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ON A MAX TYPE RECURSIVE SEQUENCE OF ORDER THREE

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Abstract. Our aim in this paper is to investigate the behavior of the solution of the following max type difference equation of order three

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two.

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1. Introduction

Our purpose in this paper is to study the behavior of the solution of the following max type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two.

Nonlinear rational difference equations are of great importance in their own right because diverse nonlinear phenomena occurring in science and engineering can be modeled by such equations. Furthermore, the results about such equations offer prototypes towards the development of the basic theory of nonlinear difference equations. See [8–10, 23, 24, 26].

The study of max-type difference equations attracted a considerable attention recently, see, for example, [1, 5, 7, 13–16] and the references therein. This type of difference equations stem from, for example, certain models in automatic control theory (see [17] and [18]). In the beginning of the study of these equations the experts have been focused on the investigation of the behavior of some particular cases of the following difference equation

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}}, \frac{A_n^{(2)}}{x_{n-2}}, \ldots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $A_n^{(i)}$, $i = 1, \ldots, k$, are real sequences (mostly constant or periodic) and the initial values $x_{-1}, \ldots, x_{-k}$ are different from zero (see, e.g., [2, 3, 6, 15] and the references therein).

Cinar et al. [4] dealt with the positive solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\}.$$

Recently, in the paper [12] it was showed that every solution of the third-order max-type difference equation (1.1), where the initial conditions are arbitrary nonzero real numbers and A_n =constant$\in R$, is eventually periodic with period three.

Also, in [11] we proved that every positive solution to the same third order non–autonomous max-type difference equation (1.1), with $\{A_n\}$ is a three-periodic sequence of positive numbers, is periodic with period three. The same result was proved for the same equation but with min-type difference equation.

Simsek et al. [19] investigated the solutions of the following difference equation

$$x_{n+1} = \max\left\{x_{n-1}, \frac{1}{x_{n-1}}\right\}.$$

Also, Simsek [20] studied the behavior of the solutions of the following system of difference equations

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{y_n}{x_n}\right\}, \quad y_{n+1} = \max\left\{\frac{A}{y_n}, \frac{x_n}{y_n}\right\}.$$

Stevic [21] studied the boundedness and global attractivity for the positive solutions of the difference equation

$$x_{n+1} = \max\left\{c, \frac{x_n^p}{x_{n-1}^p}\right\}.$$

In [25] Yalcinkaya et al. investigated the periodic nature of the solution of the max-type difference equation

$$x_{n+1} = \max\{x_n, A\}/x_n^2 x_{n-1}.$$

See also [22], [27].

Definition 1. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there is $n_0 \in \{-k, \ldots, -1, 0, 1, \ldots\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $\{x_n\}_{n=-k}^{\infty}$ is periodic with period p.

Remark 1. Note that if $A_n = 0$, then Eq. (1.1) becomes $x_{n+1} = x_{n-2}$, from which it follows that every solution is periodic with period three. Hence, in the sequel we will consider the case $A_n \neq 0$.

Remark 2. Note that if $A_n = A$, then Eq. (1.1) becomes $x_{n+1} = \max\left\{\frac{A}{x_n}, x_{n-2}\right\}$, it was shown that this equation is periodic with period three see [12]. Hence, in the sequel we will consider the case $A_n = \{A_0, A_1, A_0, A_1, ...\}$, $A_1 \neq A_0$.

2. Main results

2.1. *The Case* $A_0 > A_1$

Theorem 1. *Consider the difference equation (1.1) with $A_0 > A_1$. Then every solution of Eq. (1.1) is eventually periodic with period three.*

Proof. From Eq.(1.1), we see that

$$x_1 = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\}.$$

We consider two cases

Case (a_1) If $x_{-2}x_0 < A_0$, then $x_1 = \dfrac{A_0}{x_0}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\}.$$

(a_{11}) If $x_{-1}A_0 < A_1 x_0$, then $x_2 = \dfrac{A_1 x_0}{A_0}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, x_0\right\}.$$

(a_{111}) If $A_1 x_0^2 < A_0^2$, then $x_3 = \dfrac{A_0^2}{A_1 x_0}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2 x_0}{A_0^2}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$(\text{since } A_1 x_0^2 < A_0^2 \quad \Rightarrow \quad A_1^2 x_0^2 < A_0 A_1 x_0^2 < A_0^3)$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1 x_0}{A_0}\right\} = x_0,$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0^2}{A_1 x_0}\right\} = \frac{A_0^2}{A_1 x_0},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0^2}{A_1 x_0}\right\} = \frac{A_0^2}{A_1 x_0},$$

and the solution has the following form

$$\left\{\frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}, x_0, \frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}, x_0, \frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}, \ldots\right\}.$$

(a_{112}) If $A_1 x_0^2 > A_0^2$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1 x_0}{A_0}\right\} = x_0,$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, x_0\right\} = x_0,$$

(since $A_1 x_0^2 > A_0^2 > A_1^2$)

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, x_0\right\} = x_0,$$

(since $A_0 x_0^2 > A_1 x_0^2 > A_0^2$)

then the solution is eventually periodic with period three. Moreover, it has the following form

$$\left\{\frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, \ldots\right\}.$$

(a_{12}) If $x_{-1} A_0 > A_1 x_0$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{121}) If $x_0 < \dfrac{A_0}{x_{-1}}$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1 x_{-1}}{A_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$\left(\text{since } x_0 < \frac{A_0}{x_{-1}} \quad \Rightarrow A_1 x_0 < \frac{A_1 A_0}{x_{-1}} < \frac{A_0^2}{x_{-1}}\right),$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\},$$

(a_{1211}) If $x_0 < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_{-1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}}.$$

Clearly, the derived solution is eventually periodic with period three and takes the form

$$\left\{\frac{A_0}{x_0}, x_{-1}, \frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_{-1}, \frac{A_0}{x_{-1}}, \ldots\right\}.$$

(a_{1212}) If $x_0 > x_{-1}$, then $x_5 = x_0$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$(\text{since } x_0 > x_{-1} \quad \Rightarrow A_0 x_0 > A_0 x_{-1} > A_1 x_{-1}),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_{-1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{\frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_0, \frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_0, \ldots\right\}.$$

(a_{122}) If $A_0 < x_{-1} x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\},$$

(a_{1221}) If $x_0 < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > \frac{A_1}{x_{-1}}\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$

$$(\text{since } x_0 < x_{-1} \Rightarrow A_1 x_0 < A_0 x_0 < A_0 x_{-1}),$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

the presented solution has the form

$$\left\{x_{-1}, x_0, \frac{A_0}{x_0}, x_{-1}, x_0, \frac{A_0}{x_0}, \ldots\right\}.$$

(a_{1222}) If $x_0 > x_{-1}$, then $x_5 = x_0$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > \frac{A_1}{x_{-1}} \;\&\; x_0 > x_{-1} \;\Rightarrow x_0^2 > x_0 x_{-1} > A_1\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, x_0\right\} = x_0,$$

we obtain that the solution is eventually periodic with period three and the developed solution has the following form

$$\left\{x_0, \frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, x_0, x_0, \ldots\right\}.$$

Case (a_2) If $x_{-2}x_0 > A_0$, then $x_1 = x_{-2}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\}.$$

(a_{21}) If $x_{-1}x_{-2} < A_1$, then $x_2 = \dfrac{A_1}{x_{-2}}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\}.$$

(a_{211}) If $A_1 x_0 < A_0 x_{-2}$, then $x_3 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, x_{-2}\right\} = x_{-2},$$

$$\text{(since } A_1 x_0 < A_0 x_{-2} \;\Rightarrow \frac{1}{x_0} > \frac{A_1}{A_0 x_{-2}} \,\&\, x_{-2} > \frac{A_0}{x_0}$$

$$\Rightarrow\; x_{-2} > \frac{A_0}{x_0} > \frac{A_0 A_1}{A_0 x_{-2}} > \frac{A_1^2}{A_0 x_{-2}}),$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-2}\right\} = x_{-2},$$

(since $A_0 x_{-2}^2 > A_0 A_1$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

the resulting solution is then eventually periodic with period three in the closed form

$$\left\{\frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0}{x_{-2}}, \ldots\right\}.$$

(a_{212}) If $A_1 x_0 > A_0 x_{-2}$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\} = x_{-2},$$

(since $x_0 x_{-2} > A_0 > A_1$),

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, x_0\right\} = x_0,$$

(since $A_0x_0 > A_1x_0 > A_0x_{-2} > A_1x_{-2}$),

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\{x_{-2}, x_0\} = x_0,$$

(since $A_1x_0 > A_0x_{-2} > A_1x_{-2}$).

Thus, the solution is

$$\left\{x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, ...\right\}.$$

(a_{22}) If $x_{-1}x_{-2} > A_1$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{221}) If $x_0x_{-1} < A_0$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1x_{-1}}{A_0}, x_{-2}\right\} = x_{-2},$$
$$\left(\text{since } x_{-2} > \frac{A_0}{x_0} > \frac{x_0x_{-1}}{x_0} \Rightarrow A_0x_{-2} > A_1x_{-2} > A_0x_{-1} > A_1x_{-1}\right),$$
$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\},$$

(a_{2211}) If $\dfrac{A_0}{x_{-2}} < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\{x_{-1}, x_{-2}\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

this provides an eventually periodic with period three, which takes the form

$$\left\{x_{-2}, x_{-1}, \frac{A_0}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_0}{x_{-1}}, \ldots\right\}.$$

(a_{2212}) If $\frac{A_0}{x_{-2}} > x_{-1}$, then $x_5 = \frac{A_0}{x_{-2}}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$
$$\left(\text{since } x_{-2}x_{-1} < A_0 \Rightarrow A_1 x_{-2}x_{-1} < A_1 A_0 < A_0^2\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\{x_{-1}, x_{-2}\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

obviously, the solution is eventually periodic with period three, which can be expressed in the form

$$\left\{\frac{A_0}{x_{-1}}, x_{-2}, \frac{A_0}{x_{-2}}, \frac{A_0}{x_{-1}}, x_{-2}, \frac{A_0}{x_{-2}}, \ldots\right\}.$$

(a_{222}) If $A_0 < x_{-1}x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\} = x_{-2},$$
$$(\text{since } x_{-2}x_0 > A_0 > A_1)$$
$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\},$$

(a_{2221}) If $\frac{A_0}{x_{-2}} < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\} = x_0,$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

this in turn gives the eventually periodic solution with period three

$$\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots\}.$$

(a_{2222}) If $\frac{A_0}{x_{-2}} > x_{-1}$, then $x_5 = \frac{A_0}{x_{-2}}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > x_{-2} \Rightarrow A_0 x_0 > A_0 x_{-2} > A_1 x_{-2}\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\{x_{-2}, x_0\} = x_0,$$

this gives the solution

$$\left\{x_0, x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, \frac{A_0}{x_{-2}}, \ldots\right\}.$$

□

2.2. *The Case $A_1 > A_0$*

Theorem 2. *Consider the difference equation (1.1) for $A_1 > A_0$. Then the solution is periodic with period three or the equation has unboundedness solution takes the form:*

$$\{x_{2n-1}\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-2}\left(\frac{A_0}{A_1}\right), x_{-2}\left(\frac{A_0}{A_1}\right)^2, x_{-2}\left(\frac{A_0}{A_1}\right)^3, \ldots\right\},$$

$$\{x_{2n}\}_{n=1}^{\infty} = \left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right), \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^2, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^3, \ldots\right\}.$$

Proof. From Eq.(1.1), we see that

$$x_1 = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\}.$$

We consider the following two cases:

Case (a_1) If $x_{-2}x_0 < A_0$, then $x_1 = \frac{A_0}{x_0}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\}.$$

(a_{11}) If $x_{-1}A_0 < A_1x_0$, then $x_2 = \dfrac{A_1x_0}{A_0}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0^2}{A_1x_0}, x_0\right\}.$$

(a_{111}) If $A_1x_0^2 < A_0^2$, then $x_3 = \dfrac{A_0^2}{A_1x_0}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2x_0}{A_0^2}, \frac{A_0}{x_0}\right\},$$

(a_{1111}) If $A_1^2x_0^2 < A_0^3$, then $x_4 = \dfrac{A_0}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0^2}{A_1x_0}\right\} = \frac{A_0}{x_0},$$
$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_0, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$
$$\left(\text{since } x_0 < \frac{A_0^2}{x_0A_1} < \frac{A_0A_1}{x_0A_1} < \frac{A_0}{x_0}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1x_0}{A_0}, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

then the solution takes the form

$$\left\{\frac{A_0}{x_0}, \frac{A_1x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_0}{x_0}, \frac{A_1x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_0}{x_0}, \ldots\right\},$$

which is eventually periodic with period three.

(a_{1112}) If $A_1^2x_0^2 > A_0^3$, then $x_4 = \dfrac{A_1^2x_0}{A_0^2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^3}{A_1^2x_0}, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$
$$\left(\text{since } A_1^2x_0^2 > A_0^3 \Rightarrow A_1^3x_0^2 > A_1A_0^3 > A_0^4\right),$$

$$x_6=\max\left\{\frac{A_5}{x_5},x_3\right\}=\max\left\{\frac{A_0}{x_0},\frac{A_0^2}{A_1x_0}\right\}=\frac{A_0}{x_0},$$
$$x_7=\max\left\{\frac{A_6}{x_6},x_4\right\}=\max\left\{x_0,\frac{A_1^2x_0}{A_0^2}\right\}=\frac{A_1^2x_0}{A_0^2},$$
$$x_8=\max\left\{\frac{A_7}{x_7},x_5\right\}=\max\left\{\frac{A_0^2}{A_1x_0},\frac{A_1x_0}{A_0}\right\}=\frac{A_1x_0}{A_0},$$
$$x_9=\max\left\{\frac{A_8}{x_8},x_6\right\}=\max\left\{\frac{A_0^2}{A_1x_0},\frac{A_0}{x_0}\right\}=\frac{A_0}{x_0},$$

and the solution has the following form which is period three solution

$$\left\{\frac{A_1^2x_0}{A_0^2},\frac{A_1x_0}{A_0},\frac{A_0}{x_0},\frac{A_1^2x_0}{A_0^2},\frac{A_1x_0}{A_0},\frac{A_0}{x_0},\frac{A_1^2x_0}{A_0^2},\dots\right\}.$$

(a_{112}) If $A_1x_0^2>A_0^2$, then $x_3=x_0$ and

$$x_4=\max\left\{\frac{A_3}{x_3},x_1\right\}=\max\left\{\frac{A_1}{x_0},\frac{A_0}{x_0}\right\}=\frac{A_1}{x_0},$$
$$x_5=\max\left\{\frac{A_4}{x_4},x_2\right\}=\max\left\{\frac{A_0x_0}{A_1},\frac{A_1x_0}{A_0}\right\}=\frac{A_1x_0}{A_0},$$
$$x_6=\max\left\{\frac{A_5}{x_5},x_3\right\}=\max\left\{\frac{A_0}{x_0},x_0\right\},$$

(a_{1121}) If $x_0^2>A_0$, then $x_6=x_0$ and

$$x_7=\max\left\{\frac{A_6}{x_6},x_4\right\}=\max\left\{\frac{A_0}{x_0},\frac{A_1}{x_0}\right\}=\frac{A_1}{x_0},$$
$$x_8=\max\left\{\frac{A_7}{x_7},x_5\right\}=\max\left\{x_0,\frac{A_1x_0}{A_0}\right\}=\frac{A_1x_0}{A_0},$$
$$x_9=\max\left\{\frac{A_8}{x_8},x_6\right\}=\max\left\{\frac{A_0^2}{A_1x_0},x_0\right\}=x_0,$$

then the solution is eventually periodic with period three and the solution has the following expression:

$$\left\{\frac{A_1x_0}{A_0},x_0,\frac{A_1}{x_0},\frac{A_1x_0}{A_0},x_0,\frac{A_1}{x_0},\dots\right\}.$$

(a_{1122}) If $x_0^2 < A_0$, then $x_6 = \dfrac{A_0}{x_0}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_0, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{x_0, \frac{A_1 x_0}{A_0}\right\} = \frac{A_1 x_0}{A_0},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

then this solution can be written as

$$\left\{\frac{A_1}{x_0}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1}{x_0}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1}{x_0}, \ldots\right\},$$

which is eventually periodic with period three.

(a_{12}) If $x_{-1} A_0 > A_1 x_0$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{121}) If $x_0 < \dfrac{A_0}{x_{-1}}$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1 x_{-1}}{A_0}, \frac{A_0}{x_0}\right\},$$

(a_{1211}) If $\dfrac{A_1 x_{-1}}{A_0} < \dfrac{A_0}{x_0}$, then $x_4 = \dfrac{A_0}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$(\text{since } A_0 x_{-1} > A_1 x_0 > A_0 x_0),$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$
$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$
$$\left(\text{since } x_0 x_{-1} < A_0 < A_1 \quad \Rightarrow A_0 x_{-1} < \frac{A_1 A_0}{x_0}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{\frac{A_0}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_0}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{1212}) If $\frac{A_1 x_{-1}}{A_0} > \frac{A_0}{x_0}$, then $x_4 = \frac{A_1 x_{-1}}{A_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2}{A_1 x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} \quad \& \quad x_0 > \frac{A_0^2}{A_1 x_{-1}} \Rightarrow x_{-1} > \frac{A_1 A_0^2}{A_0 A_1 x_{-1}} > \frac{A_0^2}{A_1 x_{-1}}\right)$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1 x_{-1}}{A_0}\right\} = \frac{A_1 x_{-1}}{A_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } A_1 x_{-1}^2 > A_0 A_1\right),$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

then the solution is eventually periodic with period three. Moreover, it has the following form:

$$\left\{\frac{A_1 x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1 x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{122}) If $A_0 < x_{-1} x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0 x_0}{A_1}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} > \frac{A_0 x_0}{A_1}\right),$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\}.$$

(a_{1221}) If $A_1 < x_0 x_{-1}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} > \frac{A_0 x_0}{A_0} = x_0\right),$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

the developed solution has the following form

$$\left\{x_{-1}, x_0, \frac{A_1}{x_0}, x_{-1}, x_0, \frac{A_1}{x_0}, \ldots\right\}.$$

(a_{1222}) If $A_1 > x_0 x_{-1}$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$\left(\text{since } A_1 > x_0 x_{-1} \;\Rightarrow\; A_1^2 > A_1 A_0 > A_0 x_0 x_{-1}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} > \frac{A_0 x_0}{A_0} = x_0\right),$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we get the solution which is eventually periodic with period three. It has the following form

$$\left\{\frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

Case (a_2) If $x_{-2} x_0 > A_0$, then $x_1 = x_{-2}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\}.$$

(a_{21}) If $x_{-1} x_{-2} < A_1$, then $x_2 = \dfrac{A_1}{x_{-2}}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\}.$$

(a_{211}) If $A_1 x_0 > A_0 x_{-2}$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\}.$$

(a_{2111}) If $A_1 > x_0 x_{-2}$, then $x_4 = \dfrac{A_1}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0 x_0}{A_1}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$(\text{since } A_1 > x_0 x_{-2} \Rightarrow> A_1^2 > A_1 x_0 x_{-2} > A_0 x_0 x_{-2}),$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\{x_{-2}, x_0\}.$$

(a_{21111}) If $x_0 < x_{-2}$, then $x_6 = x_{-2}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$(\text{since } A_1 x_{-2} > A_1 x_0 > A_0 x_0),$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0 x_0}{A_1}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_{-2}\right\} = x_{-2},$$

the solution of the form

$$\left\{x_{-2}, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_{-2}, \ldots\right\},$$

is eventually periodic with period three.

(a_{21112}) If $x_0 > x_{-2}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{x_0, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\} = x_0,$$

this shows that the solution

$$\left\{x_0, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_0, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_0, \ldots\right\},$$

is just eventually periodic with period three.

(a_{2112}) If $A_1 < x_0 x_{-2}$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\{x_{-2}, x_0\}.$$

(a_{21121}) If $x_0 < x_{-2}$, then $x_6 = x_{-2}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-2}\right\} = x_{-2},$$
$$(\text{since } x_{-2} > \frac{A_1}{x_0} > \frac{A_1}{x_{-2}} > \frac{A_0}{x_{-2}}),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_{-2}\right\} = x_{-2},$$

this gives the solution

$$\left\{x_{-2}, \frac{A_1}{x_{-2}}, x_{-2}, x_{-2}, \frac{A_1}{x_{-2}}, x_{-2}, x_{-2}, \ldots\right\}.$$

(a_{21122}) If $x_0 > x_{-2}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\} = x_0,$$

then the solution is eventually periodic with period three, moreover, it has the following form

$$\left\{x_0, x_{-2}, \frac{A_1}{x_{-2}}, x_0, x_{-2}, \frac{A_1}{x_{-2}}, x_0, \ldots\right\}.$$

(a_{212}) If $A_1 x_0 < A_0 x_{-2}$, then $x_3 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, x_{-2}\right\}.$$

(a_{2121}) If $A_1^2 < A_0 x_{-2}^2$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\}.$$

(a_{21211}) If $A_1^3 < A_0^2 x_{-2}^2$, then $x_6 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-2}\right\} = x_{-2},$$

(since $A_0 x_{-2}^2 > A_1^2 > A_0 A_1$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

and the solution has the following expression

$$\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, \ldots\right\}.$$

(a_{21212}) If $A_1^3 > A_0^2 x_{-2}^2$, then $x_6 = \dfrac{A_1^2}{A_0 x_{-2}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0^2}{A_1^2} x_{-2}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1^2}{A_0 x_{-2}}\right\} = \frac{A_1^2}{A_0 x_{-2}},$$

also the the solution has the following period three as follows

$$\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_1^2}{A_0 x_{-2}}, x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_1^2}{A_0 x_{-2}}, x_{-2}, \ldots\right\}.$$

(a_{2122}) If $A_1^2 > A_0 x_{-2}^2$, then $x_4 = \dfrac{A_1^2}{A_0 x_{-2}}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2 x_{-2}}{A_1^2}, \frac{A_1}{x_{-2}}\right\}.$$

Case (i) If we take $x_5 = \dfrac{A_1}{x_{-2}}$. Then the solution has the following form:

$$\left\{\frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1^2}{A_0 x_{-2}}, \frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1^2}{A_0 x_{-2}}, \frac{A_1}{x_{-2}}, x_{-2}, \ldots\right\}.$$

Case (ii) If $x_5 = \dfrac{A_0^2 x_{-2}}{A_1^2}$, then the solution will be periodic with period three or being in the following form

$$\{x_{2n-1}\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-2}\left(\frac{A_0}{A_1}\right), x_{-2}\left(\frac{A_0}{A_1}\right)^2, x_{-2}\left(\frac{A_0}{A_1}\right)^3, \ldots\right\},$$

$$\{x_{2n}\}_{n=1}^{\infty} = \left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right), \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^2, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^3, \ldots\right\}.$$

which is unboundedness solution.

(a_{22}) If $x_{-1}x_{-2} > A_1$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{221}) If $x_0 x_{-1} < A_0$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1 x_{-1}}{A_0}, x_{-2}\right\}.$$

(a_{2211}) If $A_0 x_{-2} < A_1 x_{-1}$, then $x_4 = \dfrac{A_1 x_{-1}}{A_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2}{A_1 x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } x_{-1} > \frac{A_1}{x_{-2}} \ \ \& \ \ A_1 x_{-1} > A_0 x_{-2} \Rightarrow A_1 x_{-1}^2 > A_0 A_1 > A_0^2\right),$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1 x_{-1}}{A_0}\right\} = \frac{A_1 x_{-1}}{A_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$(\text{since } A_1 x_{-1}^2 > A_1 A_0),$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}}.$$

Obviously, the solution is eventually periodic with period three, which can be expressed in the form

$$\left\{x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1 x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1 x_{-1}}{A_0}, \ldots\right\}.$$

(a_{2212}) If $A_0x_{-2} > A_1x_{-1}$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0x_{-1}}{A_1}, x_{-2}\right\} = x_{-2},$$

(since $x_{-2} > \frac{A_1x_{-1}}{A_0} > \frac{A_0x_{-1}}{A_1}$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

this provides an eventually periodic with period three, which takes the form

$$\left\{\frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, ...\right\}.$$

(a_{222}) If $A_0 < x_{-1}x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\}.$$

(a_{2221}) If $A_1 < x_{-2}x_0$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\}.$$

(a_{22211}) If $\frac{A_1}{x_{-1}} < x_0$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

and the solution has the following form

$$\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

(a_{22212}) If $\dfrac{A_1}{x_{-1}} < x_0$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, x_{-2}\right\} = x_{-2},$$
$$(\text{since } x_{-2}x_{-1} > A_1 > x_0 x_{-1} \Rightarrow x_{-2} > x_{-1} \Rightarrow A_1 x_{-2} > A_1 x_{-1} > A_0 x_{-1}),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{2222}) If $A_1 > x_{-2}x_0$, then $x_4 = \dfrac{A_1}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0 x_0}{A_1}, x_{-1}\right\} = x_{-1},$$
$$(\text{since } x_{-2}x_{-1} > A_1 > x_0 x_{-2} \Rightarrow x_{-1} > x_0 \Rightarrow A_1 x_{-1} > A_1 x_0 > A_0 x_0),$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_{-1}\right\}.$$

(a_{22221}) If $\dfrac{A_1}{x_{-1}} > x_{-1}$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$\left(\text{since } A_1 > x_{-1}^2 > x_{-1}x_0 \Rightarrow A_1^2 x_{-1} > A_1 x_{-1} x_0 > A_0 x_{-1} x_0\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}}.$$

Then the solution is eventually periodic with period three and has the following form

$$\left\{\frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{22222}) If $\frac{A_1}{x_{-1}} < x_{-1}$, then $x_6 = x_{-1}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

we obtain that the solution is eventually periodic with period three and takes the form

$$\left\{x_{-1}, x_{-1}, \frac{A_1}{x_0}, x_{-1}, x_{-1}, \frac{A_1}{x_0}, \ldots\right\}.$$

□

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