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## CHARACTERIZATIONS OF ATOMISTIC COMPLETE FINITE LATTICES RELATIVE TO GEOMETRIC ONES

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*Abstract.* S. Radeleczki in 2002 raised the open problem of the characterization of those atomistic complete lattices whose classification lattices are geometric. This paper solves the problem for the finite cases.

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*Keywords:* geometric lattice, classification system, classification lattice, atomistic complete lattice, finite lattice

### 1. INTRODUCTION

S. Radeleczki et al. pointed out [1, 2, 6, 7] that the notion of the classification system can be applied in concept lattices. The dual of this notion is introduced by R. Wille [3, 4, 8]. In addition, S. Radeleczki et al. [1, 2, 6, 7] apply and study the properties of concept lattices in the process of construction of classification systems. Since any partition lattice is a particular geometric lattice, an open problem was arisen naturally (S. Radeleczki [7]): Characterize those atomistic complete lattices whose classification lattices are geometric. The solution of this open problem might be useful in the study of classification systems and concept lattices. Hence, this paper will characterize atomistic complete finite lattices whose classification lattices are geometric and answer the open problem for the finite case.

This paper is organized as follows. Section 2 presents some basic information relative to geometric lattices and classification systems. In Section 3, we describe certain properties on atomistic complete finite lattices related to geometric lattices. Afterwards, it answers the open problem suggested by S. Radeleczki [7] for the finite case.

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## 2. PRELIMINARIES

Some basic notions and results related to posets, lattices and classification systems are presented in this section.

### 2.1. Posets and lattices

We review some basic properties and notations of posets and lattices in this subsection. For more detail about posets and lattices, please refer to [5].

**Lemma 1.** (1)(in [5] p.232) *A lattice  $L$  is semimodular if and only if  $x \wedge y \prec x$  implies that  $y \prec x \vee y$ .*

(2)(in [5] p.234) *A lattice  $L$  is called geometric if and only if  $L$  is complete, atomistic (that is, every element of  $L$  is a join of atoms), all atoms are compact, and  $L$  is semimodular.*

(3)(in [5] p.234) *Any interval  $[y, x] = \{a \in L \mid y \leq a \leq x\}$  of a geometric lattice  $L$  is again a geometric lattice.*

We also need the following statements from [5].

(1) A finite lattice  $L$  is semimodular if for all  $x, y \in L$ : the property which  $x$  and  $y$  cover  $x \wedge y$  implies that  $x \vee y$  covers  $x$  and  $y$ .

(2) A finite lattice is geometric if it is semimodular and every element is a join of atoms.

(3) A finite semimodular lattice is characterized by the following property  $L$  is semimodular if and only if for all  $x, y \in L$ , all maximal chains between elements  $x, y$  have the same length, and the height function  $h$  of  $L$  satisfies  $h(x) + h(y) \geq h(x \wedge y) + h(x \vee y)$ .

**Some notations 1.** Let  $(P, \leq)$  be a poset. In this paper, if there is no confusion in the text, then we use the notation  $P$  instead of  $(P, \leq)$ . If  $P$  has the greatest element 1, then the height  $h(1)$  is sometimes denoted by  $h(P)$ . In  $P$ , if  $y$  covers  $x$ , it is in notation  $x \prec y$ ; if  $y$  does not cover  $x$ , it is in notation  $x \not\prec y$ ; if  $y \leq x$  and  $y \neq x$ , it is in notation  $y < x$ ; if  $y$  is not less than  $x$ , it is denoted by  $y \not\leq x$ ; if  $x$  and  $y$  are incomparable, it is denoted by  $x \parallel y$ .

### 2.2. Classification systems

Let us recall some information of classification systems from [7]. For more detail for classification systems, please see [7].

**Definition 1.** (1) A nonzero element  $p$  of a lattice  $L$  is called *completely join-irreducible* if for any system of elements  $x_i \in L$  ( $i \in I$ ), the equality  $p = \vee \{x_i : i \in I\}$  implies  $p = x_{i_0}$  for some  $i_0 \in I$ . If any nonzero element of  $L$  is a join of completely join-irreducible elements, then  $L$  is called a *CJ-generated lattice*.

(2) Let  $L$  be a complete lattice. A set  $S = \{a_j \mid j \in J\}$  ( $J \neq \emptyset$ ) of nonzero elements of  $L$  is called a *classification system* of  $L$  if the following conditions are satisfied:

(2.1)  $a_i \wedge a_j = 0$ , for all  $i \neq j$ , where  $0$  is the least element in  $L$ .

(2.2)  $x = \bigvee_{j \in J} (x \wedge a_j)$ , for all  $x \in L$ .

**Some notations 2.** (1) Let  $L$  be a CJ-generated complete lattice. The set of all completely join-irreducible elements of  $L$  is denoted by  $J(L)$ . For  $a \in L$ , let  $J(a) = \{p \in J(L) \mid p \leq a\}$  and set  $\bigvee \emptyset = 0$ .

(2) Let  $A(L)$  denote the set of atoms of a lattice  $L$ .

**Lemma 2.** (1) If  $S = \{a_i \mid i \in I\}$  is a classification system of  $L$ , then  $\pi_S = \{J(a_i), i \in I\}$  is a partition induced by  $S$  on  $J(L)$ .

(2) If  $L$  is a CJ-generated lattice, then any classification system  $S = \{a_i \mid i \in I\}$  of  $L$  is determined by the partition  $\pi_S$ , induced by  $S$  on  $J(L)$ .

(3) Any atomistic lattice  $L$  is a CJ-generated lattice with  $J(L) = A(L)$ .

Let  $L$  be an atomistic complete lattice. For any  $x \in L \setminus 0$ , the set  $S_x = \{x\} \cup \{a \in A(L) \mid a \wedge x = 0\}$  is a classification system of  $L$ .

(4) Let  $L$  be a complete lattice and  $1$  be the greatest element in  $L$ . Then  $S = \{1\}$  is a classification system.

**Definition 2.** Let  $L$  be a CJ-generated complete lattice and let  $S_p$  and  $S_q$  be two classification systems of  $L$ . We say that the system  $S_p$  is *finer* than  $S_q$  and we write  $S_p \leq S_q$ , if the partition  $\pi_{S_p}$  induced by  $S_p$  refines the partition  $\pi_{S_q}$  induced by  $S_q$ , that is, if  $\pi_{S_p} \leq \pi_{S_q}$ .

**Lemma 3.** (1) Let  $Cl_s(L)$  denote the set of all classification systems of a CJ-generated complete lattice  $L$ . Then,  $(Cl_s(L), \leq)$  is a complete lattice.

$(Cl_s(L), \leq)$  or  $Cl_s(L)$  for short, is called the *classification lattice* of the lattice  $L$ .

(2) The least element of the lattice  $Cl_s(L)$ , that is, the *finest classification system* of  $L$ , is the same as  $S_0 = \bigwedge \{S \mid S \in Cl_s(L)\}$ .

*Remark 1.* Let  $L$  be an atomistic complete lattice.

(1) In this paper,  $S_0$  stands for  $\{a \mid a \in A(L)\}$ ,  $S_1$  is  $\{1\}$ , and  $S_x$  represents  $\{x\} \cup \{a \in A(L) \mid a \not\leq x\}$  for any  $x \in L \setminus 0$ .

(2) For any  $x \in L \setminus 0$ , using the (3) in Lemma 1, we may easily know that  $S_x = \{x\} \cup \{a \in A(L) \mid a \wedge x = 0\}$  is a classification system of  $L$ . In addition, in light of the (2) in Lemma 3,  $S_0 = \{a \mid a \in A(L)\} \in Cl_s(L)$  holds, and  $S_a = S_0$  is true for any  $a \in A(L)$ .

(3) Let  $S \in Cl_s(L)$ . Then according to the (2) of Definition 1, there is  $S = S_x$  for some  $x \in L \setminus 0$ , or  $S = \{x_i \in L \setminus (A(L) \cup 0) \mid i \in I\} \cup \{a \in A(L) \mid a \not\leq x_i, i \in I\}$  where  $x_p \wedge x_q = 0$ , ( $p \neq q; p, q \in I$ ) and  $|I| \geq 2$ .

For convenient, if  $S = \{x_i \in L \setminus (A(L) \cup 0) \mid i \in I\} \cup \{a \in A(L) \mid a \not\leq x_i, i \in I\} \in Cl_s(L)$ , then it is denoted by  $S_{x_i, i \in I}$ . And further, when  $I = \{1, 2, \dots, n\}$ ,  $S_{x_i, i \in I}$  is sometimes denoted as  $S_{x_1 x_2 \dots x_n}$ .

(4) Let  $S = S_{x_1 x_2 \dots x_n}$ . By Definition 1 and the (3) above, it is easily to find  $S = S_{x_{i_1} x_{i_2} \dots x_{i_n}}$  where  $i_1, i_2, \dots, i_n$  is an arbitrary permutation of  $1, 2, \dots, n$ .

(5) Let  $x \in L \setminus (A(L) \cup 0)$ . By the discussion beyond, we easily obtain  $Cl_s([0, x]) = [S_0, S_x]$  in which  $[S_0, S_x]$  is an interval in  $Cl_s(L)$  and  $[0, x]$  is an interval in  $L$ .

**Some notations 3.** Let  $L$  be an atomistic complete finite lattice.

(1) Let  $\mathcal{F}^k$  denote  $\{x \in L \mid x \text{ has height } k \text{ in } L\}$ , ( $k = 2, \dots, h(L)$ ).

(2) For any  $x \in L$  and  $l \leq h(x)$ , let  $\mathcal{F}^l(x) = \{y \in L \mid y \in \mathcal{F}^l \text{ and } y \leq x\}$  and  $\mathcal{C}(x) = \{y \in L \mid x \text{ covers } y \text{ in } L\}$ .  $\mathcal{F}^1$  is also in notation  $A(x) = \{a \in A(L) \mid a \leq x\}$  for any  $x \in L$ .

### 3. ANSWER TO THE OPEN PROBLEM

In this section, we will discover the characterizations of atomistic complete finite lattices whose classification lattices are geometric lattices. Therefore, the open problem, which is repeated in Section 1, will be solved for finite cases.

**Theorem 1.** *Let  $L$  be an atomistic complete finite lattice. If  $|\mathcal{F}^2| = 1$ , i.e.  $L$  has only one element of height 2. Then  $Cl_s(L)$  is geometric.*

*Proof of Theorem 1.* It is easily to find  $Cl_s(L) = \{S_0, S_1\}$ . Thus,  $Cl_s(L)$  is geometric.  $\square$

In what follows,  $L$  always stands for an atomistic complete finite lattice with at least two elements of height 2. That is to say,  $\mathcal{F}^2 = \{x \in L \mid x \text{ has height } 2 \text{ in } L\}$  satisfies  $|\mathcal{F}^2| \geq 2$ .

First, we may deal with some properties related to  $Cl_s(L)$ .

**Lemma 4.**  *$Cl_s(L)$  possesses the following properties.*

(3.1) *Let  $x, y \in L \setminus 0$ . Then  $x \leq y \Leftrightarrow S_x \leq S_y$ . Further,  $x < y \Leftrightarrow S_x < S_y$ .*

(3.2) *Let  $x, y \in L \setminus (A(L) \cup 0)$ . If  $x \wedge y = 0$ , then  $S_x \vee S_y = S_{xy}$ .*

*Furthermore, let  $x_j \in L \setminus (A(L) \cup 0)$ , ( $j \in \mathcal{J}; |\mathcal{J}| \geq 2$ ). If  $x_i \wedge x_j = 0$ , ( $i, j \in \mathcal{J}; i \neq j$ ), then  $\bigvee_{j \in \mathcal{J}} S_{x_j} = S_{x_j, j \in \mathcal{J}}$ .*

(3.3) *Let  $x, y \in L \setminus (A(L) \cup 0)$ . If  $x \wedge y \neq 0$ , then  $S_x \vee S_y = S_{x \vee y}$ .*

(3.4)  *$A(Cl_s(L)) = \{S_d \mid d \in \mathcal{F}^2\}$ .*

(3.5) *Let  $S = S_{x_i, i \in I}$  with  $2 \leq |I|$ . Let  $S' \in Cl_s(L)$ . Then the following items*

*(3.5.1), (3.5.2) and (3.5.3) are true.*

(3.5.1) *If  $S' = S_y < S$  holds for some  $y \in L \setminus (A(L) \cup 0)$ , then there are  $I = \{1, 2\}$ ,  $y = x_1$  and  $h(x_2) = 2$ .*

(3.5.2) *If  $S' \neq S_y$  for any  $y \in L \setminus (A(L) \cup 0)$ ,  $S' < S$ , and  $h(x_n) = 2$  with  $x_n \notin S'$*

in which  $n \in I \setminus I_1$  for a  $I_1 \subseteq I$  with  $|I| = |I_1| + 1$ . Then  $S' = S_{x_i, i \in I_1}$ .

(3.5.3) Let  $S' = S_{x_i, i \in I_1}$  where  $I_1 \subseteq I$  with  $|I| = |I_1| + 1$ . If  $S'$  satisfies  $h(x_n) = 2$  where  $n \in I \setminus I_1$ . Then  $S' < S$  holds.

*Proof of Lemma 4.* We will demonstrate all the results step by step.

Step 1. From Definition 2 with the atomistic property of  $L$ , we may easily obtain item (3.1).

Step 2. We verify item (3.2) using steps 2.1 and 2.2 as follows.

Step 2.1. To prove: for  $x, y \in L \setminus (A(L) \cup 0)$ ,

$$x \wedge y = 0 \Rightarrow S_x \vee S_y = S_{xy}.$$

If  $x \wedge y = 0$ , then  $S_{xy} \in Cl_S(L)$  and  $S_x, S_y < S_{xy}$  according to Definition 2.

Let  $S \in Cl_S(L)$  satisfy  $S_x, S_y < S$ . Using  $x \wedge y = 0$  and Definition 2, we may find out  $A(x) \subseteq A(x_s)$  and  $A(y) \subseteq A(y_s)$  for some  $x_s, y_s \in S$ . This causes  $S_{xy} \leq S$ . Considering  $S_x, S_y < S$  with  $S_{xy} \leq S$ , we may be assured  $S_x \vee S_y = S_{xy}$ .

Step 2.2. To prove: for  $x_j \in L \setminus (A(L) \cup 0)$ , ( $j \in \mathcal{J}; |\mathcal{J}| \geq 2$ ),

$$x_i \wedge x_j = 0, (i, j \in \mathcal{J}; i \neq j) \Rightarrow \bigvee_{j \in \mathcal{J}} S_{x_j} = S_{x_j, j \in \mathcal{J}}.$$

Using induction on  $|\mathcal{J}|$  and item (3.1), we may obtain  $\bigvee_{j \in \mathcal{J}} S_{x_j} \leq S_{x_j, j \in \mathcal{J}}$  if  $x_i \wedge x_j = 0$ , ( $i, j \in \mathcal{J}; i \neq j$ ).

By the (3) in Remark 1,  $\bigvee_{j \in \mathcal{J}} S_{x_j} = S_y$  holds for some  $y \in L \setminus (A(L) \cup 0)$ , or  $\bigvee_{j \in \mathcal{J}} S_{x_j} = S_{z_1 \dots z_m}$  holds for some  $z_1, \dots, z_m \in L \setminus (A(L) \cup 0)$  with  $2 \leq m$ .

Suppose that  $\bigvee_{j \in \mathcal{J}} S_{x_j} = S_y$  holds. By item (3.1) and  $S_{x_j} \leq \bigvee_{j \in \mathcal{J}} S_{x_j}$ , ( $j \in \mathcal{J}$ ), we may determine  $x_j \leq y$  ( $j \in \mathcal{J}$ ). Furthermore,  $S_{x_j, j \in \mathcal{J}} \leq S_y$  is true by Definition 2. So, it follows  $S_y = S_{x_j, j \in \mathcal{J}}$  with  $|\mathcal{J}| \geq 2$ , which is a contradiction to the (2) of Definition 1.

Suppose that  $\bigvee_{j \in \mathcal{J}} S_{x_j} = S_{z_1 \dots z_m}$  holds for  $2 \leq m$ . Using item (3.1) and  $S_{x_j} \leq S_{z_1 \dots z_m}$  ( $j \in \mathcal{J}$ ), we point out  $x_j \leq z_{j_t}$ , ( $j \in \mathcal{J}; j_t \in \{1, \dots, m\}$ ). Utilizing  $z_{j_t} \wedge z_{i_s} = 0$  ( $j_t \neq i_s; j_t, i_s = 1, \dots, m$ ) and the result in the above case, we follow that if  $z_{j_t} \neq z_{i_s}$ , then  $x_j \neq x_i$ , ( $i, j \in \mathcal{J}$ ), where  $x_j \leq z_{j_t}$  and  $x_i \leq z_{i_s}$ . This implies  $A(x_j) \subseteq A(z_{j_t})$ , ( $j \in \mathcal{J}; j_t \in \{1, \dots, m\}$ ) since  $L$  is atomistic. Combining items (1) and (2) in Lemma 2 with Definition 2,  $\pi_{S_{x_j, j \in \mathcal{J}}} \leq \pi_{S_{z_1 \dots z_m}}$  holds. Hence,  $S_{x_j, j \in \mathcal{J}} \leq S_{z_1 \dots z_m}$  is true. Additionally, we decide  $S_{x_j} \leq S_{x_j, j \in \mathcal{J}}$  according to  $A(x_j) \subseteq A(x_j)$ , ( $j \in \mathcal{J}$ ), the definitions of  $S_{x_j, j \in \mathcal{J}}$  and  $S_{x_j}$ , and Definition 2. Thus, it follows  $\bigvee_{j \in \mathcal{J}} S_{x_j} \leq S_{x_j, j \in \mathcal{J}} \leq S_{z_1 \dots z_m}$ . Therefore, we may be assured  $S_{x_j, j \in \mathcal{J}} = \bigvee_{j \in \mathcal{J}} S_{x_j}$ .

Step 3. To prove item (3.3).

Let  $x \wedge y \neq 0$ . Then  $\{x, y\} \not\subseteq S$  is true for any  $S \in Cl_S(L)$ . If  $S_x \vee S_y = S_{z_1 z_2 \dots z_n}$  for any  $z_j \in L \setminus (A(L) \cup 0)$ , ( $j = 1, 2, \dots, n; 2 \leq n$ ). Then, the properties of  $x \leq z_1, y \leq z_2$  and  $z_1 \neq z_2$  taken together follows  $x \wedge y = 0$ . This is a contradiction to  $x \wedge y \neq 0$ . Hence, we confirm  $x, y \leq z_1$ . However,  $S_x, S_y \leq S_{z_1} < S_{z_1 \dots z_n} = S_x \vee S_y$  follows  $S_x \vee S_y \leq S_{z_1} < S_{z_1 \dots z_n} = S_x \vee S_y$ , a contradiction. Thus,  $S_x \vee S_y = S_z$  is true for some  $z \in L \setminus (A(L) \cup 0)$ . Then, we attain  $S_x, S_y \leq S_x \vee S_y$ . Furthermore, we obtain  $x, y \leq z$  using item (3.1). Moreover,  $x \vee y \leq z$  holds. So, we determine  $S_{x \vee y} \leq S_z$ .

On the other hand, in view of  $x, y \leq x \vee y$  and item (3.1), we obtain  $S_x, S_y \leq S_{x \vee y}$ . Furthermore, we attain  $S_x \vee S_y \leq S_{x \vee y}$ .

Therefore, we have demonstrated  $S_x \vee S_y = S_{x \vee y}$ .

Step 4. To prove item (3.4).

Let  $d \in \mathcal{F}^2$ . According to Remark 1, we obtain  $S_d \in Cl_s(L)$ . By item (3.1), it has  $S_0 < S_d$ . If  $S_0 < S \leq S_d$  holds for some  $S \in Cl_s(L)$ . Then we obtain  $x \in S$  and  $x \in L \setminus (A(L) \cup 0)$ . So,  $2 \leq h(x)$  holds. Furthermore, we find the existence of  $d_x \in \mathcal{F}^2(x)$ . Thus, we may obtain  $S_0 < S_{d_x} \leq S \leq S_d$ . Combining  $S_0 < S$  and  $S_0 < S_{d_x} \leq S$ , we find  $S = S_{d_x}$ . On the other hand, taking  $S \leq S_d, S_{d_x} \leq S$  with  $h(d_x) = h(d) = 2$  together, we follow  $d_x = d$ . Therefore, we attain  $S_0 < S_d$ . Hence,  $\{S_d \mid d \in \mathcal{F}^2\} \subseteq A(Cl_s(L))$  is true.

If  $A(Cl_s(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\} \neq \emptyset$ . Then, there is  $S \in A(Cl_s(L)) \setminus \{S_d \mid d \in \mathcal{F}^2\}$ . This causes  $y \in S \setminus (A(L) \cup 0)$ . So,  $2 \leq h(y)$  is real. Considered the definition of height function of a lattice in [5], we find  $d_y \in \mathcal{F}^2$  with  $d_y \leq y$ . Moreover, we obtain  $S_{d_y} \leq S$ . However,  $S_0 < S_{d_y}$  is known according to the above proof. Hence, it follows  $S = S_{d_y}$ . This is a contradiction to  $S \notin \{S_d \mid d \in \mathcal{F}^2\}$ . In other words,  $A(Cl_s(L)) = \{S_d \mid d \in \mathcal{F}^2\}$  is true.

Step 5. To prove item (3.5).

Combining  $S \in Cl_s(L)$  with Definition 1, we confirm  $\{x_i \mid i \in T\} \cup \{a \in A(L) \mid a \not\leq x_i, i \in T\} \in Cl_s(L)$  for any  $T \subseteq I$ .

Step 5.1. This step verifies items (3.5.1) and (3.5.2).

If  $S' < S$ . Then, by the (3) in Remark 1, we may find  $S' = S_y$  for some  $y \in L \setminus (A(L) \cup 0)$  or  $S' = S_{z_1 z_2 \dots z_m}$  with  $2 \leq m$  and  $z_i \in L \setminus (A(L) \cup 0), (i = 1, \dots, m)$ .

Suppose  $S' = S_y$  for some  $y \in L \setminus (A(L) \cup 0)$ . Then,  $A(y) \subseteq A(x_{i_0})$  holds for some  $i_0 \in I$ . According to  $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$ , we confirm that there is one and only one  $i_0 \in I$  satisfying  $A(y) \subseteq A(x_{i_0})$ . In addition, owing to the atomistic property of  $Cl_s(L)$ , we may find  $y \leq x_{i_0}$ . And further, we obtain  $S_y \leq S_{x_{i_0}}$  by item (3.1).

Under the assumption  $S' < S$  and the closest result above, we may infer to  $|I| = 2$ . Otherwise,  $3 \leq |I|$  will follow  $S' = S_y \leq S_{x_{i_0}} < S_{x_{i_0} x_{i_1}} < S_{x_i, i \in I} = S$ , a contradiction to  $S' < S$ .

We may assume  $I = \{1, 2\}$  and  $x_{i_0} = x_1$ . In view of  $S' = S_y \leq S_{x_1} < S, S_{x_1} \in Cl_s(L)$  and  $S' < S$ , we find  $S_y = S_{x_1}$ . Thus,  $y = x_1$  holds. Since  $x_2 \in L \setminus (A(L) \cup 0)$  causes  $h(x_2) \geq 2$ . Assume  $h(x_2) > 2$ . By the definition of height function in a lattice, we obtain  $t < x_2$  and  $h(t) = 2$ . Additionally,  $0 \leq t \wedge x_1 \leq x_2 \wedge x_1 = 0$  yields  $t \wedge x_1 = 0$ . So,  $S_{x_1 t} \in Cl_s(L)$  holds. We may easily determine  $S_{x_1} < S_{x_1 t} < S$ , a contradiction to  $S' = S_{x_1} < S$ . Therefore, we confirm  $h(x_2) = 2$ .

Suppose  $S' = S_{z_1 z_2 \dots z_m}$  with  $2 \leq m$ . No harm to assume  $I = \{1, 2, \dots, n\}$ . Taking  $S' = S_{z_1 \dots z_m} < S = S_{x_1 \dots x_n}$  and  $x_i \wedge x_j = 0, (i \neq j; i, j = 1, \dots, n)$  together, we find  $z_{i\alpha_1} \leq x_{j_1}, (\alpha_1 \in I_1), \dots, z_{i\alpha_p} \leq x_{j_p}, (\alpha_p \in I_p)$  in which  $\{i_{\alpha_t} \mid \alpha_t \in I_t\} \subseteq \{1, \dots, m\}, (t = 1, \dots, p); \{i_{\alpha_1} \mid \alpha_1 \in I_1\} \cup \dots \cup \{i_{\alpha_p} \mid \alpha_p \in I_p\} = \{1, \dots, m\}; \{i_{\alpha_k} \mid$

$\alpha_k \in I_k \} \cap \{i_{\alpha_l} \mid \alpha_l \in I_l\} = \emptyset, (k \neq l; k, l = 1, \dots, p); \{j_1, j_2, \dots, j_p\} \subseteq \{1, \dots, n\}$  and  $j_\alpha \neq j_\beta, (\alpha \neq \beta; \alpha, \beta = 1, \dots, p)$ . Thus, by the (4) of Remark 1, we may be assured  $S_{z_1 \dots z_m} = S_{z_{i_1}, i_1 \in I_1, \dots, z_{i_p}, i_p \in I_p} \leq S_{x_{j_1} \dots x_{j_p}} < S$ . Since  $S_{z_1 \dots z_m} < S$ , we obtain  $S' = S_{x_{j_1} \dots x_{j_p}} < S$ . In addition, we attain  $\{j_1, \dots, j_p\} \subset \{1, \dots, n\}$ .

If  $p < n - 1$ . Then, according to the (4) of Remark 1 and  $\{j_1, \dots, j_p\} \subset \{1, \dots, n\}$ , we may confirm  $S_{x_{j_1} \dots x_{j_p}} < S_{x_{j_1} \dots x_{j_p} x_{j_{p+1}} \dots x_{j_{n-1}}} < S_{x_{j_1} \dots x_{j_n}} = S_{x_1 \dots x_n} = S$  for a sub-arrangement  $j_1, \dots, j_p, j_{p+1}, \dots, j_{n-1}$  of  $1, 2, \dots, n$ . This causes a contradiction to  $S' < S$ . Thus,  $p = n - 1$  is true.

Certainly,  $S' = S_{x_{j_1} \dots x_{j_{n-1}}} < S$  shows that for any  $t < x_{j_n}$ , there is  $t \wedge x_j = 0$  for any  $j \in \{j_1, \dots, j_{n-1}\}$  since  $0 \leq t \wedge x_j \leq x_{j_n} \wedge x_j = 0$ .

Hence, we may obtain  $h(x_{j_n}) = 2$ . Otherwise, it follows  $S_{x_{j_1} \dots x_{j_{n-1}}} < S_{x_{j_1} \dots x_{j_{n-1}} b} < S$  for  $b < x_{j_n}$  and  $h(b) = 2$  according to Definition 2, a contradiction to  $S' < S$ .

Step 5.2. To prove item (3.5.3).

Let  $S' = S_{x_{i_1} x_{i_2} \dots x_{i_{n-1}}} = S_{x_i, i \in I_1} \in Cl_s(L)$ . Then, it is easily to find  $S' < S$ .

Suppose  $S' < S'' < S$  and  $S'' = S_y$  for some  $y \in L \setminus (A(L) \cup 0)$ . Then by item (3.1), we obtain  $x_j < y \leq x_{i_0}, (j \in I_1)$  and  $i_0 \in I$ . This follows that

if  $j \neq i_0$ , then  $x_j \wedge x_{i_0} = x_j \in L \setminus (A(L) \cup 0)$ , a contradiction to  $x_j \wedge x_{i_0} = 0$ ;

if  $j = i_0$ , then  $x_j < x_{i_0}$ , a contradiction to  $x_j = x_{i_0}$  when  $j = i_0$ .

Suppose  $S' < S'' < S = S_{x_{i_1} \dots x_{i_{n-1}} x_{i_n}}$  and  $S'' = S_{z_1 z_2 \dots z_m}$  with  $2 \leq m$ .  $S' < S''$  follows  $x_{i_{t_1}} \leq z_1, x_{i_{t_2}} \leq z_2, \dots, x_{i_{t_k}} \leq z_k$ , where  $t_1 = 1, \dots, m_1; t_2 = m_1 + 1, \dots, m_2; \dots; t_k = m_{k-1} + 1, \dots, m_k; m_1 + \dots + m_k = |I_1| = n - 1$ .  $k \leq m$  holds since  $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$ ,  $z_t \in L \setminus (A(L) \cup 0), (t = 1, \dots, m)$  and  $z_p \wedge z_q = 0, (p \neq q; p, q \in \{1, \dots, m\})$ . Meanwhile, combining  $S_{x_j, j \in T} \in Cl_s(L), (T \subseteq \{1, \dots, n\})$  with  $S'' < S = S_{x_{i_1} \dots x_{i_{n-1}} x_{i_n}}$ , we attain  $z_j \leq x_{i_j}, (j = 1, 2, \dots, n - 1)$ . Hence, in view of this result with  $x_i \wedge x_j = 0, (i \neq j; i, j \in I)$ ,  $z_t \in L \setminus (A(L) \cup 0), (t = 1, \dots, m)$  and  $x_i \in L \setminus (A(L) \cup 0), (i \in I)$ , we obtain  $z_j = x_{i_j}, (j = 1, 2, \dots, n - 1)$  and  $z_n < x_{i_n}$ . However, by the known condition, it follows  $h(x_{i_n}) = 2$ . So,  $z_n \in A(L) \cup 0$  holds. Hence, it causes  $S'' = S'$ , a contradiction to  $S' < S''$ .

Therefore, we confirm  $S' < S$ .  $\square$

Second, we will deal with some properties related to  $Cl_s(L)$  if  $Cl_s(L)$  is geometric for an atomistic complete finite lattice  $L$ .

**Lemma 5.** *Let  $L$  be an atomistic complete finite lattice with height 3. Then  $Cl_s(L)$  is geometric if and only if*

(3.6)  $|\mathcal{F}^2| \geq 2$ , that is,  $L$  has at least two elements of height 2.

(3.7) If  $d_1, d_2 \in \mathcal{F}^2$ , then  $d_1 \wedge d_2 \neq 0$ .

*Proof of Lemma 5.* We will prove with two parts.

Part I. When  $Cl_s(L)$  is geometric.

$h(L) = 3$  compels that there is  $d \in \mathcal{F}^2$  according to the definition of height function in a lattice. If  $\mathcal{F}^2 = \{d\}$ . Then, we may obtain  $Cl_s(L) = \{S_0, S_1, S_d\}$  with

$S_0 < S_d < S_1$ . It is easily found that  $Cl_s(L)$  is not geometric. This is a contradiction to the known supposition. Thus, it should have  $|\mathcal{F}^2| \geq 2$ . That is, item (3.6) is true.

Suppose that  $d_1 \wedge d_2 = 0$  holds for any  $d_1, d_2 \in \mathcal{F}^2 = \{d_j \mid j \in \mathcal{J}\}$ . Then, this causes  $S_{d_t, t \in T} \in Cl_s(L) < S_1$  for any  $T \subseteq \mathcal{J}$  in virtue of the (3) of Remark 1 and the (2) in Definition 1. In addition, we may easily find  $Cl_s(L) = \{S_0, S_1, S_{d_t, t \in T}, T \subseteq \mathcal{J}\}$ . This causes  $S_0 < S_{d_t, t \in T} \leq S_{d_j, j \in \mathcal{J}} < S_1$ . So,  $Cl_s(L)$  is not geometric since  $S_1$  is not the join of atoms in  $Cl_s(L)$ . This is a contradiction to the geometry of  $Cl_s(L)$ .

Therefore, there are  $d_6, d_7 \in \mathcal{F}^2$  satisfying  $d_6 \neq d_7$  and  $d_6 \wedge d_7 \neq 0$ .

If there is  $d_3 \in \mathcal{F}^2$  satisfying  $d_3 \wedge d_7 = 0$ . Then, we obtain  $S_0 < S_{d_7} < S_{d_7 d_3} < S_1$  and  $S_0 < S_{d_6} < S_1$  with  $S_{d_6} \parallel S_{d_7 d_3}$ . In addition,  $S_{d_7} \vee S_{d_6} = S_{d_7 \vee d_6} = S_1$  holds according to  $d_7 \wedge d_6 \neq 0$  and item (3.3). This follows  $S_{d_7} \not\leq S_1 = S_{d_7} \vee S_{d_6}$ . So,  $Cl_s(L)$  is not semimodular. This follows a contradiction to the geometry of  $Cl_s(L)$ . In other words,  $d_7 \wedge d_j \neq 0$  holds for any  $j \in \mathcal{J}$ . Analogously,  $d_6 \wedge d_j \neq 0$  holds for any  $j \in \mathcal{J}$ .

Moreover,  $S_0 < S_{d_7} < S_1$  holds. Hence, this maximal chain  $\{S_0, S_{d_7}, S_1\}$  in  $Cl_s(L)$  has length 3. If there are  $d_4, d_5 \in \mathcal{F}^2$  satisfying  $d_4 \wedge d_5 = 0$ . Then it causes  $S_0 < S_{d_4} < S_{d_4 d_5} < S_1$ . So, there is a maximal chain in  $Cl_s(L)$  with length at least 4. Hence, there are two maximal chains with different lengths in  $Cl_s(L)$ . This follows a contradiction to the geometry of  $Cl_s(L)$ .

Summing up, we obtain  $x \wedge y \neq 0$  for any  $x, y \in \mathcal{F}^2$ . That is, item (3.7) is true.

Part II. When  $L$  satisfies items (3.6) and (3.7).

Let  $d_j$  has height 2 in  $L$ , ( $j \in \mathcal{J}$ ), that is,  $\mathcal{F}^2 = \{d_j \mid j \in \mathcal{J}\}$ .

Under the suppositions of items (3.6) and (3.7) with  $h(L) = 3$ , we may easily decide that  $Cl_s(L)$  is  $\{S_0, S_1, S_{d_j}, j \in \mathcal{J}\}$  in which  $S_0 < S_{d_j} < S_1$ ,  $S_{d_i} \parallel S_{d_j}$ , ( $i, j \in \mathcal{J}; i \neq j$ ) and  $S_1 = \vee_{j \in \mathcal{J}} S_{d_j}$ . Therefore,  $Cl_s(L)$  is geometric.  $\square$

Considering the (3) in Lemma 1 with Lemma 5, we may easily express the following corollary.

**Corollary 1.** *Let  $L$  be an atomistic complete finite lattice and  $x \in L \setminus (A(L) \cup 0 \cup \mathcal{F}^2)$  with  $h(x) = 3$ . If  $Cl_s(L)$  is geometric, then there are  $|\mathcal{F}^2(x)| \geq 2$  and  $d_1 \wedge d_2 \neq 0$  for any  $d_1, d_2 \in \mathcal{F}^2(x)$  and  $x = \vee_{d \in \mathcal{F}^2(x)} d$ .*

**Lemma 6.** *Let  $L$  be an atomistic complete finite lattice. If  $Cl_s(L)$  is geometric, then  $L$  satisfies the following properties.*

(3.8)  $|\mathcal{C}(x) \setminus A(L)| \geq 2$  for any  $x \in L \setminus (A(L) \cup 0)$ .

(3.9)  $x = \vee_{d \in \mathcal{F}^2(x)} d$  for any  $x \in L \setminus (A(L) \cup 0)$ .

(3.10) Let  $x, y \in L \setminus (A(L) \cup 0)$ . If  $x \wedge y < x, y$ , then  $x, y < x \vee y$ .

(3.11) Let  $x \in L$  with  $h(x) = 3$ . Then, there is  $d_1 \wedge d_2 \neq 0$  for any  $d_1, d_2 \in \mathcal{F}^2(x)$  with  $d_1 \neq d_2$ .

(3.12) Let  $x \in L \setminus (A(L) \cup 0)$ . If  $3 \leq h(x)$ , then there are  $y, z \in \mathcal{C}(x) \setminus A(L)$  satisfying  $y \neq z$  and  $y \wedge z \neq 0$ .



(3.13) Let  $x, y \in L \setminus (A(L) \cup 0)$ . If  $2 \leq h(x \wedge y)$ ,  $x \wedge y \neq 0$ ,  $x \wedge y < x$ ,  $x \wedge y < y$  and  $z_x \wedge (x \wedge y) \neq 0$ ,  $z_y \wedge (x \wedge y) \neq 0$  for any  $z_x, z_y \in L \setminus (A(L) \cup 0)$  with  $z_x < x$ ,  $z_y < y$ . Then,  $p \wedge x \neq 0$  and  $q \wedge y \neq 0$  for any  $p, q < x \vee y$  and  $p, q \in L \setminus (A(L) \cup 0)$ .

(3.14) Let  $y_j \in L \setminus (A(L) \cup 0)$ , ( $j = 1, 2, \dots, n; 2 \leq n$ ),  $y_i \wedge y_j = 0$ , ( $i \neq j; i, j = 1, 2, \dots, n$ ).

If  $y_j \in \mathcal{F}^2$ , ( $j = 1, \dots, n$ ), then  $S < S_{y_1 \dots y_n} \Rightarrow S = S_{y_{i_1} \dots y_{i_{n-1}}}$ , for some  $\{i_1, \dots, i_{n-1}\} \subseteq \{1, \dots, n\}$  and  $|\{i_1, \dots, i_{n-1}\}| = n - 1$ .

If  $S < S_{y_1 \dots y_n}$ . We obtain that if there is  $y_{i_n} \notin \mathcal{F}^2$  for some  $i_n \in \{1, \dots, n\}$ , then  $S = S_{y_{i_1} \dots y_{i_{n-1}} z_n}$  holds where  $z_n < y_{i_n}$  and  $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ ; or if  $y_{j_n} \in \mathcal{F}^2$  for some  $j_n \in \{1, \dots, n\}$ , then  $S = S_{y_{j_1} \dots y_{j_{n-1}}}$  where  $\{j_1, \dots, j_{n-1}\} = \{1, \dots, n\} \setminus j_n$ .

(3.15) Let  $y_i \in L$ , ( $i = 1, 2, \dots, n$ ) satisfy  $y_i \wedge y_j = 0$ , ( $i \neq j; i, j = 1, 2, \dots, n; 2 \leq n$ ).

( $\alpha$ ) Setting  $x = \bigvee_{j=1}^{n-1} y_{i_j} \vee w_n$  and  $w_n < y_{i_n}$  with  $h(w_n) \geq 2$  and  $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ . Suppose that for any  $m \leq n - 1$  and  $p \in L$  with  $\bigvee_{j=1}^m y_{i_{t_j}} \leq p < x$  or  $\bigvee_{j=1}^m y_{i_{t_j}} \vee w_n \leq q < x$ , where  $\{i_{t_j} \mid j = 1, \dots, m\} \subseteq \{i_1, \dots, i_{n-1}\}$ , there exists  $s \in \{i_1, \dots, i_{n-1}\} \setminus \{i_{t_j} \mid j = 1, \dots, m\}$  satisfying  $p \wedge y_s \neq 0$ , or  $p \wedge w_n \neq 0$ , or  $q \wedge y_s \neq 0$ . Then the following property (3.15.1) is true.

( $\beta$ ) Setting  $z = \bigvee_{t=1}^{n-1} y_{j_t}$ ,  $h(y_{j_n}) = 2$  and  $\{j_1, \dots, j_n\} = \{1, \dots, n\}$ . Suppose that for any  $m \leq n - 1$  and  $p \in L$  with  $\bigvee_{j=1}^m y_{i_{t_j}} \leq p < z$  where  $\{i_{t_j} \mid j = 1, \dots, m\} \subseteq \{j_1, \dots, j_{n-1}\}$ , there exists  $s \in \{j_1, \dots, j_{n-1}\} \setminus \{i_{t_j} \mid j = 1, \dots, m\}$  satisfying  $p \wedge y_s \neq 0$ . Then the following property (3.15.2) is real.

(3.15.1) If  $x \wedge y_{i_n} \neq 0$ . Then  $x < x \vee y_{i_n} = \bigvee_{i=1}^n y_i$  holds, and in addition, for any  $m \leq n$  and  $\bigvee_{j=1}^m y_{i_{t_j}} \leq q < \bigvee_{i=1}^n y_i$  where  $\{i_{t_j} \mid j = 1, \dots, m\} \subset \{1, \dots, n\}$ , there exists  $s \in \{1, \dots, n\} \setminus \{i_{t_j} \mid j = 1, \dots, m\}$  satisfying  $q \wedge y_s \neq 0$ .

(3.15.2) If  $z \wedge y_{j_n} \neq 0$ . Then  $z < z \vee y_{j_n} = \bigvee_{j=1}^n y_j$  holds, and in addition, for any  $m \leq n$  and  $\bigvee_{j=1}^m y_{i_{t_j}} \leq q < \bigvee_{j=1}^n y_j$  where  $\{i_{t_j} \mid j = 1, \dots, m\} \subset \{1, \dots, n\}$ , there exists  $s \in \{1, \dots, n\} \setminus \{i_{t_j} \mid j = 1, \dots, m\}$  satisfying  $q \wedge y_s \neq 0$ .

(3.16) Let  $y_j \in L \setminus (A(L) \cup 0)$ , ( $j = 1, 2, \dots, n; 2 \leq n$ ),  $x_n \in L \setminus (A(L) \cup 0)$  and  $x_n \neq y_n$  satisfy  $y_i \wedge y_j = 0$ , ( $i \neq j; i, j = 1, 2, \dots, n$ ) and  $y_j \wedge x_n = 0$ , ( $j = 1, 2, \dots, n - 1$ ). Then the following statements hold.

(3.16.1) Let  $h(x_n), h(y_n) > 2$  and  $h(x_n \wedge y_n) \neq 0$ . If  $y_j \wedge (x_n \vee y_n) = 0$  holds ( $j = 1, \dots, n - 1$ ), and in addition, there is  $z_n \in L \setminus 0$  satisfying  $z_n < x_n, y_n$ . Then, for any  $p, q \in L \setminus (A(L) \cup 0)$ ,  $x_n < p < x_n \vee y_n$  and  $y_n < q < x_n \vee y_n$ , there exist  $s, t \in \{1, \dots, n - 1\}$  satisfying  $y_s \wedge p \neq 0$  and  $y_t \wedge q \neq 0$ .

Additionally, if there are  $x_n \wedge y_n \neq 0$  and  $h(x_n) = h(y_n) = 2$ , but  $y_j \wedge (x_n \vee y_n) = 0$ , ( $j = 1, \dots, n - 1$ ). Then, for any  $p, q \in L \setminus (A(L) \cup 0)$ ,  $x_n < p < x_n \vee y_n$  and  $y_n < q < x_n \vee y_n$ , there exist  $s, t \in \{1, \dots, n - 1\}$  satisfying  $y_s \wedge p \neq 0$  and  $y_t \wedge q \neq 0$ .

(3.16.2) Let  $h(x_n), h(y_n) > 2$  and  $h(x_n \wedge y_n) \neq 0$ . When  $y_j \wedge (x_n \vee \bigvee_{i=m+1}^n y_i) = 0$ , ( $j = 1, \dots, m; 1 < m < n - 1$ ), and in addition, there exists  $z_n \in L \setminus 0$  satisfying

$z_n < x_n, y_n$ . Then, for any  $p, q, f \in L \setminus (A(L) \cup 0)$ , if the expressions (i), (ii) and (iii) hold

(i)  $\bigvee_{i=t}^{t_p} y_{j_i} \leq p < x_n \vee \bigvee_{i=m+1}^n y_i$  where  $\{j_i \mid t \leq i \leq t_p\} \subseteq \{m+1, \dots, n-1\}$ ,

(ii)  $x_n \vee \bigvee_{i=t_x}^{t_q} y_{j_i} \leq q < x_n \vee \bigvee_{i=m+1}^n y_i$  where  $\{j_i \mid t_x \leq i \leq t_q\} \subseteq \{m+1, \dots, n-1\}$ ,

(iii)  $\bigvee_{i=t_1}^{t_f} y_{j_i} \leq f < \bigvee_{i=m+1}^n y_i \vee x_n$  where  $\{j_i \mid t_1 \leq i \leq t_f\} \subseteq \{m+1, \dots, n\}$ .

then, the following expressions hold:

there exists  $s_p \in \{1, \dots, n-1\} \setminus \{j_i \mid t \leq i \leq t_p\}$  satisfying  $y_{s_p} \wedge p \neq 0$  or  $x_n \wedge p \neq 0$ ;

there exists  $s_q \in \{1, \dots, n-1\} \setminus \{j_i \mid t_x \leq i \leq t_q\}$  satisfying  $y_{s_q} \wedge q \neq 0$ ;

there exists  $s_f \in \{1, \dots, n\} \setminus \{j_i \mid t_1 \leq i \leq t_f\}$  satisfying  $y_{s_f} \wedge f \neq 0$ .

Additionally, let  $x_n \wedge y_n \neq 0$  and  $h(x_n) = h(y_n) = 2$ . When  $y_j \wedge (x_n \vee \bigvee_{i=m+1}^n y_i) = 0$ , ( $j = 1, \dots, m; 1 < m < n-1$ ). Then, for any  $p, q, f \in L \setminus (A(L) \cup 0)$ , if the expressions (i), (ii) and (iii) hold, then the following expressions hold:

there is  $s_p \in \{1, \dots, n-1\} \setminus \{j_i \mid t \leq i \leq t_p\}$  satisfying  $y_{s_p} \wedge p \neq 0$  or  $x_n \wedge p \neq 0$ ;

there is  $s_q \in \{1, \dots, n-1\} \setminus \{j_i \mid t_x \leq i \leq t_q\}$  satisfying  $y_{s_q} \wedge q \neq 0$ ;

there is  $s_f \in \{1, \dots, n\} \setminus \{j_i \mid t_1 \leq i \leq t_f\}$  satisfying  $y_{s_f} \wedge f \neq 0$ .

*Proof of Lemma 6.* We will prove the needed results step by step.

Step 1. To prove item (3.8).

In view of the definition of height function in a lattice and  $h(x) = n < \infty$ , we may indicate that there is  $y \in \mathcal{F}^{n-1}(x)$ . It is easily seen  $y < x$ . So,  $y \in \mathcal{C}(x) \setminus A(L)$  holds.

If  $|\mathcal{C}(x) \setminus A(L)| = 1$ . Considered the atomistic property of  $L$ , we receive  $x = y \vee (\bigvee_{a \in A(L), a \not\leq y, a \leq x} a)$ . Hence,  $Cl_s([0, x])$  satisfies  $S \leq S_y$  for any  $S \in Cl_s([0, x]) \setminus S_x$ . This implies that  $S_x$  is not a join of atoms in  $[S_0, S_x]$ , a contradiction to the geometry of  $Cl_s(L)$ .

In another word,  $|\mathcal{C}(x) \setminus A(L)| \geq 2$  holds.

Step 2. To prove item (3.9).

It is easily found  $x = \bigvee_{d \in \mathcal{F}^2(x)} d$  if  $h(x) = 2$ .

Utilizing Corollary 1, we may attain  $x = \bigvee_{d \in \mathcal{F}^2(x)} d$  for any  $x \in L$  and  $h(x) = 3$ .

Suppose that if  $x \in L$  with  $2 \leq h(x) \leq n-1$ , then  $x = \bigvee_{d \in \mathcal{F}^2(x)} d$ .

Let  $x \in L$  and  $h(x) = n$ . In view of item (3.8), we find  $x = \bigvee_{y \in \mathcal{C}(x) \setminus A(L)} y$ . We may easily decide  $h(y) \leq n-1$  for any  $y \in \mathcal{C}(x) \setminus A(L)$ . Using the inductive on  $n$ , we obtain  $y = \bigvee_{d \in \mathcal{F}^2(y)} d$  for any  $y \in \mathcal{C}(x) \setminus A(L)$ . On the other hand, for any  $d \in \mathcal{F}^2(x)$ , we find  $d < x$  or  $d \leq y$  for some  $y \in \mathcal{C}(x) \setminus A(L)$ .

Therefore, it follows  $x = \bigvee_{y \in \mathcal{C}(x) \setminus A(L)} y = \bigvee_{d \in \mathcal{F}^2(x)} d$ .

Step 3. To prove item (3.10).

Let  $x, y \in L \setminus (A(L) \cup 0)$  and  $x \wedge y < x, y$ .

If  $x \wedge y = 0$ , then  $x, y \in A(L)$  holds since  $0 < x, y$ . This is a contradiction to  $x, y \in L \setminus (A(L) \cup 0)$ .

Thus,  $x \wedge y \neq 0$  is true.

Furthermore, by item (3.3), we may be assured  $S_x \vee S_y = S_{x \vee y}$ . Since  $Cl_s(L)$  is geometric and  $S_x, S_y \prec S_x \vee S_y$  holds, we confirm  $S_x, S_y \prec S_{x \vee y}$ . If  $x \not\prec x \vee y$ , then  $S_x < S_b < S_{x \vee y}$  for some  $b \in L$  and  $x < b < x \vee y$ . This causes a contradiction to  $S_x \prec S_{x \vee y}$ . Hence,  $x \prec x \vee y$  holds. Similarly,  $y \prec x \vee y$  holds.

Step 4. The result in item (3.11) can be produced by Corollary 1.

Step 5. To prove item (3.12).

Let  $x \in L \setminus (A(L) \cup 0)$  with  $3 \leq h(x)$ .

If  $p \wedge q = 0$  for any  $p, q \in \mathcal{C}(x) \setminus A(L)$ . Then, using (3) in Remark 1, item (3.8) and induction on  $|\mathcal{C}(x) \setminus A(L)|$ , we confirm  $S_{y, y \in \mathcal{C}(x) \setminus A(L)} = \{y \mid y \in \mathcal{C}(x) \setminus A(L)\} \cup \{a \in A(L) \mid a \not\prec y, \text{ for any } y \in \mathcal{C}(x) \setminus A(L)\} \in Cl_s(L)$  and  $S_{y, y \in \mathcal{C}(x) \setminus A(L)} \prec S_x$ , and  $S \leq S_{y, y \in \mathcal{C}(x) \setminus A(L)}$  for any  $S \in [S_0, S_x] \setminus S_x$ . Thus,  $S_x$  is not the join of atoms. This causes a contradiction to the geometry of  $Cl_s(L)$ .

Therefore, there are at least two elements  $y, z \in \mathcal{C}(x) \setminus A(L)$  satisfying  $y \wedge z \neq 0$ .

Step 6. To prove item (3.13).

From the given conditions, we follow  $S_{x \wedge y} \prec S_x, S_y$ . It is easily known  $S_{x \wedge y} \leq S_x \wedge S_y$  by item (3.1).

We will demonstrate  $S_x \wedge S_y = S_{x \wedge y}$ .

Otherwise,  $S_{x \wedge y} < S_x \wedge S_y$ . If  $S_x \wedge S_y = S_z$  for some  $z \in L$ , then  $x \wedge y < z < x, y$ . Using item (3.1), we may attain  $S_{x \wedge y} < S_z < S_x, S_y$ , a contradiction to  $S_{x \wedge y} \prec S_x, S_y$ . Thus, we find  $S_x \wedge S_y = S_{z_i, i \in \mathcal{J}} \in Cl_s(L)$  and  $2 \leq |\mathcal{J}|$ . Applying the (2) of Definition 1, the (2) of Lemma 2 with Definition 2, we may obtain  $A(x \wedge y) \subseteq A(z_{i_0})$  for some  $i_0 \in \mathcal{J}$ . Considering this result with the atomistic property of  $L$  and  $2 \leq |\mathcal{J}|$ , we follow  $x \wedge y \leq z_{i_0} < \bigvee_{i \in \mathcal{J}} z_i \leq x, y$ . And further, we find  $S_{x \wedge y} < S_{z_{i_0}} \leq S_{\bigvee_{i \in \mathcal{J}} z_i} < S_x$ , or  $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\bigvee_{i \in \mathcal{J}} z_i} \leq S_y$  since  $S_x \neq S_y$ . This causes a contradiction to  $S_{x \wedge y} \prec S_x$  if  $S_{x \wedge y} < S_{z_{i_0}} \leq S_{\bigvee_{i \in \mathcal{J}} z_i} < S_x$ , and  $S_{x \wedge y} \prec S_y$  if  $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\bigvee_{i \in \mathcal{J}} z_i} \leq S_y$ .

In other words,  $S_x \wedge S_y = S_{x \wedge y}$  is true.

Therefore, we may be assured  $S_{x \wedge y} = S_x \wedge S_y \prec S_x, S_y$ .

Furthermore, since  $Cl_s(L)$  is geometric, we may point out  $S_x, S_y \prec S_x \vee S_y$ . Since  $S_x \vee S_y = S_{x \vee y}$  holds by  $x \wedge y \neq 0$  and item (3.3), we find  $x, y \prec x \vee y$  according to the given condition  $x \wedge y \prec x, y$  with item (3.10). That is,  $S_x, S_y \prec S_z < S_{x \vee y}$  does not hold for any  $z \in L$ . Actually, if there is  $p \in L \setminus (A(L) \cup 0)$  satisfying  $p < x \vee y$  and  $p \wedge x = 0$ , then  $S_{xp} \in Cl_s(L)$  holds. We may easily find  $S_x < S_{xp} < S_{x \vee y}$ , a contradiction to  $S_x \prec S_{x \vee y}$ . Therefore, for any  $p \in L \setminus (A(L) \cup 0)$  and  $p < x \vee y$ , it has  $p \wedge x \neq 0$ . Analogously,  $q \wedge y \neq 0$  holds for any  $q \in L \setminus (A(L) \cup 0)$  and  $q < x \vee y$ .

Step 7. To prove item (3.14).

Let  $y_j \in \mathcal{F}^2$ , that is,  $y_j$  has height 2 in  $L$ , ( $j = 1, 2, \dots, n; n \geq 2$ ). If  $S \prec S_{y_1 \dots y_n}$ . Then, taken  $S \prec S_{y_1 \dots y_n}$  and the (3) in Remark 1 together, we attain  $S = S_{z_1 \dots z_m}$  for

$z_i \in L \setminus (A(L) \cup 0)$ , ( $i = 1, \dots, m$ ). We may easily obtain  $z_1 = y_{i_1}, \dots, z_{n-1} = y_{i_{n-1}}$  in which  $\{i_1, \dots, i_{n-1}\} \subseteq \{1, \dots, n\}$ ,  $|\{i_1, \dots, i_{n-1}\}| = n - 1$  and  $m \leq n$ .

If  $m = n$ . Then  $z_n \leq y_{i_n}$ ,  $z_n \in L \setminus (A(L) \cup 0)$  and  $S < S_{y_1 \dots y_n}$  taken together causes  $z_n < y_{i_n}$  in which  $i_n = \{1, \dots, n\} \setminus \{i_1, \dots, i_{n-1}\}$ . But,  $h(y_{i_n}) = 2$  follows  $z_n \in A(L) \cup 0$ , a contradiction to  $z_n \in L \setminus (A(L) \cup 0)$ .

Therefore,  $m \leq n - 1$  is true. In addition, we may easily explore that if  $p < q \leq m$ , then  $S_{z_1 \dots z_p} < S_{z_1 \dots z_p z_{p+1} \dots z_q} \leq S_{z_1 \dots z_m} \leq S_{y_{i_1} \dots y_{i_{n-1}}} < S_{y_1 \dots y_n}$ . Hence,  $n - 1 \leq m$  is true.

That is to say,  $S = S_{y_{i_1} \dots y_{i_{n-1}}}$  holds.

Next, to prove the other part in item (3.14).

Suppose  $S < S_{y_1 \dots y_n}$  and  $S = S_z$  for some  $z \in L \setminus 0$ . Since both of  $y_{i_1} \notin A(L) \cup 0$  and  $S_0 < S_{y_{i_1}} < S_{y_{i_1} y_{i_2}} < \dots < S_{y_{i_1} \dots y_{i_{n-1}}} < S_{y_{i_1} \dots y_{i_{n-1}} z_n} < S_{y_1 \dots y_n}$  are true. Additionally, according to the geometry of  $Cl_s(L)$ , we may point out  $z \notin A(L) \cup 0$ . Furthermore, we follow  $z < y_{i_0}$  for one and only one  $i_0 \in \{1, \dots, n\}$  since  $S_z < S_{y_1 \dots y_n}$  and  $y_i \wedge y_j = 0$ , ( $i \neq j; i, j = 1, \dots, n; 2 \leq n$ ). Thus, by item (3.1), we arrive at  $S_z < S_{y_{i_0}} < S_{y_1 \dots y_n}$ . This is a contradiction to  $S_z < S_{y_1 \dots y_n}$ .

Suppose  $S = S_{z_1 \dots z_m}$  with  $2 \leq m$ . Then,  $z_j \in L \setminus (A(L) \cup 0)$ , ( $j = 1, \dots, m$ ) hold according to the (3) of Remark 1.

Since  $S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$  infers that for any  $i \in \{1, \dots, m\}$ , there is  $z_i \leq y_{j_i}$  for some  $j_i \in \{1, \dots, n\}$ .

If  $z_{i_t} < y_{j_{i_1}}$ , ( $t = 1, 2, \dots, m$ ). Then  $S_{z_1 \dots z_m} < S_{y_{i_1}} < S_{y_1 \dots y_n}$  holds according to Definition 2 and  $2 \leq n$ . This causes a contradiction to  $S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$ .

Additionally, combining  $z_j \in L \setminus (A(L) \cup 0)$ , ( $j = 1, \dots, m$ ) and  $y_p \wedge y_q = 0$ , ( $p \neq q; p, q = 1, \dots, n$ ), we decide that for any  $z_{j_0}$ , where  $j_0 \in \{1, \dots, m\}$ , there is one and only one  $y_{i_{j_0}}$  satisfying  $z_{j_0} \leq y_{i_{j_0}}$ , ( $i_{j_0} \in \{1, \dots, n\}$ ). If  $z_{i_1} \neq z_{i_2}$  but  $z_{i_1} \leq y_{j_{i_1}}$ ,  $z_{i_2} \leq y_{j_{i_1}}$  and  $z_{i_3} \leq y_{j_{i_3}}$  where  $i_3 \notin \{i_1, i_2\}$ , then  $z_{i_1} \vee z_{i_2} \leq y_{j_{i_1}}$  and  $0 \leq z_{i_3} \wedge (z_{i_1} \vee z_{i_2}) \leq y_{j_{i_1}} \wedge y_{j_{i_3}} = 0$ . We may obtain  $S_{z_1 \dots z_m} < S_{(z_{i_1} \vee z_{i_2}) z_{i_3} \dots z_{i_m}} \leq S_{y_1 \dots y_n}$ . This causes a contradiction to  $S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$ . Thus,  $z_i \leq y_i$  holds ( $i = 1, \dots, m$ ). In addition, if  $z_p \neq z_q$ , then  $y_p \neq y_q$ , ( $p, q \in \{1, \dots, m\}$ ). Hence, We may assume  $z_j \leq y_j$ , ( $j = 1, \dots, m$ ). So,  $m \leq n$  is true.

If  $m = n$ . Since  $z_j \leq y_j$  ( $j = 1, \dots, n$ ) and  $S < S_{y_1 \dots y_n}$  follow that there is  $i_0 \in \{1, \dots, m\} = \{1, \dots, n\}$  satisfying  $z_{i_0} < y_{i_0}$ . We determine that if there are more than two elements  $i_1, i_2 \in \{1, \dots, m\}$  satisfying  $z_{i_1} < y_{i_1}$  and  $z_{i_2} < y_{i_2}$ , then  $S_{z_1 \dots z_{i_1-1} z_{i_1} \dots z_{i_2} \dots z_n} < S_{z_1 \dots z_{i_1-1} y_{i_1} z_{i_1+1} \dots z_{i_2-1} y_{i_2} z_{i_2+1} \dots z_n} < S_{y_1 \dots y_n}$  holds. This is a contradiction to  $S < S_{y_1 \dots y_n}$ . Hence, there is one and only one  $i_0 \in \{1, \dots, m\}$  satisfying  $z_{i_0} < y_{i_0}$ . No matter to suppose  $i_0 = n$ .

When  $m = n$  and  $h(y_n) > 2$ . We find  $S_{y_1 \dots y_{n-1} z} \in Cl_s(L)$  and  $S_{y_1 \dots y_{n-1} z} < S_{y_1 \dots y_n}$  for any  $z < y_n$ . In addition, if there is  $M \in Cl_s(L)$  satisfying  $S_{y_1 \dots y_{n-1} z} < M < S_{y_1 \dots y_n}$ , then it is easily to obtain  $y_j \in M$ , ( $j = 1, \dots, n - 1$ ) and  $b \in M$  such that  $z < b < y_n$ . This causes a contradiction to  $z < y_n$ . Therefore,  $S = S_{y_1 \dots y_{n-1} z_n}$  holds where  $z_n < y_n$ .

When  $m \leq n$  and  $h(y_n) = 2$ .

Then, we will demonstrate  $m < n$  holds.

Otherwise,  $z_n < y_n$  infers to  $z_n \in A(L) \cup 0$ , a contradiction to  $z_n \in L \setminus (A(L) \cup 0)$ .

That is to say, this case will not happen actually.

Combining

$$\begin{aligned} p < q \leq m &\Rightarrow S_{z_1 \dots z_p} < S_{z_1 \dots z_p z_{p+1} \dots z_q} \\ &\leq S_{z_1 \dots z_m} \leq S_{y_1 \dots y_m} \leq S_{y_1 \dots y_m y_{m+1} \dots y_{n-1}} \leq S_{y_1 \dots y_n} \end{aligned}$$

and  $z_j < y_j$ , ( $j = 1, \dots, m$ ) with  $S_{z_1 \dots z_m} = S < S_{y_1 \dots y_n}$  and the geometry of  $Cl_s(L)$ , we decide  $n - 1 \leq m$ .

If  $m = n - 1$ . That is,  $|\{1, \dots, m\}| = n - 1$  holds. No matter to suppose  $\{1, \dots, m\} = \{1, \dots, n - 1\}$ . Considered  $z_j \leq y_j$ , ( $j = 1, \dots, m$ ) and  $S = S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$ , we may confirm  $z_j = y_j$ , ( $j = 1, \dots, m; m = n - 1$ ).

We will demonstrate  $h(y_n) = 2$ .

Otherwise,  $h(y_n) > 2$  is true. But, using the definition of height function in a lattice, we affirm the existence of  $b \in L \setminus 0$  satisfying  $b < y_n$  and  $h(b) = h(y_n) - 1 \geq 2$ . In addition, it is easily found  $S_{z_1 \dots z_m} < S_{z_1 \dots z_m b} < S_{y_1 \dots y_n}$ . This follows a contradiction to  $S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$ .

When  $m = n - 1$  and  $h(y_n) = 2$ . Since  $S_{z_1 \dots z_m} \leq S_{y_1 \dots y_m y_{m+1} \dots y_{n-1}} < S_{y_1 \dots y_n}$  holds in light of  $z_j \leq y_j$ , ( $j = 1, \dots, m$ ) and the given condition  $S_{z_1 \dots z_m} < S_{y_1 \dots y_n}$ . We may attain  $S_{y_1 \dots y_{n-1}} \leq S$ . If  $S_{y_1 \dots y_{n-1}} < S < S_{y_1 \dots y_n}$  holds, then there is  $b \in S \setminus (A(L) \cup \{y_1, \dots, y_{n-1}\})$  satisfying  $b < y_n$ . However,  $h(y_n) = 2$  and  $b < y_n$  together causes  $b \in A(L) \cup 0$ , a contradiction to  $b \in S \setminus (A(L) \cup \{y_1, \dots, y_{n-1}\})$ . Therefore,  $S = S_{y_1 \dots y_{n-1}}$  holds.

Step 8. To prove item (3.15).

Under the supposition of  $(\alpha)$ ,  $x \wedge y_{i_n} = 0$  will not happen since  $w_n \neq 0$ ,  $w_n \leq x$  and  $w_n < y_{i_n}$ . That is to say, there is item (3.15.1) and only item (3.15.1) to happen. According to the given conditions, items (3.5) and (3.14) with (2) in Definition 1, we affirm  $S_{y_1 \dots y_{n-1} w_n} < S_{y_1 \dots y_n}, S_x$ . Thus, under the supposition of  $(\alpha)$ , we may be assured  $S_{y_{i_1} \dots y_{i_{n-1}} w_n} = S_x \wedge S_{y_1 \dots y_n} < S_x, S_{y_1 \dots y_n}$ . Furthermore, by the geometry of  $Cl_s(L)$ , we find  $S_x, S_{y_1 \dots y_n} < S_x \vee S_{y_1 \dots y_n}$ .

If the supposition of  $(\beta)$  happens. Then we find  $S_{y_{j_1} \dots y_{j_{n-1}}} = S_z \wedge S_{y_1 \dots y_n} < S_z, S_{y_1 \dots y_n}$  according to item (3.14) and the (2) in Definition 1. Thus, by the geometry of  $Cl_s(L)$ , we affirm  $S_z, S_{y_1 \dots y_n} < S_z \vee S_{y_1 \dots y_n}$ .

Step 8.1. To prove item (3.15.1).

Let  $x \wedge y_{i_n} \neq 0$ . We will prove the needed results as the following two parts.

Part I. We prove  $S_{x \vee y_{i_n}} = S_x \vee S_{y_1 \dots y_n}$ .

Let  $S_x \vee S_{y_1 \dots y_n} = S_{z_1 \dots z_m}$  for some  $z_j \in L \setminus 0$ .

The result in item (3.1) and  $x, y_j \in L \setminus (A(L) \cup 0)$  together follows  $z_i \notin A(L) \cup 0$ , ( $j = 1, \dots, n; i = 1, \dots, m$ ). This implies  $y_{i_j} \leq x \leq z_1$ , ( $j = 1, \dots, n - 1$ ) and  $y_{i_n} \leq z_2$ . If  $z_1 \neq z_2$ , then  $x \wedge y_{i_n} \leq z_1 \wedge z_2 = 0$ . This follows  $x \wedge y_{i_n} = 0$ , a

contradiction to  $x \wedge y_{i_n} \neq 0$ . Thus, we produce  $z_1 = z_2$ . According to item (3.1) and  $S_x, S_{y_1 \dots y_n} \leq S_{z_1} \leq S_{z_1 \dots z_m} = S_x \vee S_{y_1 \dots y_n}$ , we confirm  $S_x \vee S_{y_1 \dots y_n} = S_{z_1}$ . Therefore, we obtain  $S_x, S_{y_1 \dots y_n} \prec S_x \vee S_{y_1 \dots y_n} = S_{z_1}$ .

Part II. We will prove the other needed results in item (3.15.1).

In light of the result in Part I,  $x \vee (y_1 \vee \dots \vee y_n) = x \vee y_{i_n}$  and item (3.1), we find  $S_{x \vee y_{i_n}} = S_{z_1}$ . Combining this result with  $S_x \prec S_x \vee S_{y_1 \dots y_n} = S_z$  and  $w_n \prec y_{i_n}$ , we may easily affirm  $x \prec x \vee y_{i_n}$ . In addition,  $S_{y_1 \dots y_n} \prec S_{x \vee y_{i_n}} = S_{\bigvee_{j=1}^n y_j}$  illustrates that no element  $S \in Cl_s(L)$  satisfies  $S_{y_1 \dots y_n} \prec S \prec S_{\bigvee_{j=1}^n y_j}$ . According to item (3.1) and the (3) of Remark 1, we may state that for any  $m \leq n$  and  $\bigvee_{j=1}^m y_{i_{t_j}} \leq q \prec \bigvee_{j=1}^n y_j$ , there exists  $s \in \{1, \dots, n\} \setminus \{i_{t_j} \mid j = 1, \dots, m\}$  satisfying  $q \wedge y_s \neq 0$  where  $\{i_{t_j} \mid j = 1, \dots, m\} \subset \{1, \dots, n\}$ .

Step 8.2. Item (3.15.2) may be verified by similar way to that for item (3.15.1).

Step 9. To prove item (3.16).

Considering the known conditions, item (3.1), the (2) in Definition 1 with Definition 2, we obtain  $S_{y_1 \dots y_{n-1}} \prec S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_{n-1} y_n}$ . Suppose that there are  $S_2, S_3 \in Cl_s(L)$  satisfying  $S_{y_1 \dots y_{n-1}} \prec S_2 \prec S_{y_1 \dots y_{n-1} x_n}$  and  $S_{y_1 \dots y_{n-1}} \prec S_3 \prec S_{y_1 \dots y_{n-1} y_n}$ . Then, using items (3.5) and (3.14), we may obtain the following four statements.

if  $h(x_n) > 2$ , then  $S_2 = S_{y_1 \dots y_{n-1} z_2}$  where  $z_2 \prec x_n$ ;

if  $h(x_n) = 2$ , then  $S_2 = S_{y_1 \dots y_{n-1}}$ ;

if  $h(y_n) > 2$ , then  $S_3 = S_{y_1 \dots y_{n-1} z_3}$  where  $z_3 \prec y_n$ ;

if  $h(y_n) = 2$ , then  $S_3 = S_{y_1 \dots y_{n-1}}$ .

Therefore, if there is  $S \in Cl_s(L)$  satisfying  $S \prec S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_{n-1} y_n}$ , then there is  $S = S_{y_1 \dots y_{n-1} z_n}$  when  $z_n = x_n \wedge y_n \notin A(L) \cup 0$  and  $z_n \prec x_n, y_n$ ; or there is  $S = S_{y_1 \dots y_{n-1}}$  when  $x_n \wedge y_n \in A(L) \cup 0$ .

Step 9.1. To verify item (3.16.1).

Since  $z_n \prec x_n$  and  $y_n$  taken together follows  $z_n = x_n \wedge y_n$ . Hence, in view of  $h(x_n \wedge y_n) \neq 0$ , there is  $S_{y_1 \dots y_{n-1}} \prec S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n}$  if  $x_n \wedge y_n \in A(L)$ , and in addition, there is  $S_{y_1 \dots y_{n-1} z_n} \prec S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n}$  if  $z_n \in L \setminus (A(L) \cup 0)$ . No matter which of the above cases happens, using the geometry of  $Cl_s(L)$  and Lemma 1, we always obtain  $S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n} \prec S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n}$ . Utilizing the (3) in Remark 1 and Definition 2, we may easily gain  $S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$ . Thus, we arrive at  $S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n} \prec S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$ . Therefore, for any  $p, q \in L \setminus (A(L) \cup 0)$ ,  $x_n \prec p \prec x_n \vee y_n$  and  $y_n \prec q \prec x_n \vee y_n$ , there are  $s, t \in \{1, \dots, n-1\}$  satisfying  $y_s \wedge p \neq 0$  and  $y_t \wedge q \neq 0$ .

Analogously to the proof above, we may easily obtain the ‘‘additionally’’ part in item (3.16.1).

Step 9.2. To prove item (3.16.2).

Using item (3.16.1) and the induction on  $n - (m + 1)$ , similarly to the proof in Step 9.1, we may easily obtain item (3.16.2).  $\square$

*Remark 2.* In the proof of Step 7 for Lemma 6, for the case “when  $m = n - 1$  and  $h(y_n) = 2$ ”, we may use item (3.5) to obtain the same result.

But, we think that the proof in Lemma 6 for this case is useful to prove the other cases.

Third, we will reveal under what conditions,  $Cl_s(L)$  is geometric for an atomistic complete finite lattice  $L$ .

**Lemma 7.** *Let  $L$  be an atomistic complete finite lattice. If  $L$  satisfies items from (3.8) to (3.16), then  $Cl_s(L)$  is a geometric lattice.*

*Proof of Lemma 7.* Applying the information in Subsection 2.1 and Lemma 3, we only need to prove that  $Cl_s(L)$  is atomistic and semimodular.

According to Lemma 4, we may be assured  $A(Cl_s(L)) = \{S_d \mid d \in \mathcal{F}^2\}$ .

Step 1. We prove that every element in  $Cl_s(L)$  is a join of atoms using the following Step 1.1 and Step 1.2.

Step 1.1. To prove:  $S_x = \bigvee_{d \in \mathcal{F}^2(x)} S_d$  for any  $x \in L \setminus (A(L) \cup 0)$ .

Let  $x \in \mathcal{F}^2$ . The needed result is easily followed.

Let  $x \in L \setminus (A(L) \cup 0)$  and  $h(x) = 3$ . By items (3.8) and (3.11), we find  $2 \leq |\mathcal{F}^2(x)|$ . Let  $d_1, d_2 \in \mathcal{F}^2(x)$  and  $d_1 \wedge d_2 \neq 0$ . Then in light of item (3.3), we may obtain  $S_{d_1} \vee S_{d_2} = S_{d_1 \vee d_2} = S_x$ . Furthermore, we may decide  $S_x = \bigvee_{d \in \mathcal{F}^2(x)} S_d$  since  $S_d \leq S_x$  for any  $d \in \mathcal{F}^2(x)$ .

Suppose that there is  $S_x = \bigvee_{d \in \mathcal{F}^2(x)} S_d$  for any  $x \in L \setminus (A(L) \cup 0)$  and  $h(x) \leq n - 1$ .

Let  $x \in L \setminus (A(L) \cup 0)$  with  $h(x) = n$ . Let  $\mathcal{C}(x) = \{y_i \mid i \in \mathcal{I}\}$ , that is,  $\mathcal{C}(x)$  is the set of elements covered by  $x$  in  $L$ .

In view of item (3.8), we produce  $x = \bigvee_{i \in \mathcal{I}} y_i$ . Since  $L$  satisfies item (3.12). We obtain  $S_{y_1} \vee S_{y_2} = S_{y_1 \vee y_2} = S_x$  using item (3.3), where  $y_1, y_2 \in \mathcal{C}(x) \setminus A(L)$ ,  $y_1 \neq y_2$  and  $y_1 \wedge y_2 \neq 0$ . On the other hand, by item (3.1), we confirm  $S_y \leq S_x$  for any  $y \in \mathcal{C}(x) \setminus A(L)$ .

Moreover, combining the above results, we attain  $S_x = \bigvee_{y \in \mathcal{C}(x)} S_y$ .

Considered  $y \in \mathcal{C}(x)$  with  $h(y) = n$ , we find  $h(y) \leq n - 1$ . Using inductive supposition, we will obtain  $S_y = \bigvee_{d \in \mathcal{F}^2(y)} S_d$ . Additionally, we may easily find that if  $d \in \mathcal{F}^2(x)$ , then  $d \in \mathcal{C}(x) \setminus A(L)$  or  $d \leq y_d$  for some  $y_d \in \mathcal{C}(x)$ . That is to say,  $d \in \mathcal{F}^2(y)$  holds for some  $y \in \mathcal{C}(x) \setminus A(L)$ . Therefore, we arrive at  $S_x = \bigvee_{d \in \mathcal{F}^2(x)} S_d$ .

Step 1.2. To prove:  $S_{x_i, i \in T} \in Cl_s(L)$  is a join of atoms where  $|T| \geq 2$ .

Let  $S = S_{x_t, t \in \mathcal{T}} \in Cl_s(L)$  with  $2 \leq |\mathcal{T}|$ . Using (2) of Definition 1 and item (3.2), we may find  $S = \bigvee_{t \in \mathcal{T}} S_{x_t}$ . For every  $S_{x_t}$ , applying with Step 1.1, we may obtain  $S_{x_t} = \bigvee_{d_j \in \mathcal{F}^2(x_t)} S_{d_j}$  where  $\mathcal{F}^2(x_t) = \{d_j \mid j \in \mathcal{J}_t\}, (t \in \mathcal{T})$ . Therefore, we attain  $S = \bigvee_{t \in \mathcal{T}} \bigvee_{j \in \mathcal{J}_t} S_{d_j}$ .

Step 2. We prove that  $Cl_s(L)$  is semimodular.

Let  $S_2, S_3 \in Cl_s(L)$ ,  $S_2 || S_3$  and  $S_2 \wedge S_3 < S_2, S_3$ . According to the (3) of Remark 1, we may point that there are  $S_2 = S_x$  or  $S_2 = S_{x_j, j \in \mathcal{J}}$ , and in addition,  $S_3 = S_y$  or  $S_3 = S_{y_i, i \in \mathcal{I}}$ , for some  $x, y \in L \setminus 0$  and  $x_j, y_i \in L \setminus (A(L) \cup 0)$ , ( $j \in \mathcal{J}, i \in \mathcal{I}$ ) with  $|\mathcal{J}|, |\mathcal{I}| \geq 2$ . Based on this statement, we will divide different cases to prove  $S_2, S_3 < S_2 \vee S_3$  by the following Steps 2.1, 2.2 and 2.3.

Step 2.1. Assume  $S_2 = S_x$  and  $S_3 = S_y$  for some  $x, y \in L \setminus 0$ .

Since  $S_0 \leq S_2 \wedge S_3 < S_2, S_3$  and  $S_2 \wedge S_3 \in Cl_s(L)$ , we may affirm  $x, y \in L \setminus (A(L) \cup 0)$ .

We will distinguish two cases to fulfill the proof.

Case 1. Suppose  $S_2 \wedge S_3 = S_0$ .

Then,  $S_x, S_y \in A(Cl_s(L))$  holds. Thus, by item (3.4), we believe  $x, y \in \mathcal{F}^2$ .

When  $x \wedge y = 0$ . Using item (3.2), we obtain  $S_x, S_y < S_{xy} = S_x \vee S_y$ . If  $S_x < S < S_{xy}$  holds for some  $S \in Cl_s(L)$ , then  $S = S_{xp}$  holds where  $p \in L \setminus (A(L) \cup 0)$  and  $p < y$ . However,  $h(y) = 2$  and  $p < y$  taken together follows  $p \in A(L) \cup 0$ . This is a contradiction to  $p \in L \setminus (A(L) \cup 0)$ . Thus,  $S_x < S_{xy}$  holds. Analogously,  $S_y < S_{xy}$  is true.

When  $x \wedge y \neq 0$ . Using item (3.3), we affirm  $S_x \vee S_y = S_{x \vee y}$ . In light of  $x, y \in \mathcal{F}^2$  and  $S_2 \wedge S_3 = S_0$ , we find  $x \wedge y \in A(L)$ . So,  $x \wedge y < x, y$  is followed. Using item (3.10), we may affirm  $x, y < x \vee y$ . Moreover,  $h(x \vee y) = h(x) + 1 = 3$  holds. Hence, it follows  $S_x, S_y < S_{x \vee y}$ . Therefore, we may obtain  $S_x, S_y < S_x \vee S_y$ .

Case 2. Suppose  $S_x \wedge S_y \neq S_0$ .

Then,  $3 \leq h(x), h(y)$  hold since  $S_x \wedge S_y < S_x, S_y$ .

By item (3.1), it is easily to find  $S_{x \wedge y} \leq S_x \wedge S_y$ .

We will demonstrate  $S_x \wedge S_y = S_{x \wedge y}$ .

Otherwise,  $S_{x \wedge y} < S_x \wedge S_y$  holds. If  $S_x \wedge S_y = S_z$  for some  $z \in L$ , then  $x \wedge y < z < x, y$ . According to item (3.1), we may achieve  $S_{x \wedge y} < S_z < S_x, S_y$ , a contradiction to  $S_{x \wedge y} < S_x, S_y$ . That is to say, we attain  $S_x \wedge S_y = S_{z_i, i \in \mathcal{I}} \in Cl_s(L)$  and  $2 \leq |\mathcal{I}|$ . By the (2) of Definition 1, the (2) of Lemma 2, and Definition 2, we may obtain  $A(x \wedge y) \subseteq A(z_{i_0})$  for some  $i_0 \in \mathcal{I}$ . Considering this result with the atomistic property of  $L$  and  $2 \leq |\mathcal{I}|$ , we provide  $x \wedge y \leq z_{i_0} < \bigvee_{i \in \mathcal{I}} z_i \leq x, y$ , and further,  $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\bigvee_{i \in \mathcal{I}} z_i} < S_x$ , or  $S_{x \wedge y} \leq S_{z_{i_0}} < S_{\bigvee_{i \in \mathcal{I}} z_i} \leq S_y$  since  $S_x \neq S_y$ . No matter which of the above cases to happen, it causes a contradiction to  $S_{x \wedge y} < S_x$  or  $S_{x \wedge y} < S_y$ .

In other words,  $S_x \wedge S_y = S_{x \wedge y}$  is real.

In fact,  $S_x \wedge S_y \neq S_0$  and  $S_x \wedge S_y = S_{x \wedge y}$  taken together infers to  $h(x \wedge y) \geq 2$ .

Moreover,  $x \wedge y < x, y$  holds according to  $S_{x \wedge y} = S_x \wedge S_y < S_x, S_y$ . Combining with item (3.10), we may get  $x, y < x \vee y$ . Hence, since  $L$  satisfies item (3.13) and there are  $S_{x \wedge y} < S_x, S_y$ , we may determine  $S_x, S_y < S_{x \vee y}$ . Additionally,  $S_0 \neq S_x \wedge S_y$  and  $S_x \wedge S_y = S_{x \wedge y}$  follow  $x \wedge y \neq 0$ . According to item (3.3), we find  $S_{x \vee y} = S_x \vee S_y$ .

Therefore, we decide  $S_x, S_y < S_x \vee S_y$ .



Step 2.2. Assume  $S_2 = S_x$  for some  $x \in L \setminus 0$  and  $S_3 = \{y_j \mid j \in \mathcal{Y}\} \cup \{a \in A(L) \mid a \not\leq y_j, \forall j \in \mathcal{Y}\} \in Cl_s(L)$  with  $2 \leq |\mathcal{Y}|$ .

Combining items (3.5) and (3.14) with  $S_2 \wedge S_3 < S_3$ , we may state that there is  $\mathcal{Y}_3 \subset \mathcal{Y}$  satisfying  $|\mathcal{Y}_3| + 1 = |\mathcal{Y}|$  and  $S_2 \wedge S_3 \geq \{y_j \mid j \in \mathcal{Y}_3\} \cup \{a \in A(L) \mid a \not\leq y_j, \forall j \in \mathcal{Y}_3\}$ . According to  $S_2 \wedge S_3 < S_x$ , we may indicate  $x \geq \bigvee_{j \in \mathcal{Y}_3} y_j$ . No matter to assume  $\mathcal{Y} = \{1, 2, \dots, n\}$  and  $\mathcal{Y}_3 = \{1, 2, \dots, n-1\}$ . In view of item (3.14) and  $S_2 \wedge S_3 < S_{y_1 \dots y_n} = S_3$ , we may obtain that

if  $h(y_n) = 2$ , then  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1}}$ ;

if  $h(y_n) \geq 3$ , then  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1} z_n}$  where  $z_n < y_n$  and  $z_n \in L \setminus (A(L) \cup 0)$ .

Suppose  $h(y_n) = 2$ . Then  $S_{y_1 \dots y_{n-1}} < S_x, S_{y_1 \dots y_n}$  produce that  $x$ , which  $x = \bigvee_{j=1}^n y_j$ , satisfies: for any  $p \in L \setminus (A(L) \cup 0)$  with  $\bigvee_{j=1}^m y_{i_j} \leq p < x$ , there is  $y_s \wedge p \neq 0$  for some  $s \in \{1, \dots, n-1\} \setminus \{i_j \mid j = 1, \dots, m\}$  where  $\{i_j \mid j = 1, \dots, m\} \subseteq \{1, \dots, n-1\}$ . We will use the following two statuses to fulfill the proof.

Status 1. When  $x \wedge y_n = 0$ .

Then,  $S_{xy_n} \in Cl_s(L)$  is obtained from the (2) of Definition 1. Hence, it follows  $S_x, S_{y_1 \dots y_n} \leq S_{xy_n}$ . Moreover, we may be assured  $S_x \vee S_{y_1 \dots y_n} \leq S_{xy_n}$ .

Considering the (3) of Remark 1 and  $S_{y_1} < S_x < S_x \vee S_{y_1 \dots y_n}$ , we may suppose  $S_x \vee S_{y_1 \dots y_n} = S_b$  for some  $b \in L \setminus (A(L) \cup 0)$ . Evidently, this supposition will cause  $S_{xy_n} \leq S_b$ . Thus, we decide  $S_{xy_n} = S_x \vee S_{y_1 \dots y_n} = S_b$ . This is a contradiction to  $x \neq y_n$  and  $|\{b\}| = 1$ .

On the other hand, we may also suppose  $S_x \vee S_{y_1 \dots y_n} = S_{z_1 \dots z_m}$  where  $2 \leq m$  according to the (3) of Remark 1. This supposition will cause  $x \leq z_{t_1}$  and  $y_n \leq z_{t_2}$  in which  $z_{t_1}, z_{t_2} \in \{z_1, \dots, z_m\}$ . Considered both of the (1) and (2) in Lemma 2 with Definition 2, we may determine  $S_{xy_n} \leq S_{z_{t_1} z_{t_2}} \leq S_{z_1 \dots z_m}$ . Thus, it follows  $S_{xy_n} = S_{z_{t_1} z_{t_2}} = S_x \vee S_{y_1 \dots y_n}$ . If  $z_{t_1} = z_{t_2}$ , then  $S_{xy_n} = S_{z_{t_1}}$ . This expression transfers to the above case. But the above case shows that this expression is wrong. In other words,  $z_{t_1} \neq z_{t_2}$  is true.

Therefore, we obtain  $S_x \vee S_{y_1 \dots y_n} = S_{xy_n}$ .

If  $S_x < S < S_{xy_n}$  for some  $S \in Cl_s(L)$ . Then  $x \in S$  and  $b_n \in S$ , where  $b_n \in L \setminus (A(L) \cup 0)$  and  $b < y_n$ . However,  $h(y_n) = 2$  follows the non-existence of  $b_n$ . Thus, there does not exist  $S \in Cl_s(L)$  satisfying  $S_x < S < S_{xy_n}$ . That is to say,  $S_x < S_{xy_n}$  holds.

If  $S_{y_1 \dots y_n} < S < S_{xy_n}$  for some  $S \in Cl_s(L)$ . Then  $y_n \in S, \bigvee_{t=1}^u y_{j_t} \leq q \in S, \{j_t \mid t = 1, \dots, u\} \subseteq \{1, \dots, n-1\}, 2 \leq |\{j_t \mid t = 1, \dots, u\}|$  and  $A(y_j) \subseteq A(z_{j_p})$  for  $z_{j_p} \in S, (j \in \{1, \dots, n-1\} \setminus \{j_t \mid t = 1, \dots, u\})$  and  $|\{j_p \mid p = 1, \dots, v\}| \leq |\{1, \dots, n-1\} \setminus \{j_t \mid t = 1, \dots, u\}|$ . In addition, we easily find  $(\bigvee_{t=1}^u y_{j_t}) \wedge z_{j_p} = 0$  for every  $z_{j_p} \in S, (p = 1, \dots, v)$ . However, we already know that there exists  $s \in \{1, \dots, n-1\} \setminus \{j_t \mid t = 1, \dots, u\}$  satisfying  $(\bigvee_{t=1}^u y_{j_t}) \wedge y_s \neq 0$ . That is to say, it does not exist  $S \in Cl_s(L)$  satisfying  $S_{y_1 \dots y_n} < S < S_{xy_n}$ . Thus, there is  $S_{y_1 \dots y_n} < S_{xy_n} = S_x \vee S_{y_1 \dots y_n}$ .

Status 2. When  $x \wedge y_n \neq 0$ .

Actually, using item (3.15.2), we will attain  $S_{x \vee y_n} = S_x \vee S_{y_1 \dots y_n}$  and  $S_x, S_{y_1 \dots y_n} < S_{x \vee y_n}$ .

Suppose  $h(y_n) \neq 2$ . Then,  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1} z_n}$  is true where  $z_n < y_n$ . Additionally,  $S_2 \wedge S_3 < S_x$  follows  $x = \bigvee_{j=1}^{n-1} y_j \vee z_n$ . So,  $z_n = x \wedge y_n \in L \setminus (A(L) \cup 0)$  holds. This implies  $x \wedge y_n \neq 0$ .

We will prove  $S_x \vee S_{y_1 \dots y_n} = S_{x \vee y_n}$ .

It is easily found  $S_x, S_{y_1 \dots y_n} \leq S_{x \vee y_n}$  since item (3.1). Thus, we obtain  $S_x \vee S_{y_1 \dots y_n} \leq S_{x \vee y_n}$ . Assume  $S_x \vee S_{y_1 \dots y_n} = S_{b_1 \dots b_m}$  for  $b_i \in L \setminus (A(L) \cup 0)$ , ( $i = 1, \dots, m$ ). By item (3.1), it follows  $y_j \leq x < b_1$ , ( $j = 1, \dots, n-1$ ) and  $y_n \leq b_t$  for some  $t \in \{1, \dots, m\}$ . If  $t \neq 1$ , then  $0 \leq x \wedge y_n \leq b_1 \wedge b_t = 0$ . So,  $x \wedge y_n = 0$  is followed. This causes a contradiction to  $x \wedge y_n \neq 0$ . Moreover, we gain  $y_j \leq x \leq b_1$  and  $y_n \leq b_1$ , ( $j = 1, \dots, n-1$ ). Thus,  $m = 1$  holds. In addition, according to item (3.1) and  $x \vee y_n \leq b_1$ , we may present  $S_x \vee S_{y_1 \dots y_n} \leq S_{x \vee y_n} \leq S_{b_1}$ . Hence,  $S_{x \vee y_n} = S_x \vee S_{y_1 \dots y_n}$  holds.

Next, according to  $S_{y_1 \dots y_{n-1} z_n} < S_x, S_{y_1 \dots y_n}$ , we may indicate that for any  $\bigvee_{j=1}^t y_{ij} \leq p < x$  or  $\bigvee_{j=1}^t y_{ij} \vee z_n \leq q < x$  where  $\{i_j \mid j = 1, \dots, t\} \subseteq \{1, \dots, n-1\}$ , there exists  $s \in \{1, \dots, n-1\} \setminus \{i_j \mid j = 1, \dots, t\}$  satisfying  $y_s \wedge p \neq 0$  or  $z_n \wedge p \neq 0$  or  $y_s \wedge q \neq 0$ . Using items (3.15.1) and (3.1), it follows  $S_2, S_3 < S_2 \vee S_3$ .

Step 2.3. Let  $S_2 = S_{x_1 x_2 \dots x_m}$  and  $S_3 = S_{y_1 y_2 \dots y_n}$ . By item (3.5) and  $S_2 \wedge S_3 < S_2, S_3$ , we follow  $S_2 \wedge S_3 \geq S_{y_{i_1} \dots y_{i_{n-1}}} = S_{x_{j_1} \dots x_{j_{m-1}}} = S_{\{y_1, \dots, y_n\} \cap \{x_1, \dots, x_m\}}$  and  $|\{x_1, \dots, x_m\} \cap \{y_1, \dots, y_n\}| = n-1$ . No matter to suppose  $\{y_1, \dots, y_n\} \cap \{x_1, \dots, x_m\} = \{y_1, y_2, \dots, y_{n-1}\}$ . Thus, we may reveal  $n = m, x_m \neq y_n$ , and  $S_2 = S_{y_1 y_2 \dots y_{n-1} x_n}$ ,  $S_3 = S_{y_1 y_2 \dots y_{n-1} y_n}$ .

We will demonstrate

$$h(x_n) = 2 \text{ if and only if } h(y_n) = 2.$$

In fact, if  $h(y_n) = 2$ , then by item (3.14) and  $S_2 \wedge S_3 < S_{y_1 \dots y_n}$ , there is  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1}}$ . Meanwhile,  $S_{y_1 \dots y_{n-1}} = S_2 \wedge S_3 < S_2 = S_{y_1 \dots y_{n-1} x_n}$  follows  $h(x_n) = 2$  by item (3.14).

Similarly, if  $h(x_n) = 2$ , then  $h(y_n) = 2$ .

Additionally, if  $h(y_n) = 2$ , then according to items (3.5) and (3.14) and  $h(x_n) = 2$ , we obtain  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1}} < S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_{n-1} y_n}$  holds.

When  $h(y_n) = 2$  and  $x_n \wedge y_n = 0$ . We will prove  $S_{y_1 \dots y_n x_n} = S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1} x_n} = S_2 \vee S_3$ .

Suppose  $S \in Cl_S(L)$  and  $S_{y_1 \dots y_{n-1} x_n} < S < S_{y_1 \dots y_n x_n}$ . Then, by  $y_p \wedge y_q = 0, x_n \wedge y_p = 0$ , ( $p \neq q; p, q = 1, \dots, n$ ), it follows  $y_j \in S$ , ( $j = 1, \dots, n-1$ ) and  $x_n \in S$ . Additionally, by the supposition, we decide that there is  $z \in L \setminus (A(L) \cup 0)$  with  $z < y_n$  satisfying  $z \wedge y_j = z \wedge x_n = 0$ , ( $j = 1, \dots, n-1$ ). This implies  $2 \leq h(z) < h(y_n)$ , a contradiction to  $h(y_n) = 2$ . Moreover,  $S_{y_1 \dots y_{n-1} x_n} < S_{y_1 \dots y_n x_n}$  is real. Analogously,  $S_{y_1 \dots y_n} < S_{y_1 \dots y_n x_n}$  is true. Hence,  $S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1} x_n} = S_{y_1 \dots y_n x_n}$  is true and  $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1} x_n} < S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1} x_n}$  holds.

When  $h(y_n) = 2$  and  $x_n \wedge y_n \neq 0$ .

Since  $h(x_n) = h(y_n) = 2$  and  $x_n \wedge y_n \neq 0$  taken together follows  $x_n \wedge y_n < x_n, y_n$  and  $x_n \wedge y_n \in A(L)$ .

We prove  $S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$  if  $y_j \wedge (x_n \vee y_n) = 0, (j = 1, \dots, n - 1)$ .

Using the (3) in Remark 1, item (3.1) and Definition 1, it is easily to obtain  $S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n} = S_{z_1 \dots z_m}, (z_i \in L \setminus (A(L) \cup 0); i = 1, \dots, m)$ . According to  $z_t \wedge z_s = 0, (t, s = 1, \dots, m; t \neq s)$ , we follow  $y_j = z_j, (j = 1, \dots, n - 1)$ . In view of  $x_n \wedge y_n \neq 0$ , we attain  $x_n, y_n \leq z_n$ . And further, we gain  $x_n \vee y_n \leq z_n$ . Moreover, we decide  $S_{y_1 \dots y_{n-1} (x_n \vee y_n)} \leq S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1} x_n}$ . On the other hand, combining  $x_n, y_n \leq x_n \vee y_n$  with items (3.1) and (3.2), we may easily arrive at  $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1} x_n} \leq S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$ . So,

$$S_{y_1 \dots y_n} \vee S_{y_1 \dots y_{n-1} x_n} \leq S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$$

is followed.

Combining the above, we affirm  $S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$ .

Using the ‘‘Additionally’’ part in item (3.16.1), we obtain  $S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n} < S_{y_1 \dots y_{n-1} (x_n \vee y_n)}$ . Furthermore, we find  $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1} x_n} < S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n}$ .

If  $y_j \wedge (x_n \vee \bigvee_{i=m+1}^n y_i) = 0, (j = 1, \dots, m)$ . Then, using induction on  $n - (m + 1)$  and the above proof for the case of  $y_j \wedge (x_n \vee y_n) = 0, (j = 1, \dots, n - 1)$ , we may gain  $S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n} = S_{y_1 \dots y_m (x_n \vee \bigvee_{i=m+1}^n y_i)}$ . Applying the ‘‘Additionally’’ part in item (3.16.2), we gain  $S_{y_1 \dots y_n}, S_{y_1 \dots y_{n-1} x_n} < S_{y_1 \dots y_{n-1} x_n} \vee S_{y_1 \dots y_n}$ .

When  $h(y_n) > 2$ . Applying with the result above, we find  $h(x_n) > 2$ . In virtue of  $S_2 \wedge S_3 < S_{y_1 \dots y_n} = S_3$  and item (3.14), we obtain  $S_2 \wedge S_3 = S_{y_1 \dots y_{n-1} z_n}$  in which  $z_n < y_n$  and  $z_n \in L \setminus (A(L) \cup 0)$ . At the same time, we may attain  $z_n < x_n$  since  $S_2 \wedge S_3 < S_{y_1 \dots y_{n-1} x_n}$ . This follows  $z_n = x_n \wedge y_n \in L \setminus (A(L) \cup 0)$ . Furthermore, we reveal  $S_{y_1 \dots y_{n-1} z_n} < S_{y_1 \dots y_{n-1} x_n}, S_{y_1 \dots y_n}$ . Thus, for any  $\bigvee_{i=1}^{m_x} y_{j_i} \leq p < x_n, \bigvee_{i=1}^{m_y} y_{k_i} \leq q < y_n$ , and  $\{j_i \mid i = 1, \dots, m_x\}, \{k_i \mid i = 1, \dots, m_y\} \subseteq \{1, \dots, n - 1\}$ , there is  $s_x \in \{1, \dots, n - 1\} \setminus \{j_i \mid i = 1, \dots, m_x\}, s_y \in \{1, \dots, n - 1\} \setminus \{k_i \mid i = 1, \dots, m_y\}$  satisfying  $y_{s_x} \wedge p \neq 0, y_{s_y} \wedge q \neq 0$ . Owing to items (3.16.2) and (3.1), we confirm  $S_2, S_3 < S_2 \vee S_3 = S_{y_1 \dots y_m (x_n \vee \bigvee_{j=m+1}^n y_j)}$ .  $\square$

Combining Lemma 6 with Lemma 7, we may express the following theorem.

**Theorem 2.** *Let  $L$  be an atomistic complete finite lattice with  $|\mathcal{F}^2| \geq 2$ . Then  $Cl_s(L)$  is a geometric lattice if and only if  $L$  satisfies items from (3.8) to (3.16).*

#### 4. CONCLUSION

To sum up our results, we make the following remarks.

(1) Though some of conditions in items from (3.8) to (3.16) seem to be complex, they are actually expressed in a detailed and applicable way. In addition they complete the check process, since items from (3.8) to (3.16) are suitable for finite cases. Additionally, from Lemma 7, or from the results of Section 2, we can confirm that items from (3.8) to (3.16) are necessary and essential when we decide the geometry of  $Cl_S(L)$  for an atomistic complete finite lattice  $L$ .

(2) It is well known that an atomistic complete lattice is finite or infinite. Theorem 1 and Theorem 2 together answer the open problem of S.Radeleczki for finite cases. In fact, Theorem 1 is also true for infinite atomistic complete lattice. However, many preparatory works for Theorem 2 of this paper are proved with inductive method. This illustrates that Theorem 2 cannot be directly generalized to the infinite case. Even though, we may hope that the results of this paper will assist the solution of S.Radeleczki's open problem for infinite cases. We intend to pursue this line of research in the future.

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