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FIXED POINT THEOREMS IN UNIFORM SPACE ENDOWED WITH GRAPH

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Abstract. In this paper, we shall introduce the concepts of F_G-contraction and ψ_G-contraction in uniform space endowed with graph to investigate the existence of a fixed point of mappings satisfying these notions. We shall also introduce a common fixed point theorem for pair of mappings satisfying the notion of ψ_G-contraction in uniform space endowed with graph.

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1. INTRODUCTION

Very recently, Wardowski [19] categorized the family of mappings into a new family, denoted by F or $\mathfrak{F}$, to introduce a new contraction condition, the F-contraction. As it is expected, it extend the famous Banach contraction mapping principle. Inspired the result of Wardowski, [19], a number of different approaches has been reported, see [8, 13, 15–17] and related references therein.

On the other hand, Czerwik [9] extended the notion of metric by introducing a new notion, b-metric. Indeed, Czerwik [9] modified the triangle inequality that causes to several differences in the related topology, and hence in convergence of a sequence, Hausdorffness of the topology etc. We also mention that Jachymski [10] successively set-up the Banach contraction mapping principle in the frame of a complete metric space endowed with a directed graph, Jachymski [10]. Later, this interesting contribution has been appreciated by several authors [5, 6, 11, 12, 14, 18].

The purpose of this paper is to introduce and discuss some new fixed point theorems and common fixed point theorems for mappings in uniform space endowed with graph satisfying the F_G-contraction, and ψ_G-contraction. Further, we introduce and use the notion of E_s-distance in this paper.

Some basic definitions and fundamental results are recollected for the sake of completeness. Let X be a nonempty set. A nonempty family, ϑ of subsets of $X \times X$ is called the uniform structure of X, if it satisfies the following properties:

(i) if G is in ϑ, then G contains the diagonal $\{(x,x)|x \in X\}$;
(ii) if G is in ϑ and H is a subset of $X \times X$ which contains G, then H is in ϑ;
(iii) if G and H are in ϑ, then $G \cap H$ is in ϑ;
(iv) if G is in ϑ, then there exists H in ϑ, such that, whenever (x,y) and (y,z) are in H, then (x,z) is in G;
(v) if G is in ϑ, then $\{(y,x)|(x,y) \in G\}$ is also in ϑ.

The pair (X,ϑ) is called a uniform space and the element of ϑ is called entourage or neighborhood or surrounding. The pair (X,ϑ) is called a quasiuniform space (see e.g. [20]) if property (v) is omitted

Let $\Delta = \{(x,x)|x \in X\}$ be the diagonal of a non-empty set X. For $V, W \in X \times X$, we shall use the following setting in the sequel

$$V \circ W = \{(x,y)| \text{ there exists } z \in X : (x,z) \in W \text{ and } (z,y) \in V\}$$

and

$$V^{-1} = \{(x,y)|(y,x) \in V\}.$$

For a subset $V \in \vartheta$, a pair of points x and y are said to be V-close if $(x,y) \in V$ and $(y,x) \in V$. Moreover, a sequence $\{x_n\}$ in X is called a Cauchy sequence for ϑ, if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n,m \geq N$. For (X,ϑ), there is a unique topology $\tau(\vartheta)$ on X generated by $V(x) = \{y \in X|(x,y) \in V\}$ where $V \in \vartheta$.

A sequence $\{x_n\}$ in X is convergent to x for ϑ, denoted by $\lim_{n\to\infty} x_n = x$, if for any $V \in \vartheta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$. A uniform space (X,ϑ) is called Hausdorff if the intersection of all the $V \in \vartheta$ is equal to Δ of X, that is, if $(x,y) \in V$ for all $V \in \vartheta$ implies $x = y$. If $V = V^{-1}$ then we shall say that a subset $V \in \vartheta$ is symmetrical. Throughout the paper, we shall assume that each $V \in \vartheta$ is symmetrical. For more details, see e.g. [1–4]

Now, we shall recall the notions of A-distance and E-distance.

Definition 1. [2,3] Let (X,ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0,\infty)$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$.

Definition 2. [2,3] Let (X,ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0,\infty)$ is said to be an E-distance if

(i) p is an A-distance,
(ii) $p(x,y) \leq p(x,z) + p(z,y), \ \forall x,y,z \in X$.

Example 1. [2, 3] Let (X,ϑ) be a uniform space and let d be a metric on X. It is evident that (X,ϑ_d) is a uniform space where ϑ_d is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x,y) \in X^2|d(x,y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X,ϑ).

Lemma 1. *[2, 3] Let (X,ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0. Then, for $x,y,z \in X$, the following results hold:*

(a) *If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.*
(b) *If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*
(c) *If $p(x_n,x_m) \leq \alpha_n$ for all $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X,ϑ).*

Let p be an A-distance. A sequence in a uniform space (X,ϑ) with an A-distance is said to be a p-Cauchy if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n,x_m) < \epsilon$ for all $n,m \geq n_0$.

Definition 3. [2, 3] Let (X,ϑ) be a uniform space and p be an A-distance on X.

(i) X is S-complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} p(x_n,x) = 0$.
(ii) X is p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$.
(iii) $T : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n,x) = 0$ implies $\lim_{n\to\infty} p(T(x_n),T(x)) = 0$.

Remark 1. Let (X,ϑ) be a Hausdorff uniform space which is S-complete. If a sequence $\{x_n\}$ be a p-Cauchy sequence, then we have $\lim_{n\to\infty} p(x_n,x) = 0$. Regarding Lemma 1(b), We derive that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ and hence S-completeness implies p-Cauchy completeness.

Cosentino *et al*. [7] modify the $\mathfrak{F}$-family introduced by Wardowski [19], in the setting of b-metric spaces as follows:

Definition 4. [7] Let $s \geq 1$ be a real number. Denote by $\mathfrak{F}_s$ the family of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is strictly increasing, that is, for each $a_1,a_2 \in (0,\infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$;
(F_2) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$;
(F_3) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers with $\lim_{n\to\infty} \mathfrak{d}_n = 0$, there exists $k \in (0,1)$ such that $\lim_{n\to\infty} {\mathfrak{d}_\mathrm{n}}^k F(\mathfrak{d}_\mathrm{n}) = 0$.
(F_4) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers such that $\tau + F(s\mathfrak{d}_n) \leq F(\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n\mathfrak{d}_n) \leq F(s^{n-1}\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$.

Cosentino *et al*. [7] also showed that the following functions belong to $\mathfrak{F}_s$.

- $F(x) = x + \ln x$, for each $x > 0$.
- $F(x) = \ln x$, for each $x > 0$.

2. Main results

We will start this section with the following definition:

Definition 5. Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an E_s-distance if

(i) p is an A-distance,
(ii) $p(x, y) \leq s[p(x, z) + p(z, y)]$, $\forall x, y, z \in X$ and for some $s \geq 1$.

Example 2. Let (X, ϑ) be a uniform space and let d be a b-metric on X. Then clearly, (X, ϑ_d) is a uniform space where ϑ_d is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x, y) \in X^2 | d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E_s-distance on (X, ϑ).

Throughout this section, we assume that $G = (V, E)$ is a directed graph such that the set of its vertices V coincides with X (i.e., $V = X$) and the set of its edges E is such that $E \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Further, assume that G has no parallel edges. A mapping $T : X \to X$ is p_G-continuous if for each sequence $\{x_n\} \subseteq X$ with $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} p(x_n, x) = 0$, then we have $\lim_{n\to\infty} p(Tx_n, Tx) = 0$.

Definition 6. Let (X, ϑ) be a uniform space endowed with the graph G and p is an E_s-distance on X. A mapping $T : X \to X$ is a F_G-contraction, if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$, such that, for each $(x, y) \in E$, we have

$$\tau + F(sp(Tx, Ty)) \leq F(p(x, y)), \tag{2.1}$$

whenever $\min\{p(Tx, Ty), p(x, y)\} > 0$.

Theorem 1. *Let (X, ϑ) be a S-complete Hausdorff uniform space endowed with the graph G and p is an E_s-distance on X. Let $T : X \to X$ is an F_G-contraction satisfying the following conditions:*

(i) *T is edge preserving, that is, for $(x, y) \in E$, we have $(Tx, Ty) \in E$;*
(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;*
(iii) *T is p_G-continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By hypothesis (ii), there exists $x_0 \in X$ such that $(x_0, x_1) = (x_0, Tx_0) \in E$. From (2.1), we have

$$\tau + F(sp(x_1, x_2)) = \tau + F(sp(Tx_0, Tx_1)) \leq F(p(x_0, x_1)) \tag{2.2}$$

As T is edge preserving, for $(x_0, x_1) \in E$, we have $(x_1, x_2) \in E$, From (2.1), we have

$$\tau + F(sp(x_2, x_3)) = \tau + F(sp(Tx_1, Tx_2)) \leq F(p(x_1, x_2)) \tag{2.3}$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n = Tx_{n-1},\ x_{n-1} \neq x_n \text{ and } (x_{n-1}, x_n) \in E \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$\tau + F(sp(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)) \text{ for each } n \in \mathbb{N}. \tag{2.4}$$

By using the property F_4, we have

$$\tau + F(s^n p(x_n, x_{n+1})) \leq F(s^{n-1} p(x_{n-1}, x_n)).$$

Letting $p_n = p(x_n, x_{n+1})$, for each $n \in \mathbb{N}$ and after some simplification, we have

$$F(s^n p_n) \leq F(p_0) - n\tau \text{ for each } n \in \mathbb{N}. \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we get $\lim_{n\to\infty} F(s^n p_n) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} s^n p_n = 0$. From (F_3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (s^n p_n)^k F(s^n p_n) = 0.$$

From (2.5) we have

$$(s^n p_n)^k F(s^n p_n) - (s^n p_n)^k F(p_0) \leq -(s^n p_n)^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{2.6}$$

Letting $n \to \infty$ in (2.6), we get

$$\lim_{n\to\infty} n(s^n p_n)^k = 0. \tag{2.7}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^n p_n)^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$s^n p_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{2.8}$$

To show that $\{x_n\}$ is a p-Cauchy sequence, consider

$$S_n = \sum_{i=n_1}^{n} \frac{1}{i^{1/k}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, there exists $S \in [0,\infty)$ such that $\lim_{n\to\infty} S_n = S$. Consider $m,n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.8), we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &= \sum_{i=n_1}^{m-1} s^i p_i - \sum_{i=n_1}^{n-1} s^i p_i \\ &\leq S_{m-1} - S_{n-1}. \end{aligned}$$

Thus, $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. In a similar way, we show that $\lim_{n,m\to\infty} p(x_m, x_n) = 0$. Thus, $\{x_n\}$ is a p-Cauchy sequence. As (X, ϑ) is S-complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} p(x_n, x^*) = 0$. By condition (iii), when T is p_G-continuous, we have $\lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0$. As $\lim_{n\to\infty} p(x_n, x^*) = 0$ and $\lim_{n\to\infty} p(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. By condition (iii), when we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$. From (2.1), we have

$$\tau + F(sp(Tx_n, Tx^*)) \leq F(p(x_n, x^*)).$$

This implies that $sp(x_{n+1}, Tx^*) < p(x_n, x^*)$. Letting $n \to \infty$, we have $\lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0$. Again, by Lemma 1-(a) we have $x^* = Tx^*$. □

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} s^n \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Definition 7. Let (X, ϑ) be a uniform space endowed with the graph G and p is an E_s-distance on X. A mapping $T : X \to X$ is a ψ_G-contraction mapping if for each $(x, y) \in E$, we have

$$p(Tx, Ty) \leq \psi(p(x, y)) \tag{2.9}$$

where $\psi \in \Psi$.

Theorem 2. *Let (X, ϑ) be a S-complete Hausdorff uniform endowed with the graph G and p is an E_s-distance on X. Let $T : X \to X$ is a ψ_G-contraction mapping satisfying the following conditions:*

(i) *T is edge preserving, that is, for $(x, y) \in E$, we have $(Tx, Ty) \in E$;*
(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;*
(iii) *T is p_G-continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By hypothesis (ii) of theorem we have $x_0 \in X$ such that $(x_0, Tx_0) \in E$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is edge preserving, we have

$$(x_0, x_1) = (x_0, Tx_0) \in E \Rightarrow (Tx_0, Tx_1) = (x_1, x_2) \in E.$$

Inductively, we have

$$(x_n, x_{n+1}) \in E, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.10}$$

From (2.9) and (2.10), it follows that for all $n \in \mathbb{N} \cup \{0\}$, we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \psi(p(x_n, x_{n+1})). \tag{2.11}$$

Iteratively, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}.$$

Since p is an E_s-distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &\leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \end{aligned}$$

To show that $\{x_n\}$ is a p-Cauchy sequence, consider

$$S_n = \sum_{k=0}^{n} s^k \psi^k(p(x_0, x_1)).$$

Thus from (2) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{2.12}$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Thus by (2.12) we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.13}$$

Since p is not symmetric then by repeating the same argument we have $\lim_{n,m\to\infty} p(x_m, x_n) = 0$. Hence the sequence $\{x_n\}$ is a p-Cauchy in the S-complete space X. Thus, there exists $x^* \in X$ such that $\lim_{n\to\infty} p(x_n, x^*) = 0$. By condition (iii), when we have T is p_G-continuous, we get $\lim_{n\to\infty} p(Tx_n, Tx^*) = 0$ which implies that $\lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0$. Hence we have $\lim_{n\to\infty} p(x_n, x^*) = 0$ and $\lim_{n\to\infty}(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. By condition (iii), when we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$, then from (2.9)

$$p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*) \leq \psi(p(x_n, x^*)) < p(x_n, x^*). \tag{2.14}$$

Letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0$. Thus by repeating the same arguments as above we have $x^* = Tx^*$. $\square$

Example 3. Let $X = [0,1]$ be endowed with a graph $G = (V, E)$ with $V = X$ and $E = \big\{(x, y) : x, y \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\big\} \cup \{(x, x) : x \in X\}$, and b-metric $d(x, y) = (x - y)^2$ with $s = 2$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \to X$ by

$$Tx = \begin{cases} 0 \ \text{ if } x = 0 \\ \frac{1}{3n+1} \ \text{ if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} \ \text{ otherwise.} \end{cases} \tag{2.15}$$

Take $\psi(t) = \frac{t}{3}$ for all $t \geq 0$. It is easy to see that T is edge preserving and ψ_G-contraction. Also for $x_0 = \frac{1}{2}$ we have $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Moreover for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $(x_{n-1}, x_n) \in E$ for each $n \in \mathbb{N}$ we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$. Therefore by Theorem 2, T has a fixed point.

To investigate the uniqueness of a fixed point, we consider the following condition:

(H)**:** For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $(z, x) \in E$ and $(z, y) \in E$.

Here, $\mathrm{Fix}(T)$ denotes the set of all fixed points of T.

The following theorem guarantees the uniqueness of a fixed point.

Theorem 3. *Adding the condition (H) in the hypothesis of Theorem 2, we obtain the uniqueness of fixed point of T.*

Proof. Suppose, on the contrary, that $u, v \in X$ are two distinct fixed points of T. From (H), there exists $z \in X$ such that

$$(z, u) \in E \text{ and } (z, v) \in E. \tag{2.16}$$

By using the fact that T is edge preserving, from (2.16), we have

$$(T^n z, u) \in E \text{ and } (T^n z, v) \in E, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.17}$$

We define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n = T^n z_0$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_0 = z$. From (2.17) and (2.9), we have

$$p(z_{n+1}, u) = p(Tz_n, Tu) \leq \psi(p(z_n, u)), \tag{2.18}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$p(z_n, u) \leq \psi^n(p(z_0, u)), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} p(z_n, u) = 0. \tag{2.19}$$

Similarly, we have

$$\lim_{n \to \infty} p(z_n, v) = 0. \tag{2.20}$$

From (2.19) and (2.20) together with Lemma 1-(a), it follows that $u = v$. Thus, fixed point of T is unique. $\square$

Definition 8. Let (X, ϑ) be a uniform space endowed with the graph G. A pair of two self mappings $T, S : X \to X$ is said to be a ψ_G-contraction pair if for each $(x, y) \in E$, we have

$$\max\{p(Tx, Sy), p(Sx, Ty)\} \leq \psi(p(x, y)), \tag{2.21}$$

where $\psi \in \Psi$.

Theorem 4. *Let (X, ϑ) be a S-complete Hausdorff uniform space endowed with the graph G and p is an E_s-distance on X. Suppose that the pair of $T, S : X \to X$ is ψ_G-contraction pair satisfying the following conditions.*

(i) *(T, S) is edge preserving pair, that is, for each $(x, y) \in E$, we have $(Tx, Sy) \in E$ and $(Sx, Ty) \in E$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;*

(iii) *for any sequence* $\{x_n\}$ *in* X *with* $x_n \to x$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E$ *for each* $n \in \mathbb{N} \cup \{0\}$, *then* $(x_n, x) \in E$ *for each* $n \in \mathbb{N} \cup \{0\}$.

Then T *and* S *have a common fixed point.*

Proof. By hypothesis (ii) of theorem, we have $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Since (T, S) is an edge preserving pair, then we can construct a sequence such that

$Tx_{2n} = x_{2n+1}, \;\; Sx_{2n+1} = x_{2n+2}$ and $(x_n, x_{n+1}) \in E, \; (x_{n+1}, x_n) \in E, \mathbb{N} \cup \{0\}$,

for all $n \in \mathbb{N} \cup \{0\}$. From (2.21) for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &= p(Tx_{2n}, Sx_{2n+1}) \\ &\leq \max\{p(Tx_{2n}, Sx_{2n+1}), p(Sx_{2n}, Tx_{2n+1})\} \\ &\leq \psi(p(x_{2n}, x_{2n+1})). \end{aligned}$$

Hence, we conclude that

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})). \tag{2.22}$$

Similarly, we get

$$\begin{aligned} p(x_{2n+2}, x_{2n+3}) &= p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq \max\{p(Tx_{2n+1}, Sx_{2n+2}), p(Sx_{2n+1}, Tx_{2n+2})\} \\ &\leq \psi(p(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Hence, we have

$$p(x_{2n+2}, x_{2n+3}) \leq \psi(p(x_{2n+1}, x_{2n+2})). \tag{2.23}$$

Thus from (2.22) and (2.23), and by induction, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}. \tag{2.24}$$

Now we show that $\{x_n\}$ is a p-Cauchy sequence. Since p is an E_s-distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &\leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \end{aligned}$$

Now, we shall consider

$$S_n = \sum_{k=0}^{n} s^k \psi^k(p(x_0, x_1)).$$

Thus, from (2) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{2.25}$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n \to \infty} S_n = S$. Thus, by (2.25) we have

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{2.26}$$

Since p is not symmetric then by repeating the same argument we have

$$\lim_{n,m\to\infty} p(x_m, x_n) = 0. \tag{2.27}$$

Hence the sequence $\{x_n\}$ is a p-Cauchy in the S-complete space X. Thus, there exists $x^* \in X$ such that $\lim_{n\to\infty} p(x_n, x^*) = 0$ which implies $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1} = x^*$. By assumption (iii), we have $(x_n, x^*) \in E$. Thus, by using the triangular inequality and (2.21), we have

$$\begin{aligned} p(x_n, Tx^*) &\le sp(x_n, x_{2n+2}) + sp(x_{2n+2}, Tx^*) \\ &= sp(x_n, x_{2n+2}) + sp(Sx_{2n+1}, Tx^*) \\ &\le sp(x_n, x_{2n+2}) + s\max\{p(Tx_{2n+1}, Sx^*), p(Sx_{2n+1}, Tx^*)\} \\ &\le sp(x_n, x_{2n+2}) + s\psi(p(x_{2n+1}, x^*)) \end{aligned} \tag{2.28}$$

Letting $n \to \infty$ in (2.28), we have $p(x_n, Tx^*) = 0$. Hence we have $\lim_{n\to\infty} p(x_n, x^*) = 0$ and $\lim_{n\to\infty} p(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. Analogously, we can drive $x^* = Sx^*$. Therefore $x^* = Tx^* = Sx^*$. $\square$

Remark 2. Note that Theorem 4 is valid if one replace condition (ii) with

(ii)': there exists $x_0 \in X$ such that $(x_0, Sx_0) \in E$ and $(Sx_0, x_0) \in E$.

Example 4. Let $X = [0,1]$ be endowed with a graph $G = (V, E)$ with $V = X$ and $E = \big\{(x, y) : x, y \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\big\} \cup \{(x, x) : x \in X\}$, and dislocated metric space $d(x, y) = \max\{x, y\}$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$, where $U_\epsilon = \{(x, y) \in X^2 : d(x, y) < d(x, x) + \epsilon\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \to X$ by

$$Tx = \begin{cases} 0 \ \text{ if } x = 0 \\ \frac{1}{2n+1} \ \text{ if } x = \frac{1}{n} : n > 1 \\ x^2 \ \text{ otherwise} \end{cases} \tag{2.29}$$

and $S : X \to X$ by

$$Sx = \begin{cases} 0 \ \text{ if } x = 0 \\ \frac{1}{2n} \ \text{ if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} \ \text{ otherwise} \end{cases} \tag{2.30}$$

Take $\psi(t) = \frac{t}{2}$ for all $t \ge 0$. Further, it is easy to see that (T, S) is edge preserving and ψ_G-contraction pair. Also for $x_0 = \frac{1}{2}$ we have $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Moreover for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$ we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$. Therefore by Theorem 4, T and S have a common fixed point.

To investigate the uniqueness of a common fixed point, we use the following condition.

(I) For each $x, y \in CFix(T, S)$, we have $(x, y) \in E$, where $CFix(T, S)$ is the set of all common fixed points of T and S.

Theorem 5. *Adding the condition (I) in the hypothesis of Theorem 4, we obtain the uniqueness of common fixed point of T and S.*

Proof. On the contrary suppose that $u, v \in X$ are two distinct common fixed points of T and S. From (I) and (2.21) we have

$$p(u,v) = \max\{p(Tu, Sv), p(Su, Tv)\} \leq \psi(p(u,v)) < p(u,v),$$

which is impossible for $p(u,v) > 0$. Consequently, we have $p(u,v) = 0$. Analogously, one can show that $p(u,u) = 0$. Thus we have $u = v$, which is a contradiction to our assumption. Hence T and S have a unique common fixed point. □

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