $f(z)$  belonging to  $\mathcal{A}$  is said to be in the class  $\mathcal{R}_{\rho}(s, \lambda, \mu, \beta)$  if it satisfies

$$\left| \arg \left( \frac{\Theta_{\mu}^{\lambda,s+1} f(z)}{\Theta_{\mu}^{\lambda,s} f(z)} \right) \right| < \frac{\pi}{2}\beta \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \quad (1.4)$$

for some  $\beta(0 < \beta \leq 1)$  and for all  $z \in \mathbb{U}$ . Note that  $\mathcal{R}_{\rho}(0, 0, 1, \beta) = \mathcal{S}\mathcal{S}^*(\mathcal{L})$  and  $\mathcal{R}_{\rho}(1, 0, 1, \beta) = \mathcal{S}\mathcal{C}(\mathcal{L})$ .

For the class  $\mathcal{S}$  of analytic univalent functions, [6] obtained the maximum value of  $|a_3 - \tau a_2^2|$  when  $\tau$  is real. For various functions of  $\mathcal{S}$ , the upper bound for  $|a_3 - \tau a_2^2|$  is investigated by many different authors (see [1–5, 7, 8, 11–14, 16, 17] and [18]).

In this paper we obtain a sharp upper bounds for  $|a_3 - \tau a_2^2|$  when  $f$  belongs to the class of functions defined as follows:

**Definition 1.** Let  $\beta(0 < \beta \leq 1)$  and let  $f \in \mathcal{A}$ . Then  $f \in \mathcal{T}(s, \lambda, \mu, \beta)$  if and only if there exist  $g \in \mathcal{R}_{\rho}(s, \lambda, \mu, \beta)$  such that

$$\operatorname{Re} \left( \frac{\Theta_{\mu}^{\lambda,s+1} f(z)}{\Theta_{\mu}^{\lambda,s} g(z)} \right) > 0 \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}), \quad (1.5)$$

where  $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$ .

Note that  $\mathcal{T}(0, 0, 1, \beta) = \mathcal{K}(\beta)$  the class of close-to-convex functions defined by [3],  $\mathcal{T}(0, 0, 1, 1) = \mathcal{K}(1)$  is the class of normalized close-to-convex functions defined by [10].

## 2. MAIN RESULTS

In order to derive our main results, we have to recall here the following lemma.

**Lemma 1** ([17]). *Let  $h \in \mathcal{P}$  i.e.  $h$  be analytic in  $\mathbb{U}$  and be given by  $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$ , and  $\operatorname{Re} h(z) > 0$  for  $z \in \mathbb{U}$ . Then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \quad (2.1)$$

**Theorem 1.** Let  $f(z) \in \mathcal{T}(s, \lambda, \mu, \beta)$  and be given by (1.1). Then for  $\beta \geq 1$  and  $\tau \geq 1$  we have the sharp inequality

$$|a_3 - \tau a_2^2| \leq \frac{\beta^2 [(3^s \tau \mu^2 (\lambda + 2) - 2^{2s} \mu (\mu + 1) (\lambda + 1))]}{2^{2s} 3^s (\lambda + 2) (\lambda + 1)^2} + \frac{[(3^{s+1} \tau \mu^2 (\lambda + 2) - 2^{2s+1} \mu (\mu + 1) (\lambda + 1)) (2\beta + 1)]}{2^{2s} 3^{s+1} (\lambda + 2) (\lambda + 1)^2}. \quad (2.2)$$

*Proof.* Let  $f(z) \in \mathcal{T}(s, \lambda, \mu, \beta)$ . It follows from (1.5) that

$$\Theta_{\mu}^{\lambda, s+1} f(z) = \Theta_{\mu}^{\lambda, s} g(z) q(z), \quad (2.3)$$

for  $z \in \mathbb{U}$ , with  $q \in \mathcal{P}$  given by  $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots$ . Equating coefficients, we obtain

$$\frac{(\lambda + 1)}{\mu} 2^{s+1} a_2 = q_1 + \frac{(\lambda + 1)}{\mu} 2^s b_2, \quad (2.4)$$

and

$$\frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} 3^{s+1} a_3 = q_2 + \frac{(\lambda + 1)}{\mu} 2^s b_2 q_1 + \frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} 3^s b_3. \quad (2.5)$$

Also, it follows from (1.4) that

$$\Theta_{\mu}^{\lambda, s+1} g(z) = \Theta_{\mu}^{\lambda, s} g(z) (p(z))^{\beta}.$$

where for  $z \in \mathbb{U}$ ,  $p \in \mathcal{P}$  and  $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$ . Thus equating coefficients, we obtain

$$\frac{(\lambda + 1)}{\mu} 2^s b_2 = \beta p_1, \quad (2.6)$$

$$\frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} (2) 3^s b_3 = \beta (p_2 + \frac{3\beta - 1}{2} p_1^2). \quad (2.7)$$

From (2.4), (2.5), (2.6) and (2.7) we have

$$\begin{aligned} a_3 - \tau a_2^2 &= \frac{\mu(\mu + 1)}{3^{s+1} (\lambda + 2) (\lambda + 1)} (q_2 - \frac{1}{2} q_1^2) \\ &+ \frac{2^{2s+1} \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \mu^2 (\lambda + 2)}{3^{s+1} \times 2^{2s+2} (\lambda + 2) (\lambda + 1)^2} q_1^2 \\ &+ \frac{\beta \mu(\mu + 1)}{2 \times 3^{s+1} (\lambda + 2) (\lambda + 1)} (p_2 - \frac{1}{2} p_1^2) \\ &+ \frac{[2^{2s+1} \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \mu^2 (\lambda + 2)] \beta}{3^{s+1} \times 2^{2s+1} (\lambda + 2) (\lambda + 1)^2} p_1 q_1 \\ &+ \frac{3 \times 2^{2s} \beta^2 \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \beta^2 \mu^2 (\lambda + 2)}{3^{s+1} 2^{2s+2} (\lambda + 2) (\lambda + 1)^2} p_1^2. \end{aligned} \quad (2.8)$$

Assume that  $a_3 - \tau a_2^2$  is positive. Thus we now estimate  $\operatorname{Re}(a_3 - \tau a_2^2)$ , so from (2.8) and by using the Lemma 1 and letting  $p_1 = 2re^{i\theta}$ ,  $q_1 = 2Re^{i\phi}$ ,  $0 \leq r \leq 1$ ,  $0 \leq R \leq 1$ ,  $0 \leq \theta < 2\pi$  and  $0 \leq \phi < 2\pi$ , we obtain

$$\begin{aligned}
& 3^{s+1} \operatorname{Re}(a_3 - \tau a_2^2) \\
&= \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)} \operatorname{Re}(q_2 - \frac{1}{2}q_1^2) + \frac{[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+2}(\lambda+2)(\lambda+1)^2} \operatorname{Re}q_1^2 \\
&\quad + \frac{\beta\mu(\mu+1)}{2(\lambda+2)(\lambda+1)} \operatorname{Re}(p_2 - \frac{1}{2}p_1^2) \\
&\quad + \frac{\beta[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+1}(\lambda+2)(\lambda+1)^2} \operatorname{Re}p_1q_1 \\
&\quad + \frac{3 \times 2^{2s}\beta^2\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\beta^2\mu^2(\lambda+2)}{2^{2s+2}(\lambda+2)(\lambda+1)^2} \operatorname{Re}p_1^2 \tag{2.9} \\
&\leq \frac{2\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-R^2) + \frac{[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} R^2 \cos 2\phi \\
&\quad + \frac{\beta\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-r^2) \\
&\quad + \frac{2\beta[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} rR \cos(\theta + \phi) \\
&\quad + \frac{3\beta^2[2^{2s}\mu(\mu+1)(\lambda+1) - 3^s\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} r^2 \cos 2\theta \\
&\leq \frac{3^{s+1}\tau\mu^2(\lambda+2) - 2^{2s+2}\mu(\mu+1)(\lambda+1)}{2^{2s}(\lambda+2)(\lambda+1)^2} R^2 \\
&\quad + \frac{2\beta\mu[3^{s+1}\tau\mu(\lambda+2) - 2^{s+1}(\mu+1)(\lambda+1)]}{2^{2s}(\lambda+2)(\lambda+1)^2} rR \\
&\quad + \beta \left( \frac{3\beta[3^s\tau\mu^2(\lambda+2) - 2^{2s}(\mu+1)(\lambda+1)]}{2^{2s}(\lambda+2)(\lambda+1)^2} - \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)} \right) r^2 \\
&\quad + \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(\beta+2) \\
&= \Psi(r, R). \tag{2.10}
\end{aligned}$$

Letting  $\beta, \lambda, \mu$  and  $\tau$  fixed and differentiating  $\Psi(r, R)$  partially when  $\mu \geq 1, \lambda \geq 0, \beta \geq 1$  and  $\tau \geq 1$  we observe that

$$\begin{aligned}
& \Psi_{rr}\Psi_{RR} - (\Psi_{rR})^2 \\
&= \frac{2^{2s+4}\beta\mu^2(\mu+1)^2}{(\lambda+2)}[2\beta+1] - \frac{4 \times 3^{s+1}\tau\beta\mu^3(\mu+1)}{(\lambda+1)}[5\beta-1] < 0.
\end{aligned}$$

Therefore, the maximum of  $\Psi(r, R)$  occurs on the boundaries. Thus the desired inequality follows by observing that

$$\Psi(r, R) \leq \Psi(1, 1) = \frac{3\beta^2 [3^s \tau \mu^2 (\lambda + 2) - 2^{2s} \mu (\mu + 1) (\lambda + 1)]}{2^{2s} (\lambda + 2) (\lambda + 1)^2} + \frac{[(3^{s+1} \tau \mu^2 (\lambda + 2) - 2^{2s+1} \mu (\mu + 1) (\lambda + 1)) (2\beta + 1)]}{2^{2s} (\lambda + 2) (\lambda + 1)^2}.$$

The equality for (2.2) is attained when  $p_1 = q_1 = 2i$  and  $p_2 = q_2 = -2$ .  $\square$

Letting  $s = \lambda = 0$  and  $\mu = 1$  in the above Theorem, we have the result given by Jahangiri [9]:

**Corollary 1.** *Let  $f(z) \in \mathcal{K}(\beta)$  and be given by (1.1). Then for  $\beta \geq 1$  and  $\tau \geq 1$  we have the sharp inequality*

$$|a_3 - \tau a_2^2| \leq \beta^2 (\tau - 1) + \frac{(2\beta + 1)(3\tau - 2)}{3}. \quad (2.11)$$

Letting  $s = \mu = 1$  and  $\lambda = 0$  in the above Theorem, we have the following result:

**Corollary 2.** *Let  $f(z) \in \mathcal{T}(1, 0, 1, \beta)$  and be given by (1.1). Then for  $\beta \geq 1$  and  $\tau \geq 1$  we have the sharp inequality*

$$|a_3 - \tau a_2^2| \leq \frac{1}{36} [3\beta^2 (3\tau - 4) + (9\tau - 8)(2\beta + 1)].$$

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